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Correlation Receivers Minimizing Intersymbol Interference

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ABSTRACT

The problem considered here is that of binary pulse communication operating in a white noise environment wherein interfering signal pulses are generated by any of the following causes:

(1) A signaling rate larger than channel bandwidths, causing pulse spill over into succeeding bauds.

(2) Co-channel interference or cross talk on carrier systems.

(3) Signal echoes or reflections due to antenna location or mismatch of high frequency components.

The optimum correlation receiver is found to be a linear combination of the desired signal pulse plus the interfering pulse. In severe cases significant improvement can be had over a correlation receiver using only the desired signal pulse.

For the case (1) above, prior receiver decisions may be used to improve the design. In particular, the immediately preceding overall receiver decision is used to select one of two parallel component correlation operations whose designs are based on a priori knowledge of the preceding transmission. It can then be shown that this is equivalent to a single correlation operation with dual decision levels, wherein the preceding receiver output controls the selection of the decision level to be utilized next. The resulting performance of this type of receiver is superior to the correlator composed of the linear sum of signal plus interference pulses.

A similar investigation was performed for the case of Rayleigh fading on the interfering pulse. Analytical equations are established for determining the necessary solutions; however, numerical complexity precludes definitive results at this time.

CHAPTER I

INTRODUCTION

In recent years considerable engineering interest has been devoted to the use of digital data links to meet the ever increasing needs for high speed efficient communication. With suitable terminal conversion equipment, the digital communication system can convey both discrete and analogue types of information. Thus, the digital data link may be used for machine (digital) to machine communication or for the transmission of continuous signals, such as voice, via sampling, or any combination of the two (e.g., man-machine remote control). Common types of digital systems in practical use today are amplitude keying, phase shift keying, and frequency shift keying. The flexibility of digital communication design and application coupled with the growing computer technology would seem to justify the continuing interest in digital data links for modern communications.

We may visualize a communications system (Figure 1) as being composed of an information source (analogue or digital) connected to an encoder which feeds the communication channel, followed by appropriate decoding equipment which delivers the transmitted information to the user or "information sink". The communication channel is characterized by the physical medium through which we wish to transmit our message (e.g., wire, water, atmosphere, "space") terminated at each end by appropriate equipment (transmitters and receivers) designed to transmit

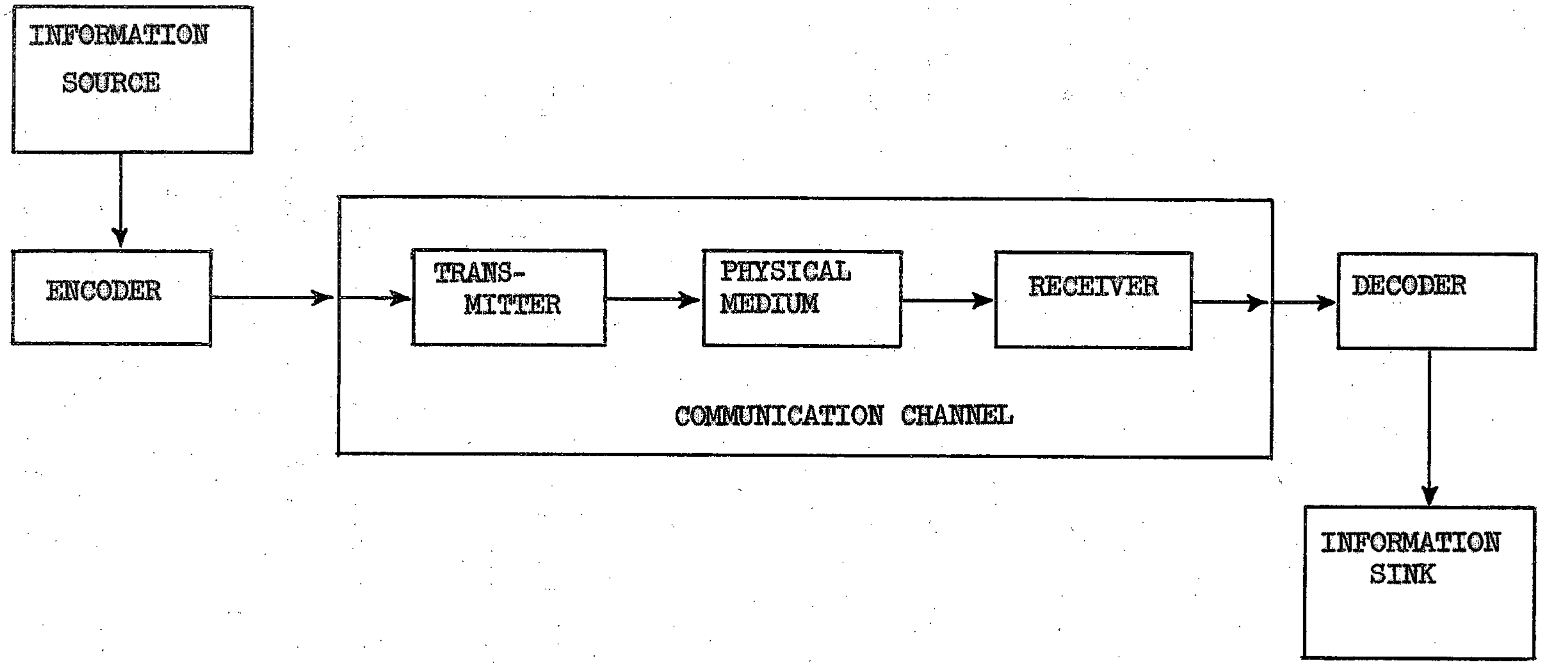


FIGURE 1 A COMMUNICATION MODEL

and receive signals pertinent to the physical medium. The purpose of the encoder is to convert the information source output to a form usable by the transmitter; similarly, the object of the decoder is to convert the receiver output to a form meaningful to the "sink". Thus the term digital data link refers to the communication channel just mentioned. The types of information sources with which a digital data link is used then determines the encoding and decoding equipment. This dissertation proposes to theoretically investigate the formal structure of the receiver in the communication channel with the following assumptions.

(1) The encoder presents to the transmitter every T seconds one of two possible symbols. The a priori probability of achieving each of these symbols is p and 1-p, respectively; and at each T instant the symbol realized is statistically independent of all other symbols. That is to say the encoder forms a sequence Z_{nT}; $n = \ldots -1, 0, 1, 2, \ldots$ of independent binary random variables where $Z_{nT} = 2$ (say) with probability p and $Z_{nT} = 1$ with probability (1-p).

(2) The transmitter has stored two possible waveforms, $a_2(t)$ and $a_1(t)$; $0 \leq t \leq T$. Both a_2 and a_1 have finite energy (square integrable) and are zero outside the interval $[0,T]$. At time nT the signal $a_2(t-nT)$ or $a_1(t-nT)$ is transmitted over the physical medium by the transmitter, the choice of a_2 or a_1 being dictated by the value of Z_{nT}.

(3) The effects of the physical medium may be abstracted to a mathematical model where the receiver is presented with data $v(t)$, which may be partitioned into a sum of signals $s_{\substack{1\\2}}(t-nT)$, $s_{\substack{1\\2}}(t-(n+1)T), \ldots,$ and noise $n(t)$. That is to say,

if $a_k(t-nT)$ $(k = 1,2)$ is transmitted in the absence of noise $n(t)$, $s_k(t-nT)$ is presented to the receiver where

$$s_k(t-nT) = 0 \qquad t < nT$$

$$\int_0^\infty s_k^2(t)dt = E_{s_k} < \infty \qquad k = 1,2$$

We note that $s_k(t-nT)$ is not necessarily zero for $t > (n+1)T$, i.e., the medium may "stretch out" or give spurious reflections of $a_k(t)$. Furthermore, we allow $s_k(t-nT)$ to be probabilisticaly derived from $a_k(t-nT)$ but require complete a priori knowledge of the necessary probability distributions and assume that they are stationary (i.e., independent of nT). As usual, $n(t)$ is assumed to be stationary white gaussion noise (WGN) of spectral intensity N_0 watts per c.p.s. (double sided).

(4) The receiver will announce at time $(n+1)T$ after examining $v(t)$ over the interval $nT \leq t \leq (n+1)T$, that Z_{nT} was either 1 or 2. Thus we are assuming perfect synchronization on nT intervals between transmitter and receiver. Since there is noise and pulse distortion present, the receiver can certainly be incorrect in its estimate of Z_{nT}. The object of our receiver design is to minimize the average error rate P_E at the receiver output, or equivalently maximize the average rate of correct reception $P_C = 1 - P_E$.

Note that as a consequence of the stationarity (assumed) P_C is independent of nT.

In terms of Statistical Hypothesis Testing the above problem is solved by a Bayes' receiver. A mathematical functional, "Likelihood Ratio", is established on the observed data, $v(t)$, from consideration of the a priori probability distributions. This functional dictates a (electronic) network which operates on any particular realization of observed data $v(t)$, producing a numerical value V_{nT} at time $(n+1)T$. This value is then compared to a stored reference level K (Bayes decision level). If V_{nT} is greater than or equal to K, the receiver announces that Z_{nT} was 2, otherwise $(V_{nT} < K)$ $Z_{nT} = 1$ is announced. It has been shown that this receiver provides an absolutely minimal error rate,[1] i.e., no other receiver structure can do better. The Bayes' receiver has achieved a great deal of engineering success in certain applications. Namely, if there is no signal pulse overlap at the receiver and $s_k(t)$, $k = 1,2$, is functionally perfectly known, then the Bayes' receiver reduces to a correlation or matched filter type of receiver. If again there is no signal pulse overlap, but only the signal envelopes and carrier frequency are known perfectly (assuming carrier phase to be uniformly distributed), then the Bayes' receiver reduces to the well known narrow band matched filter followed by a linear envelope detector. In both of the above cases the receiver performance (receiver error rate) can be predicated. For the matched filter the error rate is given by a sum of Normal Distribution functions, whereas for the envelope detector the error rate is given in terms of Marcum's "Q functions" (or "Offset circle probability distribution functions"). In general, however, predicting the performance of a Bayes' receiver is computationally intractable. Thus from an engineering point of view, the justification of fabricating such a device, is questionable.

Rather than seek the truly best receiver (Bayes'), the philosophy of this work will be to arbitrarily restrict ourselves to seeking the best receiver within a specified class of receivers (e.g., correlation, linear envelope, square law, etc.) utilizing min-max calculus. By a class of receivers we mean that the basic physical structure of each receiver in the class is the same. The receiver "class" then dictates the functional form of the receiver error rate P_E and each particular receiver in the class assigns values to the pertinent parameters of P_E. For example, under the assumptions (1) - (4) and assuming perfect knowledge of the form of $s_k(t)$ and if we choose for our receiver class the set of correlation receivers, we then have that P_E is functionally the finite sum of weighted Normal Distribution Functions whose arguments depend on the decision level K and correlating operation ($\int_0^T dt\, h(t):$) chosen. The goal is then to simultaneously choose K and h(t) so as to minimize P_E (or maximize $P_C = 1 - P_E$).

To illustrate this method in a more definitive manner we will consider a well known example.

Problem Statement

Assumptions (1) - (4) are valid. $s_k(t)$, $k = 1,2$ is zero for $t > T$, $t < 0$ and functionally perfectly well known to the receiver, i.e., the receiver may store "copies" of $s_k(t)$, $k = 1,2$.

Bayes' Solution

The best possible receiver for this problem is one which computes the number

$$V = \int_0^T (s_2(t) - s_1(t))v(t)dt$$

and compares V to the decision level

$$K = \frac{E_{s_2} - E_{s_1}}{2} + N_0 \ln \frac{p}{1 - p}$$

$$E_{s_k} = \int_0^T s_k^2(t)dt \qquad k = 1,2. \tag{1.2}$$

where 2 is announced if $V \geq K$, otherwise 1 is announced.

Since $v(t) = s_k(t) + n(t)$, and $n(t)$ is Gaussian, V (a linear operation on $v(t)$) is a Gaussian random variable. The mean and variance of V are conditioned by the prior event Z_{nT}, i.e., whether $s_1(t)$ or $s_2(t)$ was actually sent. However, the variance in either case is the same, namely

$$\sigma^2 = \left[VAR(V)\right]^2 = N_0 \int_0^T \left[s_2(t) - s_1(t)\right]^2 dt \tag{1.3}$$

Thus P_C is given by

$$P_C = 1 - P_E = p \Pr(V \geqq K/s_2) + (1 - p) \Pr(V \geqq K/s_1) \tag{1.4}$$

where

$$\Pr(V \geqq K/s_2) = \int_{\frac{K-m_2}{\sigma}}^{\infty} \exp - \frac{x^2}{2} \frac{dx}{\sqrt{2\pi}}$$

$$\Pr(V \geqq K/s_1) = \int_{-\infty}^{\frac{K-m_1}{\sigma}} \exp - \frac{x^2}{2} \frac{dx}{\sqrt{2\pi}} \tag{1.5}$$

where

$$m_k = \int_0^T s_k(t)\left[s_2(t) - s_1(t)\right] dt \qquad k = 1,2$$

$$\sigma^2 = N_o \int_0^T \left[s_2(t) - s_1(t)\right]^2 dt$$

$$K = \frac{E_{s_2} - E_{s_1}}{2} + N_o \ln \frac{p}{1 - p}$$

<u>Class Solution</u>

In this method we must select a class of receivers from which we wish to select an optimum. Choice of this class is up to the designer and so appealing to intuition and engineering artistry (besides knowing what the answers should be) we select the class of correlation receivers. Thus we form the number V (functional) from the observed data v(t) according to the prescription

$$V = V(h) = \int_0^T h(t)v(t)dt \tag{1.7}$$

and compare it to a decision level K. For each K and h(t) the correct reception rate, P_C, can be computed as

$$P_C(K;h) = p\,\Pr(V \geq K/s_2) + (1 - p)\,\Pr(V < K/s_1) \tag{1.8}$$

Once again, since V is formed by a linear operation on Gaussian data v(t), V is a Gaussian random variable, conditioned on the transmitted symbol, and the above conditional probabilities can be computed in a fashion similar to the previous method of solution. Namely,

$$P_C(K;h) = p \int_{\frac{K - \int_0^T h(t)s_2(t)dt}{\sqrt{N_o E_h}}}^{\infty} \left[\exp - x^2/2\right] dx/\sqrt{2\pi} + (1-p) \int_{-\infty}^{\frac{K - \int_0^T h(t)s_1(t)dt}{\sqrt{N_o E_h}}} \left[\exp - x^2/2\right] dx/\sqrt{2\pi} \qquad (1.9)$$

$$E_h = \int_0^T h^2(t)dt$$

Reflecting on what we have accomplished so far we see that by choosing the class of correlation receivers we immediately deduce that P_C $(= 1 - P_E)$ is a weighted sum of two normal distribution functions whose arguments depend on the quantities

$$K, \; E_h, \text{ and } \int_0^T h(t)s_k(t)dt \qquad k = 1,2.$$

We may now seek to maximize P_C through our choice of K and h(t). The details of the optimal solution for K and h(t) are carried out in Chapter II, Section 2.4. It is shown there that h(t) must have the form

$$h(t) = c_2 s_2(t) + c_1 s_1(t) \;; \quad c_2, \; c_1 \text{ constants}$$

Thus, P_C carries over to a function on three variables

$$P_C = P_C(K, c_1, c_2) \qquad (1.10)$$

and we seek values of K, c_1, c_2 which maximize P_C subject to the constraint

$$E_h = \int_0^T \left[c_2 s_2(t) + c_1 s_1(t)\right]^2 dt = E \tag{1.11}$$

Maximizing (1.10) subject to (1.11) yields the result

$$h(t) = s_2(t) - s_1(t)$$

$$V = \int_0^T \left[s_2(t) - s_1(t)\right] v(t)dt \tag{1.12}$$

$$K = \frac{E_{s_1} - E_{s_2}}{2} + N_o \ln \frac{p}{1 - p}$$

which is identical to the Bayes' receiver. Certainly a receiver designed in this fashion cannot be better than the Bayes' receiver. If the class of receivers chosen by the designer contains the Bayes' receiver, then it is reasonable to expect that this method will indeed produce the Bayes' receiver. However, this would be an extremely fortuitous circumstance and not very likely. The advantage offered by this method is a reasonably synthesizable receiver of predictable performance which is a direct consequence of pre-choosing a receiver class.

Chapter II establishes the mathematical frame-work necessary for dealing with the correlation class of receivers. In Chapter III the simplest type of intersymbol interference is considered. Here the form of the interference is assumed perfectly known. The correlation receiver utilizing no prior decisions (memoryless) is found for this situation and its performance derived. For comparative purposes, the performance of the correlation receiver just discussed is also computed. Chapter IV deals with the design of that receiver which assumes its previous decision to be absolutely correct in the environment of Chapter III.

Chapters V and VI are extensions of the efforts of Chapters III and IV, respectively, to the case of fading on the interfering signals. Chapter VII closes with an overall comparison of the results of the previous chapters together with suggestions for future work (e.g., envelope detectors and diversity systems).

Throughout this investigation, the pertinent Bayes' receivers are presented. The elements of this theory are not discussed here. For an introduction to this subject, the reader's attention is drawn to the texts of Helmstrom,[2] and, Davenport and Root.[3] The threshold behavior of the likelihood function is discussed by Middleton.[4]

CHAPTER II

GENERAL EQUATION FOR THE CORRELATION RECEIVER

In this chapter we derive in mathematical terms the general equations of the optimal correlation receiver. That is to say, our class of receivers is the one which correlates the received data $v(t)$ with a stored reference $h(t)$ and compares the result with a decision level K. Thus any particular pair $(h(t),K)$ represents a particular receiver within this class. For a particular channel model operating with the class of correlation receivers, there is a probability law, P_C, governing the performance (probability of correct reception) of any particular correlation receiver. In other words, for each pair $(h(t),K)$ there is a number $P_C(K;h(t))$ between zero and one which represents the average rate of correct reception for that particular choice of $(h(t),K)$. Furthermore, since we are dealing with correlation receivers it is reasonable to expect that $h(t)$ enters into P in the form of values $J_1,\ldots,J_n$ where

$$J_i = \int_0^T h(t)r_i(t)dt \qquad i = 1,\ldots,n.$$

and the $r_i(t)$ are n linearly independent given functions, related to the received signal alphabet.

Also, P_C may depend on the quantity

$$E_h = \int_0^T h^2(t)dt$$

However, we will assume (2.1d) that P_C is independent of receiver gain which implies invariance to the choice of a particular E_h. We may then eliminate this dependency by constraining E_h to be fixed. That is, $E_h = E$ for all h(t). Thus we seek to maximize

$$P_C(K;J_1,\ldots,J_n); \quad J_i = \int_0^T h(t)r_i(t)dt \qquad i = 1,\ldots,n$$

subject to the constraint

$$\int_0^T h^2(t)dt = E$$

by optimal choice of (h(t),K).

The general physical properties of communication media allow us the following mathematical assumptions.

2.1 Mathematical Assumptions

(a) We deal exclusively with time functions, f(t), defined on the interval, $0 \leq t \leq T$, which are real and have finite energy. i.e.,

$$\int_0^T f^2(t)dt = E_f < \infty$$

(b) P_C is a real function of n + 1 variables

$$P_C = P(K;J_1,\ldots,J_n)$$

with continuous first and second derivatives in all of its variables.

(c) The J_i are linear functionals of $h(t)$ generated by a given set of n linearly independent functions $r_i(t)$.

$$J_i = \int_0^T h(t)r_i(t)dt \qquad i = 1,\ldots,n$$

(d) P_C is invariant to receiver gain; i.e., for $\mu > 0$

$$P(K;J_1,\ldots,J_n) = P(\mu K;\mu J_1,\ldots,\mu J_n)$$

Note that P_C implicitly depends on E_h which will be held fixed.

From assumptions (2.1c and d) we may make two conclusions. Firstly, the optimal $h(t)$ is a linear combination of the $r_i(t)$, and secondly, since $h(t)$ is arbitrary to a multiplicative constant, we may eventually adjust the value of E_h for our convenience so as to normalize $h(t)$ in some sense.

We justify the conclusion that the optimal $h(t)$ is a linear combination of the $r_i(t)$.

$$h(t) = \sum_{i=1}^{n} c_i r_i(t) \qquad c_i \text{ constants} \tag{2.1.1}$$

Assumption 2.1(a) establishes that we are dealing with functions which are members of a real Inner Product Space[5] (RL_2) where the inner product is defined as the "correlation" between any two member functions. Namely,

$$\int_0^T f(t)g(t)dt$$

In particular, $h(t)$ is a member of RL_2 so that the J_i $i = 1,\ldots,n$ represent the components of $h(t)$ along the linearly independent $r_i(t)$.

Since P_C depends only on the J_i, consequently P_C depends only on the component of h(t) in the subspace spanned by the $r_i(t)$. Thus it suffices to choose h(t) as a linear combination of the $r_i(t)$. Let

$$\gamma_{ij} = \gamma_{ji} = \int_0^T r_i(t) r_j(t) dt \tag{2.1.2}$$

so that

$$J_i = \sum_{j=1}^{n} \gamma_{ij} c_j \qquad i = 1, \ldots, n$$

$$E_h = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} c_i c_j \tag{2.1.3}$$

We may now express P_C as a function on K and the "variables" $c_1, \ldots, c_n$,

$$P_C = P(K; c_1, \ldots, c_n)$$

and we seek to maximize P_C by choice of K, $c_1, \ldots, c_n$ subject to the side condition

$$E_h = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} c_i c_j = E \tag{2.1.4}$$

Note that since the J_i are linear combinations of the c_i, assumption 2.1(b) carries over to $P_C(K; c_1, \ldots, c_n)$.

2.2 Necessary Conditions

We wish here to establish the necessary conditions for maximizing

$$P_C(K; c_1, \ldots, c_n) \tag{2.2.1}$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} c_i c_j = E > 0 \qquad (2.2.2)$$

Using the usual technique of the Lagrange multipliers, form the function

$$I(K;c_1,\ldots,c_n) = P_C(K,c_1,\ldots,c_n) + \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} c_i c_j \qquad (2.2.3)$$

where α is the Lagrange multiplier. Then, as usual, the necessary conditions on α, $K, c_1,\ldots,c_n$ for maximizing P_C are that they satisfy

$$\frac{\partial I}{\partial K} = \frac{\partial I}{\partial c_1} = \ldots\ldots = \frac{\partial I}{\partial c_n} = 0 \qquad (2.2.4)$$

plus equation (2.2.2).

For the remainder of this work, the following notation conventions will be used.

(i) $K = K^*$, $c_i = c_i^*$ $\quad i = 1,\ldots,n$ represent the optimal values of K and c_i satisfying (2.2.4) and (2.2.2).

(ii) $P_C^* = P_C\,(K^*,c_1^*,\ldots,c_n^*)$

$$P_K^* = \left.\frac{\partial P_C}{\partial K}\right|_E \qquad \text{evaluated at } K = K^*,\ c_i = c_i^*,\ E \text{ fixed}$$

$$P_{c_i}^* = \left.\frac{\partial P_C}{\partial c_i}\right|_E \quad i = 1,\ldots,n \qquad \text{evaluated at } K = K^*,\ c_i = c_i^*,\ E \text{ fixed}$$

In (2.2.3) and (2.2.2) E is considered an independent quantity in terms of which we solve c_i^* and K^* after eliminating α. We should recall that the particular value of E chosen does appear in the expression for P_C. By the implicit function theorem we may solve for c_i^* and K^* as functions of E if the Jacobian of system (2.2.5) and (2.2.2) does not vanish for some $E > 0$. It is then possible to normalize h(t) so that any one non-zero c_i, say c_{i_o}, may be set to one.

2.3 The Correlation Receiver, An Example

Here we will develop the correlation receiver from the point of view of the previous discussion. With reference to the problem statement of Chapter I, pages 6, 7, 8, 9 and 10, we have that

$$I(K; c_1, c_2) = p \int_{\frac{K - c_1\gamma_{12} - c_2\gamma_{22}}{\sqrt{N_o E}}}^{\infty} \left[\exp - x^2/2\right] dx/\sqrt{2\pi} + (1-p) \int_{\infty}^{\frac{K - c_1\gamma_{11} - c_2\gamma_{12}}{\sqrt{N_o E}}} \left[\exp - x^2/2\right] dx/\sqrt{2\pi}$$

$$+ \alpha \sum_{i=1}^{2} \sum_{j=1}^{2} c_i c_j \gamma_{ij} \tag{2.3.1}$$

Denote

$$P_K = \frac{\partial P_C}{\partial K} \qquad P_{c_i} = \frac{\partial P_C}{\partial c_i} \qquad i = 1, 2. \tag{2.3.2}$$

Our three conditions on K, c_1, c_2 for maximizing P_C are then

$$\frac{\partial I}{\partial K} = \frac{\partial I}{\partial c_1} = \frac{\partial I}{\partial c_2} = 0 \tag{2.3.3}$$

or

$$\text{(a)} \qquad P_K = 0$$

$$\text{(b)} \qquad P_{c_1} + 2\alpha(\gamma_{11}c_1 + \gamma_{12}c_2) = 0 \qquad (2.3.4)$$

$$\text{(c)} \qquad P_{c_2} + 2\alpha(\gamma_{21}c_1 + \gamma_{22}c_2) = 0$$

Now

$$P_K = 0 = \frac{-p}{2\pi N_o E}\exp\left(\frac{-\,K-c_1\gamma_{12}-c_2\gamma_{22}{}^{2}}{2N_oE}\right) + \frac{(1-p)}{2\pi N_o E}\exp\left(\frac{-\,K-c_1\gamma_{11}-c_2\gamma_{21}{}^{2}}{2N_oE}\right) \qquad (2.3.5)$$

or equivalently

$$\left[K - c_1\gamma_{11} - c_2\gamma_{21}\right]^2 - \left[K - c_1\gamma_{12} - c_2\gamma_{22}\right]^2 = 2N_oE \ln\frac{p}{1-p} \qquad (2.3.6)$$

Evaluating P_{c_1} and P_{c_2}, we have

$$P_{c_1} = \frac{p\gamma_{12}}{\sqrt{2\pi N_oE}}\exp\left(\frac{-\left[K-c_1\gamma_{12}-c_2\gamma_{22}\right]^2}{2N_oE}\right) - \frac{(1-p)\gamma_{11}}{\sqrt{2\pi N_oE}}\exp\left(\frac{-\left[K-c_1\gamma_{11}-c_2\gamma_{21}\right]^2}{2N_oE}\right)$$

$$(2.3.7)$$

$$P_{c_2} = \frac{p\gamma_{22}}{\sqrt{2\pi N_oE}}\exp\left(\frac{-\left[K-c_1\gamma_{12}-c_2\gamma_{22}\right]^2}{2N_oE}\right) - \frac{(1-p)\gamma_{21}}{\sqrt{2\pi N_oE}}\exp\left(\frac{-\left[K-c_1\gamma_{11}-c_2\gamma_{21}\right]^2}{2N_oE}\right)$$

Multiply (2.3.4b) by γ_{12} and (2.3.4c) by γ_{11} and subtract, which yields

$$2\alpha c_2(\gamma_{12}^2 - \gamma_{11}\gamma_{22}) = \frac{p(\gamma_{11}\gamma_{22} - \gamma_{12}^2)}{\sqrt{2\pi N_oE}}\exp\left(\frac{-\left[K-c_1\gamma_{12}-c_2\gamma_{22}\right]^2}{2N_oE}\right)$$

after substitution of equations (2.3.7) and noting that $\gamma_{12} = \gamma_{21}$.

Since $\gamma_{11}\gamma_{22} - \gamma_{12}^2 > 0$ (Schwarz's lemma) we have

$$c_2 = \frac{-p}{2\alpha\sqrt{2\pi N_o E}} \exp\left(\frac{-\left[K - c_1\gamma_{12} - c_2\gamma_{22}\right]^2}{2N_o E}\right) \qquad (2.3.8)$$

Using this value of c_2 in (2.2.4b) and equations (2.4.7), we have that

$$c_1 = \frac{(1-p)}{2\alpha\sqrt{2\pi N_o E}} \exp\left(\frac{-\left[K - c_1\gamma_{11} - c_2\gamma_{12}\right]^2}{2N_o E}\right)$$

But by (2.3.5)

$$c_1 + c_2 = 0$$

We now choose E such that $c_2 = +1$, (i.e., $E = \gamma_{22} + \gamma_{11} - 2\gamma_{12}$)

$$h(t) = s_2(t) - s_1(t)$$

We then have for (2.3.6)

$$K = \frac{(\gamma_{22} - \gamma_{12})^2 - (\gamma_{11} - \gamma_{12})^2}{2(\gamma_{11} + \gamma_{22} - 2\gamma_{12})} + N_o \ln \frac{p}{1-p}$$

so that

$$K = \frac{\gamma_{22} - \gamma_{11}}{2} + N_o \ln \frac{p}{1-p}$$

but

$$\gamma_{22} = E_{s_2} = \int_0^T s_2^2(t)dt$$

$$\gamma_{11} = E_{s_1} = \int_0^T s_1^2(t)dt$$

Thus our receiver is given by

$$V = \int_0^T \left[s_2(t) - s_1(t)\right] v(t)dt$$

$$K = \frac{E_{s_2} - E_{s_1}}{2} + N_o \ln \frac{p}{1-p}$$

which is identical to the Bayes' receiver.

2.4 Summary: A Geometric Interpretation

We have shown that the only $h(t)$ of interest to extremize P_C is of the form

$$h(t) = \sum_{i=1}^{n} c_i r_i(t)$$

so that each receiver may be specified by the n+1 tuple $(K, c_1, \ldots, c_n)$, rather than by $(K, h(t))$. Furthermore, for a fixed value of $E_h = E$ we see that the optimal $(c_1^*, \ldots, c_n^*)$ must be a point on an n dimensional ellipsoid $\mathcal{E}_n$ centered at the origin. That is to say, we may form a n dimensional orthogonal coordinate system of the $c_1, \ldots, c_n$ and that $E_h = E$ is the equation of the ellipsoid $\mathcal{E}_n$ on which the optimal $(c_1^*, \ldots, c_n^*)$ must lie. We may adjoin to this coordinate system one more orthogonal axis, corresponding to K values, so that we have an n+1 dimensional space $\mathcal{S}$ with points given by $(K, c_1, \ldots, c_n)$. The receiver $(K, h(t))$ is then equivalent to a point in this space. Since $h(t) = 0$ is trivial, the origin of $\mathcal{S}$ is trivial as are all points lying solely on the K axis. On the other hand, note that points $(0, c_1, \ldots, c_n)$ are not trivial.

Now by assumption 2.1(d), if $(K, c_1, \ldots, c_n)$ is a non-trivial point of $\mathcal{S}$ then P is constant along the punctured line L: $(\mu K, \mu c_1, \ldots, \mu c_n)$, $\mu \neq 0$. That is to say P_C is constant along any straight line (n+1 dimensional) which passes through the origin (but excluding the origin as a point on which P_C is defined) and is not identically the K axis. Consequently, if $(K^*, c_1^*, \ldots, c_n^*)$ is an extremum of P_C then P_C is optimal over the punctured line L^*, passing through $(K^*, c_1^*, \ldots, c_n^*)$ and the origin. Whence, the antipodal intersections of L^* with $\mathcal{E}_n$ define $(K^*, c_1^*, \ldots, c_n^*)$ and $(-K^*, -c_1^*, \ldots, -c_n^*)$. A word of caution is in order: if we multiply $(K, h(t))$ by a negative constant $(\mu < 0)$ then the set of decision inequalities associated with $(K, h(t))$ must be reversed.

Conversely, each value of $E > 0$ uniquely defines an $\mathcal{E}_n(E)$ and the collection of all such $\mathcal{E}_n(E)$ forms a family of concentric ellipsoids growing monotonically on E: the locus of optimal points on each ellipsoid of this family is the orthogonal projection of L^* into the $c_1, \ldots, c_n$ sub-space. Note that if $K^* = 0$, the L^* lies wholly in the $c_1, \ldots, c_n$ subspace.

We thus are led to the conclusion that only the direction cosines of L^* need be found. In an n+1 dimensional space the orientation of a line L is given by n of its direction cosines. Consequently P_C could, as well, have been defined functionally on n variables representing the direction cosines of L and maximized over these n "direction" variables in the usual way without the use of a subsidiary constraint. Although this concept is fundamentally simpler, it is felt that the formal solutions are facilitated with the use of the Lagrange multiplier.

CHAPTER III

THE OPTIMAL CORRELATION RECEIVER FOR INTERSYMBOL INTERFERENCE

We will now apply the theory developed in the previous chapter to a simple channel model in which intersymbol interference is inherent. In particular, numerical results will be presented for the binary symmetric cases of unipolar, bipolar and orthogonal types of signals.

3.1 Channel Model

Assuming (1) through (4) of Chapter I plus the following:

(5) The receiver has complete knowledge of the channel distortion characteristics, i.e., if $a_k(t)$, $k = 1,2$ is transmitted, then $\tilde{s}_k(t)$ is received in the absence of noise and is known in full detail.

(6) Denote for $k = 1,2$. (See Figure 2)

$$s_k(t) = \tilde{s}_k(t) \qquad 0 \leq t < T$$

$$s_{T_k}(t) = \tilde{s}_k(t+T) \qquad 0 \leq t < T$$

$$\tilde{s}_k(t) = 0 \qquad \text{all other } t$$

(7) $$\int_0^T s_k^2(t)dt > \int_0^T s_{T_k}^2(t)dt \qquad k = 1,2$$

and for $k = 1,2$, $s_k(t)$ and $s_{T_k}(t)$ are linearly independent.

(8) The receiver is not to utilize any previous decisions it has made (i.e., the receiver is memory-less) with respect to T intervals.

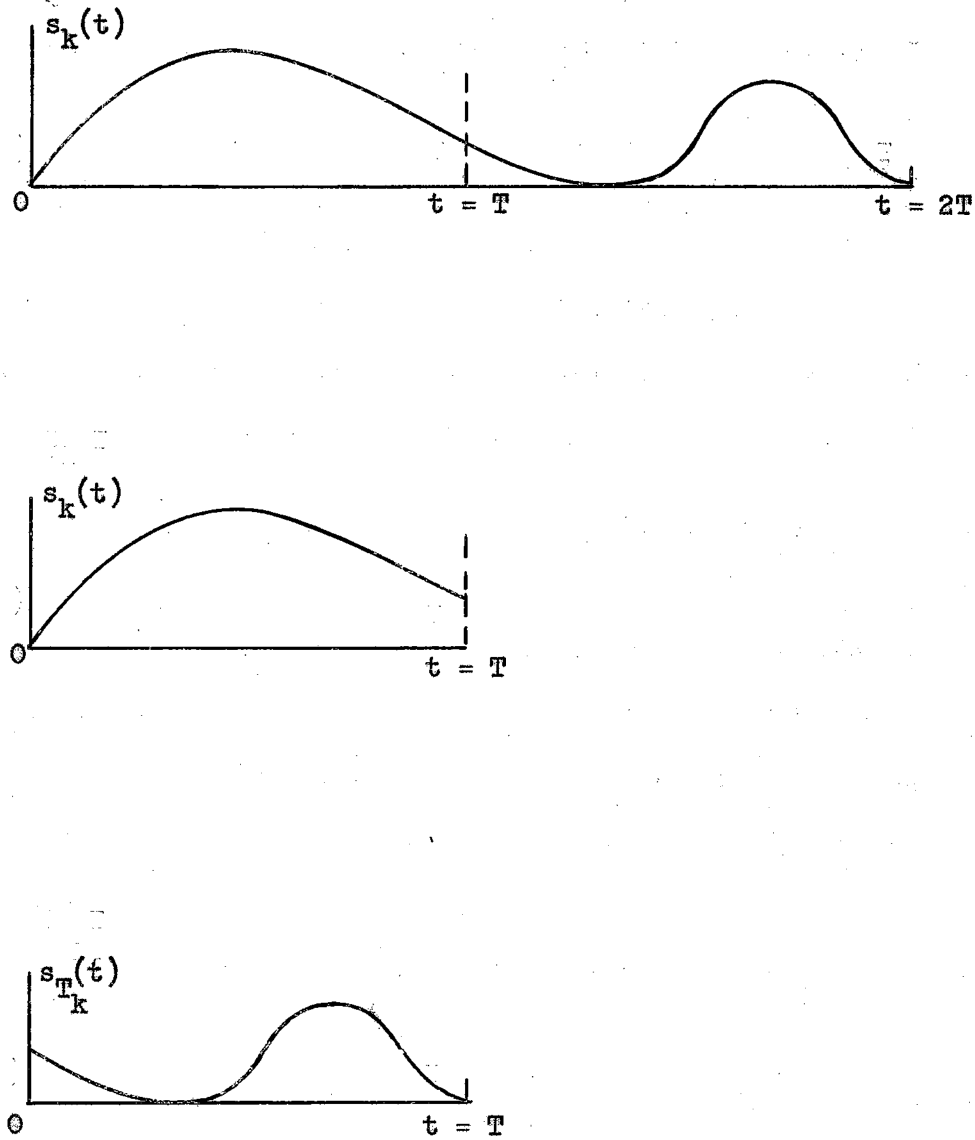


FIGURE 2 A SINGLE RECEIVED SIGNAL PULSE

A few comments are in order. (5) assumes that the channel pulse distortion is constant in time and known to the receiver. In (6), it is assumed that the pulse is "spilled over" only into the next T interval and will conflict with a transmission only on that interval. Note that the original signal pulse can be "stretched out" or spuriously "reflected" and in general may be any RL_2 function, s_{T_k} (s_T is chosen to mean signal tail). We limit ourselves to only interference on the immediately succeeding T interval because the computational details grow combinatorially whereas the basic problem remains the same. Note that (7) is reasonable in that if the opposite were true $\left(\int_0^T s_{T_k}^2 \, dt > \int_0^T s_k^2(t)dt \right)$, one would reverse the role of $s_{T_k}(t)$ and $s_k(t)$. Namely, one would delay one T interval and detect on s_{T_k} so that $s_k(t)$ would cause pre-interpulse interference as opposed to the post interpulse interference of s_{T_k} as assumed by (7).

Several physical situations come to mind which would lead to assumptions by (1) through (8). For example, a binary transmitter (e.g., high speed teletype) connected to a wire line whose effective bandwidth is slightly less than the signaling rate $(\frac{1}{T})$ of the transmitter; or receiving fixed spurious reflections or distorted echoes due to antenna location or mismatch. Another example which almost fits (5) through (8) is that of digital carrier equipment in which the channel frequency separation is not quite large enough (co-channel interference). In this context, s_{T_k} is not really generated serially (in time) but in a parallel sense. However, with assumption (8), this makes no difference in the following development; although such is not the case when (8) is relaxed (specifically Chapter IV).

3.2 The Bayes' Receiver

We now give the truly optimum receiver (Bayes') for operating in the environment of 3.1. Remembering that the only interference present is from the immediately preceding transmission which causes an $s_{T_k}(t)$ to appear additively to the $s_k(t)$ which is to be detected, and that the events are independent, we enumerate the four possible cases with their associated a priori probabilities of occurrence.

	Previous Transmission	Current Transmission	Received Signal in Absence of Noise	A Priori Probability
Case 1	$Z_{-1} = 1$	$Z_o = 1$	$s_2(t) + s_{T_1}(t)$	$(1-p)^2$
Case 2	$Z_{-1} = 1$	$Z_o = 2$	$s_2(t) + s_{T_1}(t)$	$p(1-p)$
Case 3	$Z_{-1} = 2$	$Z_o = 1$	$s_1(t) + s_{T_2}(t)$	$p(1-p)$
Case 4	$Z_{-1} = 2$	$Z_o = 2$	$s_2(t) + s_{T_2}(t)$	p^2

Table 1 Received Signal Combinations

From this table the Likelihood Ratio $\Lambda(v)$ on the received data in the presence of noise, $v(t)$ $(= s_k(t) + s_{T_1}(t) + n(t))$, can be formed.

$$\Lambda(v) = \frac{(1-p)A_1 \exp v\circ(s_2+s_{T_1})/N_o + pA_2 \exp v\circ(s_2+s_{T_2})/N_o}{(1-p)B_1 \exp v\circ(s_1+s_{T_1})/N_o + pB_2 \exp v\circ(s_1+s_{T_2})/N_o} \tag{3.2.1}$$

where

$$v\circ(s_k + s_{T_\ell}) = \int_0^T v(t)\left[s_k(t) + s_{T_\ell}(t)\right] dt \qquad k, \ell = 1,2$$

$$A_1 = \exp - \frac{E_{s_2 + s_{T_1}}}{2N_o} \qquad A_2 = \exp - \frac{E_{s_2 + s_{T_2}}}{2N_o}$$

$$B_1 = \exp - \frac{E_{s_1 + s_{T_1}}}{2N_o} \qquad B_2 = \exp - \frac{E_{s_1 + s_{T_2}}}{2N_o}$$

and $$E_{x+y} = \int_0^T (x(t) + y(t))^2 dt$$

$\Lambda(v)$ is then compared to the decision level

$$K = \frac{(1-p)}{p} \qquad (3.2.2)$$

and the Bayes' receiver is specified by (3.2.1) and (3.2.2) where the decision is 2 if $\Lambda(v) \geq K$, 1 otherwise. A block diagram for this receiver is given in Figure 3.

Now for each particular realization of $v(t)$ $(0 \leq t < T)$, $\Lambda(v)$ is simply a real number. Since $v(t)$ has random values we know that Λ is a random variable which theoretically has an associated probability density function, $p_\Lambda(\Lambda)$, derivable from the statistics of $v(t)$.

The word theoretically is used advisedly, for a glance at (3.2.1) shows that it is all but hopeless to deduce $p_\Lambda(\Lambda)$ from the statistics of $v(t)$ (actually in this case the WGN, $n(t)$) except in limiting cases.

In some cases it is possible to define a strict monotonic function on Λ; $f(\Lambda)$ (e.g., $\log \Lambda$) so that one may as well utilize the "number" $f(\Lambda(v))$ in comparison to the decision level $K_f = f(K)$ to

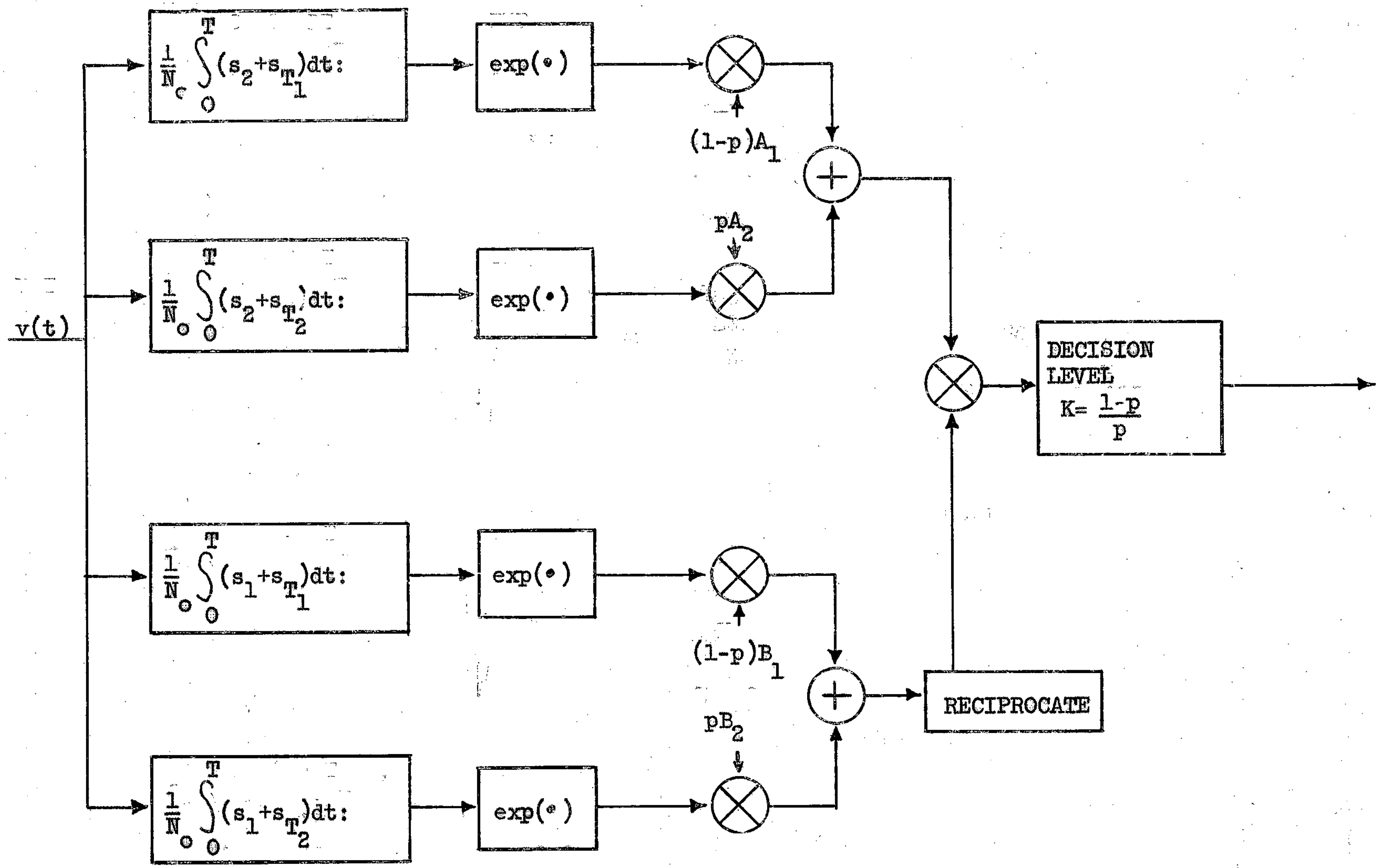


FIGURE 3 BAYES' RECEIVER

announce a reception. This receiver will have the same performance as the original (Λ, K) receiver since f is a one-one function. The purpose of this procedure is to obtain, by choice of f, a random variable, $\Gamma = f(\Lambda(v))$ with which a tractable probability density function $p_\Gamma(\Gamma)$ is associated. A classic example of this is given by setting $s_{T_1} = s_{T_2} = 0$ in (3.2.1) and taking the variable Γ to be

$$\Gamma = \ln \Lambda(v) = \ln \frac{\exp \frac{v \circ s_2}{N_o} \exp - \frac{E_{s_2}}{2N_o}}{\exp \frac{v \circ s_1}{N_o} \exp - \frac{E_{s_1}}{2N_o}} \tag{3.2.3}$$

$$= \frac{v \circ (s_2 - s_1)}{N_o} + \frac{E_{s_1} - E_{s_2}}{2N_o}$$

with decision level

$$K = \ln \frac{(1-p)}{p}$$

or comparing

$$\int_0^T v(t)\left[s_2(t) - s_1(t)\right] dt \quad \text{with} \quad \frac{E_{s_2} - E_{s_1}}{2} + N_o \ln \frac{1-p}{p} \tag{3.2.4}$$

which is recognized as the correlation receiver. That is to say, the correlation receiver is a sufficient statistic of $\Lambda(v)$ when $s_{T_1} = s_{T_2} = 0$.

At present, there does not exist a monotone function of (3.2.1), of a tractable nature, when either or $s_{T_1}(t)$, $s_{T_2}(t)$ are not zero. Furthermore, it seems unlikely that such exists. Thus we are faced with an ideal receiver about which we can make no statements pertaining to its expected performance. As seen from Figure 3, this Bayes' receiver is not particularly simple, so that an experimental approach is not justified on economic grounds. Consequently, we will design a best correlation receiver and examine its performance.

3.3 Formulation of the Correlation Class

With reference to Chapter II, we choose the class of correlation receivers $(h(t);K)$ amongst which we seek that correlation receiver which maximizes P_C. Referring to Table 1 and in a manner analogous to equations (1.1) through (1.5), it can be established that the probability of correct reception P_C for a receiver $(h(t);K)$ is

$$P_C = (1-p)^2 \int_{-\infty}^{\frac{K-h\circ(s_1+s_{T_1})}{\sqrt{N_oE_h}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} + p(1-p) \int_{-\infty}^{\frac{K-h\circ(s_1+s_{T_2})}{\sqrt{N_oE_h}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$+ p^2 \int_{\frac{K-h\circ(s_2+s_{T_2})}{\sqrt{N_oE_h}}}^{\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} + p(1-p) \int_{\frac{K-h\circ(s_2+s_{T_1})}{\sqrt{N_oE_h}}}^{\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \qquad (3.3.1)$$

where, as before

$$E_h = \int_0^T h^2(t)dt$$

$$h\circ(s_k+s_{T_\ell}) = \int_0^T h(t)\left[s_k(t) + s_{T_\ell}(t)\right]dt \qquad k, \ell = 1,2.$$

Note that if we multiply by μ, obtaining $(\mu K, \mu h(t))$, P_C is unchanged and satisfies assumption (d) of Chapter II. A further inspection of (3.3.1) shows that all the assumptions of 2.1 are valid. Defining four linear functionals corresponding to $h \circ (s_k + s_{T_\ell})$, $k, \ell = 1,2$ it follows that $h(t)$ has the form

$$h(t) = c_1\left[s_1(t)+s_{T_1}(t)\right] + c_2\left[s_1(t)+s_{T_2}(t)\right] + c_3\left[s_2(t)+s_{T_2}(t)\right] + c_4\left[s_2(t)+s_{T_1}(t)\right]$$

or taking linear combinations of these c's; obtain new c's

$$h(t) = c_1 s_1(t) + c_{T_1} s_{T_1}(t) + c_2 s_2(t) + c_{T_2} s_{T_2}(t) \tag{3.3.2}$$

As we are especially interested in the case where $p = 1/2$ and the s_k, s_{T_k} form unipolar, bipolar, or orthogonal signals, we will forego developing the general necessary conditions on (3.3.1) in favor of investigating each of these special cases separately. Our motive for specializing at this point is one of convenience. A general solution to (3.3.1) is not simple. It involves solving five coupled equations with a "$\left[\gamma_{ij}\right]$" matrix of sixteen elements. The special cases just mentioned, which are of most practical interest, considerably simplify (3.3.1) if derived separately.

3.4 Unipolar Case (On-Off Signal), $p = 1/2$

The unipolar terminology is taken to mean that the signal corresponding to $Z_{nT} = 1$ is identically zero, or that

$$\begin{aligned} s_1(t) &= s_{T_1}(t) = 0 \\ s_2(t) &= s(t) \\ s_{T_2}(t) &= s_T(t) \end{aligned} \tag{3.4.1}$$

and

$$h(t) = cs(t) + c_T s_T(t); \; E_h = E$$

Denote

$$E_s = \int_0^T s^2(t)dt$$

$$E_{s_T} = \int_0^T s_T^2(t)dt \qquad (3.4.2)$$

$$\rho = \int_0^T s(t)s_T(t)dt$$

Substituting (3.4.1) and (3.4.2) into (3.3.1) yields

$$P_C(K;c,c_T) = \frac{1}{4}\int_{-\infty}^{K/\sqrt{N_oE}} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} + \frac{1}{4}\int_{-\infty}^{\frac{K-(c\rho+c_TE_{s_T})}{\sqrt{N_oE}}} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$$

$$+\frac{1}{4}\int_{\frac{K-(c\,E_s+\rho\,+c_T\,E_{s_T}+\rho\,)}{\sqrt{N_oE}}}^{\infty} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} + \frac{1}{4}\int_{\frac{K-(cE_s-c_T\rho)}{\sqrt{N_oE}}}^{\infty} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} \qquad (3.4.3)$$

with

$$E_h = c^2E_s + 2cc_T\rho + c_T^2E_{s_T} = E \quad \text{(fixed)}$$

The necessary equations for c^*, c_T^*, K^* to maximize (3.4.3) in accordance to Chapter II are then found to be

$$\text{(a)}\quad P_K^* = 0$$

$$\text{(b)}\quad P_c^* + 2\alpha c^* E_s + 2\alpha c_T^* \rho = 0 \qquad (3.4.4)$$

$$\text{(c)}\quad P_{c_T}^* + 2\alpha c^* \rho + 2\alpha c_T^* E_{s_T} = 0$$

Define the function ψ to be

$$\psi(u) = \frac{e^{-u^2/2}}{4\sqrt{2\pi}} \qquad (3.4.5)$$

Then from (3.4.3) it follows that

$$\text{(a)}\quad P_K = \frac{\partial P_C}{\partial K} = \frac{1}{\sqrt{N_o E}}\left\{\psi\left(\frac{K}{\sqrt{N_o E}}\right) + \psi\left(\frac{K-h\circ s_T}{\sqrt{N_o E}}\right) - \psi\left(\frac{K-h\circ(s+s_T)}{\sqrt{N_o E}}\right) - \psi\left(\frac{K-h\circ s}{\sqrt{N_o E}}\right)\right\}$$

$$\text{(b)}\quad P_c = \frac{\partial P_C}{\partial K} = \frac{1}{\sqrt{N_o E}}\left\{-\rho\psi\left(\frac{K-h\circ s_T}{\sqrt{N_o E}}\right) + (E_s+\rho)\psi\left(\frac{K-h\circ(s+s_T)}{\sqrt{N_o E}}\right) + E_s\psi\left(\frac{K-h\circ s}{\sqrt{N_o E}}\right)\right\} \qquad (3.4.6)$$

$$\text{(c)}\quad P_{c_T} = \frac{\partial P_C}{\partial c_T} = \frac{1}{\sqrt{N_o E}}\left\{-E_{s_T}\psi\left(\frac{K-h\circ s_T}{\sqrt{N_o E}}\right) + (E_{s_T}+\rho)\psi\left(\frac{K-h\circ(s+s_T)}{\sqrt{N_o E}}\right) + \rho\psi\left(\frac{K-h\circ s}{\sqrt{N_o E}}\right)\right\}$$

Consider (3.4.4a); from (3.4.6a) we have

$$\psi\left(\frac{K^*}{\sqrt{N_o E}}\right) + \psi\left(\frac{K^*-h^*\circ s_T}{\sqrt{N_o E}}\right) = \psi\left(\frac{K^*-h^*\circ(s+s_T)}{\sqrt{N_o E}}\right) + \psi\left(\frac{K^*-h^*\circ s}{\sqrt{N_o E}}\right) \qquad (3.4.7)$$

where $h^* = c^* s(t) + c_T^* s_T(t)$

Now

$$K^* = \frac{1}{2} h^* \circ (s+s_T) = \frac{1}{2} c^*\left[E_s + \rho\right] + \frac{1}{2} c_T^*\left[E_{s_T} + \rho\right] \tag{3.4.8}$$

produces an identity on (3.4.7) as ψ is an even function. Moreover, this value of K^* is unique (ref. Appendix B). Thus (3.4.4a) is "decoupled" from (3.4.4b and c). We mention in passing that this happy event occurs only for $p = 1/2$.

Evaluating P_c^* and $P_{c_T}^*$, by using (3.4.8) and (3.4.6a and b) produces

$$\text{(a)}\quad P_c^* = \left(\frac{E_s - \rho}{\sqrt{N_o E}}\right)\psi\left(\frac{h^* \circ (s-s_T)}{2\sqrt{N_o E}}\right) + \left(\frac{E_s + \rho}{\sqrt{N_o E}}\right)\psi\left(\frac{h^* \circ (s+s_T)}{2\sqrt{N_o E}}\right)$$

$$\text{(b)}\quad P_{c_T}^* = \left(\frac{\rho - E_{s_T}}{\sqrt{N_o E}}\right)\psi\left(\frac{h^* \circ (s-s_T)}{2\sqrt{N_o E}}\right) + \left(\frac{\rho + E_{s_T}}{\sqrt{N_o E}}\right)\psi\left(\frac{h^* \circ (s+s_T)}{2\sqrt{N_o E}}\right) \tag{3.4.9}$$

We are now in a position to assert that αc^* is not zero.

From (3.4.4) we may algebraically obtain

$$-2\alpha c^* = \frac{E_s P_c^* - \rho P_{c_T}^*}{E_s E_{s_T} - \rho^2} \tag{3.4.10}$$

Introducing (3.4.9) into (3.4.10) produces

$$-2\alpha c^* = \frac{1}{\sqrt{N_o E}}\quad \psi\left(\frac{h^* \circ (s+s_T)}{2\sqrt{N_o E}}\right) + \psi\left(\frac{h^* \circ (s-s_T)}{2\sqrt{N_o E}}\right) \tag{3.4.11}$$

Since

$$\psi(u) = \frac{e^{-u^2/2}}{4\sqrt{2\pi}} > 0 \tag{3.4.12}$$

neither α nor c^* are zero; whence we may take $c^* = 1$. Then

$$2\alpha = -\frac{1}{\sqrt{N_o E}} \; \psi\left(\frac{h^* \circ (s+s_T)}{2\sqrt{N_o E}}\right) + \psi\left(\frac{h^* \circ (s-s_T)}{2\sqrt{N_o E}}\right) \qquad (3.4.13)$$

Solving for c_T^* in (3.4.4) produces

$$c_T^* = \frac{E_s P^*_{c_T} - \rho P^*_c}{2\alpha(\rho^2 - E_s E_{s_T})} \qquad (3.4.14)$$

Utilizing (3.4.9) and (3.4.13) gives c_T^* as the solution to

$$c_T^* = \left(1 - \frac{\psi_-^*}{\psi_+^*}\right) \Big/ \left(1 + \frac{\psi_-^*}{\psi_+^*}\right)$$

$$\psi_-^* = \psi\left(\frac{h^* \circ (s-s_T)}{2\sqrt{N_o E}}\right) \qquad (3.4.15)$$

$$\psi_+^* = \psi\left(\frac{h^* \circ (s+s_T)}{2\sqrt{N_o E}}\right)$$

where ψ is defined by (3.4.5) and

$$E = E_s + 2c_T^* \rho + (c_T^*)^2 E_{s_T}$$

Algebraic rearrangement of (3.4.15) requires that c_T^* satisfy

$$\frac{1 + c_T^*}{1 - c_T^*} = \exp - \frac{\rho}{2N_o} \exp - \frac{c_T^*(E_s E_{s_T} - \rho^2)}{2N_o(c_T^{*2} E_{s_T} + 2c_T^* \rho + E_s)} \qquad (3.4.16)$$

Let

$$\frac{1 + c_T^*}{1 - c_T^*} = u \; ; \qquad c_T^* = \frac{u - 1}{u + 1} \tag{3.4.17}$$

and seek that positive value of u which solves (see Appendix B)

$$\ln u + \frac{\rho}{2N_o} + \frac{\left(\frac{E_s E_{s_T} - \rho^2}{2N_o E_{s+s_T}}\right)(u^2 - 1)}{u^2 + 2\left(\frac{E_s - E_{s_T}}{E_{s+s_T}}\right)u + \frac{E_{s-s_T}}{E_{s+s_T}}} = 0 \tag{3.4.18}$$

We now normalize (3.4.18) in terms of

$$\frac{S}{N} = \frac{E_s}{2N_o} \quad \text{(Signal } s(t) \text{ energy/2 noise power per unit bandwidth)} \tag{3.4.19}$$

Since

$$E_{s_T} < E_s \qquad \text{and} \qquad |\rho|^2 < E_s E_{s_T}$$

define

$$\frac{E_{s_T}}{2N_o} = a\,\frac{S}{N} \qquad 0 \leq a < 1$$

$$\tag{3.4.20}$$

$$\frac{\rho}{2N_o} = b\sqrt{a}\,\frac{S}{N} \qquad -1 < b < 1$$

Then (3.4.18) reduces to

$$\ln(u) + b\sqrt{a}\,\frac{S}{N} + \frac{\frac{S}{N}\left(\frac{a\left[1-b^2\right]}{1+a+2b\sqrt{a}}\right)(u^2-1)}{u^2 + 2\left(\frac{1 - a}{1+a+2b\sqrt{a}}\right)u + \frac{1+a-2b\sqrt{a}}{1+a+2b\sqrt{a}}} = 0 \tag{3.4.21}$$

Thus we have the unique solution

$$h^*(t) = s(t) + c_T^* s_T(t)$$

$$K^* = \frac{1}{2} h^* \circ (s+s_T) = \frac{1}{2}\frac{S}{N}(1 + b\sqrt{a} + c_T^*(a + b\sqrt{a})) \tag{3.4.22}$$

$$c_T^* = \frac{(u - 1)}{u + 1}$$

where u satisfies (3.4.21)

The sufficiency of (3.4.22) is demonstrated in Appendix B. Note that K^* is just one half the filter output in the absence of noise when both $s(t)$ and $s_T(t)$ are present; and that this criterion is the same as that of the symmetric correlation receiver, $(p = 1/2, s_T = 0)$.

To evaluate $P_C(h^*,K^*)$, or more pertinently, $P_E^* = 1 - P_C(h^*,K^*)$, simply introduce (3.4.22) into (3.4.6) to obtain

$$P_E^* = \frac{1}{2}\int_{x_1^*}^{\infty} e^{-\xi^2/2}\frac{d\xi}{\sqrt{2\pi}} + \frac{1}{2}\int_{x_2^*}^{\infty} e^{-\xi^2/2}\frac{d\xi}{\sqrt{2\pi}} \tag{3.4.23}$$

$$x_1^* = \sqrt{\frac{1}{2}\frac{S}{N}}\;\frac{1 + c_T^* a + b\sqrt{a}(c_T^* + 1)}{\sqrt{1 + 2c_T^* b\sqrt{a} + c_T^{*2} a}}$$

$$x_2^* = \sqrt{\frac{1}{2}\frac{S}{N}}\;\frac{1 - c_T^* a + b\sqrt{a}(c_T^* - 1)}{\sqrt{1 + 2c_T^* b\sqrt{a} + c_T^{*2} a}}$$

In order to compare this result with a realized system we next derive the performance of a "standard" correlation receiver $(c_T = 0)$ in this environment. Since adjustment of the decision level is trivial physically, we will allow for optimum adjustment of K when $c_T = 0$. Referring to (3.4.8) it is seen that this value of K, K_o, is given by

$$K_o = \frac{1}{2} h_o \cdot (s+s_T) = \frac{1}{2} \frac{S}{N} (1 + b\sqrt{a}) 2N_o \tag{3.4.24}$$

where

$$h_o(t) = s(t).$$

With this value of K_o, $c_T = 0$, $c = 1$ (i.e., $h_o(t)$, K_o) we can compute from (3.4.3) that

$$P_E^o = \frac{1}{2} \int_{x_1^o}^{\infty} e^{-\xi^2/2} \frac{d\xi}{\sqrt{2\pi}} + \frac{1}{2} \int_{x_2^o}^{\infty} e^{-\xi^2/2} \frac{d\xi}{\sqrt{2\pi}} \tag{3.4.25}$$

$$x_1^o = \sqrt{\frac{1}{2} \frac{S}{N}} (1 + b \quad a)$$

$$x_2^o = \sqrt{\frac{1}{2} \frac{S}{N}} (1 - b \quad a)$$

which is exactly the results of (3.4.23) with c_T set to zero.

Performance curves with corresponding values of K^* and c_T^* as a function of $(\frac{S}{N})$, for various parameter values of a and b, are presented at the end of this chapter. The results were computed on an IBM 7090 computing facility. The numerical method used in solving (3.4.22) for u is explained in Appendix A.

3.5 Bipolar Case, (plus minus signals), p = 1/2

For the bipolar type of signals we require that

$$s_2(t) = - s_1(t) = s(t)$$

$$s_{T_2}(t) = - s_{T_1}(t) = s_T(t)$$

By a suitable stratagem we will reduce this case to that of the unipolar signals. Namely, if $s(t) + s_T(t)$ is always added on to the incoming data, $v(t)$, the receiver then faces a decision between the four signals.

$2s(t) + 2s_T(t)$

$2s(t)$

$2s_T(t)$

0

which is exactly the unipolar situation where the signals have been multiplied by a factor of two. Then, from the results of the previous section, the receiver is specified by

$$h'(t) = 2s(t) + 2c_T^* s_T(t)$$

$$K' = \frac{1}{2} h' \circ (2s + 2s_T) \tag{3.5.1}$$

where $\left(\frac{S}{N}\right)' = \frac{E_{2s}}{2N_o} = \frac{4E_s}{2N_o} = 4\left(\frac{S}{N}\right)$

and c_T^* solves (3.4.22) for $\left(\frac{S}{N}\right)' = 4\left(\frac{S}{N}\right)$

Now we know that the receiver gain may be scaled without affecting the performance, so we choose to divide (3.5.1) by two, obtaining

$$h^*(t) = s(t) + c_T^* s_T(t)$$

$$(K^*)' = h^* \circ (s+s_T) \tag{3.5.2}$$

and c_T^* solves (3.4.22) with $\left(\frac{S}{N}\right)$ replaced by $4\left(\frac{S}{N}\right)$. The performance P_E^*

of (3.5.2) is then given by (3.4.24) where $(\frac{S}{N})$ is replaced by $4(\frac{S}{N})$. Similarly for P_E^o, $(\frac{S}{N})$ is replaced by $4(\frac{S}{N})$ in 3.4.26). Our overall receiver is shown in Figure 4.

It is now possible to simplify the receiver of Figure 4. Consider any correlation receiver (K,h(t)) with incoming data v(t). Add to v(t) a prescribed function f(t) so that h(t) operates on v(t) + f(t). Then, no matter what signals are present in v(t), the output V of the correlator $(V = \int_0^T h(t)\left[v(t) + f(t)\right]dt)$ will always contain the constant term $\int_0^T h(t)f(t)dt$. It then follows that if f(t) is added to the data v(t) and $\int_0^T h(t)f(t)dt$ is added to the decision level K, we have equivalent receivers: where, if the performance of the receivers is identical, we call them equivalent. Whence, the receivers of Figure 5 are equivalent.

By the preceding argument, the receivers of Figure 4 can be reduced to the form of Figure 5 where K^* is given by

$$K^* = (K^*)' - h^* \circ (s+s_T) = 0 \qquad (3.5.3)$$

The optimum correlation receiver for symmetric bipolar signals is then reduced to

$$h^*(t) = s(t) + c_T^* s_T(t)$$

$$K^* = 0 \qquad (3.5.4)$$

Since (3.5.4) is equivalent to (3.5.2), P_E^* and P_E^o are given by (3.4.24) and (3.4.26), respectively, wherein $(\frac{S}{N})$ is replaced by $4(\frac{S}{N})$.

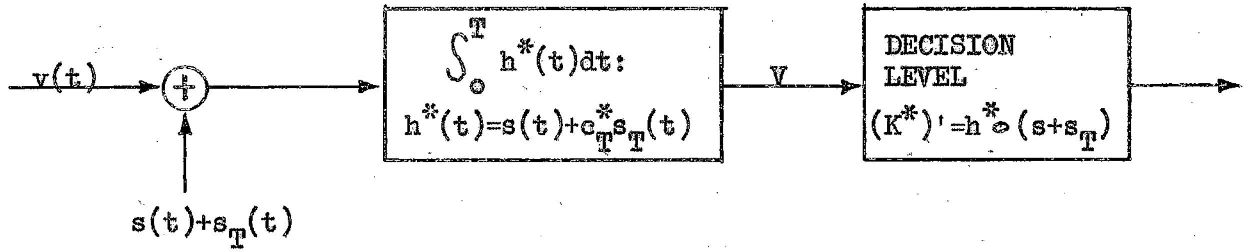


FIGURE 4 OPTIMUM CORRELATION RECEIVER FOR BIPOLAR SIGNALS

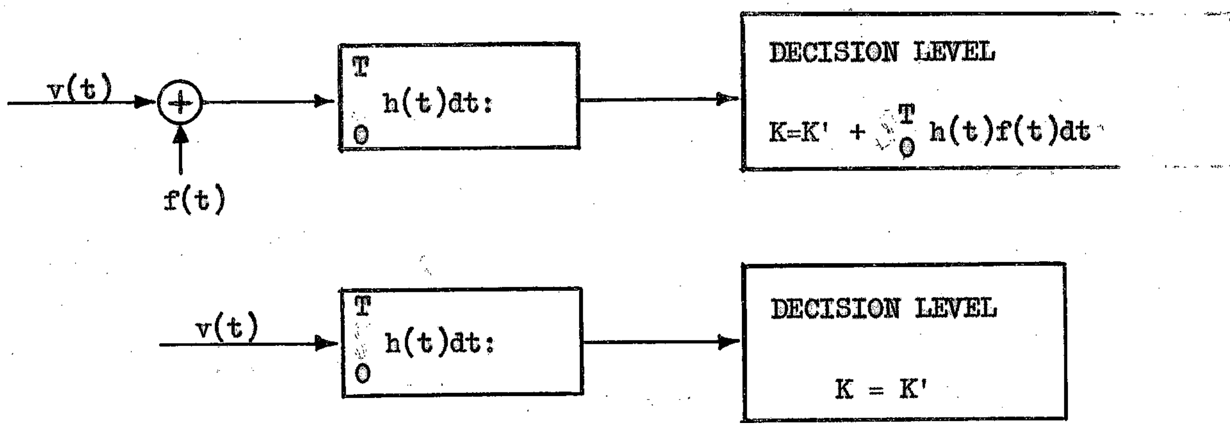


FIGURE 5 EQUIVALENT CORRELATION RECEIVERS

3.6 Orthogonal Case,(uncorrelated signals) p = 1/2

The orthogonal transmission system utilizes signals which have the following properties

$$
\begin{aligned}
&(a) \quad E_{s_1} = E_{s_2} = E_s \\
&(b) \quad E_{s_{T_1}} = E_{s_{T_2}} = E_{s_T} \\
&(c) \quad \int_0^T s_1(t)s_2(t)dt = 0 \\
&(d) \quad \int_0^T s_{T_1}(t)s_{T_2}(t)dt = 0 \\
&(e) \quad \int_0^T s_1(t)s_{T_2}(t)dt = \int_0^T s_2(t)s_{T_1}(t)dt = 0
\end{aligned}
\qquad (3.6.1)
$$

and

$$(f) \quad \int_0^T s_1(t)s_{T_1}(t)dt = \int_0^T s_2(t)s_{T_2}(t)dt = \rho$$

An example of signals which satisfy assumptions (3.6.1) (a) through (f) is the frequency shift keying (FSK) scheme.

To evaluate this case, essentially the same trick employed in the bipolar case will be used except that the orthogonal signals will be converted to bipolar signals by first subtracting

$$\frac{s_2(t) + s_1(t)}{2} + \frac{s_{T_2}(t) + s_{T_1}(t)}{2} \qquad (3.6.2)$$

from the input data v(t). The fours signals then facing the receiver are

$$\pm \frac{s_2(t) - s_1(t)}{2} \quad \pm \frac{s_{T_2}(t) - s_{T_1}(t)}{2}$$

From (3.5.4), the best correlation receiver is

$$h'(t) = \frac{s_2(t) - s_1(t)}{2} + c_T^* \frac{s_{T_2}(t) - s_{T_1}(t)}{2}$$

$$K' = 0$$

or scaling h by a factor of 2

$$h^*(t) = s_2(t) - s_1(t) + c_T^* \left[s_{T_2}(t) - s_{T_1}(t)\right]$$

$$(K^*)' = 0 \qquad (3.6.3)$$

where c_T^* solves (3.4.22) with $(\frac{S}{N})$ replaced by

$$(\frac{S}{N})' = \frac{4}{2N_o} \int_0^T (\frac{s_2(t) - s_1(t)}{2})^2 dt = 2\,\frac{S}{N}$$

Next the operation of subtracting (3.6.2) from the data may be replaced by an adjustment of the decision level. Analogous to (3.5.3), we have

$$K^* = 0 + \frac{1}{2} h^* \circ (s_2 + s_1 + s_{T_2} + s_{T_1}) \qquad (3.6.4)$$

and from the orthogonality and equi-energy properties of the signals, the integral of (3.6.4) evaluates to zero. The optimal correlation receiver is then prescribed by

$$h^*(t) = s_2(t) - s_1(t) + c_T^* \left[s_{T_2}(t) - s_{T_1}(t)\right]$$

$$K^* = 0 \qquad (3.6.5)$$

where c_T^* solves (3.4.22) with $(\frac{S}{N})$ replaced by $2(\frac{S}{N})$. Furthermore, P_E^* and P_E^o are given by (3.4.24) and (3.4.26), respectively, with $(\frac{S}{N})$ replaced by $2(\frac{S}{N})$. An equivalent physical realization of (3.6.5) is given in Figure 6.

Basically we need only one set of solutions for all three of the cases discussed. The particular case at hand is normalized by multiplying $(\frac{S}{N})$ by the appropriate doubling factor. The results are plotted for the unipolar case. Corresponding results for orthogonal and bipolar systems are obtained by multiplying $(\frac{S}{N})$ by two and four respectively, and setting K^* equal to zero.

3.7 General Observations

For the case of symmetric ($p = 1/2$) unipolar, orthogonal and bipolar signals, we seek that real root of equation (3.4.22) wherein the appropriate $\frac{S}{N}$ doubling factor is dictated by the type of above signals used. Define $G(b;u)$ as follows

$$G(b;u) = \ln(u) + b\sqrt{a}\,\frac{S}{N} + \frac{\frac{S}{N}\left[\frac{a(1-b^2)}{1+a+2b\sqrt{a}}\right](u^2-1)}{u^2 + 2\,\frac{1-a}{1+a+2b\sqrt{a}}\,u + \frac{(1+a-2b\sqrt{a})}{(1+a+2b\sqrt{a})}} \tag{3.7.1}$$

Then

$$G(-b;\frac{1}{u}) = \ln(\frac{1}{u}) - b\quad a\,\frac{S}{N} + \frac{\frac{S}{N}\;\frac{a(1-b^2)}{1+a-2b\quad a}\;(u^2-1)}{\frac{1}{u^2} + 2\,\frac{1-a}{1+a-2b\quad a}\;\frac{1}{u} + \frac{(1+a+2b\quad a)}{(1+a-2b\quad a)}} \tag{3.7.2}$$

Multiplying numerator and denominator of (3.7.2) by

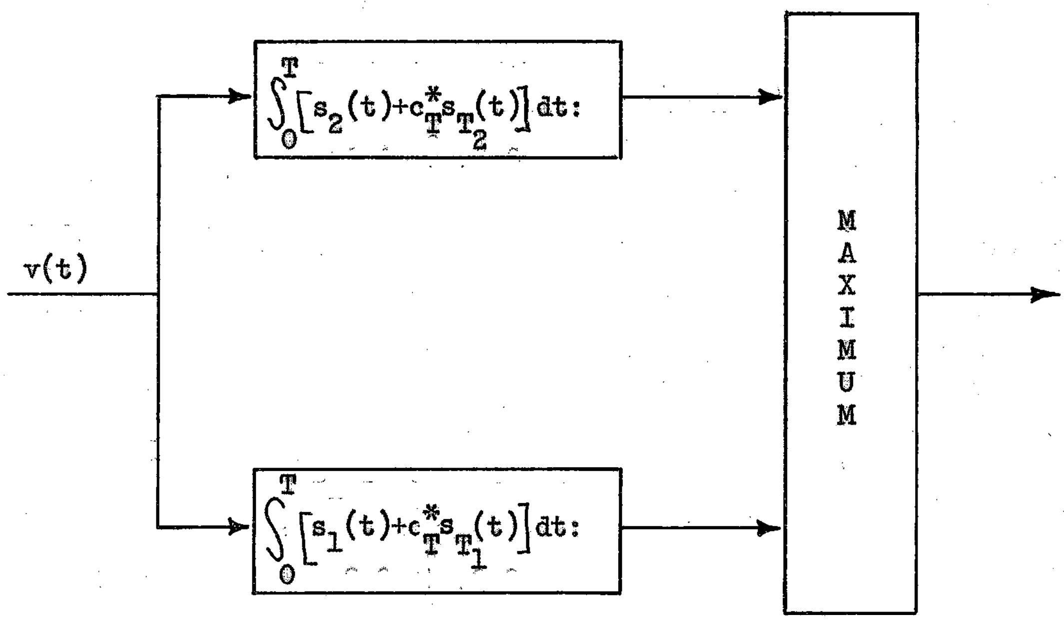


FIGURE 6 OPTIMUM CORRELATION RECEIVER FOR ORTHOGONAL SIGNALS

$$\frac{1 + a - 2b\sqrt{a}}{1 + a - 2b\sqrt{a}} u^2$$

yields the relation

$$G(-b;\frac{1}{u}) = - G(b;u) \tag{3.7.3}$$

Consequently, if $x = u$ is the real root of $G(b;x) = 0$, then $x = \frac{1}{u}$ is the real root of $G(-b;x) = 0$ and by uniqueness it is the only such root. Furthermore, if $b > 0$, it is easily seen that $0 < u < 1$. Thus by (3.4.18) c_T^* is negative for $b > 0$. If the sign of b is changed then the c_T^* corresponding to $-|b|$ is

$$c_T^* = \frac{\frac{1}{u} - 1}{\frac{1}{u} + 1} = \frac{1 - u}{1 + u} = - \frac{(u - 1)}{u + 1} \tag{3.7.4}$$

so that c_T^* is > 0 and equal in magnitude to the case in which $b > 0$. Summarizing, if $b = -|b|$ then $c_T^* = |c_T^*|$ and if $b = +|b|$ then $c_T^* = -|c_T^*|$, so that $\rho c_T^* \leqq 0$ in all cases.

With this in mind we see that P_E^* of (3.4.24), (3.5.2) and (3.6.5) depends on the absolute magnitude of $b(|b|)$. Likewise, the corresponding P_E^o depends only on $|b|$. Thus the sign of b uniquely determines the sign of c_T^* in an antisymmetrical way, whereas $|c_T^*|$ is determined by $|b|$ and P_E^* and P_E^o are independent of the sign of b.

We now examine the case for $b = 0$. Then $u = 1$ is the desired real root of (3.7.1), so that $c_T^* = 0$. Then $P_E^*\Big|_{b=o}$ is given by

$$P_E^*\Big|_{b=o} = \int_{\frac{1}{2}(\frac{S}{N})2^m}^{\infty} e^{-\xi^2/2} \frac{d\xi}{\sqrt{2\pi}} \qquad \begin{array}{l} m = 0 \text{ unipolar} \\ m = 1 \text{ orthogonal} \\ m = 2 \text{ bipolar} \end{array} \tag{3.7.5}$$

Furthermore, the corresponding P_E^o for b = o is

$$P_E^o\Big|_{b=o} = P_E^*\Big|_{b=o} \tag{3.7.6}$$

and in fact, since $c_T^* = 0$, the corresponding receivers are identical. This should not be surprising. For if b = 0 then $\rho = 0$, which means the correlation operation is insensitive to the presence of $s_T(t)$. However, if we assume a lossless (energy) medium the total transmitted energy per T interval is $(1+a)E_s$ and so the transmitted (potential) energy/noise ratio is $(1+a)(\frac{S}{N})$ where $(\frac{S}{N})$ is the actual energy/noise ratio at the receiver due only to s(t). Consequently, if b = 0, the only effect of the channel is to attenuate the "transmitted" energy noise ratio by (1+a). Thus the well known performance curves for correlation reception with no pulse distortion may be used if the energy/noise ratio is divided by (1+a).

In view of the foregoing discussion the performance curves P_E^* and P_E^o with corresponding c_T^* for the unipolar signal are plotted in Figures 7 through 15 as functions of received energy/noise ratio; $(\frac{S}{N})$, for various parameter values of a and b. The appropriate $(\frac{S}{N})$ doubling factor can then be used for obtaining the performance of the orthogonal and bipolar schemes.

There is but one subtlety left to discuss and that is the performance of the optimum receiver in the absence of noise. Certainly, if the noise, n(t), were to disappear, we would like our designed receiver to be error-less. In a sense, we require a consistent receiver. This is equivalent to requiring, for the unipolar case

$$h^*\ (s+s_T) > K^* = \frac{1}{2}\, h^* \circ (s+s_T)$$

and

$$h^* \circ s > K^*$$

with

$$h^* = s(t) + c_T^* s_T(t) \;; \qquad |c_T^*| < 1$$

which is the same as

$$h^* \circ (s \pm s_T) > 0 \tag{3.7.7}$$

Note that (3.7.7) is the same condition for errorless performance of the bipolar case.

Now suppose that $\rho = b\sqrt{a}\,E_s > 0$. Then

$$h^* \circ (s+s_T) = \left[1 - |c_T^*|a + b\sqrt{a}(1 - |c_T^*|)\right] E_s > 0 \tag{3.7.8}$$

where we recall that

$$|c_T^* \rho| < 0$$

$$|c_T^*| < 1$$

To show that

$$h^* \circ (s-s_T) > 0 \tag{3.7.9}$$

assume otherwise, i.e.,

$$h^* \circ (s-s_T) = \left[1 - b\sqrt{a} + |c_T^*|(a - b\sqrt{a})\right] E_s \leq 0 \tag{3.7.10}$$

For this event to occur

$$a - b\sqrt{a} < 0$$

and for some $0 \leq x < 1$

$$1 - b\sqrt{a} - x(b\sqrt{a} - a) = 0 \tag{3.7.11}$$

Then, since $x < 1$

$$1 - b\sqrt{a} < b\sqrt{a} - a \tag{3.7.12}$$

or, remembering that $|b| < 1$

$$\frac{1 + a}{2\sqrt{a}} < |b| < 1 \tag{3.7.13}$$

Now the function, $f(y)$

$$f(y) = \frac{\frac{1}{y} + y}{2} \geqq 1 \; ; \qquad y > 0 \tag{3.7.14}$$

with equality at $y = 1$ only.

Since $\sqrt{a} < 1$, (3.7.13) implies,

$$1 < |b| < 1, \tag{3.7.15}$$

a contradiction to assumption (3.7.8). For $\rho < 0$, the arguments are identical except reversed. That is to say, obviously

$$h^{*}\circ(s-s_T) > 0$$

and one contradicts the assumption

$$h^{*}\circ(s+s_T) \leqq 0 \; .$$

If $\rho = 0$, $c_T^{*} = 0$ and we have

$$h^{*}\circ s = E_s > 0$$

In effect we have shown that

$$h\circ(s \pm s_T) > 0$$

for any choice of c, c_T such that $\left|\frac{c_T}{c}\right| < 1$. In particular $c = 1$ and $c_T = c_T^{*}$, is a consistent receiver.

3.8 Computed Receiver Performance and Comparisons

The error rate P_E^* for the optimum correlator of the form

$$h^*(t) = s(t) + c_T^* s(t); \qquad K = K^*$$

is shown in Figures 7, 9, and 11. For comparison, the error rate P_E^o for the standard correlator of the form

$$h^o(t) = s(t); \qquad K = K^o$$

is presented in Figures 8, 10, and 12. Figures 13, 14, and 15 gives the values of c_T^* as a function of $(\frac{S}{N})$, for fixed a and b.

Examination of the performance curves of Figures 7 through 12 shows that for small a, there is negligible improvement to be had in using the optimum correlator (h^*) over the standard type (h^o). For $a = 1/4$ and $b = 0.5$ and 0.7, the optimum correlator gives a gain of 1 db in $(\frac{S}{N})$; a marginal improvement. For $a = 1/2$ and $b = 0.5$, there is a 2 db gain in $(\frac{S}{N})$, and for $b = 0.7$ there is a 3 db gain. This represents a significant improvement. Due to the steepness of the curves, over a decade decrease in P_E is to be had.

Since the receivers considered here are to be memoryless, it follows that they can obtain no a priori information about the interference pulse. Consequently, a good receiver policy would be to try to ignore the presence of the interference pulse. The optimum correlation receiver attempts to decorrelate or orthogonalize itself to the pulse interference. Thus, it is no surprise that for all cases the error curve for $b = 0$, $P_E\big|_{b=0}$, is uniformly best. For $a = 1/2$ and $b = 0.7$, $P_E\big|_{b=0}$ is 3 db better than the optimum correlator and 6 db better than the standard correlator. It then seems plausible that the best one could expect from

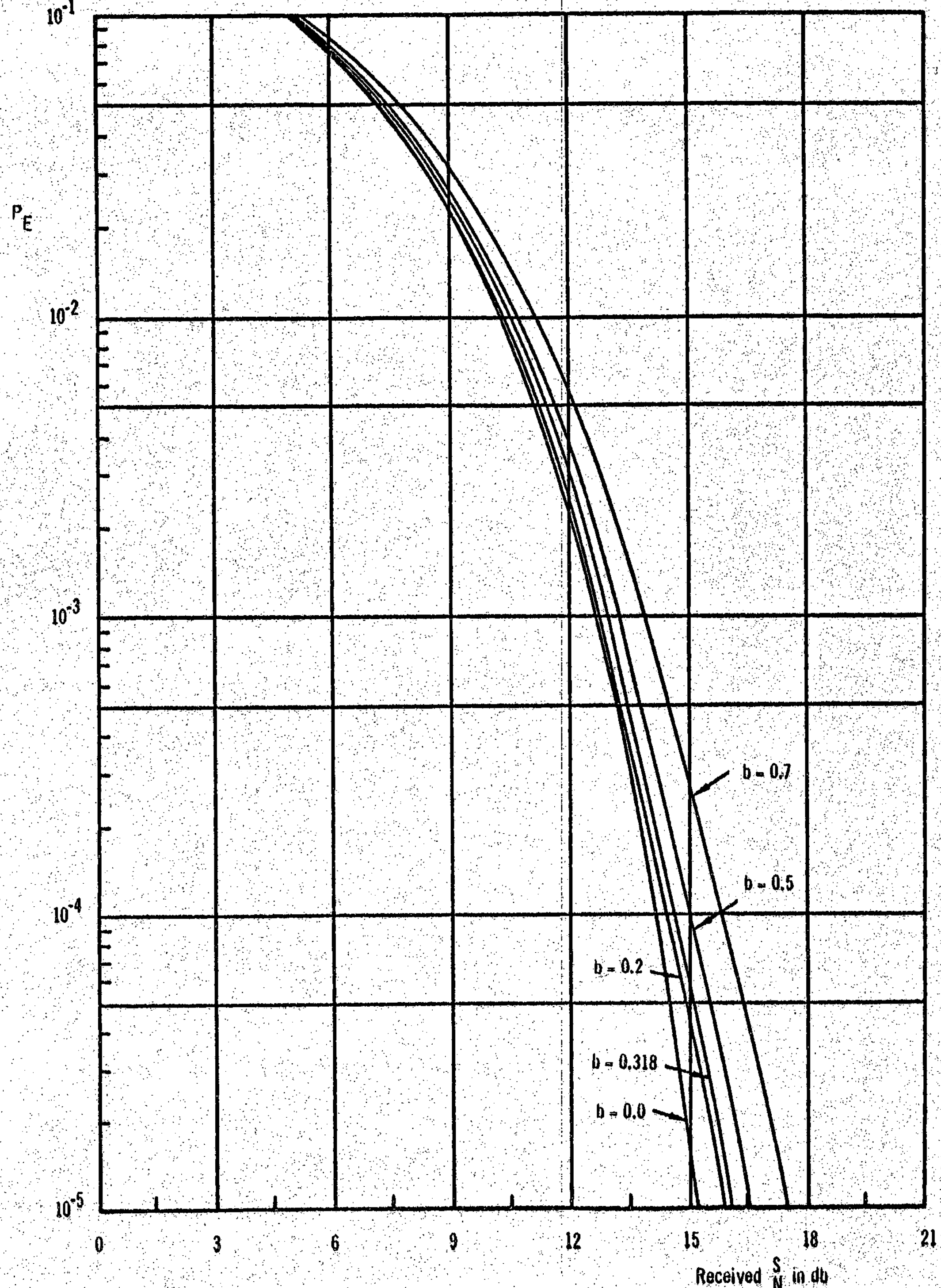


FIGURE 7 OPTIMUM CORRELATION RECEIVER ERROR RATE FOR a = 0.1

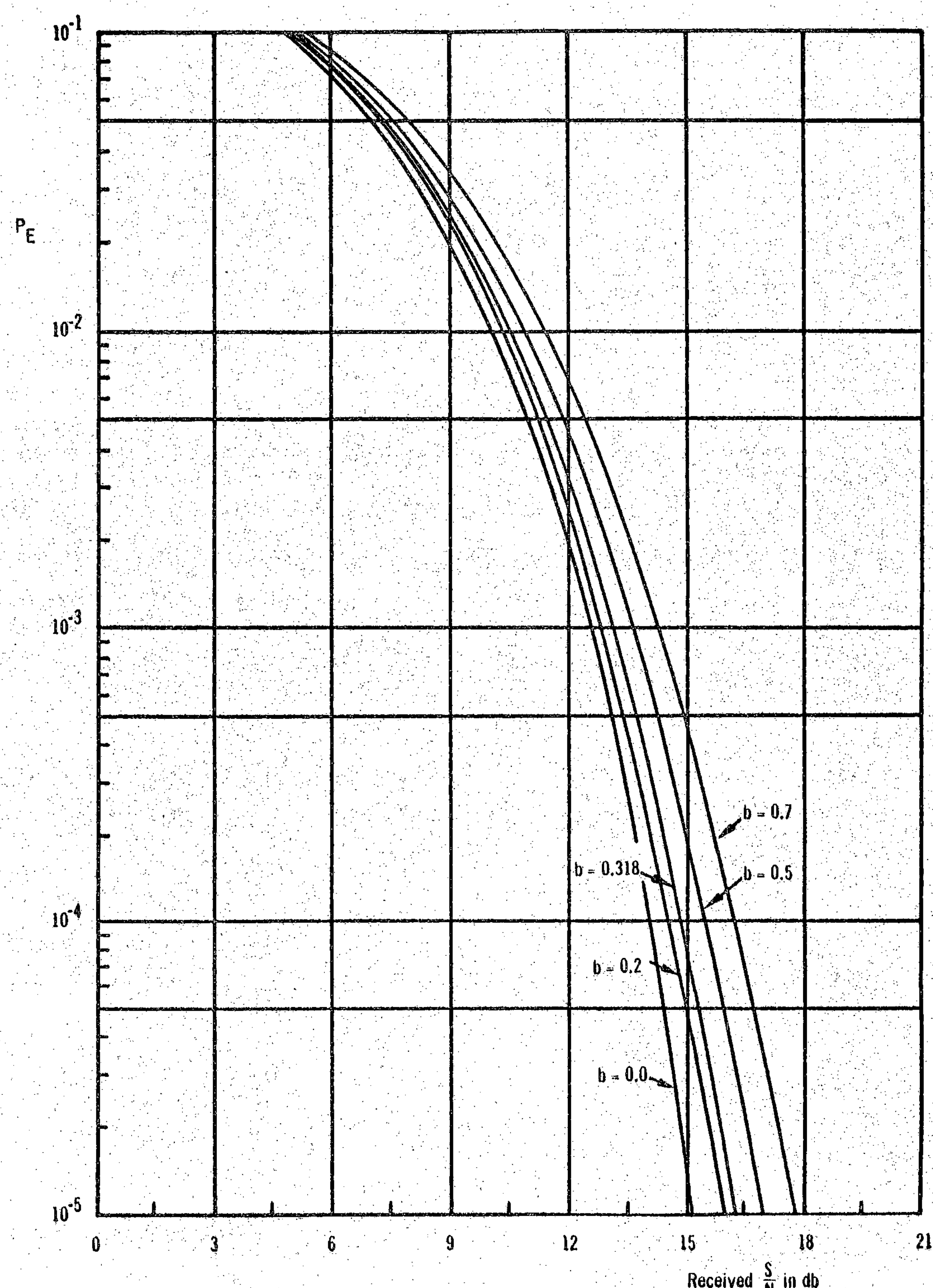


FIGURE 8 STANDARD CORRELATION RECEIVER
ERROR RATE FOR a = 0.1

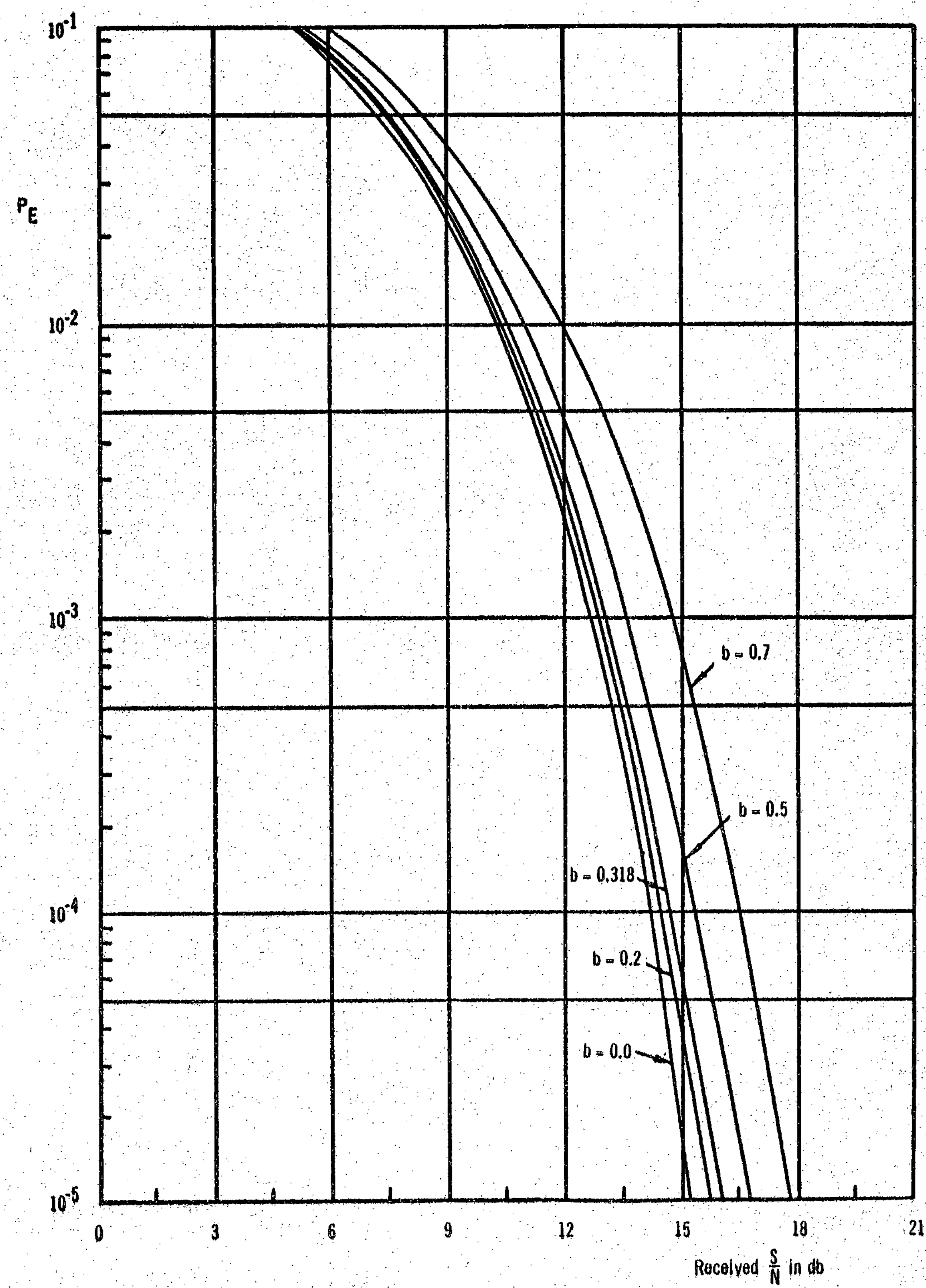


FIGURE 9 OPTIMUM CORRELATION RECEIVER
ERROR RATE FOR a = 0.25

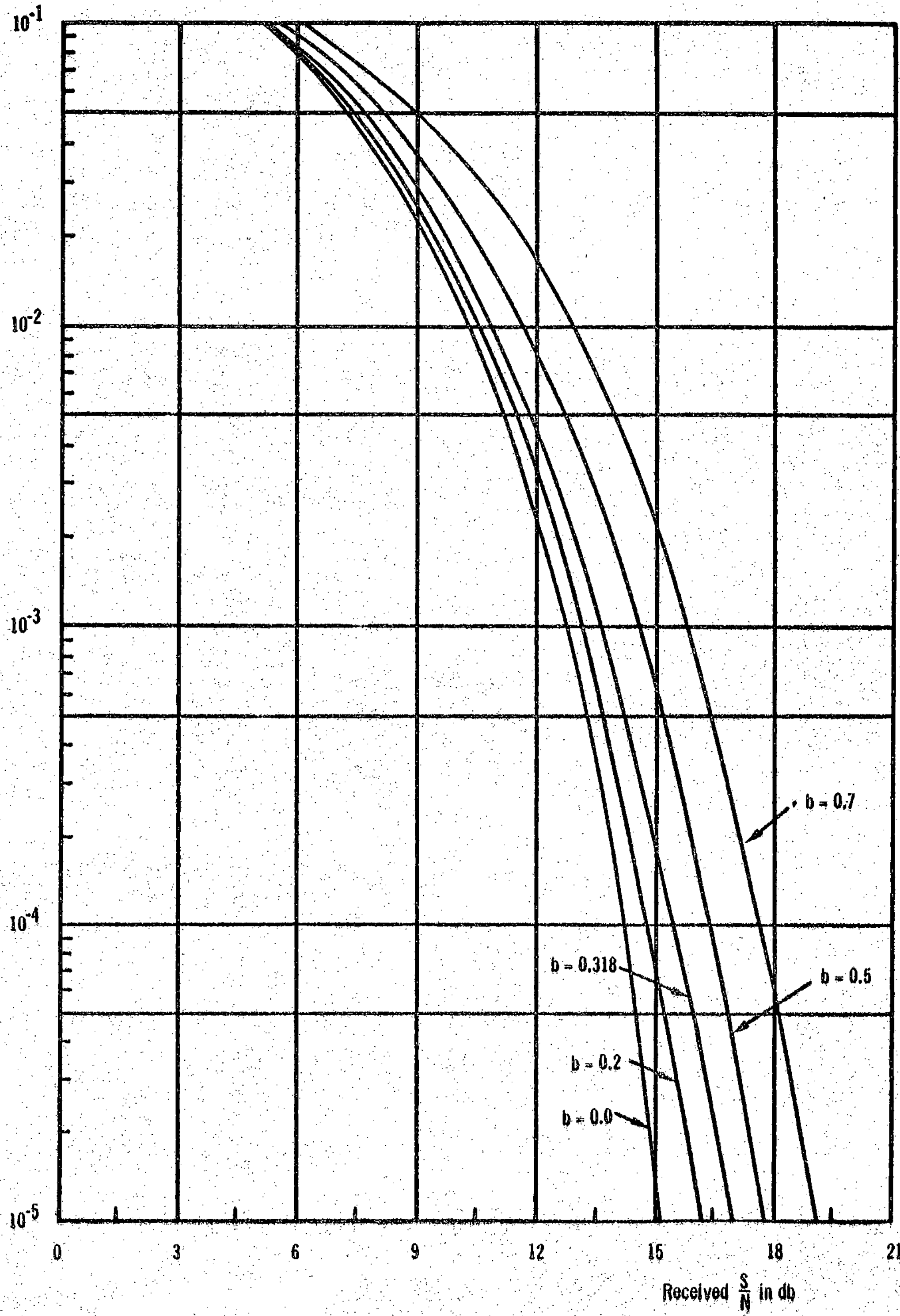


FIGURE 10 STANDARD CORRELATION RECEIVER
ERROR RATE FOR a = 0.25

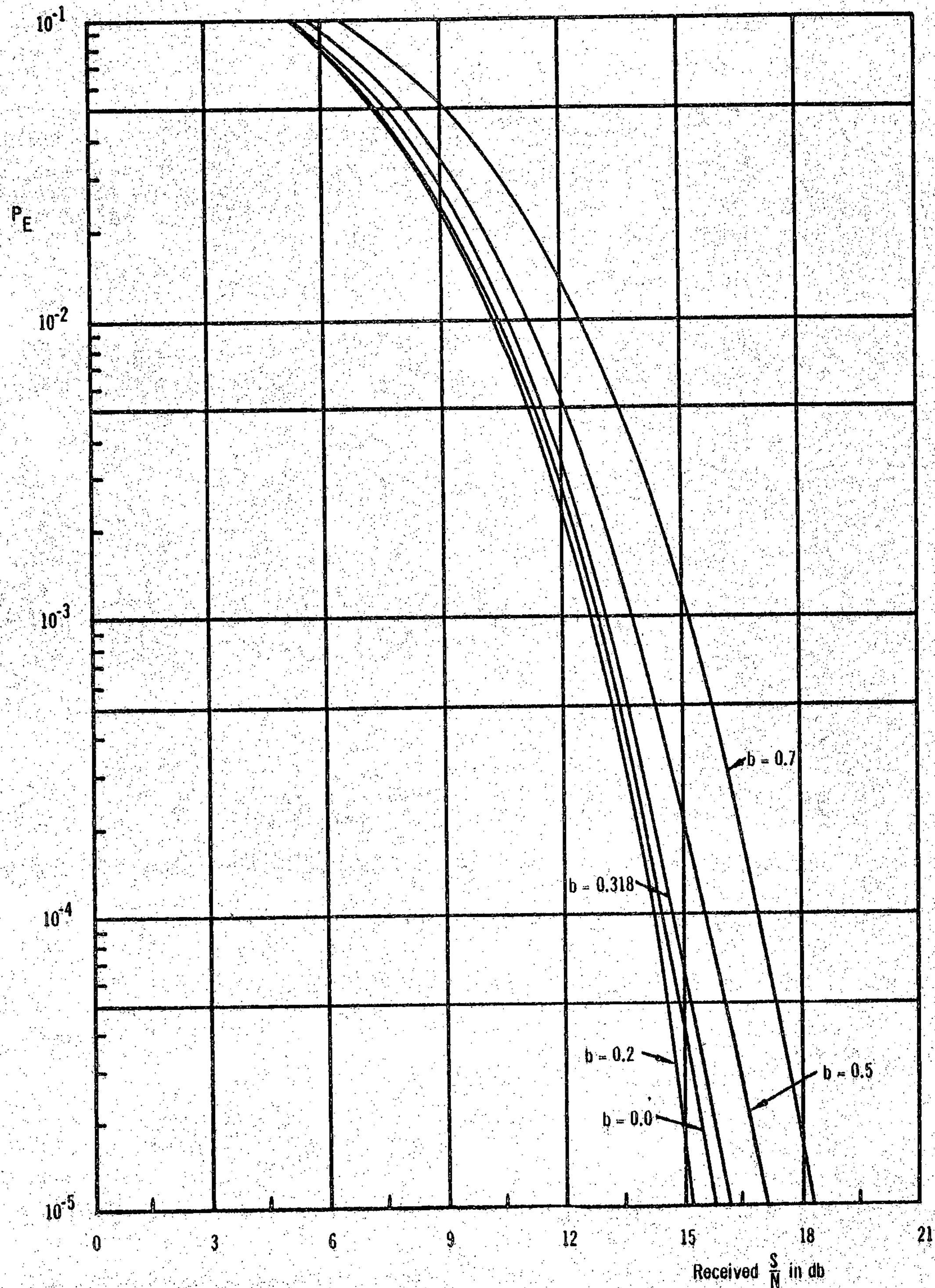


FIGURE 11 OPTIMUM CORRELATION RECEIVER ERROR RATE FOR a = 0.5

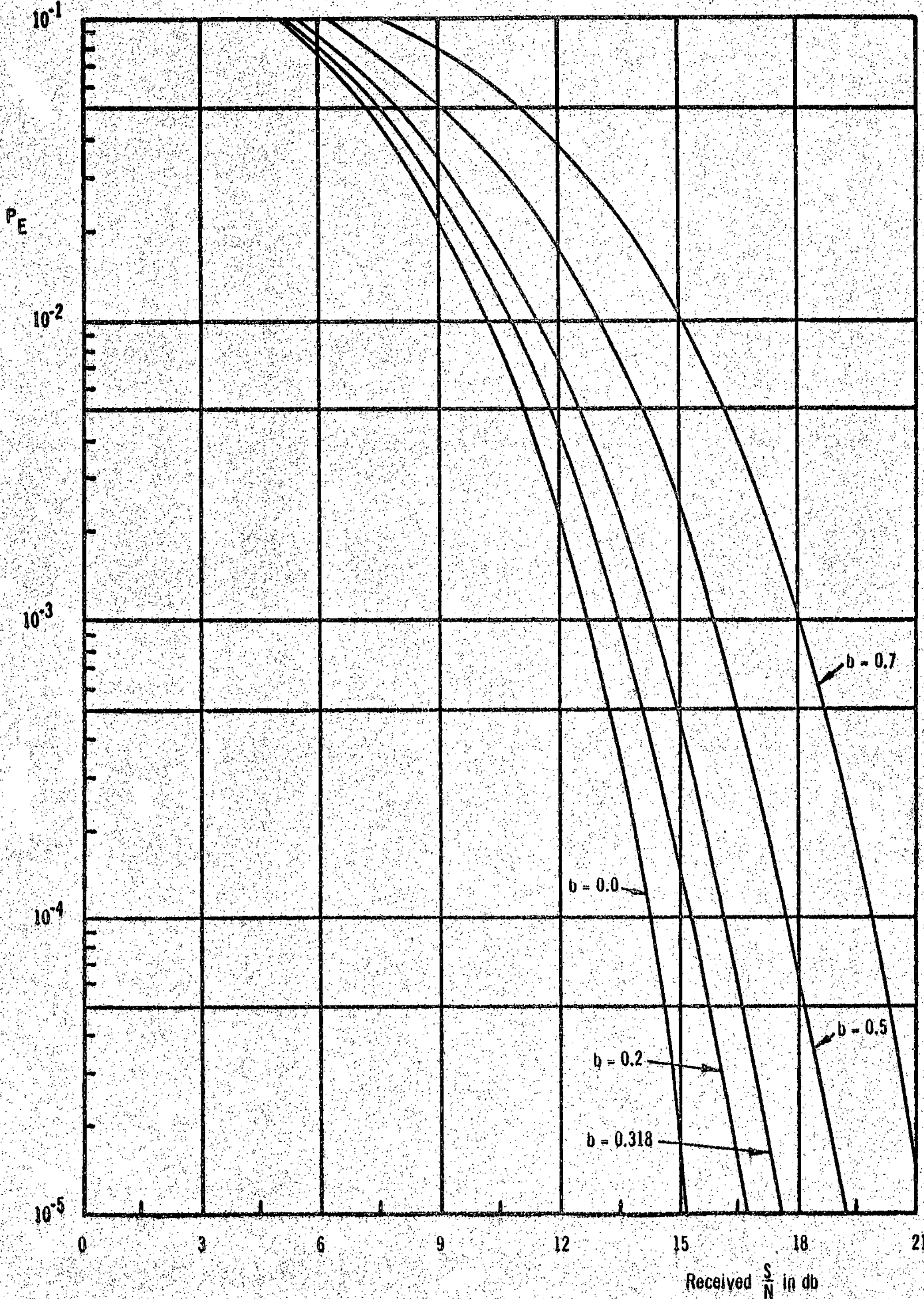


FIGURE 12 STANDARD CORRELATION RECEIVER ERROR RATE FOR a = 0.5

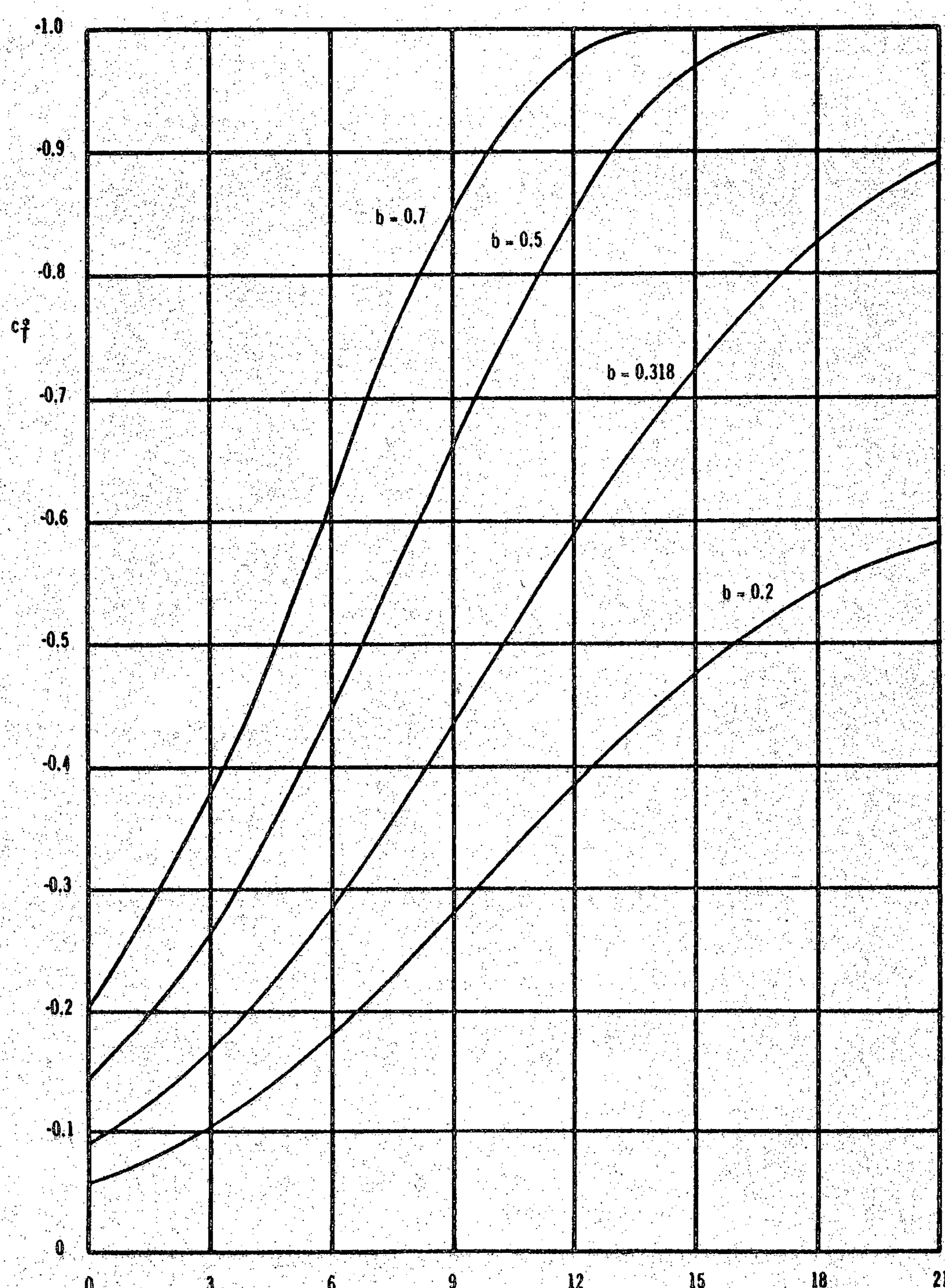


FIGURE 13 $c_{\dagger}^{*}$ VERSUS $\frac{S}{N}$ FOR a = 0.1

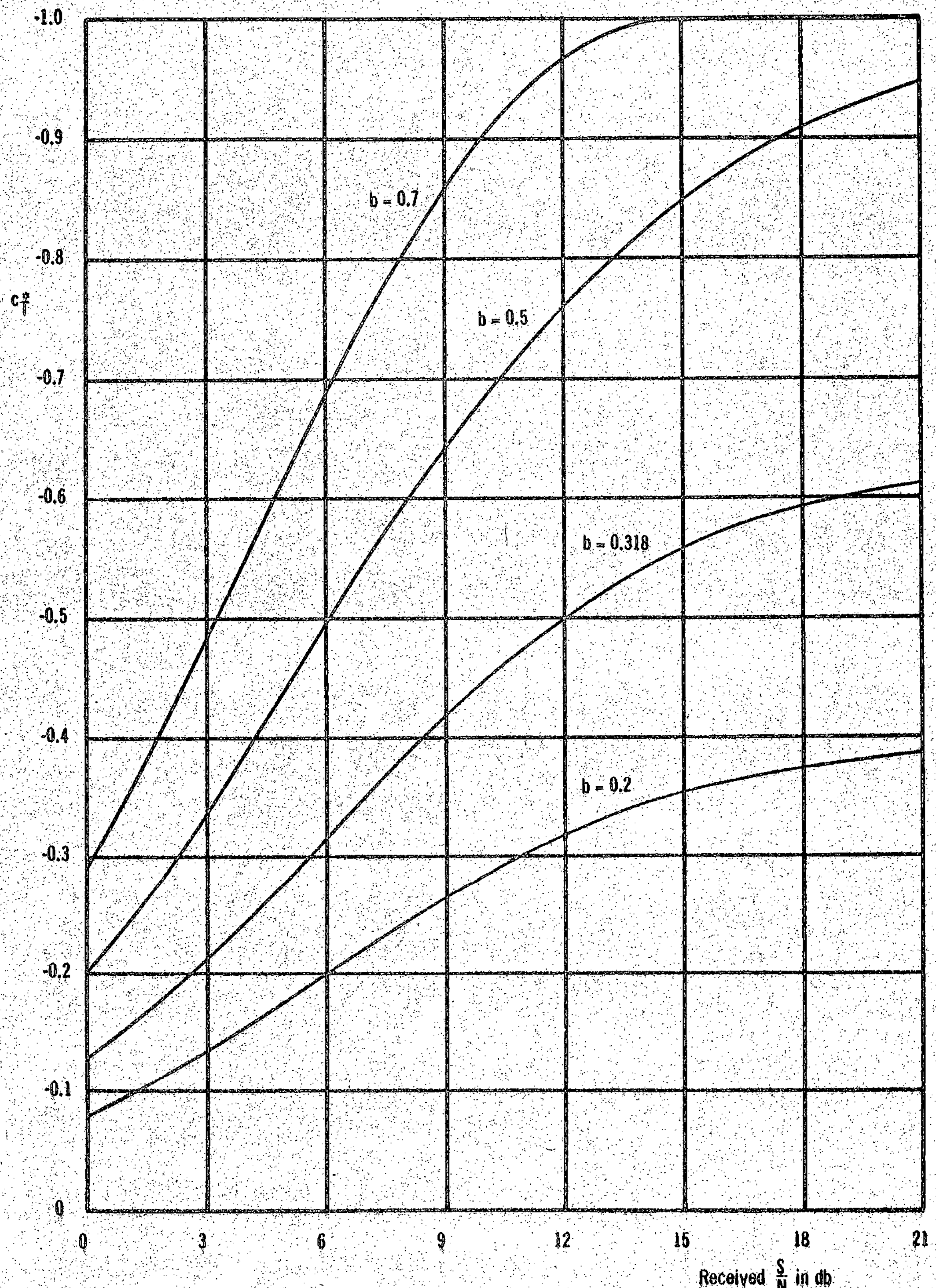


FIGURE 14 c^* VERSUS $\frac{S}{N}$ FOR a = 0.25

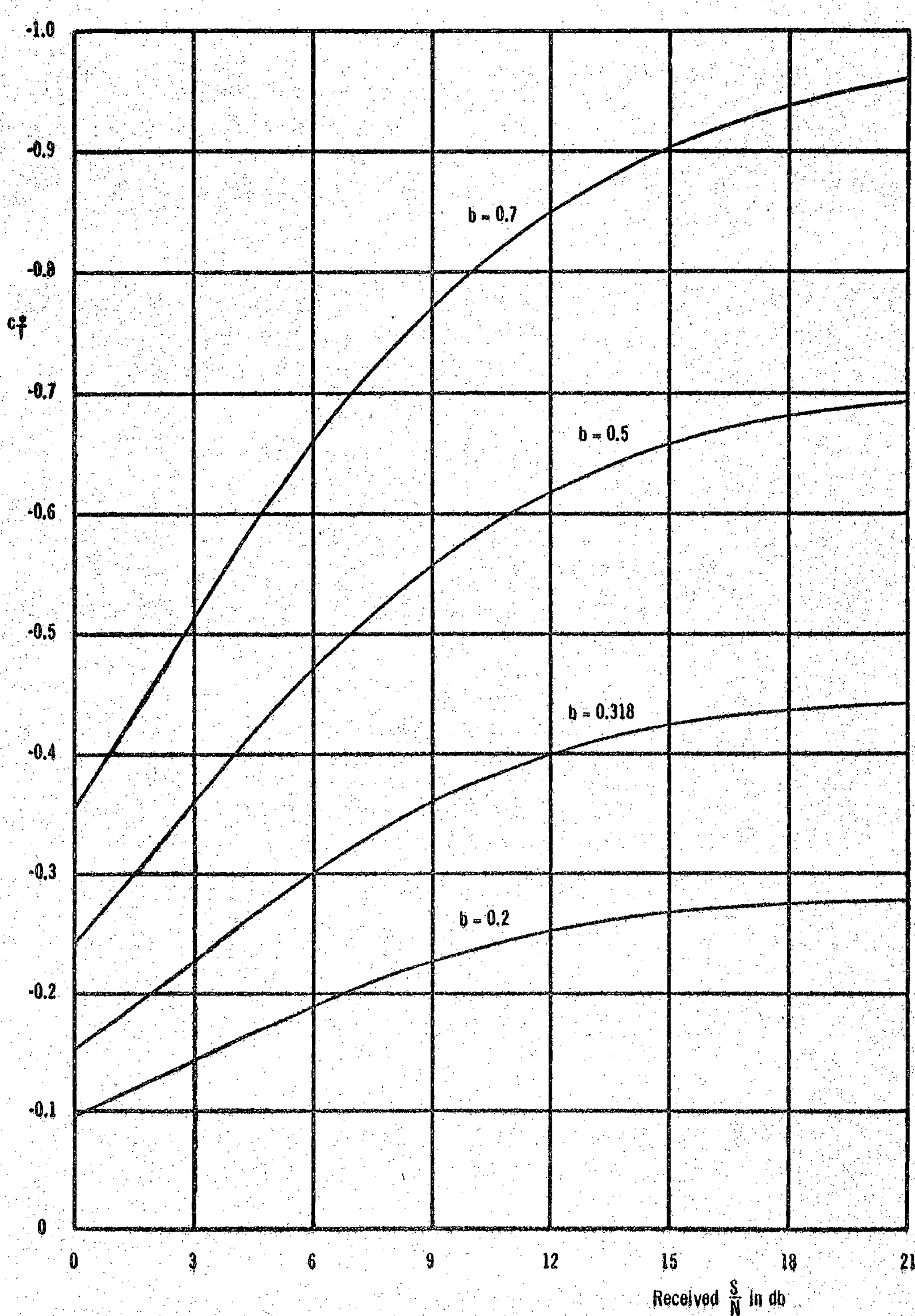


FIGURE 15 $c_{\dagger}^{*}$ VERSUS $\frac{S}{N}$ FOR a = 0.5

a memoryless receiver when both a and b are not zero is to achieve the $P_E\big|_{b=0}$ curve. If such is the case we must be willing to sacrifice $\frac{a}{1+a}$ per cent of the transmitted energy to noise ratio.

3.9 Summary: A Geometric Interpretation

As mentioned in Chapter II we are dealing with a real innerproduct space. As there are only two linearly independent vectors s, s_T in this chapter, a planar Euclidean vector representation is possible, where the length of a vector (signal) is represented by the square root of the signal energy and the "dot" product (hence angle) between two vectors is the correlation operation between the two corresponding signals, i.e.,

$$\overline{OS_1}\cdot\overline{OS_2} = \int_0^T s_1(t)s_2(t)dt$$

For convenience, a graphical description will be given for the bipolar case. The unipolar and orthogonal cases will then be shown to be simply a rotation and translation of the coordinate axes established for the bipolar situation.

In Figure 16 we have fixed the horizontal axis such that it passes through the points S and S' and is oriented in the direction of S. The vector $\overline{OS}$ represents s(t) and $\overline{OS'}$ represents -s(t). The vectors $\overline{OP}$ and $\overline{OQ}$ represent $s(t) + s_T(t)$ and $s(t) - s_T(t)$, respectively. Thus $\overline{OP'}$ and $\overline{OQ'}$ are obviously defined. So in this two dimensional signal space, the points P, Q, P', and Q' represent the four possible signal points with P, Q associated with a plus transmission and P', Q' associated with a minus transmission.

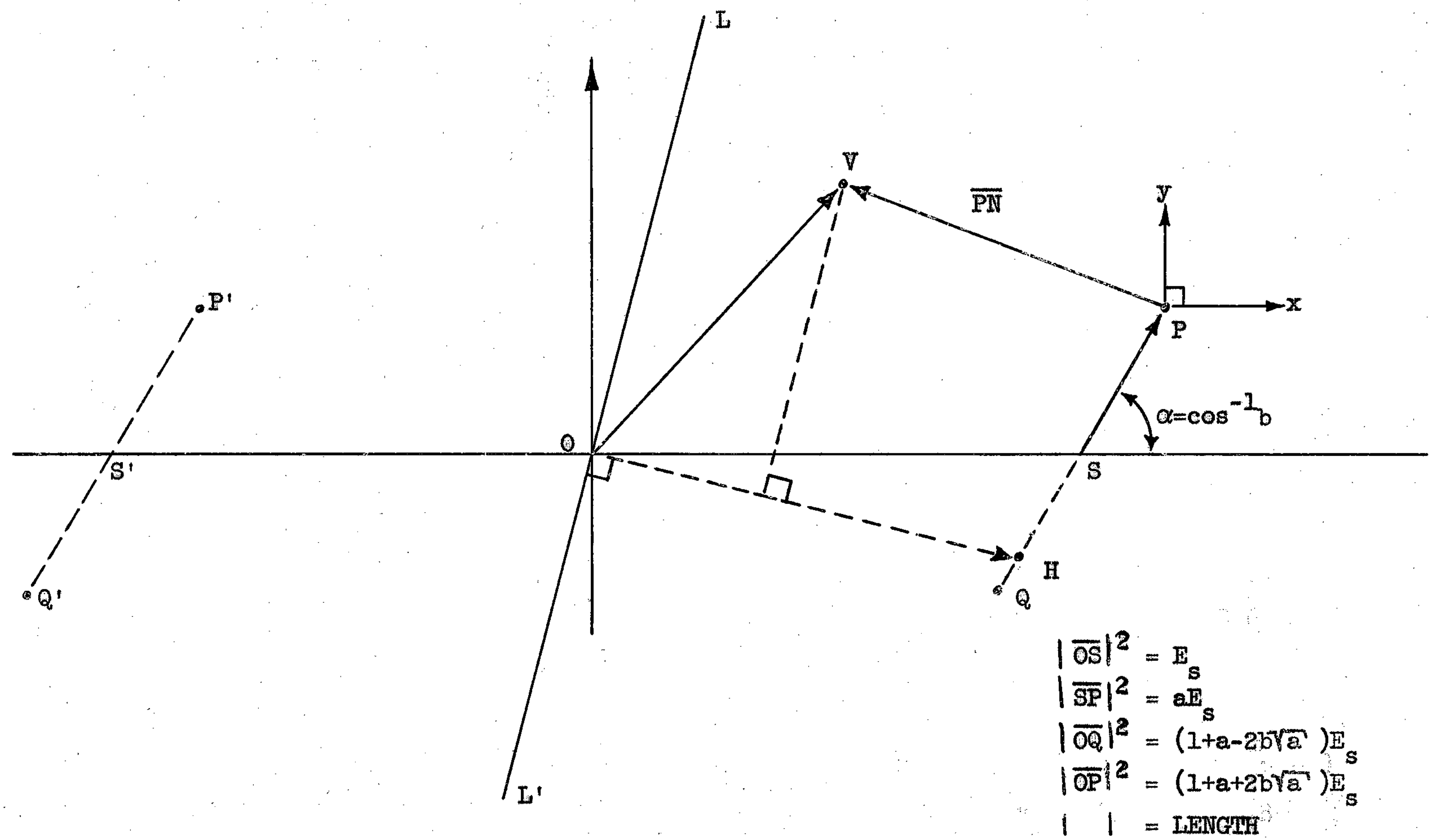


FIGURE 16 BIPOLAR RECEIVED SIGNAL SPACE

Now $\overline{SP}$ represents the signal $s_T(t)$, so that $\overline{SP} \cdot \overline{OS}$ is $\rho = b\sqrt{a}\,E_s$.
But

$$\overline{SP} \cdot \overline{OS} = |\overline{SP}|\,|\overline{OS}| \cos \alpha = b\sqrt{a}\,E_s$$

where

$| \; |$ = length

α = angle between $\overline{SP}$ and $\overline{OS}$

whence

$$\alpha = \cos^{-1} b \, .$$

Since $h(t)$ is a linear combination of $s(t)$ and $s_T(t)$, the vector $\overline{OH}$ representing $h(t)$ must lie along the line QSP. In fact, since $b > 0$ and $c_T < 1$, the point H must lie between S and Q. Now, if $\overline{OV}$ is any realized data vector corresponding to $v(t)$, then the correlator output is the projection of $\overline{OV}$ onto $\overline{OH}$. If this quantity is non-negative a plus symbol is announced, otherwise a minus symbol. It is obvious that the correlation receiver establishes a "decision line" perpendicular to $\overline{OH}$ which partitions the receiver space so that if V is to the right of LL' a plus is pronounced; if V is to the left of LL' the decision is for a minus symbol.

The unipolar case is represented by a strict translation of the axes to the point P'. The orthogonal case involves a translation and rotation. By assumptions 3.6(a) through (f), orthogonal signals need a four dimensional figure to represent their space which therefore is not shown. It is now obvious that any pair of transmitted signals whose four possible received signal points form a parallelogram may be reduced to the bipolar case by a coordinate shift.

In terms of this model (Figure 16), it is possible to give a qualitative discussion for the c_T results contained if Figures 13, 14, and 15. Notice that c_T is a direct measurement of the tilt of LL' away from vertical. The larger $|c_T|$, the closer H is to Q, and the more horizontally tilted is LL'. Consequently we will discuss the tilt in LL' as a function of $(\frac{S}{N})$, b, and a .

The probability of correct reception can be interpreted as the volume under a particular surface (probability surface) defined over the received signal plane (Figure 16). Given that v(t) is the sum of noise n(t) and s(t) + $s_T(t)$, we have that $\overline{OV}$ is the sum of the vector $\overline{OP}$ and a noise vector $\overline{PN}$. Now the vector $\overline{PN}$ may be decomposed into orthogonal components parallel and perpendicular to $\overline{OS}$. Let x and y be the respective axes of such a decomposition (Figure 16). Then the x component of $\overline{PN}$, say $\overline{PN}_x$ is given by

$$\overline{PN}_x = \frac{\overline{PN} \bullet \overline{OS}}{|\overline{OS}|} = \frac{1}{\sqrt{E_s}} \int_0^T s(t)n(t)dt$$

Since n(t) is Gaussian it follows that the probability density on $|\overline{PN}_x|$ is given by

$$p(x) = \frac{1}{\sqrt{2\pi N_o}} e^{-x^2/2N_o}$$

where x denotes $|\overline{PN}_x|$. It is also true that $|\overline{PN}_y|$ (denoted by $y = |\overline{PN}_y|$) has the identical Gaussian density function as x and furthermore, since x and y are orthogonal, x and y are statistically independent. Thus, if we wish to interpret the differential probability that $\overline{PN}$ lies within the differential area dA with coordinates x and y, as a differential volume dV, the dV is given by

$$dV = \frac{1}{2\pi N_o} e^{-(x^2+y^2)/2N_o} dxdy$$

Then the conditional probability surface σ_P, associated with the point P, is given by the Gaussian surface

$$\sigma_P = \frac{1}{2\pi N_o} e^{-(x^2+y^2)/2N_o}$$

or in circular coordinates

$$\sigma_P = \frac{1}{2\pi N_o} e^{-r^2/2N_o}$$

Consequently σ_P is generated by taking a ray from P and defining the function

$$\frac{1}{2\pi N_o} e^{-r^2/2N_o}$$

on this ray perpendicular to the plane of the receiver space and rotating the ray through 2π radians. σ_P is then the generated surface of revolution. The conditional probability that given the received point P, the data vector $\overline{OV}$ lies in a prescribed area A is given by

$$Pr(\overline{OV} \text{ in } A/P) = \int_A \sigma_P dA'$$

where dA' is the differential area element.

Thus, given that P is the received signal point, the conditional probability of correct reception is that portion of the volume under a Gaussian surface centered on P and to the right of the line LL'. Thus the total probability is one fourth (symmetric a priori probabilities) of the separate volumes under Gaussian surfaces centered on P, Q, P' and Q', respectively, and to the right of LL' for P, Q; to the left of LL' for P', Q'.

Since LL' must pass through 0 and the points P, Q, Q', P' are symmetric with respect to 0 and are symmetric with respect to their a priori probabilities of occurrences, it follows that P_C can be obtained in the manner discussed above by considering only the points P and Q. In this bipolar case then, P_C is equivalent to one half of the volume to the right of LL' and under a probability surface given by the sum of two identical Gaussian surfaces centered on P and Q, respectively. P_C is then maximized by tilting LL' such that maximum volume is contained to the right (or minimum volume to the left which represents P_E). It is now apparent that if the received signal points were not completely symmetric, then LL' need not pass through 0 so that the location and tilt of LL' are coupled and hence K^* and c_T^* would be interdependent.

For a fixed a and b we may now discuss the behavior of c_T as a function of $(\frac{S}{N})$. If $(\frac{S}{N})$ is very large, the contour lines of this probability surface in the region of the origin approximate ellipses with foci colinear with $\overline{PQ}$. It is reasonable then to expect LL' to be nearly tangent to the contour line passing through 0. This tilts the decision line LL' towards the horizontal axis. That $|c_T| < 1$ follows from $\overline{0Q}$ being less than $\overline{0P}$. The circular effect from Q slightly dominates that from P, causing the contour line through 0 to be a little more canted towards vertical than the ellipse.

For a very low $(\frac{S}{N})$ the origin is very near the hilltops of both Gaussian component surfaces so that the contour lines of the probability surface tend to be circular with center at S. This causes LL' to be directed nearly vertically. Thus $|c_T|$ approaches zero as $(\frac{S}{N})$ goes to zero.

CHAPTER IV

SWITCHED MODE RECEIVER

In this chapter we will allow our receiver to utilize its own past decisions. Except for assumption (8) our channel model is that of 3.1. Since we have signal interference on only adjacent T intervals, only the immediately preceding receiver decision will convey information about the expected interference on the next T interval. We will examine the behavior of the receiver that assumes that its previous decision is perfectly correct.

4.1 The Deterministic Switch

For the rest of this chapter we will denote by the symbol D^{-1}, the event associated with the receiver decision immediately preceding the decision, D, to be currently made. Thus D^{-1} is the receiver announcement regarding the event $Z_{(n-1)T}$ when the receiver is concerned with deciding Z_{nt}. We assign to D^{-1} two numbers, 1 and 2, which are associated with $Z_{(n-1)T}$ in the following way:

(a) if the receiver decides that $Z_{(n-1)T} = 1$, we say $D^{-1} = 1$

(b) if the receiver decides that $Z_{(n-1)T} = 2$, we say $D^{-1} = 2$.

We will design two parallel receivers wherein D^{-1} activates our choice of which one we choose to use (ref. Figure 17). Since we have a stationary channel model with statistically independent input symbols Z_{nt}, it follows that the probability of D^{-1} being correct is the same

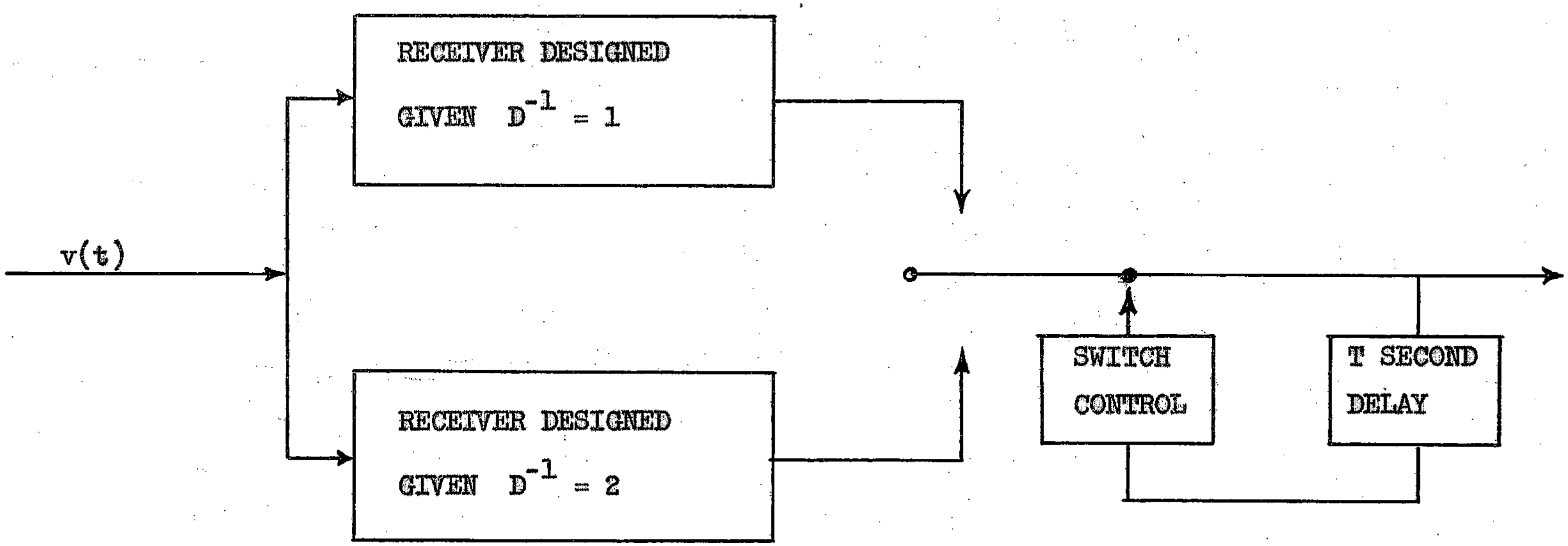


FIGURE 17 SWITCHED MODE RECEIVER

for any value of n, and, in particular, is the receiver correct reception rate or one minus the receiver error rate. As will be shown, these two receivers differ only in the decision levels, both having the same correlation operation.

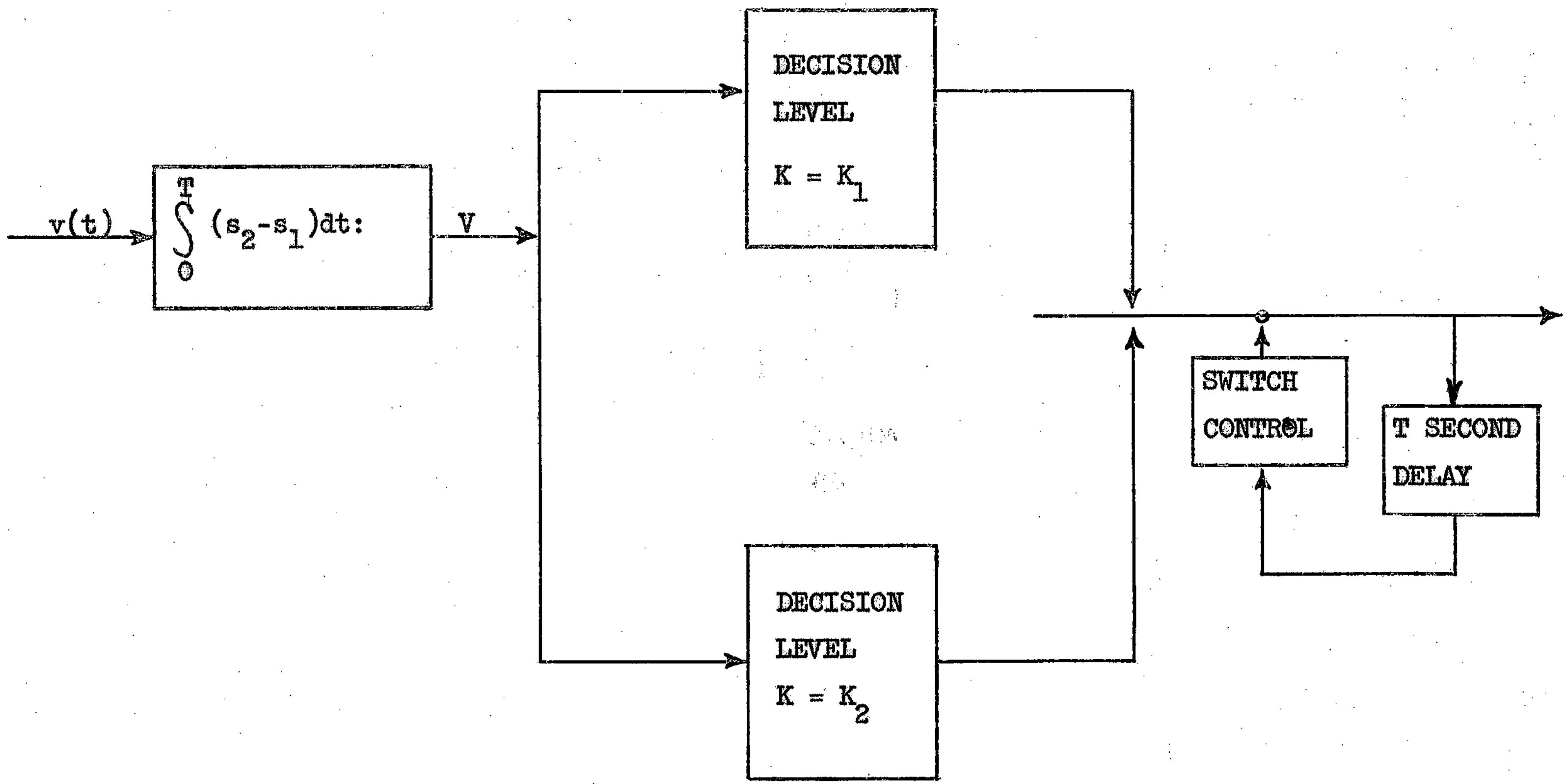
Now with reference to Table 1 of Chapter III, we see that if D^{-1} is one then the receiver presumes $s_{T_1}(t)$ to be present and since the receiver assumes D^{-1} to be correct the receiver may first subtract $s_{T_1}(t)$ from the input $v(t)$ leaving $\tilde{v}(t)$ composed only of $s_1(t)$ or $s_2(t)$ plus white gaussian noise $n(t)$. The optimal receiver operating on $v(t)$ is the correlation receiver discussed in Chapters I and II. Similarly, if D^{-1} is two, subtract $s_{T_2}(t)$ from $v(t)$ yielding $\tilde{v}(t)$. In either case the correlation operation involves the stored reference $s_2(t) - s_1(t)$. The operation of subtracting either $s_{T_1}(t)$ or $s_{T_2}(t)$ from the input may be equivalently replaced by adjusting the decision bias level. (ref. section 3.5) The receiver then has the form shown in Figure 18. The two decision levels K_1 and K_2 are given by

$$K = \frac{E_{s_2} - E_{s_1}}{2} + N_o \ln \frac{1-p}{p} + \int_0^T s_{T_\ell}(t)\left[s_2(t) - s_1(t)\right] dt \qquad \ell = 1,2 \qquad (4.1.1)$$

and

$$K_2 - K_1 = \int_0^T \left[s_{T_2}(t) - s_{T_1}(t)\right]\left[s_2(t) - s_1(t)\right] dt \qquad (4.1.2)$$

We now assume that $K_2 - K_1$ is not zero, otherwise the switch operation is superfluous. Furthermore, for the sake of argument, let $K_2 > K_1$. (We could just as easily consider $K_1 > K_2$.) For the unipolar,



$$K_j = \frac{E_{s_2} - E_{s_1}}{2} + \int_0^T s_{T_j}(t)\, s_2(t) - s_1(t)\, dt + N_o \ln \frac{1-p}{p} \; ; \quad j=1,2$$

FIGURE 18 OPTIMUM SWITCHED MODE CORRELATION RECEIVER bipolar, and orthogonal types of signals discussed in Chapter III we have the following table

	Unipolar	Orthogonal	Bipolar
$\int_0^T s_{T_1}(t)\left[s_2(t) - s_1(t)\right] dt$	0	- 1ρ	- 2ρ
$\int_0^T s_{T_2}(t)\left[s_2(t) - s_1(t)\right] dt$	ρ	+ 1ρ	+ 2ρ
$K_2 - K_1$	ρ	2ρ	4ρ

where as in Chapter III

$$\rho = \int_0^T s(t) s_T(t) dt .$$

Table 2 Switched Mode Decision Levels

Now if we draw a line (V axis) to represent the possible values of the output V of the correlator, (Figure 18), we see that K_1 and K_2 serve to define three regions A, B, and C of the V axis (see Figure 19).

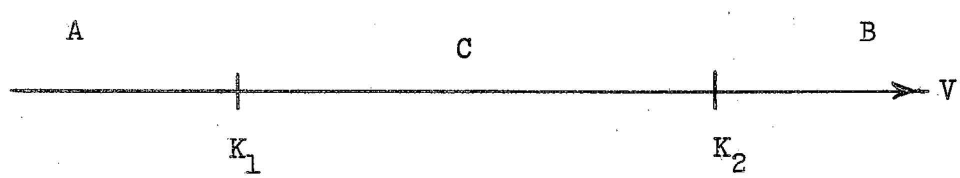


FIGURE 19 SWITCHED MODE DECISION REGIONS

Thus, if V falls into region A, then V is less than both K_1 and K_2 so that both "receivers" announce a one. Similarly, if V is in region B both receivers announce a two, consequently, if V falls in either region A or B the switching operation is unnecessary. If, however, V falls into region C one receiver says one while the other says two. The switch, governed by D^{-1}, then determines whether one or two will be announced. We point out the similarity of this arrangement to the binary erasure channel[6] wherein we use D^{-1} in a simple fashion to "write in" the symbol "erased". In this context a more detailed study from the information theory viewpoint would yield more sophisticated modes of "switch control".

We now compute the performance, P_C, of this receiver remembering that

$$\text{Prob } (D^{-1} \text{ correct}) = P_C \tag{4.1.3}$$

Listing all the possible independent events which lead to a successful decision along with their associated probabilities, we have the following (remembering that we have assumed $K_1 < K_2$);

Case I $\quad D^{-1} = 1$ and correct; $P_C(1-p)$

and (a) $\quad s_1(t)$ sent and $V < K_1$; $(1-p)\ P_I(V < K_1/s_1)$

or (b) $\quad s_2(t)$ sent and $V \geq K_1$; $p\ P_I(V \geq K_1/s_2)$

Case II $\quad D^{-1} = 2$ and correct; $P_C\ p$

and (a) $\quad s_1(t)$ sent and $V < K_2$; $(1-p)\ P_{II}(V < K_2/s_1)$

or (b) $\quad s_2(t)$ sent and $V \geq K_2$; $p\ P_{II}(V \geq K_2/s_2)$

Case III $D^{-1} = 1$ and incorrect; $(1-p)(1-P_C)$

and (a) $s_1(t)$ sent and $V < K_1$; $(1-p)\ P_{III}(V < K_1/s_1)$

or (b) $s_2(t)$ sent and $V \geq K_2$; $p\ P_{III}(V \geq K_2/s_2)$

Case IV $D^{-1} = 2$ and incorrect; $p\ (1-P_C)$

and (a) $s_1(t)$ sent and $V < K_1$; $(1-p)\ P_{IV}(V < K_1/s_1)$

or (b) $s_2(t)$ sent and $V \geq K_2$; $p\ P_{IV}(V \geq K_2/s_2)$

Now if indeed $K_2 < K_1$ we need to interchange K_1 and K_2 in Case III and Case IV.

Now each case is mutually exclusive as is each (a), (b) subcase so that P_C is the sum of the probabilities of each event.

$$
\begin{aligned}
P_C = (1-p)\ P_C & \left\{ (1-p)\ P_I(V < K_1/s_1) + p\ P_I(V \geq K_1/s_2) \right\} \\
& + p\ P_C \left\{ (1-p)\ P_{II}(V < K_2/s_1) + p\ P_{II}(V \geq K_2/s_2) \right\} \\
& + (1-p)(1-P_C) \left\{ (1-p)\ P_{III}(V < K_1/s_1) + p\ P_{III}(V \geq K_2/s_2) \right\} \\
& + p(1-P_C) \left\{ (1-p)\ P_{IV}(V < K_1/s_1) + p\ P_{IV}(V \geq K_2/s_2) \right\} \qquad (4.1.4)
\end{aligned}
$$

$K_1 < K_2$; if $K_2 < K_1$, interchange K_1 with K_2 in P_{III} and P_{IV}

The Roman numeraled conditional probabilities are the gaussian distribution functions and are computed as in equation (1.5). The gaussian distribution function Φ is defined as

$$\Phi(W) = \int_{-\infty}^{W} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

We then have

$$P_I(V < K_1/s_1) = \Phi(W_1 + W_2)$$

$$P_I(V \geq K_2/s_2) = \Phi(W_1 - W_2)$$

$$P_{II}(V < K_2/s_1) = \Phi(W_1 + W_2)$$

$$P_{II}(V \geq K_2/s_2) = \Phi(W_1 - W_2)$$

$$P_{III}(V < K_1/s_1) = \Phi(W_1 + W_2 - W_3) \qquad (4.1.5)$$

$$P_{III}(V \geq K_2/s_2) = \Phi(W_1 - W_2)$$

$$P_{IV}(V < K_1/s_1) = \Phi(W_1 + W_2)$$

$$P_{IV}(V \geq K_2/s_2) = \Phi(W_1 + W_2 - W_3)$$

and $K_1 < K_2$

where

$$W_1 = \sqrt{\frac{E_{s_2-s_1}}{4N_o}}\ ; \quad E_{s_2-s_1} = \int_0^T (s_2(t) - s_1(t))^2 dt$$

$$W_2 = \sqrt{\frac{N_o}{E_{s_2-s_1}}} \ \ln \frac{1-p}{p}$$

$$W_3 = \frac{(K_2 - K_1)}{(E_{s_2-s_1} N_o)^{1/2}} ; \ {}^{\circ}(K_2 - K_1) \text{ defined in } (4.1.2)$$

For $K_2 < K_1$ we have that

$$P_{III}(V < K_2/s_1) = \Phi(W_1 + W_2)$$

$$P_{III}(V \geq K_1/s_2) = \Phi(W_1 - W_2 + W_3)$$

$$P_{IV}(V < K_2/s_1) = \Phi(W_1 + W_2 + W_3) \qquad (4.1.5')$$

$$P_{IV}(V \geq K_1/s_2) = \Phi(W_1 + W_2)$$

One may then substitute (4.1.5) or (4.1.5'), (according to whether $K_2 > K_1$ or $K_1 > K_2$) into (4.1.4) and solve for P_C. We do this only for p = 1/2 ($W_2 = 0$).

$$4P_C = 4P_C \Phi(W_1) + 2(1-P_C)\left[\Phi(W_1) + \Phi(W_1 - W_3)\right] \qquad K_2 > K_1 \tag{4.1.6}$$

or

$$4P_C = 4P_C \Phi(W_1) + 2(1-P_C)\left[\Phi(W_1) + \Phi(W_1 + W_3)\right] \qquad K_1 > K_2$$

and since,

$$W_3 = \frac{K_2 - K_1}{\sqrt{E_{s_2-s_1} N_o}}$$

(4.1.6) is solved, irrespective of the sign of $K_2 - K_1$, as

$$P_C = \frac{\Phi(W_1) + \Phi(W_1 - |W_3|)}{2 + \Phi(W_1 - |W_3|) - \Phi(W_1)} \tag{4.1.7}$$

$$W_1 = \sqrt{\frac{E_{s_2-s_1}}{4N_o}}, \qquad W_3 = |K_2 - K_1| / \sqrt{E_{s_2-s_1} N_o}$$

We evaluate (4.1.7) for unipolar, bipolar and orthogonal signals so that they may be compared to the results of Chapter III. Using the notation for the types of signals as in Chapter III, we have the following table for W_1, W_3, K_1, and K_2 values.

	Unipolar	Orthogonal	Bipolar
W_1	$\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}$	$\sqrt{\left(\frac{S}{N}\right)}$	$\sqrt{2\left(\frac{S}{N}\right)}$
W_3	$2b\sqrt{a}\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}$	$2b\sqrt{a}\sqrt{\left(\frac{S}{N}\right)}$	$2b\sqrt{a}\sqrt{2\left(\frac{S}{N}\right)}$
$K_1/2N_o$	$-\frac{1}{2}\left(\frac{S}{N}\right)$	$-b\sqrt{a}\left(\frac{S}{N}\right)$	$-2b\sqrt{a}\;2\left(\frac{S}{N}\right)$
$K_2/2N_o$	$\frac{1}{2}\left(\frac{S}{N}\right)(1+2b\sqrt{a})$	$b\sqrt{a}\left(\frac{S}{N}\right)$	$2b\sqrt{a}\;2\left(\frac{S}{N}\right)$

Table 3 W and K Values for Unipolar, Orthogonal and Bipolar Signals

Thus for the unipolar case P_C is given by

$$P_C = \frac{\Phi\left(\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}\right) + \Phi\left(\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}\,(1-2|b|\sqrt{a})\right)}{2 + \Phi\left(\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}\,(1-2|b|\sqrt{a})\right) - \Phi\left(\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}\right)} \tag{4.1.8}$$

$$P_E = 1 - P_C = \int_{\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}}^{\infty} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} \Bigg/ \left(1 - \frac{1}{2}\int_{\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}\,(1-2|b|\sqrt{a})}^{\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}\right)$$

Once again we see that the performance of orthogonal and bipolar signals is obtained by multiplying $\left(\frac{S}{N}\right)$ by 2 and 4, respectively.

If $b = 0$, $K_1 = K_2$. Then

$$P_C = \Phi\left(\sqrt{\frac{1}{2}\left(\frac{S}{N}\right)}\right)$$

$$\begin{aligned} K_1 = K_2 &= E_s/2 \qquad \text{unipolar} \\ &= 0 \qquad \text{orthogonal or bipolar} \end{aligned} \tag{4.1.9}$$

or

$$P_E\Big|_{b=0} = \int_{\sqrt{\frac{1}{2}(\frac{S}{N})2^m}}^{\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \qquad \begin{array}{l} m = 0, \text{ unipolar} \\ m = 1, \text{ orthogonal} \\ m = 2, \text{ bipolar} \end{array} \qquad (4.1.10)$$

which is exactly the result of Chapter III, when b = 0.

Referring to the expression for P_E in (4.1.8) we may use (4.1.10) to express P_E as

$$P_E = \frac{P_E\Big|_{b=0}}{1 - \frac{1}{2}\left(\int_{(1-2|b|\sqrt{a})\sqrt{(S/N)/2}}^{\sqrt{(S/N)/2}} e^{-x^2/2}\, dx/\sqrt{2\pi} \right)} \qquad (4.1.11)$$

From this expression the following bounds on P_E are readily obtained.

$$P_E\Big|_{b=0} \leq P_E \leq 2\, P_E\Big|_{b=0} \qquad (4.1.12)$$

Furthermore, P_E is asymptotic from above to $P_E\Big|_{b=0}$ as $\frac{S}{N}$ goes to zero. If $|b|\sqrt{a} < \frac{1}{2}$ then P_E is asymptotic from above to $P_E\Big|_{b=0}$ and is bounded above by $\frac{4}{3}\, P_E\Big|_{b=0}$. Since $|b|\sqrt{a} < \frac{1}{2}$ in Chapter III, only the bounds of $P_E\Big|_{b=0}$ and $\frac{4}{3}\, P_E\Big|_{b=0}$ are presented in Figure 20.

4.2 The Probabilistic Switch

We now ask, if instead of having the switch of section 4.1 completely determined by D^{-1}, suppose the switch is probabilistically controlled. That is to say, if D^{-1} is one then the switch takes position 1 with probability ξ and position 2 with probability $(1-\xi)$. On the other hand

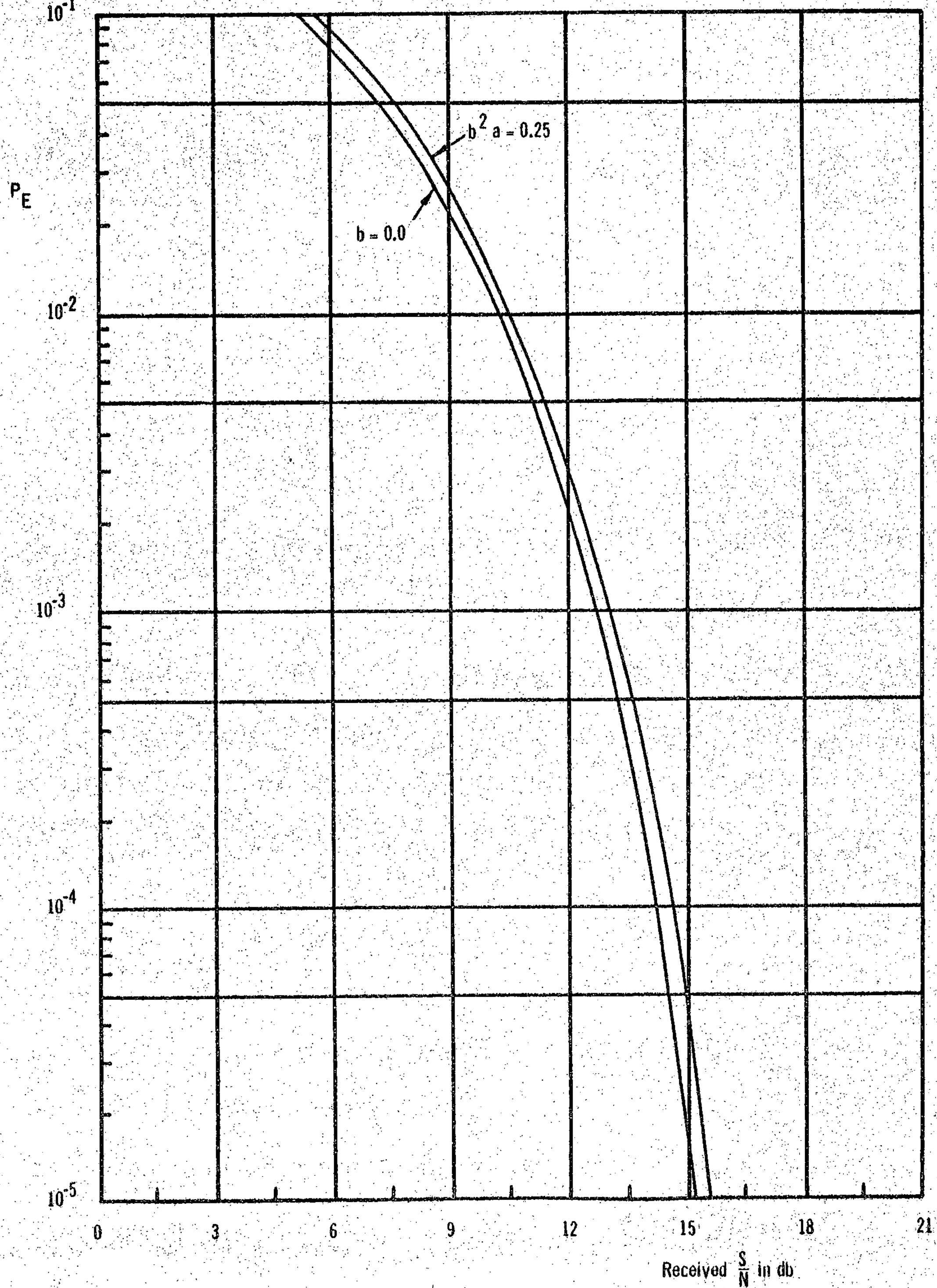


FIGURE 20 ERROR BOUNDS FOR SWITCHED MODE RECEIVER

if D^{-1} is two, position one is assumed with probability $(1-\xi)$ and position two with probability ξ. We might then choose that value of ξ which maximizes P_C.

To compute P_C we enumerate the mutually exclusive events leading to a success. Assuming $K_1 < K_2$ we have

Case I	$D^{-1} = 1$ correct and switch in position 1,
and (a)	s_1 sent, $V < K_1$; $\xi P_C(1-p)^2\ P_I(V < K_1/s_1)$
or (b)	s_2 sent, $V \geq K_1$; $\xi P_C p(1-p)\ P_I(V \geq K_1/s_2)$
Case I'	$D^{-1} = 1$ correct and switch in position 2
and (a')	s_1 sent, $V < K_1$; $(1-\xi)\ P_C(1-p)^2\ P_{IV}(V < K_1/s_1)$
or (b')	s_2 sent, $V \geq K_2$; $(1-\xi)\ P_C p(1-p)\ P_{IV}(V \geq K_2/s_2)$
Case II	$D^{-1} = 2$ correct and switch in position 2
and (a)	s_1 sent, $V < K_2$; $\xi P_C p(1-p)\ P_{II}(V < K_2/s_1)$
or (b)	s_2 sent, $V \geq K_2$; $\xi P_C p^2\ P_{II}(V \geq K_2/s_2)$
Case II'	$D^{-1} = 1$ correct and switch in position 1
and (a')	s_1 sent, $V < K_1$; $(1-\xi)\ P_C p(1-p)\ P_{III}(V < K_1/s_1)$
or (b')	s_2 sent, $V \geq K_2$; $(1-\xi)\ P_C p^2\ P_{III}(V \geq K_2/s_2)$
Case III	$D^{-1} = 1$ and incorrect and switch in position 1
and (a)	s_1 sent, $V < K_1$; $\xi(1-P_C)(1-p)^2\ P_{III}(V < K_1/s_1)$
or (b)	s_2 sent, $V \geq K_2$; $\xi(1-P_C)\ p(1-p)\ P_{III}(V \geq K_2/s_2)$
Case III'	$D^{-1} = 1$ and incorrect and switch in position 2
and (a')	s_1 sent, $V < K_2$; $(1-\xi)(1-P_C)(1-p)^2\ P_{II}(V < K_2/s_1)$
or (b')	s_2 sent, $V \geq K_2$; $(1-\xi)(1-P_C)\ p(1-p)\ P_{II}(V \geq K_2/s_2)$

Case IV $\quad D^{-1} = 2$ and incorrect and switch in position 2

and (a) $\quad s_1$ sent $\quad V < K_1;\ \xi(1-P_C)\ p(1-p)\ P_{IV}(V < K_1/s_1)$

or (b) $\quad s_2$ sent $\quad V \geq K_2;\ \xi(1-P_C)\ p^2\ P_{IV}(V \geq K_2/s_2)$

Case IV' $\quad D^{-1} = 2$ and incorrect and switch in position 1

and (a') $\quad s_1$ sent $\quad V < K_1;\ (1-\xi)(1-P_C)\ p(1-p)\ P_I(V < K_1/s_1)$

or (b') $\quad s_2$ sent $\quad V \geq K_1;\ (1-\xi)(1-P_C)\ p^2\ P_I(V \geq K_1/s_2)$

(For $K_2 > K_1$, interchange K_1 and K_2 in Cases III and IV.) The Roman numeraled conditional probabilities are identical to those found in section 4.1; namely, (4.1.5) and (4.1.5').

P_C is simply the sum of the above probabilities, and we may then solve for P_C in exactly the same fashion as in section 4.1. We do this for $p = \frac{1}{2}$ to obtain

$$P_C = \frac{2\Phi(W_1) + \xi\left[\Phi(W_1 - |W_3|) - \Phi(W_1)\right]}{2 + \Phi(W_1) - \Phi(W_1 - |W_3|) + 2\xi\left[\Phi(W_1 - |W_3|) - \Phi(W_1)\right]} \tag{4.2.1}$$

where W_1 and W_3 are defined in (4.1.6). Note that if $\xi = 1$ (4.2.1) reduces to (4.1.6). Differentiating (4.2.1) with respect to ξ yields

$$\frac{\partial P_C}{\partial \xi} = \frac{\left[\Phi(W_1) - \Phi(W_1 - |W_3|)\right]\left[3\Phi(W_1) + \Phi(W_1 - |W_3|) - 2\right]}{\left[2 + \Phi(W_1) - \Phi(W_1 - |W_3|) + 2\xi\left\{\Phi(W_1 - |W_3|) - \Phi(W_1)\right\}\right]^2} \tag{4.2.2}$$

Consequently the sign of $\frac{\partial P_C}{\partial \xi}$ is independent of ξ implying that $\xi = 1$ or 0, or P_C is invariant to ξ according to whether $\frac{\partial P_C}{\partial \xi}$ is greater than, less than, or equal to 0, respectively. Equivalently we need only examine the numerator of (4.2.2). Since Φ is a monotone increasing function we have that

$$\Phi(W_1) - \Phi(W_1 - |W_3|) > 0 \ ; \qquad W_3 \neq 0 \tag{4.2.3}$$

Whence, we need only examine the sign of the expression

$$g = 3\Phi(W_1) + \Phi(W_1 - |W_3|) - 2 \tag{4.2.4}$$

For the unipolar case (4.2.4) reduces to

$$g' = 3\Phi\left(\sqrt{\tfrac{1}{2}\left(\tfrac{S}{N}\right)}\right) + \Phi\left(\sqrt{\tfrac{1}{2}\left(\tfrac{S}{N}\right)}\,(1 - 2|b|\sqrt{a})\right) - 2 \tag{4.2.4'}$$

Now a crude sufficient condition that $\xi = 1$ or g' be greater than zero is easily derived by using (4.2.3) to obtain

$$g' > 4\Phi\left(\sqrt{\tfrac{1}{2}\left(\tfrac{S}{N}\right)}\,(1 - 2|b|\sqrt{a})\right) - 2 > 0 \tag{4.2.5}$$

or

$$1 - 2|b|\sqrt{a} > 0$$

or

$$b^2 a < \frac{1}{4}$$

and since $b^2 < 1$

$g' > 0$, if $a < 1/4$. These results are independent of $(\frac{S}{N})$ so this condition holds equally well for the orthogonal and bipolar cases. A sharper bound on (a) would involve the value of $(\frac{S}{N})$.

If $\frac{1}{2}(\frac{S}{N}) > 0.44$ or $(\frac{S}{N}) > 0.4$, then

$$3\Phi\left(\sqrt{\tfrac{1}{2}\left(\tfrac{S}{N}\right)}\right) - 2 > 0$$

and g' is guaranteed to be larger than zero. Similarly, for the orthogonal case $(\frac{S}{N}) > 0.2$ and bipolar case $(\frac{S}{N}) > 0.1$ guarantees $\xi = 1$. From an engineering point of view any system which operates at a $P_E > 10^{-1}$ ($\frac{S}{N} < 1$) is unacceptable so that, in this context, we would always choose $\xi = 1$ as is the case described in section 4.1.

4.3 Summary

In terms of the geometric model of 3.8 the switched mode scheme for the bipolar case can be depected as in Figure 21. Note that a simple translation and rotation of the coordinate axes generalizes to other than bipolar signals as in 3.8. Here it is easily seen that we have two vertical decision lines passing through K_1 and K_2, respectively. The strip between these lines defines the region in which the immediately preceding decision, D^{-1}, is used to generate the succeeding decision D.

From the relatively tight bounds on P_E in terms of $P_E\big|_{b=0}$, it appears that this receiver succeeds fairly well in its attempt to ignore the presence of the interfering pulse s_T. The smaller $b\sqrt{a}$, hence the narrower the strip between K_1 and K_2, the more successful is the receiver in obtaining the performance for $b = 0$.

There is yet to be discussed the transient problem associated with initiating communications with this receiver. As the previous discussion assumed a steady state behavior, the receiver must be turned on in a manner which insures the steady state behavior assumed. It is reasonable to expect that if the $\frac{S}{N}$ ratio is large we may use initially either decision level, K_1 or K_2 for our first transmission, as the probability of making the correct decision with either K_1 or K_2 is very close to one.

However, it would seem that the best way of initiating reception, independent of $\frac{S}{N}$, is to use the fact that there is no s_T present on

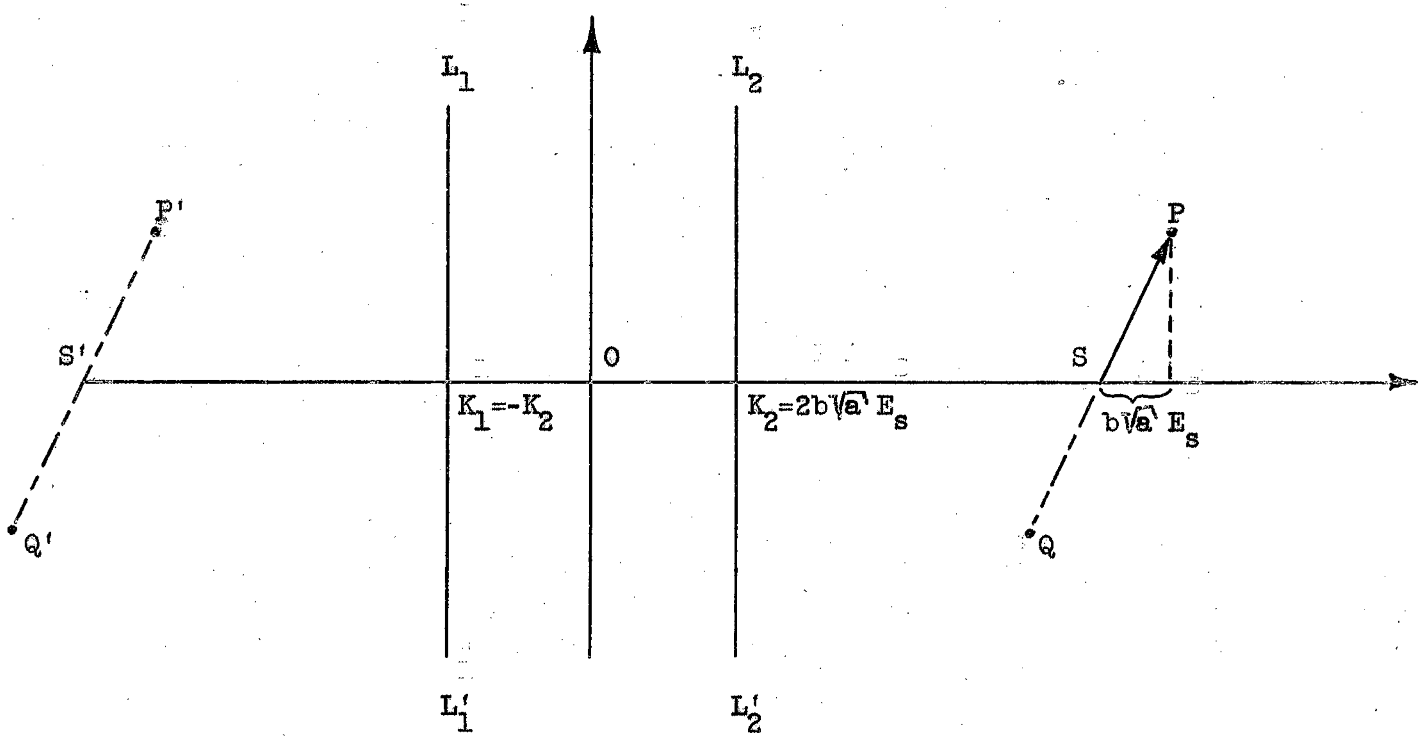


FIGURE 21 SWITCHED MODE RECEIVER SPACE

the first transmission. Hence, for this first T interval, the Bayes' receiver is the correlation receiver

$$\int_0^T s(t)v(t)dt$$

$$K = K_1 + \frac{K_2 - K_1}{2} \qquad (K_2 > K_1)$$

and this insures the highest probability that the first decision is made correctly.

This first Bayes' decision would then be used in conjunction with the two decision levels K_1 and K_2 on the second transmission and then operation would be as described in 4.1. Consequently in the practical design of this receiver three decision levels need to be incorporated besides a correlation filter, namely K_1, K_2 and a level midway between K_1 and K_2.

CHAPTER V

CHANNEL WITH FADING INTERSYMBOL INTERFERENCE

The next case to be examined involves the preceding channel model which also produces "fading" on the s_T portion of the signal. That is to say, instead of $s_T(t)$, $\sigma s_T(t)$ constitutes the interfering signal. In particular, σ is assumed to be a Rayleigh distributed random variable. A physical model for this formulation is a channel which has one spurious fading "multipath" in parallel with a direct communication link. We then seek the best correlation receiver to operate in this environment. Due to the computational complexity of the equations to be solved, they are presented without specific numerical results; furthermore, only the symmetric $(p = \frac{1}{2})$ bipolar case is examined.

5.1 Mathematical Assumptions and Associated Bayes' Receiver

The basic channel model of 3.1 with the following modifications is adopted

(1) $s_2(t) = - s_1(t) = s(t)$

$s_{T_2}(t) = - s_{T_1}(t) = s_T(t)$

(2) $s(t)$ is known perfectly at the receiver

$s_T(t) = \sigma s_T(t)$ where $s_T(t)$ is known perfectly at the receiver.

(3) σ is a random variable, statistically independent of the outcome of $s_i(t)$ and stationary over the T intervals, with the

probability density function

$$p_\sigma(\sigma) = \frac{\sigma}{d^2} e^{-\sigma^2/2d^2} \qquad \sigma \geq 0$$

$$= 0 \qquad \sigma < 0$$

where $d^2 = \frac{1}{2} E(\sigma)$; $\qquad d = \sqrt{\frac{2}{\pi}} E(\sigma)$

and E is the probability expectation operator.

(4) $d^2 < \frac{1}{2}$; in analogy to assumption (7) of Chapter III note that

$$E\left(\int_0^T \sigma^2 \, s_T^2(t)dt \right) = 2d^2 E_{s_T} < E_s$$

(5) The receiver is not to use any of its previous decisions.

The four possible combinations of signals formed from $\pm s(t) \pm \sigma s_T(t)$ constitute the equally probable combinations of received signals plus noise. The likelihood function Λ is then given as

$$\Lambda(v) = \frac{E\left[A_1 \exp(v \cdot (s-\sigma s_T)/N_o)\right] + E\left[A_2 \exp(v \cdot (s+\sigma s_T)/N_o)\right]}{E\left[A_1 \exp(-v \cdot (s-\sigma s_T)/N_o)\right] + E\left[A_2 \exp(-v \cdot (s+\sigma s_T)/N_o)\right]} \tag{5.1.1}$$

where

$$A_1 = \exp - \frac{1}{2N_o} \int_0^T \left[s(t) - \sigma s_T(t)\right]^2 dt$$

$$A_2 = \exp - \frac{1}{2N_o} \int_0^T \left[s(t) + \sigma s_T(t)\right]^2 dt$$

$$v \cdot f = \int_0^T v(t)f(t)dt$$

$$E\left[\cdot\right] = \int_0^\infty p_\sigma(\sigma)\left[\cdot\right] d\sigma$$

To perform the indicated expectations of (5.1.1) use of the following relation[7] will be made

$$\int_0^\infty \sigma \exp - (x\sigma^2 + 2y\sigma + z)d\sigma = \frac{1}{\sqrt{x}} \exp\left(\frac{y^2 - xz}{x}\right) \int_{y/\sqrt{x}}^\infty e^{-\sigma^2} \left(\frac{\sigma}{\sqrt{x}} - \frac{y}{x}\right) d\sigma$$

$$x > 0 \qquad (5.1.2)$$

To relate (5.1.2) to the Gaussian nature of the problem, define the integrated Gaussian distribution function Ⓗ as

$$Ⓗ(u) = \int_{-\infty}^{u} \left\{ \int_{-\infty}^{u} e^{-\nu^2/2} \frac{d\nu}{\sqrt{2\pi}} \right\} dv = \int_{-u}^{\infty} (v + u) e^{-v^2/2} \frac{dv}{\sqrt{2\pi}} \qquad (5.1.3)$$

Letting $\sigma = \nu/\sqrt{2}$, (5.1.2) becomes

$$\int_0^\infty \sigma \exp - (x\sigma^2 + 2y\sigma + z)d\sigma = \sqrt{\frac{\pi}{2}} \frac{Ⓗ\left(-\sqrt{\frac{2}{x}}\, y\right)}{x} \exp \frac{y^2 - xz}{x} \; ; \; x > 0 \qquad (5.1.4)$$

As an illustrative example, we compute

$$E \quad A_1 \exp(v_\theta(s - \sigma s_T)/N_o)$$

$$= \frac{1}{d^2} \int_0^\infty \sigma \exp - \left\{ \left(\frac{E_{s_T}}{2N_o} + \frac{1}{2d^2} \right) \sigma^2 - \left(\frac{\rho - v_\theta s_T}{N_o}\right)\sigma + \frac{E_s - 2v_\theta s}{N_o} \right\} d\sigma$$

where E_s, E_{s_T} and ρ are as defined in Chapter III. Referring to (5.1.4), it follows that

$$x = \frac{d^2 E_{s_T} + N_o}{2N_o d^2} \; ; \quad -y = \frac{\rho - v_\theta s_T}{2N_o} \; ; \quad z = \frac{E_s - 2v_\theta s}{2N_o}$$

and from (5.1.4)

$$E\left[A_1\exp(v\sigma(s-\sigma s_T)/N_o)\right]$$

$$=\sqrt{2\pi}\,\frac{\overline{\sigma}^2}{\lambda^2}\,Ⓗ\left\{\sqrt{\frac{\overline{\sigma}^2}{2\lambda^2}}\,\frac{(\rho-v\sigma s_T)}{N_o}\right\}\exp\left(\frac{\overline{\sigma}^2}{2N_o^2\lambda^2}\,(v\sigma s_T-\rho)^2-\frac{(E_s-2v\sigma s)}{2N_o}\right) \tag{5.1.5}$$

where $\overline{\sigma}^2 = E(\sigma^2) = 2d^2$

$\overline{\sigma} = E(\sigma) = \sqrt{\frac{\pi}{2}}\,d$

$\lambda^2 = \frac{\overline{\sigma}^2}{2}\frac{E_{s_T}}{N_o}+1$

Mathematically we may redefine $\overline{\sigma}^2 = 2d^2$ so that $E_{s_T} = E_s$; that is to say

$$\left[\overline{\sigma}^2\right] = \frac{\overline{\sigma}^2}{E_{s_T}}$$

We label $\left[\overline{\sigma}^2\right]$ as $\overline{\sigma}^2$. Letting, as before,

$$\left(\frac{S}{N}\right) = \frac{E_s}{2N_o}$$

Then;

$$\lambda^2 = 1+\overline{\sigma}^2\left(\frac{S}{N}\right)$$

Computing the expectations indicated in (5.1.1) produces

(a) $E\left[A_1\exp(v\sigma(s-\sigma s_T)/N_o)\right]$

$$=\sqrt{2\pi}\,\frac{\overline{\sigma}^2}{2\lambda^2}\,Ⓗ\left[\sqrt{\frac{\overline{\sigma}^2}{2\lambda^2}}\,\frac{(\rho-v\sigma s_T)}{N_o}\right]\exp\left\{\frac{\overline{\sigma}^2}{2\lambda^2}\left(\frac{v\sigma s_T-\rho}{N_o}\right)^2-\left(\frac{S}{N}\right)+\frac{v\sigma s}{N_o}\right\} \tag{5.1.6}$$

(b) $E\left[A_1\exp-(v\sigma(s-\sigma s_T)/N_o)\right]$

$$=\sqrt{2\pi}\,\frac{\overline{\sigma}^2}{2\lambda^2}\,Ⓗ\left[\sqrt{\frac{\overline{\sigma}^2}{2\lambda^2}}\,\frac{(\rho+v\sigma s_T)}{N_o}\right]\exp\left\{\frac{\overline{\sigma}^2}{2\lambda^2}\left(\frac{v\sigma s_T+\rho}{N_o}\right)^2-\left(\frac{S}{N}\right)-\frac{v\sigma s}{N_o}\right\}$$

(c) $E\left[A_2 \exp(v\sigma(s + \sigma s_T)/N_o)\right]$

$$= \sqrt{2\pi}\,\frac{\bar{\sigma}^2}{2\lambda^2}\,\textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2\lambda^2}}\frac{(v\sigma s_T - \rho)}{N_o}\right] \exp\left\{\frac{\bar{\sigma}^2}{2\lambda^2}\left(\frac{v\sigma s_T - \rho}{N_o}\right)^2 - \left(\frac{S}{N}\right) + \frac{v\sigma s}{N_o}\right\}$$

(5.1.6)
(cont.)

(d) $E\left[A_2 \exp - (v\,(s - \sigma s_T)/N_o\right]$

$$= \sqrt{2\pi}\,\frac{\bar{\sigma}^2}{2\lambda^2}\,\textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2\lambda^2}}\frac{(-v\sigma s_T - \rho)}{N_o}\right] \exp\left\{\frac{\bar{\sigma}^2}{2\lambda^2}\left(\frac{v\sigma s_T + \rho}{N_o}\right)^2 - \left(\frac{S}{N}\right) - \frac{v\sigma s}{N_o}\right\}$$

After dividing common factors, (5.1.1) may be expressed as

$\Lambda(v) =$

$$\exp\left[\frac{2v\sigma s}{N_o} - 2\gamma b\left(\frac{S}{N}\right)\frac{v\sigma s_T}{N_o}\right] \frac{\textcircled{H}\left[\sqrt{\gamma}\left(b\,\frac{S}{N} - \frac{v\sigma s_T}{N_o}\right)\right] + \textcircled{H}\left[-\sqrt{\gamma}\left(b\,\frac{S}{N} - \frac{v\sigma s_T}{N_o}\right)\right]}{\textcircled{H}\left[\sqrt{\gamma}\left(b\,\frac{S}{N} + \frac{v\sigma s_T}{N_o}\right)\right] + \textcircled{H}\left[-\sqrt{\gamma}\left(b\,\frac{S}{N} + \frac{v\sigma s_T}{N_o}\right)\right]} \tag{5.1.7}$$

where

$$\gamma = \frac{\bar{\sigma}^2}{2\lambda^2}$$

$$\rho = b\left(\frac{S}{N}\right) \qquad -1 < b < +1$$

We note the presence in (5.1.7) of only the even part of $\textcircled{H}$ ($\textcircled{H}_e$), if the numerator and denominator are multiplied by 1/2. The even part of $\textcircled{H}$ is given by

$$\textcircled{H}_e = \frac{1}{2}\left[\textcircled{H}(u) + \textcircled{H}(-u)\right] = \frac{1}{2}\int_{-u}^{\infty}(v+u)e^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}} + \frac{1}{2}\int_{u}^{\infty}(v-u)e^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}}$$

$$= \frac{1}{2}\int_{-u}^{\infty}e^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}} + \frac{1}{2}\int_{u}^{\infty}ve^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}} + \frac{1}{2}u\int_{-u}^{\infty}e^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}} - \frac{1}{2}u\int_{u}^{\infty}e^{-v^2/2}\,\frac{dv}{\sqrt{2\pi}},$$

which reduces to

$$\text{(H)}_e(u) = |u| \int_0^{|u|} e^{-v^2/2} \frac{dv}{\sqrt{2\pi}} + \frac{e^{-u^2/2}}{\sqrt{2\pi}} \qquad (5.1.8)$$

Taking the logarithm of (5.1.7) results in

$$\ln \Lambda(v) = \frac{2}{N_o} v \circ s - \frac{2\gamma b}{N_o} \left(\frac{S}{N}\right) v \circ s_T$$

$$+ \ln \text{(H)}_e \left[\sqrt{\gamma} \frac{v \circ s_T}{N_o} - \gamma b \left(\frac{S}{N}\right) \right]$$

$$- \ln \text{(H)}_e \left[\sqrt{\gamma} \frac{v \circ s_T}{N_o} + \gamma b \left(\frac{S}{N}\right) \right] \qquad (5.1.9)$$

Then $\ln \Lambda(v)$ given by (5.1.9) is compared to the decision level

$$\ln \frac{1-p}{p} = 0 \qquad (5.1.10)$$

Taken together, (5.1.9) and (5.1.10) define the optimum Bayes' receiver for this channel model. The essential data operations are again correlation of the input data $v(t)$ with $s(t)$ and $s_T(t)$ followed by the indicated $\ln \text{(H)}_e$ functions. The block diagram of such a receiver is given in Figure 22. Needless to say that the possibility of a more tractable form of the statistic is quite remote and, except for the limiting case, the performance of this receiver is quite unpredictable. So once again we will utilize the philosophy of pre-establishing a receiver class within which a best receiver is sought. In particular, since the pertinent operation is correlation, we choose to examine the class of correlation receivers.

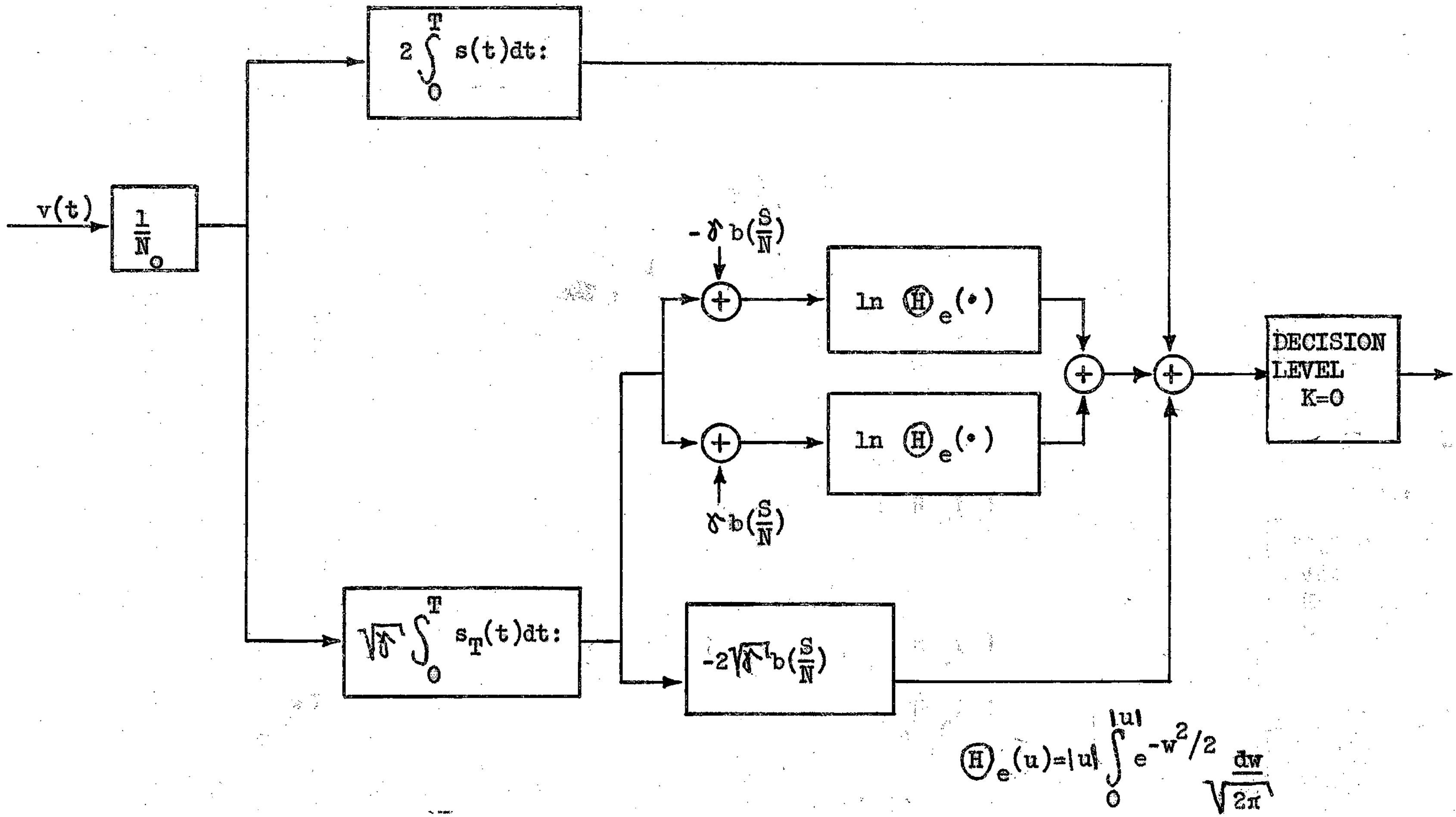


FIGURE 22 BIPOLAR BAYES' RECEIVER FOR FADING PULSE INTERFERENCE

5.2 Formulation of the Correlation Class

Let $(K,h(t))$ be a given correlation receiver. That is to say, the number V is formed as

$$V = \int_0^T h(t)v(t)dt$$

and compared to the decision level K,

$$\begin{aligned} V &\geq K \qquad \text{say } 2 \\ V &< K \qquad \text{say } 1 \end{aligned} \tag{5.2.2}$$

Then if σ is assumed known, the conditional probability of correct reception, $P_{C/\sigma}$ for the bipolar symmetric case is easily derived, as in the previous chapters, to be

$$P_{C/\sigma} = \frac{1}{4}\int_{-\infty}^{K} \exp - \frac{(V + h\circ(s+\sigma s_T))^2}{2N_oE_h} + \exp - \frac{(V + h\circ(s - \sigma s_T))^2}{2N_oE_h} \quad \frac{dV}{\sqrt{2\pi N_oE_h}}$$

$$+ \frac{1}{4}\int_{K}^{\infty} \exp - \frac{(V - h\ (s - \sigma s_T))^2}{2N_oE_h} + \exp - \frac{(V - h\circ(s+\sigma s_T))^2}{2N_oE_h} \quad \frac{dV}{\sqrt{2\pi N_oE_h}} \tag{5.2.3}$$

where

$$h\circ(s \pm \sigma s_T) = \int_0^T h(t)\left[\, s(t) \pm \sigma s_T(t)\right]dt$$

$$E_h = \int_0^T h^2(t)dt$$

It then follows from the definition of conditional probability that the average rate of correct reception P_C is found from

$$P_C = E\ P_{C/\sigma} = \int_0^\infty p_\sigma(\sigma)\ P_{C/\sigma}\ du \tag{5.2.4}$$

Operating on (5.2.3) with E and interchanging E with the indicated integrals of (5.2.3) produces integrands of the form

$$E\left(\exp - \frac{\left[V + h\sigma(\pm s \pm \sigma s_T)\right]^2}{2N_oE_h}\right) \tag{5.2.5}$$

The implied expectation integrals of (5.2.5) can be evaluated by means of relation (5.1.4). Namely,

$$E\left(\exp - \frac{\left[V + h\sigma(\pm s \pm \sigma s_T)\right]^2}{2N_oE_h}\right)$$

$$= \frac{\sqrt{2\pi}}{\eta^2}\ \textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\ \frac{(\mp h\sigma s_T)}{N_oE_h}\ (V \pm h\sigma s)\right] \exp - \frac{(V \pm h\sigma s)^2}{2\eta^2 N_oE_h}$$

where

$$\eta^2 = 1 + \frac{\bar{\sigma}^2 (h\sigma s_T)^2}{2N_oE_h}$$

P_C is then given as the sum of four integrals of the form

$$\int_{-\infty}^{K} \text{ or } \int_K^\infty \textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\ \frac{(\mp h\ s_T)}{N_oE_h}\ (V \pm h\sigma s)\right] \exp\left(-\frac{(V \pm h\ s)^2}{2\eta^2 N_oE_h}\right)\frac{dV}{\eta^2\sqrt{N_oE_h}} \tag{5.2.7}$$

which we normalize with the substitution

$$w = \frac{V \pm h\sigma s}{\sqrt{\eta^2 N_oE_h}}$$

(5.2.7) is then reduced to

$$\int_{-\infty}^{\frac{K \pm h\sigma s}{\sqrt{\eta^2 N_o E_h}}} \text{ or } \int_{\frac{K \pm h\sigma s}{\sqrt{\eta^2 N_o E_h}}}^{\infty} \textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}\,(\pm h\sigma s_T)w\right] e^{-w^2/2}\,\frac{dw}{\sqrt{\eta^2}} \qquad (5.2.7)'$$

Substituting (5.2.7)' into (5.2.4) yields P_C as

$$P_C = \frac{1}{4\sqrt{\eta^2}} \int_{-\infty}^{\frac{K + h\sigma s}{\sqrt{\eta^2 N_o E_h}}} \left\{\textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}(h\sigma s_T)w\right] + \textcircled{H}\left[-\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}(h\sigma s_T)w\right]\right\} e^{-w^2/2}\,dw$$

$$+ \frac{1}{4\sqrt{\eta^2}} \int_{\frac{K - h\sigma s}{\sqrt{\eta^2 N_o E_h}}}^{\infty} \left\{\textcircled{H}\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}(h\sigma s_T)w\right] + \textcircled{H}\left[-\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}(h\sigma s_T)w\right]\right\} e^{-w^2/2}\,dw \qquad (5.2.8)$$

Once again only the even part of $\textcircled{H}$, $\textcircled{H}_e$, is involved in determining P_C, so that P_C may be expressed as

$$P_C = \frac{1}{2\sqrt{\eta^2}} \int_{-\infty}^{\frac{K + h\sigma s}{\sqrt{\eta^2 N_o E_h}}} \textcircled{H}_e\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}\,(h\sigma s_T)w\right] e^{-w^2/2}\,dw$$

$$+ \frac{1}{2\sqrt{\eta^2}} \int_{\frac{K - h\sigma s}{\eta^2 N_o E_h}}^{\infty} \textcircled{H}_e\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}\,(h\sigma s_T)w\right] e^{-w^2/2}\,dw \qquad (5.2.9)$$

where

$$\bar{\sigma}^2 = E(\sigma^2)$$

$$h{\circ}f = \int_0^T h(t)f(t)dt$$

$$E_h = h{\circ}h$$

$$\eta^2 = 1 + \frac{\bar{\sigma}^2 (h{\circ}s_T)^2}{2N_o E_h}$$

Now if the receiver $(K, h(t))$ is scaled by λ to $(\lambda K, \lambda h(t))$, examination of (5.2.9) shows that

$$\lambda h{\circ}s_T \sqrt{\frac{\bar{\sigma}^2}{2N_o \lambda^2 E_h}} = h{\circ}s_T \sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}$$

$$1 + \frac{\bar{\sigma}^2 (\lambda h{\circ}s_T)^2}{2N_o \lambda^2 E_h} = 1 + \frac{\bar{\sigma}^2 (h{\circ}s_T)^2}{2N_o E_h} = \eta^2$$

$$\frac{\lambda K \pm \lambda h{\circ}s}{\sqrt{\eta^2 N_o \lambda^2 E_h}} = \frac{K \pm h{\circ}s}{\sqrt{\eta^2 N_o E_h}}$$

and P_C is invariant to the gain of the receiver. Thus P_C satisfies the assumptions of section 2.1, and since P_C is determined by the two linear functionals $h{\circ}s$ and $h{\circ}s_T$ we may conclude that the optimum $h(t)$ has the form

$$h(t) = cs(t) + c_T s_T(t)$$

Substituting this $h(t)$ into (5.2.9) and assuming we scale $\bar{\sigma}^2$ such that $E_{s_T} = E_s$, P_C is given by

$$P_C = \frac{1}{2\sqrt{\eta^2}} \int_{-\infty}^{\frac{K + cE_s + c_T\rho}{\sqrt{\eta^2 N_o E_h}}} \textcircled{H}_e \left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}} (c\rho + c_T E_s) w \right] e^{-w^2/2} \, dw$$

$$+ \frac{1}{2\sqrt{\eta^2}} \int_{\frac{K - cE_s - c_T\rho}{\eta^2 N_o E_h}}^{\infty} \textcircled{H}_e \left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}} (c\rho + c_T E_s) w \right] e^{-w^2/2} \, dw \qquad (5.2.10)$$

with

$$\bar{\sigma}^2 = E(\sigma^2)$$

$$E_h = (c^2 + c_T^2)E_s + 2cc_T\rho; \quad \rho = \int_0^T s(t)s_T(t)dt$$

$$\eta^2 = 1 + \frac{\bar{\sigma}^2}{2N_o E_h} (c\rho + c_T E_s)^2$$

We may now use the system of equations (2.2.5) to obtain optimal values of K^*, c^* and c_T^* such that $E_h = E$ is fixed and in particular E is that value for which $c^* = 1$ (assuming $c^* \neq 0$). Now for any particular achieved value σ_o of σ, the results of Chapter III state that K^* for σ_o is equal to zero and this value of K^* is independent of σ_o. Thus we would expect K^* to be zero for the case at hand. Now,

$$P_K = \frac{\partial P_C}{\partial K} = \frac{1}{2\sqrt{\eta^2}} \frac{1}{\sqrt{\eta^2 N_o E_h}} \left\{ \mathcal{R}_e(K + cE_s + c_T\rho) - \mathcal{R}_e(K - cE_s - c_T\rho) \right\} \qquad (5.2.11)$$

and $\mathcal{R}_e(u)$ is the even function

$$\mathcal{R}e\,(u) = \textcircled{H}_e\left[\sqrt{\frac{\bar{\sigma}^2}{2N_oE_h}}\,\frac{(c\rho + c_TE_s)u}{\sqrt{\eta^2N_oE_h}}\right]\exp - \frac{u^2}{2\eta^2N_oE_h}$$

Since $\mathcal{R}e$ is an even function, for any fixed value of c and c_T, $P_K = 0$ is solved for $K = K^* = 0$ independently of the c, c_T choice and so $P_K^* = 0$ is solved for $K^* = 0$. Furthermore, as shown in Appendix C, $K^* = 0$ is the unique solution, in the bipolar symmetric case, to $P_K = 0$ for any acceptable probability density function $p_\sigma(\sigma)$.

Setting $K = 0$ in (5.2.10) and letting $w' = -w$ in the second integral of (5.2.10) produces for P_C (noting $\textcircled{H}_e(-u) = \textcircled{H}_e(u)$)

$$P_C = \frac{1}{\sqrt{\eta^2}}\int_{-\infty}^{\frac{cE_s + c_T\rho}{\sqrt{\eta^2N_oE_h}}} \textcircled{H}_e\left[\sqrt{\frac{\bar{\sigma}^2}{2N_oE_h}}\,(c\rho + c_TE_s)w\right]e^{-w^2/2}\,dw$$

The necessary equations for maximizing (5.2.12) subject to holding $E_h = E$ fixed are then obtained from (2.2.5) as

$$\begin{aligned}
&(a)\quad P_c^* + 2\alpha(E_sc^* + \rho c_T^*) = 0\\
&(b)\quad P_{c_T}^* + 2\alpha(\rho c^* + E_sc_T^*) = 0 \qquad\qquad (5.2.13)\\
&(c)\quad (c^2 + c_T^2)E_s + 2cc_T\rho = E
\end{aligned}$$

where α is the Lagrange multiplier. It is easily verified that (5.2.13) can be put into the form

$$\begin{aligned}
&(a)\quad 2\alpha c^* = -(E_sP_c^* - \rho P_{c_T}^*)/(E_s^2 - \rho^2)\\
&(b)\quad 2\alpha c_T^* = -(E_sP_{c_T}^* - \rho P_c^*)/(E_s^2 - \rho^2)
\end{aligned} \qquad (5.2.14)$$

(c) $(c^2 + c_T^2)E_s + 2cc_T\rho = E$ (5.2.14) (cont.)

since by Schwartz's lemma $E_s^2 - \rho^2 > 0$.

We next evaluate P_c, P_{c_T}, the partial derivatives of P_C, with E_h held fixed. Put

$$\psi = c\rho + c_T E_s = h\bullet s_T$$

$$\phi = cE_s + c_T\rho = h\bullet s \qquad (5.2.15)$$

$$\eta^2 = 1 + \frac{\bar{\sigma}^2}{2N_o E}\psi^2$$

so that

$$P_C = (\eta^2)^{-1/2} \int_{-\infty}^{\phi/\sqrt{\eta^2 N_o E}} \textcircled{H}_e \left[\psi \sqrt{\frac{\bar{\sigma}^2}{2N_o E}}\, w\right] e^{-w^2/2}\, dw$$

Let ξ represent either c or c_T, so that

$$\left.\frac{\partial P_C}{\partial \xi}\right|_E = -\frac{\bar{\sigma}^2}{2\eta^2 N_o E}\psi\,\psi_\xi\, P_C$$

$$+ \frac{\left[\phi_\xi - \dfrac{\bar{\sigma}^2\phi\psi\psi_\xi}{2\eta^2 N_o E}\right]}{\eta^2\sqrt{N_o E}} \textcircled{H}_e \left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\,\frac{\phi\psi}{N_o E}\right] e^{-\phi^2/2\eta^2 N_o E}$$

$$+ \sqrt{\frac{\bar{\sigma}^2}{2\eta^2 N_o E}}\,\psi_\xi \int_{-\infty}^{\phi/\sqrt{\eta^2 N_o E}} w e^{-w^2/2}\, \textcircled{H}_e' \left[\psi\sqrt{\frac{\bar{\sigma}^2}{2N_o E}}\, w\right] dw$$

$$\xi = c,\ c_T$$

(5.2.16)

where

$$\textcircled{H}_e'(a) = \frac{d}{du}\,\textcircled{H}_e(u)\Big|_{u=a}$$

$$\psi_\xi \quad \begin{cases} \psi_c = \rho \\ \psi_{c_T} = E_s \end{cases}$$

$$\phi_\xi \quad \begin{cases} \phi_c = E_s \\ \phi_{c_T} = \rho \end{cases}$$

Using (5.2.15), $2\alpha c^*$ may be computed from (5.2.14a) as

$$2\alpha c^* = -\frac{1}{\eta^2\sqrt{N_oE}}\,\textcircled{H}_e\left[\sqrt{\frac{\overline{\sigma}^2}{2\eta^2}}\,\frac{\phi\psi}{N_oE}\right]\exp(-\phi^2/2\eta^2N_oE)\Bigg|_{\substack{c=c^*\\ c_T=c_T^*}} \tag{5.2.17}$$

Now from the definition of $\textcircled{H}_e$, (5.1.8), it follows that $\textcircled{H}_e$ is a strictly positive function, so that for any value of c, c_T and in particular c^*, c_T^* the right hand side of (5.2.17) is less than zero. Consequently, neither α nor c^* are zero and we may set c^* equal to one. Note that E is now considered a function of c_T^*. For $c^* = 1$, c_T^* may be solved from

$$c_T^* = \left\{\frac{(E_sP^*_{c_T} - \rho P^*_c)/(E_s^2 - \rho^2)}{\dfrac{1}{\eta^2\sqrt{N_oE}}\,\textcircled{H}_e\left[\sqrt{\dfrac{\overline{\sigma}^2}{2\eta^2}}\,\dfrac{\phi\psi}{N_oE}\right]\exp(-\phi^2/2\eta^2N_oE)}\right\}_{\substack{c^*=1\\ c_T=c_T^*}} \tag{5.2.18}$$

Once again using (5.2.15) it is easily established that

$$\frac{E_s P_{c_T} - \rho P_c}{E_s^2 - \rho^2} = \frac{\bar{\sigma}^2 \psi}{2\eta^2 N_o E} P_C + \frac{\bar{\sigma}^2 \phi\psi}{2(\eta^2)^2 (N_o E)^{3/2}} Ⓗ_e\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}} \frac{\phi\psi}{N_o E}\right] \exp(-\phi^2/2\eta^2 N_o E)$$

$$- \sqrt{\frac{\bar{\sigma}^2}{2\eta^2 N_o E}} \int_{-\infty}^{\phi/\sqrt{\eta^2 N_o E}} Ⓗ_e'\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E}} \psi w\right] e^{-w^2/2} w \, dw \qquad (5.2.19)$$

The integral appearing in (5.2.19) may be integrated by parts, producing

$$\frac{E_s P_{c_T} - \rho P_c}{E_s^2 - \rho^2} = \frac{\bar{\sigma}^2 \psi}{2\eta^2 N_o E} P_C + \frac{\bar{\sigma}^2 \phi\psi}{2(\eta^2)^2 (N_o E)^{3/2}} Ⓗ_e\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}} \frac{\phi\psi}{N_o E}\right] \exp(-\phi^2/2\eta^2 N_o E)$$

$$+ \sqrt{\frac{\bar{\sigma}^2}{2\eta^2 N_o E}} Ⓗ_e'\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}} \frac{\psi\phi}{N_o E}\right] \exp(-\phi^2/2\eta^2 N_o E)$$

$$- \frac{\bar{\sigma}^2}{2\sqrt{\eta^2 N_o E}} \psi \int_{-\infty}^{\phi/\sqrt{\eta^2 N_o E}} Ⓗ_e''\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E}} \psi w\right] e^{-w^2/2} \, dw \qquad (5.2.20)$$

From the definition of $Ⓗ_e(u)$

$$Ⓗ_e(u) = |u| \int_0^{|u|} e^{-v^2/2} \frac{dv}{\sqrt{2\pi}} + \frac{e^{-u^2/2}}{\sqrt{2\pi}} \qquad (5.1.8)$$

we obtain

$$Ⓗ_e'(u) = \int_{-\infty}^{u} e^{-v^2/2} \frac{dv}{\sqrt{2\pi}} - \frac{1}{2}$$

$$Ⓗ_e''(u) = \frac{e^{-u^2/2}}{\sqrt{2\pi}} \qquad (5.2.21)$$

$$\frac{E_s P_c c_T - \rho P_c}{E_s^2 - \rho^2} = \frac{\bar{\sigma}^2 \psi}{2\eta^2 N_o E} P_C + \frac{\bar{\sigma}^2 \phi\psi}{2(\eta^2)^2 (N_o E)^{3/2}} \textcircled{H}_e\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\,\frac{\phi\psi}{N_o E}\right] \exp(-\phi^2/2\eta^2 N_o E)$$

$$+\sqrt{\frac{\bar{\sigma}^2}{2\eta^2 N_o E}}\; \textcircled{H}_e'\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\,\frac{\psi\phi}{N_o E}\right] \exp(-\phi^2/2\eta^2 N_o E)$$

$$- \frac{\bar{\sigma}^2 \psi}{2\eta^2 N_o E} \int_{-\infty}^{\phi/\sqrt{N_o E}} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}} \qquad (5.2.22)$$

Whence, (5.2.18) together with (5.2.15), (5.1.8), (5.2.21), and (5.2.22) serve to establish the equation from which c_T^* is solved. Furthermore this equation involves only the well known Gaussian density and distribution functions.

Consider (5.2.18) in conjunction with (5.2.22). Suppose we change the sign of ρ (i.e., make $s_T(t) = - s_T(t)$). Letting $\rho' = - \rho$ and $c_T' = - c_T$, the following is easily verified

(i) $\phi' = E_s + c_T'\rho' = E_s + c_T\rho = \phi$

(ii) $\psi' = \rho' + c_T'E_s = -(\rho + c_T E_s) = - \psi$

(iii) η^2, E remain unchanged

(iv) P_C is even, $\textcircled{H}_e'$ is odd on ψ

Then, with regard to (5.2.22) replacing ϕ and ψ with ϕ' and ψ' is equivalent to multiplying (5.2.22) by minus one. Consequently it follows that $c_T^{*\prime} = - c_T^*$ solves (5.2.18) for $\rho' = - \rho$. So, as in Chapter III, $|c_T^*|$ is determined by $|\rho|$ and P_C is ultimately a function of $\bar{\sigma}^2$, E_s, N_o and $|\rho|$. Numerical results are not computed as the necessary solution techniques and computer time required do not seem justified for the purposes of this investigation.

5.3 Summary

In section 5.1 the ideal Bayes' receiver is developed, (5.1.9), for the bipolar symmetric signaling scheme with randomly fading (Rayleigh) but otherwise specified intersymbol interference. A glance at Figure 21 shows the complexity involved in realizing this receiver. Moreover, any attempt at computing the performance of this receiver should prove difficult in the extreme.

Section 5.2 examines the performance of the correlation receivers in this environment. The formal development is quite similar to that of Chapter III. However, we may not make the transition to the unipolar and orthogonal signaling situations with such dispatch as the precise value of σ is needed to utilize the technique used in Chapter III. We see though that for bipolar symmetric signaling the optimal decision level K^* is zero for any type of stationary, statistically independent fading. Physically speaking we would expect this result to carry over to the orthogonal scheme also and indeed such is the case as is shown in Appendix D. For the case of Rayleigh fading, the value of c_T^* involves the solution of an equation, (5.2.18), of an intrinsically Gaussian nature. So in a sense, we have traded an enormously difficult numerical analysis problem associated with the Bayes' receiver, which is of questionable practical realizability, for a much more tractable numerical problem associated with the correlation receiver, whose fabrication has already been achieved.

CHAPTER VI

SWITCHED MODE RECEIVER, FADING CHANNEL

In a fashion analogous to Chapter IV, we will examine the channel model of Chapter V, where the receiver is designed assuming its past decision was correct with probability one. In actuality we have two receivers operating simultaneously on the incoming data; the choice of which receiver decision is to be accepted is predetermined by the immediately preceding decision (ref. Figure 7, Chapter IV). Once the individual receivers have been decided on and their probability law (conditioned as in Chapter IV) established, the overall system performance is solved algebraically in exactly the manner of Chapter IV.

6.1 Switched Mode Bayes' Receiver

The Bayes' receiver is next derived wherein the assumption (hence conditioning) is made that the immediately preceding decision was absolutely correct. With the assumptions of Chapter V, excepting (5), and the notation of Chapters IV and V, we need to design two Bayes' receivers, corresponding to D^{-1} equal to one and two.

Case (1) $D^{-1} = 1$

For $D^{-1} = 1$, the receiver presumes that it must distinguish between $s(t) - \sigma s_T(t)$ and $-s(t) - \sigma s_T(t)$. Consequently, the likelihood function $\Lambda(v/D^{-1} = 1)$ is given by

$$\Lambda(v/D^{-1} = 1) = \frac{E\left[A_1 \exp(v\circ(s - \sigma s_T)/N_o\right]}{E\left[A_2 \exp(-v\circ(s + \sigma s_T)/N_o\right]} \tag{6.1.1}$$

where the notation of (6.1.1) is explained in (5.1.1).

Use of (5.1.6a and d) evaluates (6.1.1) as

$$\Lambda(v/D^{-1} = 1) = \frac{Ⓗ\left[+\sqrt{\gamma}\,(b(\frac{S}{N}) - \frac{v\circ s_T}{N_o})\right]}{Ⓗ\left[-\sqrt{\gamma}\,(b(\frac{S}{N}) + \frac{v\circ s_T}{N_o})\right]} \exp\left(2\frac{v\circ s}{N_o} - 2\gamma b(\frac{S}{N})\frac{v\circ s_T}{N_o}\right) \tag{6.1.2}$$

where the notation is that of (5.1.7).

Case (2)

Here the decision is between $s(t) + \sigma s_T(t)$ and $-s(t) + \sigma s_T(t)$. In exactly the same manner as Case (1) and with reference to (5.1.6b and c), $\Lambda(v/D^{-1} = 2)$ is obtained as

$$(v/D^{-1} = 2) = \frac{Ⓗ\left[-\sqrt{\gamma}\,(b(\frac{S}{N}) - \frac{v\circ s_T}{N_o})\right]}{Ⓗ\left[\sqrt{\gamma}\,(b(\frac{S}{N}) + \frac{v\ s_T}{N_o})\right]} \exp\left(\frac{2v\circ s}{N_o} - 2\gamma b(\frac{S}{N})\frac{v\circ s_T}{N_o}\right) \tag{6.1.3}$$

Thus, (6.1.2) and (6.1.3) demonstrate that the utilization of a switched mode in a sense separates the H 's of (5.1.7). From an analytical point of view we are in no better position than that of (5.1.9) for we have merely traded the even part of H for H itself. Consequently, we end the discussion of the switched mode Bayes' receiver.

6.2 Switched Mode Correlation Receiver

In a manner analogous to 6.1 two optimal correlation receivers, conditioned by D^{-1} are derived.

Case (1) $D^{-1} = 1$

The receiver assumes its decision is between $s(t) - \sigma s_T(t)$ and $-s(t) - \sigma s_T(t)$. Letting $(K,h(t))$ be the correlation receiver for $D^{-1} = 1$, we have from the development (5.2.1) through (5.2.8) that $P_C(h,K/D^{-1} = 1) = P_C(1)$ is given by

$$P_C(1) = \sqrt{\frac{\pi}{2\eta^2}} \int_{-\infty}^{(K+h\circ s)/\sqrt{\eta^2 N_o E_h}} Ⓗ \left(h\circ s_T \sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}\, w\right) e^{-w^2/2} \frac{dw}{\sqrt{2\pi}}$$

$$+ \sqrt{\frac{\pi}{2\eta^2}} \int_{(K-h\circ s)/\sqrt{\eta^2 N_o E_h}}^{\infty} Ⓗ \left(h\circ s_T \sqrt{\frac{\bar{\sigma}^2}{2N_o E_h}}\, w\right) e^{-w^2/2} \frac{dw}{\sqrt{2\pi}} \qquad (6.2.1)$$

Strictly speaking, we should subscript $h(t)$ and K as $h_1(t)$ and K_1 to separate them from the receiver corresponding to $D^{-1} = 2$. As we will deal with $D^{-1} = 2$ in a summary manner, there will be no confusion if the subscripts are deleted.

The reader's attention is drawn to the similarity of (6.2.1) to (5.2.8). The difference between the two lies in omitting the Ⓗ's of (5.2.8) which have minus signs in their argument. Since the case $D^{-1} = 2$ corresponds to using $-s_T(t)$, $P_C(2)$ is of the exact form as $P_C(1)$, except that minus signs need be inserted in the Ⓗ arguments. There is only a small difference "formally" in the necessary conditions on (K^*, c^*, c_T^*) from those of 5.2. K^* cannot be solved independently of (c^*, c_T^*), a difference which adds a great deal of computational difficulty. The condition

$$\left.\frac{\partial P_C(1)}{\partial K}\right|_{K^*} = 0$$

becomes

$$Ⓗ\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\ \frac{h^*\circ s_T(K^* + h^*\circ s)}{N_o E}\right] \exp(-[K^* + h^*\circ s]^2/2N_o E)$$

$$= Ⓗ\left[\sqrt{\frac{\bar{\sigma}^2}{2\eta^2}}\ \frac{h^*\circ s_T(K^* - h^*\circ s)}{N_o E}\right] \exp(-[K^* - h^*\circ s]^2/2N_o E) \qquad (6.2.2)$$

As Ⓗ has no particular odd or even properties, obtaining K^* directly from h^* is not obvious. Note the following effect. If $s_T(t)$ is replaced by $-s_T(t)$ and K by minus K in (6.2.1); and then the variable of integration set to -w, (6.2.1) remains unchanged. From this one may conclude that if K^* solves (6.2.2) for $s_T(t)$, then $-K^*$ solves (6.2.2) for $-s_T(t)$.

Now changing the variable of integration to -w in the second integral, (6.2.1) sets $P_C(1)$ equal to

$$P_C(1) = \frac{1}{2}(\eta^2)^{-1/2} \int_{-\infty}^{\phi_+/\sqrt{\eta^2 N_o E}} Ⓗ\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E}}\ \psi w\right] e^{-w^2/2}\, dw$$

$$+ \frac{1}{2}(\eta^2)^{-1/2} \int_{-\infty}^{\phi_-/\sqrt{\eta^2 N_o E}} Ⓗ\left[\sqrt{\frac{\bar{\sigma}^2}{2N_o E}}\ \psi w\right] e^{-w^2/2}\, dw \qquad (6.2.3)$$

where

$$\psi = h\circ s_T = \int_0^T h(t)s_T(t)dt; \qquad h\circ s = \int_0^T h(t)s(t)dt$$

$$\phi_+ = h\circ s + K$$

$$\phi_- = h\circ s - K$$

The great similarity of (6.2.3) to (5.2.5) is seen if $\textcircled{H}$ is substituted for $\textcircled{H}_e$ and ϕ_+ and ϕ_- are accounted for. Thus, the development for demonstrating that $c^* \neq 0$ (say $c^* = 1$) and obtaining c_T^* follows directly (5.2.16), through (5.5.22). In particular, the following are easily derived

$$2\alpha c^* = -\frac{1}{2\eta^2\sqrt{N_oE}}\left[\textcircled{H}\left[\sqrt{\frac{\overline{\sigma}^2}{2\eta^2}}\frac{\phi_+\psi}{N_oE}\right]\exp(-\phi_+^2/2\eta^2N_oE) + \textcircled{H}\left[\sqrt{\frac{\overline{\sigma}^2}{2\eta^2}}\frac{\phi_-\psi}{N_oE}\right]\exp(-\phi_-^2/2\eta^2N_oE)\right]_{\substack{c = c^* \\ c_T = c_T^* \\ K = K^*}} \tag{6.2.4}$$

so that $\alpha c^* < 0$. Consequently,

$$c_T^* = -(E_sP_{c_T}^* - \rho P_c^*)/(E_s^2 - \rho^2)(2\alpha c^*) \tag{6.2.5}$$

Now

$$2(E_sP_{c_T} - \rho P_c)/(E_s^2 - \rho^2$$

can be read directly from (5.2.22) in the following manner

(i) For P_C substitute $P_C(1)$

(ii) Replace the term $\left\{\phi\ \textcircled{H}_e(\ldots\phi)e^{-\phi^2/\ldots}\right\}$ with

$$\left\{\phi_+\ \textcircled{H}(\ldots\phi_+)e^{-\phi_+^2/\ldots} + \phi_-\ \textcircled{H}(\ldots\phi_-)e^{-\phi_-^2/\ldots}\right\}$$

(iii) Similarly, replace $\left\{\textcircled{H}_e'(\ldots\phi)e^{-\phi^2/\ldots}\right\}$ by

$$\left\{\textcircled{H}'(\ldots\phi_+)e^{-\phi_+^2/\ldots} + \textcircled{H}'(\ldots\phi_-)e^{-\phi_-^2/\ldots}\right\}$$

(iv) Change $\left\{ \int_{-\infty}^{\phi/\ldots} \right\}$ to $\left\{ \int_{-\infty}^{\phi_+/\ldots} + \int_{-\infty}^{\phi_-/\ldots} \right\}$

From the discussion following (5.2.22) and the preceding remarks concerning the sign of K^*, it follows that if $s_T(t)$ is replaced by $-s_T(t)$ then K^* is replaced by $-K^*$ and c_T^* by $-c_T^*$.

Case (2) $D^{-1} = 2$

As already pointed out, $D^{-1} = 2$ is equivalent to the case of $D^{-1} = 1$ if $s_T(t)$ is replaced by $-s_T(t)$ so that K^* and c_T^* need only be negated.

The resulting receiver is depicted in Figure 23.

6.3 Summary

We see in the immediately preceding two sections the great analytical similarity to Chapter V. Ⓗ is substituted for $Ⓗ_e$ and the sign of $s_T(t)$ comes into some play. The actual performance of the receiver so designed requires the probability law $P_C(1)$ (which will be of the same form as $P_C(2)$ because, effectively, only $s_T(t)$ has been negated). With this in hand, one simply proceeds as in Chapter IV, enumerating the events leading to a successful decision and performing the necessary algebra.

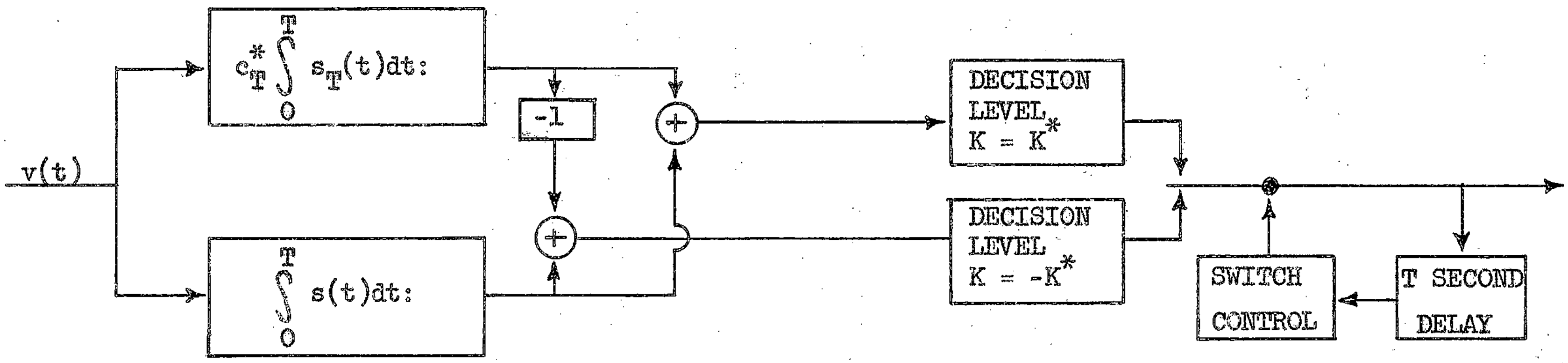


FIGURE 23 BIPOLAR SWITCHED MODE CORRELATION RECEIVER
FOR FADING PULSE INTERFERENCE

CHAPTER VII

CONCLUSIONS

7.1 System Comparison

In this section we compare the performance of the receivers of Chapters III and IV. That is to say, the memoryless correlation receivers of optimum type ($h^* = s + c_T^* s_T$, $K = K^*$) and standard ($h^o = s$, $K = K^o$) type and the dual decision level, switch controlled correlator ($h = s; K_1, K_2$), where the immediately preceding receiver output activates the switch to select the next decision level, K_1 or K_2, to be used. Figures 24, 25 and 26 plot the error performance, P_E, of these receivers for the extreme case of $b = 0.7$, $a = 0.1$, 0.25 and 0.5, as functions of the received $(\frac{S}{N})$ ratio. Also plotted on these figures is the P_E curve for $b = 0$. In all cases for $P_E < 10^{-1}$, the switched mode correlator provides the best performance. In fact, in terms of $(\frac{S}{N})$, the switched mode is negligibly different from the $b = 0$ curve. Even for as small an a as 0.1 the switched mode correlator represents a $(\frac{S}{N})$ gain of 1.5 db over the optimal memoryless correlator. For $a = 0.25$, the switched mode correlator represents a gain of approximately 2 db over the optimal correlator and more than 3 db over the standard correlator. For $a = 0.5$, the switched mode correlator represents an improvement of 3 db and 6 db, respectively.

The switched mode correlator has another advantage over the memoryless correlators in its ease of construction and that existing systems

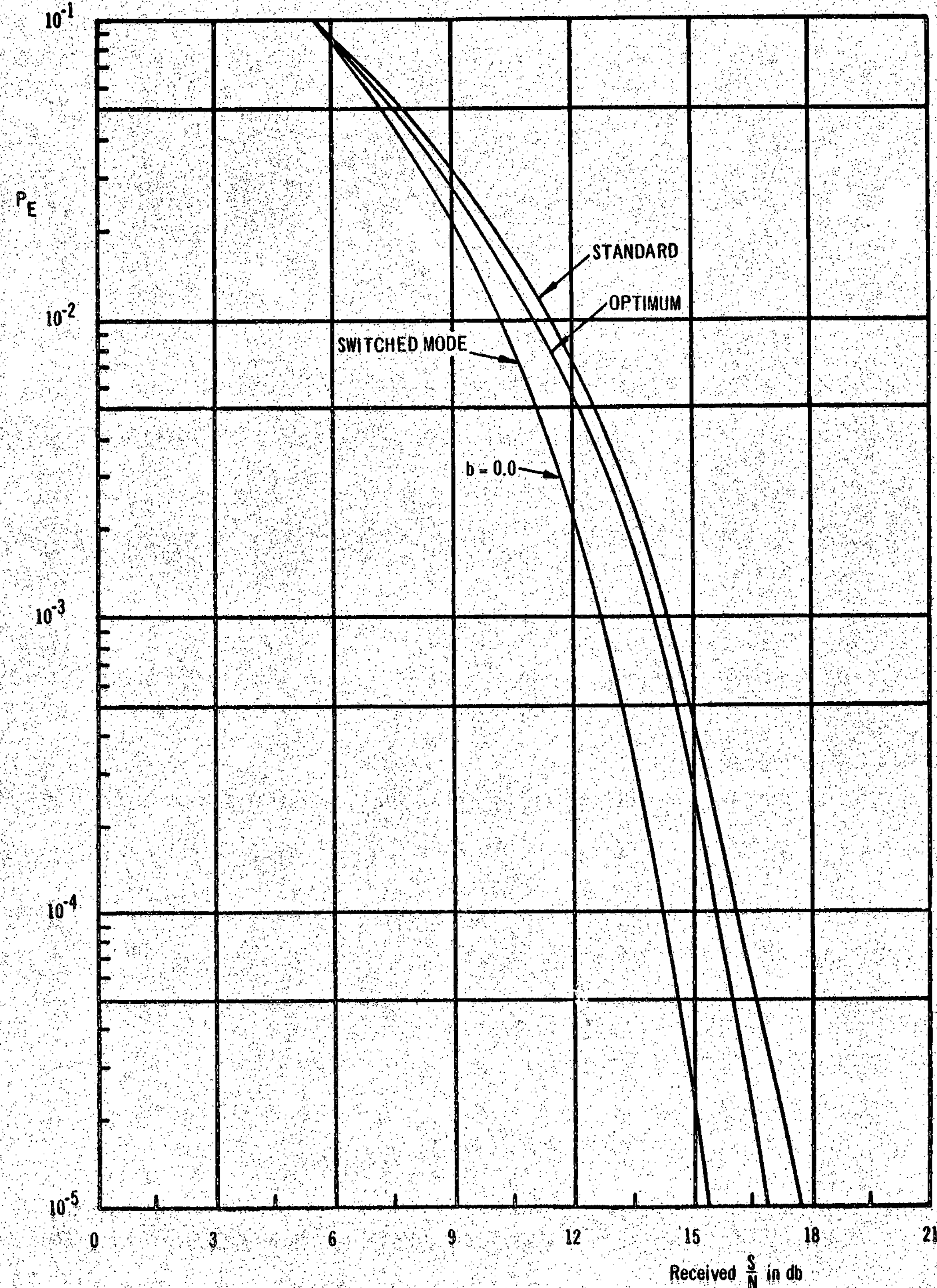


FIGURE 24 ERROR RATE COMPARISON OF CORRELATION RECEIVERS FOR a = 0.1, b = 0.7

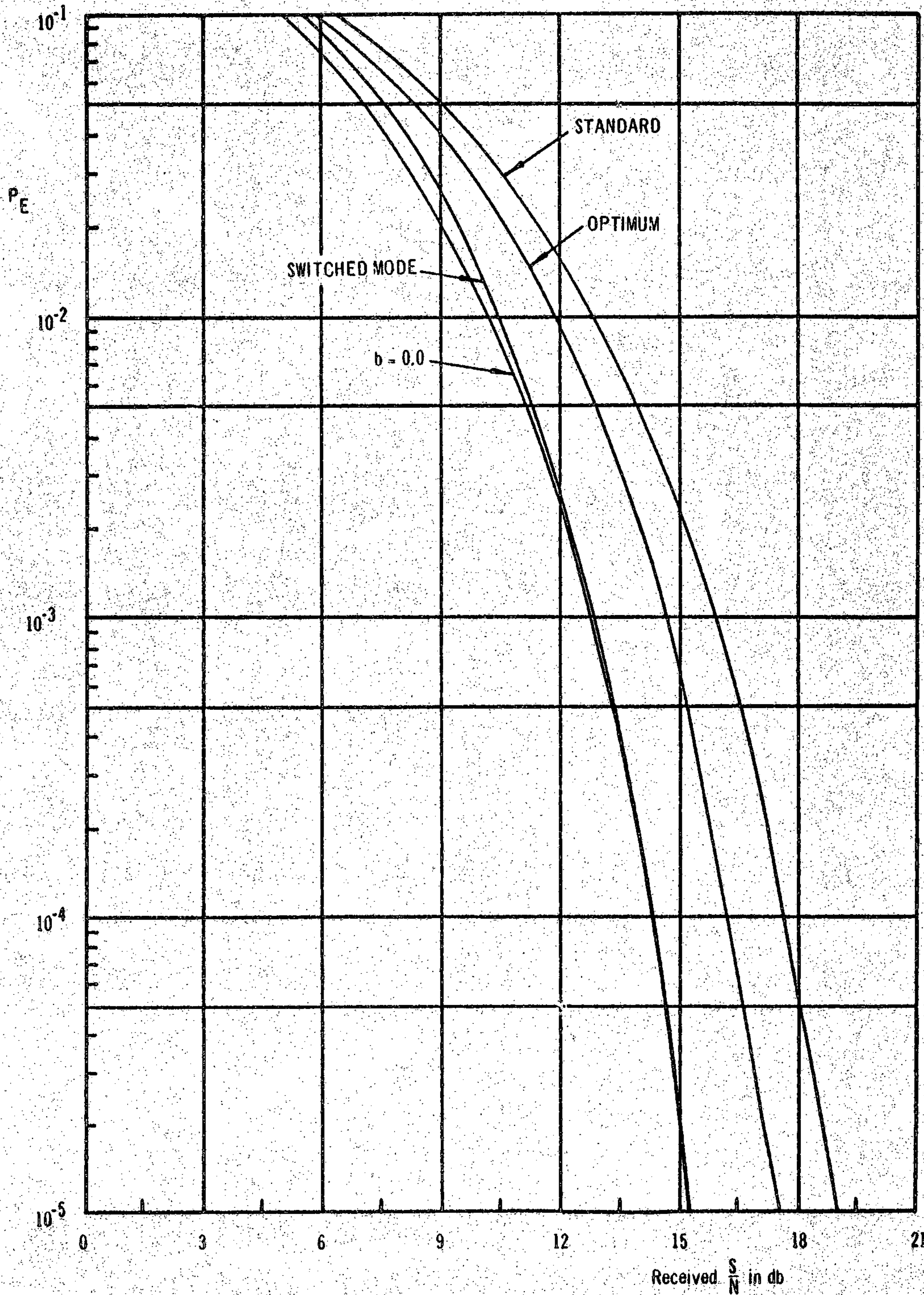


FIGURE 25 ERROR RATE COMPARISON OF CORRELATION RECEIVERS FOR a = 0.25, b = 0.7

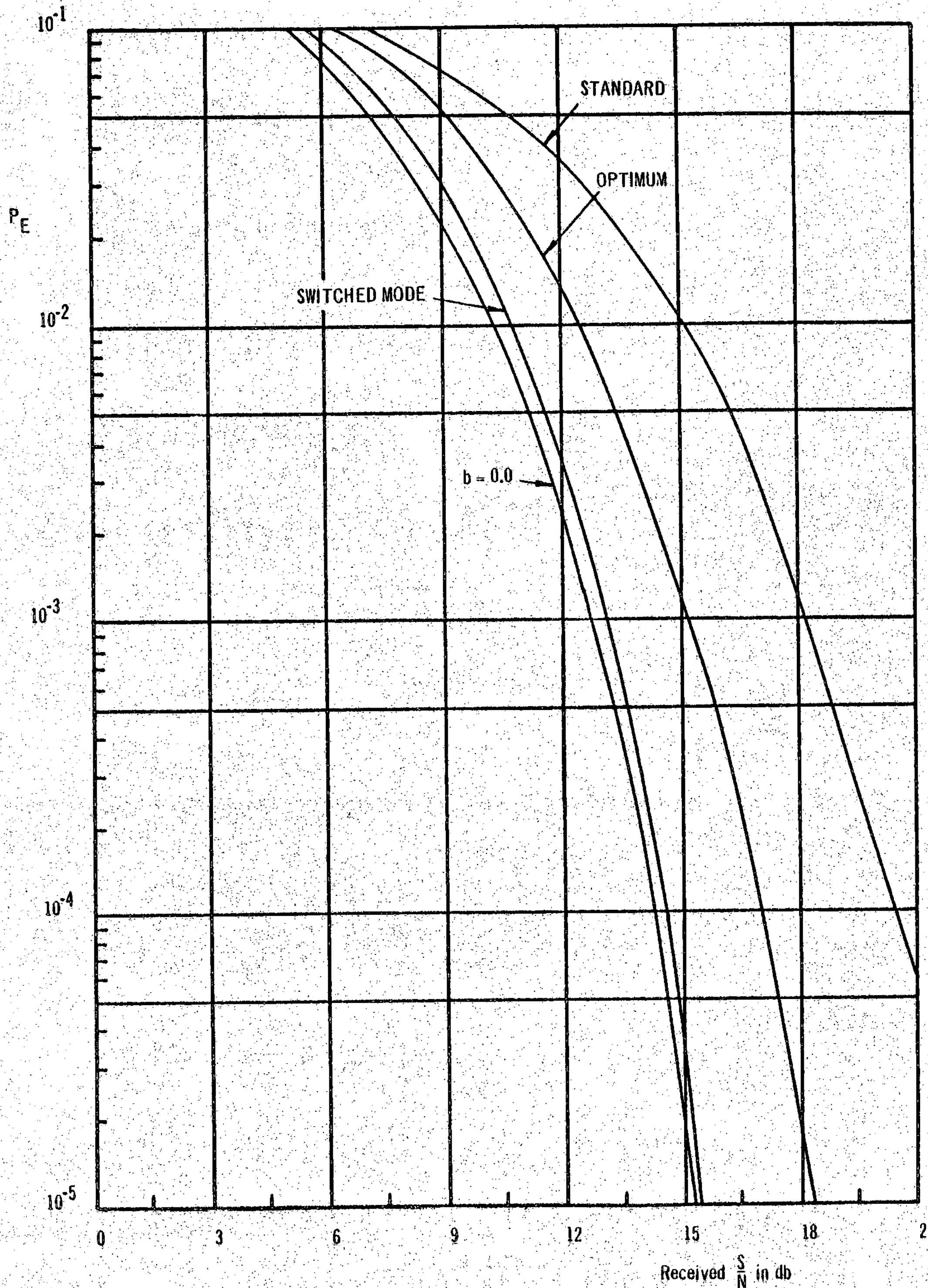


FIGURE 26 ERROR RATE COMPARISON OF CORRELATION RECEIVERS FOR a = 0.5, b = 0.7

may be easily modified to a switched mode operation. The prescription for implementing this modification is to take the usual correlator, $\int_0^T h(t)dt$, or "matched filter", and modify it for two decision levels.

Thus the switched mode correlator is much the preferred receiver. However, the pulse interference must be serial in time so that memory may be employed. If on the other hand the pulse interference is caused by a parallel communication channel, e.g., cross talk on multichannel carrier equipment, then memory may not be employed and the optimum memoryless correlator should be employed if a and b are sufficiently large to justify the cost of improving the error performance.

7.2 Analytical Approach

At the onset of this investigation a receiver class (correlation) was postulated and the probability law over this class was maximized. This represents a functional approach to the problem of reception. Involved in this method was extremizing a function of functionals (linear functionals). In this context, Andreev,[8] in a recent paper, mathematically discusses the necessary and sufficient conditions for extremizing a function of functionals subject to side conditions. Both the functionals and side conditions are fairly general in that they need not be linear nor holonomic respectively. The orientation in his paper is towards automatic control in that the motivation is to extremize a "performance index" associated with a control problem. This performance index is mathematically described as a function of functionals. In a communications context, our performance index is the probability law associated with the receiver class.

On the other hand, with the development of Chapters II and III, a geometric approach has emerged which is equivalent to the functional method. There is a received signal space in which one seeks the best straight line decision curve. The linear decision curve is equivalent to the correlation receiver class. As has been pointed out[2,3,4] the Bayes' receiver represents a decision curve (in two dimensions; in general this is referred to as a decision surface) in the received signal space. The difficulty in obtaining the probability law associated with the Bayes' receiver is precisely the difficulty in relating definitively, the liklihood equation

$$\Lambda(v) = K \qquad K = \text{decision level}$$

to a decision curve in the received signal space. Thus the decision line represents a zero order "approximation" to an unknown curve.

This idea suggests successively approximating the Bayes' decision curve with perhaps polynomial curves. It would thus be of theoretical interest to establish a few theorems of mathematical statistics regarding the convergence of a sequence of decision curves to an optimum curve. Furthermore, given a particular decision curve, what is its physical or functional representation in general? e.g., a linear decision line implies a correlation receiver.

In Chapters V and VI the received signal points in the absence of noise are not fixed. In general, if the noiseless signal points fall in a region of the plane with a given probability distribution conditioned by the transmission of one of two possible information states, the conditional probability surface, given the transmitted state, is obtained by averaging the individual probability surfaces (Gaussian if the noise is Gaussian) with respect to the signal point probability distribution associated with the given transmitted state. Consequently, the geometric interpretation associated with Chapters III and IV carries over to Chapters V and VI. In particular, the decision curves are linear. However, the probability surface associated with those chapters is not obvious.

The philosophy contained in these two approaches may be summarized briefly. If the functional approach is taken, one chooses a receiver class for which there is associated a well known probability law. His choice of class is arbitrary; perhaps intuitive or this class may represent the pertinent physical operations involved in the Bayes' receiver. Applying extremal calculus, the extremizing receiver in the prechosen class is found. On the other hand, if the geometric approach is taken, one must first construct the appropriate probability surfaces. Given the necessary a priori probability distribution for noise and signals, this construction is theoretically straightforward, if practically difficult. By consideration of the surface, a choice of decision curve is made. Perhaps this choice is arrived at by volumetric extremization with respect to a parametric family of decision curves. For example, extremize with respect to the two parameter family of decision curves $y = ax + b$. Having obtained a decision curve there yet remains the problem of relating said curve to a physical device.

7.3 Suggestions for Further Investigation

It should be of some interest to initiate a detailed computer study to obtain the necessary solutions of the equations of Chapters V and VI. The results of this study could then be compared in a manner similar to that performed for Chapters III and IV. It would then be possible to see if, and to what extent, simple memory utilization improves the correlation receiver in the presence of fading interference pulses. Obviously, a further investigation should be made for the case where the fading is present on both the desired and interference signal pulses. Also, of importance is the extension of the results obtained to the case of M-ary signal alphabets. The approach to these problems may be through either or both the functional and geometric viewpoints.

Of particular interest would be an investigation of diversity schemes. Consider M channels for which M correlation receivers $(h_j(t)K_j)$ $j = 1,\ldots,M$ are to be selected. For each channel, in the absence of noise, there is associated a set of N_j of linearly independent signals, s_{ji}; $i = 1,\ldots,N_j$; $j = 1,\ldots,M$, which may be received. The M receiver outputs are diversity combined with which there is an associated probability law P_C. The problem is then to choose the M optimal correlators $(h_j^*(t),K_j^*)$, $j = 1,\ldots,M$. The side conditions imposed for extremizing P_C will be of paramount importance. For example, there comes to mind the two possibilities

(1) $E_{h_j} = E_j \qquad j = 1,\ldots,M$ and $E_1:E_2:\ldots:E_M$ is prescribed.

(2) $\sum_{j=1}^{M} \alpha_j E_{h_j} = E$,

where the ratios of α_j are fixed or the α_j themselves may be further adjusted to maximize P_C.

Previous results on diversity combination should be obtainable from the above model and would serve as a check on this approach.

In the realm of nonlinear extensions to this dissertation would be an optimization of (say) the bipolar symmetric case of Chapter III with respect to the family of decision curves $y = ax + bx^3$. Having found the best "cubic" decision curve it is then necessary to obtain its physical realization.

Analogous to the receiver problems (baseband) so far discussed is the detection of interfering r. f. signal pulses. Assuming the phases of different r. f. signals to be independent and uniformly distributed, the pertinent receiver class would seem to be the set of linear envelope detectors, i.e., a passband filter followed by an ideal linear envelope detector and an associated decision level. The probability law, P_C associated with this class is the distribution function generated by the Modified Rayleigh density function. The problem is then to maximize P_C over the class of envelope detectors.

In general, it should be possible to apply the approach outlined in Chapter I to any situation for which Bayes' criterion is applicable. In fact, a reasonable choice of receiver class may be made by deducing what pertinent physical operation is involved in the Likelihood Ratio and using that operation to establish the receiver class over which P_C is maximized. Of course, for the receiver class chosen, the designer must functionally have knowledge of P_C.

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APPENDIX

APPENDIX A

NUMERICAL METHOD OF SOLVING EQUATION (3.4.22)

Define G(x) as

$$G(x) = \ln x + b\sqrt{a}\,(\frac{s}{N}) + \frac{(\frac{s}{N})(\frac{a[1 - b^2]}{1 + a + 2b\sqrt{a}})(x^2 - 1)}{x^2 + 2\frac{1 - a}{1 + a + 2b\sqrt{a}}x + \frac{1 + a - 2b\sqrt{a}}{1 + a + 2b\sqrt{a}}} \tag{A1}$$

where, $0 \leqq a < 1$

$0 \leqq b < 1$

Then (3.4.22) requires that real value of x, x = u, so that G(u) = 0. As shown in Appendix B, u is unique and positive. It is also shown in Appendix B that

$$x^2 + 2\frac{1 - a}{1 + a + 2b\sqrt{a}}x + \frac{1 + a - 2b\sqrt{a}}{1 + a + 2b\sqrt{a}} \tag{A2}$$

has no real roots, so that $G(x) - \ln(x)$ is bounded. Then for sufficiently small x, $G(x) < 0$; since $G(1) > 0$, u must lie between zero and one.

The following method of halving was used to approximate u. Set

$x_0 = 0$

$x_1 = 1$

Take

$$x_{k+1} = x_k + |x_k - x_{k-1}|/2 \quad \text{if } G(x_k) < 0$$

$$x_{k+1} = x_k - |x_k - x_{k-1}|/2 \quad \text{if } G(x_k) > 0;\ k = 1,2,\ldots\ldots$$

In the unlikely event that $G(x_k) = 0$, we have found u exactly. Assume that this does not occur. We then choose to stop the reiteration process when

$$|x_k - x_{k-1}| < 10^{-6}$$

and u is taken to be the last computed x_k value. Now,

$$|x_k - x_{k-1}| = |x_{k-1} - x_{k-2}|/2$$

so that

$$|x_k - x_{k-1}| = |x_1 - x_0|/2^{k-1} = 2^{1-k}, \quad k = 1,2,\ldots\ldots$$

Consequently, for k = 20

$$|x_{20} - x_{19}| = 2^{-19} < 10^{-6}$$

and the process is terminated.

Suppose $0 < u < 2^{-19}$, then $G(x_k) > 0$ for $k = 1,2,\ldots\ldots,19$, so that $x_{20} = 2^{-20}$ and $|x_{20} - u| \leq 10^{-6}$. Suppose $2^{-19} < u < 1$, then for some $k = k_1$, $G(x_k) > 0$ for $k = 1,\ldots\ldots,k_1-1$ and $G(x_{k_1}) < 0$. Thus, the x_k step to the left by halves until $k = k_1$.

$$x_{k_1-1} = 2^{1-k_1}$$

$$x_{k_1} = 2^{-k_1}$$

Since $G(x_{k_1}) < 0$, u must lie between x_{k_1} and x_{k_1-1} and

$$x_{k_1+1} = x_{k_1} + 2^{-k_1-1}$$

The x_k then step to the right by halves until $k = k_2$; at which point $x_{k_2} > u$, so that $G(x_{k_2}) > 0$ and the x_k proceed to move left by halves;

and so on until k = 20. Consequently,

$$\left| x_{20} - u \right| \leq 10^{-6}$$

and since

$$c_T^* = \frac{u - 1}{u + 1}$$

the error in c_T^* is at most one part in one million.

APPENDIX B

SUFFICIENCY OF THE OPTIMAL CORRELATION RECEIVER OF SECTION 3.4

B.1 K^* Uniqueness

Rather than show that K^* is the unique solution to equation (3.4.7a)

$$P_K^* = 0 \tag{3.4.7a}$$

we will show that for a certain set of choices of c and c_T

$$K = \hat{K} = \frac{1}{2} h \circ (s + s_T); \qquad h(t) = cs(t) + c_T s_T(t)$$

uniquely maximizes P_C over K, for c and c_T fixed. In particular, if h(t) is of the form

$$h(t) = cs(t) + c_T s_T(t) \tag{B.1.1}$$

and has the property that

$$h \circ (s \pm s_T) > 0 \tag{B.1.2}$$

Then P_C considered as a function of K, with c and c_T fixed, is maximized absolutely for the value

$$K = \hat{K} = \frac{1}{2} h \circ (s + s_T) \tag{B.1.3}$$

This is demonstrated easily by writing

$$K = \hat{K} + \epsilon \tag{B.1.4}$$

and substituting this K into equation (3.4.3). With this substitution,

$$P_C(\hat{K}+\epsilon) = \frac{1}{4}\left\{1 + \int_{\frac{\epsilon - h\circ(s+s_T)}{2\sqrt{N_o E_h}}}^{\frac{\epsilon + h\circ(s+s_T)}{2\sqrt{N_o E_h}}} e^{-u^2/2}\frac{du}{\sqrt{2\pi}} + \int_{\frac{\epsilon - h\circ(s-s_T)}{2\sqrt{N_o E_h}}}^{\frac{\epsilon + h\circ(s-s_T)}{2\sqrt{N_o E_h}}} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\right\}$$

Since $e^{-u^2/2}$ is even and monotone decreasing for positive u and since

$$h\circ(s \pm s_T) > 0$$

the desired result follows. Now if h is of the form

$$h(t) = s(t) + c_T s_T(t); \quad |c_T| < 1, \tag{B.1.6}$$

then by the discussion in section 3.7, h has property (B.1.2) and

$$\hat{K} = \frac{1}{2}\, h\circ(s + s_T)$$

maximizes P_C for fixed c_T, where $c = 1$ and $|c_T| < 1$. Thus, when $c_T = c_T^*$, $K = K^*$ and K^* is unique. Physically speaking, (B.1.2) and (B.1.3), assure errorless performance if the noise were to disappear.

It ought to be emphasized, that these results are valid only for symmetric ($p = 1/2$) transmission of information. In essence, the above argument is directly tied to the fact that (3.4.7a) can be solved independently of (3.4.7b and c). If we have a non-symmetric channel, such is not the case.

B.2 c_T^* Uniqueness

If we let

$$(1 + c_T^*)/(1 - c_T^*) = u; \qquad c_T^* = (u - 1)/(u + 1) \tag{B.2.1}$$

then from (3.4.16) u must be that real number which satisfies

$$ue^{\rho/2N_o} = \exp - \frac{\left(\frac{E_s E_{s_T} - \rho^2}{2N_o E_{s+s_T}}\right)(u^2 - 1)}{u^2 + 2\,\frac{E_s - E_{s_T}}{E_{s+s_T}}\,u + \frac{E_{s-s_T}}{E_{s+s_T}}} \tag{B.2.2}$$

Now the right hand side of (B.2.2) is positive for all $-\infty < u < \infty$, so the solution, u, of (B.2.2) must certainly be greater than zero. Furthermore, it is shown below that the function g(u),

$$g(u) = \frac{\frac{\left(E_s E_{s_T} - \rho^2\right)}{2N_o E_{s+s_T}}(u^2 - 1)}{u^2 + 2\,\frac{E_s - E_{s_T}}{E_{s+s_T}}\,u + \frac{E_{s-s_T}}{E_{s+s_T}}} \tag{B.2.3}$$

is monotone increasing for $u > 0$. Consequently, for $u > 0$, the right hand side of (B.2.2) is monotone decreasing on u; whereas the left hand side is monotone increasing. Since the range of the right hand side of (B.2.2) intersects that of the left hand side, for $u > 0$, we are guaranteed of a unique, positive u solution to (B.2.2). Taking the log of (B.2.2) produces (3.4.18).

Differentiating g, (B.2.3), produces

$$\frac{N_o E_{s+s_T}\, g'(u)}{(E_s E_{s_T} - \rho^2)} = \frac{\frac{E_s - E_{s_T}}{E_{s+s_T}}\,u^2 + 2\,\frac{E_s + E_{s_T}}{E_{s+s_T}}\,u + \frac{E_s - E_{s_T}}{E_{s+s_T}}}{u^2 + 2\,\frac{E_s - E_{s_T}}{E_{s+s_T}}\,u + \frac{E_{s-s_T}}{E_{s+s_T}}} \tag{B.2.4}$$

and since

$$E_s > E_{s_T},\ E_s E_{s_T} > \rho^2$$

$$g'(u) > 0,\ u > 0$$

hence g is monotone increasing on $(0, \infty)$. Note that

$$u^2 + 2\ \frac{E_s - E_{s_T}}{E_{s+s_T}}\ u + \frac{E_{s-s_T}}{E_{s+s_T}}$$

has no real u roots. This is certainly verified if

$$\left(\frac{E_s - E_{s_T}}{E_{s+s_T}}\right)^2 < \frac{E_{s-s_T}}{E_{s+s_T}}$$

or

$$(E_s - E_{s_T})^2 < E_{s-s_T}\ E_{s+s_T} = (E_s + E_{s_T})^2 - 4\rho^2$$

Equivalently,

$$-2E_s E_{s_T} < 2E_s E_{s_T} - 4\rho^2$$

whence

$$\rho^2 < E_s E_{s_T}$$

which is guaranteed by Schwar inequality.

B.3 Sufficiency

It was shown in section 3.4 that $c^* \neq 0$ and by the nature of P_C, c^* could be set to one. Under B.1 we saw that K is an absolute maximum with $c = 1$ and $|c_T| < 1$. Then it follows that

$$\max_{K,c,c_T} P_C(K,c,c_T) = \max_{\substack{c=1 \\ |c_T|<1}} P_C(\hat{K},1,c_T)$$

Now $P_C(\hat{K},1,c_T)$ is a function of one variable, c_T. Consequently $c = 1$, $c_T = c_T^*$, $K = K^*$ is a local maximum if

$$\left.\frac{d^2 P_C(\hat{K},1,c_T)}{dc_T^2}\right|_{c_T=c_T^*} < 0 \qquad \text{(B.3.1)}$$

where it must be remembered that E_h is now a function of c_T

$$E_h = E_s + 2\rho c_T + c_T^2 E_{s_T} \qquad \text{(B.3.2)}$$

Denote by the following

$$h^*(t) = s(t) + c_T^* s_T(t) \qquad \text{(B.3.3)}$$

$$E^* = h^* \circ h^* = E_s + 2\rho c_T^* + (c_T^*)^2 E_{s_T}$$

$$h^*(+) = h^* \circ (s+s_T)$$

$$h^*(-) = h^* \circ (s-s_T)$$

$$\psi(u) = e^{-u^2/2}/4\sqrt{2\pi}$$

$$\psi^*(+) = \psi(h^*(+)/2\sqrt{N_o E^*})$$

$$\psi^*(-) = \psi(h^*(-)/2\sqrt{N_o E^*})$$

It can then be shown that

$$-\left.\frac{d^2P_C(\hat{K},1,c_T)}{dc_T^2}\right|_{c_T=c_T^*} = \frac{\left[E_sE_{s_T} - \rho^2\right]^2}{4E^*(N_oE^*)^{3/2}}\left\{(1+c_T^*)^2\, h^*(-)\psi^*(-) + (1-c_T^*)^2 h^*(+)\psi^*(+)\right\}$$

$$+ \frac{\left[E_sE_{s_T} - \rho^2\right]}{E^*(N_oE^*)^{1/2}}\left\{\psi^*(-) + \psi^*(+)\right\}$$

$$- \frac{3(\rho + c_T^* E_{s_T})}{(E^*)^2(N_oE^*)^{1/2}}\left\{\left[1+c_T^*\right]\psi^*(-) - \left[1-c_T^*\right]\psi^*(+)\right\}$$

(B.3.4)

But equation (3.4.16) states

$$\left[1 - c_T^*\right]\psi^*(+) - \left[1 + c_T^*\right]\psi^*(-) = 0 \qquad (3.4.16)$$

and since,

$$h^*(\pm) = h^*\circ(s \pm s_T) > 0$$

$$\psi^*(\pm) > 0 \qquad (B.3.5)$$

$$E_sE_{s_T} - \rho^2 > 0$$

the right hand side of (B.3.4) is positive, and (B.3.1) is proven. Thus the receiver

$$h^*(t) = s(t) + c_T^*\, s_T(t)$$

$$(B.3.6)$$

$$K^* = \frac{1}{2} h^*\,(s + s_T)$$

is locally maximizing; the point $K = K^*$, $c = 1$, $c_T = c_T^*$ is unique amongst all choices of c, c_T such that

$$\int_0^T \left[cs(t) + c_T s_T(t)\right]^2 dt = E^*$$

Furthermore, for any fixed choice of c, c_T satisfying (B.3.7), it is easily seen from (3.4.3) that

$$\lim_{K \to \pm\infty} P_C(K, c, c_T) = \frac{1}{2}$$

which represents the minimal value of P_C. Thus, we have established that (B.3.6) absolutely maximizes P_C.

APPENDIX C

DECISION LEVEL FOR BIPOLAR SYMMETRIC CHANNEL WITH FADING INTERFERENCE

With reference to (5.2.11) it is here shown that $K^* = 0$ is the unique extremum corresponding to (5.2.11) for any choice of c, and c_T, and for any acceptable probability density function (distribution) $p_\sigma(\sigma)$. Consider (5.2.5)

$$P_C(K, c, c_T) = E(P_{C/\sigma}(K, c, c_T, \sigma)) \tag{5.2.5}$$

where $$E(\cdot) = \int_{-\infty}^{\infty} p_\sigma(\sigma)(\cdot)d\sigma$$

using expression (5.2.3) for $P_{C/\sigma}$ and interchanging the expectation operation with the indicated integrals of (5.2.3) produces

$$P_C(K, c, c_T) = \frac{1}{4}\int_{-\infty}^{K} E\left(e^{-[u+x+\sigma y]^2/2\lambda^2} + e^{-[u+x-\sigma y]^2/2\lambda^2}\right)\frac{d}{\sqrt{2\pi N_o E}}$$

$$+ \frac{1}{4}\int_{K}^{\infty} E\left(e^{-[u-x+\sigma y]^2/2\lambda^2} + e^{-[u-x-\sigma y]^2/2\lambda^2}\right)\frac{d}{\sqrt{2\pi N_o E}} \tag{C1}$$

where

$$x = h\circ s = cE_s + c_T\rho$$

$$y = h\circ s_T = c\rho + c_T E_s$$

$$\lambda^2 = N_o E$$

Note that P_C is even on K. Then differentiating (C1) with respect to K, (c, c_T) held constant, gives

$$\frac{\partial P_C}{\partial K}\left(4\sqrt{2\pi N_o E}\right) = E\left(f_{x,y,K}(\sigma)\right) \tag{C2}$$

$$f_{x,y,K}(\sigma) = e^{-\left[K+x+\sigma y\right]^2/2\lambda^2} + e^{-\left[K+x-\sigma y\right]^2/2\lambda^2}$$

$$- e^{-\left[K-x+\sigma y\right]^2/2\lambda^2} - e^{-\left[K-x-\sigma y\right]^2/2\lambda^2}$$

Now note that the argument of the expectation, $f_{x,y,K}(\sigma)$, is an even function of σ so that the expectation may be equivalently carried out only for $\sigma \geq 0$ (denoted by $E_{\sigma\geq 0}(\bullet) = \int_0^\infty p_\sigma(\sigma)(\bullet)d\sigma$). Rearranging $f_{x,y,K}(\sigma)$ algebraically, we obtain

$$f_{x,y,K}(\sigma) = g_{x,y,K}(\sigma)\ e^{-(\sigma^2 y^2 + [K-x]^2)/2\lambda^2}$$

$$g_{x,y,K}(\sigma) = e^{-\sigma yx/\lambda^2}\left[e^{K(2x-\sigma y)/\lambda^2} - e^{\sigma yK/\lambda^2}\right] \tag{C3}$$

$$+ e^{\sigma yx/\lambda^2}\left[e^{K(2x+\sigma y)/\lambda^2} - e^{-\sigma yK/\lambda^2}\right]$$

Thus, if we show that $f_{x,y,K}(\sigma) \underset{(<)}{>} 0$, $\sigma \geq 0$, for $K \neq 0$, then

$$\frac{\partial P_C}{\partial K} = 0$$

is solved uniquely by $K^* = 0$, because the expectation of a definite quantity is itself not zero. Equivalently, we need only examine $g_{x,y,K}(\sigma)$, as the exponential function of a real argument is always positive. Clearly, for $K = 0$,

$$g_{x,y,0}(\sigma) = 0 \tag{C4}$$

and K = 0 is a solution to (5.2.11). Also, for x = 0

$$g_{0,y,K}(\sigma) = 0 \tag{C5}$$

However, x = 0 makes no physical sense as it implies that our receiver is insensitive to the presence of s(t). Consequently, we assume $x \neq 0$ and $K > 0$. Since P is even on K, this represents no restriction. If $y = 0$,

$$g_{x,0,K}(\sigma) = 2e^{2Kx/\lambda^2} > 0 \tag{C6}$$

There are now four possible cases to be examined, corresponding to the four combinations of $\pm x$, $\pm y$.

Case (1) $x > 0,\ y > 0$

$$\sigma = 0; \qquad g_{x,y,K}(0) = 2e^{2Kx/\lambda^2} > 0 \tag{C7}$$

$$0 < \sigma < \frac{x}{y}; \qquad g_{x,y,K}(\sigma) \geq e^{-x^2/\lambda^2}\left[e^{Kx/\lambda^2} - e^{Kx/\lambda^2}\right] + \left[e^{2Kx/\lambda^2} - e^{-Kx/\lambda^2}\right] > 0 \tag{C8}$$

$$\sigma = \frac{x}{y}; \qquad g_{x,y,K}(x/y) = e^{x^2/\lambda^2}\left[e^{3Kx/\lambda^2} - e^{-Kx/\lambda^2}\right] > 0 \tag{C9}$$

For $\sigma \geq \frac{x}{y}$, let $\sigma y = ax$, $a > 1$, then

$$g_{x,y,K}(\sigma) = e^{-ax^2/\lambda^2}\left[e^{Kx(2-a)/\lambda^2} - e^{Kxa}\right] + e^{ax^2/\lambda^2}\left[e^{Kx(2+a)/\lambda^2} - e^{-Kxa}\right] \quad a > 1 \tag{C10}$$

Clearly, for a > 1 (x > 0, y > 0, K > 0)

$$g_{x,y,K}(\sigma) \geq e^{-ax^2/\lambda^2}\left[1 - e^{axK/\lambda^2}\right] + e^{ax^2/\lambda^2}\left[e^{axK/\lambda^2} - 1\right] \tag{C11}$$

or,

$$g_{x,y,K}(\sigma) \geq e^{axK/\lambda^2}\left[e^{ax^2/\lambda^2} - e^{-ax^2/\lambda^2}\right] + \left[e^{-ax^2/\lambda^2} - e^{ax^2/\lambda^2}\right]$$

$$(a > 1,\ x > 0,\ K > 0) \tag{C12}$$

Equivalently,

$$g_{x,y,K}(\sigma) \geq \left[e^{ax^2/\lambda^2} - e^{-ax^2/\lambda^2}\right]\left[e^{axK/\lambda^2} - 1\right]$$

$$> \left[e^{ax^2/\lambda^2} - e^{-ax^2/\lambda^2}\right] > 0 \tag{C13}$$

So $K = 0$ is unique and, moreover, since P_C is even on K and $\partial P/\partial K > 0$, $(K > 0)$ it follows that $K = 0$ is the absolute minimum for an fixed c, c_T. If $K < 0$, then $\partial P/\partial K < 0$ as the derivative of an even function is odd.

Case (2) $x < 0,\ y < 0$

From (C2) note that

$$f_{-x,-y,-K}(\sigma) = f_{x,y,K}(\sigma) \tag{C14}$$

Consequently, if K is replaced by its negative $(-K < 0)$

$$f_{x,y,+K}(\sigma) = f_{|x|,|y|,-K}(\sigma) < 0, \text{ by Case (1)} \tag{C15}$$

$x < 0$

$y < 0$

And, as in Case (1), $K = 0$ is the absolute minimum for c, c_T fixed.

There is a physical interpretation of this argument. It is simply "multiplying" the receiver by minus one (which also causes a reversing of decision inequalities).

Case (3) $x > 0,\ y > 0$

This easily reduces to Case (1) by noting that the same effects may be had if y is considered positive and σ negative. That is to say

$$\underset{\sigma \leq 0}{E(f_{x,|y|,K}(\sigma))} = \underset{\sigma \geq 0}{E(f_{x,y,K}(\sigma))} \qquad x > 0,\ y < 0 \tag{C15}$$

But $f_{x,y,K}(\sigma)$ is even on σ, so that

$$\underset{\sigma \leq 0}{E(f_{x,|y|,K}(\sigma))} = \underset{\sigma \geq 0}{E(f_{x,|y|,K}(\sigma))} > 0, \text{ by Case (1)}$$

Case (4) $x < 0,\ y > 0$

By an argument identical to Case (3), we may reduce this to Case (2) and thence to Case (1).

APPENDIX D

DECISION LEVEL FOR ORTHOGONAL SYMMETRIC CHANNEL WITH FADING INTERFERENCE

Here we will show that K = 0 is the optimal decision level for orthogonal signals by reducing this case to a form where Appendix C applies. Consider a channel model given by assumptions 5.1 (3), (4) and (5) and 5.1 (1) is replaced by (3.6.1 (a) through (d)) and with an obvious extension of 5.1 (2). Then h(t) is of the form

$$h(t) = c\left[s_2(t) - s_1(t)\right] + c_T\left[s_{T_2} - s_{T_1}(t)\right] + d\left[s_2(t) + s_1(t)\right] + d_T\left[s_{T_2}(t) + s_{T_1}(t)\right] \quad \text{(D1)}$$

and a simple calculation shows that P_C is given by (C1) except that K, x, and y are replaced respectively by

$$\begin{aligned} M &= K - d(E_s + \sigma\rho) - d_T\sigma(\rho + E_s) \\ x &= h\sigma(s_2 - s_1) \\ y &= h\sigma(s_{T_2} - s_{T_1}) \end{aligned} \quad \text{(D2)}$$

Careful examination of the indicated x and y shows that they are independent of d, d_T values. Consequently, the only effect of d and d_T is an adjustment of the decision level. From the argument of Appendix C it follows that M = 0 for all allowed values of σ, otherwise

$$\frac{\partial P_C}{\partial K} \neq 0$$

Suppose there is only one allowed value of σ, say σ_o (i.e., $p_\sigma(\sigma) = \delta(\sigma - \sigma_o)$). Then,

$$M = 0 = K - d(E_s + \sigma_o \rho) - d_T \sigma_o (\rho + E_s) \tag{D3}$$

or

$$K = dE_s + \sigma_o \left[(d + d_T)\, \rho + d_T E_s \right] \tag{D4}$$

Suppose, next, that there is more than one allowed value of σ. Consequently, rearranging (D3)

$$M = 0 = K - dE_s - \sigma \left[d\rho + (E_s + \rho) d_T \right] \tag{D5}$$

and for this to be so, for more than one value of σ, the coefficient of σ must vanish, namely,

$$d_T = - \frac{d\rho}{E_s + \rho} \qquad (E_s + \rho > 0\ ) \tag{D6}$$

Thus, to satisfy

$$\frac{\partial P_C}{\partial K} = 0$$

for any choice of c and c_T, either (D4) or (D5) and (D6) must hold as a relation between d, d_T and K. We remark that (D4) is the special case when no fading is present. Then, d and d_T may be arbitrarily specified. When, indeed, there is fading, we have from (D5) and D6) that only d may be specified arbitrarily, namely

$$K = + dE_s$$

$$d_T = \frac{-d\rho}{(E_s + \rho)}$$

From (D2), (D5), and (D7) it follows that having chosen consistent values of K, d, and d_T, P_C is no longer dependent on these variables. Thus, no generality is lost by taking $d = d_T = 0$, so that $K = 0$.