7-1-1962

Electromagnetic Propagation Velocities in an Inhomogeneous or Random Atmosphere

G. R. Cooper
Purdue University

Robert Eugene Harrison
Purdue University

TR-EE 62-13
NATIONAL SCIENCE FOUNDATION CONTRACT G-18997
PRF 2974

PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

Electromagnetic Propagation Velocities in an Inhomogeneous or Random Atmosphere

G. R. Cooper, Principal Investigator

R. E. Harrison

July, 1962

Lafayette, Indiana

SUPPORTED BY
NATIONAL SCIENCE FOUNDATION
WASHINGTON, D.C.

TR-EE62-13

National Science Foundation Contract G-18997

PRF 2974

ELECTROMAGNETIC PROPAGATION VELOCITIES

IN AN INHOMOGENEOUS OR RANDOM ATMOSPHERE

SUPPORTED BY

NATIONAL SCIENCE FOUNDATION

WASHINGTON, D. C.

by

G. R. Cooper, Principal Investigator

R. E. Harrison

School of Electrical Engineering
Purdue University
Lafayette, Indiana

August, 1962

ACKNOWLEDGMENTS

It is difficult for the degree candidate to thank the many persons and organizations to whom he owes the ultimate completion of the doctoral studies and research reported in this thesis.

To his wife and sons he owes very much for enduring patience and encouragement.

He owes a special debt of gratitude to his major professor, George R. Cooper, for continued guidance and inspiration extending from 1951 through the present time. Valuable background for his research has been received in courses taught by Professor Cooper and Professor Kenneth Andrew. He wishes to thank Professor C. M. Evans for his valuable comments on the thesis.

He is especially grateful to Mrs. Miriam Smith for special help in preparation of this report, Miss Linda Zarse for courage and perseverance in typing a difficult manuscript, and Mrs. Ann Callcott for considerable help in obtaining library materials.

He is particularly grateful for the expressions of confidence and cooperation afforded him by Messrs. Robert Grizetti, Charles Clarke and Earle Mullen of the General Electric Company.

The author wishes to express his thanks to the National Science Foundation for support of this research.

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS iv

ABSTRACT. v

INTRODUCTION. 1

REVIEW OF PERTINENT LITERATURE. 6

PHASE AND ABSORPTION CHARACTERISTICS IN STEADY STATE. 11

3.1 Magnetio-ionic and Collision Absorption of the Ionosphere. 11
3.2 Elementary Theory of Absorption in a Medium Consisting of Nonpolar and Polar Molecules. 18
3.3 Brief Survey of Quantum-Mechanical Theory of Dispersion. 29

VELOCITIES OF PROPAGATION 31

4.1 Solutions of the Wave Equation for Nondispersive Media 31
4.2 Phase Velocity for a Nondispersive Medium 36
4.3 Group Velocity for a Nondispersive Medium 38
4.4 Solutions of the Wave Equation for Dispersive Media--Ionosphere 42
4.5 The Signal and Group Velocities 70
4.6 Solutions of the Wave Equation for Dispersive Media--Troposphere. 75

PROPAGATION IN AN INHOMOGENEOUS OR RANDOM MEDIUM. 81

5.1 Spectral Characteristics of Turbulence. 81
5.2 Propagation Velocities in an Inhomogeneous Troposphere 92
5.3 Propagation Velocities in a Random Troposphere 95

SUMMARY OF RESULTS AND CONCLUSIONS. 101

BIBLIOGRAPHY. 103

APPENDIX 107

LIST OF ILLUSTRATIONS

Figure		Page
3.1	Dispersion of a Nonpolar Gas Showing Anomalous Dispersion.	24
4.1	Paths of Integration in the s-Plane for Equation (4.88).	49
4.2	Final Paths of Integration of Equation (4.88).	51
4.3	Contour Plot of the Exponent of the Kernel $K(s,t_p,t/t_p)$.	57
4.4	Typical Contours for a Refractive Index Function of the Form $\alpha(s) = \left[1 + 1/(s^2+\omega_o^2)\right]^{1/2}$.	62
4.5	Relation of Integration Path to Signal Velocity.	71
4.6	Relations Between Signal Group and Phase Velocity in a Dispersive Medium (BRI 2).	73
5.1	Convolved Spectra.	89
5.2	Pattern of Successive Motion of Viscous Eddies	90
A.1	Path for Integration on Contour C_2 Around a Branch Cut	109
A.2	Initial Growth of the Received Wave.	119

ABSTRACT

Harrison, Robert Eugene, Ph.D., Purdue University, August, 1962. Electromagnetic Propagation Velocities in an Inhomogeneous or Random Atmosphere. Major Professor: George R. Cooper.

This thesis is concerned primarily with determination of statistics for the velocities of propagation of an electromagnetic wave in a dispersive medium. The velocities of propagation are discussed in terms of a plane travelling wave solution of Maxwell's equations obtained using the multiple Laplace transformation and complex inversion integrals. The types of dispersion discussed correspond to magneto-ionic, electron displacement and polar resonances of the ionosphere and troposphere. The physical nature of the randomness of the dispersive index of refraction is derived from considerations of statistical turbulence theory. Expressions are then obtained for determining the mean, mean square and variance of the signal, group and phase velocity of an electromagnetic wave.

It is proposed by S. M. Harris (IRE Trans. Vol. AP-9, No. 2, pp. 207-210, Mar., 1961) that the group velocity and phase velocity of an electromagnetic wave propagated in the ionosphere may be averaged to obtain a velocity estimate free of refraction to within second order refractive effects. The basis for this procedure is that for an operating frequency considerably above the critical frequencies of the ionospheric medium, the group velocity is slightly less than the velocity of light.

CHAPTER 1

Introduction

Arnold Sommerfeld and Leon Brillouin (SOM 1, BRI 1) first discussed the velocities of propagation of electromagnetic energy in a dispersive medium in 1914. At that time, a controversy existed as to the validity of the statement in Einstein's theory that the velocity of light could not be exceeded for electromagnetic wave propagation. W. Wien had proposed that, since the group velocity in a dispersive medium can be infinite, a contradiction to Einstein's theory existed. Sommerfeld was able to show by means of a complex plane integration that solutions of Maxwell's equations were zero until the vacuum propagation time, equal to the distance from the source divided by the velocity of light, had elapsed. In his dissertation under Sommerfeld, Brillouin completed the wave equation solutions and obtained relations between the phase and group velocities, and a new quantity representing the actual velocity of propagation of energy at the signal frequency. He called this quantity the signal velocity.

For many years after the controversy was settled, the results described above were mainly of academic interest and were mentioned in only a few texts on electromagnetic theory. In particular, the nonrealistic character of the group velocity was seldom discussed. The student obtains the impression that the physical velocity of propagation of a wave packet is the group velocity without a description of the restrictions of this approximation. In the past few years, interest in velocity measurements using electromagnetic energy has revived interest in the original work of Sommerfeld and Brillouin with the result that the work is now available in English (BRI 2).

In a recent article, S. M. Harris (HAR 1) exploits the fact that in an ionospheric type of dispersive medium at an operating frequency well above the frequency range of strong dispersive effects, the group velocity is as much less as the phase velocity is greater than the velocity of light in vacuum to within second order refractive effects. He then proposes that simple averaging of the phase and group velocities enables one to determine the vacuum velocity of the wave independent of the dispersive effects of the medium.

The problem is then suggested that one may desire to relate the various velocities in other types of dispersive media than the ionoshere and perform compensation for the dispersive effects at operating frequencies within the dispersive range. Also one desires to know the mean value and variance of the velocity measurements in an inhomogeneous or random medium which may be dispersive. This thesis proposes a method of analysis for obtaining answers to these aspects of the problem.

In Chapter 2, the theory of refractive compensation proposed by Harris is reviewed, and the pertinent results of his work are derived and discussed. Some of the work of Brillouin and Sommerfeld, who first discussed the nature of propagation velocities in a dispersive medium is considered briefly.

In Chapter 3, the steady-state response equations of media such as are encountered in the atmosphere are derived. The equations for the dielectric constant of a magneto-ionic medium are obtained and discussed. Assuming a certain mode of propagation, the formula for the complex refractive index function for steady-state propagation of a sine wave in a magneto-ionic medium is obtained for use in development of the thesis. This equation is typical for a propagation situation in the ionosphere. Following this treatment for an ionized medium a derivation is given for the complex refractive index function for non-polar and polar molecules. This equation is descriptive of the dispersive nature of the troposphere. The relations of this chapter are obtained using classical-mechanical models of linear oscillators. A brief survey is given of the changes required in the equations of the complex refractive function for a quantum-mechanical analysis of dispersion.

In Chap. 4, a technique for solving Maxwell's equations using the multiple Laplace transformation and inversion integrals is presented. The solution of Maxwell's equations for a plane wave is a simple and useful example which is employed throughout the report. The multiple Laplace transform applied to partial differential equations consists of transforming with respect to each positional coordinate as well as time, and seeking the solution of the resulting algebraic equations in a sequence of inversions. The partial differential equation may be transformed if the coordinate system is separable.

Initial and boundary conditions are introduced as their transforms, and it is not necessary to assume the form of the solution. It is shown that the characteristic impedance of the medium must necessarily relate the electric field strength and magnetic intensity vectors in order to obtain a nontrivial or travelling wave solution (the standing wave solution is a special case not discussed). It is also shown that expressions for the phase and group velocity may be obtained simply by imposing invariance conditions on the phase or amplitude of the envelope of the propagated wave and solving the resulting equations for the velocity. Complete and detailed solutions of wave equations by the multiple Laplace transform for a plane wave solution are given and the author feels that this is a contribution which may have instructional value. The conditions for a travelling wave, use of initial and boundary conditions, application of invariance methods for derivation of group and phase velocity expressions, and rigorous incorporation of dispersion characteristics are felt to be contribution in techniques of solution of wave equations. The advantage of the Laplace transform in permitting easy adaptation to spectral formulations has long been recognized and is exploited in the thesis. The Laplace transform solution also expedites rigorous derivation of quantities such as the characteristic impedance and dielectric constant of complex media.

Maxwell's equations for a dispersive medium are formulated using the equation for the complex refractive index functions of the ionosphere and troposphere. The solution for propagation of a plane wave in a dispersive medium is given in terms of a complex inversion integral. The integration of this integral is considered in detail, and due to the irrational nature of the exponential kernel, it is necessary to obtain approximate solutions by replacing the original contour integration with integration on contours passing over saddle-points of the exponent of the kernel. The manner in which the saddle-points depend on the time of propagation and the running time variable is discussed along with the effects of saddlepoint location upon the characteristics of the wave.

At this point the concept of signal velocity as defined by Brillouin is introduced and expressions obtained for the signal velocities for the magneto-ionic, electron displacement and polar resonances of atmospheric media. The methods for evaluation are indicated with means for obtaining necessary constants for other types of molecular resonances. The refraction compensation scheme of Harris is now discussed in terms of its extension to regions of dispersion and constitutes a contribution to this theory.

In Chap. 5, a review is given of the theory of statistical turbulence as it applies to the randomness of the tropospheric refractive index. The physical nature of the turbulence is discussed as it affects the density of distribution of molecular species. Expressions are then derived for the mean square values, mean value, and the variance of the various velocities of propagation in terms of the refractive index spectral density functions. The latter results are the primary contribution of the thesis.

CHAPTER 2

Review of Pertinent Literature

In a recent paper by Harris (HAR 1) it is proposed to obtain an estimate of the true velocity of a wave propagated through a spherically stratified ionosphere by calculation of the arithmetic average of the phase and group velocities of the wave. In the analysis of the problem by that author, repeated in brief form below, it is shown that this estimate is accurate to within second-order refractive effects. The possibility of generalizing the compensation of refractive effects for other media and conditions in the media was considered and is treated in this thesis.

The complex refractive index of a region populated with free electrons is given by the Appleton-Hartree equation (MIT 1). For frequencies well above the cyclotron frequency $\omega_\odot$ of the ions, we may write

$$n^2 = (1-\omega_\odot^2/\omega^2), \tag{2.1}$$

where ω is the angular frequency of the incident radiation. The phase velocity v_p of a wave propagated on a path ST may be expressed in terms of the refractive index as

$$v_p = c/n \tag{2.2}$$

where n is the index of refraction on the path and c is the velocity of propagation of the radiation in free space. The vacuum phase range of the path ST is proportional to the total phase shift and, assuming 2π radians phase shift per vacuum wavelength, is given by

$$R_p(\omega) = ct_p = \int_S^T n(\omega)ds, \tag{2.3}$$

where ds is an element of path length and t_p is defined as the phase time delay. $R_p(\omega)$ is also known as the electrical length. It is well known (BOR 1, Section 3.3) that the integral (2.3) is an extremal by Fermat's principle.

The value of ω_o^2/ω^2 is approximately 0.01 or smaller at the frequency range considered by Harris. Consequently a power series expansion for n is valid, thus

$$n(\omega) = \sqrt{1-\omega_o^2/\omega^2}$$
$$= 1-\omega_o^2/2\omega^2 - 1/8\,(\omega_o^4/\omega^4) - \ldots$$

from which we have, substituting this result into (2.3),

$$R_p(\omega) = \int_S^T ds - (1/2\omega^2)\int_S^T \omega_o^2(x,y,z,)ds - (1/8\,\omega^4)\int_S^T \omega_o^4(x,y,z)ds \tag{2.4}$$

In the integral relation (2.3), $n(\omega)$ is a function of the coordinates of the path. In the integral relations of (2.4), this dependence is born by $\omega_o(x,y,z)$.

The integral of ds over the path ST is the true, or geometrical length of the path R_o, that is,

$$\int_S^T ds = R_o, \tag{2.5}$$

Thus, from (2.4) electrical length of the path becomes

$$R_p = R_o - \frac{A}{\omega^2} - \frac{C}{\omega^4} - \ldots \tag{2.6}$$

where

$$A = (1/2)\int_S^T \omega_o^2(x,y,z)ds \tag{2.7}$$

and

$$C = (1/8)\int_S^T \omega_o^4 (x,y,z)ds \qquad (2.8)$$

It may be shown (BOR], Sect. 1.3) and will be discussed in later sections, that the delay time of a group or wave packet is given by

$$t_g = \left[\partial g(\vec{r})/\partial\omega\right]_{\bar{\omega}} \qquad (2.9)$$

where $\bar{\omega}$ is the mean frequency of the wave group and $g(\vec{r})$ expresses the variation of the phase for a wave packet of the form

$$V(\vec{r},t) = \mathrm{Re}\int_{(\Delta\omega)} a_\omega(\vec{r})\, e^{-[\omega t - g_\omega(\vec{r})]}\, d\omega \qquad (2.10)$$

An explicit method for obtaining the group delay is developed in Section IV. The phase is expressed as a function of the mean frequency approximately as follows

$$g(\vec{r}) = \bar{\omega}\, t_p \qquad (2.11)$$

where t_p, the phase delay, is associated with the mean frequency $\bar{\omega}$

Applying (2.9) to (2.11), and since $t_p = R_p/c$, we have from (2.4) that

$$t_g = d(\omega t_p)/d\omega = (1/c)\int_S^T ds + (1/2c\omega^2)\int_S^T \omega_o^2(x,y,z)ds$$
$$+ (3/8c\,\omega^4)\int_S^T \omega_o^4(x,y,z)ds \qquad (2.12)$$

The group range R_g is given by

$$R_g = ct_g = \int_S^T ds + (1/2\omega^2)\int_S^T \omega_o^2(x,y,z)ds + (3/8\ \omega^4)\int_S^T \omega_o^4(x,y,z)ds,$$
$$= R_o + A/\omega^2 + 3C/\omega^4 \qquad (2.13)$$

From (2.6) and (2.13) the arithmetic average of the group and phase measurements is then

$$R = (R_g + R_p)/2 = R_o + C/\omega^4, \qquad (2.14)$$

so that the average will be free of the first-order, $(1/\omega^2)$, refraction term. Harris considers other effects in the ionosphere such as magneto-ionic effects which, because they add a conductivity effect, require an imaginary component in the index of refraction. The expression for the index then has both even and odd components, and the expansion involves the odd powers of $1/\omega^2$ as well. It cannot be expected that these will cancel, since the cancellation noted above is due to the evenness of the index of refraction when it can be considered to be a real function. The general conditions for such cancellation will be considered in more detail below.

The restriction of the compensation scheme proposed by Harris to an operating frequency well above the reasonace band of the medium permits use of the particular functional forms given above for the phase characteristic. In the study to be presented, the subject of the effect of resonances of the medium is considered in detail, and more general conditions for obtaining compensation developed. In general, any material medium may exhibit resonances, and hence refractive compensation may be considered for media other than the ionosphere.

The properties of the group and phase velocity as a function of frequency was studied for optical media by Sommerfeld and Brillouin in 1914 (BRI 2). Brillouin notes that the group velocity differs from the actual velocity with which the signal propagates, and, in the range of the resonances of the medium, the group velocity is equal to the signal velocity only for certain frequencies. The nonphysical nature of the group velocity is quite evident in the resonance region. The signal and phase (Doppler) velocities are the only quantities which may be of use in a general compensation scheme. It is apparent from the work of Sommerfeld and Brillouin that other schemes than the arithmetic averaging of phase and group velocity may be used at operating frequencies closer to the resonance band.

In the following discussion, the nature of resonances of the medium is first determined. The resonances are related to the physical properties of the material comprising the medium, and in the case of magneto-ionic effects, to the direction of propagation with respect to the magnetic field. Transforms of the wave equations are employed throughout in order to facilitate discussion of random effects in the medium

CHAPTER 3

Phase and Absorption Characteristics of Media in Steady State

3.1 Magneto-ionic and Collision Absorption of the Ionosphere

We shall consider characteristics of the ionosphere and troposphere as they affect propagation of an electromagnetic wave. The difference between these two media is due to the concentration of ionized versus un-ionized varieties of material comprising the media. Consider first (JOR 1, Sec. 17.03) the ionospheric region which is a dielectric region containing free electrons and ions. In the absence of these free charges the constants of this region would be essentially those of free space, that is $\epsilon = \epsilon_v$, $\mu = \mu_v$, and $\sigma = 0$, where ϵ_v and μ_v are the free space values of the dielectric constant and permeability, and σ is the conductivity of the medium. As the electromagnetic wave passes, the charges have imparted to them an oscillatory motion that absorbs some of the incident energy which they reradiate as a source.

For an analysis of the ionosphere we consider an ion or electron density of N ions or electrons per cubic meter. If $\vec{E}$ is the field strength in volts per meter of the electromagnetic wave, the force on the charged particle having a charge q will be $q\vec{E}$. For an effective damping due to collisions of $R_e\vec{v}$, the equation of motion for the particle will be

$$q\vec{E} = m(d\vec{v}/dt) + R_e\vec{v} \tag{3.1}$$

where m is the mass of the particle, $\vec{v}$ is the velocity, and R_e is an effective frictional resistance. The actual average frictional force due to collisions is given by $mv\nu$, where mv is the average momentum lost lost on collision and ν is the frequency of collision. Thus

$$R_e = m\nu . \tag{3.2}$$

For a sinusoidal variation of the field strength with time, i.e.,

$$\vec{E} = \vec{E}_o e^{j\omega t}, \tag{3.3}$$

the velocity of the particle is of the form $\vec{v} = \vec{v}_o e^{j\omega t}$ (3.4)

where

$$\vec{v}_o = \vec{E}_o q/(R_e + j\omega m). \tag{3.5}$$

Now the current density for a flux of such ions is expressed by

$$\vec{i} = Nq\vec{v}, \tag{3.6}$$

where N is the number of charges per unit volume. The current density is also sinusoidal from (3.4), i.e.,

$$\vec{i} = i_o e^{j\omega t} = \left[\vec{E}_o Nq^2/(R_e + j\omega m)\right] e^{j\omega t}.$$

Or

$$\vec{i}_o = \left[N_q{}^2 \vec{E}_o R_e/(R_e{}^2 + \omega^2 m^2)\right] - j\omega m \left[\vec{E}_o N_q{}^2/(R_e{}^2 + \omega^2 m^2)\right].$$

Substituting from (3.2)

$$\vec{i}_o = \left[N_q{}^2 \vec{E}_o \nu / m(\nu^2 + \omega^2)\right] - j\omega \left[\vec{E}_o N_q{}^2/m(\nu^2 + \omega^2)\right]. \tag{3.7}$$

For these sinusoidal fields, Maxwell's equation for the electromotive force may be written

$$\begin{aligned} \text{curl } \vec{H}_o &= j\omega\epsilon_v \vec{E}_o + \vec{i}_o \\ &= j\omega\epsilon_v \vec{E}_o\left[1 - Nq^2/\epsilon_v m(\nu^2+\omega^2)\right] + Ne^2\nu\vec{E}_o/m(\nu^2+\omega^2) \\ &= (j\omega\epsilon_r\epsilon_v + \sigma)\vec{E}_o \end{aligned} \tag{3.8}$$

where the dielectric constant of the medium relative to vacuum is

$$\epsilon_r = 1 - Nq^2/\epsilon_v m(\nu^2+\omega^2) \tag{3.9}$$

and the conductivity of the medium is

$$\sigma = Nq^2\nu/m(\nu^2+\omega^2). \tag{3.10}$$

We note that for a given frequency, the conductance σ is maximum when ω equals the collision frequency ν. The presence of the charged particles reduces the dielectric constant below that of free space and reults in a conductivity that is maximum at the collision frequency. The extent of the effect is a function of the collision frequency and the density of the particles.

The presence of the Earth's magnetic field couples the motion of the charged particles along coordinates transverse to the direction of the magnetic field. Consider the simplified case of a constant magnetic field $\vec{B}_o$ aligned along the positive axis of propagation. The non-relativistic vector equation for the Lorentz force on the ions or electrons may be equated to the mechanical forces,

$$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B}_o) = m(d\vec{v}/dt) + R_e\vec{v}. \tag{3.11}$$

We are assuming that

$$\vec{B}_o = B_o\vec{k} \tag{3.12}$$

where $\vec{k}$ is the unit vector along the axis of propagation (the z-axis), hence

$$B_x = B_y = 0 \tag{3.13}$$

Again assume the incident field is of constant amplitude and frequency, then the velocity will also be sinusoidal since the equation of motion (3.11) is linear. Then

$$\vec{v} = \vec{v}_o e^{j\omega t} \tag{3.14}$$

The rectangular components of Eq. (3.11) may be written

$$E_{ox} + v_{oy}B_o = j\omega v_{ox}\, m/q + R_e v_{ox}/q \tag{3.15}$$

$$E_{oy} - v_{ox}B_o = j\omega v_{oy}\, m/q + R_e v_{oy}/q \tag{3.16}$$

$$E_{oz} = j\omega\, m/q\, V_{oz} + R_e V_{oz}/q \tag{3.17}$$

Equations (3.15) and (3.16) contain only terms in the velocity components in the transverse plane and may be solved simultaneously for these, thus

$$v_{ox} = \left[qE_{ox}(R_e + j\omega m) + q^2 E_{oy} B_o\right] \Big/ \left[(q^2 B_o{}^2 + R_e{}^2 - \omega^2 m^2) + 2j\omega\, mR_e\right] \tag{3.18}$$

and

$$v_{oy} = \left[qE_{oy}(R_e + j\omega m) - q^2 E_{ox} B_o\right] \Big/ \left[(q^2 B_o{}^2 + R_e{}^2 - \omega^2 m^2) + 2j\omega m R_e\right] \tag{3.19}$$

For a general orientation of the magnetic field, the velocities along each axis may contain terms of each type.

If the effect of the collisions can be neglected ($R_e = 0$), it is seen that the expressions reduce to

$$v_{ox} = \left[j\omega(m/q)E_{ox} + B_o E_{oy}\right] / \left[B_o{}^2 - \omega^2 (m/q)^2\right], \tag{3.20}$$

and

$$v_{oy} = \left[j\omega(m/q)E_{oy} - B_o E_{ox}\right] / \left[B_o{}^2 - \omega^2 (m/q)^2\right]. \tag{3.21}$$

There is a resonant frequency at

$$\omega_o = B_o(q/m) \tag{3.22}$$

From Jordan (JOR 1), assuming the earth's magnetic field intensity is 0.5 gauss (0.5X10^{-4} webers/m^2 and q/m for the electron is 1.77x10^{11}

coulombs/kg, the resonant or gyrofrequency is 1.4 Mcps for electrons and 800cps for the hydrogen ion. The effect of the collision frequency (and thus R_e) upon the gyrofrequency is to broaden the response and to lower it slightly, in the usual manner for the damping term of the characteristic equation for a linear system. The effect of such a resonant frequency upon radiation incident upon the ionosphere is to cause absorption at this frequency. (Emission due to excited quantum states is discussed briefly at the close of the chapter.) It is well established for daytime propagation that a broad absorption band does exist in the ionosphere with a center frequency in the neighborhood of 1400 kcps (JOR 1, MIT 1).

Equations (3.20), (3.21), (3.22) may be substituted into (3.6) to obtain an expression for the components of the current density, assuming the charge is the electronic charge e, thus

$$i_{ox} = E_{ox}\left\{\left[j\omega Ne(m/e)\right]/\left[B_o^2-\omega^2(m/e)^2\right]\right\} + E_{oy}\left\{NeB_o/\left[B_o^2-\omega^2(m/e)^2\right]\right\} \quad (3.23)$$

$$i_{oy} = E_{ox}\left\{NeB_o/\left[B_o^2-\omega^2(m/e)^2\right]\right\} + E_{oy}\left\{\left[j\omega(m/e)Ne\right]/\left[B_o^2-\omega^2(m/e)^2\right]\right\} \quad (3.24)$$

$$i_{oz} = E_{oz}Ne/j\omega(m/e) \quad (3.25)$$

For a region having such a current density, Maxwell's equation for current density expressed in rectangular cartesian coordinates is

$$\mathrm{curl}_x H = i_{ox} + \epsilon_V E_{ox} \quad (3.26)$$

$$\mathrm{curl}_y H = i_{oy} + \epsilon_V E_{oy} \quad (3.27)$$

$$\mathrm{curl}_z H = i_{oz} + \epsilon_V E_{oz} \quad (3.28)$$

Assuming sinusoidal variations, we replace E_{ox}, E_{oy}, E_{oz} in (3.23), (3.24), (3.25) by $E_{ox}/j\omega$, $E_{oy}/j\omega$, and $E_{oz}/j\omega$, respectively, and substitute the results into (3.26), (3.27), and (3.28), obtaining

$$\text{curl}_x H = \epsilon_v \dot{E}_{ox} \left\{ 1+Ne^2/[\epsilon_v m(\omega_o^2-\omega^2)] \right\} - j\left\{Ne^2\omega_o/[\omega m(\omega_o^2-\omega^2)]\right\} \dot{E}_{oy} \tag{3.29}$$

$$\text{curl}_y H = \epsilon_v \dot{E}_{oy} \left\{1+Ne^2/[\epsilon_v m(\omega_o^2-\omega^2)]\right\} + j\left\{Ne^2\omega_o/[\omega m(\omega_o^2-\omega^2)]\right\} \dot{E}_{ox} \tag{3.30}$$

$$\text{curl}_z H = \epsilon_v [1-Ne^2/\epsilon_v \omega^2 m] \dot{E}_{oz} \tag{3.31}$$

If we define the critical angular frequency of the conducting medium as

$$\omega_c = \sqrt{Ne^2/\epsilon_v m} \tag{3.32}$$

we may write

$$\text{curl}_x H = \epsilon_2 E_{ox} - j \epsilon_3 E_{oy} \tag{3.33}$$

$$\text{curl}_y H = \epsilon_2 E_{oy} + j \epsilon_3 E_{ox} \tag{3.34}$$

$$\text{curl}_z H = -\epsilon_1 E_{oz} \tag{3.35}$$

where

$$\epsilon_1 = \epsilon_v(1-\omega_c^2/\omega^2) \tag{3.36a}$$

$$\epsilon_2 = \epsilon_v \left[1-\omega_c^2/(\omega^2-\omega_o^2)\right] \tag{3.37a}$$

$$\epsilon_3 = -\epsilon_v \left[\omega_o\omega_c^2/\omega(\omega^2-\omega_o^2)\right] \tag{3.38a}$$

The relative dielectric constants are then

$$\epsilon_{r1} = 1-\omega_c^2/\omega^2 \tag{3.36b}$$

$$\epsilon_{r2} = 1 -\omega_c^2/(\omega^2-\omega_o^2) \tag{3.37b}$$

$$\epsilon_{r3} = - \omega_o\omega_c^2/\omega(\omega^2-\omega_o^2) \tag{3.38b}$$

Thus there are essentially three distinct dielectric constants ϵ_1, ϵ_2, and ϵ_3. The presence of the magnetic field has caused the medium to exhibit different responses with respect to the propagation direction. The effect along the axis is contained in the ϵ_1 term, and it is this frequency dependence which is utilized by Harris (HAR 1) in deriving his refraction compensation scheme described in Chapter 2. We shall also typify the resonance characteristics of ionospheric propagation by an equation of this form in the following sections.

If the collision damping effects cannot be ignored, we have upon substituting (3.18) into (3.6)

$$i_{ox} = \left[Nq^2(R_e + j\omega m)E_{ox} + Nq^3 B_o E_{oy}\right] / \left[(q^2B_o^2 + R_e^2 - \omega^2 m^2) + 2j\omega m R_e\right].$$

Since the principle absorption is due to the motion of electrons, q = e, the electron charge. Also from (3.2), (3.22), (3.32),

$$R_e = m\nu$$

$$\omega_o = B_o e/m$$

$$\omega_c^2 = Ne^2/\epsilon_v m,$$

so that, as a function of ω,

$$i_{ox}(\omega) = \epsilon_v \omega_c^2 \left\{ \left[(\nu + j\omega)E_{ox}(\omega) + \omega_o E_{oy}(\omega)\right] / \left[(\omega_o^2 + \nu^2 - \omega^2 + 2j\omega\nu\right] \right\}$$

$$= \epsilon_v \omega_o^2 \left[(\nu + j\omega)\ E_{ox}(\omega) + \omega_o E_{oy}(\omega)\right] / \left[(j\omega + \nu + j\omega_o)\ (j\omega + \nu - j\omega_o)\right] \quad (3.39)$$

Whereas the ionosphere exhibits effects principally due to the response of free electrons to the incident radiation, such charged particles are of negligible importance to propagation phenomena in the troposphere. This is due to the fact that those electrons and ions that are formed by high energy collisions in the lower altitudes quickly recombine due to the much lower mean free path. However dispersive effects still exist in media that do not contain such free charged particles. These are due to the interaction of the electromagnetic wave with the electronic charges of nonpolar and polar molecules. Such effects are of minor importance in the ionosphere. The following section considers the response to electromagnetic waves of a medium containing only polar and nonpolar molecules.

3.2 Elementary Theory of Absorption in a Medium Consisting of Nonpolar and Polar Molecules

When a dielectric is placed in an electric field it acquires surfaces charges on its faces, proportional to the strength of the field. These surface charges contribute to the field just as do any other charges which may be applied by external means. Our theory shall treat all charges in like manner for their effect upon the medium. In applying Maxwell's equations to the dielectric, it shall be considered different from free space only due to the presence of these polarizable electrons. The polarization charge must be produced in the originally uncharged dielectric by the motion of positive charges in the direction of the applied field and of the negative charges in the opposite direction, depending upon the relative mobilities.

We shall consider the interaction of the electric field upon the medium by expressing the flux densities $\vec{D}$ and $\vec{B}$ as the sum of two terms (BOR 1). Of these, one is taken to be the vacuum field and one is regarded as arising from the influence of matter. One is thus lead to the introduction of two new vectors for describing the effects of matter: the electric polarization $\vec{P}$ and the magnetic polarization or magnetization $\vec{M}$. Relations involving $\vec{P}$ and $\vec{M}$ replace the usual material relations of Maxwell's equations:

$$\vec{D} = \epsilon\vec{E} \tag{3.40}$$

$$\vec{B} = \mu\vec{H} \tag{3.41}$$

The use of $\vec{P}$ amd $\vec{M}$ results in a more direct physical meaning of the interaction. Thus an electromagnetic field produces in a given volume element of medium an amount of polarization proportional to the field.

Each volume element becomes the source of a new secondary or scattered wavelet whose strength is related in a simple way to $\vec{P}$ amd $\vec{M}$. All the secondary wavelets combine with the incident field and with each other to form the total field. Instead of the relations (3.40), (3.41), we shall now describe the interaction of matter and field by means of the relations

$$\vec{D} = \epsilon_v \vec{E} + \vec{P} \tag{3.42}$$

$$\vec{B} = \mu_v \vec{H} + \vec{M}, \tag{3.43}$$

where ϵ_v and μ_v are the dielectric constant and permeability of vacuum. $\vec{P}$ amd $\vec{M}$ vanish in a vacuum and therefore represent the influence of the matter. One regards matter as composed of interacting particles embedded in the vacuum, producing fields which have large local variations in the interior of the matter. This internal field is modified by any field which is applied externally, and the properties of the matter are then derived by averaging over the whole field within it. As long as the region over which the average is taken is large compared with the linear dimensions of the particles the electromagnetic properties of each can be represented as those of an electric and magnetic dipole. The secondary fields are then just the retarded fields of these dipoles. It is assumed then that

$$\vec{P} = \eta \vec{E} \tag{3.44}$$

$$\vec{M} = \chi \vec{H} \tag{3.45}$$

The factor η is called the electric susceptibility and χ the magnetic susceptibility, where from (3.40), (3.41), (3.42), and (3.43), these quantities are related to the dielectric constant and the magnetic permeability by

$$\epsilon = \epsilon_v + \eta \tag{3.46}$$

$$\mu = \mu_v + \chi \tag{3.47}$$

It is necessary to distinguish between the effective fields, $\vec{E}'$ or $\vec{H}'$, acting on a molecule and the mean or observed field, $\vec{E}$ or $\vec{H}$, obtained by averaging over a region which contains a great number of molecules. The difference between the two fields is due to the gaps between the molecules and depends on the number of molecules per unit volume.

To estimate the difference $\vec{E}' - \vec{E}$ between the effective field $\vec{E}'$ and mean field $\vec{E}$, consider a particular molecule centered within a sphere of radius large compared to its linear dimensions. Following Slater and Frank (SLA 1), we replace the effects of the external fields by the polarization on this sphere. We may calculate the force at the center of the sphere using the system of induced charges on the surface of the sphere. The surface density of induced charge on a spherical ring at an angle θ to the direction of the field is $\vec{P}\cos\theta$. The area of the ring is $2\pi R^2 \sin\theta \, d\theta$. The charge on the ring is therefore $2\pi\vec{P}R^2 \sin\theta \cos\theta \, d\theta$. This charge produces a field at the center of the sphere whose component parallel to $\vec{E}$ is

$$d\vec{E}' = \left\{\left[2\pi\vec{P}R^2\cos^2\theta\sin\theta\right]/\left[4\pi\epsilon_v R^2\right]\right\}d\theta \tag{3.48}$$

The total charge on the spherical surface produces a field at the center equal to

$$\vec{E}' = (\vec{P}/2\epsilon_v)\int_0^{\pi} \cos^2\theta \sin\theta d\theta = (\vec{P}/3\epsilon_v) \qquad (3.49)$$

It has been shown that the force exerted upon the central molecule by other molecules within the sphere is zero (BOR 1). The total field within the sphere, which is the effective field acting on the central molecule, is obtained by adding to this the mean field $\vec{E}$, given by

$$\vec{E}' = \vec{E} + \vec{P}/3\epsilon_v \qquad (3.50)$$

If we now assume that the molecule is a dipole with one electronic charge, the force on the electron is $e\vec{E}'$. When the electron undergoes displacement $\vec{x}$ due to polarization of the applied field, the restoring force is $-a\vec{x}$. Therefore

$$-a\vec{x} + e\vec{E}' = 0 \qquad (3.51)$$

and

$$\vec{x} = e\vec{E}'/a \qquad (3.52)$$

The dipole moment so induced is

$$e\vec{x} = e^2\vec{E}'/a. \qquad (3.53)$$

The quantity e^2/a is called the mean polarizability α of the molecule. It expresses the proportionality between the applied field and the electric dipole moment. The total electric moment per unit volume is

$$\vec{P} = N\vec{p} = N\alpha\vec{E} \qquad (3.54)$$

Eliminating $\vec{E}'$ between (3.50 and (3.54), the dielectric susceptibility is

$$\eta = N\alpha/\left[1-(N\alpha/3\epsilon_v)\right]. \qquad (3.55)$$

Substituting for η from (3.47), we obtain the following expression for the dielectric constant

$$\epsilon = \epsilon_v + N\alpha/\left[1-(N\alpha/3\epsilon_v)\right] \tag{3.56}$$

or, solving for α, we obtain the Lorenz-Lorentz relation (mks system), using $\epsilon = \epsilon_r\epsilon_v$,

$$\alpha = 3\epsilon_r(\epsilon_r-1)/N(\epsilon_r+2) \tag{3.57}$$

where ϵ_r is the dielectric constant of the medium relative to vacuum. Finally employing Maxwell's relation $n^2 = \epsilon_r\mu_r$, or $n^2 = \epsilon_r$ for non-magnetic media

$$\alpha = 3\epsilon_v(n^2-1)/N(n^2+2) \tag{3.58}$$

In order to determine the dependence of the polarization and the refractive index of nonpolar molecules upon the frequency of the field we must find the displacement $\vec{r}$ of each charged particle from its equilibrium position. We assume that each electron is acted on by the Lorentz force $\vec{F}$

$$\vec{F} = e(\vec{E}'+\vec{v}\times\vec{B}) \tag{3.59}$$

where e is the charge of the electron and $\vec{v}$ is its velocity. It will be assumed that $|\vec{v}|$ is small compared to the velocity of light. The following derivation assumes a very elementary model for the interaction of the charged particle upon its force center and neglects quantum-mechanical effects. Even so, the result is useful for describing the principle nature of dispersion. We shall assume that an electron behaves as if it were bound to an equilibrium position by a quasi-elastic restoring force $-q\vec{r}$ where $\vec{r}$ is the displacement, and that the damping force due to collisions is $g\dot{\vec{r}}$. The equation of motion is then

$$m\ddot{\vec{r}} + g\dot{\vec{r}} + q\vec{r} = e\vec{E}' \tag{3.60}$$

If ω is the angular frequency of the incident field,

$$\vec{E}' = \vec{E}'_o \, e^{-j\omega t} \tag{3.61}$$

The displacement is also harmonic and the steady state solution is

$$\vec{r} = e\vec{E}'/[m(\omega_o^2-\omega^2)-j\omega g] \tag{3.62}$$

where the natural frequency

$$\omega_o = \sqrt{q/m} \tag{3.63}$$

Each electron contributes to the polarization a moment $\vec{p} = e\vec{r}$. The contributions of the motions of the heavy nuclei are neglected, since their displacements are several orders of magnitude below $\vec{r}$. Assuming also that there is only one effective electron in a molecule with resonant frequency ω_o, the total polarization $\vec{P}$ is

$$\vec{P} = N\vec{p} = Ne\vec{r} = Ne^2\vec{E}'/m\,(\omega_o^2-\omega^2)-j\omega g \tag{3.64}$$

where N is the number of molecules per unit volume. $P = N\alpha$, therefore

$$N\alpha = Ne^2/[m(\omega_o^2-\omega^2)-j\omega g \tag{3.65}$$

so that $N\alpha$ is a complex quantity related to the index of refraction by (3.58), so that

$$(N\alpha/3\epsilon_v) = (n^2-1)/(n^2+2) \tag{3.66}$$

The behavior of $N\alpha$ is shown in Figure 3.1. Between the region of the maxima is the region $\omega_1 - \omega_2$ of anomalous dispersion.

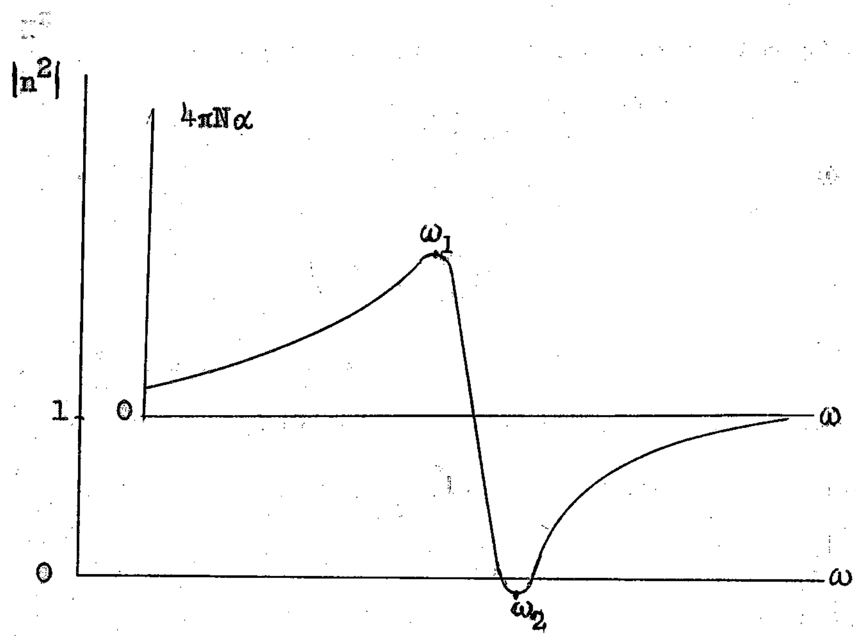

Figure 3.1

Dispersion of a Nonpolar Gas Showing Anomulous Dispersion

We have so far considered that the resonating molecule has only one resonant frequency. In general there will be many resonant frequencies even in systmes with the same kind of molecules. A more general expression, neglecting damping and the motion of the nuclei, is

$$(N\alpha/3\epsilon_v) = (n^2-1)/(n^2+2) = (Ne^2/3\epsilon_v m) \sum_i \left[h_i/(\omega_i^2-\omega^2)\right] \quad (3.67)$$

where Nh_i is the number of electrons corresponding to the resonant frequency ω_i. For gases, the index of refraction is approximately unity, and we may rewrite (3.67) in the form

$$n^2-1 = (N\alpha/\epsilon_v) = (Ne^2/\epsilon_v m) \sum_i \left[h_i/(\omega_i^2-\omega^2)\right] \quad (3.68)$$

For polar molecules it is necessary to account for the energy absorbed in the angular orientation of the molecule as the polarity of the field reverses. Debye (DEB 1) considers the analysis of this case in detail. For the present purposes, some simplification is permissible. The following analysis of the effect of a permanent polarization in the molecule is adapted from the presentation of Loeb (LOE 1).

Certain molecules, such as water, have permanent dipoles present, and their dielectric constant is composed of two types of action. There is the usual oscillatory separation of the charges by the field as analyzed above. Also because the fixed dipoles in the molecules are oriented in all directions because of thermal agitation, they suffer torques as they tend to align themselves with the electrical field. This torque is temperature dependent because the alignment is being continually destroyed by the random impacts of neighboring molecules under thermal agitation. But, on the average, there is a resultant component of these dipoles in the field and they act to increase the dielectric strength of the material, or, contribute to the polarizability of the molecules. Since the action expressed by the Lorenz-Lorentz law involves linear coordinates and the present effect involves angular coordinates, they are additive in the sense of energies, and may be analyzed separately. Accordingly the present analysis will be solely concerned with the effect of permanent electric dipoles upon the dielectric constant, the effects of displacement frequencies having been considered above.

Assuming the molecule as a whole is uncharged, the potential energy is given by

$$u = -\vec{p}\cdot\vec{E}$$

$$= -\ pE \cos \theta \qquad (3.69)$$

where θ is the angle between the dipole moment $\vec{p}$ and the electric field strength $\vec{E}$. Thus the number of molecules which have a potential energy u in the field is given by the Maxwell-Boltzmann law (LOE 1) as

$$Ae^{-(u/kT)}d\Omega = Ae^{(pE\cos\theta)/kT}d\Omega \qquad (3.70)$$

where k is Boltzmann's constant , T is the absolute temperature, and $d\Omega$ represents the element of volume surrounding the point where the potential energy is u. The average moment $\overline{p}$ of the dipole in the field is given by

$$\overline{p} = \left[\int_0^{\pi} e^{(pE\cos\theta/kT)} p\cos\theta d\Omega\right] \Big/ \left[\int_0^{\pi} e^{(pE\cos\theta/kT} d\Omega\right] \qquad (3.71)$$

The elementary volume $d\Omega = 2\pi\sin\theta\ d\theta$. Making the substitutions $\xi = \cos\theta$, and $x = pE/kT$, (3.72) becomes

$$(\overline{p}/p) = \left[\int_{-1}^{+1} e^{x\xi}\ \xi\ d\xi\right] \Big/ \left[\int_{-1}^{+1} e^{x\xi}\ d\xi\right] \qquad (3.72)$$

which integrates to the Langevin function

$$(\overline{p}/p) = \left[(e^{x}+e^{-x})/(e^{x}-e^{-x})\right] - (1/x) = L(x) \qquad (3.73)$$

By dividing through by $e^{x}-e^{-x}$ the following asymptotic expansion results which is good for large values of x:

$$L(x) = 1 - (1/x) + 2e^{-2x} + \dots \qquad (3.74)$$

Powers of x and exponents of x in the expansion for this function are negative and it is seen that this function approaches unity as a saturation value at very high fields. For the weak fields of our concern we may use the approximations

$$L(x) = (x/3) - (x^3/45) + \dots,$$

or even only

$$L(x) = x/3. \tag{3.75}$$

$$\bar{p}/p = pE/3kT, \tag{3.76}$$

or

$$\bar{p} = p^2E/3kt, \tag{3.77}$$

which is the contribution of a permanent dipole to the measureable polarizability. A unit volume contains N polar molecules. Hence the polarization for a unit volume is

$$N\bar{p} = Np^2E/3kT \tag{3.78}$$

and the general expression for magnitude of the molecular polarizability of polar molecules of gases is then, including the effect given in (3.64)

$$P = (Ne^2E)/\left[m(\omega_o^2 - \omega^2) - j\omega g\right] + (Np^2E)/(3kT)$$

$$= (EN)\left\{\left[(e^2)/[m(\omega_o^2-\omega^2)-j\omega g]\right] + p^2/3kT\right\}. \tag{3.79}$$

The frequency response of the dipole population must also be considered. There is a definite time required for the majority of the population of the disoriented molecules to return to their previous state of thermal equilibrium. Debye (DEB 1) considers this situation in detail and characterizes the process by a simple exponential decay of the excited population. The complete expression, taking this relaxation time into account is then,

$$P = EN\left\{e^2/[m(\omega_o^2-\omega^2)-j\omega g] + (p^2/3kT)/(1+j\omega\tau)\right\} \tag{3.80}$$

But from (3.54)

$$\vec{P} = N\alpha\vec{E}$$

Hence

$$N\alpha = N\left\{e^2/[m(\omega_o^2-\omega^2)-j\omega g] + p^2/3kT(1+j\omega\tau)\right\} \tag{3.81}$$

From

$$(n^2-1)/(n^2+2) = (\epsilon_r-1)/(\epsilon_r+2) = N\alpha/3\epsilon_v$$
$$= (N/3\epsilon_v)\left\{e^2/[m(\omega_o^2-\omega^2)-j\omega g]\right.$$
$$\left. + (p^2/3kT)/(1+j\omega\tau)\right\} \quad (3.82)$$

which is the final expression for the dielectric constant for polar molecules, including the effects of displacing charges linearly and changing the angular orientation of the electric dipole of the molecule. Since the index of refraction for gases is near unity, i.e., $n^2 = \epsilon_r \approx 1$,

$$\epsilon_r-1 \approx (N/3\epsilon_v)\left\{e^2/[m(\omega_o^2-\omega^2)-j\omega g] + (p^2/3kT)/(1+j\omega\tau)\right\} \quad (3.83)$$

For all possible states,

$$\epsilon_r-1 \approx (N/\epsilon_v) \sum_i \left\{ h_i e^2/[m(\omega_{oi}^2-\omega^2)-j\omega g_i] + [(h_i p_i^2)/3kT(1+j\omega\tau)]\right\}. \quad (3.84)$$

In summary of the brief treatment above, it is seen that the frequency dependence of dielectrics may be represented by linear oscillator models which yield second or third order characteristic equations with constant coefficients for the cases of the ionosphere, Eqs. (3.36), (3.37), (3.39), and nonpolar and polar dielectrics, Eqs. (3.67), (3.68), and (3.84). According to Born and Wolf (BOR 1., p. 97), the classical mechanical model is quite sucessful in predicting experiments and, in fact, the solution for a quantum mechanical model differs only in requiring an infinitude of virtual oscillators instead of a finite number, as was obtained above. However in the quantum mechanical model the weighting of the oscillators is such that only a finite number of the terms may need to be considered. We shall now consider briefly the differences due to quantum-mechanical considerations.

3.3 Brief Survey of Quantum-Mechanical Theory of Dispersion

In the quantum-mechanical theory of dispersion, there is a term of the form of (3.65) associated with each possible quantum transition in the molecule. The coefficient N is replaced with quantities known as the oscillator strengths f_{mn} associated with each resonance frequency by the formula

$$f_{mn} = (2m\omega_{mn}|r_{mn}|^2)/3\hbar \qquad (3.85)$$

in which ω_{mn} is the angular frequency associated with the quantum jump between the m^{th} and n^{th} levels and r_{mn} is the matrix component of the vector coordinates connecting these two states. There is also a damping term associated with each quantum transition, $\gamma_{mn}\omega_{mn}$.

According to Condon (CON 1, Part 6, Chapter 6), there is a feature of the quantum-mechanical theoretical formula of negative dispersion that has no classical analog. In (3.85) it is assumed that the quantum transition m-n is one corresponding to a jump from a lower to a higher level, that is, an absorption. However, if the initial state m is an excited state, then there will be a lower state n, for which the associated quantum jump will be an emission. For this transition, the frequency ω_{mn} and hence the oscillator strength f_{mn} are considered negative. Hence the contribution of dispersion due to molecules in excited states is negative at frequencies associated with emission transitions and subtracts from the total absorption.

If N is the total number of molecules in a unit volume and N_m is the total number of molecules in the m^{th} quantum state, a particular frequency will occur twice in the formula for α, once positively as an absorption, and once negatively as an emmission frequency. The total contribution to α is given by

$$\alpha = (e^2/Nm)\sum_m \sum_n{}' \left[N_m - N_n\right] f_{mn} / \left[(\omega_{mn}^2 - \omega^2) + j\gamma_{mn}\omega\right] \quad (3.86)$$

where the prime on the summation indicates that this sum is to be taken only for levels which are higher in energy than the m^{th} level. The values of f_{mn} may be determined by experiment. It turns out that only certain of these have appreciable strength in a given case, so that a finite number of terms of the summation is sufficient. We shall assume that the details of determination of the coefficients may be settled from quantum-mechanical considerations, if necessary. However, we are most interested in the mathematical form of the resonance term and shall use the classical-mechanical form for convenience.

CHAPTER 4

Velocities of Propagation

4.1 Solutions of the Wave Equation for Nondispersive Media

We shall establish certain results basic to our discussion by considering solutions of the wave equation for a linear medium obtained through the use of multiple Laplace transformations. The single application of the Laplace transform to partial differential equations can be found in the presentations of several authors (see for example CHU 1, STR 1). The application of the multiple Laplace transformation to partial differential equations is treated by van der Pol and Bremmer (POL 1) who gives credit to Heaviside. There is also a suggestion of the existence of such methods in the promise of the second volume of Gardner and Barnes (GAR 1, Preface and p. 320). An elementary discussion of the details of the types of manipulations involved in using one-sided multiple transforms is given by Estrin and Higgins (EST 1). Manipulation of single two-sided transforms is discussed by Truxal (TRU 1) and van der Pol and Bremmer (POL 1).

The performance of waves in a linear medium is described by Maxwell's equations(mks system of units):

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}, \tag{4.1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{4.2}$$

$$\nabla \cdot \vec{D} = \rho, \tag{4.3}$$

$$\nabla \cdot \vec{B} = 0; \tag{4.4}$$

together with the material relations

$$\vec{D} = \epsilon \vec{E}, \tag{4.5}$$

and

$$\vec{B} = \mu \vec{H}, \tag{4.6}$$

where vectorial quantities are indicated by the arrow. We make the additional assumptions that there is no distributed charge and that the medium is non-conducting, hence $\rho = 0$, and $\vec{j} = 0$, resp.

We shall first obtain solutions for a simplified model and coordinate system. The direction of propagation will be along the positive z-axis by a transverse wave whose components are given by the equations

$$E_x = f(z,t) \tag{4.7}$$

$$H_y = g(z,t) \tag{4.8}$$

where f and g are each Laplace-transformable with respect to t and z. Applying equations (4.1), (4.2), (4.5), and (4.6) to the components E_x and H_y, we have

$$-\frac{\partial H_y}{\partial z} = \epsilon \frac{\partial E_x}{\partial t} \tag{4.9}$$

and

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \tag{4.10}$$

Following Estrin and Higgins we define the single and multiple Laplace transforms of a function of two variables in the following manner. The Laplace transform of f(z,t) with respect to t is

$$L_t\left[f(z,t)\right] = F(z,s) = \int_0^\infty e^{-st} f(z,t)\, dt, \tag{4.11}$$

where we assume that $f(z,t)$ is transformable if a real number γ exists such that

$$\lim_{T\to\infty}\int_0^T \left| e^{-\gamma T} f(z,t)\right| dt < \infty \; . \qquad (4.12)$$

This single transform $F(z,t)$ can be transformed again with respect to z by the relation

$$F(r,s) = L_z\left[F(z,s)\right] = \int_0^\infty e^{-rz} F(z,s)\, dz \qquad (4.13)$$

where again convergence of the integral is assumed in the sense equivalent to (4.12). The multiple transform operations indicated are assumed to be interchangeable with respect to the order of transformation. That is, the multiple operation

$$L_z L_t\left[f(z,t)\right] = \int_0^\infty e^{-rz}\int_0^\infty e^{-st} f(r,z)dt\, dz \qquad (4.14)$$

may be written in either order, thus

$$F(r,s) = \int_0^\infty e^{-rz}\int_0^\infty e^{-st} f(z,t)dt\, dz = \int_0^\infty e^{-st}\int_0^\infty e^{-rz} f(z,t)dz\, dt \qquad (4.15)$$

which implies that the integrand is uniformly convergent. The inversion integral is given by

$$f(z,t) = \frac{1}{(2\pi j)^2}\int_{\gamma - j\infty}^{\gamma + j\infty} e^{ts}\int_{\xi - j\infty}^{\xi + j\infty} e^{zr} F(r,s)dr\, ds, \; \text{Re } s > \gamma$$

$$\text{and Re } r > \xi . \qquad (4.16)$$

We now transform (4.9) and (4.10) with respect to both z and t, using (4.7) and (4.8). This operation is called simultaneous transformation, or transposition, by van der Pol and Bremmer (POL 1). Thus

$$-r\, \mathbf{G}\,(r,s) + G(,0,s) = \; s\, \mathbf{F}\,(r,s) - \; F(r,+0) \qquad (4.17)$$

$$r\,\mathbf{F}\,(r,s) - F\,(,0,s) = -\mu s\,\mathbf{G}\,(r,s) + \mu G\,(r,+0) \qquad (4.18)$$

(Bold capital letters indicate double transforms, light capital letters indicate single transforms). Solving (4.17) for $\mathbf{G}$ (r,s) and substituting into (4.18),

$$r\,\mathbf{F}\,(r,s) = F,(+0,s) + \mu G(r,+0) + \frac{\epsilon\mu s^2}{r}\mathbf{F}(r,s) - \frac{\epsilon\mu s}{r}F(r,+0) - \frac{\mu s}{r}G(+0,s) \qquad (4.19)$$

or

$$(r^2 - \epsilon\mu\, s^2)\,\mathbf{F}(r,s) = r\left[F(+0,s) + \mu G(r,+0)\right] - \mu s\left[\epsilon F(r,+0) + G(+0,s)\right] \qquad (4.20)$$

The initial conditions correspond to an unexcited medium at t = +0, i.e.,

$$f(z,+0) = E_x(z,+0) = 0, \quad g(z,+0) = H_y(z,+0) = 0. \qquad (4.21)$$

Hence

$$F(r,+0) = 0, \quad G(r,+0) = 0. \qquad (4.22)$$

Now from (4.17) and (4.18),

$$r\,\mathbf{G}(r,s) + \epsilon\, s\,\mathbf{F}(r,s) = \epsilon F(r,+0) + G(+0,s) \qquad (4.23)$$

$$r\,\mathbf{F}(r,s) + \mu\, s\,\mathbf{G}(r,s) = F(+0,s) + \mu G(r,+0). \qquad (4.24)$$

We shall be interested in solutions for which time and the z-coordinate are positive. Hence the frequency variable s and wave number variable r must also be positive from the definition of the Laplace transform of f(z,t), (4.15). Then the condition for a non-trivial solution is, from (4.23) and (4.24)

$$\mu\left[\mathbf{G}(r,s)\right]^2 - \left[\mathbf{F}(r,s)\right]^2 = 0$$

or

$$\mathbf{G}(r,s) = \sqrt{\frac{\epsilon}{\mu}}\;\mathbf{F}(r,s). \qquad (4.25)$$

The quantities ϵ and μ have positive real parts in absorptive media and the square root $\sqrt{\epsilon/\mu}$ is taken to be a positive quantity. (4.25) constitutes a relation from which conditions on $G(H_y)$ or $F(E_x)$

may be found if only one or the other is specified. $\sqrt{\epsilon/\mu}$ has the dimensions of an electrical impedance and is the characteristic impedance of the medium.

The sinusoid applied at the origin at time t = +0 is

$$f(+0,t) = E_x(+0,t) = E_o \sin\omega t \qquad (4.26)$$

Hence the Laplace transform of the forcing function defined by (4.26) is

$$F(+0,s) = E_o\,\omega/\ (s^2+\omega^2). \qquad (4.27)$$

From (4.25) the condition on H_y is given by

$$G(+0,s) = \sqrt{\epsilon/\mu}\ E_o\,\omega/\ (s^2+\omega^2) \qquad (4.28)$$

Substituting (4.22), (4.27), and (4.28) into (4.20), and solving for $\mathbf{F}(r,s)$

$$\mathbf{F}(r,s) = E_o\,\omega/\ (r-\sqrt{\epsilon\mu}\ s)\ (s^2+\omega^2). \qquad (4.29)$$

Taking the inverse transform with respect to z,

$$F(z,s) = (1/2j)\ E_o\ e^{\sqrt{\epsilon\mu}\ z}\left[\ 1/(s+j\omega) + 1/(s-j\omega)\right] \qquad (4.30)$$

The inverse transform with respect to t is,

$$f(z,t) = E_o\ \sin\,\omega\,(t-\sqrt{\epsilon\mu}\ z) \qquad (4.31)$$

This equation indicates a wave travelling from the origin and arriving at time $t = \sqrt{\epsilon\mu}\ z$. The quantity $1/\sqrt{\epsilon\mu}$ has the dimensions of a velocity and is the velocity with which the wave propagates. The quantities ϵ and μ, the dielectric constant, permeability of the medium, respectively, are related to the values ϵ_v, μ_v for vacuum by the relations

$$\mu = \mu_r\,\mu_v, \qquad (4.32)$$

$$\epsilon = \epsilon_r\,\epsilon_v. \qquad (4.33)$$

The quantities μ_r and ϵ_r are the relative permeability and dielectric constant of the medium. We have the relation between ϵ_v and μ_v and the velocity of light c

$$c^2 = \frac{1}{\mu_v \epsilon_v}, \tag{4.34}$$

so that

$$\begin{aligned} \mu\epsilon &= \mu_r \mu_v \epsilon_r \epsilon_v, \\ &= \frac{\mu_r \epsilon_r}{c^2} = \frac{n^2}{c^2}, \end{aligned} \tag{4.35}$$

where n, the index of refraction of the medium is given by Maxwell's relation,

$$n = \sqrt{\mu_r \epsilon_r}. \tag{4.36}$$

4.2 Phase Velocity for a Nondispersive Medium

We shall now seek the condition for which the phase of the positive-going wave is constant. This may be accomplished by examining the conditions under which the wave is totally invariant. Since the wave is sinusoidal, variation in amplitude is the result of variation of the phase argument. We therefore obtain, from considering invariance of the wave, a condition for invariance of phase. The total differential of the wave is

$$d\left[E_o \sin \omega\,(t-\sqrt{\mu\epsilon}\;z)\right] = 0, \tag{4.37}$$

or

$$\left\{\cos\left[\omega(t-\sqrt{\mu\epsilon}\,z)\right]\right\}\, d\left[\omega(t-\sqrt{\mu\epsilon}\;z)\right] = 0, \tag{4.38}$$

which is satisfied if

$$d\left[\omega(t-\sqrt{\mu\epsilon}\,z)\right] = 0, \tag{4.39}$$

or if

$$\omega\ (dt - \sqrt{\mu\epsilon}\, dz) = 0,$$

whence from (4.31)

$$\frac{dz}{dt} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} = v_p. \qquad (4.40)$$

Since this is the condition for constant phase, v_p is called the phase velocity of the sinusoid. This is the velocity with which surfaces of constant phase are propagated, and as a concept, is discussed in many books on electromagnetic theory (see for example BOR 1 and JOR 1). It has a precise meaning only for sinusoids of infinite extent.

If we apply a time function

$$f(z,t) = E_x(+0,t)$$

at the origin at time t = +0, and express its transform as

$$F(+0,s) = E(s)$$

it follows from (4.20) and (4.25) that

$$F(r,s) = E(s)\ /(r + \sqrt{\epsilon\mu}\ s), \qquad (4.41)$$

so that

$$F(z,s) = E(s)\ \epsilon^{-\sqrt{\epsilon\mu}\, sz} \qquad (4.42)$$

Then, taking the inverse transform with respect to t,

$$f(z,t) = (1/2\pi j)\int_{\gamma - j\infty}^{\gamma + j\infty} e^{s\left[t - \sqrt{\epsilon\mu}\ z\right]} E(s)ds. \qquad (4.43)$$

Under conditions of narrow bandwidth of the spectrum E(s) discussed below, the solution is of the form

$$f(z,t) = E(t - z/v_g),$$

where v_g is the velocity of propagation and depends upon E(s). The velocity v_g associated with the narrow band spectrum E(s) is called the velocity of the wave group E(s), or the group velocity.

4.3 Group Velocity for a Nondispersive Medium

We may write the transformed solution (4.41) as

$$F(r_r+jr_i, \sigma+j\omega) = E(\sigma+j\omega)/\left[(r_r+jr_i) + \sqrt{\epsilon\mu}\,(\sigma+j\omega)\right] \tag{4.44}$$

Where r_r, r_i, σ, ω are the real and imaginary parts of r and s, respectively. Assuming now that the real parts, σ and r_r, are zero (this corresponds to steady state and vanishingly small change of the wave with distance), then (4.44) becomes

$$F(jr_i, j\omega) = E(j\omega)/\ \ j(r_i + \sqrt{\epsilon\mu}\,\omega\ )$$

The inversion integral now becomes a Fourier integral

$$f(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-z/v_p)}\ E(\omega)d\omega \tag{4.45}$$

where $v_p = 1/\sqrt{\epsilon\mu}$. Assume now that the spectral components of $E(\omega)$ are limited to the narrow band $\Delta\omega = \omega_1 - \omega_2$, so that (4.45) may be written

$$f(z,t) = (1/2\pi)\int_{\Delta\omega} e^{j\omega(t-z/v_p)}\ E(\omega)d\omega \tag{4.46}$$

We further restrict the bandwidth so that $\Delta\omega$ is much smaller in magnitude than the mean frequency $\bar{\omega}$, that is,

$$\frac{\Delta\omega}{\bar{\omega}} = \frac{\omega_2 - \omega_1}{\dfrac{\omega_1 + \omega_2}{2}} \ll 1 \tag{4.47}$$

It will now be convenient to define the wave number of the wave with frequency ω. The wave number k gives the phase shift in radians per meter traveled in the medium, that is

$$k = \omega / v_p \qquad (4.48a)$$

$$= n\omega / c \qquad (4.48b)$$

where n is the index of refraction. Inserting (4.48a) into (4.46),

$$f(z,t) = (1/2\pi) \int_{\Delta\omega} e^{j(\omega t - kz)} E(\omega) d\omega \qquad (4.49)$$

Now the index of refraction is a function of frequency, as was discussed in Chapter 3. So that

$$k = k(\omega)$$

For a suitable small frequency range we may expand $k(\omega)$ in a Taylor's series about the mean frequency $\bar{\omega}$:

$$k(\omega) = k(\bar{\omega}+\delta\omega) = k(\bar{\omega}) + \delta\omega\left[\partial k(\omega)/\partial\omega\right]_{\bar{\omega}} + (\delta\omega)^2/2\left[\partial^2 k(\omega)/\partial\omega\right]_{\bar{\omega}} + \ldots, \qquad (4.50)$$

which for sufficiently small $\delta\omega$ may be simplified to

$$k(\omega) - k(\bar{\omega}) \approx \delta\omega\left[\partial k(\omega)/\partial\omega\right]_{\bar{\omega}} . \qquad (4.51)$$

Equation (4.46) may now be expressed as

$$f(z,t) = W(z,t)\, e^{j\left[\omega t - k(\bar{\omega})z\right]} \qquad (4.52)$$

where

$$W(z,t) = (1/2\pi) \int_{\Delta\omega} E(\omega) e^{j\left\{(\omega - \bar{\omega})t - \left[k(\omega) - k(\bar{\omega})\right]z\right\}} d\omega. \qquad (4.53)$$

Inserting (4.51) into (4.53),

$$W(z,t) \approx (1/2\pi) \int_{\Delta\omega} E(\omega)\, e^{j\left\{(\omega - \bar{\omega})t - \delta\omega\left[\delta k(\omega)/\partial\omega\right]_{\bar{\omega}} z\right\}} d\omega. \qquad (4.54)$$

The exponential factor of (4.52) represents the central frequency of the wave group (4.46). This frequency is called the carrier frequency of an electrical communication system. The W(z,t) factor is a slowly varying modulation of the amplitude of the carrier having frequency components in the range ω_1, ω_2. From (4.48a) the phase velocity varies inversely with $k(\omega)$ and the propagation velocity is not equal for all frequency components. Consequently the shape of the wave, which depends upon the phase of the various components, changes as the wave travels.

The exponential factor of (4.52) represents a sinusoidal factor of constant amplitude. Therefore the factor $W(z,t)$ propagates the shape of the wave group and is called the envelope. For $\Delta\omega$ sufficiently small, the product $[\partial k(\omega)/\partial\omega]_{\bar{\omega}} z$ is essentially constant and may be termed the propagation time t_g of the group, that is,

$$t_g = [\partial k(\omega)/\partial\omega]_{\bar{\omega}}\, z \tag{4.55}$$

We may solve for the average velocity

$$v_g = \frac{z}{t_g} = \frac{1}{[\partial k(\omega)/\partial\omega]_{\bar{\omega}}}. \tag{4.56}$$

This derivation is essentially that given by Born and Wolf (BOR 1). An alternate method of derivation of the group velocity may be given in terms of invariance conditions applied to the envelope. In (4.54), $\delta\omega = \omega - \bar{\omega}$, and therefore

$$W(z,t) = \frac{1}{2\pi}\int_{\Delta\omega} E(\omega)\, e^{j(\omega - \bar{\omega})\left[t - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]} d\omega. \tag{4.57}$$

or

$$W(z,t) = e^{-j\bar{\omega}\left[t - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]} \left(\frac{1}{2\pi}\right)\int_{\Delta\omega} E(\omega)\, e^{j\omega\left[t - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]} d\omega$$

$$= e^{-j\bar{\omega}\left[t - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]} E\left[t - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]. \tag{4.58}$$

From (4.52) we then have

$$f(z,t) = e^{j\left[\omega\frac{\partial k(\omega)}{\partial\omega} - k(\omega)\right]_{\bar{\omega}} z}\, E\left[t - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]. \tag{4.59}$$

The factor bearing the shape characteristics of the group is $E\left[t-\frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]$, the envelope. We see that the argument of the envelope depends only upon the independent variables z and t for a chosen center frequency. If we examine the condition for which the amplitude of the envelope is invariant, we will obtain an expression involving the propagation velocity of the group directly. This is true so long as the shape does not change. Points on the wavefront travel with the propagation velocity by definition. Proceeding with the condition for invariance of the envelope, we set the total differential equal to zero, thus

$$d\left\{E\left[t-\frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} z\right]\right\} = \frac{\partial E}{\partial t}\,dt + \frac{\partial E}{\partial z}\,dz = 0 \qquad (4.60)$$

or

$$dt - \frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}} dz = 0$$

so that,

$$v_g = \frac{dz}{dt} = \frac{1}{\frac{\partial k(\omega)}{\partial\omega}\Big|_{\bar{\omega}}} \,. \qquad (4.61)$$

which agrees with the previous result. The additional phase factor given in (4.59), $\exp j\left[\omega\,\partial k(\omega)/\partial\omega \quad -k(\omega)\right]_{\bar{\omega}} z$, results from the lack of symmentry of the onesided transform and the odd forcing function. It shows that there is a phase shift of the carrier of the amount

$$\phi = \left[\omega\,\partial k(\omega)/\partial\omega \quad -k(\omega)\right]_{\bar{\omega}} z \text{ radians} \qquad (4.62)$$

over a path z meters long.

4.4 Solutions of the Wave Equation for Dispersive Media--Ionosphere

The rectangular cartesian components of Maxwell's equations (4.1) and (4.2), are

$$\mathrm{curl}_x E = \partial E_z/\partial y - \partial E_y/\partial z = -\partial B_x/\partial t \tag{4.63}$$

$$\mathrm{curl}_y E = \partial E_x/\partial z - \partial E_z/\partial x = -\partial B_y/\partial t \tag{4.64}$$

$$\mathrm{curl}_z E = \partial E_y/\partial x - \partial E_x/\partial y = -\partial B_z/\partial t \tag{4.65}$$

$$\mathrm{curl}_x H = \partial H_z/\partial y - \partial H_y/\partial z = i_x + \partial D_x/\partial t \tag{4.66}$$

$$\mathrm{curl}_y H = \partial H_x/\partial z - \partial H_z/\partial x = i_y + \partial D_y/\partial t \tag{4.67}$$

$$\mathrm{curl}_z H = \partial H_y/\partial x - \partial H_x/\partial y = i_z + \partial D_z/\partial t \tag{4.68}$$

For our case, only E_x and H_y are nonzero, so that we have need for only two of the above equations, (4.64) and (4.66). As shown in Chapter 3, the dielectric constant is frequency dependent. The work of Chapter 3 applies to steady state solutions for sinusoidal waves. For waves of more general types of time variation, it is necessary to express the dielectric constant as a time varying coefficient in Maxwell's equations. Rather than seek a solution of the wave equations in terms of the time variation, we shall utilize the expressions developed in Chapter 3 for the frequency variation of the dielectric constant. These expressions will find use as the equivalent Laplace transforms. The differential equations (4.64) and (4.66) will be transformed and solutions obtained by inversion of the transform.

Equation (4.66) may be rewritten as

$$-\partial H_y(z,t)/\partial z = i_x(z,t) + \partial D_x(z,t)/\partial t. \tag{4.69}$$

For the assumed nonzero wave components E_x, H_y, and alignment of magnetic field along the z-axis we have from (3.39),

$$i_{ox}(\omega) = \epsilon_v \omega_c^2 E_{ox} \left[(\nu + j\omega)/(\omega_o^2 + \nu^2 - \omega^2 + 2j\omega\nu)\right] \qquad (4.70)$$

Now, the subscript "o" in the variables of (4.70) was used in Chap. 3 in order to designate the amplitude of the components i and E which were assumed to have a sinusoidal time variation, i.e., from (3.3),

$$E_x = E_{ox} e^{j\omega t}.$$

The assumption of sinusoidal solutions of a differential equation is equivalent to imposing conditions for the steady-state response, or mathematically speaking, obtaining the Fourier transform of the solution.

Further, the Fourier transform may be obtained from the Laplace transform by substituting $j\omega$ for s. Since we have expressions for the Fourier transform of the conduction current (4.70), we desire to proceed in the reverse direction, i.e., to obtain the Laplace transform from the Fourier transform. This we can do by the substitution of s for $j\omega$, s^2 for $-\omega^2$, providing we can assume the initial conditions of the Laplace transform are all identically zero.

In the case of the conduction current (4.70), we are assuming that the medium is initially at rest, so that the latter restriction is satisfied. Hence we form the Laplace transform of the conduction current from (4.70) by the substitutions given above. The result is

$$\mathbf{I}_x(r,s) = \epsilon_v \omega_c^2 \mathbf{F}(r,s) \left[(\nu + s)/(\omega_o^2 + \nu^2 + s^2 + 2\nu s)\right] \qquad (4.71)$$

where the "o" subscript is now no longer necessary since the time variation of $i(z,t)$ and $E(z,t)$ is not restricted to sinusoids.

Taking the Laplace transform of (4.69) with respect to t and z , letting $E_x = f$, $H_y = g$,

$$-r\mathbf{G}(r,s) + G(+0,s) = \mathbf{I}_x(r,s) + s\epsilon_v \mathbf{F}(r,s) - \epsilon_v F(r,+0). \qquad (4.72)$$

Substituting from (4.71) and rearranging

$$r\mathbf{G}(r,s) + s\left[\frac{\epsilon_v\ \omega_c^{\ 2}}{s^2+2\nu s+\omega_o^2+\nu^2} + \epsilon_v\right]\mathbf{F}(r,s) = \epsilon_v F(r,+0) + G(+0,s) - \frac{\epsilon_v\omega_c^2\mathbf{F}(r,s)}{s^2+2\nu s+\omega_o^2+\nu^2}. \quad (4.73)$$

Now (4.64) may be rewritten as

$$\partial E_x(z,t)/\partial z = -\ \mu\left[\partial H_y(z,t)/\partial t\right] \quad (4.74)$$

Taking the Laplace transform with respect to t and z, with notation consistent with expressions above, rearranging the order of terms,

$$r\mathbf{F}(r,s) + \mu s\mathbf{G}(r,s) = F(+0,s) + \mu G(r,+0) \quad (4.75)$$

Now, since r and s are not identically zero, and the right hand members of (4.73) and (4.75) are arbitrary, the condition for a nontrivial solution of (4.73) and (4.75) requires that the determinant

$$\begin{vmatrix} \mathbf{G}(r,s) & \left[\dfrac{\epsilon_v\,\omega_c^{\ 2}}{s^2+2\nu s+\omega_o^2+\nu^2} + \epsilon_v\right]\mathbf{F}(r,s) \\ \mathbf{F}(r,s) & \mu\,\mathbf{G}(r,s) \end{vmatrix} = 0 \quad (4.76)$$

for the mode and medium that we have assumed. This condition yielded (4.25) in the case of a nondispersive medium and gave the familiar expression relating E_x and H_y by the characteristic impedance $\sqrt{\epsilon/\mu}$ of the medium. In the present case the characteristic impedance of the medium is modified by the damped resonance of the electrons and presence of the magnetic field. We therefore have from (4.76) a new relation which must exist between $\mathbf{G}(r,s)$ and $\mathbf{F}(r,s)$ for nontrivial solutions,

$$r^2 - \epsilon_v \mu_v \left[s^2 \frac{\mu_v}{\epsilon_v} Z_o^2(s) + s \left(\frac{\nu\omega_c^2}{s^2+2\nu s+\omega_o^2+\nu^2} \right) \right] F(r,s)$$

$$= \left[r - \mu_v s Z_o(s) \right] F(+0,s) \qquad (4.81)$$

Letting

$$s^2 q^2(s) = s^2(\mu_v/\epsilon_v) Z_o^2(s) + s \left[\nu\omega_c^2/(s^2+2\nu s+\omega_o^2+\nu^2) \right]$$

$$= s^2 \left[\frac{s^3+s2\nu s^2+(\omega_o^2+\omega_c^2+\nu^2)s+\nu\omega_c^2}{s(s^2+2\nu s+\omega_o^2+\nu^2)} \right]$$

or

$$q(s) = \sqrt{\frac{s^3+2\nu s^2+(\omega_o^2+\omega_c^2+\nu^2)\ s+\nu\omega_c^2}{s(s^2+2\nu s+\omega_o^2+\nu^2)}} \qquad (4.82a)$$

But $\sqrt{\epsilon_v \mu_v} = 1/c$, and

$$s\mu_v Z_o(s) = (s/c) \sqrt{\frac{s^2+2\nu s+\omega_o^2+\omega_c^2+\nu^2}{s^2+2\nu s+\omega_o^2+\nu^2}} \qquad (4.82b)$$

We may write (4.81) in the form

$$F(r,s) = \frac{\left[r - s\mu_v Z_o(s) \right] F(+0,s)}{\left[r+sq(s)/c \right] \left[r-sq(s)/c \right]}$$

$$= \left[e^{sq(s)z/c} + e^{-sq(s)z/c} \right] F(+0,s)$$

$$- \left[c\mu_v Z_o(s)/q(s) \right] \left[e^{sq(s)z/c} - e^{-sq(s)z/c} \right] F(+0,s). \qquad (4.83)$$

where

$$\left[c\mu_v Z_o(s) \right] / q(s) = \sqrt{\frac{s^3+2\nu s^2+(\omega_o^2+\omega_c^2+\nu^2)s}{s^3+2\nu s^2+(\omega_o^2+\omega_c^2+\nu^2)s+\nu\omega_c^2}} \qquad (4.84)$$

Now $f(z,t)$ is given by the inversion of (4.83). Even with considerable simplification the inversion of (4.83) may be quite complicated. The process may be illustrated by considering a much simplified case. Assuming that the effects of collision damping are negligible,

$R_e = m\nu = 0$, or $\nu = 0$, and

$$Z_o(s) = \sqrt{\frac{\epsilon_v}{\mu_v}\left(1 + \frac{\omega_c^2}{s^2+\omega_o^2}\right)},$$

$$q(s) = s\left[1 + \omega_c^2/(s^2+\omega_o^2)\right]^{1/2},$$

$$\mathbf{F}(r,s) = \frac{F(+0,s)}{r+s\mu_v Z_o(s)}.$$

Then

$$F(z,t) = e^{-\left[\mu Z_o(s)\right] sz} F(+0,s) \qquad (4.85)$$

Now for $\nu = 0$

$$\mu_v Z_o(s) = \mu_v\sqrt{\frac{\epsilon_v}{\mu_v}\left[1 + \omega_c^2/(s^2+\omega^2)\right]}$$

$$= \sqrt{\mu_v\epsilon_v}\sqrt{1 + \omega_c^2/(s^2+\omega^2)}$$

The quantity $\mu_v\epsilon_v$ $=(1/c^2)$, where c is the velocity of propagation of electromagnetic radiation in vacuum. Therefore

$$\mu_v Z_o(s) = (1/c)\sqrt{1+ \omega_c^2/(s^2+\omega^2)}$$

As will be shown, the quantity under the square root sign determines the characteristics of propagation of the signal $f(+0,t)$. In subsequent discussions we shall have occasion to discuss the radicand in some detail. It shall be designated as $\alpha(s)$ where

$$\mu Z_o(s) = \frac{\alpha(s)}{c}$$

or

$$\alpha(s) = \sqrt{1 + \frac{\omega_c^2}{s^2+\omega^2}} \tag{4.86}$$

We rewrite (4.85) in terms of $\alpha(s)$,

$$F(z,t) = e^{-[\alpha(s)sz/c]}F(+0,s) \tag{4.87}$$

The inverse transform with respect to t is given by the inversion integral

$$f(s,t) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} e^{s[t-\alpha(s)z/c]} F(+0,s)ds \qquad \gamma < \text{Re } s \tag{4.88}$$

Consider now the evaluation of the contour integral (4.88). As $s \to \infty$, $\alpha(s) \to 1$. Therefore, starting at $s = \gamma - j\infty$, if $(t-z/c) < o$, the contour may be closed by an infinite semi-circle to the right encircling the right-half s-plane. Since $\gamma < \text{Re}(s)$, if there are no poles of $F(+0,s)$ in the right-half plane, $f(z,t) = 0$. Thus at a point z in the medium the field is zero for $t < z/c$. Hence the velocity of the wave front cannot exceed the velocity of light c.

When $(t-z/c) > 0$, the exponential factor $\to \infty$ as $s \to \infty$ and hence the contour can be closed only in the negative half plane. The singularities enclosed include poles of $F(+0,s)$ in the left half s-plane and also the branch points of $e^{[t-\alpha(s)\ z/c]}$. Neglecting damping We write $\alpha(s)$ in the form

$$\alpha(s) = \sqrt{\frac{s^2+\omega_o^2+\omega_c^2}{s^2+\omega_o^2}}, \tag{4.89}$$

where the positive root only is indicated by an unsigned radical.

We see that

$$\alpha(s) = \infty \text{ when } s = \pm j\omega_o$$

$$\alpha(s) = 0 \text{ when } s = +j\sqrt{\omega_o^2 + \omega_c^2}.$$

The location of the singularities is indicated in Fig. 4.1, where the branch points are denoted by

$$a_\pm = \pm j\omega_o$$

$$b_\pm = \pm j\left(\omega_o^2 + \omega_c^2\right)^{1/2}.$$

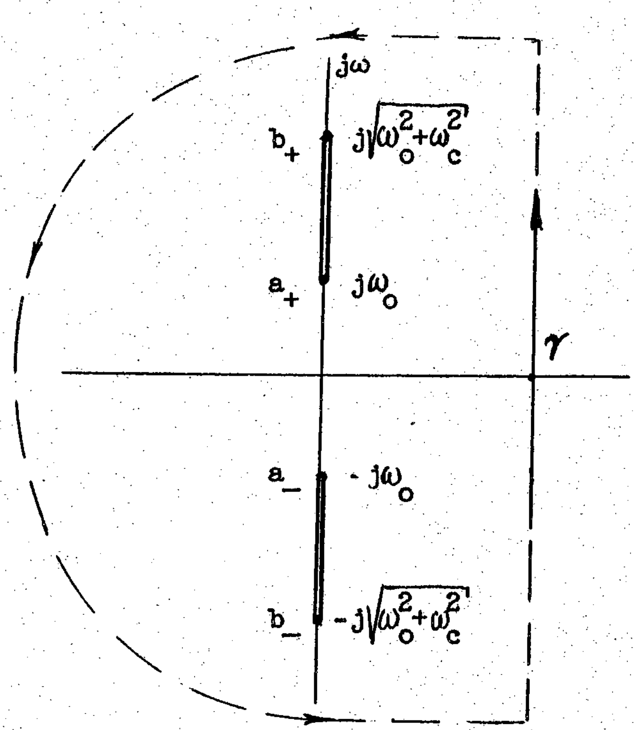

Figure 4.1

Paths of Integration in the s-Plane for Equation (4.88)

The double lines connecting a_+ to b_+, a_- to b_- represent the branch cuts. Crossing a branch cut so that a single branch point is encircled results in a change of sign of the integral. This source of ambiguity is avoidable by devising an integration contour which does not cross over the branch cuts, and thus branch points are encircled in pairs with no resultant sign change. Along the branch cuts the path of the contour may be so arranged that the branches are traversed as if the function were single-valued. The details of branch point integration are discussed in detail in Morse and Feshbach (MOR 1, Section 4.4). As the equivalence of single-valued branches we use a contour such as shown in Fig. 4.1. The path of integration for (4.88), which follows the imaginary axis from $\gamma - j\infty$ to $\gamma + j\infty$, is closed by an infinite semicircle to the left. It may now be deformed in any manner on the cut plane without altering the value of the integral, provided only that in the process of deformation the contour does not exclude the poles of $F(+0,s)$ or cross either of the branch cuts. The path may be shrunk to the form indicated in Fig. 4.2a, where it is assumed in the discussion of phase velocity in the present case that $f(+0,t)$ is a sinusoid and hence $F(+0,s)$ has a pair of poles on the $j\omega$ axis at $\pm j\omega$. The contributions arising from a passage back and forth along the straight lines connecting each of the encircling contours cancel each other since there are no singularities enclosed. Consequently Eq. (4.88) reduces to four integrals about the closed contours C_1, C_2, C_3, and C_4 as shown in Fig. 4.2b. The integrals around the poles of $F(+0,s)$ over contours C_2 and C_3 may be evaluated at once by summing the residues. We may

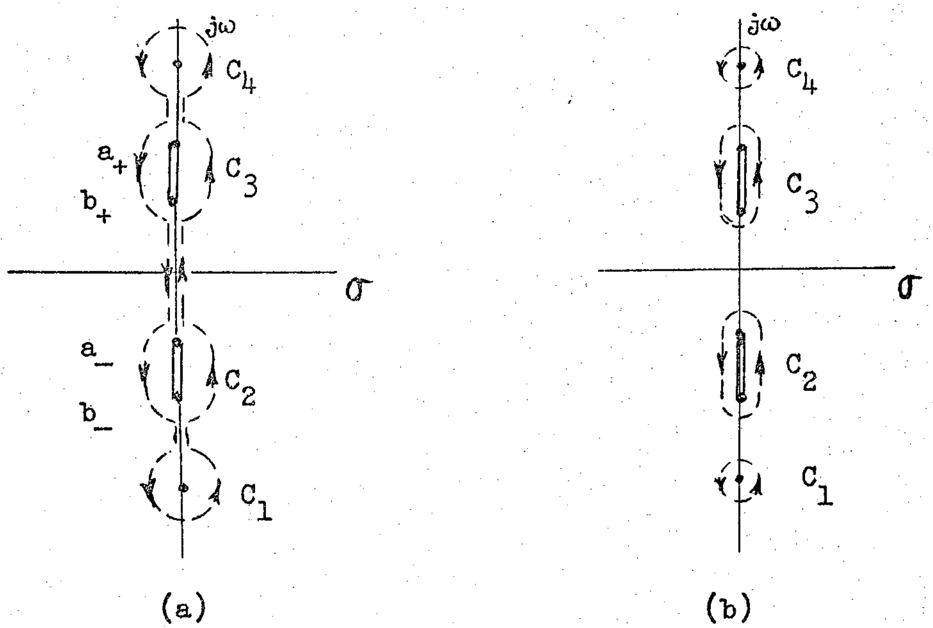

Figure 4.2

Final Paths of Integration of Equation (4.88)

represent their contribution as

$$f_{14}(z,t) = \frac{1}{2\pi j}\int_{C_1+C_4} e^{s\left[t-\frac{\alpha(s)}{c}z\right]}F(+0,s)ds \tag{4.90}$$

The remaining two integrals around contours C_1 and C_4 surrounding the branch cuts may be represented by

$$f_{23}(z,t) = \frac{1}{2\pi j}\int_{C_2+C_3} e^{s\left[t-\alpha(S)\ \ t_p\right]}F(,0,s)ds \tag{4.91}$$

where $t_p = z/c$, the vacuum propagation time. Physically, the components of $f(z,t)$ resulting from integration around C_1 and C_4 represent the forced response of the medium to the incident radiation at the origin $f(+0,t)$. The components $f_{23}(z,t)$ represent the free or transient response of the charges of the medium at the natural frequencies ω_o and ω_c.

It is shown in the Appendix that the solution for the positive traveling wave is of the form

$$f(z,t) = \frac{E_o}{2}\left\{\sin \omega\left(t-\sqrt{1+\frac{\omega_c^2}{\omega_o^2+\omega^2}}\; t_p\right)\right.$$

$$+\frac{\omega}{\omega^2-\omega_o^2}\left\{\cos\omega_o(t-t_p)\left[-\frac{\omega_c^2 t_p}{2}+\frac{\omega_c^4(t-t_p)}{16\omega_o}-\frac{\omega_c^6 t_p^3}{192\omega_o^2}\left(\frac{(\omega_o^2t_p^2-3)(t-t_p)^2}{2!t_p^2}-\frac{3(t-t_p)}{2t_p}\right)\right.\right.$$

$$+\frac{\omega_c^8 t_p^4}{384\omega_o^4}\left(\frac{(\omega_o^4t_p^4-15\omega_o^2t_p^2(t-t_p)^3}{48t_p^4}-\frac{(18\omega_o^2t_p^2+75)\,(t-t_p)^2}{16t_p^3}+\frac{(8\omega_o^2t_p^2+15)\,(t-t_p)}{t_p^2}+\ldots\right]$$

$$+\sin\omega_o(t-t_p)\left[\frac{\omega_c^4}{16\omega_o}-\frac{\omega_c^6t_p^3}{192\omega_o^2}\left(\frac{3\omega_o(t-t_p)^2}{2!}+\frac{(3\omega_o^2t_p^2+1)\,(t-t_p)}{2\omega_o t_p{}^2}+\frac{3}{2\omega_o t_p}\right)\right.$$

$$+\frac{\omega_c^8 t_p^4}{384\omega_o^4} - \frac{(6\omega_o^3 t_p^3+15\,\omega_o t_p)\,(t-t_p)^3}{48t_p^4}$$

$$-\frac{(7\omega_o^3 t_p^3+15\,\omega_o t_p)\,(t-t_p)^2}{16t_p^3}$$

$$+\frac{(8\omega_o^2 t_p^2+18\omega_o t_p+15)\,(t-t_p)}{16t_p^2}$$

$$+\frac{(8\omega_o^2 t_p^2+15)}{\omega_o t_p}$$

$$+\ldots\Big]\Big\} \qquad (4.92)$$

The first term is the steady state and the succeeding terms are those of the transient. It may be noted that all but the first transient term contains a factor of $(t-t_p)$. Up to the time it takes the wave to propagate a distance z in vacuum, the field is zero. At time $t = t_p$, the response is still zero to the first order in t_p as it can be seen by using the small angle approximation on the steady state term and by expanding the radical. Thus for time slightly greater than t_p,

$$\sin\,\omega\left[t-\sqrt{1+(\omega_c^2)/(\omega_o^2-\omega^2)}\;t_p\right] \approx \omega\left[t-\sqrt{1+(\omega_c^2/(\omega_o^2-\omega^2)}\;t_p\right]$$

$$\approx \omega\,(t-t_p) + (\omega\,\omega_c^2 t_p)/2(\omega^2-\omega_o^2)$$

which, combined with the first transient term yields

$$f(z,t) \approx (E_o/2)\,\omega\,(t-t_p), \qquad t = t_p+\delta t_p$$

$$\delta t_p \ll t_p.$$

Thus for a very small initial instant after the wave would have reached the point z in vacuum, the response in the medium grows linearly from zero. Quantitative discussion of the transient in subsequent instants depends upon the plasma frequencies ω_c and ω_o and the distance z and is better indicated by an asymptotic form of the series solution below.

After decay of the transient the steady state response prevails. The phase velocity may be derived as was done previously, and is

$$v_p = \frac{c}{\sqrt{1 + \frac{\omega_c^2}{\omega_o^2 - \omega^2}}} \tag{4.93}$$

The complex nature of the wave transient after $t = t_p$ cannot be visualized from the solution given in (4.92), but by the use of asymptotic relations and approximation techniques it is possible to determine certain characteristic features of the response.

For very small values of time in excess of the vacuum propagation time t_p, we may apply the initial value theorem (see the Appendix) to obtain as the response in the instants of time for which t is slightly greater than t_p,

$$\lim_{t \to 0} f(z,t) = \frac{E_o \omega}{2} \left(\frac{2(t-t_p)}{\omega_c^2 t_p} \right)^{1/2} J_1 \left[\sqrt{2 \omega_c^2 t_p (t-t_p)} \right] \tag{4.94}$$

where J_1 is the first order Bessel function of the first kind. A plot of Eq. (4.94) is presented in the Appendix in Fig. A.2. It can be seen from this that the response is a growing oscillation. The spacing of the zeros is that of the first order Bessel function having an increasing argument given in (4.94). The oscillations decrease in frequency with time at this particular epoch of the response. The frequency approaches the critical frequency of the electrons. The response given in (4.94) is eventually damped and is replaced by that of succeeding epochs which will be described below. This first part of the response which arrives with velocity of light is called the first forerunner or precursor of the signal (BRI 2).

For the response for times much in excess of the vacuum propagation time, it is necessary to consider details of integration of the contour integral using approximate methods. We note that the exponential kernel of (4.88) has the form

$$K(s,t_p,t/t_p) = e^{st_p(t/t_p - \alpha(s))} .$$

In general $\alpha(s)$ is complex and therefore the exponent has both real and imaginary parts which vary with s,t and t_p. If the real part is zero, we may use the approximate method known as the method of stationary phase (BRI 2). If both real and imaginary parts are present, as in our case, the method of steepest descents may be employed (MOR 1).

The oscillations of the integrand due to the imaginary part of the exponent of the kernel cause the algebraic sign and magnitude of the integrand to fluctuate as the path of integration is traversed in the s-plane. For a given absolute value of the kernel, the convergence of the integral is affected adversely by oscillations in sign. However the oscillations are a function of the contour of integration. Since the contour may be chosen anywhere to the right of the real part of the exponent of the kernel, there is the possibility that the integral may be made to converge rapidly to the greatest part of its value in a relatively short path. The value of the integral determined on such a path would then be taken as the approximate value of the integral. If necessary or feasible the path of integration may be closed in a vicinity where the absolute value of the kernel is very small, relatively speaking, and where oscillations due to traversing the remaining part of the contour thus have small effect.

It is apparent that the best path to choose is one where the rate of change with respect to path length is maximum and at the same time oscillations of the kernel are minimized or held to a minimum over this portion of the path. From the theory of functions of a complex variable, an analytic function possesses neither true maxima or minima but instead, exhibits minimax or saddlepoints. At these points, due to the orthogonal character of the real and imaginary parts of the analytic function, the lines along which the function changes at a maximum rate are also lines along which the rate of change of phase angle is zero and hence are lines of constant phase. It is thus possible to use a line of maximum descent as the desired path of integration. Whereas a line of maximum ascent connects the high points of the function, such as poles, a line of maximum descent connects the low points or valleys of the function. These lines cross orthogonally at saddlepoints. Since the original integral is presumed convergent, the path along the ridge line, or line of maximum ascent causes the integral to become infinite at the poles and thus cannot be the required path. The path of integration thus leads from one valley region of negatively infinite values for the complex exponent to another valley region of negatively infinite values, see Fig. 4.3.

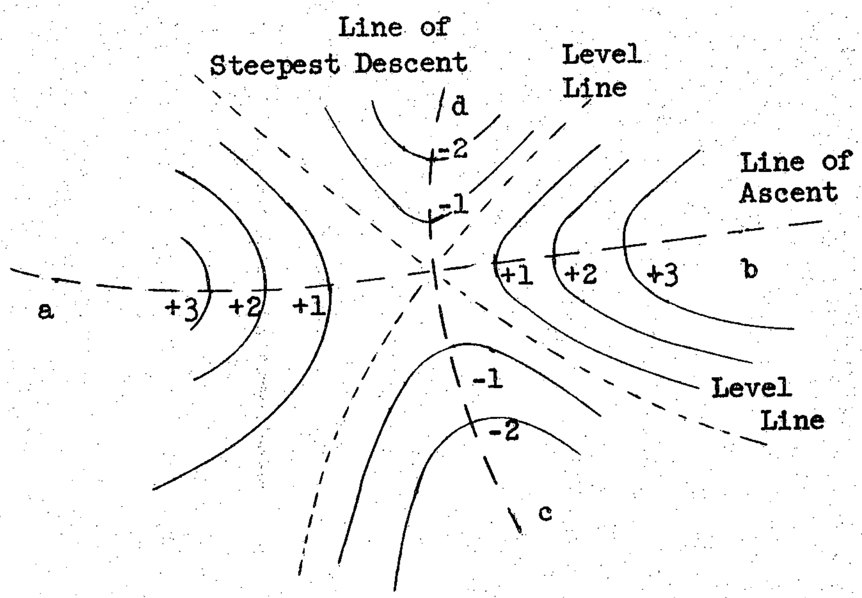

Figure 4.3

Contour Plot of the Exponent of the Kernel $K(s, t_p, t/t_p)$

If in Fig. 4.3, curve ab represents the line of ascent along which the real part of the exponent increases from the saddlepoint as fast as possible, and curve cd represents the line of steepest descent along which the exponent decreases from the saddlepoint at maximum rate, then curve cd is the desired path of integration.

We shall now derive a general formula for the first term in the asymptotic expansion for f (z,t). Let the contour integral be represented by

$$f(z,t) = (1/2\pi j)\int_C e^{t_p w(s)} F(+0,s)ds \tag{4.95}$$

In the neighborhood of the saddlepoint at $s = s_o$ we may represent the exponent of the exponential kernel $w(s)$ by a Taylor's series

$$w(s) = w(s_o) + (s-s_o)\, w'(s_o) + (1/2!)\,(s-s_o)^2\, w''(s_o) + \ldots$$

Since we are at a saddlepoint of $w(s)$, $w'(s_o) = 0$. Therefore

$$w(s) = w(s_o) + (1/2!)\,(s-s_o)^2\, w''(s_o) + \ldots$$

The approximation for the integral becomes

$$f(z,t) = (1/2\pi j)\int_c^d e^{t_p\left[w(s_o) + (s-s_o)^2\, w''(s_o)/2!\right]} F(+0,s)ds$$

$$= (e^{t_p w(s_o)}/2\pi j)\int_c^d e^{t_p (s-s_o)^2\, w''(s_o)/2!}\; F(+0,s)ds$$

where the path is taken in the direction of the original contour, say from c to d. Now $w(s)$ takes on large negative values at the extremities of the path of steepest descent, so that the quantity

$$\left[(s-s_o)^2\, w''(s_o)\right]/2! < 0.$$

Furthermore as either t_p or t in the exponent takes on large values the exponent becomes a larger negative number and the convergence of the integral on the path of steepest descent is even more pronounced. If the path is sufficiently short due to the effect of these factors on the convergence, then $F(+0,s)$ is relatively constant and may be factored outside of the integral. We therefore write the integral in the form

$$f(z,t) = \left(\sqrt{|w''(s_o)|} \quad e^{t_p w(s_o)}/2\pi\right) \int_c^d e^{-t_p S^2/2} \, dS,$$

where

$$S^2 = \left|(s-s_o)^2 \, w''(s_o)\right|,$$

and

$$dS = j\sqrt{|w''(s_o)|} \quad ds$$

For sufficiently large t_p or t, the integrand will effectively become zero outside the range in which the Taylor's series for w(s) is valid. We may then replace the contour integral by a real integral over the range $-\infty$ to $+\infty$. The direction of integration on the path is chosen to be consistent with the direction of the original contour. For these conditions the approximate value of the integral is then

$$f(z,t) = \sqrt{|w''(s_o)/2\pi t_p|} \quad \left[e^{t_p w(s_o)} F(+0,s_o)\right] /2 \quad (4.96)$$

We note that in this method the phase of f(s) on the path of integration is taken to be the phase of $f(s_o)$.

In order to obtain the rest of the terms of the asymptotic expansion, we shall use a different expansion than that of (4.94) (MOR 1). Let

$$f(s) = f(s_o) - \xi^2. \qquad (4.97)$$

Since the phase of f(s) is that of $f(s_o)$, then ξ is real. The integral now becomes

$$f(z,t) = \left[e^{t_p f(s_o)}/2\pi j\right] \int_C e^{t_p \xi^2} F(+0,s) \, (ds/d\xi) \, d\xi$$

$$f(z,t) = \left[F(+0,s_o)e^{t_p f(s_o)}/2\pi j\right] \int_{-\infty}^{\infty} e^{-t_p \xi^2} (ds/d\xi)\, d\xi \quad (4.98)$$

We now compute $ds/d\xi$ by inverting the power series for $f(s)$, (MOR 1),

$$(ds/dw) = \sum_{n=0}^{\infty} a_n \xi^n, \quad (4.99)$$

where

$$a_n = (1/n!)\,(d^n/ds^n)\left[(s-s_o)/g(s)\right]^{n+1} \quad (4.100)$$

and

$$\xi = g(s) = \sum_{n=1}^{\infty} (a_{n-1}/n)\,(s-s_o)^n \quad (4.101)$$

We may now investigate the response of the medium after the instant of arrival of the first part of the wave transient at $t = t_p$ as given in (4.94). From (4.86) we have the square of the complex refractive index function

$$\alpha^2(s) = 1 + \omega_c^2\left[1/(s^2+\omega_o^2)\right]$$

with singularities located in the complex plane as shown in Fig. 4.2. The factor $w(s)$ of the exponent of the exponential kernel is of the form

$$w(s) = s\left[t/t_p - \alpha(s)\right] \quad (4.102)$$

We now investigate $w(s)$ for the location of saddlepoints, that is where $dw(s)/ds = 0$. Taking derivatives,

$$\left[dw(s)/ds\right] = t/t_p - \alpha(s) - s\left[d\alpha(s)/ds\right] \quad (4.103)$$

$$\left[d^2w(s)/ds^2\right] = -\left[d\alpha(s)/ds\right] - s\left[d^2\alpha(s)/ds^2\right] \quad (4.104)$$

For $\alpha(s)$ as given above

$$d\alpha(s)/ds = -\left\{ s\,\omega_c^2 / \left[(s^2+\omega_o^2)^{3/2} \, (s^2+\omega_o^2+\omega_c^2)^{1/2} \right] \right\} \qquad (4.105)$$

and

$$d^2\alpha(s)/ds^2 = \frac{\omega_c^2 \left[3s^4 + 2s^2(\omega_o^2+\omega_c^2) - \omega_o^2(\omega_o^2+\omega_c^2) \right]}{(s^2+\omega_o^2)^{7/2} \, (s^2+\omega_o^2+\omega_c^2)^{1/2}} \qquad (4.106)$$

For exact determination of the location of the saddlepoints, Eq. (4.102) may be substituted into Eq. (4.100) and the roots located as t/t_p varies. However it is readily perceived that the resulting algebraic equation is of fourth degree, and in general the roots are all complex. This projected task is therefore not a light one, and is one best accomplished with computational facilities. This approach is not the one employed here, since there are other more simple methods which may be employed to trace the loci of the saddlepoints as a function of t/t_p.

It is possible to solve the equivalent of the subject algebraic equation by a combination of computational and graphical techniques. There is still a substantial amount of labor required in this procedure, but it has the feature not afforded by the pure computational method that the topology and the motion of the saddlepoints is more easily visualized. This method first requires a plot of the contours of the real and imaginary parts of $\alpha(s)$ on the complex plane. Computational and crossplotting techniques are employed for this step. A plot results such as given in Fig. 4.4. Eq. (4.101) is now employed to compute the real part of $w(s)$ for a chosen value of t/t_p. The contour plot made as a result of this step resembles Fig. 4.3 in the neighborhood of the saddlepoint. While this method does enhance the understanding of the motion of

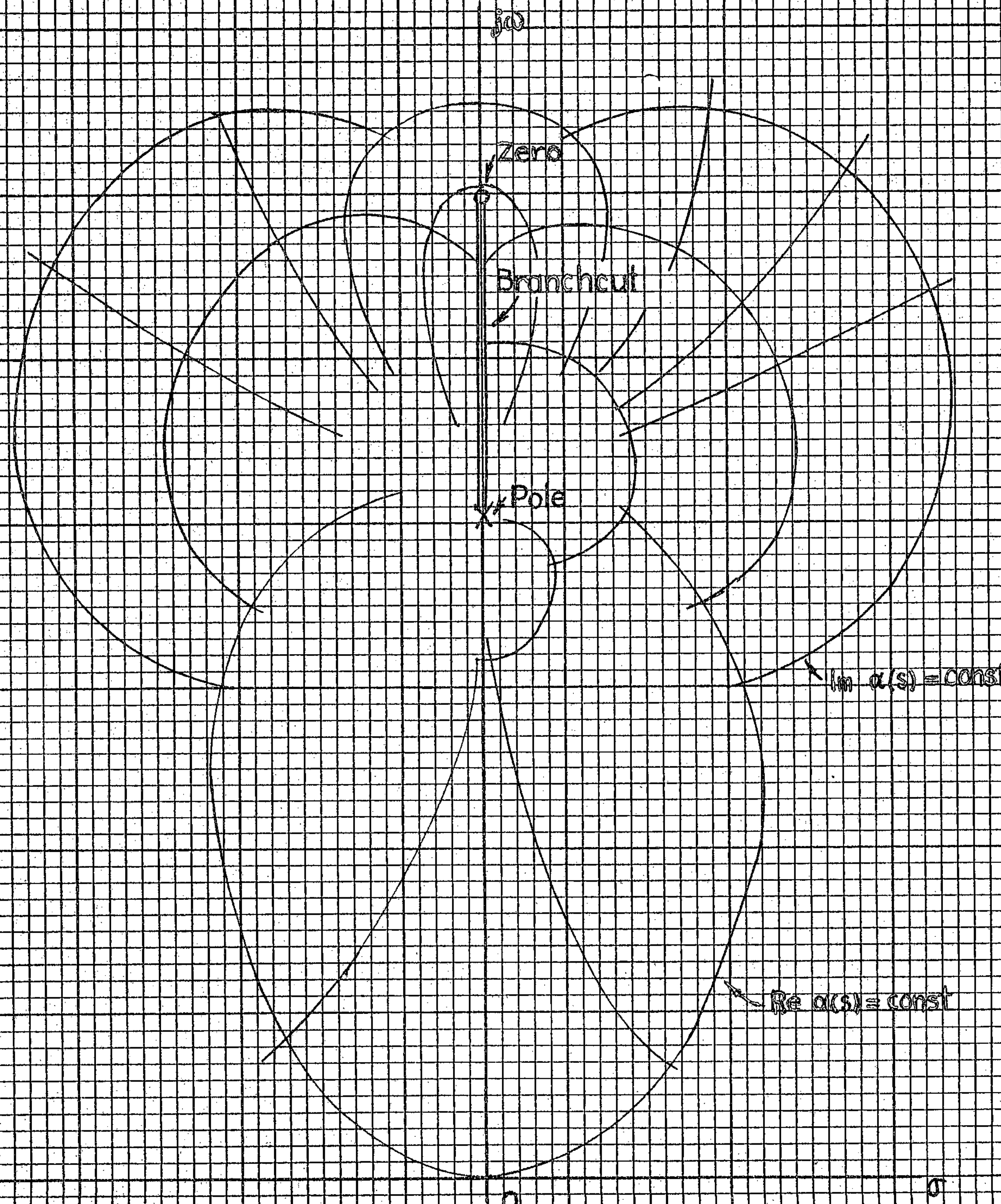

Figure 4.4

Typical Contours for a Refractive Index Function of the Form $\alpha(s)=\sqrt{1+\frac{1}{s^2+\omega_0^2}}$

the saddlepoints, and like the computational method yields results which may be made arbitrarily more accurate, it is still quite lengthy compared to the following more approximate method.

It is learned through experience with the plots of the graphical method just described that the locations of the saddlepoints for which there is the greatest change in contour values are those closest to the singularities of $\alpha(s)$. This immediately suggests an approximate method since only factors associated with the singularity of a function of a complex variable change rapidly in neighborhood of the singularity. The motion of a saddlepoint near a singularity may then be traced by studying the behavior of an increment in s for the condition that the first derivative must be zero. This procedure will now be applied to the problem at hand.

We have

$$\alpha(s) = \left[1 + \omega_c^2/ (s^2 + \omega_o^2)\right]^{1/2}$$

$$= \left[(s^2+\omega_o^2+\omega_c^2) / (s^2+\omega_o^2)\right]^{1/2}$$

with singularities located at $s = \pm j(\omega_o^2+\omega_c^2)^{1/2}$ and $s = \pm j\omega_o$. Near $s = + j\omega_o$, $\alpha(s)$ is an even function of s. Because of this symmetry we shall deal only with the upper half plane in this discussion, let

$$s = re^{j\theta} \tag{4.107}$$

Then

$$\alpha(s) = \left[1 + \omega_c^2/ (r^2e^{j2\theta}+\omega_o^2)\right]^{1/2}.$$

Now for r very small, we have

$$\alpha(s) = (1 + \omega_c^2/\omega_o^2)^{1/2},$$

and then

$$w(s) = s\left[t/t_p - \alpha(s)\right]$$

$$= s\left[t/t_p - (1+\omega_c^2/\omega_o^2)^{1/2}\right]$$

so that

$$dw/ds = \left[t/t_p - (1+\omega_c^2/\omega_o^2)^{1/2}\right].$$

We have then that a saddlepoint or inflection point for which $dw(s)/ds = 0$ is located at $t/t_p = (1+\omega_c^2/\omega_o^2)^{1/2}$. Since $d^2w/ds^2 = 0$, this is an inflection point rather than a saddlepoint. Therefore this point does not locate the summit across which the path of integration must pass.

We next consider r quite large and since our approximation technique is being employed to reduce algebraic complexity, we consider only the predominant term for the other extreme for which $r^2e^{j2\theta} \gg \omega_o^2$, so that

$$\alpha(s) = (1+\omega_c^2/r^2e^{j2\theta})^{1/2}.$$

We now consider two cases, first the case for which $|\omega_c^2| > |r^2e^{j2\theta}|$.

Then

$$\alpha(s) \approx \omega_c r^{-1} e^{-j\theta}$$

and

$$w(s) = re^{j\theta}(t/t_p - \omega_c r^{-1}e^{-j\theta})$$

$$= (t/t_p)\, re^{j\theta} - \omega_c,$$

so that

$$dw = (t/t_p)\, e^{j\theta}\, dr + j(t/t_p) re^{j\theta}\, d\theta$$

For a saddlepoint, dw = 0, so that

$$(t/t_p)\ e^{j\theta}(dr+jd\theta) = 0$$

which yields the conditions,

$$dr/r = -j\theta$$

or

$$r = Ce^{-j\theta}$$

and

$$\theta = +j\infty, \text{ since } t/t_p = 1.$$

The result, $r = +\infty$, is incompatible with the original premise of $|\omega_c^2| > |r^2|$. The indication however that r is always quite large, is consistent between cases. In the "vicinity" of $\pm j\,\omega_o$ we have left the case of $|r^2 e^{j2\theta}| > \omega_c^2$. Then

$$\alpha(s) \approx (1+\omega_c^2/r^2e^{j2\theta})^{1/2}$$

$$\approx 1 + (1/2)\omega_c^2 r^{-2} e^{-j2\theta}$$

Therefore

$$w(s) = re^{j\theta}\left[(t/t_p) - 1 - (1/2\,\omega_c^2 r^{-2} e^{-j2\theta}\right]$$

$$= (t/t_p - 1)\ re^{j\theta} - (1/2)\omega_c^2 r^{-1} e^{-j\theta}$$

For a saddlepoint, dw = 0, hence

$$\left[(t/t_p-1)\ e^{j\theta} + \omega_c^2 r^{-2} e^{j2\theta}/2\right] dr + jr\left[(t/t_p-1)\ e^{j\theta} + \omega_c^2 r^{-2} e^{-j\theta}/2\right] d\theta = 0$$

from which we obtain the conditions

$$dr/r = -jd\theta$$

or

$$re^{j\theta} = s = C$$

where C is a constant independent of r and θ, and

$$r^2 e^{j2\theta} = \omega_c^2/2(t/t_p - 1) = s^2.$$

The final form for s is

$$s = +\,\omega_c/\sqrt{2(t/t_p - 1)} \qquad (4.108)$$

where the positive root is chosen consistent with the direction of the contour of integration. We note from this, that for $t/t_p = 1$, the saddlepoints are located at $\pm\infty$ on the real axis. Since our solution given in (4.94) is exact for this case, it is not necessary to obtain an approximate solution by the saddlepoint method.

From (4.108) it is indicated that the location of the saddlepoints approach zero as $t/t_p \to \infty$. This result is due to the incomplete nature of our approximation since we are considering the effect of only one singularity at a time. We next assume $s = j\omega_o + re^{j\theta}$, and since $\alpha(s)$ is an even function of s we need only to consider only the upper half plane. We have then

$$\alpha(s) = (1 + \omega_c^2/r^2 e^{j2\theta})^{1/2}$$

This result is identical to the preceding case, so that we may conclude that saddlepoints significant for operating frequencies near $j\omega_o$ are on the real axis between $\pm\infty$ and 0.

We now let $s = j(\omega_o^2 + \omega_c^2)^{1/2} + re^{j\theta}$ to locate the approximate path of saddlepoints near this singularity. We have

$$\alpha(s) = \left[\frac{j2(\omega_o^2 + \omega_c^2)^{1/2}\, re^{j\theta} + r^2 e^{j2\theta}}{r^2 e^{j2\theta} + j2(\omega_o^2 + \omega_c^2)^{1/2} re^{j\theta} - \omega_c^2}\right]^{1/2}.$$

First consider the case for which $\left|j(\omega_o^2+\omega_c^2)^{1/2}re^{j\theta}\right| > \left|r^2e^{j2\theta}\right|$.

Then

$$\alpha(s) = 1/\left[1+(j\omega/2)\,(\omega_o^2+\omega_c^2)^{-1/2}r^{-1}e^{-j\theta}\right]^{1/2}$$

and therefore

$$w = \left[j(\omega_o^2+\omega_c^2)^{1/2}+re^{j\theta}\right]\left[t/t_p - \sqrt{2}\,(\omega_o^2+\omega_c^2)\,e^{-j\pi/4}r^{1/2}e^{j\theta/2}/\omega_c\right]$$

or

$$w \approx j(\omega_o^2+\omega_c^2)^{1/2}\left[t/t_p - \sqrt{2}(\omega_o^2+\omega_c^2)e^{-j\pi/4}r^{1/2}e^{j\theta/2}/\omega_c\right].$$

Taking the derivative and solving yields the requirement that r be infinite, which is in contradiction to the hypothesis for this case. We are thus led to considering a case for which r may be large. Allowing such a condition results in obtaining the functional variation for r. Let

$$\left|j(\omega_o^2+\omega_c^2)^{1/2}re^{j\theta}\right| < \left|r^2e^{j2\theta}\right|.$$

Then

$$\alpha(s) \approx \left[r^2e^{j2\theta}/(r^2e^{j2\theta}-\omega_c^2)\right]^{1/2}$$

$$= \left[(1/1-\omega_c^2r^{-2}e^{-j2\theta})\right]^{1/2}$$

Now require that

$$\left|\omega_c^2r^{-2}e^{-j2\theta}\right| > 1.$$

We have

$$\alpha(s) = j\,\omega_c r^{-1}e^{-j\theta}$$

and

$$w = j(t/t_p)\,(\omega_o^2+\omega_c^2)^{1/2} + \omega_c(\omega_o^2+\omega_c^2)^{1/2}r^{-1}e^{-j\theta}+re^{j\theta}(t/t_p) - j\,\omega_c$$

Then for a saddlepoint, dw = 0, and

$$\left[-\omega_c(\omega_o^2+\omega_c^2)^{1/2}r^{-2}e^{-j\theta}+(t/t_p)e^{j\theta}\right](dr+jrd\theta) = 0$$

which yields the result

$$s = \omega_c^{1/2}(\omega_o^2+\omega_c^2)^{1/2}/\sqrt{t/t_p} + j(\omega_o^2+\omega_c^2)^{1/2}. \qquad (4.109)$$

as the location of the predominant saddlepoint. Finally consider the case for which

$$\left|\omega_c^2 r^{-2}e^{-j2\theta}\right| < 1.$$

We have

$$\alpha(s) \approx 1 + \omega_c^2 r^{-2}e^{-j2\theta}/2$$

and

$$w = \left[j(\omega_o^2+\omega_c^2)^{1/2}+re^{j\theta}\right]\left[t/t_p-1-\omega_c^2 r^{-2}e^{-j2\theta}/2\right]$$

From dw = 0, we have the condition

$$(t/t_p-1)\,r^3e^{j3\theta}+(\omega_c^2/2)\,re^{j\theta}+j\omega_c^2(\omega_o^2+\omega_c^2)^{1/2} = 0.$$

For $re^{j\theta}$ very large the approximate roots of this equation are

$$s_1 = re^{j\theta} = \left[\omega_c^2(\omega_o^2+\omega_c^2)^{1/2}/(t/t_p-1)\right]^{1/3}e^{-j\pi/6},$$

$$s_2 = \left[\omega_c^2(\omega_o^2+\omega_c^2)^{1/2}/(t/t_p-1)\right]^{1/3}e^{j\pi/2},$$

and

$$s_3 = \left[\omega_c^2(\omega_o^2+\omega_c^2)^{1/2}/(t/t_p-1)\right]^{1/3}e^{j\pi/6}$$

Only the second of these locations is along the positive imaginary axis and hence will contribute most to the response.

We may now summarize our observations concerning the location of the saddlepoints. When the operating frequency is in the vicinity of the origin or near or less than $s = j\omega_o$, the location of the saddlepoint is given by

$$s_s = +\omega_c/\sqrt{2(t/t_p-1)} \tag{4.110}$$

When the operating frequency is near $s = j(\omega_o^2+\omega_c^2)^{1/2}$, the location of the saddlepoints is given by

$$s = \left(\omega_c^{1/2}(\omega_o^2+\omega_c^2)^{1/4}/\sqrt{t/t_p}\right) + j(\omega_o^2+\omega_c^2)^{1/2} \tag{4.111}$$

Then when the operating frequency is beyond $s = j(\omega_o^2+\omega_c^2)^{1/2}$, the location of the saddlepoint is given approximately by

$$s = j\left\{(\omega_o^2+\omega_c^2)^{1/2} + \left[\omega_o^2(\omega_o^2+\omega_c^2)^{1/2}/(t/t_p-1)\right]^{1/3}\right\} \tag{4.112}$$

We shall now discuss the characteristics of the transient response of the medium in terms of the motion of the saddlepoints and the frequency range. From (4.93) the first term of the asymptotic expansion for the response is given by

$$f(z,t) = \left\{\sqrt{|w''(s_o)|\ /2\pi t_p}\, F(+0,s_o)e^{t_p w(s_o)}\right\}/2.$$

or

$$f(z,t) = A(s_o)e^{t_p s_o\left[t/t_p - \alpha(s_o)\right]}$$

We note that

$$A(s_o) = \sqrt{|w''(s_o)|\ /2\pi t_p}\ F(+0,s_o)/2$$

is an amplitude factor dependent upon time through the location of the saddlepoint s_o. From the discussion above there are three frequency ranges in which the apparent motion of the saddlepoints differ somewhat with respect to t/t_p. In general each frequency range sees the saddlepoint in the same proper direction but due to the nature of the approximations used for discussing the location of the saddlepoints, the exact locations and weighting are not equal. For the range $0 < \omega < \omega_o$, the saddlepoint starts in from $+\infty$ at $t/t_p = 1$ and approaches 0 as $t/t_p \to \infty$. Then (4.108) takes the form

$$f(z,t) = A(s_o)e^{t_p \omega_c [t/t_p - \alpha(s_o)] / \sqrt{2(t/t_p - 1)}},$$

which for $t/t_p >> 1$ becomes

$$f(z,t) = A(s_o)e^{t_p \omega_c [\sqrt{t/t_p} - \alpha(s_o)\sqrt{t_p/t}] / \sqrt{2}}.$$

4.5 The Signal and Group Velocities

It will be recalled from (4.94) that at $t/t_p = 1$, the response begins with a very small high frequency oscillation and grows steadily in amplitude with a decrease in frequency. As t/t_p becomes larger, the response given by (4.113) grows as a simple exponential since the exponent is real. If there had been damping included in the complex refractive index function, the initial oscillatory response would begin to be damped at this point. The remaining part of the response is given by additional terms in the asymptotoc expansion of (4.99). While mathematical evaluation of the inversion of the power series given in (4.99) is quite involved, a good understanding of the overall result may be obtained by referring to Fig. 4.5 showing contours of $w(s)$ on the complex plane. Since at the poles of $\alpha^2(s)$, $w(s)$ becomes $\pm\infty$, the lines of steepest descent pass from one pole of $\alpha^2(s)$, over the saddle-point to the other pole. On Fig. 4.5 we see that as the saddlepoint moves inward toward the origin with increasing t/t_p, the integration path along the line of steepest descent is brought closer to the imaginary axis. Therefore as the path comes into proximity with

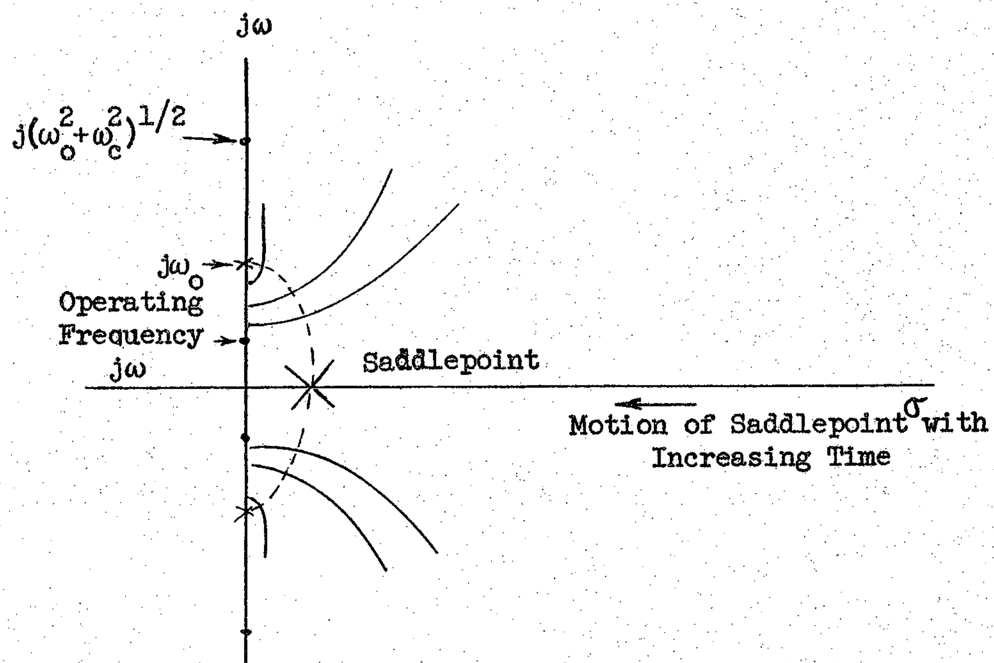

Figure 4.5

Relation of Integration Path to Signal Velocity

the operating frequency $j\omega$, a term begins to appear in the response having the frequency of the signal or operating frequency. The growth of the signal is quite rapid as the contour approaches the origin in the limit, as can be determined by the dependence of the saddlepoint location upon the time t. At a particular time the signal has grown to an arbitrary fraction of final value. This is the arrival time t_s. The distance z and the arrival time may be related by a velocity called the signal velocity v_s, for which an expression may be obtained as follows. The function w(s) is defined by

$$w(s) = t_p s \left[t/t_p - \alpha(s) \right]$$

The condition for a saddlepoint is

$$dw(s)/ds = t_p \left[t/t_p - \alpha(s) - s d\alpha(s)/ds \right] = 0$$

or since $t_p = z/c$,

$$z/t = c/\left[\alpha(s) + s d\alpha(s)/ds \right]$$

Now the actual velocity of propagation of the signal to the point z is z/t. If at a time t_s the signal has attained detectable proportions and is considered to have arrived, we may define the associated velocity as the signal velocity v_s

$$v_s = z/t_s = c/\left[\alpha(s) + s d\alpha(s)/ds \right]_{s = s_s} \qquad (4.114)$$

where s_s is the position of the saddlepoint at time t_s. The quantity $c/[s\, d\alpha(s)/ds]$ is known as the group velocity (BOR 1, Section 1.3), and was discussed earlier. It is apparent from (4.114) that, depending upon the relative sizes of $\alpha(s)$, $d\alpha(s)/ds$, and the detecting level for the signal, the signal velocity may be significantly different from the group velocity.

Our discussion of the effects of saddlepoint motion upon v_s may now be carried out for each frequency range using the equations for the path of the saddlepoints derived above. The general effect in each range may be computed with no difference in technique from that given above. We have established the key result that there may be a significant difference between the velocity with which a signal is propagated and the group velocity. This effect varies with the operating frequency relative to the critical frequencies.

Figure 4.6 presents a plot (BRI 2) showing variation of these three types of velocity with operating frequency. The relative ordinates of the actual curves vary in a given choice of critical frequencies, detection level, and form of $\alpha(s)$, which itself depends (at least in determination of the location of the zeros) upon the propagation mode. General features common to the curves are noted. In particular the group velocity is equal to either the phase or signal velocity at their respective maxima.

The basis for the compensation method proposed by Harris (HAR 1) is now evident. For frequencies much above the critical frequency,

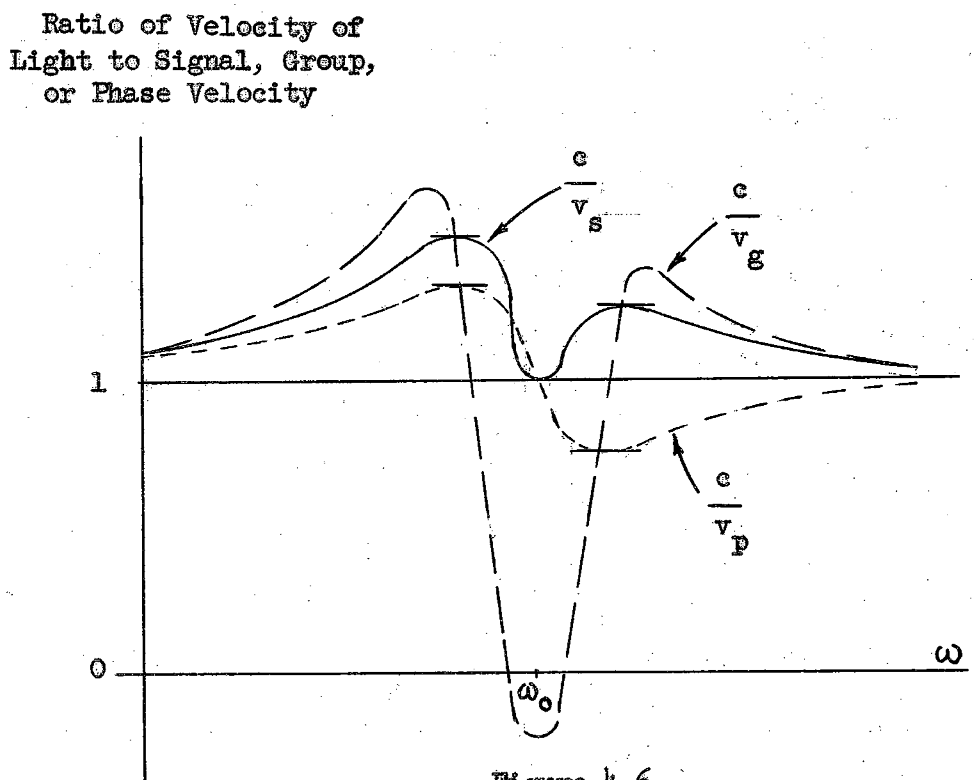

Figure 4.6

Relations Between Signal Group and Phase Velocity in a Dispersive Medium (BRI 2) the group velociy is very nearly equal to the signal velocity, and in addition, is approximately as much less than the velocity of light as the phase velocity is greater. We note that this relation does not exist at frequencies near the critical frequency. However if two or more propagation links may be used, many other schemes are apparent, such as using other weights than simple averaging of group and phase velocity. Propagation velocities at other frequencies may also be combined with certain weights to effect compensation for dispersion. However effects not studied here, such as the averaging of signals received over slightly different paths by the aperture of an antenna, adds variations which require a transverse dimension for their specification and analysis.

Referring to (4.85), complete discussion of the velocity effects including collision damping requires discussion of the singularities of $e^{s\,t+q(s)t_p}\,F(+0,s)$, $e^{s\,t-q(s)t_p}\,F(+0,s)$, etc. The task becomes much more formidable than the previous case because (4.84) and the defining relation for $q(s)$, (4.82a) contain a cubic factor which can only be studied numerically. The discriminant of the cubic factor in (4.82) and (4.84) is not a perfect square. Thus factors of the cubic equation can not be indicated (DIC 1) in a way that expedites a general discussion of the saddlepoints for any of the exponential kernels involving $q(s)$. It would appear that a numerical integration would be the most expedient means for studying solutions of (4.85). In general there are more saddlepoints than in the previous case. The contribution of the integration over each saddlepoint must be summed. At some operating frequencies the transient is more complex than indicated above, there being additional predominant parts in the transient as the saddlepoints approach the critical frequencies of the exponential kernel.

We shall now consider briefly the nature of propagation in the troposphere in which the resonances of polar and nonpolar molecules determine the critical frequencies.

4.6 Solutions of the Wave Equation for Dispersive Media--Troposphere

As discussed in Chap. 3, we consider the tropospheric medium to consist only of nonconducting nonpolar and polar molecules. Maxwell's general current density equation including polarization effects is

$$\nabla \times \vec{H} = \partial \vec{D}/\partial t + \vec{i} + \vec{i}_p \tag{4.115}$$

where $\vec{i}$ is the current density due to conduction processes in the medium and $\vec{i}_p$ is the displacement current density which accounts for polarization. The polarization $\vec{P}$ is a charge density due to the shift of charges and reorientation of molecules in the medium. The displacement current density is then (SLA 1, Section 170),

$$\vec{i}_p = \partial \vec{P}/\partial t \tag{4.116}$$

Since the medium is assumed nonconducting, $\vec{i} = 0$, and Maxwell's current density equation becomes

$$\nabla \times \vec{H} = \partial \vec{D}/\partial t + \partial \vec{P}/\partial t$$

From (3.80), neglecting collision damping, the polarization is expressed as a function of the frequency of an applied steady state electric wave $\vec{E} = \vec{E}_o e^{j\omega t}$ in the form

$$\vec{P}(\omega) = \vec{E}(\omega) N \left[e^2/m(\omega_o^2 - \omega^2) + p^2/3kT(1 + j\omega\tau) \right] \tag{4.117}$$

This may be converted to the equivalent Laplace transform by the substitution $s = j\omega$, assuming the medium is devoid of waves at $t = +0$. Hence

$$\vec{P}(s) = \vec{E}(s)N\left[e^2/m(s^2+\omega_o^2) + p^2/3kT(1+s\tau)\right],$$

or

$$\vec{P}(s) = \left[\vec{E}(s)N/3kmT\tau\right] \quad (mp^2s^2+3kT\tau e^2s+mp^2\omega_o^2+3kTe^2)/(s+1/\tau)(s^2+\omega_o^2) \tag{4.118}$$

The component of Maxwell's equation in rectangular coordinates which is nonzero for the transverse wave under consideration is

$$-\partial H_z/\partial z = \epsilon_v \partial E_x/\partial t + \partial P_x/\partial t \tag{4.119}$$

Taking multiple Laplace transform,s with the notation of (4.7) and (4.8)

$$-r\mathbf{G}(r,s) + G(+0,s) = \epsilon_v s\mathbf{F}(r,s) - \epsilon_v F(r,+0) + s\mathbf{P}_x(r,s) - P_x(r,+0) \tag{4.120}$$

From (4.118)

$$\mathbf{P}_x(r,s) = \left[\mathbf{F}(r,s)N/3kmT\tau\right](mp^2s^2+3kT\tau e^2s+mp^2+3kTe^2)/(s+1/\tau)(s^2+\omega_o^2). \tag{4.121}$$

Substituting (4.121) into (4.120) and using the initial conditions that the medium is at rest, i.e., both $F(r,+0)$ and $P_x(r,+0)$ are zero, we have upon rearranging,

$$r\mathbf{G}(r,s) + s\left[\epsilon_v+Np(s)/3kmT\tau\right]\mathbf{F}(r,s) = G(,0,s) + \epsilon_v F(r,+0) + P_x(r,+0) \tag{4.122a}$$

where

$$p(s) = (mp^2s^2+3kT\tau e^2s+mp^2\omega_o^2+3kTe^2)/(s+1/\tau)(s^2+\omega_o^2) \tag{4.122b}$$

Also from (4.75)

$$r\mathbf{F}(r,s) + \mu s\mathbf{G}(r,s) = F(+0,s) + \mu G(r,+0) \tag{4.123}$$

Since $r,s \neq 0$, and $G(+0,s)$, $F(r,+0)$, $P_x(r,+0)$, $F(+0,s)$, $G(r,+0)$ may be chosen arbitrarily, the condition for nontrivial solutions is (assuming $\mu_r = 1$)

$$G(r,s) = \sqrt{\epsilon_v/\mu_v}\,\sqrt{1+Np(s)/3kmT\tau\,\epsilon_v}\,F(r,s) \qquad (4.124)$$

from which we have the transformed boundary condition at the origin

$$G(+0,s) = \sqrt{\epsilon_v/\mu_v}\,\sqrt{1+Np(s)/3kmT\tau\,\epsilon_v}\,F(+0,s) \quad (4.125)$$

Substituting from (4.122) and (4.125) into (4.74) and solving for $F(r,s)$, we have

$$F(r,s) = F(+0,s)/\left[r+(s/c)\sqrt{1+Np(s)/3kmT\tau\,\epsilon_v}\right]$$

from which

$$F(z,s) = e^{-(sz/c)\sqrt{1+Np(s)/3kmT\tau\,\epsilon_v}}\,F(+0,s).$$

Therefore

$$f(z,t) = (1/2\pi j)\int_{\gamma-j\infty}^{\gamma+j\infty} e^{s\left[t-\alpha_t(s)t_p\right]}\,F(+0,s)\,ds, \qquad \gamma < \mathrm{Re}\ s \qquad (4.126)$$

where for tropospheric electron and polar resonances,

$$\alpha_t(s) = \sqrt{1+Np(s)/3kmT\tau\,\epsilon_v} \qquad (4.127)$$

with $p(s)$ defined in (4.122b).

The inversion integral (4.126) may be approximated by integrating over the saddlepoints of the exponent of the kernel exp $s\left[t-\alpha_t(s)t_p\right]$. $\alpha(s)$ is somewhat more complex than the example given earlier for the ionosphere.

For the radio frequency portion of the electromagnetic spectrum, the resonances of the tropospheric refractive anomalies are due principally to the presence of water vapor and oxygen. The resonance bands of water vapor and oxygen are located approximately at 22.5 kM cps and 60 kM cps respectively (USA 1, BUR 1). These are not the electron resonance phenomena defined by the coefficients of the frequency-sensitive term $p(s)$ of the polarization discussed above.

The wavelengths for the electron displacement resonances usually fall in the ultraviolet or optical wavelengths. It is therefore necessary, in using an expression for p(s), to choose a form and coefficients which represent the spectral region of interest. The various types of molecular resonance have frequency variations described by the general form of p(s) but the coefficients must be generalized to represent the relative strength of the absorption; somewhat in the fashion indicated in the discussion of quantum effects at the close of Chap. 3.

The radio frequency absorption bands of water vapor and oxygen were predicted and found to exist in work conducted in the last twenty years (see references cited by BUR 1, p. 51). The 23.5 kM cps water vapor absorption is due to a rotational spectral line of relatively small strength having a half width of 3000 M cps. For a concentration of 1% water molecules in air (density of 7.5 gm per meter^3) the peak absorption is only 0.17 db per km. We also note that the relaxation time τ for water vapor (DEB 1, p. 85) corresponds to a critical frequency of about 40 kM cps. It can be shown (TOW 1) that the collisions cause pressure broadening of the spectral line at 23.5 kM cps and accounts for its shape.

For oxygen present in normal amounts in air at 76 cm pressure there are two absorption regions in the millimeter wavelength region. The resonance band at 60 kM cps has a half-width of 600 M cps and a peak absorption of 14 db/km. The spectral line at approximately 120 kM cps is much sharper and has a peak absorption of about 3.5 db/km. Meteorological conditions cause significant variation in the shape and magnitude of the absorption. Heavy rainfall (BUR 1) causes a sharp general absorption rise at all frequencies in the general region above 6 kM cps. The absorption in db/km may be increased a hundred fold at 6 kM cps by a cloudburst. The absorption levels off and even decreases at certain millimeter wave lengths. The equivalent coefficients for use in the above solutions may be obtained by the following method.

The complex dielectric constant is given as

$$\epsilon = n^2(1-j\kappa)^2 \qquad (4.128)$$

where n is the index of refraction and κ is the absorption index. In terms of the absorption index, the exponent w(s) may be written,

$$w(s) = s\left[t-n(1-j\kappa)t_p\right] \qquad (4.129)$$

The absorption data is indicated for a sinusoid of frequency ω.

For $s = j\omega$,

$$w(\omega) = -\omega n\kappa t_p + j\omega(t-nt_p) \qquad (4.130)$$

The amplitude of a sinusoid travelling in the medium is proportional to $e^{-\omega n\kappa t_p}$. The intensity is therefore proportional to $e^{-2\omega n\kappa t_p}$. The attenuation is the ratio of incident to received intensity and thus

db attenuation = 10 $\log_{10}$ (Incident power/Received power)

$$= 10 \log_{10} e^{-2\omega n\kappa t_p}$$

$$= -8.7\,\omega n\kappa t_p \qquad (4.131)$$

Since the index of refraction of air is essentially unity in these weak dispersion bands, and

$$t_p = z/c$$

$$= 1000/(3\cdot 10^8),$$

$$t_p = (0.33)\ 10^{-5} \text{ sec for 1 km}$$

$$\kappa \approx -(\text{db attenuation/km})/(8.7)(0.33 \times 10^{-5})\omega$$

$$\approx -(\text{db attenuation/km})/2.9 \times 10^{-5}\omega \qquad (4.132)$$

As an example, for the oxygen absortion band at 60 kM cps,

$$\kappa \approx 14/(2.9 \times 10^{-5} \times 60 \times 10^{9} \times 2\pi$$

$$\approx 1.3 \times 10^{-6}$$

The coefficients to be associated with the real and imaginary parts of w(s) may thus be determined from measured values of the refractive index and absorption at the frequency of maximum absorption by use of (4.131) or (4.132) and substituting for κ into (4.129).

The range of operating frequencies in the electromagnetic spectrum has broadened with developments in the state of the art. For frequencies under 6000 kM cps, there is very little effect upon propagation time in the troposphere due to dispersion. However, the nonuniformity in composition normally encountered due to inhomogeneity of airborne moisture (WHE 1), is sufficient to cause effects of some significance. This is especially true if the signal is used for velocity measurement. Temporal variations in density of the medium causes fluctuations which distort precise target velocity measurements. A variation in signal propagation time directly affects measurements of the velocity of motion of an object by radar techniques. We shall consider in the last chapter the relation between the inhomogeneity and temporal fluctuations of the medium and variations in the propagation time and signal velocity.

CHAPTER 5

Propagation in an Inhomogeneous or Random Medium

In this section, expressions will be derived for the velocities of propagation of electromagnetic waves in an inhomogeneous or random dispersive or nondispersive medium. While the results are particularized to the troposphere, the method of analysis can be extended to the ionosphere by use of the ionospheric refractive index function discussed earlier. We shall begin with a discussion of the nature of turbulence and its effect upon the medium.

5.1 Spectral Characteristics of Turbulence

The randomness encountered in the troposphere is a variation in both time and spatial coordinates. The general theory of statistical turbulence has been applied over the past decade to improve the understanding of electromagnetic propagation in a random medium. This work has been reviewed by Wheelon (WHE 1). Results from analysis of turbulence using correlation techniques have been applied to predict the shape of the high frequency end of the velocity spectra.

Large amounts of energy at low wave numbers are fed into the atmospheric turbulence processes by meteorological phenomena. The nonlinear Navier-Stokes equation, the partial differential equation for the forces on each particle of gas, indicates that low wave number energy is transferred toward high wave numbers. Complete solutions for the Navier-Stokes equation, especially for turbulent situations have not been obtained.

Taylor (TAY 1), von Karman and Howarth (KAR 1), Kolmogorov (KOL 1), Obukhov (OBU 1), and Batchelor (BAT 1) have discussed spectral concepts of turbulent velocities. Wheelon (WHE 2) has related velocity distributions to distributions of a passive additive, such as pressure or humidity, which may have a direct connection to refractive index variations for the troposphere. By use of the equation of continuity of fluid dynamics and the dimensional analyses of statistical turbulence theory, he derives spectral formulations for the dielectric constant in the troposphere.

The literature of statistical turbulence theory is growing rapidly and there are many differences in the approaches to be found in the literature. An assumption often made is that the fluid is incompressible. The simplication resulting from this assumption is considerable and appears to be essential to dimensional analysis applied to the shorter wavelength end of the spectrum. At the large scale wavelengths at which energy is derived from solar heating cycles and Coriolis forces, density differences must be assumed (CON 1, Chap. 2, Part 3), and the analysis of the spectral form for these wavelengths is incomplete. It is indicated below that for processes of small scale in which the dominant forces are associated with motions along streamlines, the assumption of incompressibility is quite valid.

We shall now consider the general equations of motion of fluids. In particular we are interested in indicating how energy introduced at low wavenumbers is changed into energy at high wavenumbers. Since the wavelengths at the high wave number end of the spectrum are commensurate with distances and wavelengths in radio frequency communication, it will be of interest to examine the physical origin of the inhomogeneous structure of the troposphere.

The equation of motion for fluids is the Navier-Stokes equation. A discussion of the derivation of this equation is found in practically any text or treatment of general fluid mechanics (see, for example, BIR 1). This equation and the equation of continuity bear certain common features which may be interpreted by concepts of spectral analysis to explain why the low wave number kinetic energy of the general drift or bulk motion of a body of gas is converted to high wave number energy even in the absence of turbulence.

The equation of motion is applied in statistical turbulence theory to obtain relations involving correlations and spectra of gas velocity or momentum distributions. The equation of continuity connects derivatives of the gas density and velocity and, as mentioned, has been applied by Wheelon (WHE 2) to relate the spatial spectra of the material density in the medium to the input of convective energy using the results of the dimensional analysis of velocity spectra. We shall briefly examine the nonlinear terms of these equations and discuss the physical processes involved from spectral concepts. The derivations below follow the very lucid presentation of Bird, et al (BIR 1).

The equation of continuity expresses the mass balance over the stationary volume element $\Delta x \, \Delta y \, \Delta z$ through which the fluid is flowing. The mass balance may be expressed as

$$\begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{entering}\\ \text{of mass}\end{bmatrix} - \begin{bmatrix}\text{rate of}\\ \text{exit of}\\ \text{mass}\end{bmatrix} \qquad (5.1)$$

The left-hand side involves the time rate of change of the density times unit volume at a fixed point in space, that is x,y,z = constant. Therefore

$$\begin{bmatrix}\text{rate of mass}\\ \text{accumulation}\end{bmatrix} = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} . \qquad (5.2)$$

where ρ is the density of the fluid. The right hand member can be expressed in terms of the negative of the divergence of the product ρv where $\vec{v}$ is the velocity of the fluid in the infinitesimal element of volume. Thus

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = - \left[\nabla\cdot(\rho\vec{v})\right] \Delta x \Delta y \Delta z \qquad (5.3)$$

Dividing this equation by $\Delta x \Delta y \Delta z$ and taking the limit as the dimensions of the volume approaches zero, we have

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z , \qquad (5.4)$$

the equation of continuity. Performing the indicated differentiation and collecting all derivatives of ρ on the left side:

$$\frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \qquad (5.5)$$

The left side is the "substantial" derivative (BIR 1) of density and expresses the time rate of change of density at a point moving with the fluid. We may rewrite this result as

$$\frac{D\rho}{Dt} = -\rho(\nabla\cdot\vec{v}) \qquad (5.6)$$

where the capital D denotes the substantial derivative. For an incompressible fluid, the density is constant and the latter relation yields the following equation for that condition

$$(\nabla\cdot\vec{v}) = 0$$

This result enables considerable simplification in discussion of the equation of motion. Since this condition is valid if ρ remains constant as the fluid element moves along a streamline for which $D\rho/dt = 0$, it is equivalent to requiring that diffusion effects are nil and that mixing can be neglected. This condition is only partly met in the turbulence processes of the lower atmosphere. The useful results of Batchelor's prediction of the form of the spatial spectra (BAT 1) depend on this assumption.

The equation of motion is written from a momentum balance for the unit volume $\Delta x \, \Delta y \, \Delta z$. Thus

$$\begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{accumulation}\end{bmatrix} = \begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{rate of}\\ \text{momentum}\\ \text{out}\end{bmatrix} + \begin{bmatrix}\text{sum of}\\ \text{forces acting}\\ \text{on system}\end{bmatrix} \quad (5.7)$$

Momentum flows in and out of the volume element by two mechanisms, convection, or bulk fluid flow, and molecular transfer by velocity gradients (pressure and viscosity effects). Considering the x-component of the equation of motion, the rate at which momentum enters the face at x by convection is $\rho v_x v_x \big|_x \Delta y \, \Delta z$ and the rate at which it leaves at $x + \Delta x$ is $\rho v_x v_x \big|_{x+\Delta x} \Delta y \, \Delta x$. The rate at which it enters at y is $\rho v_y v_x \big|_y \Delta x \, \Delta z$, etc. Including the terms for all six faces of a unit cube at x,y,z, we have the x-component of convective momentum flow into the volume element is

$$\Delta y \Delta z(\rho v_x v_x \big|_x - \rho v_x v_x \big|_{x+\Delta x}) + \Delta x \Delta z(\rho v_y v_x \big|_y - \rho v_y v_x \big|_{y+\Delta y}) + \Delta x \Delta y(\rho v_z v_x \big|_z - \rho v_z v_x \big|_{z+\Delta z}) \quad (5.8)$$

The forces due to molecular transport (not just viscosity) are as follows. The shear stress exerted in the x-direction on a fluid surface having coordinate $y + \Delta y$ by the fluid along the surface having the coordinate y is designated as τ_{yx} for fluid having a velocity component

v_x. By the definition of the coefficient of viscosity, the shearing stress (defined as the force in the direction of the relative velocity divided by the area normal to the velocity gradient) is proportional to the negative of the local velocity gradient, that is

$$\tau_{yx} = -\mu_v \frac{dv_x}{dy} \tag{5.9}$$

where μ_v is the coefficient of viscosity. We note that the force $\tau_{yx}\Delta x \Delta z$ is in the + x direction. Then the rate at which the x-component of momentum enters the face at x by molecular transport is $\tau_{xx}|_x \Delta_y \Delta_x$, and the rate of which it leaves at $x + \Delta x$ is $\tau_{xx}|_{x+\Delta x} \Delta y \Delta x$. The rate at which it enters at y is $\tau_{yx}|_y \Delta_x \Delta_z$, etc. Summing all such terms, we get

$$\Delta y \Delta z(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}) + \Delta x \Delta z(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}) + \Delta x \Delta y (\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z}) \tag{5.10}$$

The component τ_{xx} is the normal stress on the x-face resulting from pressure or potential energy, and τ_{yx} is the x-directed tangential or shear stress on the y-face resulting from viscous forces.

The pressure and gravitational forces in the x-direction are

$$\Delta y \Delta z \ (p|_x - p|_{x+\Delta x}) + \rho g_x \Delta x \Delta y \Delta z$$

where p is the pressure and g_x is the gravitation constant in the x-direction.

The rate of accumulation of the x-component of momentum within the element is

$$\Delta x\, \Delta y\, \Delta z \ \ (\partial \rho\, v_x / \partial t). \tag{5.11}$$

The balance equation for the x-component of momentum may now be written, and the limit taken of the quotient of each of the above terms as the dimensions approach zero. The result is

$$\frac{\partial(\rho v_x)}{\partial t} = -\left(\frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_y v_x)}{\partial y} + \frac{\partial(\rho v_z v_x)}{\partial z}\right) - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial z} - \frac{\partial \tau_{zx}}{\partial z}\right) - \frac{\partial p}{\partial x} - \rho g \tag{5.12}$$

The equation of continuity may be applied to the latter result and similar ones for the y- znd z-components to obtain component equations of the form

$$\rho \frac{DV_x}{\partial t} = -\frac{\partial p}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \tag{5.13}$$

This is essentially the x-component of the Navier-Stokes equation. Again the left member involves the substantial derivative which gives the acceleration of a small volume of fluid which is moving with the fluid. Equations may be obtained in terms of a point fixed in space if desired (BIR 1). The following discussion of the resulting spectra will apply in either case.

The left hand side of the force equation, and the equation of continuity, both involve substantial derivatives, that is, terms of the form

$$\begin{aligned} Dv_x/Dt &= \partial v_x/\partial t + (\partial v_x/\partial x)\,(\partial x/\partial t) + (\partial v_x/\partial y)\,(\partial y/\partial t) \\ &\quad + (\partial v_x/\partial z)\,(\partial z/\partial t) \\ &= \partial v_x/\partial t + v_x(\partial v_x/\partial x) + v_y(\partial v_x/\partial y) + v_z(\partial v_x/\partial z) \end{aligned} \tag{5.14}$$

and

$$D\rho/Dt = \partial \rho/\partial t + v_x(\partial \rho/\partial x) + v_y(\partial \rho/\partial y) + v_z(\partial \rho/\partial z) \tag{5.15}$$

Taking Laplace transforms of the product terms of the right hand side of (5.14) yields expressions involving multiple complex convolutions of the velocity spectrum with itself. For example, transforming first with respect to t yields

$$L_t\left[Dv_x/Dt\right] = L_t\left[\partial v_x/\partial t\right] + L_t\left[v_x(\partial v_x/\partial x)\right] + L_t\left[v_y(\partial v_x/\partial y)\right] + L_t\left[v_z(\partial v_x/\partial z)\right]$$

$$= sV_x(x,y,z,s) + V_y(x,y,z,s) \otimes_x \left[\partial V(x,y,z,s)/\partial x\right] + \ldots \quad (5.16)$$

where the symbol $\otimes_s$ denotes the complex convolution (GAR 1) with respect to s defined by a term of the form

$$V_x \otimes_s (\partial V_x/\partial x) = (1/2\pi j)\int_{C-j\infty}^{C+j\infty} V_x(x,y,z,s-s')\left[\frac{\partial V_x(z,y,z,s')}{\partial x}\right] ds$$

$$\max\left[\sigma_{V_x},\ \sigma_{\frac{\partial V_x}{\partial x}},\ \sigma_{V_x + \frac{\partial V_x}{\partial x}}\right] < \sigma,\quad \sigma_{V_x} < C < \sigma - \sigma_{V_x},\quad \sigma + \mathrm{Re}\ s \quad (5.17)$$

etc. for the other velocity and density components. There is a set of complex convolutions for the transform of each product term with respect to each of the coordinate variables x,y,z. The use of spectra in this case does not aid in getting a solution. However the following interpretation is instructive.

The complex convolution integrals may be interpreted graphically, as follows. Consider only the time convolution given above at a fixed point in space. Then if the spectrum $V_x(x,y,z,s)$ is contained in the frequency band $0 \leq s' \leq s_1$, and the spectrum $\partial V_x(x,y,z,s)/\partial x$ is contained in the frequency band $0 \leq s' \leq s_2$, the convolved product is zero except for values of s where the spectra overlap, Fig. 5.1, that is for $0 \leq s \leq s_1 + s_2$.

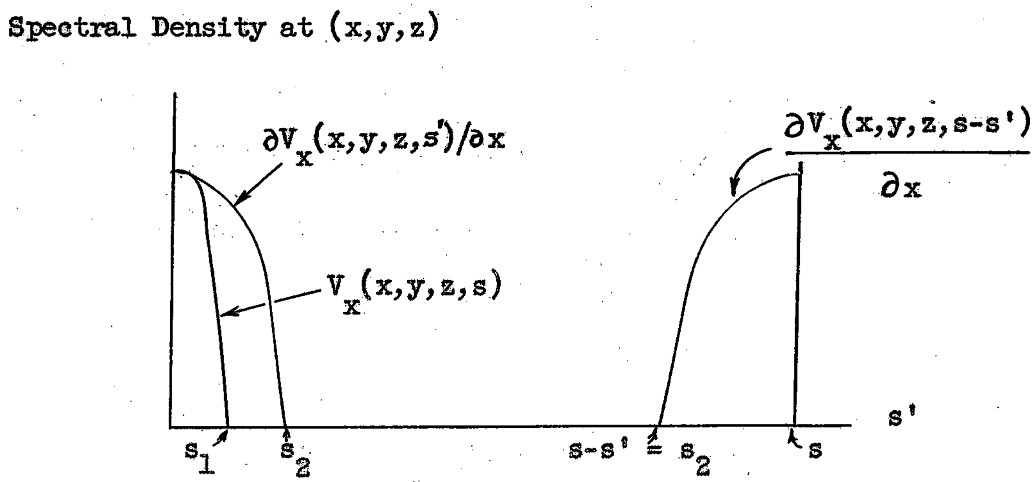

Figure 5.1

Convolved Spectra

The convolved spectrum in the variable s thus has a wider range than either of the original spectra. Since the substantial derivative is merely the result of writing a force equation involving an inertial (acceleration) coordinate system and a moving coordinate system for expressing relative motion or shear forces, we see that the upscaling (or increasing the range) of energy wave numbers is a consequence of the combination of inertial and viscous forces.

It is a simple matter to illustrate this process if one assumes a turbulent flow, Fig. 5.2. From a study of the figure it may be seen that energy in the incident gas stream entering with a very low wave number at A is deflected as the gas stream nears the low velocity interface; and due to conservation of momentum begins vortex motion, providing the ratio of inertial to viscous forces (Reynold's number) is large enough. Part of the stream turns toward B and forms a smaller interface at which momentum is exchanged with the surrounding gas. As each vortex continues its motion, additional interfaces are encountered

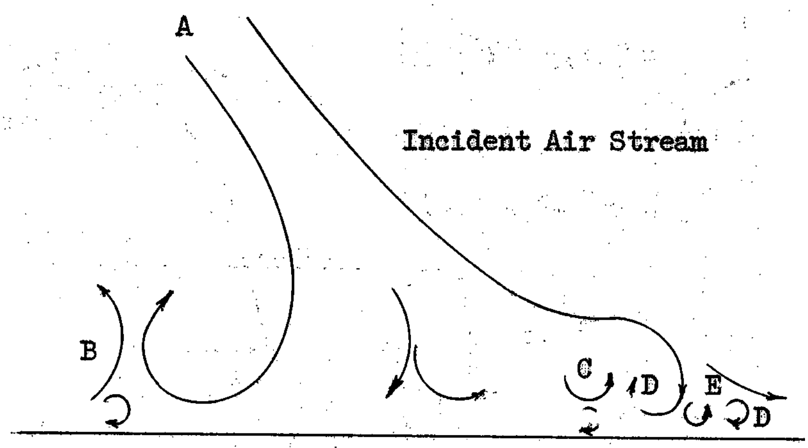

Figure 5.2

Pattern of Successive Motion of Viscous Eddies

at which momentum is interchanged, such as C,D,E, etc. We note that each interchange of momentum occurs with the creation of a pair of smaller vortex centers where the stream splits. This process continues until the motion is completely dissipated in thermal agitation of molecules.

This particular viscous process of a chaotic distribution of vortices is not necessary to the up-scaling of the wavenumbers. As shown above, up-scaling occurs due simply to the necessity of having to use a relative coordinate system in conjunction with an inertial coordinate system in order to describe the balance of momentum of inertial, viscous and other types of forces. The presence of any viscous process, even a laminar one, would not eliminate the necessity of the complex convolutions given above. We might expect however that a purely laminar process would be typified perhaps by a spectrum that is less gradual in the fall off at higher wavenumbers. That is, the turbulence would appear to hasten the energy dissipation and hence the fall-off of high wave numbers due to damping. This is indeed the case and, in fact, it has been suggested (DRY 1) that turbulence may be defined as a viscous process having a certain characteristic rate of fall-off at high wave numbers.

It is obvious from the transformed left-hand member of the Navier-Stokes equation above that the general solution of the component spectra of the velocity fluctuations may be impossible. A similar remark may be made for the continuity equation for which the left member written above transforms in the same manner.

The useful results so far obtained (KOL 1, OBU 1, BAT 1, VIL 1, WHE 2) rely upon dimensional analysis to predict the general shape of the spectra of various physical quantities. A number of "mixing" models have been developed as reviewed by Wheelon (WHE 1). Perhaps the one of these which is most useful for predictions of communication phenomena is "the mixing-in-gradient" model (WHE 2) which predicts the turbulence spectrum $S(k)$ of dielectric fluctuation to be of the form

$$S(k) = 6\pi^2 \langle \Delta\epsilon^2 \rangle l_0^3/(1+k^2 l_0^2)^{5/2} \qquad (5.18a)$$

where $\langle \Delta\epsilon^2 \rangle$ is the mean square value of the dielectric constant ϵ, l_0 is a characteristic length of the turbulence and k is the wave-number. Results of tropospheric scatter and line of sight propagation experiments (NOR 1) confirms the shape (-5/2 power) of this spectrum quite well at the high wave number end of the spectra. The spatial correlation function $C(R)$ equivalent to (5.18) contains the first order Bessel Function of the second kind, K_1,

$$C(R) = (R/l_o)\, K_1\, (R/l_o), \tag{5.18b}$$

where

$$\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$$

5.2 Propagation Velocities in an Inhomogeneous Troposphere

From (4.118) we have the polarization vector for a homogeneous troposphere expressed in the form

$$\vec{P}(r,s) = \vec{E}(r,s)\ (N/3kmT\tau)\ p(s) \tag{5.19}$$

where p(s) is given by (4.122b) for electron displacement and polar resonances or by similar terms with appropriate coefficients as determined from absorption and index of refraction data at the close of Chapter 4. Now in (5.19), for an inhomogeneous troposphere, N becomes a function of z and t. We wish to consider the manner in which this dependence on z,t should be introduced.

The particle density N must be considered a parametric function of z in (5.19). It is not proper to consider N as a transformed function of z such that $\vec{P}(r,s)$ should contain N as the spatial spectrum N(r) or its convolution with $\vec{E}(r,s)$. This can be reasoned in the following way.

Consider a region having a particle density N_1 in which the wave is being propagated. The signal velocity v_s is given by (4.114) as

$$v_{s_1} = c/\left[\alpha_1(s)+s\,d\alpha_1(s)/ds\right]_{s\,=\,s_s} \tag{5.20}$$

where s_s is the location of the saddlepoint in the complex plane at which the signal is first detectable. We also are given that the actual propagation time t_s for the signal is

$$t_{s_1} = z/v_{s_1} = (z/c)\left[\alpha_1(s)+sd\,_1(s)/ds\right]_{s\,=\,s_s} \qquad (5.21)$$

where z is the length of the path. If the dispersive effects are negligible at the chosen operating frequency, we have from (4.127)

$$\alpha_1 \approx 1 + KN_1/2,\ d\alpha_1/ds = 0 \qquad (5.22)$$

where K is a constant. We may then conclude by substituting (5.22) into (5.21), that there is an increment of t_s proportional to the density of the gas in the medium. If we then have a continuation of the path through a second section of medium of density N_2 and having the same thickness, the total signal propagation time is $t_{s_1} + t_{s_2}$.

Now consider an equivalent path of the same total length through several layers of density N_1 and N_2, but thinner in proportion to the numbers of each layer. There is to be the same total thickness of medium of densities N_1 and N_2. Under these conditions the signal propagation time is still $t_{s_1} + t_{s_2}$.

If we had considered it necessary to express $\vec{P}(r,s)$ as the product of N(r), the spatial spectrum of the density, with other factors of (5.19), it is easy to see that we are faced with a difficult inversion integral with respect to z (see 4.126). The spectra for the two different arrangements of the layers of densities N_1 and N_2 are different. The inversion integral would therefore not yield the same exponential kernel for the z inversion and a different expression for (5.20) would result for rearrangement of the layered medium. The signal propagation times would therefore not be the same in the two cases. We see then that N must be a z-parametric factor which typifies the density over a path which is of length z. In the limit, N becomes a continuous function of z, N(z), for which the applicable path is of incremental length dz. The total time of propagation then becomes a matter of integration of the incremental contributions to the signal propagation time for each increment of path length, i.e.,

$$t_s = \int_0^z dt_s(z) = (1/c)\int_0^z \left\{\alpha(s)+s\left[d\alpha(s)/ds\right]\right\}_{s_s} dz \qquad (5.23)$$

The z dependence of α is, as explained above, the factor N(z), which expresses the inhomogeneous distribution of density. The average signal velocity over the path of length z is then

$$\bar{v}_s = z/t_s = zc/\left\{\int_0^z \left\{\alpha(s)+s\left[d\alpha(s)/ds\right]\right\}_{s_s} dz\right\}. \qquad (5.24)$$

By a similar line of reasoning, the time of propagation of a wave front through on incremental thickness of a medium is given by

$$dt_{ph} = dz/v_p$$

$$v_p = c/\alpha(s)$$

$$dt_{ph} = (1/c)\ \alpha\ (s)\ dz \qquad (5.25)$$

The total time of propagation of a phase front over distance d is then

$$t_{ph} = (1/c)\int_0^z \alpha(s,z)\ dz \qquad (5.26)$$

The average velocity of propagation of the phase front over the whole distance z is then

$$\overline{v}_p = z/t_{ph} = zc\left[\int_0^z (s,z)\ dz\right]^{-1} \tag{5.27}$$

Similarly for the group velocity

$$v_g = c/(s d\alpha/ds)$$

$$v_g = zc\left[\int_0^z \frac{s d\alpha(s)}{ds}\ dz\right]^{-1} \tag{5.28}$$

5.3 Propagation Velocities in a Random Troposphere

Due to the convection of the air mass, the particle density as a function of z is also a slowly varying random function of time. Thus the signal propagation time at $t = t_1$ is, from (5.23),

$$t_s(t_1) = (1/c)\int_0^z \left\{\alpha(s,z,t_1) + s\left[d\alpha(s,z,t_1)/ds\right]\right\}_{s_s} dz \tag{5.29}$$

where the time variation of α is due to $N(z,t)$. There is negligible interaction upon the signal frequency because the signal frequencies are generally much higher than those of the randomness of $N(z,t)$. The autocorrelation function of $t_s(t)$ is

$$R_{t_s t_s}(t_1,t_2) = E\left[t_s(t_1)t_s(t_2)\right]$$

$$= E\left\{(1/c)\int_0^z \alpha(s,z_1,t_1) + s\ \ d\alpha(s,z_1,t_1)/ds\right]_{s_s} dz_1$$

$$\times (1/c)\int_0^z\left[\alpha(s,z_2,t_2) + s\ \ d\alpha(s,z_2,t_2,)/ds\right]_{s_s} dz_2\left.\right\}$$

$$= (1/c^2)\left[\int_0^z dz_1\int_0^z dz_2\left\{\alpha(s,z_1,t_1)\ \alpha(s,z_2,t_2)\right.\right.$$

$$+ s\alpha(s,z_1,t_1)\ \ d\alpha(s,z_2,t_2)/ds\ \ + s\alpha(s,z_2,t_2)$$

$$\times d\alpha(s,z_1,t_1)/ds\ \ +$$

$$+ s^2 \left[d\alpha(s,z_1,t_1)/ds \right] \left[d\alpha(s,z_2,t_2)/ds \right] \Big\} \Big]_{s=s_s} \qquad (5.30)$$

Now for gases, we have from (5.22)

$$\alpha(s,z,t) \Big|_{s_s} \approx 1 + K(s)\ N(z,t)/2 \Big|_{s_s}$$

$$d\,\alpha(s,z,t)/ds \Big|_{s_s} \approx \left[N(z,t)/2 \right] dK(s)/ds \Big|_{s_s}$$

where K(s) expresses the product of conversion factors and the function of s, and N(z,t) expresses the gas density as a function of position and time. Then

$$R_{t_s t_s}(t_1,t_2) = (1/c^2)\ E \Big\{ \int_0^z dz_1 \int_0^z dz_2 \Big\{ 1 + [K(s)/2]\ [N(z_1,t_1) + N(z_2,t_2)] + [K^2(s)/4]\ N(z_1,t_1)\ N(z_2,t_2) + s[dK(s)/ds]\ [N(z_1,t_1) + N(z_2,t_2)]/2 + s[K(s) \times dK(s)/ds]\ N(z_1,t_1)\ N(z_2,t_2)/2 + s^2 [dK(s)/ds]^2 \times N(z_1,t_1)\ N(z_2,t_2)/4 \Big]_{s_s}$$

$$= (1/c^2) \Big\{ \int_0^z dz_1 \int_0^z dz_2 + [K(s)/2]_{s_s}\ E[\bar{N}(t_1) + \bar{N}(t_2)] + (1/4c^2)\ [K(s) + s\,dK(s)/ds]^2_{s_s}\ E \Big\{ \int_0^z dz_1 \int_0^z dz_2 \times N(z_1,t_1) \times N(z_2,t_2) \Big\}$$

$$= (z^2/c^2) \Big\{ 1 + K(s) \Big|_{s_s} \langle \bar{N}(z,t) \rangle \Big\} + (1/4c^2) \times \left[K(s) + s\,dK(s)/ds \right]_{s_s} E \Big\{ \int_0^z dz_1 \int_0^z dz_2\ N(z_1,t_1)$$

$$\times N(z_2, t_2) \tag{5.31}$$

where

$$\overline{N}(z,t) = (1/z)\int_0^z N(z',t)\, dz'$$

and $\langle\overline{N}\rangle$ is the time average of $\overline{N}$. If $\overline{N}(z,t)$ is stationary and ergodic

$$R_{t_s t_s}(t_1, t_2) = R_{t_s t_s}(\tau)$$

$$= t_p^{\,2}\left[1+K(s)\Big|_{s_s} \langle N(z,t)\rangle\right]$$

$$+ (t_p^2/4)\left[K(s)+s dK(s)/ds\right]_{s_s} R_{\overline{N}\overline{N}}(\tau) \tag{5.32}$$

The mean square value of t_s, $\langle t_s^2\rangle$ is

$$\langle t_s^{\,2}\rangle = R_{t_s t_s}(0) = t_p^2\left\{1 + \left[K(s)+s dK(s)/ds\right]_{s_s} R_{\overline{N}\overline{N}}(0)\right\}/4. \tag{5.33}$$

In order to apply this equation we must establish the relation of the autocorrelation function of the z-averaged density function to the available data. To obtain a spatial spectrum function for the density, an instantaneous measurement of the density versus position must be made over a length of path sufficiently great to define the spectrum to the necessary degree of precision. This being a practical impossibility, the method used (NOR 1, KUB 1) in most line of sight propagation experiments consists of measuring the time fluctuation of the phase of the signal between two points in the medium. The process is then assumed to be stationary so that time averages may be assumed to be equivalent to spatial averages. This assumption is under question at the present time with regard to very low frequency disturbances (WHE 1). The spectrum which is measured is then $R_{\overline{N}\overline{N}}(\tau)$

providing the paths are of the same length. Changing the length of the path changes the length of the z-average. In case the spatial spectrum for the dielectric constant is given, as in (5.18a), the z-averaging operation becomes equivalent to multiplying the spectrum of (5.18) by $(1+e^{-kz})/k$, where z is averaging length and k is the z-wave number, equivalent to the variable r in Chap. 4.

From (WHE 1), the space-time correlation of the dielectric constant at low radio frequencies, neglecting dispersion, may be written in the form

$$\langle \Delta\epsilon(\vec{d},t)\, \Delta\epsilon(\overrightarrow{d+R}, t+\tau)\rangle = (1/8\pi^3)\int d^3\vec{k} S(\vec{k}) e^{d\vec{k}\cdot(\overrightarrow{R+U\tau})}\, \eta(\vec{k},\tau). \quad (5.34)$$

$\vec{R}$ is the spatial correlation separation corresponding to the temporal lag τ, $\vec{k}$ is the vector wave number, $\vec{U}$ is the air mass velocity and $\eta(\vec{k},\tau)$ is the time correlation of fluctuations contained in a fixed wavenumber interval $\vec{k}$. $\eta(\vec{k},\tau)$ is unity for zero time displacement and is often taken as unity for greater time displacements due to lack of knowledge concerning its functional form. Some work has been done on dimensional analysis of significant variables for η, (see WHE 1). Assuming $\eta(\vec{k},\tau) = 1$, and the value of $\vec{U}$ being known, $S(\vec{k})$ may be obtained from (5.18) modified by the factor $(1+e^{-kz})/k$ to represent the average over z; hence the correlation (5.33) is known from (5.33) and the proportionality constant relating ϵ and the density N (see BEA 1). The mean square value of t_s may then be computed from the relations above.

The mean value of t_s may be obtained from the mean value of $t_s(t)$ given in (5.29) and hence depends on $\langle \bar{N}(z,t)\rangle$ since the time variation is due to $N(z,t)$. The mean value of the index of refraction (hence N) has been studied in detail (BEA 1), and is given in terms of measurable meteorological conditions by the following standardized expression, with the special meaning of symbols noted,

$$n - 1 = \left[77.6P/T + 3.73 \times 10^5 e/T^2 \right] \times 10^{-6} \tag{5.35}$$

where

n = index of refraction,

T = air temperature in degrees Kelvin,

P = total air pressure in millibars, and

e = partial water-vapor pressure in millibars.

From the mean square $\langle t_s{}^2 \rangle$ and mean $\langle t_s \rangle$, the variance of t_s is given by

$$\sigma^2_{t_s} = \langle t_s{}^2 \rangle - \langle t_s \rangle^2 . \tag{5.36}$$

Since the signal velocity is given by

$$v_s = z/t_s, \tag{5.37}$$

then

$$dv_s = -(z/t_s^2) \quad dt_s . \tag{5.38}$$

The variance of v_s is therefore given by

$$\sigma^2_{v_s} = (z^2/t_s^4) \; \sigma^2_{t_s} . \tag{5.39}$$

A similar procedure may be followed for the phase velocity, where (5.31) is replaced by the autocorrelation function of the propagation time t_{ph} of the phase front,

$$R_{t_{ph} t_{ph}} (t_1, t_2) = (z/c)^2 \; \left[1 + K^2(s)/4 \right] R_{\overline{N}\overline{N}}(t_1, t_2) \tag{5.40}$$

The procedure is otherwise the same and may be followed as well for the group velocity.

The procedures given in this chapter enable calculation of the mean, mean square and variance of the signal, phase and group velocities in a random atmosphere. The computation is quite involved in the case of a dispersive medium, since it is necessary to first determine the location of the predominant saddlepoint in the complex s-plane for the time at which the signal becomes detectable as the time parameters are varied. The spatial spectral functions for the dielectric constant, or gas density, may then be applied in the above relations to obtain the desired statistical quantities.

CHAPTER 6

Summary of Results and Conclusions

In this thesis, a study is made of the velocities of propagation of electromagnetic waves in the atmosphere. The solutions are obtained for phase, group and signal velocities in steady state and random dispersive media. In the case of dispersive media, the solutions are indicated in the form of certain contour integrations which must be performed to obtain approximations for complex inversion integrals. The investigation of the conditions for arrival of the signal involve trial and error solutions and are of a tedious nature for all but the simplest types of media. Computer solutions are recommended for obtaining specific values.

Methods for solving the wave equations of electromagnetic theory are developed using the multiple Laplace transform. The application of multiple transform methods would appear to have applications in formulating solutions to a number of problems in electromagnetic theory. Derivations for the characteristic impedance and dielectric constant of complex media are simple and include the assumptions of the mode of propagation and boundary conditions and therefore are rigorous.

The physical nature of turbulence and its effect upon factors which influence electromagnetic wave propagation is discussed. A study is made of the random variables in the wave equation solution and the manner in which they are related to the turbulence model. The expressions are then developed for the mean, mean square and variance of the propagation velocities. For the signal velocity the result is indicated in terms of the contour integral used to define the arrival time of the signal.

The practical aspects of the work presented above can be extended by investigation of the signal and target velocity statistics for various specific media and statistical models. The effects mentioned herein will be of special interst for operating frequencies in the infra-red and optical regions of the electromagnetic spectrum for which dispersive effects are more pronounced and numerous than in the radio-frequency portion of the spectrum.

The statistical optimization of refractive compensation methods was an early objective of this work, and at this point, it can be said that the analytic and computational tools are now defined for such a study. It is noted that in the report that the signal group and phase velocities are quite distinct mathematically and there should be considerable independence in their statistics. Accordingly it should be possible to combine these, if measurable, with properly chosen weighting functions in order to achieve more accurate velocity measurement.

BIBLIOGRAPHY

BAT 1. Batchelor, G. K., The Theory of Homogeneous Turbulence, Cambridge University Press, Cambridge, Mass., 1953.

BEA 1. Bean, B. R., "The Radio Refractive Index of Air," Proceedings of the Institute of Radio Engineers, Vol. 50, No. 3, pp. 260-273, Mar. 1962.

BIR 1. Bird, R. B. Stewart, W. E., and Lightfoot, E. N., Transport Phenomena, John Wiley and Sons, Inc., New York, N. Y. 1960.

BOR 1. Born, M., and Wolf, E., Principles of Optics, Pergamon Press, New York, N. Y., 1959.

BRI 1. Brillouin, L., "Uber die Fortpflanzung des Lichtes in dispergierenden Medien, Part 2." Annalen der Physik, Vol 44, No. 10, pp. 203-240, 1914. English translation in BRI 2, Chapter 3.

BRI 2. Brillouin, Wave Propagation and Group Velocity, Academic Press, New York, N. Y., 1960.

BUR 1. Burrows, C. S., and Attwood, S. S., Radio Wave Propagation, Academic Press Inc., New York, N. Y., 1949.

CHE 1. Chernov, L. A., Wave Propagation in a Random Medium, McGraw-Hill Book Company, Inc., New York, N. Y., 1960.

CHU 1. Churchill, R. V., Modern Operational Mathematics in Engineering, McGraw-Hill Book Company, Inc., New York, N. Y., 1944.

CON 1. Condon, E. V., Handbook of Physics, McGraw-Hill Book Company, Inc., New York, N. Y., 1958.

DEB 1 Debye, P., Polar Molecules, Chemical Catalog Company, Inc., 1928, Reprinted by Dover Publications, Inc., New York, N. Y.

DIC 1 Dickson, L. E., New First Course in the Theory of Equations, John Wiley and Sons, Inc., New York, N. Y., 1939.

DRY 1 Dryden, H. L., "The Turbulence Problem Today", Proceedings of the First Midwestern Conference on Fluid Dynamics, U. of Ill., May 12-13, 1950, Edwards Brothers, Inc., Ann Arbor, Michigan, 1951.

EST 1. Estrin, Thelma A., and Higgins, T. J., "The Solution of Boundary Value Problems by Multiple Laplace Transformations," Journal of the Franklin Institute, Vol. 252, No. 2, pp. 153-167, Aug. 1951.

GAR 1. Gardner, M. F., and Barnes, J. L., Transients in Linear Systems, Vol. I, John Wiley and Sons, Inc., 1942.

HAR 1. Harris, S. M., "Refraction Compensation in a Spherically Stratified Ionosphere," I. R. E. Transactions on Antennas and Propagation, Vol. AP-9, No. 2, pp. 207-210, Mar. 1961.

JOR 1. Jordan, E. C., Electromagnetic Waves and Radiating Systems, Prentice-Hall, Inc., New York, N. Y., 1950.

KAR 1. von Karman, T. and Howarth, L. "On the Statistical Theory of Isotropic Turbulence," Proceedings of the Royal Society (London), Series A, Vol. 164, 192-195, 1938.

KOL 1. Kolmogorov, A. N., "The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers," Comptes Rendus (Doklady) of the Academy of Sciences, USSR, Vol. 30, No. 4, 301-305. 1941

KUB 1. Kubasov, P. Ye., "Investigation of the Space-time Correlations of the Field Level of Centimeter Waves Propagating in a Turbulent Atmosphere," Radio Engineering and Electronic Physics, pp. 1092-1097, No. 8, Aug. 1961.

LOE 1. Loeb, L. B., The Kinetic Theory of Gases, Dover Publications Inc., New York, N. Y., 1961

MEN 1. Men', A. V., "Time (Spectral) Characteristics of Phase-Difference Fluctuations Occurring in the Propagation of Radio Waves in the Troposphere, Radio Engineering and Electronic Physics, No. 10, Oct. 1961

MIT 1. Mitra, S. K., The Upper Atmosphere, The Asiatic Society, Calcutta, India, 1952.

MOR 1. Morse, P. M., and Feshbach, H., Methods of Theoretical Physics, 2 Volumes, McGraw-Hill Book Company, Inc., New York, N. Y., 1953.

MUC 1. Muchmore, R. B., and Wheelon, A. D., "Line-of-Sight Propagation Phenomena-I. Ray Treatment," Proceedings of the Institute of Radio Engineers, Vol. 43, No. 10, pp. 1937-1449, Oct. 1955.

MOR 1. Norton, K. A., et al., An Experimental Study of Phase Variations in Line-of-Sight Microwave Transmission, National Bureau of Standards Monograph 33, Washington, D. C., 1961

OBU 1. Obukhov A., "On the Energy Distribution in the Spectrum of a Turbulent Flow," Comptes Rendos (Doklady) de l'Academie des Sciences de l' URSS, Vol. 32, No. 1, 1941.

POL 1. Van der Pol, B., and Bremmer, H., Operational Calculus based on the Two-sided Laplace Integral, Cambridge, University Press, 1950.

SLA 1. Slater, J. C., and Frank, N. H., Introduction to Theoretical Physics, McGraw-Hill Book Company, Inc., New York, N. Y., 1933.

SOM 1. Sommerfeld, A., "Uber die Fortpflanzung des Lichtes in dispergierenden Medien, Part 1," Annalen der Physik, Vol. 44, No. 10, pp. 177-202, 1914. English translation in BRI 2, Chapter 2.

STR 1. Stratton, J. A., Electromagnetic Theory, McGraw-Hill Book Company, Inc., New York, N. Y., 1941.

TAT 1. Tatarski, V. I., Wave Propagation in a Turbulent Medium, translated from the Russian by R. A. Silverman, McGraw-Hill Book Company, Inc., New York, N. Y., 1961.

TOW 1. Townes, C. H., and Schawlow, A. L., Microwave Spectroscopy, McGraw-Hill Book Company, Inc., New York, N. Y., 1955.

TRU 1. Truxal, J. G., Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, Inc., New York, N. Y., 1955.

USA 1. United States Air Force Air Research and Development Command, Handbook of Geophysics, Rev. Ed., MacMillan Company, New York, N. Y., 1960.

VIL 1. Villars, F., and Weisskopf, V. F., "On the Scattering of Radio Waves by Turbulent Fluctuations of the Atmosphere," Proceedings of the Institute of Radio Engineers, Vol. 43, No. 10, pp. 1232-1239, Oct. 1955.

WHE 1. Wheelon, A. D., "Radio-wave Scattering by Tropospheric Irregularities, "Journal of Research, National Bureau of Standards, Vol. 63D, No. 2, pp. 205-233, 1959.

WHE 2. Wheelon, A. D., "Spectrum of Turbulent Fluctuations Produced by Convective Mixing of Gradients," Physical Review, Vol. 105, No. 6, Mar. 15, 1957.

APPENDIX

APPENDIX

Evaluation of a Certain Complex Inversion Integral Encountered in Solution of the Wave Equation with Time-Varying Coefficients

We consider the evaluation of the complex inversion integral Eq. (4.91)

$$f_{23}(z,t) = (1/2\pi j)\int_{C_2+C_3} e^{s[t-\alpha(s)(z/c)]} F(+0,s)ds \qquad (A.1)$$

which occurs in the solution of the wave equation for dispersive media, (see Chapter 4).. In this case, for the ionosphere, neglecting collision damping (see Chapter 3),

$$\alpha(s) = +\left[1+\omega_c^2/(s^2+\omega_o^2)\right]^{1/2}$$

and the exponential kernel displays branch points at

$$a_{\pm} = \pm j\omega_o \qquad (A.2)$$

$$b_{\pm} = \pm j\sqrt{\omega_o^2+\omega_c^2} \qquad (A.3)$$

We shall first assume that $F(+0,s)$ is the transform of a sinusoid

$$F(+0,s) = E_o\omega/(s^2+\omega^2), \qquad (A.4)$$

so that

$$f_{23}(z,t) = (E_o\omega/2\pi j)\int_{C_2+C_3} \left\{e^{s[t-\alpha(s)(z/c)]}/(s^2+\omega^2)\right\} ds \qquad (A.5)$$

Assuming the singularities of the exponential kernel and denominator are distinct, they occur on the imaginary axis as shown in Fig. 4.2.

The residues of f (z,t) for $s = +j\omega$ present no problem. However the integration around the contour encircling the branch points must be examined in detail. In order to determine the nature of the contributions of the branch points to the contour integration, consider the contour C_2 in Fig. A.1, encircling one pair of the branch points. The shrinking of the contours to this particular one was discussed in Chap. 4 .

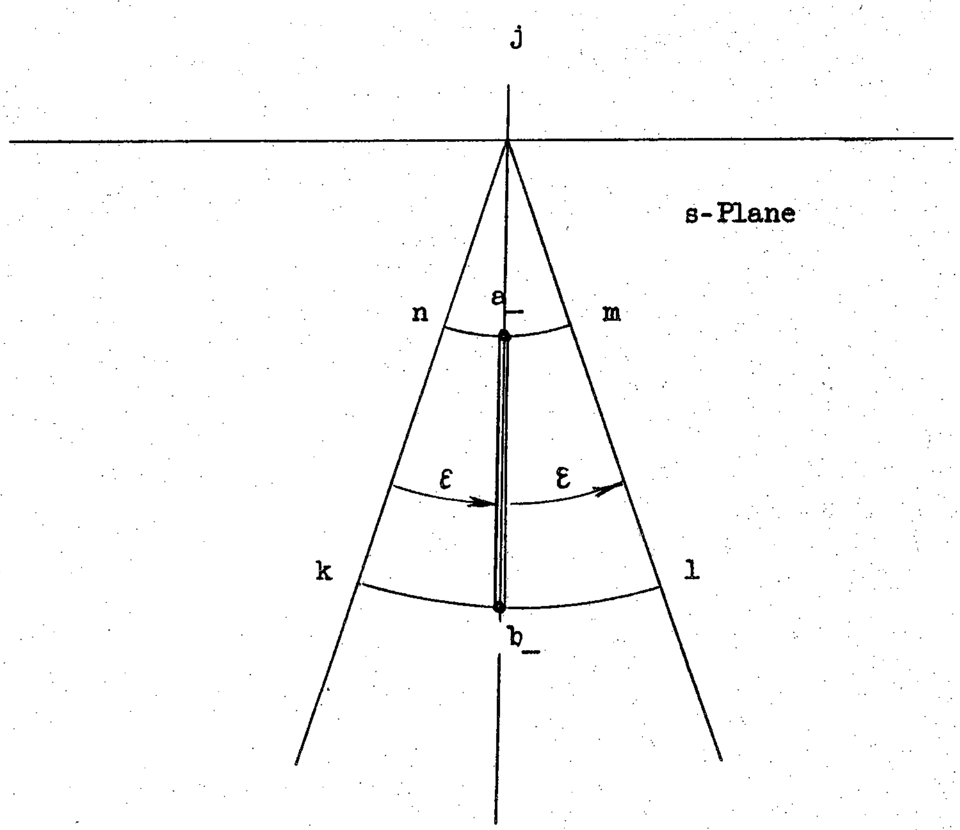

Figure A.1

Path for Integration on Contour C_2 Around a Branch Cut

Now consider the integration along contour C_2 element by element. Refering to Fig. A.1 we have for segment k-1:

$$s = \rho e^{j\theta},\ -(\pi/2)-\varepsilon < \theta < -(\pi/2)+\varepsilon\ ,\ \rho = (\omega_o^2+\omega_c^2)^{1/2} \qquad (A.6)$$

$$s = \rho e^{j\theta},\ -(\pi/2)-\varepsilon < \theta < -(\pi/2)+\varepsilon,\ \rho= (\omega_o^2+\omega_c^2)^{1/2}. \qquad (A.7)$$

For segment l-m:

$$s = \rho e^{j(-\pi/2-\varepsilon)},\quad (\omega_o^2+\omega_c^2)^{1/2} > \rho > \omega_o\,. \qquad (A.8)$$

For segment m-n:

$$s = \rho e^{j\theta},\quad -(\pi/2)+\varepsilon > \theta > -(\pi/2)-\varepsilon\ ,\ \rho = \omega_o \qquad (A.9)$$

For segment n-k:

$$s = \rho e^{j(-\pi/2-\varepsilon)},\quad \omega_o < \rho < (\omega_o^2+\omega_c^2)^{1/2} \qquad (A.10)$$

Evaluation of $f_{23}(z,t)$ on segment k-l:

$$s = \rho e^{j\theta},\quad -\pi/2-\varepsilon < \theta < -\pi/2+\varepsilon\ ,\quad \rho = (\omega_o^2+\omega_c^2)^{1/2}$$

Let $\theta = -\pi/2 + \phi\ ,\ -\varepsilon < \phi < \varepsilon$

Then $s = \rho e^{j(-\pi/2+\phi)} = -j\rho e^{j\phi}$

$$= -j(\omega_o^2+\omega_c^2)^{1/2} e^{j\phi},$$

or for ϕ small,

$$s = -j(\omega_o^2+\omega_c^2)^{1/2}(1+j\phi)$$

Then

$$\alpha(s) = \sqrt{\left[1 + \omega_c^2/(s^2+\omega_o^2)\right]}$$

$$= \sqrt{\left[(\omega_o^2+\omega_c^2)\ (2\phi^2-2j\phi)\right] / \left[(\omega_o^2+\omega_c^2)\ (2\phi^2-2j\phi)-\omega_c^2\right]}$$

and

$$e^{s\left[t-\alpha(s)(z/c)\right]} = e^{-j(\omega_o^2+\omega_c^2)^{1/2}\left\{t-(z/c)\sqrt{\frac{(\omega_o^2+\omega_c^2)\;(2\phi^2-2j\phi)}{(\omega_o^2+\omega_c^2)\;(2\phi^2-2j\phi)-\omega_c^2}}\right\}}$$

Also

$$F(+0,s) = E_o\omega/(s^2+\omega^2)$$

$$= E_o\omega/\left[\omega(\omega_o^2+\omega_c^2)\;(1+2j\phi)+\omega^2\right]$$

From these expressions, upon substituting into (A.1) for the portion of the path k-l and taking the limit as ε approaches zero, we have

$$I_1 = \lim_{\substack{\epsilon\to 0\\ \phi\to 0}} \int_{-\varepsilon}^{\varepsilon}\left[e^{-j(\;_o^2+\;_c^2)^{1/2}\left\{t-(z/c)\sqrt{\frac{(\omega_o^2+\omega_c^2)\;(2\phi^2)\;(2\phi^2-2j\phi)}{(\omega_o^2+\omega_c^2)\;(2\phi^2-2j\phi)\;-\omega_c^2}}\right\}}\right]$$

$$\left[E_o\omega/\left[(\omega_o^2+\omega_c^2)\;(1+2j\phi)\;+\omega_c^2\right]\right]\;\rho(1+j\phi)d\phi$$

$$= \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} e^{-j(\omega_o^2+\omega_c^2)^{1/2}t}\left[E_o\omega/(\omega_o^2+\omega_c^2+\omega^2)\right]d\phi = 0 \qquad \text{(A.11)}$$

Evaluation of $f_{23}(z,t)$ on segment l-m:

$$s = \rho e^{j(-\pi/2+\varepsilon)}, \qquad (\omega_o^2+\omega_c^2)^{1/2} > \rho > \omega_o$$

$$= -j\rho e^{j\varepsilon}$$

$$\approx -j\rho(1+j\varepsilon)$$

$$\alpha(s) = \sqrt{\left[\omega_o^2+\omega_c^2-\rho^2(1+j2\varepsilon)\right]\;\left[\omega_o^2-\rho^2(1+j2\varepsilon)\right]}$$

$$e^{s\;[t-\alpha(s)(z/c)]} = e^{jp(1+j\varepsilon)\left[t-(z/c)\sqrt{\frac{\omega_o^2+\omega_c^2-\rho^2(1+j2\epsilon)}{\omega_o^2-\rho^2(1+j2\epsilon)}}\right]}$$

$$F(+0,s) = E_o\omega/\left[\omega^2-\rho^2(1+j2\varepsilon)\right]$$

whence, taking the limit as $\varepsilon\to 0$,

$$I_2 = -j\int_{(\omega_o^2+\omega_c^2)^{1/2}}^{\omega_o} \left\{ e^{-j\rho\left[t-\sqrt{(\omega_o^2-\omega_c^2-\rho^2)/(\omega_o^2-\rho^2)}\,(z/c)\right]} \right\} \left\{E_o\omega/(\omega^2-\rho^2)\right\} d\rho$$

which becomes improper at $\rho = \omega_o$, in the same fashion as the original integral.

Evaluation of $f_{+23}(z,t)$ on segment m-n:

$$s = \rho e^{j\phi}, \quad -\pi/2+\varepsilon > \theta > -\pi/2-\varepsilon \quad , \ \rho = \omega_o$$

Let $\theta = -\pi/2+\phi \quad \varepsilon > \phi > -\varepsilon$

then $s \approx -j\omega_o(1+j\phi)$

$$\alpha(s) = \frac{1}{\phi^{1/2}}\sqrt{\frac{\omega_c^2+j2\phi\omega_o^2}{j2\omega_o^2}} \ , \ F(+0,s) = \frac{E_o\omega}{\omega^2-\omega_o^2-j2\phi\omega_o^2}.$$

Therefore

$$I_3 = \lim_{\substack{\varepsilon\to 0\\ \phi\to 0}} \int_{\varepsilon}^{-\varepsilon} \exp\left\{-j\omega_o(1+j\phi)\left[t-\frac{1}{\phi^{1/2}}\sqrt{\frac{\omega_c^2+j2\phi\omega_o^2}{j2\omega_o^2}}\right]\left(\frac{z}{c}\right)\right\} \frac{\omega E_o\omega_o(1+j\phi)d\phi}{\omega^2-\omega_o^2+j2\phi\omega_o^2} \quad \text{(A.12)}$$

which becomes improper as ϕ and ω_o approach zero.

Evaluation of $f_{23}(z,t)$ on segment n-k:

$$\alpha(s) = \sqrt{\frac{\omega_o^2+\omega_c^2-\rho^2(1-j2\varepsilon)}{\omega_o^2-\rho^2(1-j2\varepsilon)}}$$

$$e^{s\left[t-\alpha(s)(z/c)\right]} = e^{-j\rho\left\{t-\frac{z}{c}\sqrt{\left[\omega_o^2+\omega_c^2-\rho^2(1-j2\varepsilon)\right]/\left[\omega_o^2-\rho^2(1-j2\varepsilon)\right]}\right\}}$$

$$F(+0,s) = E_o\omega/\left[\omega^2-\rho^2(1-j2\varepsilon)\right]$$

Thus,

$$I_4 = \lim_{\varepsilon\to 0}\int_{\omega_o}^{(\omega_o^2+\omega_c^2)^{1/2}} e^{-j\rho\left[t-\sqrt{\frac{\omega_o^2+\omega_c^2-\rho^2(1-j2\varepsilon)}{\omega_o^2-\rho^2(1-2\varepsilon)}}(z/c)\right]} \frac{E_o\omega[-j(1-j\varepsilon)]}{\omega^2-\rho^2(1-2\varepsilon)} d\rho. \quad \text{(A.13)}$$

which becomes improper at $\rho = \omega_o$.

We see that the integrand of the contour integral around C_2 exhibits a singularity at the branch point a_-, and otherwise is zero. We may therefore collapse the contour to encircle this singularity only.

In order to evaluate the contour integral we must expand the exponential kernel about the singularities a_+ and a_-. $\alpha(s)$ is an even function of s and the expansion will enable evaluation of the integral for both singularities. We abbreviate slightly and write $\alpha(s)$ as

$$\alpha(s) = \left[1+\omega_c^2/(s^2+\omega_o^2)\right]^{1/2} = (1+W)^{1/2}$$

where W represents $\omega_c^2/(s^2+\omega_o^2)$ in the algebraic steps to follow. Then

$$\alpha(s) = 1+(W/2) - (W^2/2^2 2!) + (2W^3/2^3 3!) - (3\cdot 5W^4/2^4 4!) + \ldots$$

Letting $z/c = t_p$, the vacuum propagation time,

$$e^{-s\alpha(s)\frac{z}{c}} = e^{-s\alpha(s)t_p}$$

Or

$$e^{-s\alpha(s)\,(z/c)} = e^{-st_p(1+W/2-W^2/2^2 2!-3W^3/2^3 3!-\ldots)}$$

$$=e^{-st_p}\left[1-st_pW/2+s^2t_p^2W^2/2^2 2!-\ldots\right]\left[1+st_pW^2/2^2 2!+ s^2t_p^2W^4/2^4(2!)^3+\ldots\right]\left[1-3st_pW^3/2^3 3!+ +3^2s^2t_p^2W^6/2^6(3!)^2 2!-\ldots\right]\ \ldots$$

$$= e^{st_p}\left[1-st_pW/2+(s^2t_p^2+st_p)W^2/2^2 2!-(s^3t_p^3+3st_p)\,W^3/2^3 3! +(s^4t_p^4+6s^3t_p^3+15s^2t_p^2+15st_p)W^4/2^4 4!\right.$$

$$-(s^5t_p^5+10s^4t_p^4+45s^3t_p^3+105s^2t_p^2+105st_p)W^5/2^5 5!+\ldots\Big].$$

Now from (A.5),

$$f_{23}(z,t) = \frac{\omega E_o}{2\pi j}\int_{C_2+C_3} e^{st}\frac{e^{-st_p}}{s^2+\omega^2}\Bigg[1-\frac{st_p\omega_c^2}{2(s+\omega_o^2)}+\frac{(s^2t_p^2+st_p)\omega_c^4}{2^2 2!\,(s^2+\omega_o^2)^2}$$

$$-\frac{(s^3t_p^3+3s^2t_p^2+3st_p)\omega_c^6}{2^3 3!\,(s^2+\omega_o^2)^3}$$

$$+\frac{(s^4t_p^4+6s^3t_p^3+15s^2t_p^2+15st_p)\omega_c^8}{2^4 4!\,(s^2+\omega_o^2)^4}$$

$$-\frac{(s^5t_p^5+10s^4t_p^4+45s^3t_p^3+105s^2t_p^2+105st_p)\omega_c^{10}}{2^5 5!(s^2+\omega_o^2)^5}$$

$$+\ldots\Bigg]ds \qquad \text{(A.14a)}$$

$$=\omega E_o\sum_{s=\pm j\omega_o}\text{Res}\,\frac{e^{-st_p}}{s^2+\omega^2}\Bigg[1-\frac{st_p\omega_c^2}{2(s^2+\omega^2)}+\frac{(s^2t_p^2+st_p)\omega_c^4}{2^2 2!\,(s^2+\omega_o^2)^2} \qquad \text{(A.14b)}$$

But

$$s/(s^2+\omega_o^2) = (1/2)\Big[1/(s-j\omega_o)+1/(s+j\omega_o)\Big], \qquad \text{(A.15)}$$

$$\frac{s^2+s/t_p}{(s^2+\omega_o^2)^2}=\frac{1}{4\omega_o^2}\;\frac{\omega_o^2-j\omega_o/t_p}{(s-j\omega_o)^2}-\frac{j\omega_o}{s-j\omega_o}+\frac{\omega_o^2+j\omega_o/t_p}{(s+j\omega_o)^2}+\frac{j\omega_o}{s+j\omega_o} \qquad \text{(A.16)}$$

$$\frac{s^3+3s^2/t_p+3s/t_p^2}{(s^2+\omega_o^2)^3}=\frac{1}{8\omega_o^3}\Bigg\{\left[\frac{\omega_o^3-3\omega_o/t_p^2-j3\omega_o^2/t_p}{(s-j\omega_o)^3}\right]+\frac{3}{2}\left[\frac{-\omega_o/t_p-j(\omega_o^2+1/t_p^2)}{(s-j\omega_o^2)^2}\right]-$$

$$- \frac{j3}{2t_p(s-j\omega_o)} + \left[\frac{\omega_o^3 - 3\omega_o/t_p^2 + j3\omega_o^2/t_p}{(s+j\omega_o)^3}\right] + \frac{3}{2}\left[\frac{-\omega_o/t_p + j(\omega_o^2 + 1/t_p^2)}{(s+j\omega_o^2)^2}\right] + \frac{j3}{2t_p(s+j\omega_o)}\Bigg\}, \tag{A.17}$$

$$\frac{s^4 + \frac{6s^3}{t_p} + \frac{15s^2}{t_p^2} + \frac{15s}{t_p^3}}{(s^2+\omega_o^2)^4} = \frac{1}{16\omega_o^4}\left[\frac{\omega_o^4 - \frac{15\omega_o^2}{t_p} + j\left(-\frac{6\omega_o^3}{t_p} + \frac{15\omega_o}{t_p^3}\right)}{(s-j\omega_o)^4}\right.$$

$$-\left[\frac{\frac{18\omega_o^2}{t_p} + \frac{75}{t_p^3} + j\left(7\omega_o^3 + \frac{15\omega_o}{t_p^3}\right)}{(s-j\omega_o)^3}\right] + \frac{1}{2}\left[\frac{8\omega_o^2 + \frac{15}{t_p^2} + j\frac{18\omega_o}{t_p}}{(s-j\omega_o)^2}\right]$$

$$+ \frac{j}{2\omega_o}\left[\frac{8\omega_o^2 + \frac{15}{t_p^2}}{s-j\omega_o}\right] + \ldots \tag{A.18}$$

etc.

Substituting (A.5), (A.16), (A.17), (A.18), etc., into (A.14) summing the residues, and including those for $s = \pm j\omega$, we have

$$f(z,t) = E_o\left[\sin\ \omega\left(t - \sqrt{1+\omega_c^2/(\omega_o^2-\omega^2)}\ t_p\right)\right.$$

$$+ \frac{\omega}{\omega^2-\omega_o^2}\ \cos\omega_o(t-t_p)\left[-\frac{\omega_c^2 t_p^2}{2t_p} + \frac{\omega_c^4(t-t_p)}{16\omega_o}\right.$$

$$-\frac{\omega_c^6 t_p^3}{192\omega_o^2}\left(\frac{(\omega_o^2 t_p^2 - 3)(t-t_p)^2}{2!\ t_p^2} - \frac{3(t-t_p)}{2t_p}\right) +$$

$$+\frac{\omega_c^8 t_p^4}{384\omega_o^4}\left(\frac{(\omega_o^4 t_p^2-15\omega_o^2)(t-t_p)^3}{48t_p^4}-\frac{(18\omega_o^2 t_p^2+75)(t-t_p)^2}{16t_p^3}+\frac{8\omega_o^2 t_p^2+15)(t-t_p)}{t_p^2}\right)$$

$$\left.+\ \ldots\right]$$

$$+\sin\omega_o(t-t_p)\left[\frac{\omega_c^4}{16\omega_o}-\frac{\omega_c^6 t_p^3}{192\omega_o^2}\left(\frac{3\omega_o(t-t_p)}{2!}+\frac{(3\omega_o^2 t_p^2+1)(t-t_p)}{2\omega_o t_p^2}+\frac{3}{2\omega_o t_p}\right)\right.$$

$$+\frac{\omega_c^8 t_p^4}{384\omega_o^4}\left(-\frac{(6\omega_o^3 t_p^2+15\omega_o t_p)(t-t_p)^3}{48t_p^4}-\frac{(7\omega_o^3 t_p^2+15\omega_o +_p)(t-t_p)}{16t_p^2}\right)$$

$$\left.+\frac{(8\omega_o^2 t_p^2+18\omega_o t_p+15)(t-t_p)}{16t_p^2}+\frac{(8\omega_o^2 t_p^2+15)}{{}_o t_p^2}+\ldots\right] \qquad \text{(A.19)}$$

Since the damping term was not included, it would be expected that this solution would diverge as time becomes infinite. Addition of the damping term can be made and the same steps followed to obtain the appropriate series solution. This will not be done at present since quantitative discussions of propagation effects in the ionosphere will not be given.

It is of interest to determine the behavior of the solution for small values of time. As noted in the discussion of the contour integral in Chapter 4, there is no response at z prior to the vacuum propagation time. This result, using the contour integral existence concepts was first obtained by Sommerfeld (SOM 1, represented in Sec. 22, BRI 2). The first term of (A.19) is the steady state term and the succeeding terms are those of the transient. It may be noted that all but the first transient term contains a factor of $(t-t_p)$. These terms are definitely zero at $t = t_p$. At this instant the other terms sum to zero as can be seen by using the small angle approximation on the steady state term and by expanding the radical. Thus for a time slightly in excess of t_p,

$$\sin \omega \left[t - \sqrt{1+\omega_c^2/(\omega_o^2+\omega^2)}\; t_p \right] \approx \omega\left[t - \sqrt{1+\omega_c^2/(\omega_o^2+\omega^2)}\; t_p\right]$$

$$\approx \omega(t-t_p) + \omega\omega_c t_p/2(\omega^2-\omega_o^2)$$

which combined with the first transient term yields

$$f\,(z,t) \approx E_o\omega(t-t_p)/2, \qquad \text{for } t = t_p + \delta t_p, \text{ where } \delta t_p \ll t_p.$$

We may determine a good deal more about the behavior of the solution for small values of $t-t_p$ by reducing it to an asymptotic form (POL 1). In essence, this was done by Sommerfeld (op. cit.), however the simplicity of the approach using the Lapalce transform may be of interst. In order to obtain an asymptotic series valid for small values of t, we consider the value of the inverse transform for large values of s. The series of (A.14a) or (A.5) may be transformed term by term to obtain a series in the variable t. However a more direct procedure from (A.5) is to write

$$\lim_{t\approx 0} f\,(z,t) = \lim_{s\approx\infty} \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} e^{s\left[t-\alpha(s)\; t_p\right]}\, F(+0,s)ds,$$

where by the limiting processes we mean that each side will assume asymptotic forms as t vanishes and s increases without bound. It is shown by saddlepoint integration that the initial response is determened for the values of s beyond ω_o. Therefore for $s > \omega_o$;

$$\alpha(s) \approx 1 + \omega_c^2/2s^2,$$

and

$$F(+0,s) \approx E_o\omega/s^2.$$

Then

$$\lim_{t\approx 0} f(z,t) = \lim_{s\approx\infty} \frac{E_o\omega}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty}\left[e^{s(t-t_p)}\, e^{-\omega_c^2 t_p/2s}\right]/s^2\, ds$$

From a table of transforms (CHU 1, No. 80), we then have

$$\lim_{t\approx 0} f(z,t) = \lim_{s\approx\infty} E_o\left(\frac{2(t-t_p)}{\omega_c^2 t_p}\right)^{1/2}\left[J_1\sqrt{2\omega_c^2 t_p(t-t_p)}\right] \qquad (A.20)$$

The series obtained by expanding this solution and transforming the limiting series obtained from (A.14a) agree. The initial part of the transient starting at time $t = t_p = z/c$ and represented by (A.20) is plotted in Fig. A. 2.

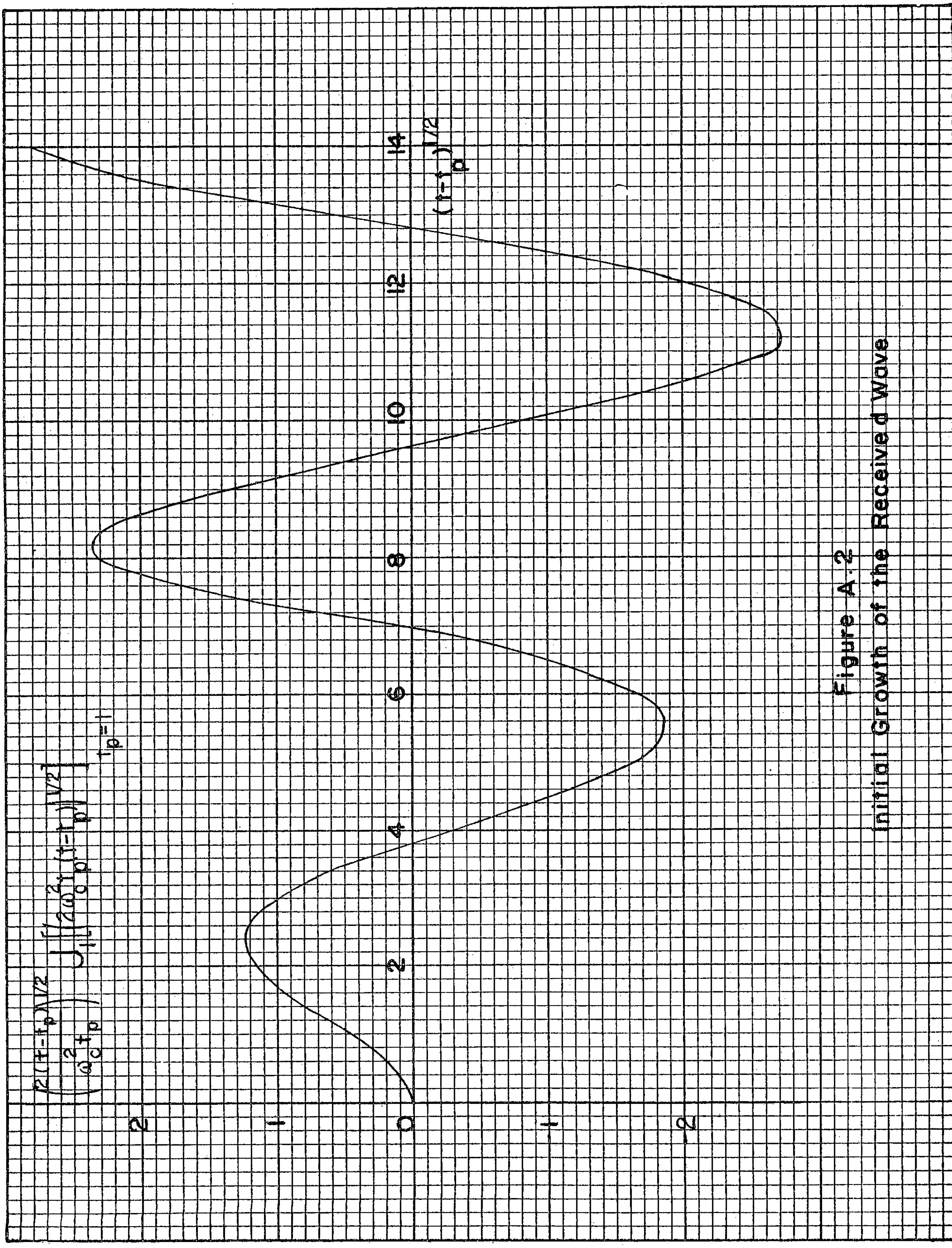

Figure A.2
Initial Growth of the Received Wave