Purdue University
Purdue e-Pubs

Department of Electrical and Computer Engineering Technical Reports | Department of Electrical and Computer Engineering

3-1-1963

Estimation of Time-Varying Correlation Functions

G. R. Cooper
Purdue University

Helmut Berndt
Purdue University

Follow this and additional works at: https://docs.lib.purdue.edu/ecetr

Cooper, G. R. and Berndt, Helmut, "Estimation of Time-Varying Correlation Functions" (1963). *Department of Electrical and Computer Engineering Technical Reports.* Paper 516.
https://docs.lib.purdue.edu/ecetr/516

This document has been made available through Purdue e-Pubs, a service of the Purdue University Libraries. Please contact epubs@purdue.edu for additional information.

TR-EE 63-1
NATIONAL SCIENCE FOUNDATION CONTRACT G-18997
PRF 2974

PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

Estimation of Time-Varying Correlation Functions

G. R. Cooper, Principal Investigator
H. Berndt

March, 1963
Lafayette, Indiana

SUPPORTED BY
NATIONAL SCIENCE FOUNDATION
WASHINGTON, D.C.

TR-EE63-1

National Science Foundation Contract G-18997

PRF 2974

ESTIMATION OF
TIME-VARYING CORRELATION FUNCTIONS

SUPPORTED BY
NATIONAL SCIENCE FOUNDATION
WASHINGTON, D. C.

by

G. R. Cooper, Principal Investigator

Helmut Berndt

School of Electrical Engineering

Purdue University

Lafayette, Indiana

March, 1963

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

ABSTRACT

Berndt, Helmut, Ph.D., Purdue University, June 1963. Estimation of Time-Varying Correlation Functions. Major Professor: George R. Cooper.

The need for estimating the auto- or crosscorrelation functions of nonstationary random processes frequently arises in communication and self-adaptive systems. In most situations only one sample function can be observed over finite time. It is the purpose of this work to establish a firm theoretical basis for such a measurement of time-varying correlation functions, and the emphasis here is on suitable estimation procedures rather than specific measurement techniques.

Second order stochastic processes are used as a mathematical model. The minimum mean square error between a weighted time average and the true (ensemble average) correlation function is investigated. This procedure leads to an optimum weighting function which can be obtained numerically under the Gaussian assumption. The results of such an analysis justify the much simpler finite integration-time average as an estimate.

By employing a bilinear approximation in time t and delay τ to the true correlation function, the mean value and variance of the simple finite-time average are found. A minimum upper bound on the mean square error is used as a criterion for an optimum observation time of such an estimate. Explicit results, however, require again the Gaussian assumption.

These approaches are based mainly on a strict error analysis. A more direct approach, that leads to approximants which are random variables

with unknown properties, is also outlined. The restrictions and difficulties are discussed.

Examples to support the proposed estimation procedures are presented and explicit results appear mostly in graphical form.

CHAPTER 1

INTRODUCTION

With the rapid advances in space technology and automation, the design of time-varying systems such as space communication or observation and adaptive control systems became a great challenge to present day engineering. A statistical description of the unwanted random disturbances was already essential for proper design of any modern, reliable, and efficient time-invariant system. In the time-varying case even more emphasis must be placed on the statistical parameters of the nondeterministic fluctuations, if the design problem is to be solved successfully. The term "random noise" is often used to describe such fluctuations and this terminology will be followed here.

The theory of describing random noise as a stochastic process based on the laws of probability theory has been well developed for the case of stationary random processes. It allows us to estimate the statistical parameters of the process which are of engineering interest [10] and there has been much emphasis in the late 1940's and through the 1950's on developing measuring devices for estimating correlation functions and spectral densities. The literature on these subjects is extensive, e.g., [2], [6], [25], to cite some of the earlier work, and applications in the design of time-invariant systems are numerous.

While, in general, an analysis of stationary random noise leads to satisfactory and sufficient results for the design of time-invariant systems, the time-varying situation, because of its dynamic nature, requires the consideration of nonstationary random processes, i.e., those whose probability laws change with time.

At present, time-varying system design is still largely based upon assumptions of specific noise characteristics, simply because the analysis of nonstationary stochastic processes and the estimation of their statistical parameters as time functions are not developed sufficiently to yield applicable results. It is hoped that this work might help fill this gap. In the general situation of nonstationary noise a much higher degree of difficulty is encountered than in the stationary case. A mathematical theory, when existing, is still fragmentary and limited to either specific classes of nonstationary stochastic processes or is so general that it is not readily applied to develop practical measuring techniques for the estimation of desired statistical parameters.

For this reason, hardly any work has been done in this area and we are also forced to limit ourselves to just a specific sub-class of nonstationary random processes or noise signals. Aside from technical requirements, we have to rely upon some mathematical properties of the stochastic processes considered in order to derive useful results. It is fortunate that the class of random processes chosen is large enough to include the most likely situations in communications or control systems.

CHAPTER 2

ESTIMATION PROBLEM

The need for estimating the auto- or crosscorrelation functions of nonstationary random processes frequently arises in communication and self-adaptive systems. Among the statistical parameters of unwanted disturbances, correlation functions are by far the most important quantity for design purposes, since they are necessary for system optimization, prediction and signal detection, signal-to-noise ratio, etc. [1] [20].

2.1 Problem Statement

An estimation of auto- or crosscorrelation functions of nonstationary random processes is desired. In most situations only one sample function of the process can be observed over a finite time interval. Thus, the problem has to be considered under this technical requirement.

It is assumed that these correlation functions always exist, and this restriction corresponds to the mathematical assumption of a second order random process, i.e., one with finite second moments. The first moment, or mean value, of such a process is considered to be identically zero. The reason for this simplifying assumption is twofold. In many applications it is not necessary to know the mean value, or it is known that the mean is zero because of physical or technical design features.

It follows, therefore, that the mean $m_x(t)$ and variance $\sigma_x^2(t)$ of a random process are here:

$$m_x(t) = E[x(t)] \equiv 0, \tag{2-1}$$

$$\sigma_x^2(t) = E[x^2(t)] < \infty, \tag{2-2}$$

where E indicates the mathematical expectation. The autocorrelation function may be denoted by

$$\mathcal{R}_{xx}(t_1,t_2) = E[x(t_1)\ x(t_2)], \tag{2-3}$$

and, if two processes x(t) and y(t) are considered, there exist two crosscorrelation functions dependent on the time instants t_1 and t_2:

$$\mathcal{R}_{xy}(t_1,t_2) = E[x(t_1)\ y(t_2)]; \tag{2-4}$$

$$\mathcal{R}_{yx}(t_1,t_2) = E[y(t_1)\ x(t_2)]. \tag{2-5}$$

The emphasis is on the estimation of autocorrelation functions. When permissible, however, the approach will be extended to include crosscorrelation functions. In this case the function symbol will be used without subscripts and the term correlation function refers to either one or both types of correlation functions.

2.2 Definitions

From a technical viewpoint, it is desirable to introduce a delay variable τ, $\tau > 0$, together with a single time variable t. This leads to two more definitions of correlation functions, and both will have to be used.

First, one defines

$$t = t_1, \tag{2-6}$$

$$\tau = t_1 - t_2, \tag{2-7}$$

such that the autocorrelation function becomes

$$R_{xx}(t,\tau) = E[x(t)\ x(t-\tau)], \tag{2-8}$$

and the crosscorrelation functions are

$$R_{xy}(t,\tau) = E[x(t)\ y(t-\tau)]; \qquad (2\text{-}9)$$

$$R_{yx}(t,\tau) = E[y(t)\ x(t-\tau)]. \qquad (2\text{-}10)$$

This definition is advantageous, since in this form the correlation functions depend only on past values of $x(t)$ and $y(t)$.

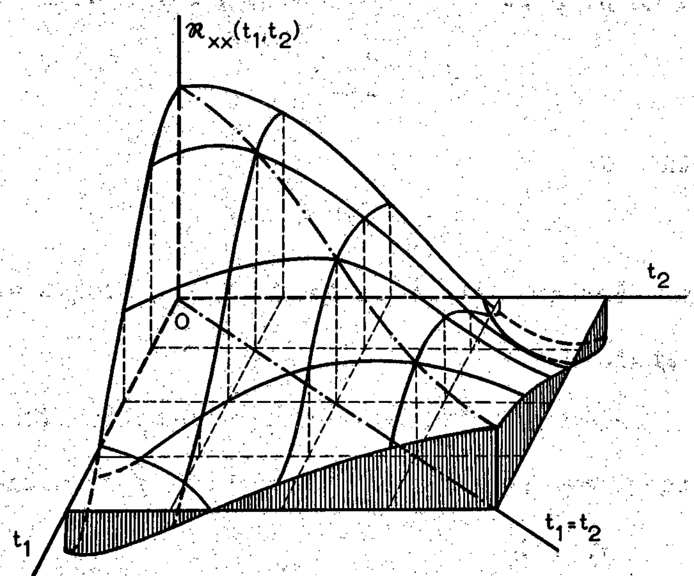

Figure 2-1

Sketch of an autocorrelation function $\mathcal{R}_{xx}(t_1,t_2)$ showing the symmetry in t_1 and t_2

However, autocorrelation functions possess the so-called Hermitian property [11] [22] whose significance, in the case of real-valued stochastic processes, is symmetry in t_1 and t_2. This behavior is demonstrated in figure 2-1 for an arbitrary autocorrelation function. It means that

$$\mathcal{R}_{xx}(t_1,t_2) = E[x(t_1)\ x(t_2)] = E[x(t_2)\ x(t_1)] = \mathcal{R}_{xx}(t_2,t_1). \qquad (2\text{-}11)$$

This property is significant as well as useful and one would like to use a notation that expresses it simply, while still using the delay variable τ. Writing the autocorrelation function as

$$\tilde{R}_{xx}(t,\tau) = E[x(t+\tfrac{\tau}{2})\ x(t-\tfrac{\tau}{2})], \tag{2-12}$$

one has

$$\tilde{R}_{xx}(t,\tau) = \tilde{R}_{xx}(t,-\tau) \tag{2-13}$$

as the Hermitian property. In this second definition, the midpoint is

$$t = \frac{t_1+t_2}{2}, \tag{2-14}$$

and the spacing is

$$\tau = t_1-t_2, \tag{2-7}$$

as previously. For crosscorrelation functions such a definition has no particular significance.

These different definitions are related by the following identities. They are written for the general case and apply to auto- as well as crosscorrelation functions.

$$\mathcal{R}(t_1,t_2) = R(t_1,t_1-t_2) = \tilde{R}(\frac{t_1+t_2}{2},\ t_1-t_2) \tag{2-15}$$

$$\mathcal{R}(t,t-\tau) = R(t,\tau) = \tilde{R}(t-\tfrac{\tau}{2},\tau) \tag{2-16}$$

$$\mathcal{R}(t+\tfrac{\tau}{2},t-\tfrac{\tau}{2}) = R(t+\tfrac{\tau}{2},\tau) = \tilde{R}(t,\tau) \tag{2-17}$$

It would be desirable to use only one definition throughout this thesis. Unfortunately, however, $\tilde{R}(t,\tau)$ offers some important notational advantages, while $R(t,\tau)$ is the more sensible definition of a correlation function from an engineering viewpoint. It is, therefore, in the interest of a clearer presentation, to use both forms side by side. The transition from one definition to the other is easily achieved by the identities stated above. When possible, the use of $R(t,\tau)$ is preferred.

2.3 Reasonable Approximants

Based on the time-varying nature of the random process, other engineering requirements arise. It has already been stated that only one sample function is usually available for examination over a time interval T. Therefore, any estimation scheme must be adapted to this restriction. Furthermore, the correlation function at any desired observation point t_o should be known as soon after t_o as possible.

Thus, it is assumed that a sample function x(t) - using the same symbol for the stochastic process as for a particular sample function - is given over an interval of at least length T (or the longest possible period Θ over which data can be obtained, $T \ll \Theta$). This interval contains, in general, only past values including the observation point t_o at the end of that time period. If the observation point is chosen to be in the middle of the observation interval, it will be denoted by θ_o. Since t_o and θ_o refer to an arbitrary time origin inside or outside the interval, the frame of reference may easily be shifted to an interval [-T,0]. These assumptions and definitions are illustrated by figure 2-2.

The problem, as stated, is to estimate $R_{xx}(t_o,\tau)$ from past values of x(t). Clearly, the only reasonable operations which can be performed on this data result in some sort of time average within the given interval of observation [4].

While avoiding any discussion of ergodic properties [12] [13] and specific classes of nonstationary second order processes [3] [17], the ergodic situation, although invalid for the general nonstationary case, suggests approximants for the desired correlation function $R_{xx}(t,\tau)$ which are of the form,

$$^{a}R_{xx}(t_o,\tau,T) = \frac{1}{T}\int_{t_o-T}^{t_o} x(t)\,x(t-\tau)\,dt, \qquad (2\text{-}18)$$

evaluated at t_o; see figure 2-2. Equation (2-18) can also be written:

$$^{a}R_{xx}(t_o,\tau,T) = \frac{1}{T}\int_{-T}^{0} x(t+t_o)\,x(t+t_o-\tau)\,dt \qquad (2\text{-}19)$$

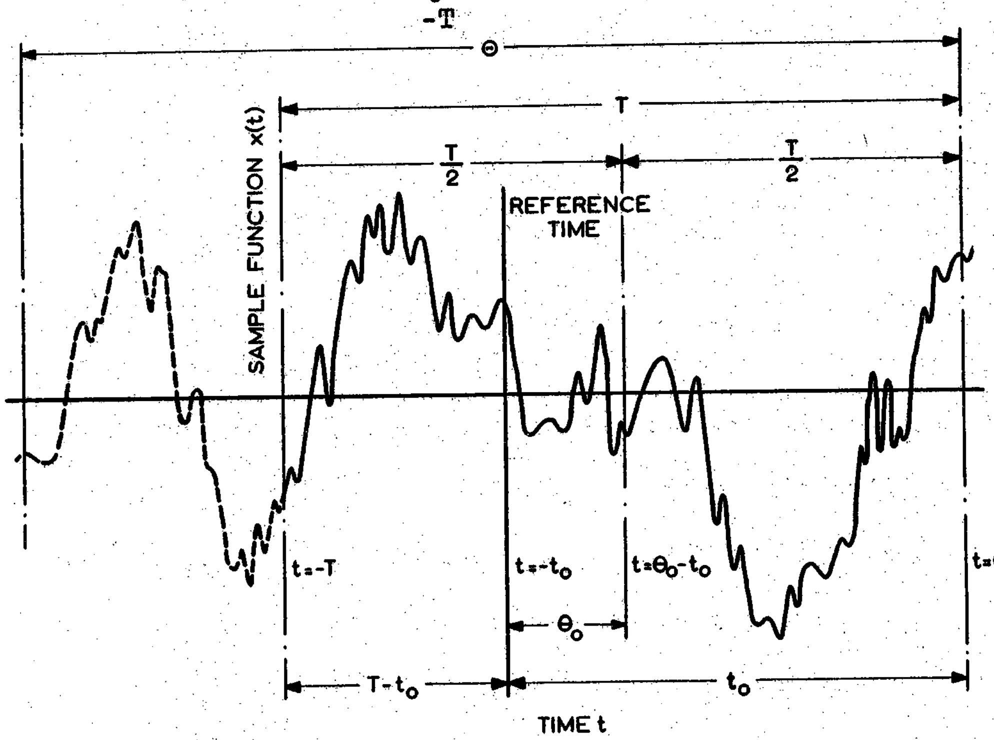

Figure 2-2

Given sample function x(t)

The observation interval T may possibly be determined to be some optimum value T_o so that a specific error criterion is satisfied.

For $\tilde{R}_{xx}(t_o,\tau)$ the following approximant would be appropriate:

$$^{a}\tilde{R}_{xx}(t_o,\tau,T) = \frac{1}{T}\int_{t_o-T}^{t_o} x(t+\tfrac{\tau}{2})\,x(t-\tfrac{\tau}{2})\,dt \qquad (2\text{-}20)$$

This form is suggested by eq. (2-12), but violates the assumption that only past values are available for the estimation, where t_o is the present. Either $\tilde{R}_{xx}(t,\tau)$, which also depends on future values, is estimated only at $t_o-\frac{\tau}{2}$ instead of t_o, or the restrictions on the observation interval must be relaxed. It is precisely for this reason that the form $R_{xx}(t,\tau)$ is preferred technically. Yet, when it becomes necessary to use $\tilde{R}_{xx}(t,\tau)$, the observation interval of length T will be shifted to the region $[-T+\frac{\tau}{2},\frac{\tau}{2}]$ and the time $t_o+\frac{\tau}{2}$ might then be regarded as the present. This difference in definitions will be understood. Equation (2-20) can also be written

$$^{a}\tilde{R}_{xx}(t_o,\tau,T) = \frac{1}{T}\int_{-T}^{0} x(t+t_o+\tfrac{\tau}{2})\ x(t+t_o-\tfrac{\tau}{2})\ dt. \qquad (2\text{-}21)$$

The assumption of such approximants is a reasonable extension from the ergodic case to the slowly time-varying case. Yet, without anticipating certain results, such an approximant has to be considered as too specific. A more general approach can be taken, by multiplying the integrand of (2-19) by an appropriate weighting function $h(t,T)$, where T is now a parameter, and the following estimate formed:

$$^{a}R_{xx}(t_o,\tau,h) = \int_{-\infty}^{0} h(t,T)\ x(t+t_o)\ x(t+t_o-\tau)\ dt \qquad (2\text{-}22)$$

The approximant (2-19) is the special case in which, for example,

$$h(t,T) = \frac{1}{T}[u(t-T) - u(t)], \qquad (2\text{-}23)$$

with

$$u(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases} \qquad (2\text{-}24)$$

In this particular situation, $h(t,T)$ is the finite-time integrator and is referred to as the uniform weighting function $h_u(t,T)$. An expression analogous to (2-22) could, of course, be formed from ${}^{a}\tilde{R}_{xx}(t,\tau,T)$.

One of the advantages of such an approximant is that none of the variables appear in the limits of integration. The main objective is, however, to weight the values of the single sample function in such a way over T that the desired ensemble average at t_o, over all sample functions, is best resembled by this weighted time average.

2.4 Error Considerations

With an emphasis on estimation procedures which are suitable for practical measuring techniques, the behavior of the mean and of the variance of the approximant are of interest. In particular, the mean square error of the approximant is most important. In the stationary situation the mean of an approximant according to (2-19) is the true value. Therefore, the variance and mean square error are identical and they vanish for $T \to \infty$ [10]. Here, however, T should be as short as possible, in general, since the mean of (2-19) yields a time averaged autocorrelation function in the nonstationary situation. The mean value of the more general approximant (2-22) is

$$E[{}^{a}R_{xx}(t_o,\tau,h)] = \int_{-\infty}^{0} h(t,T)\, R_{xx}(t+t_o,\tau)\, dt, \qquad (2\text{-}25)$$

and the variance is

$$\begin{aligned} \sigma_R^2 &= E[{}^{a}R_{xx}^2(t_o,\tau,h)] - E^2[{}^{a}R_{xx}(t_o,\tau,h)] \\ &= \int_{-\infty}^{0}\int_{-\infty}^{0} h(t_a,T)\, h(t_b,T)\, \mu_{xx}^2(t_a,t_b,t_o,\tau)\, dt_b\, dt_a \qquad (2\text{-}26) \\ &\quad - \left[\int_{-\infty}^{0} h(t,T)\, R_{xx}(t+t_o,\tau)\, dt\right]^2, \end{aligned}$$

where $\mu_{xx}^2(t_a,t_b,t_o,\tau)$ stands for the fourth mixed moment:

$$\mu_{xx}^2(t_a,t_b,t_o,\tau) = E[x(t_a+t_o)\ x(t_a+t_o-\tau)\ x(t_b+t_o)\ x(t_b+t_o-\tau)] \qquad (2\text{-}27)$$

Since the mean is not the true value, the variance differs from the mean square error of the estimate. The latter is given by

$$\begin{aligned} s^2(R,{}^aR) &= E\left[\{R_{xx}(t_o,\tau) - {}^aR_{xx}(t_o,\tau,h)\}^2\right]; \\ &= \int_{-\infty}^{0}\int_{-\infty}^{0} h(t_a,T)\ h(t_b,T)\ \mu_{xx}^2(t_a,t_b,t_o,\tau)\ dt_b\ dt_a \\ &\quad - 2R_{xx}(t_o,\tau) \int_{-\infty}^{0} h(t,T)\ R_{xx}(t+t_o,\tau)\ dt \\ &\quad + R_{xx}^2(t_o,\tau). \end{aligned} \qquad (2\text{-}28)$$

A constraint forcing the mean of the estimate to be the true value, eq. (2-25), would impose a restriction on $h(t,T)$ as well as on $R_{xx}(t,\tau)$ which is too strong. Hence, only a minimization of the mean square error $s^2(R,{}^aR)$ can be employed to arrive at an optimum weighting function.

CHAPTER 3

OPTIMUM WEIGHTING FUNCTION

In this chapter, the best or optimum estimate with respect to the minimum mean square error criterion will be considered. The mean square error of an approximant to the desired correlation function was previously defined. Standard minimization techniques lead to a condition for an optimum weighting function.

3.1 Integral Equation for the Optimum Weighting Function

From the definition of the mean square error, eq. (2-28), of an approximant according to eq. (2-22), the following relationship can be obtained by taking the first variation:

$$\delta s(R, {}^{a}R) = 2\int_{-\infty}^{0}\int_{-\infty}^{0} \delta h(t_a,T)\, h(t_b,T)\, \mu_{xx}^{2}(t_a,t_b,t_o,\tau)\, dt_b\, dt_a - 2R_{xx}(t_o,\tau)\int_{-\infty}^{0} \delta h(t_a,T)\, R_{xx}(t_a+t_o,\tau)\, dt_a \tag{3-1}$$

In deriving (3-1), use was made of the symmetry of the fourth mixed moment with respect to t_a and t_b. Setting the first variation to zero yields the desired condition for the minimum mean square error.

$$\delta s(R, {}^{a}R) = 0$$

$$0 = 2\int_{-\infty}^{0} \delta h(t_a,T)\left[\int_{-\infty}^{0} h(t_b,T)\, \mu_{xx}^{2}(t_a,t_b,t_o,\tau)\, dt_b - R_{xx}(t_o,\tau)\, R_{xx}(t_a+t_o,\tau)\right] dt_a \tag{3-2}$$

Since the integral (3-2) has to vanish for all $\delta h(t_a,T)$, the integrand itself must vanish; thus

$$\int_{-\infty}^{0} h(t_b,T)\, \mu_{xx}^2(t_a,t_b,t_o,\tau)\, dt_b - R_{xx}(t_o,\tau)\, R_{xx}(t_a+t_o,\tau) = 0, \qquad (3\text{-}3)$$

for all t_a, $t_a \in (-\infty,0)$.

The $h(t_a,T)$ which satisfies this first order integral equation of Fredholm type is the optimum weighting function $h_o(t_a,T)$ for time t_o and lag τ, and it is the one which minimizes the mean square error.

3.2 Numerical Solution of the Integral Equation

This linear integral equation for $h(t_a,T)$ is indeed very similar to the Wiener - Hopf equation [11] in prediction theory. Unfortunately, the various elegant methods for a direct solution of this equation cannot be applied in this nonstationary situation. Of course, a solution in series form, in terms of the eigenvalues and eigenfunctions of the kernel, is always possible [8]. The condition, that correlation functions as well as the fourth mixed moment are of integrable square, is implied in the basic definitions. However, finding these characteristic functions and values imposes difficulties. Numerical iteration schemes [15] have to be applied and certain problems of accuracy and convergence arise. An immediate, approximate numerical solution is much more feasible.

The method chosen is that of undetermined coefficients or collocation [15]. The weighting function $h(t_a) \equiv h(t_a,T)$ is approximated by

$$h(t_a) \approx \sum_{k=1}^{n} a_k\, \Phi_k(t_a), \qquad (3\text{-}4)$$

where the $\Phi_k(t_a)$ are n suitably chosen functions. The coefficients a_k could be determined from a set of linear equations, if the integration in (3-3) is replaced by a weighted sum. But instead of attempting a collocation at specific points, the $\mathring{a}_k$ are evaluated in such a manner that the resulting weighting function is a least square approximation to the true solution. The reasons for this approach and its advantages for numerical analysis are numerous [16]. Hence, the unknown coefficients a_k satisfying the least square error criterion,

$$\int_{-\infty}^{0}\left[\int_{-\infty}^{0}\sum_{k=1}^{n} a_k \, \Phi_k \, \mu_{xx}^2(t_a,t_b,t_o,\tau)\, dt_b - R_{xx}(t_o,\tau)\, R_{xx}(t_a+t_o,\tau)\right]^2 dt_a,$$

have to be found [15]. Standard minimization procedures lead to n integral equations. By using a numerical integration scheme, a system of n linear algebraic equations can be solved instead. Matrix notation makes this approach well suited for the use of digital computer methods in order to proceed to an actual, least square solution of the integral equation (3-3).

This numerical solution procedure was programmed for an IBM 7090 computer. As a matter of convenience in evaluating and handling the various matrices, an observation interval [0,T] with positive time was used instead of [-T,0] as in the theoretical considerations of this thesis. It is for this reason that tables and figures dealing with optimum weighting functions $h(t_a)$ have a positive time scale for t_a and the observation point is at the beginning of the interval. Replacing t_a by $-t_a$ brings the figures and tables into agreement with the theory. Since the tables were printed directly by the computer, it would have introduced a certain degree of ambiguity to present the illustrations differently.

3.3 Assumptions and Tests

In order to proceed to explicit results, it is not only necessary to choose autocorrelation functions of interest, but the corresponding fourth product moment must be known. Only for a Gaussian stochastic process is the knowledge of the autocorrelation function sufficient and in that case the fourth mixed moment is given by

$$\begin{aligned}\mu^2_{xx}(t_a,t_b,t_o,\tau) = {} & R_{xx}(t_a+t_o,\tau)\, R_{xx}(t_b+t_o,\tau) \\ & + R_{xx}(t_a+t_o,t_a-t_b)\, R_{xx}(t_a+t_o-\tau,t_a-t_b) \qquad (3\text{-}5) \\ & + R_{xx}(t_a+t_o,t_a-t_b+\tau)\, R_{xx}(t_a+t_o-\tau,t_a-t_b-\tau).\end{aligned}$$

It is an important property of autocorrelation functions that there is always a Gaussian process having the same autocorrelation function as the process under consideration [22]. Then it is possible to obtain results for $h(t_a)$ by substituting the equivalent Gaussian process. From a comparison with information theoretical concepts, it is felt that this procedure may lead, in general, to an upper bound on the mean square error. But neither a sufficient proof nor a counter example to this conjecture has been found.

For the series expansion to the weighting function, eq. (3-4), the first 10 orthonormal Laguerre functions were used after extensive tests with the following stationary autocorrelation functions:

$$R_1(\tau) = e^{-\alpha|\tau|} \qquad (3\text{-}6)$$

$$R_2(\tau) = e^{-\alpha|\tau|}\cos\tau \qquad (3\text{-}7)$$

These Laguerre functions are an orthonormalized version of the Laguerre polynomials. Their important properties and the numerical expressions (from 0th through 9th order) are given explicitly in appendix B.

In a stationary situation, e.g., the cases above, one would expect a finite time integrator for the optimum weighting function, i.e., uniform weight, at first sight. However, it can be shown that such a solution does not satisfy the integral equation (3-4) for any finite T (except in degenerate cases). Only an infinite integration time leads to the familiar result. Actual optimum weighting functions, which were obtained numerically for (3-6), are given in figures 3-1 and 3-2. These graphs are scaled differently to demonstrate better the differences and fluctuations caused by different choices for $1/\gamma$, the time constant of the Laguerre functions (see appendix B). Tables, together with a graphical representation of $h(t_a)$, may also be found in appendix C. The optimum weighting function for (3-6) is given in table and figure C-2 for the best choice of γ. The same results for the stationary correlation function (3-7) appear in table and figure C-3. These graphs show significant increases in $h(t_a)$ at the beginning and at the end of the chosen observation interval. For different T the behavior is essentially the same, if the series expansion for $h(t_a)$ is chosen appropriately. Unfortunately, the weighting function is not so accurately determined in these interesting regions of the interval as it is in the center. This situation is due to the numerical integration method and the possible error cannot be eliminated.

It should also be emphasized, that the choice of "appropriate" functions in the assumed expansion of $h(t_a)$, eq. (3-4), is a difficult one. Laguerre functions were found to be more advantageous than others because

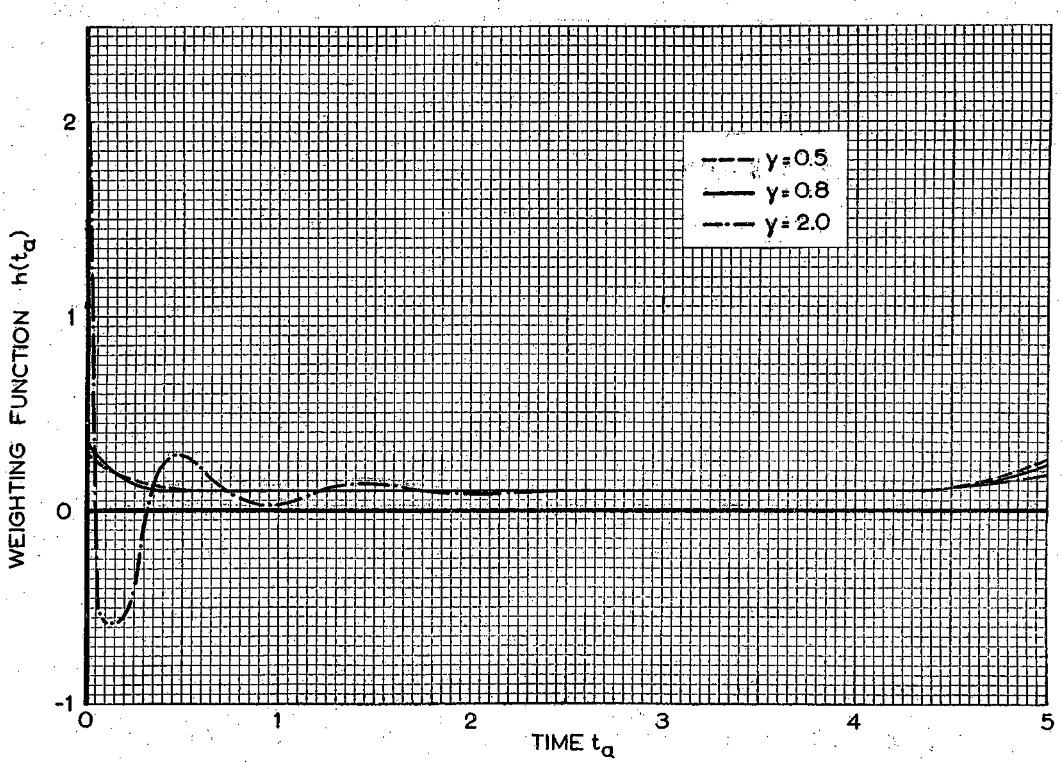

Figure 3-1

Optimum weighting function for $R_1 = e^{-|\tau|}$ and γ = 0.5, 0.8, 2.0

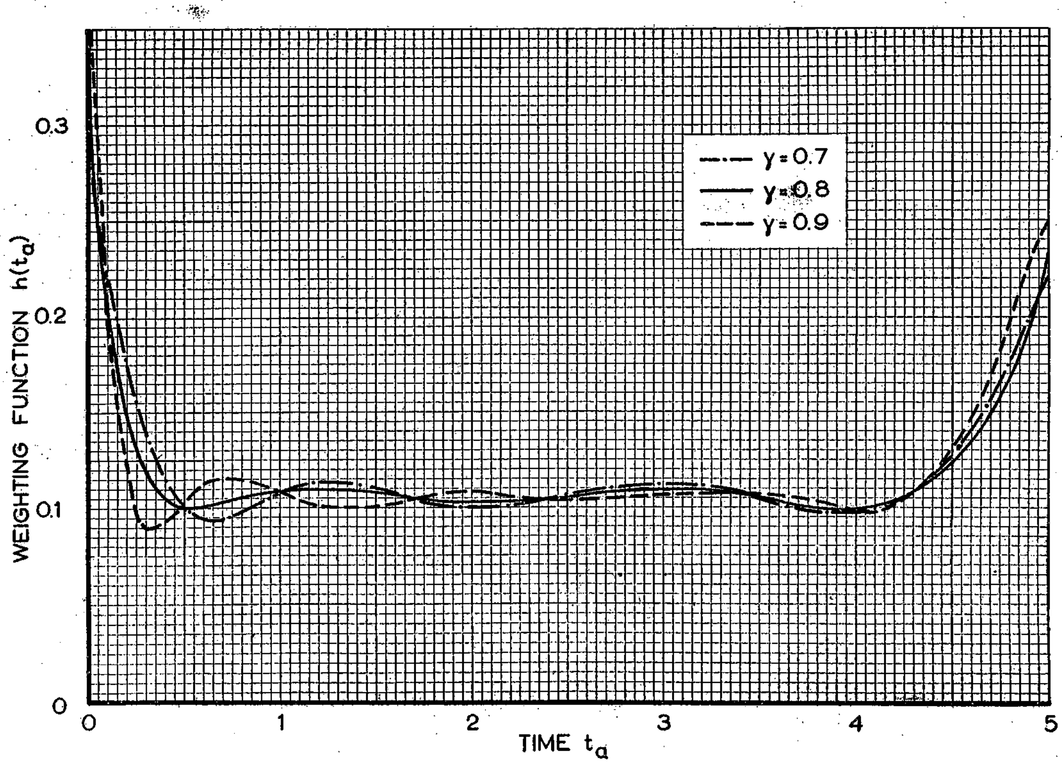

Figure 3-2

Optimum weighting function for $R_1 = e^{-|\tau|}$ and γ = 0.7, 0.8, 0.9

of the special property that they lead to simple physically realizable networks which have the corresponding weighting function as their impulse response [21]. The number of terms necessary in eq. (3-4) and the time constant $1/\gamma$ of the chosen Laguerre functions are critical as may be seen from figure 3-1. For an interval, T = 5 (in some relative time scale) and 51 equally spaced points, the first 10 Laguerre functions were sufficient in all investigated cases. Their time constants should be chosen approximately equal to the time constant which determines the decay of the autocorrelation function. A ratio of 5/4 seems to be optimum, but equal time constants lead also to a close approximation to the true optimum weighting function. This can be seen easily in figure 3-2 where γ = 0.7, 0.8, or 0.9 while the time constant of the correlation function is unity. Only the enlarged scale in comparison to figure 3-1 reveals the differences. Figure 3-1 shows an extremely unfitting choice in γ = 2.0 while a longer time constant, γ = 0.5, is less critical.

3.4 Examples

Based upon these tests with stationary correlation functions, three classes of nonstationary correlation functions were considered (making essentially the Gaussian assumption).

(a) $$R_{xx}(t,\tau) = \left[1 - \frac{t-\frac{\tau}{2}}{A}\right] e^{-\alpha|\tau|} \qquad \text{for } t \leq A \tag{3-8}$$

(b) $$R_{xx}(t,\tau) = \left[e^{-(t-\frac{\tau}{2}) + A}\right] e^{-\alpha|\tau|} \tag{3-9}$$

(c) $$R_{xx}(t,\tau) = \frac{1}{2}\left[\cos(2t-\tau) + \cos\tau\right] e^{-\alpha|\tau|} \tag{3-10}$$

These time-varying correlation functions showing (a) linear, (b) exponential, and (c) periodic time dependence were used to evaluate optimum weighting functions for a wide variety of different parameter choices. Characteristic results for cases (a) and (c) with 51 points over T = 5 are given in the tables and illustrations in appendix C. Tables and figures C-4 through C-9 are some examples of optimum weighting functions to (3-8) for A = 10 and different t_o spaced over a whole observation interval T = 5. Other parameter choices show the same general behavior. Also (3-9), case (b), leads to quite similar optimum weighting functions and specific examples have been omitted. Examples for case (c), eq. (3-10), are given in tables and figures C-10 through C-14. While the first examples, case (a), are characteristic of the usual behavior of $h(t_a)$, the latter, case (c), show a more diversive character. However, the main difference lies in larger gain variations and fluctuations.

The examples for case (c) are of special interest. Note that the periodic correlation function (3-10), evaluated for $\tau = 0$, vanishes at odd multiples of $\frac{\pi}{2}$, $t = \frac{2n+1}{2}\pi$, but has a maximum at even multiples of $\frac{\pi}{2}$, $t = n\pi$, $(n = 1, 2, \ldots)$. The weighting function reflects this periodicity. See, e.g., figure C-10 for $t_o = 0$ and compare it with figure C-13 at $t_o = 3$. The gain changes over one period are considerable, but the weighting function itself also shows large oscillations.

As might be concluded from the preceding statements, the computer program, which was used to evaluate $h(t_a)$, was written for a maximum of 51 points over the closed observation interval. The maximum number of terms in the series expansion to the weighting function is 10. These limitations are mainly of an economical nature. This program uses 30,527

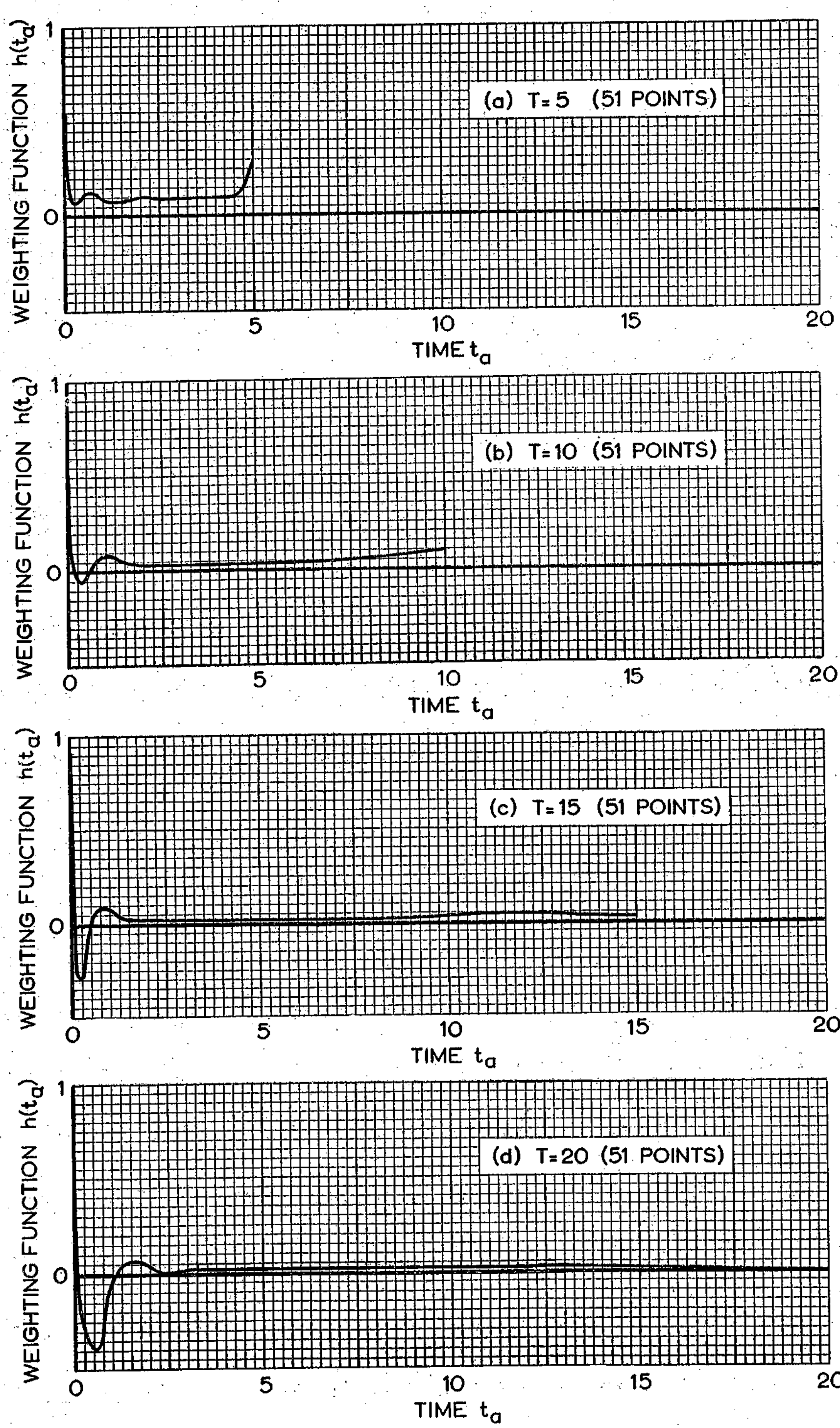

Figure 3-3

Optimum weighting function for case (a) with A = 20 and for four different values of the observation time T out of the 32,768 (= 2^{15}) core memory locations of an IBM 7090. It takes on the average 0.44 minutes to evaluate the a_k and $h(t_a)$, if the assumed autocorrelation function is not too complicated. The solution can be carried out for any desired τ, but it was examined only for some small values of the delay variable. The fluctuations in $h(t_a)$ for these values of τ were found to be small. In general, $\tau = 0$ was used and all examples presented in appendix C are calculated with this value of τ.

Figure 3-3 might serve as an example of what happens, if one demands too much of such a specific computer setup. It is for this reason that the results presented in this thesis should be regarded as a first study of optimum weighting functions and further numerical investigations should be encouraged. In figure 3-3 "optimum" weighting functions are shown, when the observation interval T is increased from 5 up to 20 without increasing the number of points or terms in the series expansion. Case (a) is chosen for this demonstration.

3.5 Discussion and Conclusions

Certainly the investigation of only three cases, even if they are chosen to resemble a wide variety of possible situations, is insufficient to arrive at final conclusions. Furthermore, the limited amount of numerical analysis constitutes only a preliminary study when compared with the scope of the possibilities. However, most of the least square approximations to the optimum weighting function seem to indicate that a uniform weight, i.e., finite time integrator - as in the stationary situation - does not appear to be "too bad", if the observation interval is relatively short. Thus, an approximant according to eq. (2-19) is well justified and will be considered more closely.

In making use of the method discussed in this chapter for an actual estimation scheme, the estimate or approximant with the optimum weighting function,

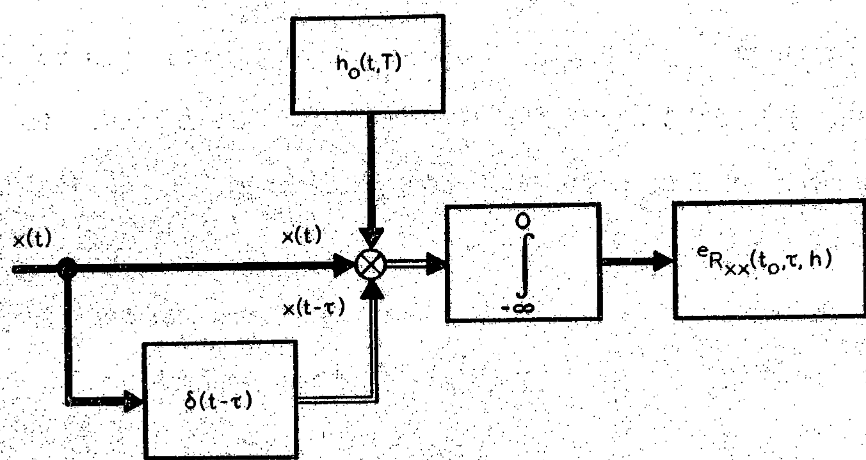

Figure 3-4

Autocorrelator for the minimum mean square error estimate using an optimum weighting function

$$^eR_{xx}(t_o,\tau,h) = \int_{-\infty}^{0} h_o(t,T)\; x(t+t_o)\; x(t+t_o-\tau)\; dt, \qquad (3\text{-}11)$$

is known instantaneously at t_o, if $h_o(t,T)$ can be pre-determined. This estimate has the minimum mean square error. The measuring procedure which would apply is outlined in the block diagram of figure 3-4.

CHAPTER 4

OPTIMUM OBSERVATION TIME

The estimation method of the preceding chapter is based solely on a minimization of the mean square error. It was noted that this analysis does not rule out the use of a finite time integrator as a weighting function. As a matter of fact, the mean and variance of the general estimate, eq. (2-22), can only be found after the weighting function is determined numerically. For the finite integration time approximant, eq. (2-19) or eq. (2-21), these statistical parameters can be determined directly and only under a few restrictive assumptions. While this type of approximant is not optimum with respect to minimizing the actual mean square error, a different criterion can be formulated by finding a minimum on the upper bound on the mean square error of the estimate. Thus, an optimum observation time can be obtained. Such a quantity suggested itself already in the early discussions of chapter 2.

4.1 Bilinear Approximation

It is a classical problem in approximation theory to approximate a function of two variables by products of functions of a single variable [14] [23]. This bilinear approximation, when applied to correlation functions, is a least square approximation of the form,

$$\tilde{R}(t,\tau) \approx \sum_{i=1}^{n} r_{i1}(t)\, r_{i2}(\tau), \qquad (4\text{-}1)$$

and it can be shown, if n increases without limit, that

$$\tilde{R}(t,\tau) = \underset{n \to \infty}{\text{l.i.m.}} \sum_{i=1}^{n} r_{i1}(t)\, r_{i2}(\tau). \tag{4-2}$$

The bilinear approximation is here applied to an autocorrelation function in the definition of eq. (2-12) in order that the symmetry condition may be given entirely as a condition on the functions $r_{i2}(\tau)$. Crosscorrelation functions can be handled either way.

A bilinear approximation with n terms involves the solution of a system of 2n linear integral equations [14]. For the purpose of this analysis, it may be assumed that the functions $r_{i1}(t)$ and $r_{i2}(\tau)$, $(i = 1, 2, \ldots, n)$, both forming orthogonal systems [23], have been found. Various methods are known for solving these integral equations in practice [7].

The whole analysis presented in this chapter is based upon the possibility of separating the dependence of a time-varying correlation function on t from the dependence on τ in some functional form. Considerable simplifications arise, if an approximation by just one product term is sufficient or correct. It is also important to note that the bilinear approximation is unique [14].

4.2 Expected Value of the Estimate

The mean value of an approximant according to eq. (2-21) can be determined with the use of a bilinear approximation. It is here that the definition of $\tilde{R}(t,\tau)$ proves advantageous.

If - and this is the only restrictive assumption - a Taylor series expansion exists for the $r_{i1}(t)$, $(i = 1, 2, \ldots, n)$, in the neighborhood of an observation point t_o such that

$$r_{i1}(t) = \sum_{k=0}^{\infty} a_{ik}(t-t_o)^k, \tag{4-3}$$

where the Taylor coefficients are given by

$$a_{ik} = \frac{r_{i1}^{(k)}(t_o)}{k!}; \tag{4-4}$$

then correlation functions can be approximated by

$$\tilde{R}(t,\tau) \approx \sum_{i=1}^{n} \sum_{k=0}^{m} a_{ik}(t-t_o)^k \, r_{i2}(\tau). \tag{4-5}$$

Truncation after the first m terms yields the usual truncation error. Equation (4-5) gives, at the observation point itself,

$$\tilde{R}(t_o,\tau) \approx \sum_{i=1}^{n} r_{i2}(\tau), \tag{4-6}$$

if all constant factors are included in the $r_{i2}(\tau)$, $(i = 1, 2, \ldots, n)$.

Since the functions $r_{i1}(t)$ are only of integrable square, like the correlation function itself, they do not necessarily possess a Taylor series expansion for all t. Thus, eq. (4-3) imposes a restriction on the problem exceeding the basic definitions. However, in the practical situation, this requirement hardly matters. The likelihood of encountering a case where (4-3) does not hold is fairly small. In any lumped circuit, for instance, all derivatives in (4-4) will always exist and the series expansion is possible for all t.

It follows from the definitions of correlation function approximants in the case of uniform weight, eqs. (2-19) and (2-21), that the expectation of the approximant to $\tilde{R}(t_o,\tau)$ becomes, in accordance with eq. (2-25),

$$E[{}^a\tilde{R}(t_o,\tau,T)] = \frac{1}{T}\int_{-T}^{0}\tilde{R}(t+t_o,\tau)\,dt, \tag{4-7}$$

Now the following approximation can be obtained with eq. (4-5) for $t_o = 0$:

$$E[{}^a\tilde{R}(t_o,\tau,T)] \approx \frac{1}{T}\int_{-T}^{0}\sum_{i=1}^{n}\sum_{k=0}^{m} a_{ik}\, t^k\, r_{i2}(\tau)\,dt \tag{4-8}$$

Integrating on the right hand side of (4-8) yields:

$$\begin{aligned} E[{}^a\tilde{R}(t_o,\tau,T)] &\approx \sum_{i=1}^{n}\sum_{k=0}^{m} a_{ik}\frac{(-T)^k}{k+1} r_{i2}(\tau) \\ &\approx \tilde{R}(t_o,\tau) + \sum_{i=1}^{n}\sum_{k=1}^{m} a_{ik}\frac{(-T)^k}{k+1} r_{i2}(\tau) \end{aligned} \tag{4-9}$$

In this form the second double summation term is seen to be the mean value of the estimation error or the bias.

The estimation point θ_o in the middle of the observation interval may lead to a smaller error term in the mean value of the approximant. A Taylor series expansion is here made around θ_o such that

$$\begin{aligned} E[{}^a\tilde{R}(\theta_o,\tau,T)] &\approx \frac{1}{T}\int_{-T/2}^{T/2}\sum_{i=1}^{n}\sum_{k=0}^{m} b_{ik}\, t^k\, r_{i2}(\tau)\,dt \\ &\approx \frac{1}{T}\int_{-T/2}^{T/2}\sum_{i=1}^{n}\left[1 + \sum_{k=1}^{m} b_{ik}\, t^k\right] r_{i2}(\tau)\,dt \\ &\approx \tilde{R}(\theta_o,\tau) + \sum_{i=1}^{n}\sum_{k=1}^{\mu} b_{i2k}\frac{T^{2k}}{2^{2k}(2k+1)} r_{i2}(\tau), \end{aligned} \tag{4-10}$$

where the Taylor coefficients in this expansion are denoted by b_{ik}, and

$$\mu = \begin{cases} \frac{m}{2} & \text{if m is even;} \\ \frac{m-1}{2} & \text{if m is odd.} \end{cases} \qquad (4\text{-}11)$$

This expression contains only even powers of T in the mean estimation error. Each term is also smaller by a factor 2^{2k} than in (4-9). However, the Taylor series expansion is made at a different point and a direct comparison of the bias terms is not possible.

With eqs. (4-2) and (4-3) the exact form of (4-9) and (4-10) can be deduced in the same fashion. For instance (4-9) would become

$$E[{}^a\tilde{R}(t_o,\tau,T)] = \tilde{R}(t_o,\tau) + \underset{n\to\infty}{\text{l.i.m.}} \sum_{i=1}^{n} \sum_{k=1}^{\infty} a_{ik} \frac{(-T)^k}{k+1} r_{i2}(\tau). \qquad (4\text{-}12)$$

4.3 An Upper Bound on the Variance and the Mean Square Error of the Estimate

For the variance of the approximant at t_o, eq. (2-21) yields in analogy to eq. (2-26),

$$\sigma^2_{\tilde{R}} = E[{}^a\tilde{R}^2(t_o,\tau,T)] - E^2[{}^a\tilde{R}(t_o,\tau,T)]. \qquad (4\text{-}13)$$

The mean square error of the estimate, on the other hand, becomes

$$s^2(\tilde{R},{}^a\tilde{R}) = \sigma^2_{\tilde{R}} + \left\{ E[{}^a\tilde{R}(t_o,\tau,T)] - \tilde{R}(t_o,\tau) \right\}^2. \qquad (4\text{-}14)$$

An upper bound on $\sigma^2_{\tilde{R}}$ which occurs for $\tau = 0$ can be found in the Gaussian case. It is derived in appendix A under some simplifying assumptions. If the same reasoning is applied in this situation, which was previously used in section 3.3, then this upper bound might be substituted into eq.

(4-14). Equation (A-5) constitutes this bound which reads in this case:

$$\sigma^2_{\tilde{R}} \leq \frac{2}{T} \int_{-\infty}^{\infty} \tilde{R}^2(t_o,\lambda)\, d\lambda \tag{4-15}$$

This assumption, together with eq. (4-14), leads to the following conjectured bound on the mean square error of the approximant, evaluated at t_o:

$$s^2(\tilde{R},{}^a\tilde{R}) \leq \frac{2}{T} \int_{-\infty}^{\infty} \tilde{R}^2(t_o,\lambda)\, d\lambda + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{m} a_{ik}\, a_{j\ell} \frac{(-T)^{k+\ell}}{(k+1)(\ell+1)} r_{i2}(0)\, r_{j2}(0) \tag{4-16}$$

The corresponding expression for θ_o is:

$$s^2(\tilde{R},{}^a\tilde{R}) \leq \frac{2}{T} \int_{-\infty}^{\infty} \tilde{R}^2(\theta_o,\lambda)\, d\lambda + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{\mu} \sum_{\ell=1}^{\mu} b_{i2k}\, b_{j2\ell} \frac{T^{2(k+\ell)}}{(2k+1)(2\ell+1)2^{2(k+\ell)}} r_{i2}(0) r_{j2}(0) \tag{4-17}$$

Equations (4-16) and (4-17) depend only on the observation point and the observation interval length T. They can, therefore, be utilized to define an optimum observation time which minimizes the maximum of this mean square error.

4.4 Condition for the Optimum Observation Time

The conjectured maximum of the mean square error, as expressed by the right hand side of (4-16) or (4-17), may be denoted by $q^2(\tilde{R},{}^a\tilde{R})$. This error is minimized with respect to the observation time T for

$$\frac{\partial}{\partial T} q^2(\tilde{R}, {}^a\tilde{R}) = 0, \tag{4-18}$$

subject to the constraint that

$$\frac{\partial^2}{\partial T^2} q^2(\tilde{R}, {}^a\tilde{R}) > 0. \tag{4-19}$$

It should be emphasized that this criterion can only be applied when T is expected to be small enough, such that the truncation of the Taylor series does not eliminate significant higher order terms in (4-16) or (4-17). This situation has to be kept in mind together with the Gaussian assumption. Unfortunately, a mathematically more satisfying criterion than (4-16) or (4-17) cannot be constructed easily. Examples indicate that a minimization of $q^2(\tilde{R}, {}^a\tilde{R})$ yields also fairly large values of this optimum interval T_o which are hardly affected by a choice of m.

With this criterion for the optimum observation time T_o, the actual condition, which follows from (4-18), becomes:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{\ell=1}^{m} a_{ik}\, a_{j\ell} \frac{(k+\ell) r_{i2}(0) r_{j2}(0)}{(k+1)(\ell+1)} (-T)^{(k+\ell)+1} - 2\int_{-\infty}^{\infty} \tilde{R}^2(t_o,\lambda) d\lambda = 0 \tag{4-20}$$

Equation (4-20) is the minimum condition on $q^2(\tilde{R}, {}^a\tilde{R})$ for t_o. The corresponding expression for an observation point θ_o is:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{\mu} \sum_{\ell=1}^{\mu} b_{i2k} b_{j2\ell} \frac{(k+\ell) r_{i2}(0) r_{j2}(0)}{(2k+1)(2\ell+1) 2^{2(k+\ell)}} T^{2(k+\ell)+1} - \int_{-\infty}^{\infty} \tilde{R}^2(\theta_o,\lambda)\, d\lambda = 0 \tag{4-21}$$

These conditions are polynomials in T of at least third order in (4-20) but of fifth order in (4-21) since $m \geq 1$.

Considerable simplifications occur when the Taylor expansion can be truncated after the second term. If the observation point is θ_o, the optimum observation time in this case is simply

$$T_o = \sqrt[5]{\frac{72 \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{-\infty}^{\infty} r_{i2}(\lambda)\, r_{j2}(\lambda)\, d\lambda}{\sum_{i=1}^{n} \sum_{j=1}^{n} b_{i2}\, b_{j2}\, r_{i2}(0)\, r_{j2}(0)}} \,. \tag{4-22}$$

If, furthermore

$$\tilde{R}(t,\tau) \simeq r_1(t)\, r_2(\tau), \tag{4-23}$$

then

$$T_o = \sqrt[5]{\frac{72}{b_2^2} \int_{-\infty}^{\infty} \tilde{\rho}^2(\theta_o,\lambda)\, d\lambda} \,, \tag{4-24}$$

where $\tilde{\rho}(\theta_o,\tau)$ is the normalized correlation function [9] for the midpoint observation time. These expressions are presented for θ_o, since they are less complicated in this case.

4.5 Examples

The aspects of actually finding the optimum observation time as defined in the preceding section will be illustrated on three simple examples where a bilinear approximation is not even necessary.

The optimum observation time T_o in the case of a correlation function varying linearly with time, case (a), as well as for a quadratic, case (b), and a periodic time dependence, case (c), will be discussed.

(a) For a correlation function like the autocorrelation function (3-8) which depends linearly on time,

$$\tilde{R}(t,\tau) = \left[1 - \frac{t}{A}\right] e^{-\alpha|\tau|}, \tag{4-25}$$

a finite optimum observation time cannot be found for θ_o. This situation arises because

$$E[{}^a\tilde{R}(\theta_o,\tau,T)] = \frac{1}{T}\int_{-T/2}^{T/2}\left[1 - \frac{t+\theta_o}{A}\right] e^{-\alpha|\tau|}\, dt$$

$$= \left[1 - \frac{\theta_o}{A}\right] e^{-\alpha|\tau|} \tag{4-26}$$

$$= \tilde{R}(\theta_o,\tau).$$

In this specific case the expected value of the estimate is equal to the true value as in the stationary situation. But this result is only due to the position of the integration interval with respect to the estimation point θ_o. By using eq. (4-9) the result would be

$$E[{}^a\tilde{R}(t_o,\tau,T)] = \frac{1}{T}\int_{-T}^{0}\left[1 - \frac{t+t_o}{A}\right] e^{-\alpha|\tau|}\, dt$$

$$= \tilde{R}(t_o,\tau) - \frac{T}{2A}\, e^{-\alpha|\tau|}, \tag{4-27}$$

and a finite optimum observation time exists in this case, where the estimation point is at the end of the interval. The value for T_o can be evaluated explicitly using (4-20). The first approach leads consequently to a variance of a measurement which would vanish as $T_o \to \infty$. However, for finite T_o, the variance of a measurement is not zero, but has a minimum upper bound.

(b) For a slightly more complex situation than in the first example, one might choose:

$$\tilde{R}(t,\tau) = (1 + c_1 t + c_2 t^2)\, e^{-\alpha|\tau|} \tag{4-28}$$

The expectation of the time average estimate is here at the midpoint θ_o

of the observation interval:

$$E[{}^{a}\tilde{R}(\theta_o,\tau,T)] = \tilde{R}(\theta_o,\tau) + \frac{c_2T^2}{12}\, e^{-\alpha|\tau|} \qquad (4\text{-}29)$$

The optimum observation time is given by

$$T_o = \left\{ \frac{72}{c_2^2} \int_{-\infty}^{\infty} \tilde{R}^2(\theta_o,\lambda)\, d\lambda \right\}^{\frac{1}{5}}. \qquad (4\text{-}30)$$

Note that for $c_1 = 0$ and $c_2 = 0.01\alpha^2$ the chosen correlation function changes roughly 1% over a time interval $1/\alpha$. This numerical example yields

$$T_o \approx \left\{ \frac{72}{c_2^2\alpha} \right\}^{\frac{1}{5}} = \frac{14.85}{\alpha}. \qquad (4\text{-}31)$$

For this optimum observation time the upper bound for the bias, denoted by $e^2(\tilde{R},{}^{a}\tilde{R})$ becomes

$$e^2(\tilde{R},{}^{a}\tilde{R}) = \left[\frac{c_2}{12} \right]^2 T_o^4 = 0.03366. \qquad (4\text{-}32)$$

With this value, the total rms error becomes ca. 41%. This example is indeed very specific, but it indicates that it is difficult to obtain a good correlation function estimate, i.e., one with a small mean square error. Yet here the correlation function varies slowly with time.

(c) A correlation function of the type

$$\tilde{R}(t,\tau) = \cos\alpha t \; r_2(\tau), \qquad (4\text{-}33)$$

where $r_2(\tau)$ can be a simple exponential as in the previous examples, but does not need to be specified, will be considered. In this last case different approximate solutions for T_o will be carried through. A comparison with an exact solution for T_o is also possible.

The expectation of the estimate at θ_o by integration, without Taylor series expansion, is:

$$E[{}^{a}\tilde{R}(\theta_o,\tau,T)] = \frac{2\cos\alpha\theta_o\ r_2(\tau)}{\alpha T}\ \sin\frac{\alpha}{2}T \qquad (4\text{-}34)$$

Integration, <u>after</u> expanding $\cos(\alpha t+\alpha\theta_o)$ in its series representation, yields:

$$E[{}^{a}\tilde{R}(\theta_o,\tau,T)] = \cos\alpha\theta_o\ r_2(\tau) \sum_{k=1}^{\infty} \frac{(-1)^k\ \alpha^{2k}\ T^{2k}}{2^{2k}\ (2k+1)!} \qquad (4\text{-}35)$$

Both equations, (4-34) and (4-35), are of course identical as can easily be verified. The condition for the optimum observation time is here

$$\cos^2\alpha\theta_o \left[r_2^2(0) \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} (-1)^{k+\ell}\ \frac{2(k+\ell)(\alpha T)^{2(k+\ell)+1}}{2^{2(k+\ell)}(2k+1)!(2\ell+1)!} - 2\gamma\right] = 0, \qquad (4\text{-}36)$$

with

$$2\int_{-\infty}^{\infty} \cos^2\alpha\theta_o\ r_2^2(\lambda)\ d\lambda = 2\gamma\ \cos^2\alpha\theta_o. \qquad (4\text{-}37)$$

Truncating after the first 5 terms leads to the following 13th order polynomial in T

$$A_{13}(\alpha T)^{13} + A_{11}(\alpha T)^{11} + A_9(\alpha T)^9 + A_7(\alpha T)^7 + A_5(\alpha T)^5 - 2\beta = 0, \qquad (4\text{-}38)$$

where

$$\beta = \frac{\gamma}{r_2^2(0)}. \qquad (4\text{-}39)$$

The coefficients in this polynomial are numerically, according to eq. (4-36),

$$A_5 = 6.94444466\text{x}10^{-3},$$

$$A_7 = -2.60416678\text{x}10^{-4},$$

$$A_9 = 4.23693800\text{x}10^{-6},$$

$$A_{11} = -4.12642187\text{x}10^{-8},$$

$$A_{13} = 2.74356958\text{x}10^{-10}.$$

The 15^{th} order term would have a coefficient $A_{15} = -1.29120581\text{x}10^{-12}$. And the next higher terms have coefficients $A_{17} = 4.28109688\text{x}10^{-15}$, $A_{19} = -9.48077679\text{x}10^{-18}$, and $A_{21} = 1.19706777\text{x}10^{-20}$.

A standard numerical solution for the first positive, real root of (4-38) yields values for the optimum observation time T_o as given in table 4-1 for different values of α and β. The notation in this table corresponds to the rules for FORTRAN statements as given in table C-1.

In this particular example, an exact solution for T_o can also be carried through, since the error term in (4-35) can be rewritten as

$$e^2(\tilde{R}, {}^a\tilde{R}) = \cos^2\alpha\theta_o \; r_2^2(\tau) \left[\frac{2}{\alpha T} \sin\frac{\alpha}{2}T - 1\right]^2. \qquad (4\text{-}40)$$

A numerical solution for the same sets of parameters as reported in table 4-1 gave values of T_o which coincide sufficiently well with the solutions of (4-38). In both cases the numerical results were obtained by successive iterations. The differences in the resulting T_o were always less than 5%.

4.6 Discussion and Conclusions

The examples of the preceding section should have elucidated the problems which are encountered in this approach. For simplicity, the observation point in the middle of the interval was preferred in the latter examples.

The whole approach in finding an optimum observation time could be

TABLE 4-1

VALUES FOR THE OPTIMUM OBSERVATION TIME

ALPHA	BETA/2.0	TERMS IN POLYNOMIAL	OPTIMUM OBSERVATION TIME
0.100	1.000	05	40.5620
		09	40.5660
0.100	10.000	05	78.8419
		09	78.8439
1.000	1.000	05	4.0560
		09	4.0560
1.000	10.000	05	7.8840
		09	7.8842

carried out without particular reference to either auto- or crosscorrelation functions, but this was only possible since references could be made to the earlier developments. The optimum weighting function approach is also applicable to crosscorrelation functions and the basic integral equation can be rewritten for that case simply by dropping the subscripts.

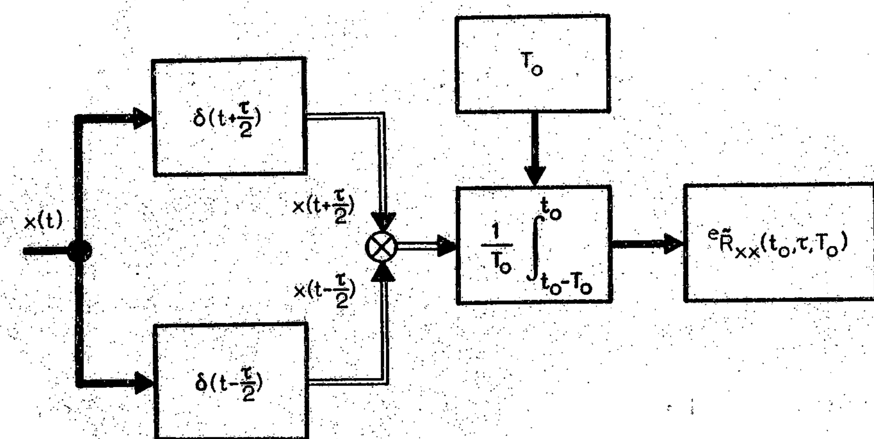

Figure 4-1

Autocorrelator for the optimum observation time estimate

This second approach, based on the conclusion that a finite time integrator is also a reasonable approximant, refers, for mathematical reasons, to $\tilde{R}(t,\tau)$. This makes a direct comparison with the optimum weighting function approach difficult, since the results can only be obtained by numerical methods.

The approximant using the optimum observation time is, in the autocorrelation case, the estimate:

$$^{e}\tilde{R}_{xx}(t_o,\tau,T_o) = \frac{1}{T_o} \int_{-T_o}^{0} x(t+t_o+\frac{\tau}{2})\, x(t+t_o-\frac{\tau}{2})\, dt \qquad (4\text{-}41)$$

The block diagram of figure 4-1 shows an autocorrelator which could perform the measurement (4-41). The difference in the definition of the estimate can easily be seen by comparing figure 4-1 with figure 3-4.

CHAPTER 5

FURTHER POSSIBILITIES AND CONCLUSIONS

The preceding discussions and results are based on error analyses which involve the true value. The mean and variance of the estimates are considered, since they are deterministic quantities. Neither an optimum weighting function nor an optimum observation time can be determined without knowledge or certain assumptions about the true value. This method might be termed indirect, and it would be very desirable to find also a more direct approach to this estimation problem.

5.1 Prospects of a Different Approach

There are certain possibilities in the direction of a direct approach. However, the major difficulties are quite severe. First, an error analysis of such an approximant is almost impossible because of the high degree of complexity. But, even if an engineering mind might be willing to accept this as a fact and rely on actual measurements and experiments, a second difficulty arises. The pure existence of such an approximant cannot be guaranteed.

One method which is closely related to the preceding discussions and for which at least one range of possible application has been found will be discussed briefly. There are also certain possibilities for an extension of this approach.

5.2 Comparison of τ-Dependence

All correlation functions can be approximated in a bilinear fashion which separates the time from the delay dependence. Then it must be possible to base an estimation method on a comparison of either the t- or τ-dependent functions with the corresponding dependence of a reference function.

Such a reference for a τ-comparison is readily found in a quantity which might be called the mean correlation function, defined as

$$\overline{R_{\Theta}(t,\tau)} = \frac{1}{\Theta} \int_{-\Theta}^{0} R(t,\tau)\, dt, \qquad (5\text{-}1)$$

where Θ is a large, but finite averaging time, in general the largest possible observation time, $\Theta \gg T$. Mean correlation functions are used often in the Russian literature [5], where the problems of their measurement are discussed [18]. Since a mean correlation function depends only on Θ and τ, the functions $r_{i2}(\tau)$ can be determined from $\overline{R_{\Theta}(t,\tau)}$ because of the uniqueness of the bilinear approximation.

Then a correlation function estimate could be constructed from the mean correlation function by comparing the τ-dependence. The block diagram of figure 5-1 might serve as an illustration for such an estimation scheme.

Unfortunately, the existence of a better estimate than $\overline{R_{\Theta}(t,\tau)}$ cannot be established for the general case as indicated earlier. However, in the case of a random process termed by Silverman [24] as "locally stationary", a simple comparison method can be applied and some examples have been constructed for which a good approximant to $R(t,\tau)$ could be obtained.

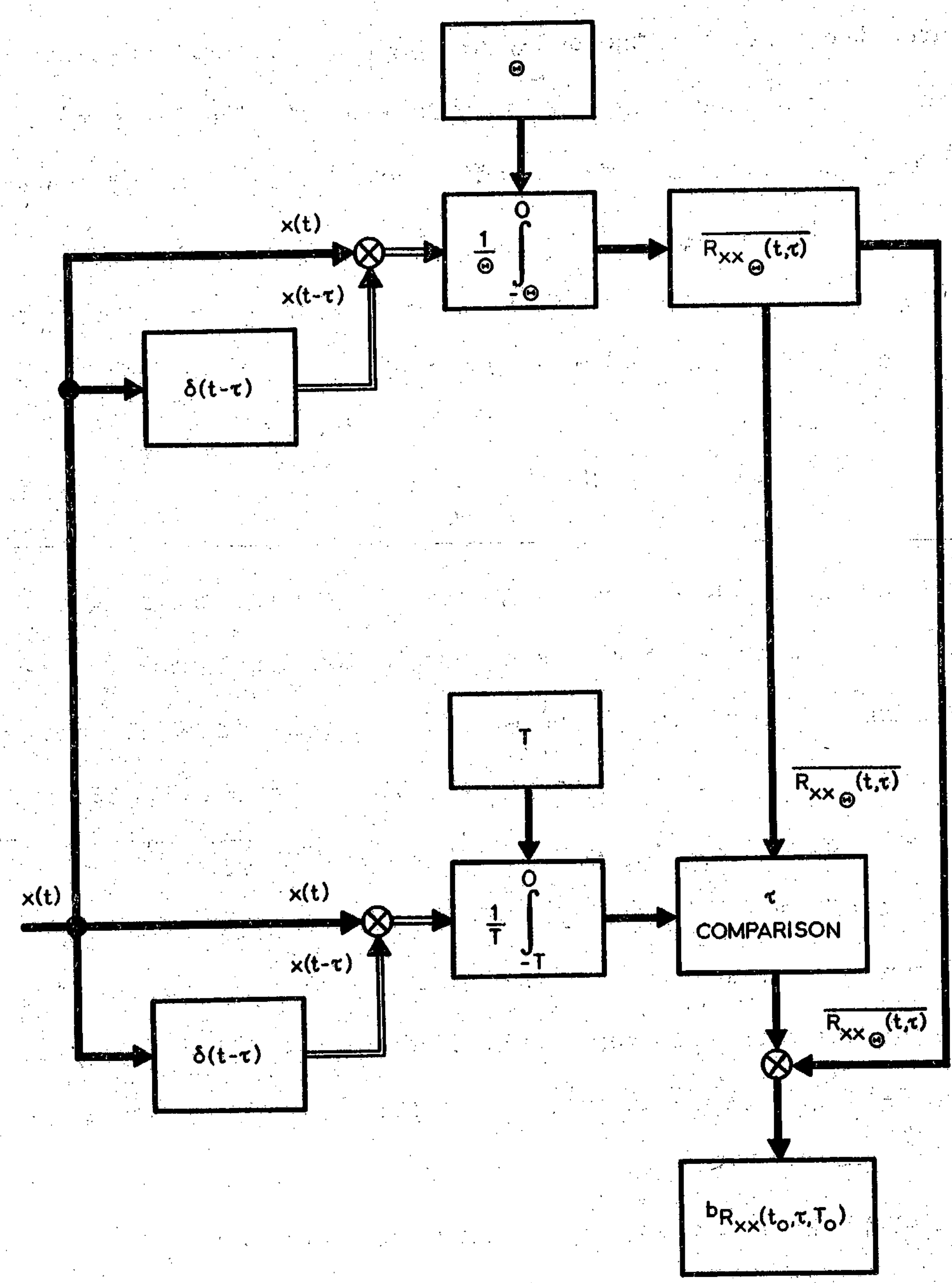

Figure 5-1

Autocorrelator comparing the τ-dependence of the mean autocorrelation function with an approximation to the true value in order to determine a better estimate at t_o

It is felt that further investigations in this area might reveal a mathematically acceptable estimation procedure which possesses some generality.

5.3 General Conclusions

The estimation of time-varying correlation functions from a single sample function has been considered. A weighted finite time-average over this data has been proposed as a reasonable approximant to the desired correlation functions.

Minimizing the mean square error of this approximant has led to an optimum weighting function. This analysis has shown, that even in the ergodic case, the simple finite time average is not optimum with respect to a minimum mean square error criterion. More surprising, however, is the similarity between weighting functions in the stationary and nonstationary situation. Aside from a rather specific example, this has led to the conclusion that, while not optimum with regard to this criterion, a finite time average is not unreasonable.

By making a bilinear approximation to the desired correlation function - in a slightly different definition - the mean value of a finite time-average approximant has been found. A conjectured upper bound on the variance has also been established. A minimization of this maximum squared error has been used as a convenient criterion for an optimum observation time. However, some examples have indicated that this simplified approach may lead to relatively large rms errors.

Explicit results have only been obtained essentially by making the assumption of a nonstationary Gaussian process. It is felt that a removal of this restriction would be the most valuable extension of this work. The criteria and equations which have been derived are valid for non-Gaussian processes as well.

Furthermore, additional numerical investigations - especially of optimum weighting functions - should be encouraged. The results presented here are only representative examples out of a large number which have been considered. But the study of certain classes of correlation functions may well lead to certain patterns in the optimum weighting functions which were not observed in this first investigation.

These approaches to the estimation problem have been based on a strict error analysis. A more direct approach, which would lead to approximants which are random variables with unknown properties, has been outlined, too. The restrictions and difficulties are discussed.

Examples to support the proposed estimation procedures have been included for all cases and explicit results appear mostly in graphical form.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Anderson, G. W., Buland, R. W., Cooper, G. R., "Use of Crosscorrelation in an Adaptive Control System," Proceedings of the National Electronics Conference, vol. 15, pp. 34-45; 1959

[2] Bennett, W. R., "The Correlatograph - A Machine for Continuous Display of Short Term Correlation," Bell System Technical Journal, vol. 32, pp. 1173-1185; 1953

[3] Blanc-Lapierre, A., Fortet, R., Théorie des fonctions aléatoires, Masson et Cie., Éditeurs, Paris, France; 1953

[4] Brennan, D. G., "Probability Theory in Communication System Engineering," Lectures on Communication System Theory, Baghdady, E. J., Ed., ch. 2, pp. 7-31, McGraw-Hill Book Company, Inc., New York, N. Y.; 1961

[5] Bunimovich, V. I., Fliuktuatsionnye protsessy v radiopriemnykh ustoĭstvakh, Izdatel'stvo "sovetskoe radio", Moskva, USSR; 1951

[6] Cheatham, T. P., Jr., The Experimental Evaluation of Correlation Functions and Their Use in the Statistical Theory of Communications, Doctor of Science Thesis, Massachusetts Institute of Technology; 1949

[7] Collatz, L., Eigenwertprobleme und ihre numerische Behandlung, Akademische Verlagsgesellschaft Becker und Erler, Berlin, Germany; 1944, Chelsea Publishing Company, New York, N. Y.; 1948

[8] Courant, R., Hilbert, D., Methods of Mathematical Physics, vol. 1, Interscience Publishers, New York, N. Y.; 1953

[9] Cramér, H., The Elements of Probability Theory, John Wiley and Sons, Inc., New York, N. Y.; 1955

[10] Davenport, W. B., Johnson, R. A., Middleton, D., "Statistical Errors in Measurements on Random Time Functions," Journal of Applied Physics, vol. 23, pp. 377-388; 1952

[11] Davenport, W. B., Root, W. L., An Introduction to the Theory of Random Signals and Noise, McGraw-Hill Book Company, Inc., New York, N. Y.; 1958

[12] Doob, J. L., Stochastic Processes, John Wiley and Sons, Inc., New York, N. Y.; 1950

[13] Gnedenko, B. W., Lehrbuch der Wahrscheinlichkeitsrechnung (Kurs teorii veroiatnosteĭ), tr., Akademie-Verlag, Berlin, Germany; 1957

[14] Golomb, M., "Approximation by Functions of Fewer Variables," On Numerical Approximation, Langer, R. E., Ed., pp. 275-327, University of Wisconsin Press, Madison, Wis.; 1959

[15] Hildebrand, F. B., Methods of Applied Mathematics, Prentice Hall, Inc., New York, N. Y.; 1952

[16] Hildebrand, F. B., Introduction to Numerical Analysis, McGraw-Hill Book Company, Inc., New York, N. Y.; 1956

[17] Kampé de Fériet, J., Frenkiel, F. N., "Estimation de la corrélation d'une fonction aléatoire non stationaire," Comptes rendus hebdomadaires des séances de l'Academie des Sciences, vol. 249, pp. 348-351; 1959

[18] Kharkevich, A. A., Spectra and Analysis (Spektry i analiz), tr., Consultants Bureau, New York, N. Y.; 1960

[19] Laning, J. H., Jr., Battin, R. H., Random Processes in Automatic Control, McGraw-Hill Book Company, Inc., New York, N. Y.; 1956

[20] Lee, Y. W., Cheatham, T. P., Jr., Wiesner, J. B., "Applications of Correlation Analysis to the Detection of Periodic Signals and Noise," Proceedings of the Institute of Radio Engineers, vol. 38, pp. 1165-1171; 1950

[21] Lee, Y. W., Statistical Theory of Communication, John Wiley and Sons, Inc., New York, N. Y.; 1960

[22] Loève, M., Probability Theory, 2nd ed., D. Van Nostrand Company, Inc., New York, N. Y.; 1960

[23] Schmidt, E., "Zur Theorie der linearen und nichtlinearen Integralgleichungen, I. Teil: Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener," Mathematische Annalen, vol. 63, pp. 433-476; 1907

[24] Silverman, R. A., "Locally Stationary Random Processes," IRE Transactions on Information Theory, vol. IT-3, no. 3, pp. 182-187; September, 1957

[25] Singleton, H. E., "A Digital Electronic Correlator," Proceedings of the Institute of Radio Engineers, vol. 38, pp. 1422-1488; 1950

APPENDICES

APPENDIX A

UPPER BOUND ON THE VARIANCE OF AN APPROXIMANT TO CORRELATION FUNCTIONS IN THE GAUSSIAN CASE

An upper bound on the variance of an autocorrelation function measurement can easily be derived for a stationary Gaussian process [19]. An extension of this result to the nonstationary situation is not only possible, but based upon a rather simple argument.

If an approximant with uniform weight, eq. (2-19), is considered, the variance (2-26) becomes with (2-23)

$$\sigma_R^2 = \frac{1}{T^2} \int_{-T}^{0} \int_{-T}^{0} \mu_{xx}^2(t_a, t_b, t_o, \tau)\, dt_b\, dt_a - \frac{1}{T^2} \left[\int_{-T}^{0} R_{xx}(t+t_o, \tau)\, dt \right]^2 . \tag{A-1}$$

Equation (A-1) yields for a nonstationary Gaussian process with (3-5):

$$\sigma_R^2 = \frac{1}{T^2} \int_{-T}^{0} \int_{-T}^{0} \Big[R_{xx}(t_a+t_o, t_a-t_b)\, R_{xx}(t_a+t_o-\tau, t_a-t_b) + R_{xx}(t_a+t_o, t_a-t_b+\tau)\, R_{xx}(t_a+t_o-\tau, t_a-t_b-\tau) \Big]\, dt_b\, dt_a \tag{A-2}$$

It is now assumed that the observation interval can be made small enough, such that the time average of the autocorrelation function over T does not differ appreciably from the true value at t_o. Then, by analogy to the stationary case, eq. (A-2) can be approximated by

$$\sigma_R^2 \approx \frac{1}{T^2} \int_{-T}^{0} \int_{-T}^{0} \Big[R_{xx}^2(t_o, t_a-t_b) + R_{xx}(t_o, t_a-t_b+\tau)\, R_{xx}(t_o, t_a-t_b-\tau) \Big] dt_b\, dt_a . \tag{A-3}$$

An upper bound on the variance may be obtained, if τ is small compared to T, but T is large compared to the significant duration of the autocorrelation function. With $\lambda = t_a - t_b$, relation (A-3) can then be simplified as follows:

$$\sigma_R^2 \approx \frac{1}{T^2}\left\{ \int_{-T}^{0}\int_{-T}^{\lambda} \left[R_{xx}^2(t_o,\lambda) + R_{xx}(t_o,\lambda+\tau)\,R_{xx}(t_o,\lambda-\tau)\right] dt_a\, d\lambda \right.$$
$$\left. + \int_{0}^{T}\int_{-T+\lambda}^{0} \left[R_{xx}^2(t_o,\lambda) + R_{xx}(t_o,\lambda+\tau)\,R_{xx}(t_o,\lambda-\tau)\right] dt_b\, d\lambda \right\} \tag{A-4}$$

$$\approx \frac{1}{T}\int_{-T}^{T} \left(1-\frac{|\lambda|}{T}\right)\left[R_{xx}^2(t_o,\lambda) + R_{xx}(t_o,\lambda+\tau)\,R_{xx}(t_o,\lambda-\tau)\right] d\lambda$$

From this form the desired upper bound can be obtained [19]:

$$\sigma_R^2 \leq \frac{1}{T}\int_{-T}^{T} \left|1-\frac{|\lambda|}{T}\right|\left|R_{xx}^2(t_o,\lambda) + R_{xx}(t_o,\lambda+\tau)\,R_{xx}(t_o,\lambda-\tau)\right| d\lambda$$

$$\leq \frac{1}{T}\left[\int_{-T}^{T} R_{xx}^2(t_o,\lambda)\,d\lambda + \int_{-T}^{T}\left|R_{xx}(t_o,\lambda+\tau)\,R_{xx}(t_o,\lambda-\tau)\right| d\lambda\right]$$

$$\sigma_R^2 \leq \frac{2}{T}\int_{-\infty}^{\infty} R_{xx}^2(t_o,\lambda)\,d\lambda \tag{A-5}$$

In deriving this final result, use was made of the Schwarz inequality and the condition that an autocorrelation function is of integrable square over the infinite interval.

APPENDIX B

LAGUERRE FUNCTIONS

Among the sets of orthonormal functions which can be derived from the classical orthogonal polynomials, functions of Laguerre type have an interesting property. The electrical network corresponding to such a set is much simpler than for other standard orthonormal functions and orthogonal polynomials [21].

The Laguerre functions over the range $(0,\infty)$ in which we are interested are obtained by orthonormalization of the sequence

$$(\gamma t)^n e^{-\gamma t} \qquad \text{for } n = 0, 1, 2, \ldots \tag{B-1}$$

just as the Laguerre polynomials are obtained by simply orthogonalizing this sequence. The resulting set of functions $\{L_n(t)\}$ has the property:

$$\int_0^\infty L_m(t)\, L_n(t)\, dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{B-2}$$

with [21]

$$L_n(t) = \sqrt{2\gamma} \left[\frac{(2\gamma)^n}{n!} t^n - \frac{n(2\gamma)^{n-1}}{(n-1)!} t^{n-1} + \frac{n(n-1)(2\gamma)^{n-2}}{2!(n-2)!} t^{n-2} - \frac{n(n-1)(n-2)(2\gamma)^{n-3}}{3!(n-3)!} t^{n-3} + - \ldots \ldots + (-1)^n \right] e^{-\gamma t} \tag{B-3}$$

Substitution of $n = 0, 1, 2, \ldots, 9$ leads to the first ten Laguerre functions which are:

$$L_0(t) = \sqrt{2\gamma}\, e^{-\gamma t} \tag{B-4a}$$

$$L_1(t) = \sqrt{2\gamma}\left[2\gamma t - 1\right] e^{-\gamma t} \tag{B-4b}$$

$$L_2(t) = \sqrt{2\gamma}\left[2\gamma^2 t^2 - 4\,\gamma t + 1\right] e^{-\gamma t} \tag{B-4c}$$

$$L_3(t) = \sqrt{2\gamma}\left[\frac{4}{3}\gamma^3 t^3 - 6\gamma^2 t^2 + 6\gamma t - 1\right] e^{-\gamma t} \tag{B-4d}$$

$$L_4(t) = \sqrt{2\gamma}\left[\frac{2}{3}\gamma^4 t^4 - \frac{16}{3}\gamma^3 t^3 + 12\gamma^2 t^2 - 8\gamma t + 1\right] e^{-\gamma t} \tag{B-4e}$$

$$L_5(t) = \sqrt{2\gamma}\left[\frac{4}{15}\gamma^5 t^5 - \frac{10}{3}\gamma^4 t^4 + \frac{40}{3}\gamma^3 t^3 - 20\gamma^2 t^2 + 10\gamma t - 1\right] e^{-\gamma t} \tag{B-4f}$$

$$L_6(t) = \sqrt{2\gamma}\left[\frac{4}{45}\gamma^6 t^6 - \frac{8}{5}\gamma^5 t^5 + 10\gamma^4 t^4 - \frac{80}{3}\gamma^3 t^3 + 30\gamma^2 t^2 - 12\gamma t + 1\right] e^{-\gamma t} \tag{B-4g}$$

$$L_7(t) = \sqrt{2\gamma}\left[\frac{8}{315}\gamma^7 t^7 - \frac{28}{45}\gamma^6 t^6 + \frac{28}{5}\gamma^5 t^5 - \frac{70}{3}\gamma^4 t^4 + \frac{140}{3}\gamma^3 t^3 - 42\gamma^2 t^2 + 14\gamma t - 1\right] e^{-\gamma t} \tag{B-4h}$$

$$L_8(t) = \sqrt{2\gamma}\left[\frac{2}{315}\gamma^8 t^8 - \frac{64}{315}\gamma^7 t^7 + \frac{112}{45}\gamma^6 t^6 - \frac{224}{15}\gamma^5 t^5 + \frac{140}{3}\gamma^4 t^4 - \frac{224}{3}\gamma^3 t^3 + 56\gamma^2 t^2 - 16\gamma t + 1\right] e^{-\gamma t} \tag{B-4i}$$

$$L_9(t) = \sqrt{2\gamma}\left[\frac{16}{10,715}\gamma^9 t^9 - \frac{4}{70}\gamma^8 t^8 + \frac{32}{35}\gamma^7 t^7 - \frac{112}{15}\gamma^6 t^6 + \frac{168}{5}\gamma^5 t^5 - 84\gamma^4 t^4 + 112\gamma^3 t^3 - 72\gamma^2 t^2 + 18\gamma t - 1\right] e^{-\gamma t} \tag{B-4j}$$

APPENDIX C

SOME RESULTS FOR THE WEIGHTING FUNCTION $h(t_a)$

The numerical results as obtained for $h(t_a)$ by solving the integral equation (3-3) on an IBM 7090 digital computer are given in this appendix. The complete set of tables as obtained for the stationary test cases is reproduced in order to show the small oscillations in $h(t_a)$ with more significant digits than a graphical representation allows. Nevertheless, the corresponding graphs are added. For the non-stationary situation, the resulting weighting functions are presented for the examples discussed in section 3.4 where three groups of time-varying autocorrelation functions were considered. In this case, however, only the a_k, $(k = 0, 1, 2, \ldots, 9)$, the coefficients of the series expansion of $h(t_a)$ in Laguerre functions, are given in table form. The optimum weighting function itself is only presented graphically.

All tables are reproduced as they were prepared by the IBM 7090 (including text and scale designations for the figures). The programming of the problem was done in FORTRAN and the resulting tables follow the output specifications of this programming language. Table C-1 defines all appearing variables with respect to the usual notation throughout this thesis. The necessary mathematical statements and number format (when different) are also defined in this table.

TABLE C-1

FORTRAN VARIABLES, STATEMENTS, AND EXPONENTIAL NUMBER FORMAT

H(TA)	$h(t_a)$
R	$R_{xx}(t,\tau)$
T	t
ABST	$\|t\|$
TAU	τ
ABSTAU	$\|\tau\|$
T1	t_o
L(K)	L_k
A(K)	a_k
X = Y	$x = y$
X + Y	$x + y$
X - Y	$x - y$
X * Y	xy
X / Y	$\frac{x}{y}$
X ** Y	x^y
ABSF(X)	$\|x\|$
EXPF(X)	e^x
SINF(X)	$\sin x$
COSF(X)	$\cos x$
3.1560452E-01	3.1560452×10^{-1}
-6.5115054E 00	-6.5115054×10^{0}

TABLE C-2

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

R = EXPF (-ABSTAU)

EVALUATED AT T1 = 0.000 WITHIN THE OBSERVATION TIME INTERVAL
(TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K)
HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0)	=	3.1560452E-01	FOR L(0)
A(1)	=	1.1946575E 00	FOR L(1)
A(2)	=	4.3682929E 00	FOR L(2)
A(3)	=	9.1514664E 00	FOR L(3)
A(4)	=	1.0659068E 01	FOR L(4)
A(5)	=	3.8835135E 00	FOR L(5)
A(6)	=	-6.5115054E 00	FOR L(6)
A(7)	=	-1.0454051E 01	FOR L(7)
A(8)	=	-6.2928140E 00	FOR L(8)
A(9)	=	-1.5144896E 00	FOR L(9)

TABLE C-2
(CONTINUED)

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

RESULTING VALUES FOR H(TA)

1	TA =	0.		H =	3.5107622E-01
2	TA =	0.1000		H =	2.2202556E-01
3	TA =	0.2000		H =	1.5294351E-01
4	TA =	0.3000		H =	1.1904185E-01
5	TA =	0.4000		H =	1.0478440E-01
6	TA =	0.5000		H =	1.0073900E-01
7	TA =	0.6000		H =	1.0140350E-01
8	TA =	0.7000		H =	1.0373899E-01
9	TA =	0.8000		H =	1.0619611E-01
10	TA =	0.9000		H =	1.0809544E-01
11	TA =	1.0000		H =	1.0923101E-01
12	TA =	1.1000		H =	1.0963178E-01
13	TA =	1.2000		H =	1.0943162E-01
14	TA =	1.3000		H =	1.0879928E-01
15	TA =	1.4000		H =	1.0790082E-01
16	TA =	1.5000		H =	1.0687656E-01
17	TA =	1.6000		H =	1.0584507E-01
18	TA =	1.7000		H =	1.0492457E-01
19	TA =	1.8000		H =	1.0417413E-01
20	TA =	1.9000		H =	1.0367993E-01
21	TA =	2.0000		H =	1.0344740E-01
22	TA =	2.1000		H =	1.0350794E-01
23	TA =	2.2000		H =	1.0383684E-01
24	TA =	2.3000		H =	1.0443720E-01
25	TA =	2.4000		H =	1.0522687E-01
26	TA =	2.5000		H =	1.0615721E-01
27	TA =	2.6000		H =	1.0713558E-01
28	TA =	2.7000		H =	1.0805723E-01
29	TA =	2.8000		H =	1.0885252E-01
30	TA =	2.9000		H =	1.0941270E-01
31	TA =	3.0000		H =	1.0964423E-01
32	TA =	3.1000		H =	1.0952063E-01
33	TA =	3.2000		H =	1.0896058E-01
34	TA =	3.3000		H =	1.0801433E-01
35	TA =	3.4000		H =	1.0670019E-01
36	TA =	3.5000		H =	1.0508301E-01
37	TA =	3.6000		H =	1.0329329E-01
38	TA =	3.7000		H =	1.0153822E-01
39	TA =	3.8000		H =	1.0003660E-01
40	TA =	3.9000		H =	9.9037719E-02
41	TA =	4.0000		H =	9.8908077E-02
42	TA =	4.1000		H =	9.9959046E-02
43	TA =	4.2000		H =	1.0261562E-01
44	TA =	4.3000		H =	1.0726563E-01

TABLE C-2
(CONTINUED)

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

RESULTING VALUES FOR H(TA)

45	TA =	4.4000	H =	1.1436130E-01
46	TA =	4.5000	H =	1.2438896E-01
47	TA =	4.6000	H =	1.3776129E-01
48	TA =	4.7000	H =	1.5494442E-01
49	TA =	4.8000	H =	1.7634870E-01
50	TA =	4.9000	H =	2.0236644E-01
51	TA =	5.0000	H =	2.3342182E-01

FIGURE C-2

GRAPHICAL REPRESENTATION OF H(TA)

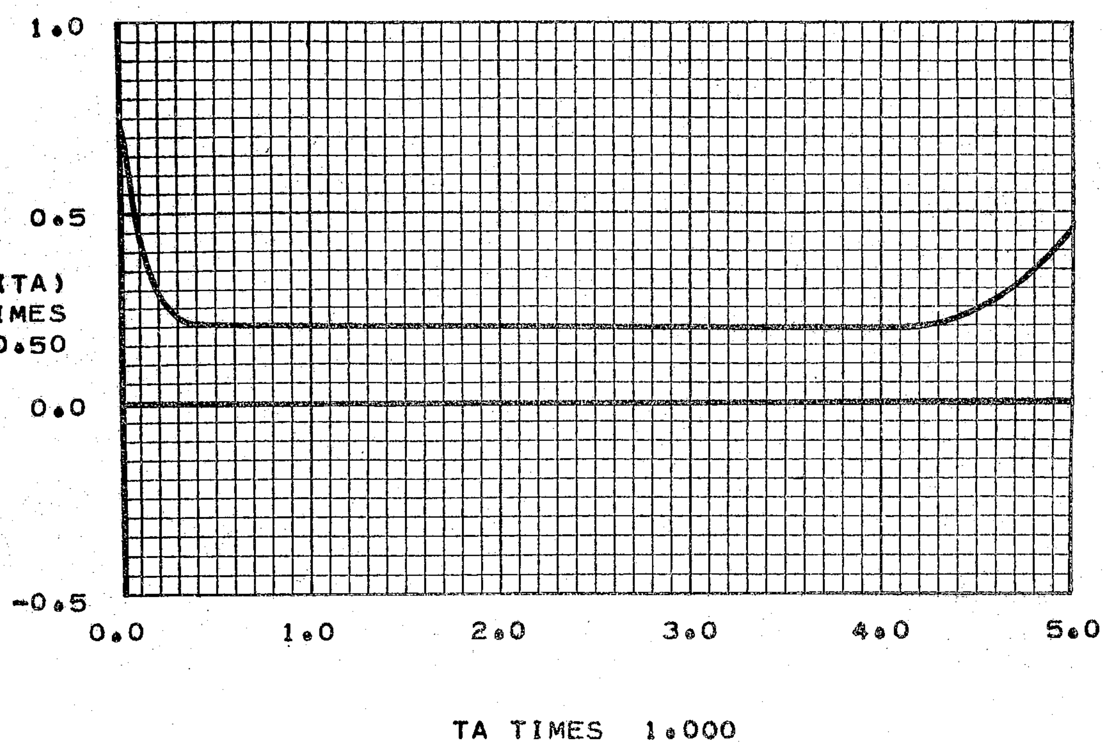

TABLE C-3

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = EXPF(-ABSTAU) * COSF(TAU)
```

EVALUATED AT T1 = 0.000 WITHIN THE OBSERVATION TIME INTERVAL
(TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K)
HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0)	=	3.6905789E-01	FOR	L(0)
A(1)	=	1.5423765E 00	FOR	L(1)
A(2)	=	5.0628045E 00	FOR	L(2)
A(3)	=	8.6303000E 00	FOR	L(3)
A(4)	=	4.8661839E 00	FOR	L(4)
A(5)	=	-9.8910680E 00	FOR	L(5)
A(6)	=	-2.4410959E 01	FOR	L(6)
A(7)	=	-2.4411445E 01	FOR	L(7)
A(8)	=	-1.2508087E 01	FOR	L(8)
A(9)	=	-2.7490088E 00	FOR	L(9)

TABLE C-3
(CONTINUED)

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

RESULTING VALUES FOR H(TA)

1	TA	=	0.	H	=	3.2615209E-01
2	TA	=	0.1000	H	=	2.0228832E-01
3	TA	=	0.2000	H	=	1.4106251E-01
4	TA	=	0.3000	H	=	1.1455713E-01
5	TA	=	0.4000	H	=	1.0605627E-01
6	TA	=	0.5000	H	=	1.0597920E-01
7	TA	=	0.6000	H	=	1.0914072E-01
8	TA	=	0.7000	H	=	1.1295220E-01
9	TA	=	0.8000	H	=	1.1628366E-01
10	TA	=	0.9000	H	=	1.1877607E-01
11	TA	=	1.0000	H	=	1.2042161E-01
12	TA	=	1.1000	H	=	1.2134430E-01
13	TA	=	1.2000	H	=	1.2168921E-01
14	TA	=	1.3000	H	=	1.2159879E-01
15	TA	=	1.4000	H	=	1.2118881E-01
16	TA	=	1.5000	H	=	1.2054917E-01
17	TA	=	1.6000	H	=	1.1973942E-01
18	TA	=	1.7000	H	=	1.1888569E-01
19	TA	=	1.8000	H	=	1.1802605E-01
20	TA	=	1.9000	H	=	1.1730031E-01
21	TA	=	2.0000	H	=	1.1670527E-01
22	TA	=	2.1000	H	=	1.1635199E-01
23	TA	=	2.2000	H	=	1.1624103E-01
24	TA	=	2.3000	H	=	1.1644994E-01
25	TA	=	2.4000	H	=	1.1692141E-01
26	TA	=	2.5000	H	=	1.1765168E-01
27	TA	=	2.6000	H	=	1.1855321E-01
28	TA	=	2.7000	H	=	1.1952205E-01
29	TA	=	2.8000	H	=	1.2050961E-01
30	TA	=	2.9000	H	=	1.2137356E-01
31	TA	=	3.0000	H	=	1.2198181E-01
32	TA	=	3.1000	H	=	1.2230115E-01
33	TA	=	3.2000	H	=	1.2216406E-01
34	TA	=	3.3000	H	=	1.2163673E-01
35	TA	=	3.4000	H	=	1.2066934E-01
36	TA	=	3.5000	H	=	1.1925999E-01
37	TA	=	3.6000	H	=	1.1751625E-01
38	TA	=	3.7000	H	=	1.1562127E-01
39	TA	=	3.8000	H	=	1.1376214E-01
40	TA	=	3.9000	H	=	1.1211329E-01
41	TA	=	4.0000	H	=	1.1106860E-01
42	TA	=	4.1000	H	=	1.1085289E-01
43	TA	=	4.2000	H	=	1.1187846E-01
44	TA	=	4.3000	H	=	1.1446807E-01

TABLE C-3
(CONTINUED)

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

RESULTING VALUES FOR H(TA)

45	TA =	4.4000	H =	1.1901776E-01
46	TA =	4.5000	H =	1.2599736E-01
47	TA =	4.6000	H =	1.3567477E-01
48	TA =	4.7000	H =	1.4847840E-01
49	TA =	4.8000	H =	1.6464587E-01
50	TA =	4.9000	H =	1.8442184E-01
51	TA =	5.0000	H =	2.0813426E-01

FIGURE C-3

GRAPHICAL REPRESENTATION OF H(TA)

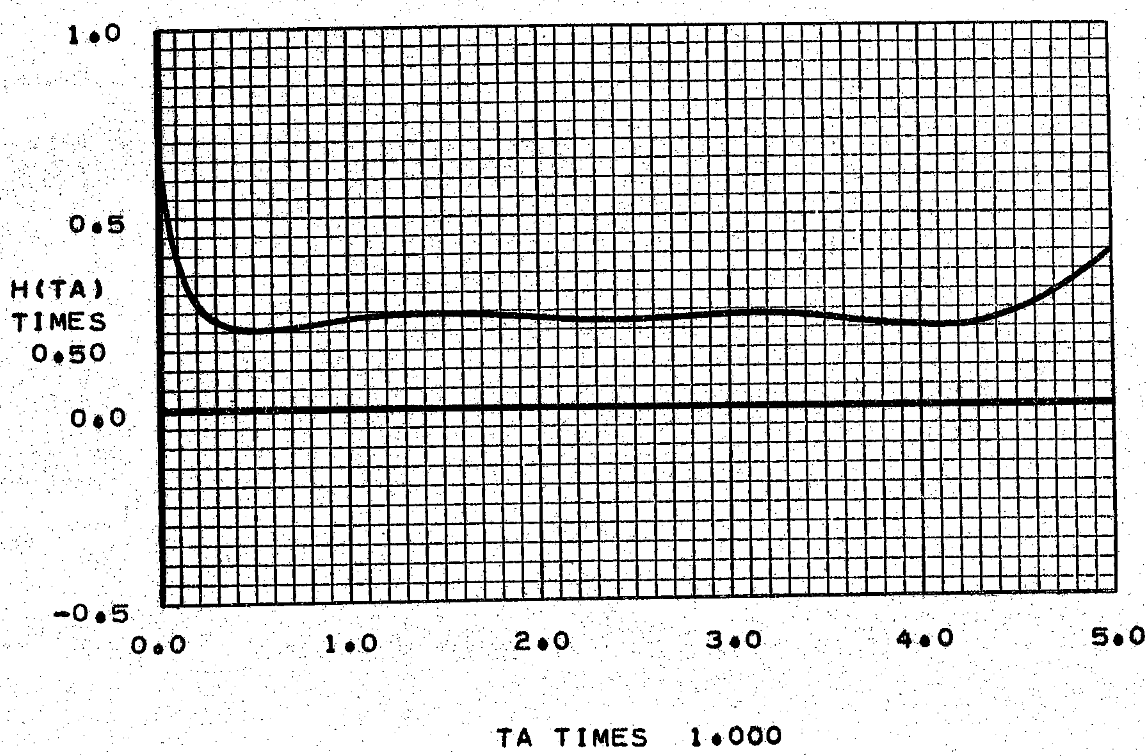

TABLE C-4

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS. WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = (1.0-ABSF((T+0.5*TAU)/10.0   )) * EXPF(-1.0*ABSF(TAU))
```

EVALUATED AT T1 = 0.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 1.0000

A(0) =	2.8357137E-01	FOR L(0)
A(1) =	1.2465865E 00	FOR L(1)
A(2) =	5.5821312E 00	FOR L(2)
A(3) =	1.6234379E 01	FOR L(3)
A(4) =	3.1756814E 01	FOR L(4)
A(5) =	4.1316443E 01	FOR L(5)
A(6) =	3.4473548E 01	FOR L(6)
A(7) =	1.5915680E 01	FOR L(7)
A(8) =	2.1414945E 00	FOR L(8)
A(9) =	-9.3405449E-01	FOR L(9)

FIGURE C-4

GRAPHICAL REPRESENTATION OF H(TA)

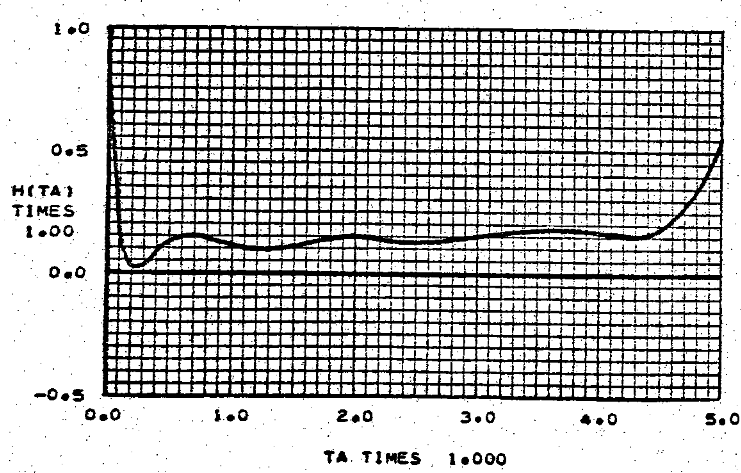

TABLE C-5

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS. WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = (1.0-ABSF((T+0.5*TAU)/10.0   )) * EXPF(-1.0*ABSF(TAU))
```

EVALUATED AT T1 = 1.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 1.0000

A(0) =	3.2460424E-01	FOR L(0)
A(1) =	1.5911035E 00	FOR L(1)
A(2) =	6.2061215E 00	FOR L(2)
A(3) =	1.4401067E 01	FOR L(3)
A(4) =	2.0997136E 01	FOR L(4)
A(5) =	1.7683338E 01	FOR L(5)
A(6) =	4.9646825E 00	FOR L(6)
A(7) =	-6.0276093E 00	FOR L(7)
A(8) =	-6.8818397E 00	FOR L(8)
A(9) =	-2.4477375E 00	FOR L(9)

FIGURE C-5

GRAPHICAL REPRESENTATION OF H(TA)

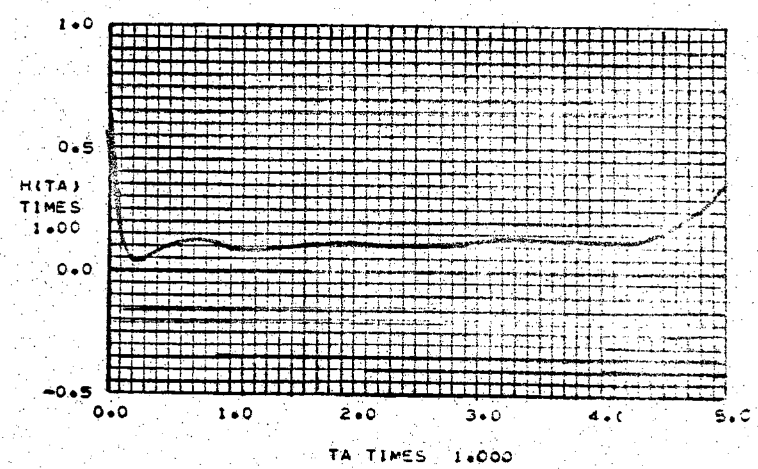

TABLE C-6

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = (1.0-ABSF((T+0.5*TAU)/10.0   )) * EXPF(-1.0*ABSF(TAU))
```

EVALUATED AT T1 = 2.000 WITHIN THE OBSERVATION TIME INTERVAL
(TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K)
HAVING AN INVERSE TIME CONSTANT OF 1.0000

A(0) =	2.4417574E-01	FOR L(0)
A(1) =	9.0095128E-01	FOR L(1)
A(2) =	3.7293214E 00	FOR L(2)
A(3) =	9.5665578E 00	FOR L(3)
A(4) =	1.6305871E 01	FOR L(4)
A(5) =	1.7965023E 01	FOR L(5)
A(6) =	1.1742157E 01	FOR L(6)
A(7) =	2.6481322E 00	FOR L(7)
A(8) =	-1.6595899E 00	FOR L(8)
A(9) =	-1.0989082E 00	FOR L(9)

FIGURE C-6

GRAPHICAL REPRESENTATION OF H(TA)

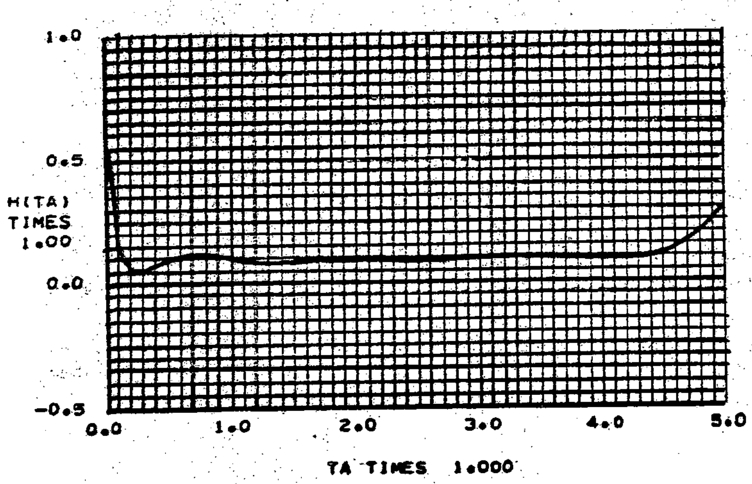

TABLE C-7

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = (1.0-ABSF((T+0.5*TAU)/10.0   )) * EXPF(-1.0*ABSF(TAU))
```

EVALUATED AT T1 = 3.000 WITHIN THE OBSERVATION TIME INTERVAL
(TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K)
HAVING AN INVERSE TIME CONSTANT OF 1.0000

A(0) =	1.9819326E-01	FOR L(0)
A(1) =	5.6109758E-01	FOR L(1)
A(2) =	2.6498452E 00	FOR L(2)
A(3) =	8.0942253E 00	FOR L(3)
A(4) =	1.6766138E 01	FOR L(4)
A(5) =	2.3060006E 01	FOR L(5)
A(6) =	2.0642213E 01	FOR L(6)
A(7) =	1.0677947E 01	FOR L(7)
A(8) =	2.2375239E 00	FOR L(8)
A(9) =	-3.1312859E-01	FOR L(9)

FIGURE C-7

GRAPHICAL REPRESENTATION OF H(TA)

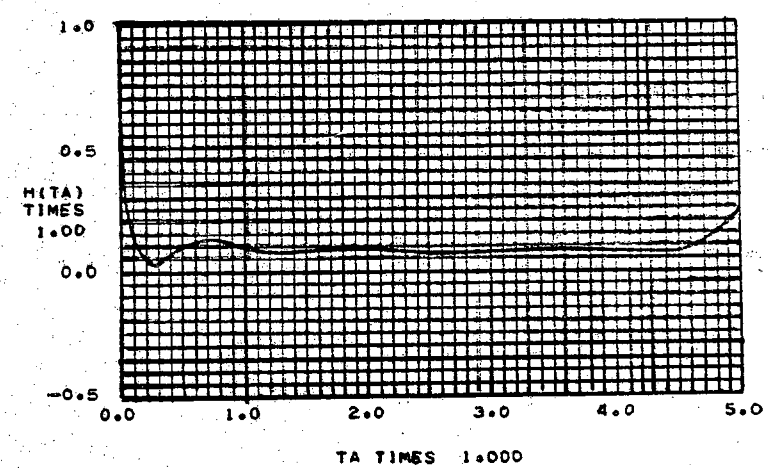

TABLE C-8

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = (1.0-ABSF((T+0.5*TAU)/10.0   )) * EXPF(-1.0*ABSF(TAU))
```

EVALUATED AT T1 = 4.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 1.0000

A(0)	=	-1.0900430E-01	FOR L(0)
A(1)	=	-2.2359981E 00	FOR L(1)
A(2)	=	-7.5285708E 00	FOR L(2)
A(3)	=	-1.1562772E 01	FOR L(3)
A(4)	=	-1.3751622E 00	FOR L(4)
A(5)	=	2.6549732E 01	FOR L(5)
A(6)	=	5.1116283E 01	FOR L(6)
A(7)	=	4.8310114E 01	FOR L(7)
A(8)	=	2.4706832E 01	FOR L(8)
A(9)	=	5.5821851E 00	FOR L(9)

FIGURE C-8

GRAPHICAL REPRESENTATION OF H(TA)

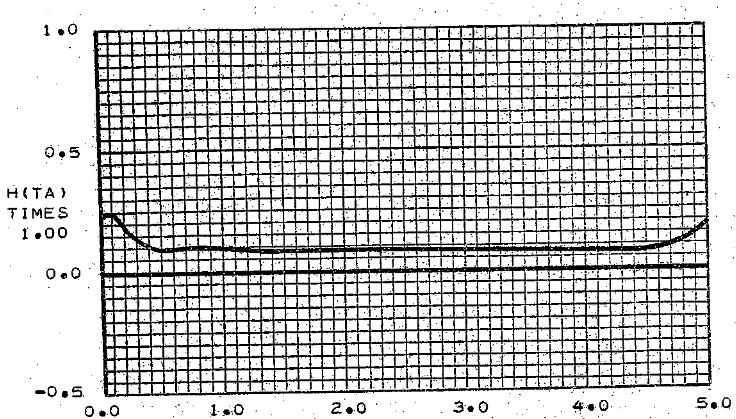

TABLE C-9

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = (1.0-ABSF((T+0.5*TAU)/10.0   )) * EXPF(-1.0*ABSF(TAU))
```

EVALUATED AT T1 = 5.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 1.0000

A(0)	=	1.7553805E-01	FOR L(0)
A(1)	=	2.6120899E-01	FOR L(1)
A(2)	=	1.1382175E 00	FOR L(2)
A(3)	=	3.3505098E 00	FOR L(3)
A(4)	=	6.9447904E 00	FOR L(4)
A(5)	=	9.5015571E 00	FOR L(5)
A(6)	=	8.4809418E 00	FOR L(6)
A(7)	=	4.3653911E 00	FOR L(7)
A(8)	=	9.2593682E-01	FOR L(8)
A(9)	=	-1.2034984E-01	FOR L(9)

FIGURE C-9

GRAPHICAL REPRESENTATION OF H(TA)

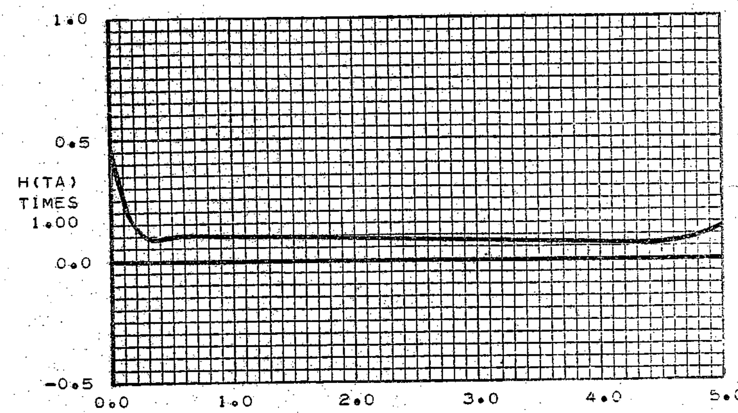

TABLE C-10

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = 0.5*EXPF(-ABSTAU)*(COSF(2.0*T+TAU)+COSF(TAU))
```

EVALUATED AT T1 = 0.000 WITHIN THE OBSERVATION TIME INTERVAL
(TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K)
HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0) =	4.9277425E-01	FOR L(0)
A(1) =	2.7126286E-01	FOR L(1)
A(2) =	-1.9240588E 00	FOR L(2)
A(3) =	-7.5234541E 00	FOR L(3)
A(4) =	-1.5705344E 01	FOR L(4)
A(5) =	-1.7998555E 01	FOR L(5)
A(6) =	-8.4828066E 00	FOR L(6)
A(7) =	1.6621083E 00	FOR L(7)
A(8) =	3.3025208E 00	FOR L(8)
A(9) =	7.8525869E-01	FOR L(9)

FIGURE C-10

GRAPHICAL REPRESENTATION OF H(TA)

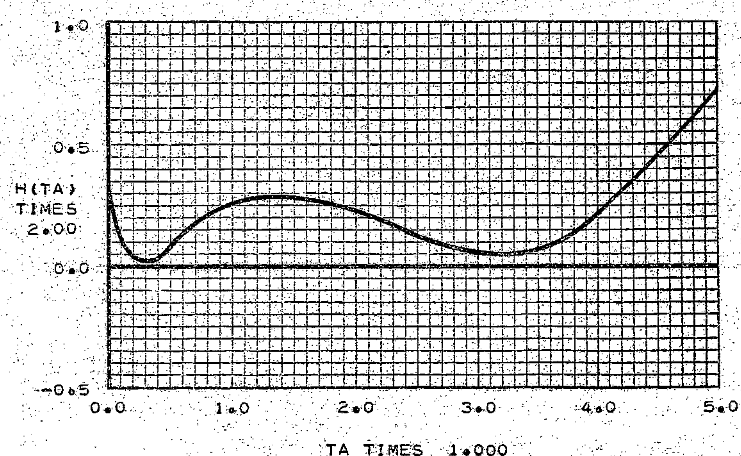

TABLE C-11

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = 0.5*EXPF(-ABSTAU)*(COSF(2.0*T+TAU)+COSF(TAU))
```

EVALUATED AT T1 = 1.000 WITHIN THE OBSERVATION TIME INTERVAL
(TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K)
HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0) =	1.4601877E-01	FOR L(0)
A(1) =	2.0613372E-01	FOR L(1)
A(2) =	4.0539711E-01	FOR L(2)
A(3) =	3.2641858E-01	FOR L(3)
A(4) =	5.9866104E-02	FOR L(4)
A(5) =	-5.1126018E-01	FOR L(5)
A(6) =	-3.3522630E-01	FOR L(6)
A(7) =	2.9529682E-01	FOR L(7)
A(8) =	2.2433019E-01	FOR L(8)
A(9) =	-1.2276771E-01	FOR L(9)

FIGURE C-11

GRAPHICAL REPRESENTATION OF H(TA)

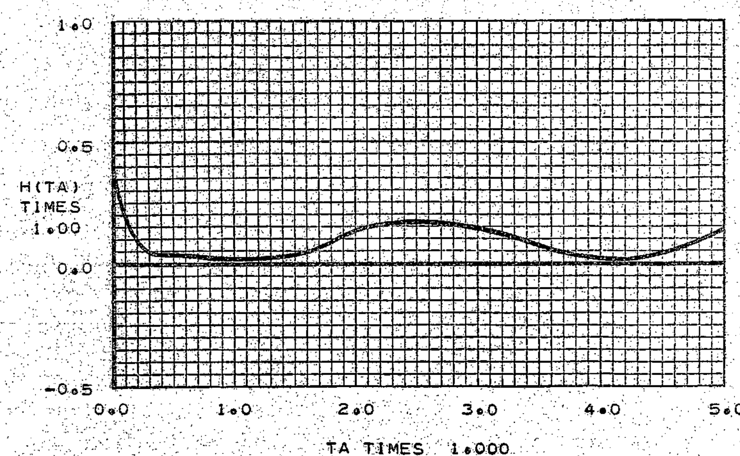

TABLE C-12

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = 0.5*EXPF(-ABSTAU)*(COSF(2.0*T+TAU)+COSF(TAU))
```

EVALUATED AT T1 = 2.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0)	=	6.2119632E-01	FOR L(0)
A(1)	=	-2.3774191E-01	FOR L(1)
A(2)	=	4.1788879E-01	FOR L(2)
A(3)	=	1.2730863E 00	FOR L(3)
A(4)	=	1.9388168E 00	FOR L(4)
A(5)	=	6.8296146E-01	FOR L(5)
A(6)	=	-2.4919308E 00	FOR L(6)
A(7)	=	-2.4238864E 00	FOR L(7)
A(8)	=	-1.1652991E-01	FOR L(8)
A(9)	=	9.1672815E-01	FOR L(9)

FIGURE C-12

GRAPHICAL REPRESENTATION OF H(TA)

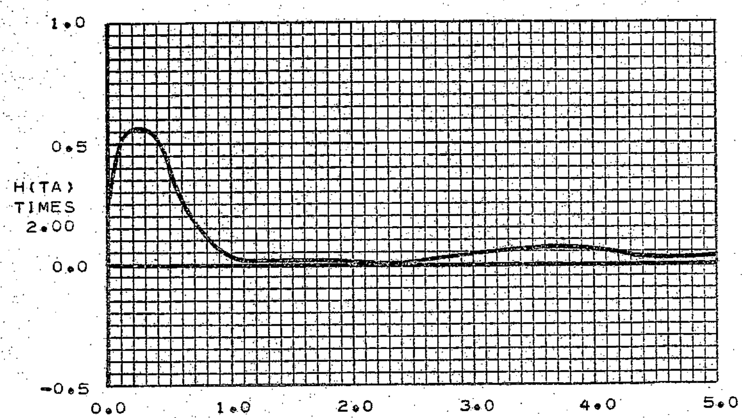

TABLE C-13

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS, WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = 0.5*EXPF(-ABSTAU)*(COSF(2.0*T+TAU)+COSF(TAU))
```

EVALUATED AT T1 = 3.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0)	=	5.4687141E-01	FOR L(0)
A(1)	=	3.8548826E-01	FOR L(1)
A(2)	=	-9.3052193E-01	FOR L(2)
A(3)	=	-3.8180895E 00	FOR L(3)
A(4)	=	-9.5489272E 00	FOR L(4)
A(5)	=	-1.0919385E 01	FOR L(5)
A(6)	=	2.3088270E-01	FOR L(6)
A(7)	=	1.2791865E 01	FOR L(7)
A(8)	=	1.2756815E 01	FOR L(8)
A(9)	=	4.1060761E 00	FOR L(9)

FIGURE C-13

GRAPHICAL REPRESENTATION OF H(TA)

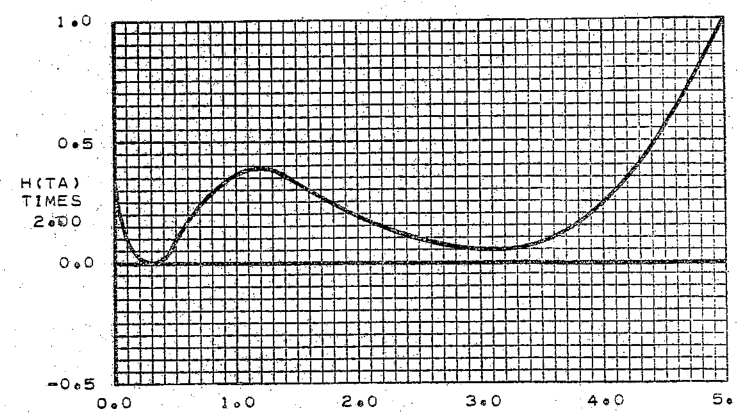

TABLE C-14

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS. WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = 0.5*EXPF(-ABSTAU)*(COSF(2.0*T+TAU)+COSF(TAU))
```

EVALUATED AT T1 = 4.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0) =	3.3204969E-01	FOR L(0)
A(1) =	1.5766258E 00	FOR L(1)
A(2) =	5.4674563E 00	FOR L(2)
A(3) =	1.1200368E 01	FOR L(3)
A(4) =	1.3234342E 01	FOR L(4)
A(5) =	5.6046904E 00	FOR L(5)
A(6) =	-6.2998371E 00	FOR L(6)
A(7) =	-1.1488023E 01	FOR L(7)
A(8) =	-7.8721487E 00	FOR L(8)
A(9) =	-2.3531046E 00	FOR L(9)

FIGURE C-14

GRAPHICAL REPRESENTATION OF H(TA)

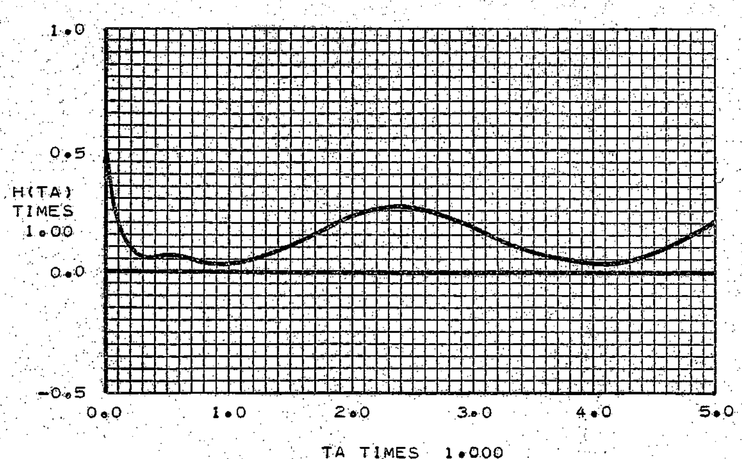

TABLE C-15

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION MINIMIZING THE MEAN SQUARE ERROR OF AN AUTOCORRELATION FUNCTION MEASUREMENT

SOLVED FOR THE WEIGHTING FUNCTION H(TA) EXPANDED IN LAGUERRE FUNCTIONS. WHEN THE TRUE AUTOCORRELATION FUNCTION IS ASSUMED TO BE GIVEN BY THE FOLLOWING FORTRAN STATEMENT

```
R = 0.5*EXPF(-ABSTAU)*(COSF(2.0*T+TAU)+COSF(TAU))
```

EVALUATED AT T1 = 5.000 WITHIN THE OBSERVATION TIME INTERVAL (TA = 0.000, ..., 5.000)

RESULTING COEFFICIENTS FOR THE LAGUERRE FUNCTIONS L(K) HAVING AN INVERSE TIME CONSTANT OF 0.8000

A(0) =	3.3405423E-01	FOR L(0)
A(1) =	-3.7556396E-01	FOR L(1)
A(2) =	-4.5662506E-01	FOR L(2)
A(3) =	-1.3123676E 00	FOR L(3)
A(4) =	-1.5819117E 00	FOR L(4)
A(5) =	-1.5657655E 00	FOR L(5)
A(6) =	-8.6767368E-01	FOR L(6)
A(7) =	4.1061714E-01	FOR L(7)
A(8) =	1.1430570E 00	FOR L(8)
A(9) =	6.5601937E-01	FOR L(9)

FIGURE C-15

GRAPHICAL REPRESENTATION OF H(TA)