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Binary Communication Systems Using Wideband Signals (Final Report v.II)

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Binary Communication Systems Using Wideband Signals

J. C. Hancock, Principal Investigator
W. D. Wade

May, 1962
Lafayette, Indiana

COMMUNICATIONS LABORATORY
AERONAUTICAL SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE

FINAL REPORT

VOLUME II

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BINARY COMMUNICATION SYSTEMS USING WIDEBAND SIGNALS

for

U. S. AIR FORCE

WRIGHT AIR DEVELOPMENT DIVISION

WRIGHT-PATTERSON AIR FORCE BASE

DAYTON, OHIO

by

J. C. Hancock, Principal Investigator

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May, 1962

FOREWORD

This report was prepared by J. C. Hancock and W. D. Wade on Air Force Contract AF 33(616)-8283 under Task No. 40621 of Project No. 4335 at the Communication Sciences Laboratory of Purdue University. This represents the second in a series of three volumes presented under contract title "Improved Information Transfer Efficiency". The work was carried out under the direction of the Communication Laboratory, Aeronautical Systems Division of the Wright Air Development Center. The assistance of task engineer B. W. Russell is gratefully acknowledged. The work presented began in June, 1961 and concluded June, 1962.

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ABSTRACT

It has been pointed out that communication systems having wideband signal waveforms have certain advantages over the conventional narrowband systems. This report describes the results of a research program which examined in detail the information efficiency of wideband systems.

The results presented in this report fall into three major categories: (1) analysis of systems utilizing linear receivers (i.e. synchronous receivers, etc.), (2) analysis of system utilizing non-linear receivers, and (3) analysis and description of methods which can be used for realizing certain optimum and sub-optimum wideband receivers. The development of these topics is based on the geometrical concept of a signal space. These signal space concepts along with certain definitions are discussed in Chapter II.

In Chapters III, IV and V wideband systems utilizing linear receivers are analyzed and the performance of these systems is determined for certain important types of noise which may be added in this communication channel. It is shown that in most cases, optimum performance cannot be obtained through the use of linear receivers.

Non-linear techniques which are capable of supplying improved performance are discussed in detail in Chapters VI and VII.

From the consideration of the types of noise which one may expect to encounter when using wideband systems, it is apparent that a relatively sophisticated approach is required in order to obtain the required physical realizations. Certain methods which appear to be useful in obtaining these realizations are discussed in Chapter VIII and IX. Because the nature of the interference, in general, will not be known in

detail to the designer, certain adaptive techniques are incorporated.

CHAPTER I

INTRODUCTION

The subject of wideband communications systems has received a great deal of attention in the past few years[1,2,3,4]. As Costas[1] has pointed out these systems have certain advantages over narrowband systems. The most notable advantage of the wideband systems is that their information efficiency[5] can be expected to remain relatively constant when the noise environment is characterized by a jagged, highly variable spectrum. Under these conditions the performance of a narrowband link can be expected to vary from extremely good to extremely poor with the result that reliable communications can be obtained from a wideband system with a smaller expenditure of signal energy. Thus, one of the important aspects of the application of wideband systems is their performance in a non-white and possibly non-gaussian noise environment.

In this report, the performance of linear wideband systems is analyzed for two important classes of noise, viz., interference from a large number of narrowband stations and impulse noise. In order to achieve better performance than is available from linear receivers, certain non-linear techniques are investigated. One non-linear technique which appears quite promising is analyzed in detail.

The performance of wideband systems can be improved by using a prewhitening filter at the receiver. In the case where the noise spectrum is variable, a receiver which automatically adapts its filter to the spectrum is desirable. Two receivers which perform this function are described in Chapter IX along with a more sophisticated version which also analyzes the noise statistics and adapts itself accordingly.

CHAPTER II

THE ANALYSIS OF BINARY SYSTEMS USING SIGNAL SPACE CONCEPTS

2.1 Introduction

A binary communications system is generally thought of as being a communications link over which sequences of binary digits are transmitted. Each of these binary digits, which are usually referred to as bauds, is represented in the transmission channel by one of two possible waveforms. These waveforms generally have a (nominal) time duration T and a (nominal) bandwidth W and can uniquely be represented by 2WT numbers. Thus it is possible to conceive of these waveforms as being points in a 2WT dimensional signal space. In subsequent chapters the concepts of n-dimensional geometry will be used in the analysis of various types of wideband binary systems. As a foundation for this material certain definitions and fundamental notions pertaining to signal space representations will be discussed in this chapter.

2.2 The signal space

The sampling theorem[6] states that a waveform which has a nominal bandwidth W and nominal time duration T is uniquely specified by 2WT properly taken samples. Furthermore the theorem states that the waveform can be reconstructed by multiplying the sample values by appropriate time functions and summing the resultant products. Since the waveform can be represented as 2WT numbers, the waveform can be designated as the position vector $\vec{E}$ of a particular point in an orthogonal n-dimensional space and there is a one-to-one correspondence between each point in the space and each possible waveform having the specified duration and bandwidth.

There are two extremely useful basic forms. In both cases, the sample values are the coefficients of orthonormal waveforms. In the first of these forms, the samples are obtained by sampling the amplitude of the waveform e(t) at intervals of 1/2W seconds. If these samples are represented by the symbols $E_1/\sqrt{2W}$, $E_2/\sqrt{2W}$, $\cdots E_n/\sqrt{2W}$ (where n = 2WT) then the waveform is given by

$$e(t) = \sum_{i=1}^{2WT} E_i \, g_i(t) \tag{2.2-1}$$

where the functions

$$g_i(t) = \sqrt{2W} \, \frac{\sin \pi(2Wt - i)}{\pi(2Wt - i)} \tag{2.2-2}$$

are orthonormal functions and have the properties

$$\int_{-\infty}^{\infty} g_i(t) \, g_j(t) = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases} \tag{2.2-3}$$

The quantities E_1, E_2 $\cdots$ E_n are the coordinate values of the vector $\vec{E}$ which in terms of these values is

$$\vec{E} = \sum_{i=1}^{2WT} E_i \vec{I}_i \tag{2.2-4}$$

where $\vec{I}_i$ is a unit vector in the positive direction of the i-th coordinate axis.

It is of interest to note that the form given by Eq. (2.2-1) describes a waveform e(t) which has a nominal duration of T seconds but is strictly band limited to a bandwidth of W.

Another useful form is the following expression which has a strictly limited time duration but is only nominally band limited:

$$e(t) = \sum_{i=1}^{2WT} E_i f_i(t) \qquad (2.2\text{-}5)$$

The time functions $f_i(t)$ are defined as

$$f_i(t) = \begin{cases} \left[u(t) - u(t - T)\right]\sqrt{2/T} \cos \frac{2\pi i t}{T}; & i = 1, 3, 5 \cdots \\ \left[u(t) - u(t - T)\right]\sqrt{2/T} \sin \frac{2\pi (i - 1) t}{T}; & i = 2, 4, 6 \cdots \end{cases} \qquad (2.2\text{-}6)$$

and satisfy the following relationships

$$\int_0^T f_i(t)\, f_j(t)\, dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad (2.2\text{-}7)$$

and are, therefore, orthonormal. Thus, their coefficients are the coordinate values of a point of n-dimensional space having an orthogonal coordinate system. The values of the various E_i's can be obtained by sampling the voltage spectrum of the baud waveform at frequencies 1/T, 3/T, 5/T, etc. These samples are complex quantities and the real parts are $\sqrt{T}\,E_1$, $\sqrt{T}\,E_3$,

$\sqrt{T}\,E_5$ etc. and the imaginary parts are $\sqrt{T}\,E_2$, $\sqrt{T}\,E_4$, $\sqrt{T}\,E_6$, etc. at the frequencies 1/T, 3/T, 5/T, etc.

Either of the above signal spaces are useful in the analysis of binary systems using wideband waveforms and the type of space which is selected to be used in a particular analysis will, in general, depend on the type of noise.

2.3 The transmitter

A binary communications system consists of a transmitter and a receiver, both of whose parameters are under the control of the designer, and the channel whose parameters are not. In Fig. 2.3-1 is shown a binary communications link. The purpose of the transmitter is to produce a suitable waveform $e_1(t)$ depending on the binary input which is recognizable to the receiver. The noise $n(t)$ which is added in the channel, however, causes the waveform which arrives at the receiver to be somewhat different with the resulting possibility that the receiver may make an incorrect interpretation. Therefore, it is important that the waveforms produced by the transmitter be designed to produce as small a probability of error as possible under the constraints that may be present.

The waveforms for each baud which are produced by the transmitter can be represented by 2WT-dimensional vectors. Fig. 2.3-2 shows the signal vectors corresponding to a "mark" and to a "space" for one such baud. In order to minimize the probability that the receiver will misinterpret the received signal it is clear that the two signal points be placed as far apart as possible, i.e., the mark vector $\vec{M}$ and the space vector $\vec{S}$ should be diametrically opposed,

$$\vec{S} = -k\,\vec{M} \qquad (2.3\text{-}1)$$

If the transmitter has an average power constraint it can easily be shown that the separation between the signal points is maximum when

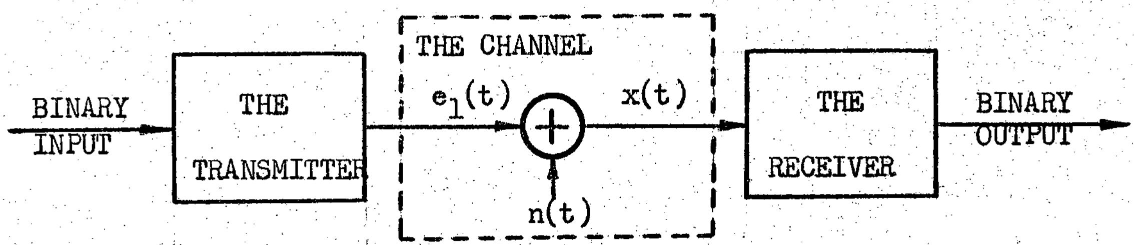
$$k = \frac{P(M)}{P(S)} \qquad (2.3\text{-}2)$$

where P(M) is the probability that the baud will be a "mark" and P(S) is the probability that it will be a "space".

2.4 <u>The receiver</u>

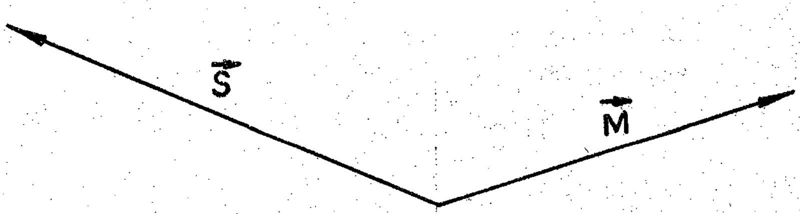
The purpose of a receiver in a binary communication system is to determine and indiciate whether a "mark" or a "space" has been transmitted. Only in the absence of noise or at least the presence of a very special kind of noise is it possible to perform this function with complete certainty, and when noise is added in the channel we may expect the receiver to make errors from time to time. In general, subject to the constraints of high cost and reduced equipment reliability which usually accompany complexity, a receiver's utility is measured in terms of its capability of detecting with the least probability of error which signal was sent. This process is accomplished by dividing the signal space of the channel into a "mark" region and a "space" region. If the received signal plus noise vector falls in the "mark" region R_m the receiver will decide that a "mark" has been transmitted and if it falls into the "space" region R_s then the decision is that a "space" was sent. The hypersurface separating these two regions is referred to as the decision surface.

Although the decision process can be performed directly on the signal space of the channel it is more convenient to first transform the multidimensional space of the channel to a one-dimensional decision space. This process will be referred to as demodulation. In this way the decision



A BINARY COMMUNICATIONS SYSTEM

FIGURE 2.3-1



THE OUTPUT SIGNAL SPACE OF A TRANSMITTER

FIGURE 2.3-2

process is reduced to the relatively simply task of detecting whether the output of the demodulator is above or below a certain decision threshold value. For this type of receiver the decision surface in the channel signal space is determined by mapping the decision threshold value back into the channel space with the aid of the inverse demodulation transformation.

The probability of error P_e (on a per baud basis) for any binary receiver is the probability of the following event: that the signal plus noise vector $\vec{X}$ lies in the "space" region R_s and that a "mark" was sent (i.e. $\vec{E}_1 = \vec{M}$), or that the signal plus noise vector $\vec{X}$ lies in the "mark" region R_m and that a "space" was sent (i.e. $\vec{E}_1 = \vec{S}$). Since the events are mutually exclusive

$$P_e = \text{Prob}\left\{\vec{X} \text{ in } R_s, \vec{E}_1 = \vec{M}\right\} + \text{Prob}\left\{\vec{X} \text{ in } R_m, \vec{E}_1 = \vec{S}\right\} \tag{2.4-1}$$

which can also be written

$$P_e = P(M)\, \text{Prob}\left\{\vec{X} \text{ in } R_s \middle| \vec{E}_1 = \vec{M}\right\} + P(S)\, \text{Prob}\left\{\vec{X} \text{ in } R_m \middle| \vec{E}_1 = \vec{S}\right\} \tag{2.4-2}$$

where P(M) and P(S) are the respective probabilities that a "mark" and a "space" were transmitted and where Prob $\left\{\vec{X} \text{ in } R_s \middle| \vec{E}_1 = \vec{M}\right\}$ is the conditional probability that "$\vec{X}$ is in R_s given $\vec{E}_1 = \vec{M}$", etc. The conditional probability density functions over the appropriate regions

$$\text{Prob}\left\{\vec{X} \text{ in } R_s \middle| \vec{E}_1 = \vec{M}\right\} = \int_{R_s} p_{xm}(\vec{X}|M)\, d\vec{X} \qquad (2.4\text{-}3)$$

$$\text{Prob}\left\{\vec{X} \text{ in } R_m \middle| \vec{E}_1 = \vec{S}\right\} = \int_{R_m} p_{xs}(\vec{X}|S)\, d\vec{X} \qquad (2.4\text{-}4)$$

The above conditional probability density functions can be related to the joint density function of the noise $p_n(\vec{N})$ with the aid of the fact that $\vec{N} = \vec{X} - \vec{E}_1$. Therefore,

$$p_{xm}(\vec{X}|M) = p_n(\vec{X} - \vec{M}) \qquad (2.4\text{-}5)$$

and

$$p_{xs}(\vec{X}|S) = p_n(\vec{X} - \vec{S}) \qquad (2.4\text{-}6)$$

By substituting Eq. (2.4-3), (2.4-4), (2.4-5) and (2.4-6) into Eq. (2.4-2) the following expression for the probability of error is obtained.

$$P_e = P(M) \int_{R_s} p_n(\vec{X} - \vec{M})\, dX$$

$$+ P(S) \int_{R_m} p_n(\vec{X} - \vec{S})\, dX \qquad (2.4\text{-}7)$$

As mentioned above this is the probability of error of a particular baud. If any of the quantities in the right hand side of Eq. (2.4-7) vary from baud to baud then, in general, the probability of error may be expected to vary also. In such an instance the average probability of error P_e must be computed in order to evaluate the performance of the receiver.

2.5 Optimum receivers

In the analysis of binary communications systems where the design of the transmitter has already been fixed and the joint probability density of the noise is known, it is often of interest to determine the parameters of the optimum receiver where an optimum receiver is defined as one in which the probability of error is minimum, i.e., the output is a "mark" for the case

$$P(M|\vec{X}) > P(S|\vec{X}) \tag{2.5-1}$$

and the output is a "space" if

$$P(M|\vec{X}) < P(S|\vec{X}) \tag{2.5-2}$$

where $P(M|\vec{X})$ is the conditional probability that a "mark" was sent given the waveform $\vec{X}$ is received and $P(S|\vec{X})$ is the conditional probability that a "space" was sent given the waveform $\vec{X}$ was received.

By applying Bayes' Theorem to inequalities (2.5-1) and (2.5-2) the following decision rules are obtained: If

$$p_n(\vec{X} - \vec{M})\, P(M) > p_n(\vec{X} - \vec{S})\, P(S) \tag{2.5-3}$$

then the optimum decision is "mark" and if

$$p_n(\vec{X} - \vec{M})\, P(M) < p_n(\vec{X} - \vec{S})\, P(S) \tag{2.5-4}$$

then the optimum decision is "space".

The surface which separates the region where the optimum decision is a "mark" and the region where the optimum decision is a "space" is given by

$$p_n(\vec{X} - \vec{M})\, P(M) = p_n(\vec{X} - \vec{S})\, P(S) \qquad (2.5\text{-}5)$$

Quite often the likelihood ratio[7] is used in the specification of the rules for optimum detection.

The likelihood ratio* is defined as

$$L = \frac{P(M|\vec{X})}{P(S|\vec{X})} \qquad (2.5\text{-}6)$$

which can also be written

$$L = \frac{P(\vec{X}|M)\, P(M)}{P(\vec{X}|S)\, P(S)} = \frac{p_n(\vec{X} - \vec{M})\, P(M)}{p_n(\vec{X} - \vec{S})\, P(S)} \qquad (2.5\text{-}7)$$

Clearly the optimum decisions are "mark" if $L > 1$ and "space" if $L < 1$ and the decision surface is given by the equation

$$L = 1. \qquad (2.5\text{-}8)$$

2.6 Signal waveform generation

In order to produce either the "mark" or "space" waveforms at the transmitter it may be desirable to perform a suitable operation on a signal carrier waveform. This operation will produce a change in the carrier waveform such that the required "mark" or "space" waveform results.

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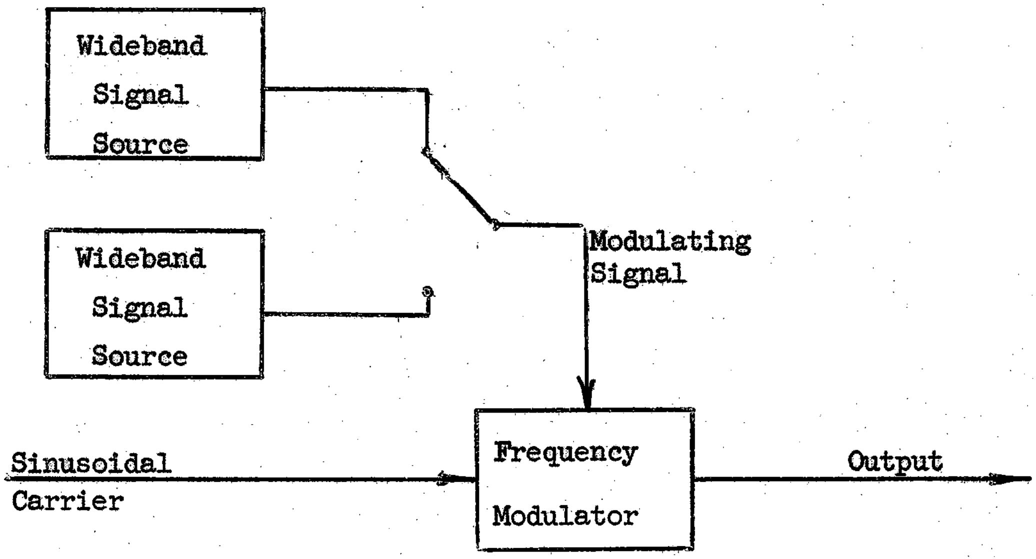
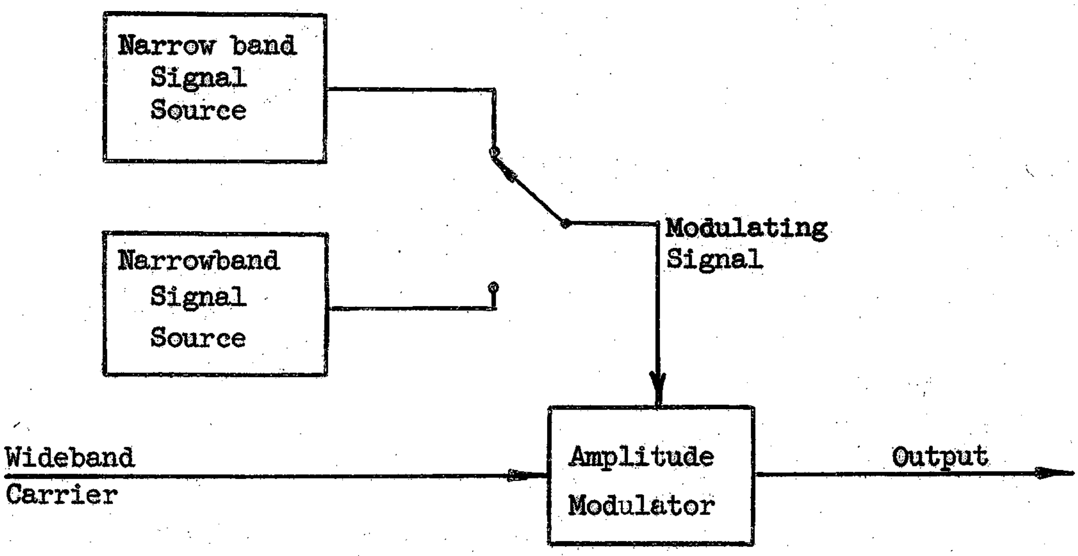
* It should be noted that this is a more general form than used by many authors. For the case $P(M) = P(S)$ this form reverts to the more usual $p(\vec{X}|M) \,/\, p(\vec{X}|S)$.



A METHOD OF PRODUCING A WIDEBAND SIGNAL PAIR BY THE

USE OF FREQUENCY MODULATION

FIGURE 2.6-1



A METHOD OF PRODUCING A WIDEBAND SIGNAL PAIR BY THE

USE OF AMPLITUDE MODULATION

FIGURE 2.6-2

This type of operation will be refered to as modulation.

Many different configurations are possible. For example, a sinusoidal carrier may be frequency modulated by a wideband signal as is shown in Fig. 2.6-1. On the other hand, the signal to be transmitted might be the result of amplitude modulating a relatively wideband carrier signal with a narrowband modulation signal (see Fig. 2.6-2).

In spite of the many ways in which the modulation can take place it is possible to make use of a generalized approach.

Let the unmodulated carrier be written in the following form

$$c(t) = A_c(t) \cos\left[\omega_c t + \phi_c(t) \right] \tag{2.6-1}$$

and the modulated carrier be written

$$e(t) = B(t) \cos\left[\omega_c t + \phi(t) \right] \tag{2.6-2}$$

It is evident from Eq. (2.6-1) and Eq. (2.6-2) that modulation can produce a change in the multiplying amplitude factor or in the magnitude of the angle and therefore it follows that the two basic classes of modulation are: (1) amplitude modulation where

$$A_m(t) = B(t) / A_c(t) \tag{2.6-3}$$

is the amplitude modulation waveform and

(2) angle modulation where

$$\phi_m = \theta(t) - \phi_c(t) \tag{2.6-4}$$

is the angle modulation waveform. Thus, the "mark" or "space" waveforms can be written in the following form.

$$m(t) = A_m(t)\, A_c(t) \cos\left[\omega_c t + \phi_c(t) + \phi_m(t) \right] \tag{2.6-5}$$

$$s(t) = A_s(t)\, A_c(t) \cos\left[\omega_c t + \phi_c(t) + \phi_s(t) \right] \tag{2.6-6}$$

These waveforms can also be expressed in the following equivalent forms.

$$\begin{aligned} m(t) &= A_m(t)\, A_c(t) \cos\left[\omega_c t + \phi_c(t) \right] \cos \phi_m(t) \\ &\quad - A_m(t)\, A_c(t) \sin\left[\omega_c t + \phi_c(t) \right] \sin \phi_m(t) \end{aligned} \tag{2.6-7}$$

$$\begin{aligned} s(t) &= A_s(t)\, A_c(t) \cos\left[\omega_c t + \phi_c(t) \right] \cos \phi_s(t) \\ &\quad - A_s(t)\, A_c(t) \sin\left[\omega_c t + \phi_c(t) \right] \sin \phi_s(t) \end{aligned} \tag{2.6-8}$$

It is evident that the waveforms given by Eq. (2.6-7) and Eq. (2.6-8) can be produced through the use of amplitude modulation by the scheme shown in Fig. 2.6-3 where the carrier waveform is given by Eq. (2.6-1) and the modulation waveform is given by

$$A'_m(t) = A_m(t) \cos \phi_m(t) \tag{2.6-9}$$

if a "mark" is to be transmitted and

$$A'_s(t) = A_s(t) \cos \phi_s(t) \tag{2.6-10}$$

if a "space" is to be transmitted. The waveforms given in Eq. (2.6-9) and Eq. (2.6-10) will be referred to as composite modulation waveforms.

It may be expected that, in general, the probability of error of the receiver output will depend on the nature of the signal waveforms. As was shown in Section 2.2, these waveforms are completely specified by points in an appropriate signal space. The important characteristic of a binary signal pair is the vector which is the difference of the position vectors of the signal points, viz., $\vec{M} - \vec{S}$. In fact, it is evident that this difference vector is the only characteristic of the signal pair which influences the probability of error and the probability of error will be determined both by its magnitude and its direction. Furthermore, if the equiprobable surfaces of the probability density function of the noise are spherical (as would be the case if the noise were white gaussian noise) then it is clear that the probability of error depends only on the magnitude of the difference vector, i.e. $|\vec{M} - \vec{S}|$.

In some applications it may be desirable to use a non-repeating carrier waveform and, therefore, it is of interest to determine the conditions under which the probability of error is independent of the shape of the carrier waveform (i.e. the direction of the carrier vector). For this reason the following theorem is important.

Theorem If (1) the equiprobable surfaces of the joint probability density of the noise are spherical and if (2) the spectra of the carrier waveform and the composite modulation waveform do not overlap and if (3) the energy of the carrier waveform is uniformly distributed over the duration of the baud then the probability of error of the output of an optimum receiver is independent of the carrier waveform.

Since by hypothesis, the equiprobable surfaces of the joint probability density of the noise are spherical, it is sufficient to prove that the square of the magnitude of the difference vector $|\vec{M} - \vec{S}|^2$ is independent of the shape of the carrier waveform.

This quanity can be written in the following form

$$|\vec{M} - \vec{S}|^2 = \vec{M}\cdot\vec{M} + \vec{S}\cdot\vec{S} - 2\vec{M}\cdot\vec{S} \qquad (2.6\text{-}11)$$

The terms on the right hand side of Eq. (2.6-11) can be evaluated directly from the time functions given in Eq. (2.6-5) and Eq. (2.6-6).

$$\vec{M}\cdot\vec{M} = \int_0^T A_m^2(t)\, A_c^2(t)\, \cos^2\left[\omega_c t + \phi_c(t) + \phi_m(t)\right] dt$$

which can be written

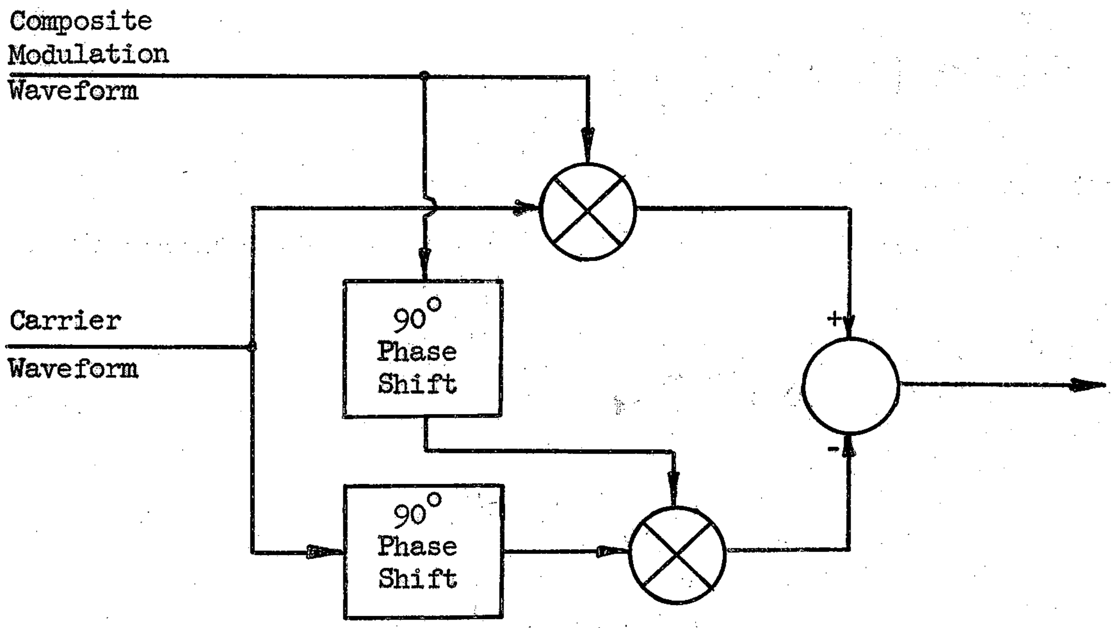
$$\vec{M}\cdot\vec{M} = 1/2 \int_0^T A_m^2(t)\, A_c^2(t)\, dt$$

$$+ 1/2 \int_0^T A_m^2(t)\, A_c^2(t)\, \cos 2\left[\omega_c t + \phi_c(t) + \phi_m(t)\right] dt \qquad (2.6\text{-}12)$$

The second term of the right hand side of Eq. (2.6-12) can be written in the following form.

$$\frac{1}{2}\int_0^T \left[A_c^2(t)\cos 2\left[\omega_c t + \phi_c(t)\right]\right]\left[A_m^2(t)\cos 2\phi_m(t)\right] dt$$

$$+ \frac{1}{2}\int_0^T \left[A_c^2(t)\sin 2\left[\omega_c t + \phi_c(t)\right]\right]\left[A_m^2(t)\sin 2\phi_m(t)\right] dt$$



A METHOD OF OBTAINING A GENERALIZED MODULATION

FIGURE 2.6-3

Since, according to the hypothesis, the spectra of the carrier and the composite modulation do not overlap, the spectra of two factors in the integrands of the two integrals in the above expression do not overlap. Thus, the factors, in the integrands are orthogonal and, therefore, the values of the integrals are zero.

Thus, Eq. (2.6-12) becomes

$$\vec{M}\cdot\vec{M} = 1/2 \int_0^T A_m^2(t)\, A_c^2(t)\, dt \qquad (2.6\text{-}13)$$

In a similar fashion the values of $\vec{S}\cdot\vec{S}$ and $\vec{M}\cdot\vec{S}$ can be determined.

$$\vec{S}\cdot\vec{S} = 1/2 \int_0^T A_s^2(t)\, A_c^2(t)\, dt \qquad (2.6\text{-}14)$$

$$\vec{M}\cdot\vec{S} = 1/2 \int_0^T A_m(t)\, A_s(t)\, A_c^2(t) \cos\left[\phi_m(t) - \phi_s(t)\right] dt \qquad (2.6\text{-}15)$$

By substituting Eqs. (2.6-13), (2.6-14) and (2.6-15) into Eq. (2.6-11) the following expression is obtained

$$|\vec{M} - \vec{S}|^2 = 1/2 \int_0^T A_m^2(t)\left[A_m^2(t) + A_s^2(t) - 2 \cos\left[\phi_m(t) - \phi_s(t)\right]\right] dt \qquad (2.6\text{-}16)$$

Since the energy of the carrier is evenly distributed over the duration of the baud, the quantity $A_c^2(t)$ is a constant and, thus, it is evident that the quantity given by Eq. (2.6-16) is independent of the shape of the carrier waveform.

Eq. (2.6-16) is also generally useful in determining the effect of waveform variation when different schemes of modulation are used under the conditions where the equiprobable surfaces of the noise are spherical and where the carrier and modulation spectra do not overlap. For, example, if an amplitude modulation scheme is used where

$$A_m(t) = 1 \tag{2.6-17}$$

and

$$A_s(t) = 1 \tag{2.6-18}$$

then, according to Eq. (2.6-16)

$$|\vec{M} - \vec{S}|^2 = 2 \int_0^T A_c^2(t)\, dt \tag{2.6-19}$$

From Eq. (2.6-19) it is clear that the probability of error does not depend on the carrier waveshape in any way and, therefore, the carrier energy need not be evenly distributed over the duration of the baud.

It should be noted that the hypothesis of the theorem consists of a set of conditions which are sufficient to insure that the probability of error is independent of the carrier waveform. It is not necessary that all three conditions be fulfilled in every case as is shown by the above example.

2.7 Information efficiency

The information efficiency[5] has been defined as

$$\eta = \frac{H(x) - H(x|y)}{H(x)} \times 100$$

where H(x) is the data rate of the source and H(x|y) is the equivocation of the channel. It has been shown that this quantity depends only on source statistics (i.e. P(M) and P(S)) and on the transitional error probabilities (i.e. the probability that a "mark" is received as a "space" and probability that a "space" is received as a "mark").

In general, the results which will be obtained will be for symmetrical systems. Under these circumstances

$$H(x) = 1$$

and

$$H(x|y) = -\left[P_e \log_2 P_e + (1 - P_e) \log_2 (1 - P_e) \right]$$

and the information efficience is uniquely related to the probability of error.

In order to be able to compare the performance of wideband systems with narrow band systems the following definition will be made.

Definition Two binary communications signal pairs will be said to correspond if the energy of the mark vectors are equal and the energy of the space vectors are equal and if the inner product of the "mark" and the "space" vectors (i.e. $\vec{M}\cdot\vec{S}$) are equal. To have the inner product $\vec{M}\cdot\vec{S}$ equal at the same time that the mark energies are equal and the space energies are equal is equivalent to having the mark-space correlation coefficients equal.

If white gaussian noise is added in the channel the efficiency is affected by the placement of the signal points according to the magnitude of the vector difference $|\vec{M}-\vec{S}|$. It is evident from Eq. (2.6-11) that, under these conditions a wideband signal pair offers no advantage over the corresponding narrowband signal pair.

2.8 Summary

In Chapter II the concepts of the signal space have been reviewed and the way in which they apply to a binary communications system has been discussed. Methods by which the "mark" and "space" waveform may be generated from a single carrier have been discussed and the criterion for making the system performance independent of the carrier waveshape has been derived.

The efficiency of wideband communications systems are discussed and it is shown that, under the condition that the channel noise is white and gaussian, that a wideband signal pair offers no inherent advantage over the corresponding narrowband pair.

CHAPTER III

LINEAR RECEIVERS

3.1 Introduction

Synchronous receivers and matched filter receivers have been extensively employed where superior performance has been desired. It has been shown[8] that these types of receivers given optimum performance when the noise is gaussian. In this chapter a more general theory for the performance of these receivers will be developed. This theory will be applied in subsequent chapters to determine the performance of these receivers when the noise is non-gaussian.

3.2 Signal space analysis of the synchronous receiver

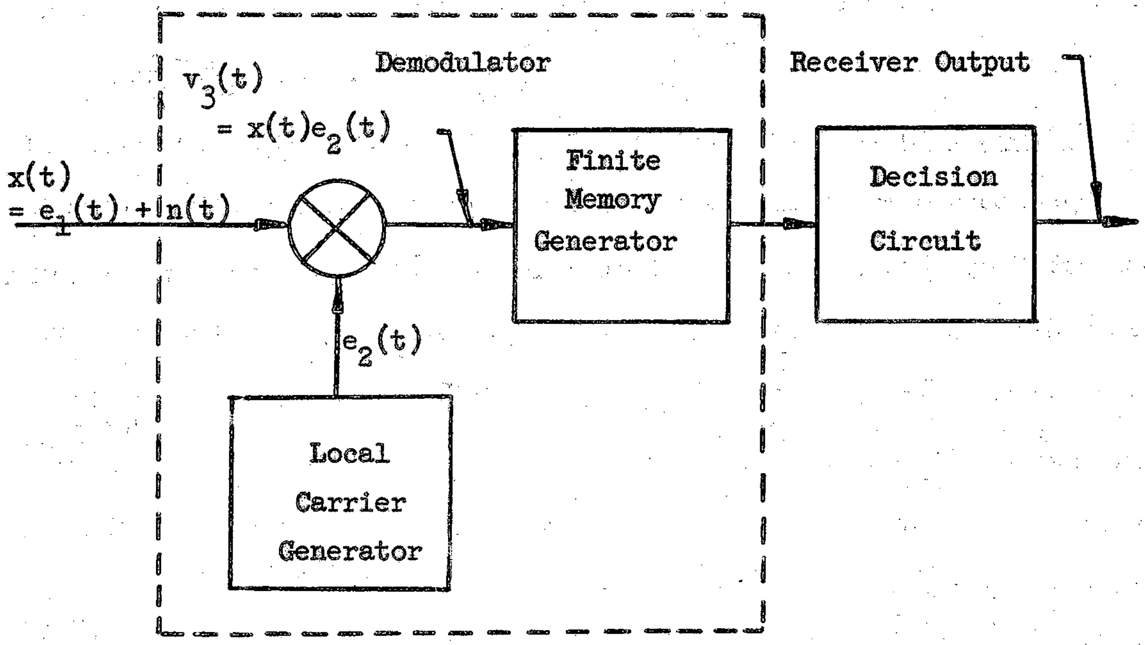
A typical synchronous receiver is shown in Fig. 3.2-1. In this receiver the input $x(t) = e_1(t) + n(t)$ (where $e_1(t)$ is the signal component and $n(t)$ is the noise component of the input) is multiplied by the local carrier $e_2(t)$ producing the input to the finite memory integrator $v_3(t) = x(t)\, e_2(t)$. The output of the finite memory integrator as a function of its input is

$$v_4(t) = \int_{t-T}^{t} v_3(u)\, du \qquad (3.2\text{-}1)$$

Here the lenth of the memory T_m has been chosen equal to the baud length T. One method of realizing the input-output characteristic given in Eq. (3.2-1) is shown in Fig. 3.2-2.

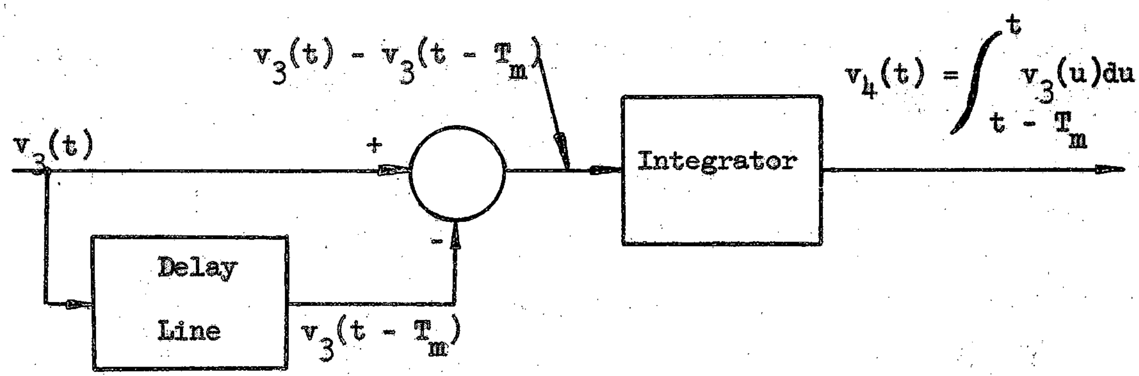
At $t = T$ the decision circuit samples its input which is

$$V_d = v_4(T) = \int_0^T (e_1(t) + n(t))\, e_2(t)\, dt \qquad (3.2\text{-}2)$$



A SYNCHRONOUS RECEIVER

FIGURE 3.2-1



A FINITE MEMORY INTEGRATOR

FIGURE 3.2-2

If this input is greater than some threshold level V_{do} then the decision circuit will indicate for example, that a "mark" had been transmitted. If, however, V_d is less than V_{do} then a "space" will be indicated.

If the received signal, received noise and local carrier waveforms are bandlimited to a bandwidth W then they each can be represented by the coefficients of the 2WT orthonormal functions $f_i(t)$ in the following series representations.

$$e\ (t) = \sum_{i=1}^{2WT} E_{1i}\ f_i(t) \tag{3.2-3}$$

$$n(t) = \sum_{i=1}^{2WT} N_i\ f_i(t) \tag{3.2-4}$$

$$e_2(t) = \sum_{i=1}^{2WT} E_{2i}\ f_i(t) \tag{3.2-5}$$

where $e_1(t)$ $n(t)$ and $e_2(t)$ are assumed to be strictly time limited.

The following expression for the output of the integrator V_d is obtained by substituting Eq. (3.2-3), (3.2-4) and (3.2-5) into Eq. (3.2-2).

$$V_d = \int_0^T \left[\sum_{i=1}^{2WT} (E_{1i} + N_i)\ f_i(t) \right] \left[\sum_{j=1}^{2WT} E_{2j}\ f_j(t) \right] dt$$

$$= \sum_{i=1}^{2WT} \sum_{j=1}^{2WT} (E_{1i} + N_i)\ E_{2j} \int_0^T f_i(t)\ f_j(t)\ dt \tag{3.2-6}$$

According to the definition given in Chapter II the orthonormal functions $f_i(t)$ has the following property

$$\int_0^T f_i(t)\, f_j(t)\, dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

which in conjunction with Eq. (3.2-6) yields the following expression for V_d

$$V_d = \sum_{i=1}^{2WT} (E_{1i} + N_i)\, E_{2i} \qquad (3.2\text{-}7)$$

From Eq. (3.2-7) it is clear that V_d is the inner product between the received signal plus noise vector $\vec{X} = \vec{E}_1 + \vec{N}$ and the local carrier vector $\vec{E}_2$ and can be written

$$V_d = (\vec{E}_1 + \vec{N})\cdot\vec{E}_2 = \vec{X}\cdot\vec{E}_2 \qquad (3.2\text{-}8)$$

Eq. (3.2-8) is the mathematical expression for the demodulation process by which the multidimensional variable $\vec{X}$ is transformed into the one-dimensional variable V_d. Since this is a linear transformation, this type of receiver will be referred to as a linear receiver.

Eq. (3.2-8) can be written in the following form

$$\frac{V_d}{|\vec{E}_2|} = X\cdot\left\{\frac{\vec{E}_2}{|\vec{E}_2|}\right\} \qquad (3.2\text{-}9)$$

where $|\vec{E}_2|$ is the magnitude of the vector $\vec{E}_2$. From Eq. (3.2-9) it is evident that $V_d/|\vec{E}_2|$ is the length of the projection of the vector $\vec{X}$ onto the vector $\vec{E}_2$. From this it is clear that the equation

$$V_{do} = \vec{X}\cdot\vec{E}_2 \qquad (3.2\text{-}10)$$

where V_{do} is an arbitrary constant is satisfied by the set of vectors

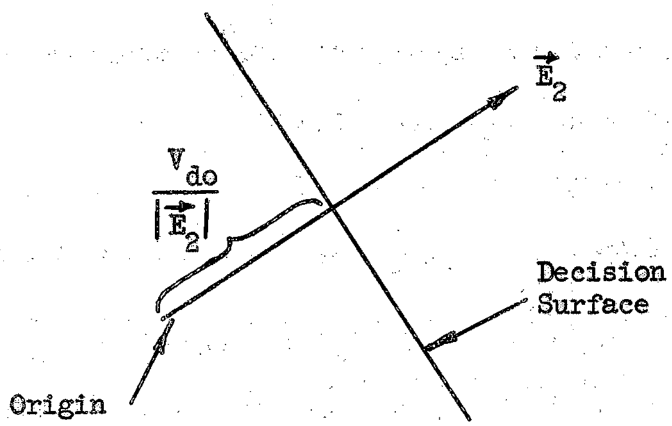
$\vec{X} = \vec{X}_d$ whose end points lie on the hyperplane which is orthogonal to $\vec{E}_2$ and intersects $\vec{E}_2$ at a distance $V_{do}/|\vec{E}_2|$ from the origin. If V_{do} is the decision threshold voltage then the hyperplane determined by Eq. (3.2-10) is the decision surface which is generated by the synchronous receiver. Fig. 3.2-3 shows the location of this decision surface in a typical 2-dimensional subspace containing the vector $\vec{E}_2$.

If, for example, $V_d > V_{do}$ is interpreted as a "mark" then the "mark" decision region R_m will be to the right of the decision surface in Fig. 3.2-3 and if $V_d < V_{do}$ is a "space" then the "space" decison region R_s is to the left of the decision surface.

For a given binary system the waveforms which are transmitted to represent a "mark" (i.e. $\vec{E}_1 = \vec{M}$) and a "space" (i.e. $\vec{E}_1 = \vec{S}$) will be known along with their probabilities of transmission P(M) and P(S). If, in addition to the above information the joint probability density of the noise in the channel $p_n(\vec{N})$ is known, it is possible to calculate the probability of error P_e with the aid of Eq. (2.4-7), viz.

$$P_e = P(M) \int_{R_s} p_n\,(\vec{X} - \vec{M})\, d\vec{X} + P(S) \int_{R_m} p_n\,(\vec{X} - \vec{S})\, d\vec{X} \qquad (2.4\text{-}7)$$

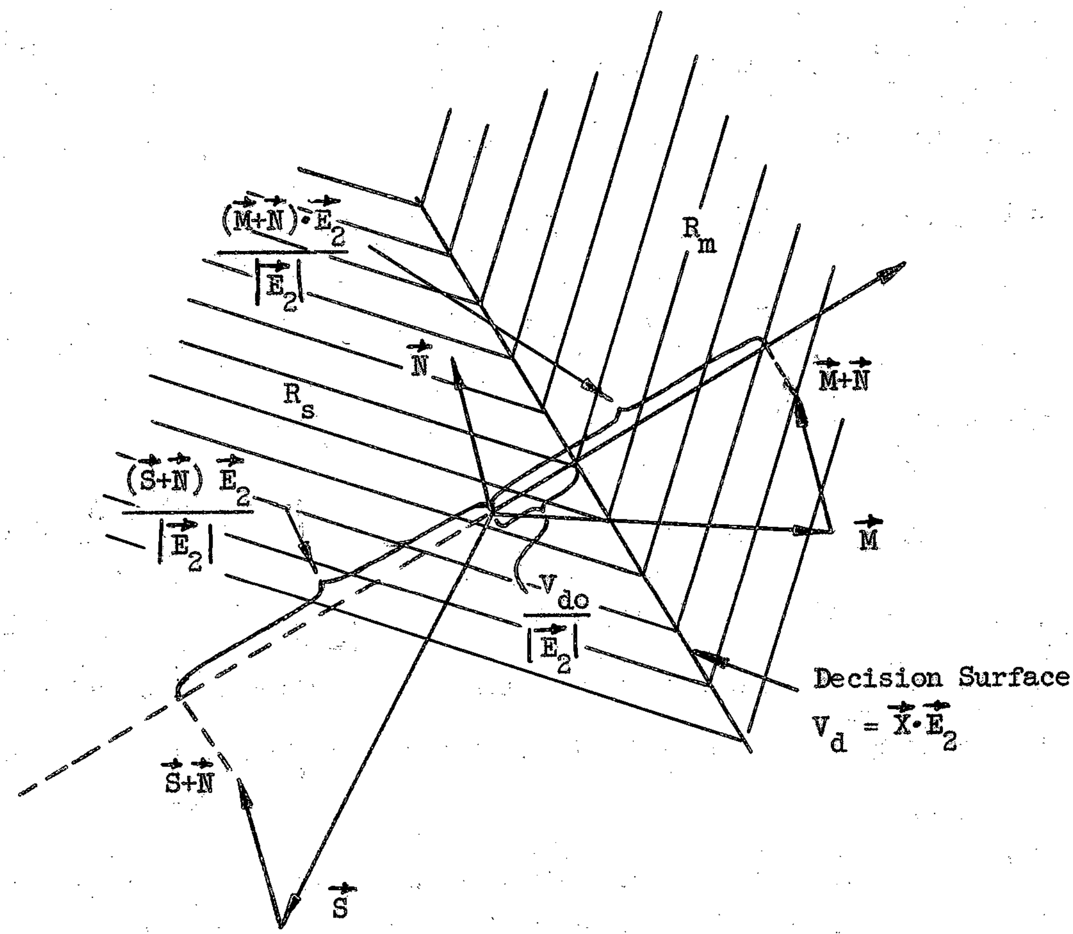
The evaluation of the integrals in the right hand side of Eq. (2.4-7) is tedious and involved if the dimensionality of the signal is very large. Fortunately, however, there is an alternative approach which can be used to obtain a simplified solution for certain important special cases.



THE GEOMETRICAL REPRESENTATION OF THE DECISION SURFACE

GENERATED BY A SYCHRONOUS RECEIVER

FIGURE 3.2-3



THE GEOMETRICAL REPRESENTATION OF THE OPERATION OF A SYNCHRONOUS RECEIVER

FIGURE 3.2-4

According to Eq. (2.4-2) the probability of error is

$$P_e = P(M)\ \text{Prob}\ \left\{ \vec{X} \text{ in } R_s \mid \vec{E}_1 = \vec{M} \right\}$$

$$+ P(S)\ \text{Prob}\ \left\{ \vec{X} \text{ in } R_m \mid \vec{E}_1 = \vec{S} \right\} \qquad (2.4\text{-}2)$$

In terms of the decision space "$\vec{X}$ in R_s" corresponds to $V_d < V_{do}$ and "$\vec{X}$ in R_m" corresponds to $V_d > V_{do}$ and the expression for the probability of error becomes

$$P_e = P(M)\ \text{Prob}\ \left\{ V_d < V_{do} \mid \vec{E}_1 = \vec{M} \right\}$$

$$+ P(S)\ \text{Prob}\ \left\{ V_d > V_{do} \mid \vec{E}_1 = \vec{S} \right\} \qquad (3.2\text{-}11)$$

which with the aid of Eq. (3.2-7) becomes

$$P_e = P(M)\ \text{Prob}\ \left\{ \vec{N}\cdot\vec{E}_2 < V_{do} - \vec{M}\cdot\vec{E}_2 \right\}$$

$$+ P(S)\ \text{Prob}\ \left\{ \vec{N}\cdot\vec{E}_2 > V_{do} - \vec{S}\cdot\vec{E}_2 \right\} \qquad (3.2\text{-}12)$$

The quantity

$$\vec{N}\cdot\vec{E}_2 = \sum_{i=1}^{2WT} N_i E_{2i} \qquad (3.2\text{-}13)$$

is a random variable which is the sum of 2WT random variables and will have a gaussian distribution under either of the following conditions:

(1) That each of the random variables N_i has a gaussian distribution, or

(2) That each of the random variables $N_i E_{2i}$ is independent and has a common distribution (here it is assumed that the number of random variables $N_i E_{2i}$ is sufficiently large for the central limit theorem to apply).

Clearly if $\vec{N} \cdot \vec{E}_2$ is the sum of several groups of $N_i E_{2i}$ such that each group falls into one or the other of the two categories described above then the sum of each group is a gaussian random variable and therefore $\vec{N} \cdot \vec{E}_2$ will also have a gaussian distribution. The mean $\overline{V_n}$ and variance σ_n^2 of this random variable will be the respective sums of the means and variances of the individual random variables $N_i E_{2i}$.

$$\overline{V}_n = \sum_{i=1}^{2WT} \overline{N}_i \, E_{2i} \tag{3.2-14}$$

$$\sigma_n^2 = \sum_{i=1}^{2WT} \sigma_i^2 \tag{3.2-15}$$

where

$$\sigma_i^2 = \left[\overline{N_i^2} - (\overline{N}_i)^2 \right] E_{2i}^2 \tag{3.2-16}$$

The probability of error is

$$P_e = \frac{P(M)}{\sqrt{2\pi}\,\sigma_n} \int_{-\infty}^{V_{do} - \vec{M}\cdot\vec{E}_2} \exp\left[- \frac{(v - \overline{V}_n)^2}{2\,\sigma_n^2} \right] dv + \frac{P(S)}{\sqrt{2\pi}\,\sigma_n} \int_{V_{do} - \vec{S}\cdot\vec{E}_2}^{\infty} \exp\left[- \frac{(v - \overline{V}_n)^2}{2\,\sigma_n^2} \right] dv \tag{3.2-17}$$

which in terms of error functions can be written

$$P_e = \frac{P(M)}{2} \left\{ 1 + \operatorname{erf}\left[\frac{V_{do} - \vec{M}\cdot\vec{E}_2 - \overline{V}_n}{\sqrt{2}\,\sigma_n} \right] \right\} + \frac{P(S)}{2} \left\{ 1 + \operatorname{erf}\left[\frac{V_{do} - \vec{S}\cdot\vec{E}_2 - \overline{V}_n}{\sqrt{2}\,\sigma_n} \right] \right\} \tag{3.2-18}$$

3.3 Equivalent linear receivers

In Section 3.2 the synchronous receiver was analyzed. In this section the matched filter receiver and hybrid combinations of the matched filter and synchronous receivers will be considered.

Fig. 3.3-1 shows a matched filter receiver. The input signal plus noise is filtered and at the end of the baud the output of the filter $v_4(t)$ is sampled and the decision circuit determines the appropriate output depending on whether the sample is above or below a certain threshhold value.

The output of the filter is

$$v_4(t) = \int_{-\infty}^{t} x(u)\, h(t - u)\, du \qquad (3.3\text{-}1)$$

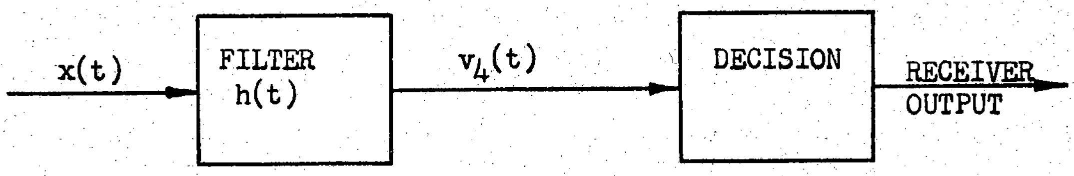
which at the end of the baud (i.e. t = T) when the filter output is sampled is

$$V_d = v_4(T) = \int_{-\infty}^{T} x(u)\, h(T - u)\, du \qquad (3.3\text{-}2)$$

It may be assumed that in order to avoid intersymbol interference the duration of the filter impluse response is made equal to the baud length. This along with the fact that the second factor in the integral of Eq. (3.3-2) is equal to the waveform to which the filter is matched allows Eq. (3.3-2) to be rewritten to the following form

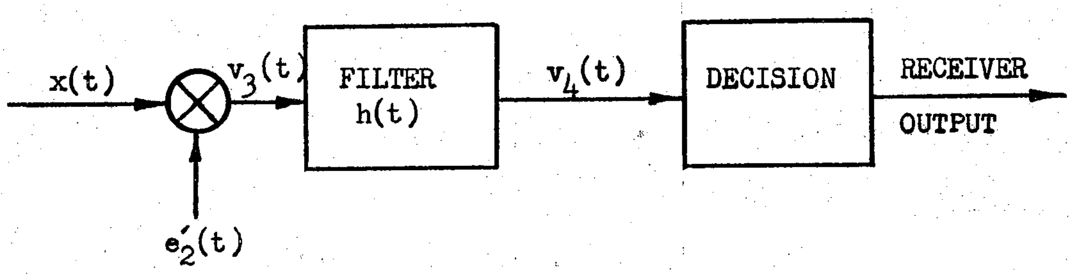
$$V_d = \int_{0}^{T} x(u)\, e_2(u)\, du \qquad (3.3\text{-}3)$$

where $e_2(u) = h(T - u)$.



MATCHED FILTER RECEIVER

FIGURE 3.3-1



HYBRID RECEIVER

FIGURE 3.3-2

Eq. (3.3-3) is of the same form as Eq. (3.2-2) which is the equation describing the performance of the demodulator of a synchronous receiver. Clearly the development which follows Eq. (3.2-2) also applies to Eq. (3.3-3) and the results which apply to the synchronous receiver also applies to the matched filter receiver.

In Fig. 3.3-2 is shown the block diagram of the hybrid receiver in which the process of matched filter reception and synchronous reception are combined. The output of the filter $v_4(t)$ is related to the receiver input x(t) by the following equation

$$v_4(t) = \int_{t-T}^{t} x(u)\, e_2'(u)\, h(t-u)\, du \qquad (3.3\text{-}4)$$

where the value of the lower limit of the integral is the result of limiting the length of the impulse response of the filter to a duration T. At the end of the baud the output of the filter is sampled by the decision circuit just as in the other receivers and the value of this sample is

$$V_d = v_4(T) = \int_0^T x(u)\, e_2'(u)\, h(T-u)\, du \qquad (3.3\text{-}5)$$

Designating the waveform to which the filter is matched by the function

$$\frac{e_2(t)}{e_2'(T)} = h\,(T-t) \qquad (3.3\text{-}6)$$

Eq. (3.3-5) can be reduced to

$$V_d = \int_0^T x(u)\, e_2(u)\, du\,. \qquad (3.3\text{-}7)$$

Eq. (3.3-7) is of the same form as Eq. (3.2-2) and the results which apply to the synchronous receiver are also applicable to the hybrid receiver.

Thus, it has been demonstrated that on a per baud basis the theoretical performance of the synchronous, matched filter and hybrid receivers are equivalent.

3.4 Summary

In this chapter the general performance of linear receivers has been derived. The class of linear receivers is that class of receivers which performs a linear transformation on the n - dimensional input signal in order to reduce it to a one dimensional signal. A typical example of a linear receiver, the synchronous receiver, is analyzed in detail and the results are extended to include matched filter receivers and other equivalent types.

CHAPTER IV

THE APPLICATION OF LINEAR RECEIVERS (Part I)

4.1 Introduction

In Chapter III the general theory of linear receivers has been developed with the aid of signal space concepts. These results will be used in this chapter as a basis for the analysis of the performance which can be expected from linear receivers for various kinds of interfering noise. Although some of the material which is developed in this chapter has more general application, the emphasis will be on noise which has independent spectral components.

4.2 Optimum reception with linear receivers

In Chapter III, it was shown that the decision surface generated by a linear receiver is a hyperplane. Thus a linear receiver can provide optimum reception only if the optimum decision surface is also a hyperplane. Optimum reception can then be accomplished with a linear receiver whose local carrier vector and decision threshold have been appropriately chosen. Clearly, if the optimum decision surface is not a hyperplane, optimum reception with a linear receiver is not possible.

In this section the types of noise which result in a hyperplane decision surface will be determined. In order to simplify the analyses it will be assumed that the source is symmetric and that the "space" waveform is the negative of the "mark", i.e.,

$$P(S) = P(M)$$

$$\vec{S} = -\vec{M} \qquad (4.2\text{-}1)$$

A binary communications system having the properties given by Eq. (4.2-1) will be referred to as a symmetric binary system. From Eq. (2.5-5) and Eq. (4.2-1) the equation for the optimum decision surface may be obtained, viz.,

$$p_n(\vec{X} - \vec{M}) = p_n(\vec{X} + \vec{M}) \tag{4.2-2}$$

Definition - a function $f(\vec{Z})$ is said to be symmetrical about the hyperplane

$$(\vec{Z} - \vec{B}) \cdot \vec{A} = 0 \tag{4.2-3}$$

if the variable vector $\vec{Z}$ in Eq. (4.2-3) also satisfied the Eq.

$$f(\vec{Z} + u\vec{A} - \vec{B}) = f(\vec{Z} - u\vec{A} - \vec{B}) \tag{4.2-4}$$

where $\vec{A}$ and $\vec{B}$ are fixed vectors and u is a variable scalar.

An example of this type of symmetry is given in Fig. 4.2-1 where typical curve of $f(\vec{Z})$ set equal to an arbitrary constant is shown as an illustration. Although the example is only two dimensional it is typical of any two dimensional cross section containing the vector $\vec{A}$ in an n-dimensional space.

Definition - A closed surface is said to be convex if there is no straight line which intersects the surface at more than two points.

With the aid of the above definition it is possible to state the following theorem

Theorem 4.2-1 The optimum surface of a symmetric binary system is a hyperplane if the joint density of the noise $p_n(\vec{N})$ is monotone decreasing with increasing noise magnitude $|\vec{N}|$ (while holding the angle of $\vec{N}$

constant) and if the equiprobable surfaces of $p_n(\vec{N})$ are every where convex and are symmetrical about the hyperplane $\vec{X}\cdot\vec{M} = 0$ where $\vec{M}$ is the "mark" signal vector. The optimum decision surface is the hyperplane of symmetry.

From the hypothesis and the definition for symmetry the following equation can be written

$$p_n(\vec{X} + \vec{M}) = p_n(\vec{X} - \vec{M}) \qquad (4.2\text{-}5)$$

where $\vec{X}$ satisfies the equation

$$\vec{X}\cdot\vec{M} = 0 \qquad (4.2\text{-}6)$$

Eq. (4.2-5) is identical to Eq. (4.2-2) which is the equation of the optimum decision surface and according to the above statement is satisfied when $\vec{X} \cdot \vec{M} = 0$, i.e., when $\vec{X}$ is a point on the hyperplane orthogonal to $\vec{M}$ and passing through the origin. It is now necessary to prove that Eq. (4.2-6) is the only solution to (4.2-5). Since $p_n(\vec{N})$ is monotone decreasing with increasing $|\vec{N}|$ there can be only one equiprobably surface for a given value of $p_n(\vec{N})$. Therefore the two sides of Eq. (4.2-5) must refer to points on the same equiprobably surface.

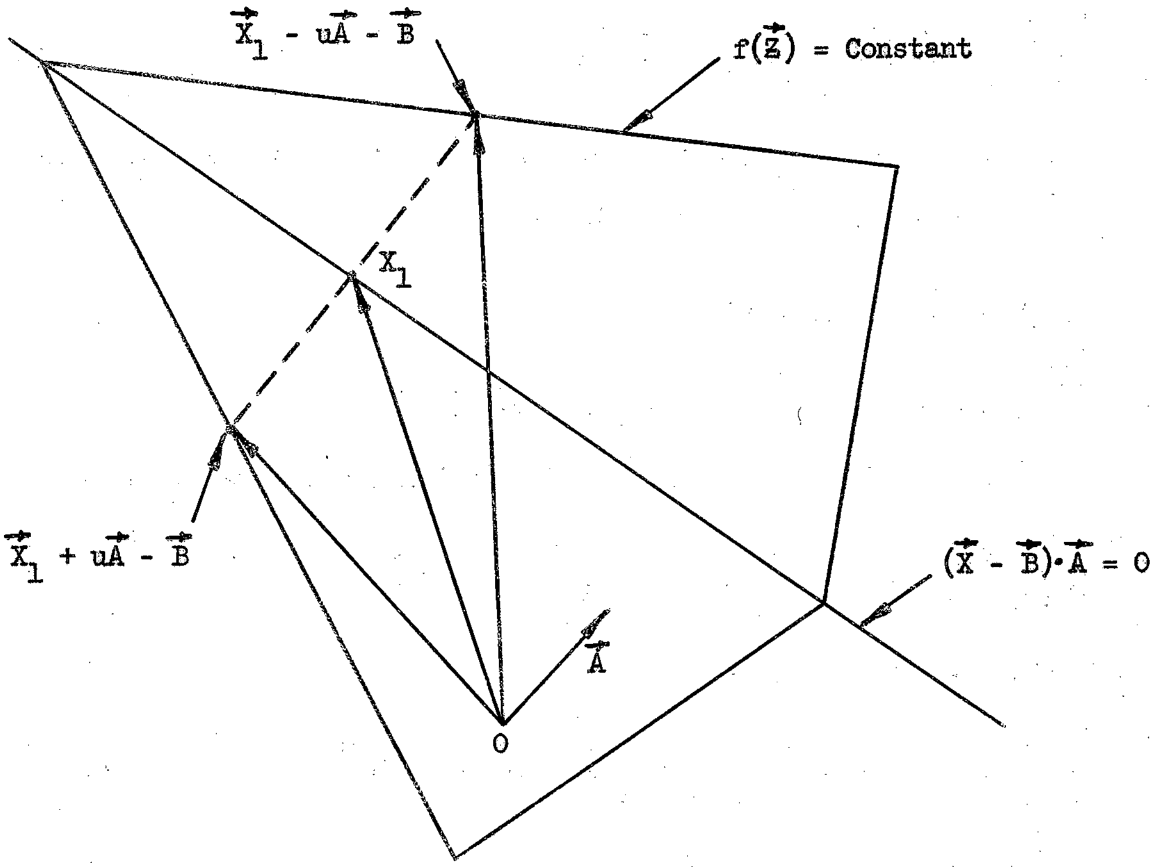
The rest of the proof is by contradiction. Suppose there is a point $\vec{C}$ not on the hyperplane $\vec{X} \cdot \vec{M} = 0$ which is a solution of Eq. (4.2-5), i.e.,

$$p_n(\vec{C} + \vec{M}) = p_n(\vec{C} - \vec{M}) \qquad (4.2\text{-}7)$$

where $\vec{C} \cdot \vec{M} \neq 0$. See Fig. 4.2-2

Let the scalar a and the vector $\vec{X}_1$ be chosen such that

$$\vec{C} = \vec{X}_1 + a\vec{M} \qquad (4.2\text{-}8)$$



AN EXAMPLE OF SYMMETRY ABOUT A PLANE

FIGURE 4.2-1

where $\vec{X}_1$ satisfies the relation $\vec{X}_1 \cdot \vec{M} = 0$.

Substituting Eq. (4.2-8) into Eq. (4.2-7) gives the following result

$$p_n(\vec{X}_1 + a\vec{M} + \vec{M}) = p_n(\vec{X}_1 + a\vec{M} - \vec{M}) \tag{4.2-9}$$

As a result of the symmetry which was specified in the hypothesis, the following equation must be satisfied

$$p_n(\vec{X} + u\vec{M}) = p_n(\vec{X} - u\vec{M}) \tag{4.2-10}$$

for all u and for all $\vec{X}$ satisfying the Eq. $\vec{X} \cdot \vec{M} = 0$.

Setting $\vec{X} = \vec{X}_1$ and $u = (1 + a)$, Eq. (4.2-10) can be written

$$p_n\left[\vec{X}_1 + (1 + a)\vec{M}\right] = p_n\left[\vec{X}_1 - (1 + a)\vec{M}\right] \tag{4.2-11}$$

and similarly with $u = (1 - a)$

$$p_n\left[\vec{X}_1 + (1 - a)\vec{M}\right] = p_n\left[\vec{X}_1 - (1 - a)\vec{M}\right] \tag{4.2-12}$$

From Eqs. (4.2-9), (4.2-11) and (4.2-12) it is evident that the points

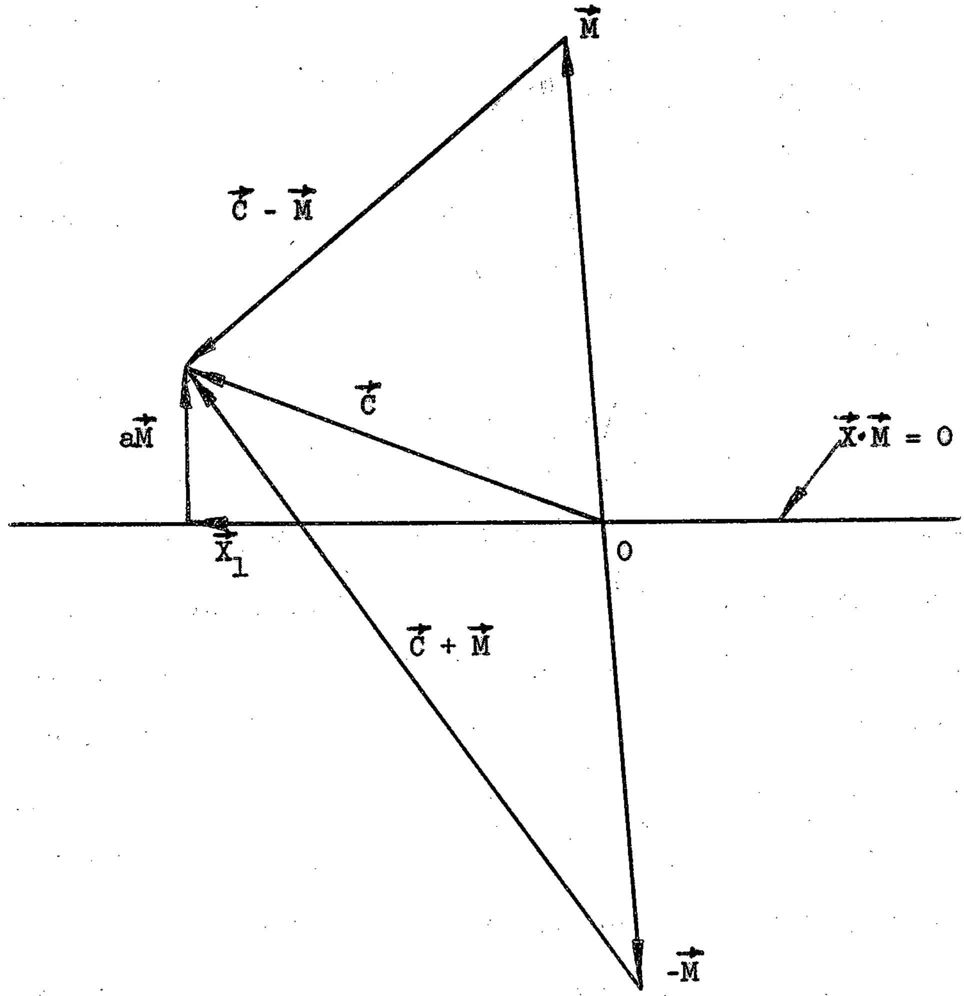
$$\vec{P} = \vec{X}_1 + (1 + a)\vec{M} \tag{4.2-13}$$

$$\vec{Q} = \vec{X}_1 + (1 - a)\vec{M} \tag{4.2-14}$$

$$\vec{R} = \vec{X}_1 - (1 + a)\vec{M} \tag{4.2-15}$$

and

$$\vec{S} = \vec{X}_1 - (1 - a)\vec{M} \tag{4.2-16}$$



THE GEOMETRICAL REPRESENTATION OF THE SIGNAL PLUS NOISE SPACE WHEN A SOLUTION NOT ON THE HYPERPLANE $\vec{X} \cdot \vec{M} = 0$ IS ASSUMED

FIGURE 4.2-2

are on the same equiprobable surface. The location of these points are shown on Fig. 4.2-3.

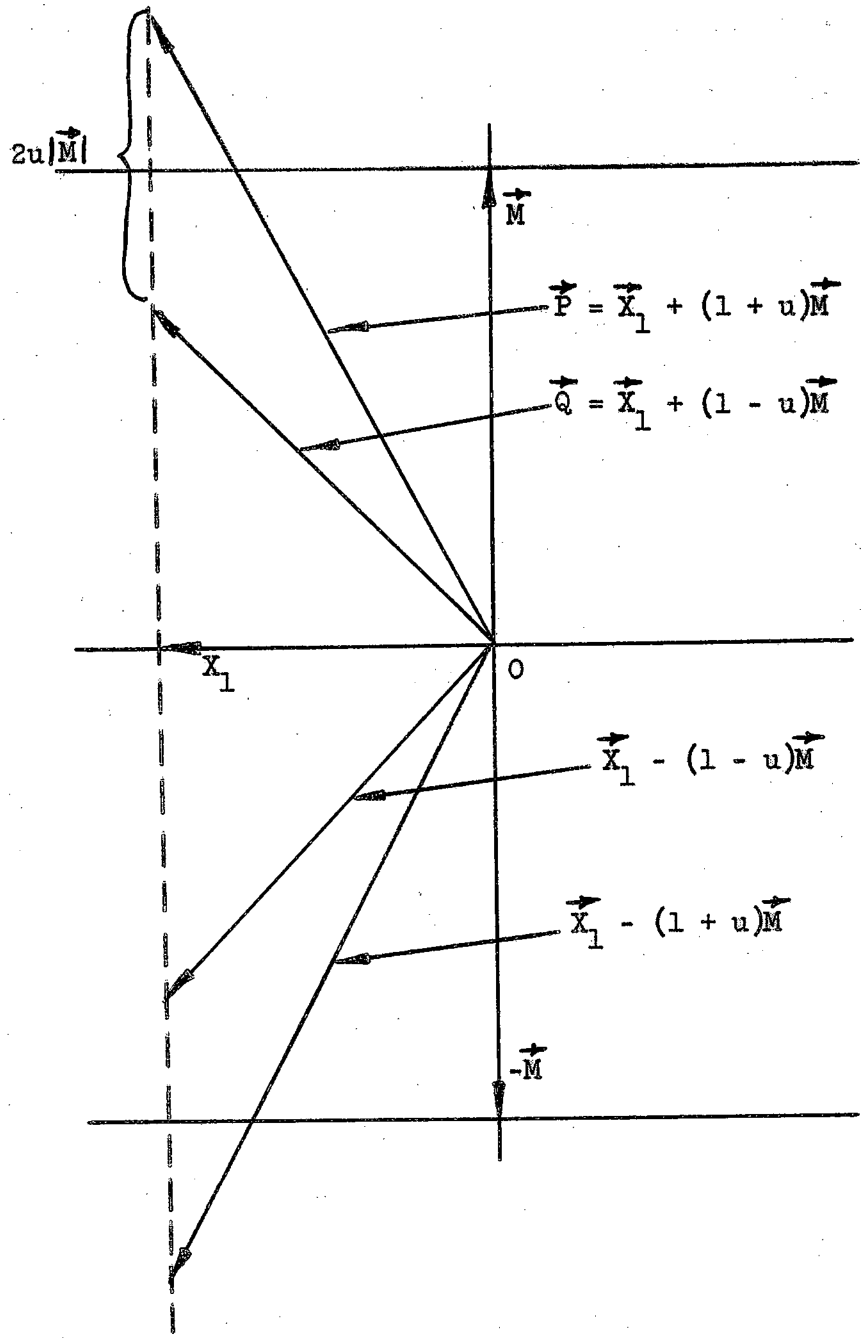
From Eqs. (4.2-13), (4.2-14), (4.2-15) and (4.2-16) it is clear that the end points of $\vec{P}$, $\vec{Q}$, $\vec{R}$ and $\vec{S}$ fall on a straight line. This is contrary to the hypothesis that the equiprobable surfaces are convex and therefore the only solutions of (4.2-5) are given by (4.2-6) thereby proving the theorem.

4.3 <u>Types of noise encountered</u>

The types of noise which may be encountered can be divided into two main catagories: (1) natural interference and (2) man-made interference. Natural interference consists mainly of thermal noise which has a gaussian distribution. Other types of natural noise include impulse noise due to lightning and VLF whistlers. Since there seems to be little likelihood that wideband binary systems would be used at the lower frequencies it may be assumed that interference from VLF whistlers will be of no interest.

For the most part, man made noise consists of radio stations which generally have relatively narrow bandwidths. The presence of these narrow band stations will cause the spectrum to be very jagged. It appears likely that this type of noise will be the most significant in many applications of wideband systems.

As an example of this type of noise, consider the case where the interference consists mainly of a large number of conventional A.M. radio telephone links of equal bandwidth. The stations may be assumed to be independent of each other and to have a wide variation of average powers. If the data rate (1/T) of the wideband system is equal to the bandwidth of the interferring A.M. stations then the instantaneous signal level for



THE GEOMETRICAL REPRESENTATION OF THE NOISE SPACE OF $\vec{N}$ FOR THE CASE WHERE A SOLUTION NOT ON THE HYPERPLANE $\vec{X}\cdot\vec{M} = 0$ IS ASSUMED

FIGURE 4.2-3

each station can be represented as a single sample for each baud period and, as a first approximation, it may be assumed that the carrier frequency of each interferring station is the same as the frequency of one of the coordinate axes. If the interferring links are of a similar nature then it may be assumed that the instantaneous carrier levels follow the same statistical law and, therefore, the only differences between the probability densities associated with each of these carrier levels is their variances.

The effect of this noise can be determined with the aid of the signal space representation given in Eqs. (2.2-5) and (2.2-6). These equations can be written in the following form

$$e_1(t) = \sum_{i=1}^{WT} E_{1i}\, f_{1i}(t) \tag{4.3-1}$$

where

$$f_{1i}(t) = \sqrt{2/T}\, \left[u(t) - u(t - T) \right] \cos\left(\frac{2\pi i t}{T} - \theta_{1i}\right) \tag{4.3-2}$$

(and where E and f (t) are not the same quantities used previously). Here both E_i and θ_i are variables and the total number of variables is the required 2WT.

The following relationships can easily be verified

$$\int_0^T f_{1i}(t)\, f_{1j}(t)\, dt = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{4.3-3}$$

and

$$\int_0^T f_{1i}(t)\, f_{2j}(t)\, dt = \begin{cases} \cos\,(\theta_{2i} - \theta_{1i}) & i = j \\ 0 & i \neq j \end{cases} \qquad (4.3\text{-}4)$$

If the signal plus noise is given by

$$x(t) = n(t) + e_1(t) = \sum_{i=1}^{WT} N_i f_{ni}(t) + \sum_{i=1}^{WT} E_{1i} f_{1i}(t) \qquad (4.3\text{-}5)$$

and the local carrier of the synchronous receiver is given by

$$e_2(t) = \sum_{i=1}^{WT} E_{2i} f_{2i}(t) \qquad (4.3\text{-}6)$$

The output of the integrator of the synchronous receiver at the end of a baud will be

$$V_d = \int_0^T x(t)\, e_2(t)\, dt$$

$$= \sum_{i=1}^{WT} \sum_{j=1}^{WT} \left\{ \int_0^T N_i E_{2i} f_{ni}(t)\, f_{2i}(t)\, dt + \int_0^T E_{1i} E_{2i} f_{1i}(t)\, dt \right\}$$

which in accordance with Eq. (4.3-4) reduces to

$$V_d = \sum_{i=1}^{WT} N_i E_{2i} \cos(\theta_{2i} - \theta_{ni})$$

$$+ \sum_{i=1}^{WT} E_{1i} E_{2i} \cos(\theta_{2i} - \theta_{1i}) \qquad (4.3\text{-}7)$$

It is apparent that, for this situation, it is adequate to represent the various waveforms as WT-dimensional vectors as follows.

$$E_1 = \sum_{i=1}^{WT} E_{1i} \cos(\theta_{2i} - \theta_{1i}) \qquad (4.3\text{-}8)$$

$$N = \sum_{i=1}^{WT} N_i \cos(\theta_{2i} - \theta_{ni}) \qquad (4.3\text{-}9)$$

and

$$E_2 = \sum_{i=1}^{WT} E_{2i} \qquad (4.3\text{-}10)$$

The quantity $N_i \cos(\theta_{2i} - \theta_{ni})$ is a random variable which is the product of the random variables N_i and $\cos(\theta_{2i} - \theta_{ni})$ the latter one depending on the random variable $(\theta_{2i} - \theta_{ni})$. The quantity N_i is the interferring carrier level and $(\theta_{2i} - \theta_{ni})$ is phase difference between the interferring carrier and the i - th component of $\vec{E}_2$.

It is reasonable to assume that the probability density of $(\theta_{2i} - \theta_{ni})$ is uniformly distributed over the range 0 to 2π. For this case, it can easily be determined that the probability density of

$$v = \cos(\theta_{2i} - \theta_{ni}) \tag{4.3-11}$$

is given by

$$p_v(v) = \begin{cases} \dfrac{1}{\sqrt{1 - v^2}} & v^2 < 1 \\ 0 & v^2 > 1 \end{cases} \tag{4.3-12}$$

The distribution function of a random variable

$$w = u\,v \tag{4.3-13}$$

can be obtain by evaluating the following expression

$$P_w(w) = \iint_R p_u(u)\, p_v(v)\, du\, dv \tag{4.3-14}$$

where the region of integration R is given by the inequality $w \leqslant uv$.

Therefore

$$P_w(w) = \int_{-\infty}^{0} \int_{w/u}^{\infty} p_u(u)\, p_v(v)\, dv\, du$$

$$+ \int_{0}^{\infty} \int_{-\infty}^{w/u} p_u(u)\, p_v(v)\, dv\, du \tag{4.3-15}$$

The probability density function of the random variable w can be obtained by differentiating Eq. (4.3-15)

$$p_w(w) = \int_{-\infty}^{\infty} \frac{1}{|u|} p_v\left(\frac{w}{u}\right) p_u(u)\, du \tag{4.3-16}$$

In particular if $u = N_i$ and v is the quanity given in Eq. (4.3-11) then the density function of the random variable

$$w = N_i \cos(\theta_{2i} - \theta_{ni}) \tag{4.3-17}$$

is given by

$$p_w(w) = \int_{|w|}^{\infty} \frac{p_u(u)\, du}{\pi\sqrt{u^2 - w^2}} \tag{4.3-18}$$

where the probability density function of the random variable N_i is $p_u(n_i)$.

If the probability density $p_u(N_i)$ is known then it is possible to make a complete analysis of the performance of the system.

A similar approach is possible for other types of interferring narrow-band station, e.g. DSB, SSB, FSK and to a first approximation the only difference will be the shape of the density function $p_u(N_i)$.

Better approximations are possible by taking into consideration the relationship between the baud length T and the bandwidth of the interferring stations W_i.

If, for example, $1/T = 3W_i$ then there will be room for three adjacent channel stations for each coordinate of the signal vector E_1. The coordinate values of the w_i' can be considered (to a first approximation) to be the sum of three independent random variable, i.e.

$$w_i' = w_{i1} + w_{i2} + w_{i3} \tag{4.3-19}$$

where w_{i1}, w_{i2} and w_{i3} are determined with the aid of Eq. (4.3-18).

If, on the other hand, $3/T = W_i$, then the bandwidth of an interferring station corresponds to three of the coordinate axes of the signal. This situation can be handled with the aid of the following signal space representation.

$$e_1(t) = \sum_{i=1}^{WT} E_{1i} f_{1i}(t) \tag{4.3-20}$$

where

$$f_{1i}(t) = \begin{cases} \sqrt{6/T}\left[u(t) - u(t - T/3)\right]\cos\left(\frac{6\pi i t}{T} - \theta_{1i}\right), & i = 1,4,7\cdots\cdots \\ \sqrt{6/T}\left[u(t - T/3) - u(t - 2T/3)\right]\cos\left(\frac{6\pi i t}{T} - \theta_{1i}\right), & i = 2,5,8\cdots\cdots \\ \sqrt{6/T}\left[u(t - 2T/3) - u(t - T)\right]\cos\left(\frac{6\pi i t}{T} - \theta_{1i}\right), & i = 3,6,9\cdots\cdots \end{cases} \tag{4.3-21}$$

The noise from the k-th interferring station can be represented as the three random variables w_k, w_{k+1}, and w_{k+2} whose densities are determined from Eq. (4.3-18). If the spectrum of the interferring station is white then these random variables are independent.

From the analysis given above, it is apparent that a good approximation of the interference resulting from many similar narrow band stations is that the signal, represented by a WT-dimensional vector, is perturbed by WT random variables with similar density functions given by (4.3-18). In many cases the random variables can be considered independent.

4.4 Reception of signals perturbed by narrow band interfering stations

In the preceding section it was pointed out that one of the most important types of interference which a wideband system may encounter will be caused by narrow band stations. In this and the following sections, the type of performance which can be obtained from linear receivers when used in this environment will be determined.

As was pointed out in Section 4.2, optimum reception with a linear receiver is posssible only when the optimum decision surface is a hyperplane. In general, this is not the case for the noise which has been assumed as indicated by the following example.

Suppose the noise is a result of WT adjacent channel stations and that each station corresponds to a coordinate axis of the signal space. Further, suppose the noise density as determined by Eq. (4.3-16) has the following form

$$p_w(w_i) = \exp\,(a_i - b_i w^4) \qquad (4.4\text{-}1)$$

Since the stations may be assumed to be independent the coordinate values of the noise will be assumed to be independent. The joint density can therefore be written

$$p_n(N) = \prod_{i=1}^{WT} \exp(a_i - b_i w^4) \qquad (4.4\text{-}2)$$

For the sake of simplicity it will be assumed that the system is symmetric.

The optimum decision surface according to (2.5-11) can be written

$$\prod_{i=1}^{WT} \exp\left[a_i - b_i (X_i - M_i)^4 \right]$$

$$= \prod_{i=1}^{WT} \exp\left[a_i - b_i (X_i + M_i)^4 \right] \qquad (4.4\text{-}3)$$

By taking the logarithm of Eq. (4.4-3) the following equation for the optimum decision surface may be obtained.

$$\sum_{i=1}^{WT} a_i - b_i (X_i - M_i)^4 = \sum_{i=1}^{WT} a_i - b_i (X_i + M_i)^4 \qquad (4.4\text{-}4)$$

which can be simplified to read

$$\sum_{i=1}^{WT} 8\, b_i\, X_i\, M_i\, (X_i^2 + M_i^2) = 0 \qquad (4.4\text{-}5)$$

Since Eq. (4.4-5) is clearly not a hyperplane then optimum reception cannot be obtained with a linear receiver.

Although optimum reception with a linear receiver may not be possible in a certain situation, it is, never-the-less, important to determine what criterion may be used to obtain maximum performance, i.e., least probability of error. Although it is possible to determine the probability of error from Eq. (2.4-7), the process involved is tedious and no general results are possible. For wideband systems it will be shown in the next section that the signal-to-noise ratio criterion can be used for an important subclass of this type noise to obtain useful knowledge of performance.

4.5 The signal-to-noise ratio

The signal-to-noise ratio is a factor which is often used in the evaluation of communications systems. When the statistics of the noise are known a complete analysis of a binary communications system will have as its most important result the relationship between the signal-to-noise ratio at the input to the receiver and the probability of error of its output. The relationship between the signal-to-noise ratio at the demodulator output and the signal-to-noise ratio at the receiver input is also useful, either when used as an intermediate step to obtain the relation between the output probability of error and the receiver input signal-to-noise ratio or when used directly to evaluate receiver performance.

From Eq. (3.2-9) the output of the demodulator can be written

$$V_d = \vec{E}_1 \cdot \vec{E}_2 + \vec{N} \cdot \vec{E}_2 \qquad (4.5\text{-}1)$$

In Eq. (4.5-1) the quantity $\vec{E}_1 \cdot \vec{E}_2$ is the signal component of V_d and $\vec{N} \cdot \vec{E}_2$ is the noise component. The output signal to noise power ratio $(SNR)_o$ is given by

$$(SNR)_o = \frac{(\vec{E}_1 \cdot \vec{E}_2)^2}{E\left[\vec{N} \cdot \vec{E}_2\right]^2} \qquad (4.5\text{-}2)$$

where $E\left[\vec{N} \cdot \vec{E}_2\right]^2$ is the expected value of $(\vec{N} \cdot \vec{E}_2)^2$. As a general rule the noise will have a zero mean value and $E\left[\vec{N} \cdot \vec{E}_2\right]^2$ becomes equal to

$$E\left[\vec{N} \cdot \vec{E}_2\right]^2 = E\left[\sum_{i=1}^{n} N_i E_{2i}\right]^2 = \sum_{i=1}^{n} \sigma_i^2 \, E_{2i}^2 \qquad (4.5\text{-}3)$$

where σ_i^2 is the variance of the random variable N_i and where $n = 2WT$. Substituting Eq. (4.5-3) into Eq. (4.5-2) results in

$$(SNR)_0 = \frac{\left[\vec{E}_1 \cdot \vec{E}_2\right]^2}{\sum_{i=1}^{n} \sigma_i^2 E_{2i}^2} \qquad (4.5\text{-}4)$$

In order to facilitate the further analysis of Eq. (4.5-4) it will be assumed that the noise is white, i.e.

$$\sigma_i^2 = \frac{\sigma^2}{n} \qquad (4.5\text{-}5)$$

where σ^2 is the variance of the noise $\vec{N}$. For this case Eq. (4.5-4) becomes

$$(SNR)_0 = \frac{n\left[\vec{E}_1 \cdot \vec{E}_2\right]^2}{\sigma^2 \sum_{i=1}^{n} E_{2i}^2} \qquad (4.5\text{-}6)$$

It is apparent from Eq. (4.5-6) that for fixed values of $\vec{E}_1$, σ^2 and n the quantity $(SNR)_0$ is a function only of the direction of $\vec{E}_2$. If only the direction of the vector E_2 is allowed to vary then the denominator of the right hand side of Eq. (4.5-6) is a constant and $(SNR)_0$ is a maximum when the numerator is a maximum. Clearly this is the case when E_2 has either the same or the opposite direction of E_1, i.e. the maximum value of $(SNR)_0$ is obtained when

$$\vec{E}_2 = k\,\vec{E}_1 \tag{4.5-7}$$

where k may be positive or negative.

Substituting Eq. (3.4-7) into Eq. (4.5-6) gives the following result.

$$\text{Max}(SNR)_0 = \frac{n\left|E_1\right|^2\left|kE_1\right|^2}{\sigma^2\left|kE_1\right|^2} = \frac{n\left|E_1\right|^2}{\sigma^2} \tag{4.5-8}$$

and

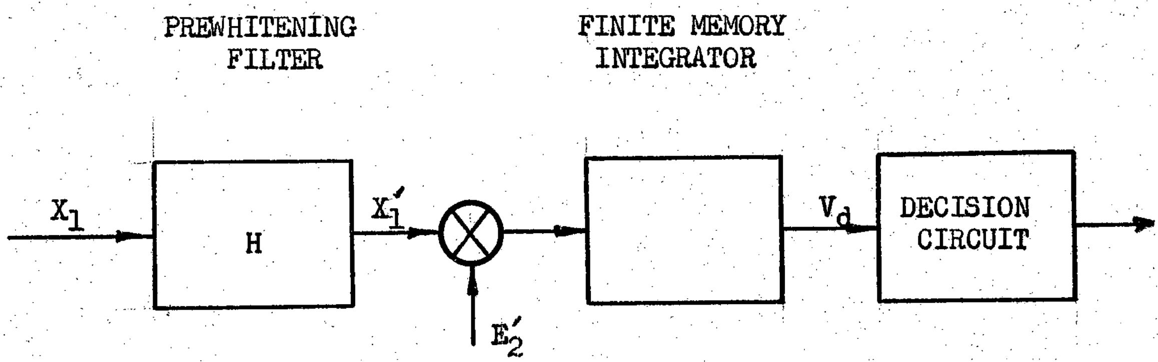
$$V_d = k\,\vec{X}\cdot\vec{E}_1 = k\left|\vec{E}_1\right|^2 + k\,\vec{E}_1\cdot\vec{N} \tag{4.5-9}$$

Since $\left|E_1\right|^2/\sigma^2$ is simply the input signal-to-noise ratio $(SNR)_i$ Eq. (4.5-8) can be written

$$\text{Max}(SNR)_0 = n(SNR)_i = 2WT(SNR)_i \tag{4.5-10}$$

which is identical to the results obtained by Turin.[9] If the noise at the input to the receiver is not white then a prewhitening filter can be placed ahead of the demodulator (see Fig. 4.5-1). In order to simplify reception it is desirable that the prewhitening filter have a constant delay over the input signal band. In this case the filter can be represented as the following linear transformation of the input signal space

$$\vec{X}' = \sum_{i=1}^{n} H_i\, X_i\, \vec{I}_i \tag{4.5-11}$$



A SYNCHRONOUS RECEIVER WITH A PREWHITENING FILTER

FIGURE 4.5-1

where the X_i correspond to frequency domain samples of the input signal plus noise and H_i is the filter gain at the sample frequencies. In order to transform the noise at the input of the filter to white noise, H_i must satisfy the following equation

$$H_i^2 \ \sigma_i^2 = \frac{A\sigma^2}{n} \tag{4.5-12}$$

where the σ_i^2's are the variance of the coordinate values of the input noise and σ^2 is the variance of the noise at the output. The gain factor A is an arbitrary constant and will be taken equal to unity for the sake of convenience. The signal at the output of the filter is

$$E_1' = \sum_{i=1}^{n} \frac{\sigma}{\sqrt{n}\ \sigma_i} \ E_{1i}\vec{I}_i \tag{4.5-13}$$

and the signal-to-noise ratio at the filter output is

$$(SNR)_i' = \frac{|E_1'|^2}{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} \frac{E_{1i}^2}{\sigma_i^2} \tag{4.5-14}$$

The expression on the right hand side of Eq. (4.5-14) is the average of the coordinate signal-to-noise ratios.

By combining Eq. (4.5-10) and (4.5-14) the following general expression for the maximum output signal-to-noise ratio of a linear receiver is obtained.

$$\text{Max}(SNR)_0 = \sum_{i=1}^{n} \frac{E_{1i}^2}{\sigma_i^2} \tag{4.5-15}$$

It is evident from Eq. (4.5-15) that the maximum output signal-to-noise is equal to the sum of the coordinate signal-to-noise ratios.

In this case the local carrier must have the same direction as E_1' to realize maximum signal-to-noise ratio at the output, i.e.,

$$\vec{E}_2' = k\,\vec{E}_1'$$

For this case the output becomes

$$V_d = \sum_{i=1}^{n} X_i'\,E_{2i}' = \sum_{i=1}^{n} k\,X_i'\,E_{1i}' \qquad (4.5\text{-}16)$$

Eq. (4.5-13) is also valid for the signal plus noise vector $\vec{X}'$, i.e.

$$\vec{X}' = \sum_{i=1}^{n} \frac{\sigma}{\sqrt{n}\;\sigma_i}\,X_i\,\vec{I}_i \qquad (4.5\text{-}17)$$

and therefore Eq. (4.5-16) can be written

$$V_d = \sum_{i=1}^{n} \frac{k\,\sigma^2}{n\,\sigma_i^2}\,X_i E_{1i} \qquad (4.5\text{-}18)$$

From Eq. (4.5-18) it is evident that identical performance to the linear receiver having the prewhitening filter described above can be obtained with a simple synchronous receiver (which does not have a pre-whitener) having a local carrier

$$\vec{E}_2 = \sum_{i=1}^{n} \frac{k\sigma^2}{n\,\sigma_i^2}\,E_{1i}\,\vec{I}_i \qquad (4.5\text{-}19)$$

In many cases it is possible to easily determine the output probability of error from the signal-to-noise ratio at the output of the demodulator. If the random variables $E_{2i}N_i$ are independent with common distributions then, according to the central limit theorem, the output of the demodulator V_d will have a gaussian distribution.

This will also be true if there is some variation between the quantities $E_{2i}N_i$ so long as they can be grouped such that each group contains a relatively large number of random variables whose distributions are the same. In such a case the probability of error can easily be determined. If the system is symmetrical and the receiver has a zero threshold then it can easily be shown that

$$P_e = 1/2 \left[1 - \text{erf} \sqrt{(1/2)\ (SNR)_0} \right] \qquad (4.5\text{-}20)$$

If, for example, the noise consists of a very large number, say one thousand, narrow band stations then it appears reasonable to assume that Eq. (4.5-20) will give a good approximation to the probability of error which will be encountered.

This approach is extremely useful since, in most cases, the probability densities of the interference will not be known, thus, making a more exact analysis impossible.

4.6 Summary

In Chapter IV it has been shown that, in general, optimum reception cannot be obtained with a linear receiver. Optimum reception can be obtained, however, in certain cases and a sufficient condition for such a case to exist has been derived.

The types of noise which may be encountered in practice have been discussed including gaussian noise, interference from narrow band stations and impulse noise. The nature of the noise which results from a large number of similar interfering narrow band stations has been analyzed in detail.

In general, great difficulty is encountered in determining the configuration of the linear receiver which results in the minimum probability of error for a given type of noise. For this reason the signal-to-noise ratio criterion is of importance. In this chapter the configuration which results in maximum signal-to-noise ratio has been derived using geormetrical methods based on the signal space concept.

CHAPTER V

THE APPLICATION OF LINEAR RECEIVERS (Part II)

5.1 Introduction

In Chapter IV the performance of linear receivers was analyzed when the interference was from narrow band stations. In this chapter the case where the signal is perturbed by impulse noise will be examined. The results will be extended to include both impulse noise and noise from narrow band stations.

5.2 Types of impulse noise

Impulse noise can be caused by natural phenomina such as lightning or it can be man-made such as, for example, switching transients. Although usually it is not periodic, in some cases it has a periodic nature as, for example, commutation noise. In many cases the occurrance of a pulse of noise depends to some degree on the time of occurrence of previous pulses. Thus, it is evident that to properly specify the statistical nature of the noise it is necessary to refer to the specific type.

In many cases, however, impulse noise which is objectionable is characterized by very strong, short duration pulses which are separated by relatively long intervals. If the data rate of the communication system is sufficiently high the probability of more than one impulse occuring in a single baud will be negligable. In such cases, it is sufficient to specify the amplitude distribution and average rate of occurrance. The affects of this type of noise will be analyzed in the next section.

5.3 The analysis of systems utilizing quasi-random binary carriers when perturbed by impulse noise

As was indicated in the previous section, impulse noise can often be characterized by high amplitude, short duration bursts of noise which occur infrequently.

The affects of this type of noise on the following types of systems will be analyzed: (1) systems with low pass signals, (i.e., signal spectra centered about zero) and (2) systems with band pass signals (i.e., signals whose spectra are centered about $f_o > W/2$ where W is the signal bandwidth).

The Low Pass Case. Since the results depend to a certain extent on the nature of the signal waveform it is necessary to assume a specific waveform characterization in order to proceed with the analysis. Of most interest are the waveforms consisting of sequence of quasi-random binary pulses which are produced by shift register generators.

$$e_1(t) = \sum_{i=1}^{2WT} E_{1i}\, f_i(t) \qquad (5.3\text{-}1)$$

where

$$f_i(t) = \sqrt{2W}\left[\,u(t - (i - 1)/2W) - u(t - i/2W)\,\right] \qquad (5.3\text{-}2)$$

and where the amplitude of the i-th pulse of the sequence A_{ij} is

$$A_{1i} = E_{1i}\,\sqrt{2W} \qquad (5.3\text{-}3)$$

For a typical binary system where the "space" waveform is the negative of "mark" waveform the receiver local carrier $e_2(t)$ will have the same shape as the "mark" waveform. These waveforms can be written in vector form as follows:

$$\vec{M} = \sum_{i=1}^{2WT} M_i \vec{I}_i \tag{5.3-4}$$

$$\vec{S} = -\vec{M} \tag{5.3-5}$$

$$\vec{E}_2 = k \vec{M} \tag{5.3-6}$$

where k is a constant which for convenience is taken greater than zero and where

$$M_i^2 = A^2/2W \tag{5.3-7}$$

In the absence of noise the output of the finite memory integrator at the end of the baud is

$$V_{dm} = \vec{M}\cdot\vec{E}_2 = \sum_{i=1}^{2WT} k M_i^2 = k A^2 T \tag{5.3-8}$$

if the input to the receiver was a "mark" waveform, and the output is

$$V_{ds} = \vec{S}\cdot\vec{E}_2 = \sum_{i=1}^{2WT} - k M_i^2 = - k A^2 T \tag{5.3-9}$$

if the input was a "space".

If, for a particular baud, the input signal is perturbed by an impulse having the form of a Dirac delta function

$$n(t) = C \, \delta \, (t - t_1) \tag{5.3-10}$$

where t is measured from the beginning of the baud and t_1 is the time of occurence of the impulse then the output due to the impulse is

$$V_{dn} = \int_0^T n(t)e_2(t)dt = \int_0^T C\,\delta(t - t_1)\, k \sum_{i=1}^{2WT} M_i f_i(t)dt$$

$$= C\, k\, M_i \sqrt{2W} \tag{5.3-11}$$

where the subscript i is given by

$$(i - 1)/2W < t_1 < i/2W \tag{5.3-12}$$

If, as is usually the case, the waveform $\vec{M}$ is chosen such that it has a zero dc component then the probability that a particular M_i picked at random will be positive will be equal to the probability that it is negative. If the time which the impulse occurs is a random variable which is uniformly distributed over the duration of the baud then from Eq. (5.3-7) and (5.3-11) it is evident that the probability density function of the output of the integrator due to the presence of the impulse is

$$p_d(V_{dn}) = \frac{1}{2kA}\left[p_c(V_{dn}/kA) + p_c(-V_{dn}/kA)\right] \tag{5.3-13}$$

where $p_c(C)$ is the probability density function of the impulse amplitude C.

If the decision threshold is set at zero then the probability of error P_e is given by

$$P_e = \text{Prob}\left\{V_{dm} + V_{dn} < 0\right\}\ P(M)$$

$$+ \text{Prob}\left\{V_{ds} + V_{dn} > 0\right\}\ P(S) \tag{5.3-14}$$

where P(M) is the probability that a mark was transmitted and P(S) in the probability that a space was transmitted. From Eq. (5.3-8), (5.3-9) and (5.3-13) it is evident that

$$\text{Prob}\left\{V_{dm} + V_{dn} < 0\right\} = \text{Prob}\left\{V_{ds} + V_{dm} > 0\right\} \tag{5.3-15}$$

which reduces Eq. (5.3-14) to

$$P_e = \text{Prob}\left\{V_{dn} < -V_{dm}\right\} \tag{5.3-16}$$

The right hand side of Eq. (5.3-16) is the probability distribution function of V_{dm} evaluated at $-V_{dm}$ and therefore with the aid of Eq. (5.3-13) the probability of error can be written

$$P_e = \int_{-\infty}^{-V_{dm}} \frac{1}{2kA}\left[p_c(V_{dn}/kA) + p_c(-V_{dn}/kA)\right] dV_{dn} \tag{5.3-17}$$

With the aid of Eq. (5.3-8) the above Eq. can be written

$$P_e = \int_{-\infty}^{-AT} \frac{1}{2}\left[p_c(z) + p_c(-z)\right] dz$$

which in terms of distribution function of C becomes

$$P_e = \frac{1}{2}\left[P_c(-AT) + 1 - P_c(AT)\right]$$

If the impulses have an average occurrence of $\overline{0}$ impulses per baud than the average probability of error due to impulse noise is

$$\overline{P}_e = \frac{1}{2}\,\overline{0}\left[1 + P_c(-AT) - P_c(AT)\right] \tag{5.3-19}$$

or in terms of the average signal power $P_s = A^2$

$$\bar{P}_e = \frac{1}{2}\,\bar{O}\left[\,1 + P_c(\,-\sqrt{P_s}\,T) - P_c(\sqrt{P_s}\,T)\,\right] \qquad (5.3\text{-}20)$$

Further results depend, of course, on a knowledge of P (C) and $\bar{O}$. If, for example, the impulse noise has a constant amplitude $C = C_1AT$ then from Eq. (5.3-19) it is clear that

$$\bar{P}_e = \begin{cases} \bar{O}/2; & c_1 > 1 \\ 0; & c_1 < 1 \end{cases} \qquad (5.3\text{-}21)$$

There is almost always some sort of filter between the input of the receiver and the demodulator. For this reason the noise pulses at the input to the demodulator will generally not be impulses but pulses of finite amplitude and non-zero duration. The length of the pulses will depend on the nature of the input filter and since the input filter is usually made approximately equal to the bandwidth W of the signal the length of the noise pulses will be in the order of 1/2W. Such a noise pulse C n(t) is shown in Fig. 5.3-1. For this particular example the pulse occurs at time t_1 and has, for all practical purposes decayed to zero at a time $\frac{1}{2W}$ later. Furthermore, the waveform of the pulse n(t) is everywhere non-negative and C is a positive multiplier. The output of the finite memory integrator at the end of the baud due to the presence of the pulse at the input of the demodulator during the i-th pulse of the local carrier $e_2(t)$ is

$$V_{dn} = \int_0^T Cn(t - t_1)\, e_2(t)\, dt \qquad (5.3\text{-}22)$$

where t_1 has the probability density function

$$p_t(t_1) = \begin{cases} 2W; & \frac{i-1}{2W} \leqslant t_1 < \frac{i}{2W} \\ 0; & \text{otherwise} \end{cases} \qquad (5.3\text{-}23)$$

Since, in general, the noise pulse overlaps two of the pulses of $e_2(t)$, then there are four possible forms which Eq. (5.3-22) can take. These forms correspond to the four following cases

Case (1) $A_i = A_{(i+1)} = A$

Case (2) $A_i = -A_{(i+1)} = A$

Case (3) $A_i = A_{(i+1)} = -A$

Case (4) $A_i = -A_{(i+1)} = -A$

where kA_i and $k_{(i+1)}$ are the amplitudes of i-th and (i+1)-th pulses of $e_2(t)$ and kA is the positive square-root of the average power of $e_2(t)$.

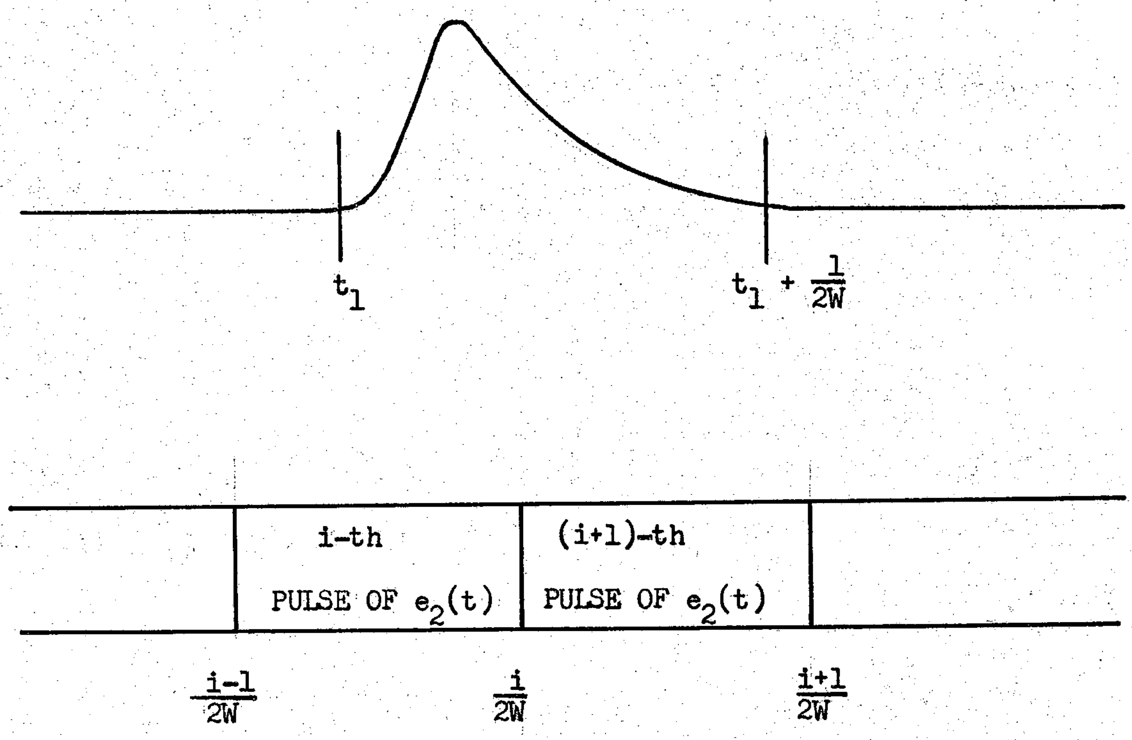
For case (1) the output of the integrator V_{dn_1} is independent of t_1 and is given by

$$V_{dn_1} = kAC \int_0^{\frac{1}{2W}} n(t)\, dt \qquad (5.3\text{-}24)$$

The integrator output for case (2) depends on the value of t_1, however, and is

$$V_{dn_2} = kAC \int_{t_1}^{\frac{1}{2W}} n(t - t_1)\, dt - kAC \int_{i/2W}^{t_1 + \frac{1}{2W}} n(t - t_1) dt \qquad (5.3\text{-}25)$$

which can be written



A TYPICAL NOISE PULSE

FIGURE 5.3-1

$$V_{dn_2} = 2kAC \int_0^{y} n(t)\, dt - V_{dn_1} \tag{5.3-26}$$

where

$$y = \frac{i}{2W} - t_i \tag{5.3-27}$$

For cases (3) and (4) the outputs are

$$V_{dn_3} = -V_{dn_1} \tag{5.3-28}$$

and

$$V_{dn_4} = -V_{dn_2} \tag{5.3-29}$$

Designating the conditional density functions of V_{dn} for cases (1), (2), (3) and (4) as $p_1(V_{dn}|1)$, $p_2(V_{dn}|2)$, $p_3(V_{dn}|3)$, and $p_4(V_{dn}|4)$ respectively, the density function of the output can be written

$$p(V_{dn}) = \frac{1}{4}\, p_1(V_{dn}|1) + \frac{1}{4}\, p_2(V_{dn}|2) + \frac{1}{4}\, p_3(V_{dn}|3) + p_4(V_{dn}|4) \tag{5.3-30}$$

The distribution function of the output is

$$P(V_{dn}) = \frac{1}{4} \int_{-\infty}^{V_{dn}} p_1(x|1)dx + \frac{1}{4} \int_{-\infty}^{V_{dn}} p_2(x|2)dx + \frac{1}{4} \int_{-\infty}^{V_{dn}} p_3(x|3)dx$$

$$+ \frac{1}{4} \int_{-\infty}^{V_{dn}} p_4(x|4)dx \tag{5.3-31}$$

From Eq. (5.3-23), (5.3024), (5.3-25), (5.3-28) and (5.3-29) it is evident that

$$p_3(V_{3n}|3) = p_1(-V_{dn}|1) \tag{5.3-32}$$

and

$$p_4(V_{dn}|4) = p_1(-V_{dn}|1) \tag{5.3-33}$$

and therefore

$$P(V_{dn}) = \frac{1}{4}\int_{-\infty}^{V_{dn}} p_1(x|1)\,dx + \frac{1}{4}\int_{-\infty}^{V_{dn}} p_2(x|2)\,dx + \frac{1}{4}\int_{-\infty}^{V_{dn}} p_1(-\,z|1)dz$$

$$+ \frac{1}{4}\int_{-\infty}^{V_{dn}} p_2(-\,z|2)\,dz \tag{5.3-34}$$

Letting $-z = x$, Eq. (5.3-34) becomes

$$P(V_{dn}) = \frac{1}{2} + \frac{1}{4}\int_{-V_{dn}}^{V_{dn}} p_1(x|1)\,dx + \frac{1}{4}\int_{-V_{dn}}^{V_{dn}} p_2(x|2)\,dx \tag{5.3-35}$$

From Eq. (5.3-24) it is clear that

$$p_1(V_{dn}|1) = \delta(V_{dn_1} - V_{dn}) \tag{5.3-36}$$

Since $n(t) \geqslant 0$ it is also evident from Eq. (5.3-23), (5.3-24), (5.3-26) and (5.3-27) that

$$-V_{dn_1} \leq p_2(V_{dn}|2) \leq V_{dn_1} \tag{5.3-37}$$

With the aid of relationships (5.3-36) and (5.3-37), Eq. (5.3-35) can be written

$$P(V_{dn}) = \begin{cases} 0; & -\infty < V_{dn} < -V_{dn_1} \\ \frac{1}{2} + \frac{1}{4}\displaystyle\int_{-V_{dn}}^{V_{dn}} p_2(x|2)\,dx; & -V_{dn_1} \leq V_{dn} < V_{dn_1} \\ 1; & V_{dm_1} \leq V_{dn} < \infty \end{cases} \tag{5.3-38}$$

From Eq. (5.3-26) and the fact that $C\,n(t) \geq 0$ it is clear that

$$\frac{dy}{dx} > 0 \tag{5.3-39}$$

and therefore

$$P(V_{dn}) = \begin{cases} 0; & -\infty < V_{dn} < -V_{dn_1} \\ \frac{1}{2} + \frac{1}{4}\displaystyle\int_{y(-V_{dn})}^{y(V_{dn})} p(y)\,dy; & -V_{dn_1} \leq V_{dn} < V_{dn_1} \\ 1; & V_{dn_1} \leq V_{dn} < \infty \end{cases} \tag{5.3-40}$$

where

$$p(y) = p\left(\frac{i}{2W} - t_1\right) = 2W \tag{5.3-41}$$

By substituting Eq. (5.3-41) into the right hand side and performing the indicated integration, Eq. (5.3-40) becomes

$$P(V_{dn}) = \begin{cases} 0; & -\infty < V_{dn} < -V_{dn_1} \\ \frac{1}{2} + \frac{W}{2}\left[y(V_{dn}) - y(-V_{dn}) \right]; & -V_{dn_1} \leqslant V_{dn} < V_{dn_1} \\ 1; & V_{dn_1} \leqslant V_{dn} < \infty \end{cases} \qquad (5.3\text{-}42)$$

where $y(V_{dn})$ is given implicity by Eq. (5.3-26), i.e.,

$$V_{dn} = 2kAC \int_0^{y(V_{dn})} n(t)\, dt - V_{dn_1} \qquad (5.3\text{-}43)$$

For a symmetrical binary communications systems (i.e., $P(M) = P(S)$ and $\vec{S} = -\vec{M}$) the probability of error is

$$P_e = P(-V_{dm}) \qquad (5.3\text{-}44)$$

where V_{dm} is the output of the finite memory integrator when the receiver input is a "mark" and is given by

$$V_{dm} = k\, A^2 T \qquad (5.3\text{-}45)$$

and where it has been assumed that the input filter has not appreciably distored the input waveform.

Therefore, from Eq. (5.3-42) the probability of error is found to be

$$P_e = \begin{cases} 0; & \frac{V_{dn}}{kT} < A^2 < \infty \\ \frac{1}{2} + \frac{W}{2}\left[y(-kA^2T) - y(kA^2T) \right]; & 0 < A^2 \leqslant \frac{V_{dn1}}{kT} \end{cases} \qquad (5.3\text{-}46)$$

If C is a negative multiplier the above results can be shown to apply by interchanging case (1) with case (3) and case (2) with case (4).

If C is a random variable with a probability density $p_c(C)$ the probability of error can be obtained by taking the expected value of the right hand side of Eq. (5.3-46) and, therefore,

$$P_e = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{W}{2}\left[y(-k\ A^2T) - y(k\ A^2T) \right] p_c(C)\ dC \qquad (5.3\text{-}47)$$

is the probability of error if C is unbaounded. If C is bounded then Eq. (5.3-47) is valid for the range $0 < A^2 \leq (V_{dn})_{max}/kT$ and zero over the range $(V_{dn})_{max}/kT < A^2 \leq \infty$ where $(V_{dn})_{max}$ is the maximum magnitude of V_{dn} and is given by

$$(V_{dn})_{max} = k\ A(C)_{max} \int_{0}^{1/2W} n(t)\ dt \qquad (5.3\text{-}48)$$

and where $(C)_{max}$ is the maximum magnitude of the random variable C.

The average probability of error can then be determined by multiplying P_e by the average occurrence of the pulses per baud $\overline{O}$, i.e.,

$$\overline{P}_e = \overline{O}\ P_e \qquad (5.3\text{-}49)$$

Although the above analysis is restricted to noise waveforms $n(t) \geq 0$ which have a duration 1/2W, similar analyses can be made of waveforms of longer duration. It is also possible to make analyses of specific waveforms having zero crossings by using the same methods.

To illustrate the use of the above results consider impulse noise consisting of square pulses, i.e.

$$n(t) = \begin{cases} 1; & 0 \leqslant t < \frac{1}{2W} \\ 0; & t < 0 \text{ and } t > \frac{1}{2W} \end{cases} \tag{5.3-50}$$

substituting Eq. (5.3-50) into Eq. (5.3-43) yields

$$V_{dn} = 2k\,AC \int_0^{y(V_{dn})} dt - V_{dn_1} \tag{5.3-51}$$

which, when solved for y, results in

$$y(V_{dn}) = \frac{V_{dn} + V_{dn_1}}{2KAC} \tag{5.3-52}$$

The average probability of error can now be obtained by substituting Eq. (5.3-52) and (5.3-47) into Eq. (5.3-49)

$$P_e = \overline{0}\left\{\frac{1}{2} + \int_{-\infty}^{\infty} \frac{W}{2}\left[\frac{-kA^2\;T + V_{dn_1}}{2kAC} - \frac{kA^2\;T + V_{dn_1}}{2kAC}\right] p_c(C)\;dC\right\} \tag{5.3-53}$$

which becomes

$$P_e = \overline{0}\left\{\frac{1}{2} - \frac{AWT}{2}\int_{-\infty}^{\infty} \frac{p_c(C)}{C}\;dC\right\} \tag{5.3-54}$$

If the pulses have a constant amplitude then

$$P_e = \begin{cases} \overline{0}\left[\frac{1}{2} - \frac{AWT}{2C}\right]; & 0 < 2AWT \leqslant C \\ 0 \; ; & 2AWT > C \end{cases} \qquad (5.3\text{-}55)$$

It is evident from Eq. (5.3-55) that P_e is linearly related to the product of the signal dimensionality times the ratio of the signal amplitude to the noise pulse amplitude in the range where this product is less than unity and that P_e is zero in the range where this product is greater than unity.

The Band Pass Case

When a radio link is used, the usual procedure is to translate a low pass signal to some convenient point in the rf spectrum before the signal is radiated. To demodulate this signal, the signal waveform must be multiplied by $e_2(t)\cos(2\pi f_1 t + \phi_1)$ where $e_2(t)$ is the low pass waveform previously referred to and f_1 is the translation frequency (see Fig. 5.3-2).

If the noise consists of impulses, where it is assumed that no more than one impulse occurs during a given baud, then the noise n(t) at point "A" due to a particular impulse $U\,\delta(t-t_1)$ is

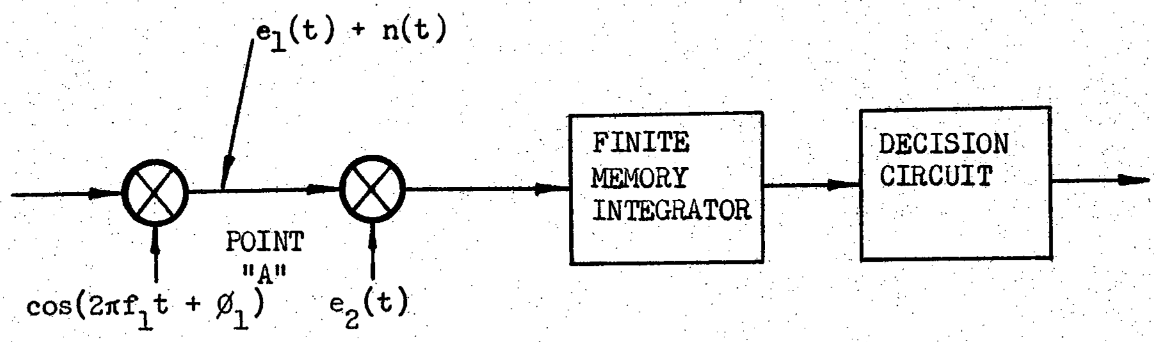
$$n(t) = \cos(2\pi f_1 t + \phi_1)\, U\, \delta(t - t_1) \qquad (5.3\text{-}56)$$

where t_1 is a random variable with a probability density which is uniformly distributed over the baud interval

Eq. (5.3-56) can be written

$$n(t) = C\, \delta(t - t_1) \qquad (5.3\text{-}57)$$

where



A BANDPASS SIGNAL RECEIVER

FIGURE 5.3-2

$$C = U \cos(2\pi f_1 t_1 + \phi_1) \tag{5.3-58}$$

The random variable C is the product of the two random variables U and $v = \cos(2\pi f_1 t_1 + \phi_1)$. If the translation frequency f_1 is large compared with the reciprocal of the baud length 1/T then it can be assumed with little error that $\theta = (2\pi f_1 t_1 + \phi_1)$ is uniformly distributed over the range $0 \quad \theta \quad 2\pi$. From this hypothesis it can easily be shown that the random variable v has the probability density function

$$p_v(v) = \begin{cases} \dfrac{1}{\pi\sqrt{1 - v^2}} \; ; & v^2 < 1 \\ 0 \; ; & v^2 > 1 \end{cases} \tag{5.3-59}$$

From Eq. (4.3-15) the probability distribution function of the random variable C is

$$P_c(C) = \int_{-\infty}^{0} \int_{C/U}^{\infty} p_u(U) p_v(v) dv \, d\,U + \int_{0}^{\infty} \int_{-\infty}^{C/U} p_u(U) p_v(v) dv \, d\,U \tag{5.3-60}$$

The following result is obtained when Eq. (5.3-59) is substituted into Eq. (5.3-60) and the integrations with respect to v are performed.

$$P_c(C) = \frac{1}{2} - \frac{1}{\pi} \int_{-\infty}^{-|C|} \sin(C/U) \, p_u(U) \, dU$$

$$+ \frac{1}{\pi} \int_{|C|}^{\infty} \sin\,(C/U) \, p_u(U) \, dU \tag{5.3-61}$$

Eq. (5.3-57) is the same as Eq. (5.3-10) and therefore the resulting average probability of error is given by Eq. (5.3-20). The substitution of Eq. (5.3-61) into Eq. (5.3-20) yields the following expression for the probability of error.

$$\overline{P}_e = \overline{0} \left\{ \frac{1}{2} + \frac{1}{\pi} \int_{-\infty}^{-|AT|} \sin^{-1}(AT/U)\, p_u(U)\, dU \right.$$

$$\left. - \frac{1}{\pi} \int_{|AT|}^{\infty} \sin^{-1}(AT/U)\, p_u(U)\, dU \right\} \qquad (5.3\text{-}62)$$

For the case where U is a constant quantity, Eq. (5.3-62) reduces to

$$\overline{P}_e = \overline{0} \left[\frac{1}{2} - \frac{1}{\pi} \sin^{-1} (AT/U) \right] \qquad (5.3\text{-}63)$$

Usually there is a filter at the input to the receiver. The effect of this filter is to cause the waveform at point "A" resulting from an impulse at the input to have a finite amplitude and a length of the order of the reciprocal of the filter bandwidth.

In general for any noise waveform $n_1(t)$ at the receiver input, the noise waveform at point "A" is

$$n(t) = \cos(2\pi f_1 t + \phi_1) \int_{-\infty}^{\infty} h_1(u)\, n_1(t - u)\, du \qquad (5.3\text{-}64)$$

where $h_1(t)$ is the impulse response of the input filter. If the input filter is assumed to be summetrical about some frequency f_o then the transfer function $H_1(j\omega)$ of the filter can be expressed in terms of a particular low pass filter $H(j\omega)$ by the following relationship.

$$H_1(j\omega) = H(j\omega + j\omega_0) + H(j\omega - j\omega_0) \qquad (5.3\text{-}65)$$

The impulse response of the filter can be determined by taking the Fourier transform of Eq. (5.3-65).

$$h_1(t) = \int_{-\infty}^{\infty} H(j\omega + j\omega_0) + H(j\omega - j\omega_0) \exp(j\omega t)\, df \qquad (5.3\text{-}66)$$

which can be written

$$h_1(t) = \exp(-j2\pi f_0 t) \int_{-\infty}^{\infty} H(j2\pi u) \exp(j2\pi ut)\, du$$

$$+ \exp(j2\pi f_0 t) \int_{-\infty}^{\infty} H(j2\pi v) \exp(j2\pi vt)\, dv \qquad (5.3\text{-}67)$$

where $u = f + f_o$ and $v = f - f_o$.

The two integrals in the right hand side of Eq. (5.3-67) are simply the expression for the impulse response of the low pass prototype filter and therefore

$$h_1(t) = 2h(t) \cos(2\pi f_0 t) \qquad (5.3\text{-}68)$$

which can be written in the following form

$$h_1(t) = 2h(t) \cos(2\pi f_0 t) \qquad (5.3\text{-}69)$$

If the transmissions are double sideband, i.e. symmetrical about f_1, then the center frequency of the filter f_o will be made equal to f_1. For this case according to Eq. (5.3-64) and (5.3-69), the noise waveform at point A will be

$$n(t) = \cos(2\pi f_1 t + \phi_1) \int_{-\infty}^{\infty} 2h(u) \cos(2\pi f_1 u) n_1(t - u) du \quad (5.3\text{-}70)$$

If the input noise waveform n (t) is the impulse $U \delta(t-t_1)$ then Eq. (5.3-70) reduces to

$$n(t) = U h(t - t_1) \left[\cos(\phi_1 + 2\pi f_1 t_1) + \cos(4\pi f_1 t + \phi_1 - 2\pi f_1 t_1) \right] \quad (5.3\text{-}71)$$

Since the finite memory filter is a type of low pass filter then there will be little contribution from the second term of the sum in the right hand side of Eq. (5.3-71) and for all practical purposes n(t) is given by

$$n(t) = C h(t - t_1) \quad (5.3\text{-}72)$$

where C is the random variable

$$C = U \cos(\phi_1 + 2\pi f_1 t_1)$$

and has the probability function given in Eq. (5.3-61)

The output of the finite memory integrator at the end of the baud is

$$V_{dn} = \int_0^T C h(t - t_1) e_2(t) dt \quad (5.3\text{-}73)$$

which is of the same form as Eq. (5.3-22). If the same assumptions are made about the length of the impulse response h(t), viz., the length of the waveform has essentially died out at t = 1/2W then the results of the development following Eq. (5.3-22) apply. Thus, the probability of error is given by

$$P_e = \frac{1}{2} + \int_{-\infty}^{\infty} \frac{W}{2} \left[y(-kA^2 T) - y(kA^2 T) \right] p_c(C)\, dC \qquad (5.3\text{-}74)$$

where $y(V_{dn})$ is given implicitly by

$$V_{dn} = 2kAC \int_0^{y(V_{dn})} h(t)\, dt - \int_0^{1/2W} h(t)\, dt \qquad (5.3\text{-}75)$$

If the transmissions are single sideband then $f_o = f_1 \pm W/2$ where the plus sign is chosen for upper sideband transmission and the minus sign corresponds to lower sideband transmission. It can easily be shown that the noise waveform which results from an impulse at the input is

$$2Uh(t - t_1) \cos(2\pi f_1 t + \phi_1) \cos \left[2\pi(f_1 \pm W/2)\, (t - t_1) \right]$$

and the low frequency component is found to be

$$n(t) = Uh(t - t_1) \cos(Wt/2 - Wt_1/2 \pm \phi_1 \pm 2\pi f_1 t_1) \qquad (5.3\text{-}76)$$

The probability of error can be found by a method similar to the one used previously (i.e. Eq. (5.3-24) to (5.3-49) taking into consideration that n(t) as given by Eq. (5.3-76) will change polarity for certain values of the quantity

$$-\frac{W}{2}t_1 \pm \phi_1 \pm 2\pi f_1 t_1$$

5.4 The analysis of systems having carriers consisting of quasi-random impulses when perturbed by impulse noise

In the analysis of systems which are perturbed by impulse noise, the following question arises. What is the nature of the signal waveform which is least vulnerable to impulse noise? If the energy of the signal waveform is held constant, it appears that in many cases the probability of error can be reduced (in comparison to the case of the last section) by selecting a signal waveform which is zero throughout the major part of the duration of the baud. In this manner the probability that the noise pulse occurs at a time that the local carrier is non-zero is made small.

In this section, the case will be examined where the carrier consists of pulses, each corresponding to a baud and each located in a quasi-random fashion during its baud period and having a quasi-random polarity (i.e. if a carrier pulse is picked at random then the probability that it is positive is one-half). It will be assumed that the signal pulses at the input to the receiver are square with a duration of $1/2W$ and have an amplitude A_1.

Since the ratio of the time during which the signal waveform is non-zero to the baud period is $1/2WT$ then the probability of the integrator output noise voltage V_{dn} being non-zero is $\overline{\Theta}/2WT$, where $\overline{\Theta}$ is the average occurrence per baud of the noise pulses. If the noise pulses have the waveform given in Eq. (5.3-10) i.e.

$$n(t) = C\ \delta(t - t_1)$$

If the amplitude of the locally generated carrier is kA_1 then the signal component of the integrator output at the end of the baud will be

$$V_{dm} = \int_0^T e_1(t)\, e_2(t)\, dt \qquad (5.4\text{-}1)$$

where

$$e_1(t) = \pm A_1 \left[u(t - t_0) - u(t - t_0 - 1/2W) \right] \qquad (5.4\text{-}2)$$

and

$$e_2(t) = \pm k A_1 \left[u(t - t_0) - u(t - t_0 - 1/2W) \right] \qquad (5.4\text{-}3)$$

Therefore,

$$V_{dm} = k A_1^2/2W \qquad (5.4\text{-}4)$$

The noise output, when it occurs, will have an amplitude

$$|V_{dn}| = |k A_1 C| \qquad (5.4\text{-}5)$$

Thus it is evident that the derivation following Eq. (5.3-10) can be made to fit this situation if AT is replaced by $A_1/2W$ in the equation following Eq. (5.3-17).

The probability of error obtained from Eq. (5.3-19) will be the conditional probability given that the noise pulse occurred during the interval that the signal was non-zero, i.e. $\overline{O}/2WT$, and therefore the average probability of error is

$$\overline{P}_e = \frac{1}{2} \left(\frac{\overline{O}}{2WT}\right) \left[1 + P_c(-A_1/2W) - P_c(A_1/2W) \right] \qquad (5.4\text{-}6)$$

In terms of the average signal power

$$P_s = A_1^2/2WT \tag{5.4-7}$$

The probability of error is

$$\overline{P}_E = \frac{1}{2} \left(\frac{\overline{0}}{2WT}\right) \left[1 + P_c(-T\sqrt{P_s} \;/\sqrt{2WT}) - P_c(T\sqrt{P_s}\;/\sqrt{2WT})\right] \tag{5.4-8}$$

Similarly, the resulting probability of error for the bandpass case can be obtained by replacing AT by $A_1/2W$ in Eq. (5.3-62) and $\overline{0}$ by $\overline{0}/2WT$.

$$\overline{P}_e = \frac{\overline{0}}{2WT} \left\{ \frac{1}{2} + \frac{1}{\pi} \int_{-\infty}^{-|A_1/2W|} \sin^{-1}\left[\frac{A_1}{2WU}\right] p_u(U)\, dU \right.$$

$$\left. - \frac{1}{\pi} \int_{|A_1/2W|}^{\infty} \sin^{-1}\left[\frac{A_1}{2WU}\right] p(U)\, dU \right\} \tag{5.4-9}$$

Of particular interest is the case where the impulses have been filtered previous to demodulation. From Eq. (5.4-3) it is evident that for the low pass case the noise component of the output of the integrator at the end of the baud will be

$$V_{dn} = \pm k\, A_1\, C \int_0^z h(t)\, dt \tag{5.4-10}$$

where h(t) is the impulse response of the filter and z is the random variable

$$z = t_0 + \frac{1}{2W} - t_1 \qquad (5.4\text{-}11)$$

having a probability density function

$$p(z) = \begin{cases} 2W; & 0 \leq y < 1/2W \\ 0; & \text{otherwise} \end{cases} \qquad (5.4\text{-}12)$$

The probability distribution function can be determined in the usual manner and is

$$P(V_{dn}) = 2W\, z(V_{dn}) \qquad (5.4\text{-}13)$$

where $z(V_{dn})$ is given implicitly by Eq. (5.4-10).

$$\overline{P}_e = \frac{1}{2} \left[\frac{\overline{0}}{2WT} \right] \left[P(- V_{dm}) - P(V_{dm}) + 1 \right] \qquad (5.4\text{-}14)$$

which with the aid of Eq. (5.4-13) becomes

$$\overline{P}_e = \frac{\overline{0}}{2WT} \left[\frac{1}{2} + Wz(-V_{dm}) - Wz(V_{dm}) \right] \qquad (5.4\text{-}15)$$

If C is a random variable, the value of $\overline{P}_e$ as obtained in Eq. (5.4-15) must be averaged with respect to C, i.e.

$$\overline{P}_e = \frac{\overline{0}}{2WT} \left\{ \frac{1}{2} + W \int_{-x}^{x} \left[y(-V_{dm}) - y(V_{dm}) \right] p_c(C)\, dC \right\} \qquad (5.4\text{-}16)$$

Eq. (5.4-16) also applies to the bandpass case since the random variable C can be related to the input impulse amplitude U by the method described in the last section.

5.5 The affects of noise which is a combination of impulse noise and interference from narrow band stations

In this section, the method by which the probability of error can be determined will be outlined for the case where the noise is a combination of interfering narrowband stations and impulse noise. Since, in general, the spectrum of the interference from the narrow band stations will be very jagged, it will be assumed that there will be a filtering operation on both the input signal plus noise waveform and on the locally generated carrier waveform (as shown in Fig. 5.5-1(a)). The affect of this filtering on the output component resulting from impulse noise will now be analyzed. The output of the input filter H_1 is

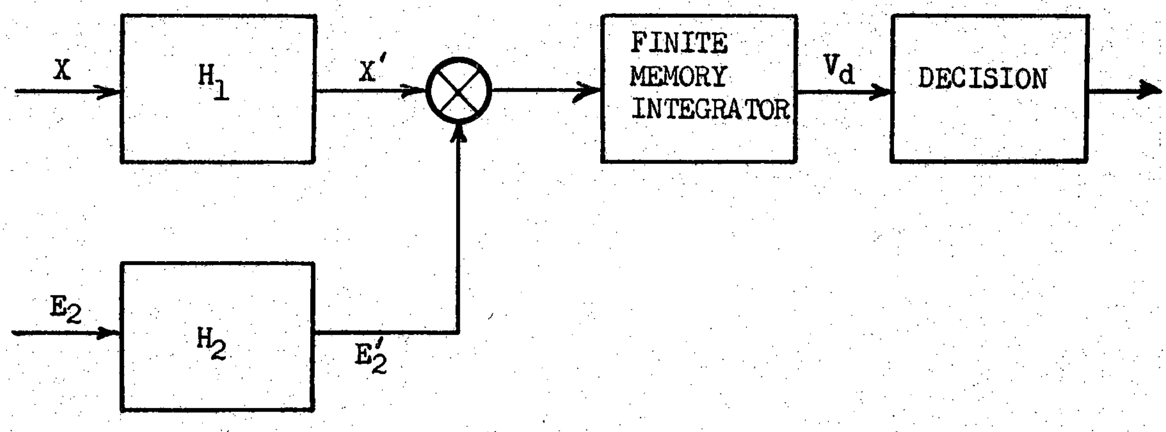
$$\vec{X}' = \sum_{i=1}^{n} X_i H_{1i} \vec{I}_i \qquad (5.5\text{-}1)$$

and the output of H_2 is

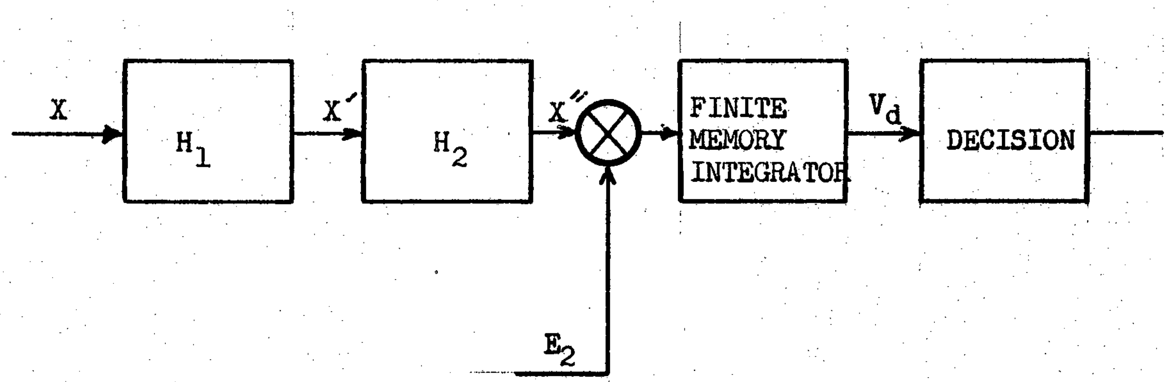
$$\vec{E}_2' = \sum_{i=1}^{n} E_{2i} H_{2i} \vec{I}_i \qquad (5.5\text{-}2)$$

The output of finite memory integrator at the end of the baud is

$$V_d = \vec{X}' \cdot \vec{E}_2' = \sum_{i=1}^{n} X_i E_{2i} H_{1i} H_{2i} \qquad (5.5\text{-}3)$$



(a)



(b)

EQUIVALENT WIDEBAND RECEIVERS

FIGURE 5.5-1

For the system shown in figure 5.5-1(b) the output of filter H_2 is

$$\vec{X}'' = \sum_{i=1}^{n} X_i\, H_{1i}\, H_{2i}\, \vec{I}_i \tag{5.5-4}$$

and the output of the integrator at the end of the baud is

$$V_d = \vec{X}''\cdot E_2 = \sum_{i=1}^{n} X_i\, E_{2i}\, H_{1i}\, H_{2i} \tag{5.5-5}$$

From Eq. (5.5-3) and (5.5-5) it is evident that the two systems are equivalent.

If $e_2(t)$ is either a quasi-random binary sequency or quasi-random impulses then the material which was previously developed in this chapter can be used where the impulse response referred to is the impulse response of the combination of H_1 and H_2, i.e.

$$h(t) = \int_{-\infty}^{\infty} H_1(j\omega)\, H_2(j\omega)\, e^{\omega t}\, df \tag{5.5-6}$$

If interference from narrow band stations is present as well as impulse noise then it is necessary to determine the probability density function of

$$V_{dn} = V_{dni} + V_{dns} \tag{5.5-7}$$

where V_{dni} is the component due to the impulse noise and V_{dns} is the component due to the interference from the narrow band stations. V_{dni} and V_{dns} are independent random variables and their density functions have been determined (or approximated) in Chapter IV and Chapter V. The density function of V_{dn} can be determined in the usual manner by convolution

$$p(V_{dn}) = \int_{-\infty}^{\infty} p_i(x)\, p_s(V_{dn} - x)\, dx \tag{5.5-8}$$

where $p_i(V_{dni})$ and $p_s(V_{dns})$ are the density functions of V_{dni} and V_{dns}.

The probability of error can now be determined in the usual way and for a symmetrical system

$$P_e = \frac{1}{2} \int_{-\infty}^{-V_{dm}} p(V_{dn})\, d\, V_{dn} + \frac{1}{2} \int_{V_{dm}}^{\infty} p(V_{dn})\, dV_{dn} \tag{5.5-9}$$

where V_{dm} is the signal component of the output of the integrator when a "mark" is transmitted.

The presence of a prewhitening filter will cause the synchronous demodulator output which results from impulse noise at the input to be smeared over a relatively long interval. If, for example, the prewhitener consists of a bank of filters where the bandwidth of each component filter is 1/T then the impulse response of the prewhitening filter will be in the order of T seconds. It is evident that, in this case, the advantage of using a quasi-random impulse carrier to reduce the systems vulnerability to impulse noise will be nullified. Any attempt to reduce the vulnerability to impulse noise by increasing the bandwidth of the component filters will result in an increased vulnerability to narrow band interference. Thus, it is apparent that a system cannot be designed to have minimum vulnerability to narrowband interference.

5.6 Summary

In this chapter, the performance of linear receivers has been analyzed for the case where the interference consists of impulse noise. Two types of linear receivers have been considered. The first type of receiver employs a quasi-random binary noise carrier and the second uses a carrier consisting of quasi-random impulses. The results obtained are extended to include interference which consists of narrowband noise as well as impulse noise. It was shown that minimum vulnerability to impulse noise cannot be achieved while at the same time maintaining minimum vulnerability to narrowband interference.

CHAPTER VI

NON-LINEAR RECEIVERS

6.1 Introduction

In the last three chapters it has been shown that a properly designed linear receiver can be expected to perform relatively well in the large class of situations where the optimum decision surface could be approximated by a hyperplane. In this chapter several non-linear methods will be discussed which can be used to obtain optimum or at least nearly optimum reception for a much more general class of situation.

6.2 Computation of the likelihood ratio

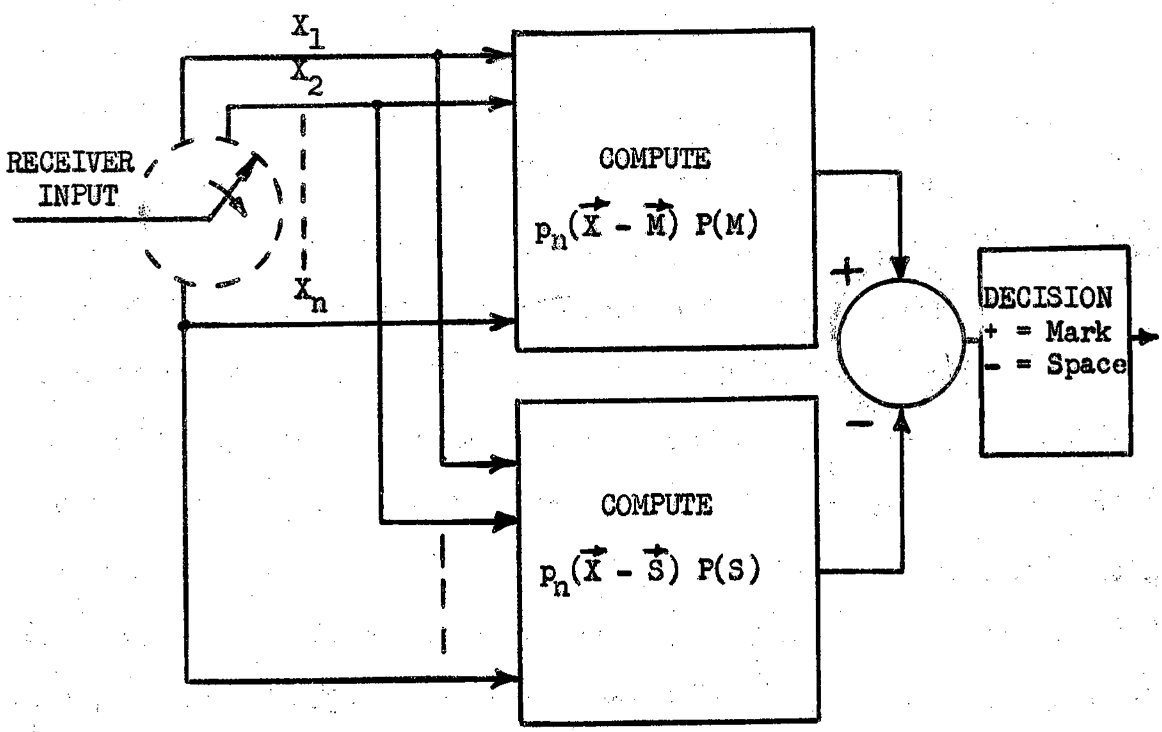
In section 2.5 the likelihood ratio was shown to be equal to

$$L = \frac{p_n(\vec{X} - \vec{M})\, P(M)}{p_n(\vec{X} - \vec{S})\, P(S)} \tag{6.2-1}$$

where $p_n(N)$ is the probability density of the noise vector $\vec{N}$, and when $P(M)$ and $P(S)$ are the probabilities of having transmitted a mark and a space respectively. Here, as usual $\vec{X}$, $\vec{M}$ and $\vec{S}$ are respectively the received signal plus noise, "mark", and "space" vectors. It was shown that the equation of the optimum decision surface is

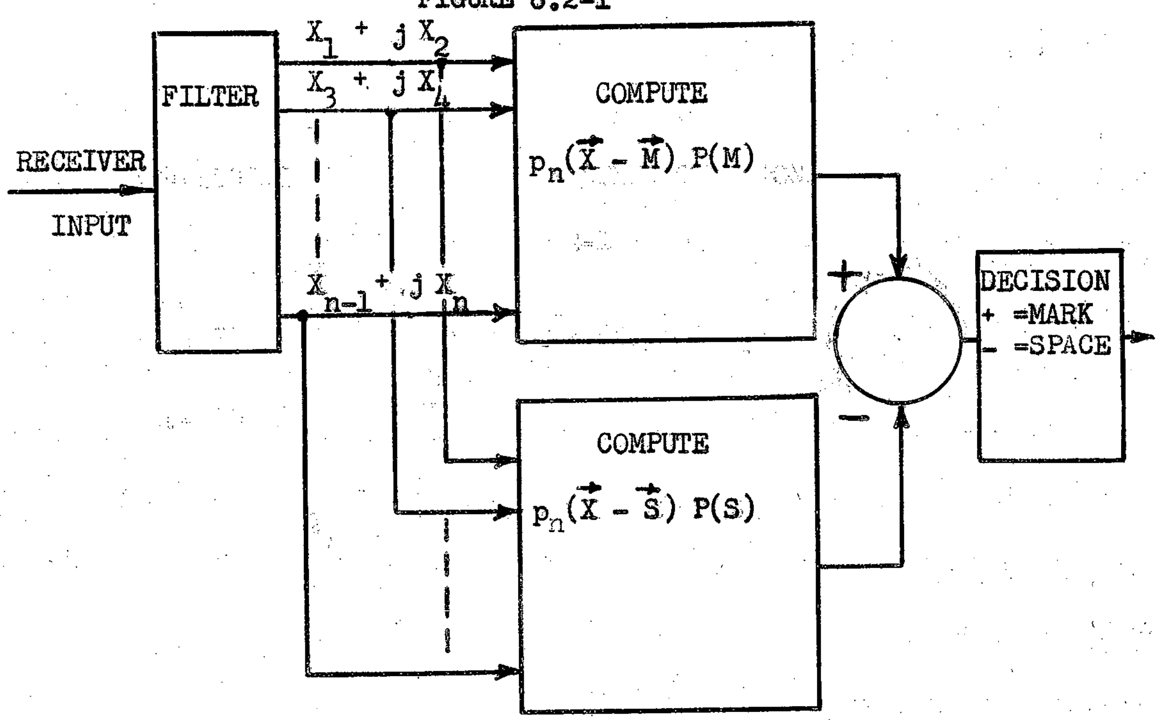
$$L = 1 \tag{6.2-2}$$

and that the optimum decision is "mark" for $L > 1$ and "space" for $L < 1$. Clearly, one method of obtaining optimum reception is to design the receiver to perform the computation indicated by Eq. (6.2-1) and then put out a "mark" or "space" depending on whether this value exceeds or is less than unity. Two possible configurations of such a receiver are



A NON-LINEAR RECEIVER WHICH SAMPLES THE SIGNAL IN TIME

FIGURE 6.2-1



A NON-LINEAR RECEIVER WHICH SAMPLES THE SIGNAL IN FREQUENCY

FIGURE 6.2-2

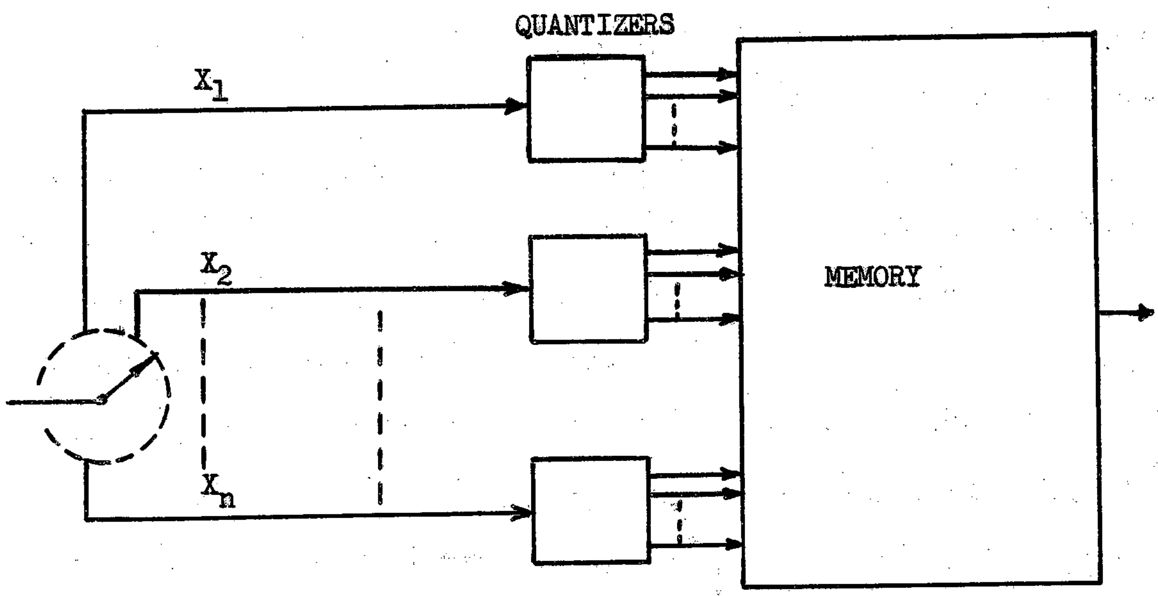
shown in Fig. 6.2-1 and 6.2-2. In Fig. 6.2-1 the input to the receiver is sampled at time intervals of 1/2W to get an orthogonal set of coordinate values of the signal plus noise vector $\vec{X}$. The numerator and denominator of the right hand side of Eq. (6.2-1) are computed separately and the value of the denominator is subtracted from the numerator. If this difference is greater than zero then L is greater than unity and the optimum decision is "mark". If the difference is less than zero the optimum decision is "space". This is accomplished with a decision circuit having a zero threshold level.

In Fig. 6.2-2 is an alternative configuration. The input is sampled in frequency at intervals of 1/T to obtain an orthogonal set of coordinate values of the signal plus noise vector $\vec{X}$. The method in which these coordinate values are further processed is essentially the same as with the configuration of Fig. 6.2-1.

If all the design information is known, (i.e. $p_n(N)$, $\vec{M}$, $\vec{S}$, P(M) and P(S)) then it is not difficult to conceive of the realization of such a receiver. For example, the coordinate values of $\vec{X}$ can be converted to digital form and processed by special purpose computers to perform the indicated operations. A difficulty arises, however, in that in general we would not expect to know the magnitudes of the received signal waveforms $\vec{M}$ and $\vec{S}$. This implies that the receiver must also contain some mechanism which measures the magnitudes of the $\vec{M}$ and $\vec{S}$ vectors at its input and then adapts its computation process accordingly.

6.3 Subdivision of the signal space.

With this method the signal space is divided into subdivisions and each subdivision is appropriately tagged with the corresponding optimum



A NON-LINEAR RECEIVER WHICH SUBDIVIDES THE SIGNAL SPACE

FIGURE 6.3-1

decision. This information is stored in the memory of the receiver (see Fig. 6.3-1). The signal plus noise is sampled (either in frequency or in time) in the same manner as was described in section 6.2 to obtain its coordinate values. These coordinate values are then quantized and the combined output of the quantizers (which in each case uniquely corresponds to a subdivision of the signal space) is used to retrieve the correct decision from the memory.

Here, as was the case with the systems described in section 6.2, it is necessary to know the magnitudes of the signal vectors $\vec{M}$ and $\vec{S}$ beforehand. Furthermore a change in the magnitudes of $\vec{M}$ and $\vec{S}$ may require a major revision of the information stored in the memory which would make it particularly difficult to incorporate a suitable adaptive process.

6.4 Non-linear coordinate transformations

In this section a method will be described by which a non-linear transformation can be determined which, when applied to the coordinate values of the received signal plus noise vector, will cause the decision surface to be transformed into a hyperplane. Subsequent processing with a suitable linear receiver will be shown to result in optimum reception.

It will be assumed that the noise vector $\vec{N}$ which has been added to the signal has coordinate values which are independent random variables and therefore the joint probability density of the noise $p_n(\vec{N})$ will have form

$$p_n(\vec{N}) = \prod_{i=1}^{n} p_i(N_i) \qquad (6.4\text{-}1)$$

where $p_i(N_i)$ is the probability density function of the i-th coordinate value of the noise.

As was shown in Section 2.5 the decision surface (see Eq. 2.5-5) is given by

$$p_n(\vec{X} - \vec{M})\, P(M) = p_n(\vec{X} - \vec{S})\, P(S) \qquad (6.4\text{-}2)$$

The optimum decision will be a "mark" if

$$p_n(\vec{X} - \vec{M})\, P(M) > p_n(\vec{X} - \vec{S})\, P(S) \qquad (6.4\text{-}2a)$$

and a "space" if

$$p_n(\vec{X} - \vec{M})\, P(M) < p_n(\vec{X} - \vec{S})\, P(S) \qquad (6.4\text{-}2b)$$

For the sake of simplicity it will be assumed that the system is symmetric, i.e. that $\vec{S} = -\vec{M}$ and that $P(M) = P(S) = 1/2$ which reduces Eq. (6.4-2) to

$$p_n(\vec{X} - \vec{M}) = p_n(\vec{X} + \vec{M}) \qquad (6.4\text{-}3)$$

Substituting (6.4-1) into (6.4-3) we obtain

$$\prod_{i=1}^{n} p_i(X_i - M_i) = \prod_{i=1}^{n} p_i(X_i + M_i) \qquad (6.4\text{-}4)$$

which can also be expressed in the form

$$\sum_{i=1}^{n} \log p_i(X_i - M_i) = \sum_{i=1}^{n} \log p_i(X_i + M_i) \qquad (6.4\text{-}5)$$

Now suppose the logarithm of each of the density function is expanded into a Maclaurin series

$$\log p_i(N_i) = a_{i0} + a_{i1}N_i + a_{i2}N_{i2}^2 + \cdots \qquad (6.4\text{-}6)$$

which when substituted into Eq. (6.4-5) yields

$$\sum_{i=1}^{n} a_{i0} + a_{i1}(X_i - M_i) + a_{i2}(X_i - M_i)^2 + \cdots$$

$$= \sum_{i=1}^{n} a_{i0} + a_{i1}(X_i + M_i) + a_{i2}(X_i + M_i)^2 + \cdots \qquad (6.4\text{-}7)$$

which can be simplified to

$$- \sum_{i=1}^{n} (2a_{i1}M_i) + (4a_{i2}X_iM_i) + (6a_{i3}X_i^2M_i + 2a_{i3}M_i^3) \qquad (6.4\text{-}8)$$

$$+ (8a_{i4}X_i^3M_i + 8a_{i4}X_iM_i^3) + \cdots = 0$$

Thus the surface can be represented as

$$- \sum_{i=1}^{n} \sum_{j=0}^{\infty} \alpha_{ij}X_i^j = 0 \qquad (6.4\text{-}9)$$

where

$$\alpha_{io} = a_{i1}M_i + a_{i3}M_i^3 + a_{i5}M_i^5 + \cdots$$

$$= \sum_{k=1}^{\infty} a_{i,(2k-1)}M_i^{(2k-1)}$$

$$\alpha_{i1} = 2a_{i2}M_i + 4a_{i4}M_i^3 + 6a_{i6}M_i^5 + \cdots$$

$$= \sum_{k=1}^{\infty} 2ka_{i,2k} M^{(2k - 1)}$$

$$\alpha_{i2} = 3a_{i3} M_i + 10a_{i5} M_i^3 + 21a_{i7} M_i^5 + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(2k + 1)!}{(2k - 1)!2!} a_{i,(2k + 1)} M_i^{(2k - 1)}$$

$$\alpha_{ij} = \sum_{k=1}^{\infty} \frac{(j + 2k - 1)!}{j!(2k - 1)!} a_{i,(j + 2k - 1)} M_i^{(2k - 1)} \qquad (6.4\text{-}10)$$

A set of non-linear coordinate transformations can be choosen such that

$$Y_i = \frac{1}{\beta_i} \sum_{j=0}^{\infty} \alpha_{ij} X_i^j \qquad (6.4\text{-}11)$$

where the β_i's are arbitrary constants. When Eq. (6.4-11) is substituted into Eq. (6.4-9) we have

$$- \sum_{i=1}^{n} \beta_i Y_i = 0 \qquad (6.4\text{-}12)$$

which is the equation of the hyperplane which passes through the origin and is orthogonal to the vector

$$\vec{B} = \sum_{i=1}^{n} \beta_i \vec{I}_i \qquad (6.4\text{-}13)$$

Thus, the non-linear coordinate transformation given in equation (6.4-11) transforms the decision surface given by Eq. (6.4-4) into the hyperplane given by Eq. (6.4-12).

It is important to determine the affect that the transformation has on the "mark" region specified by Eq. (6.4-2a) and "space" region specified by Eq. (6.4-2b). The affect on the "mark" region can be determined by employing a line of reasoning on Eq. (6.4-2a) similar to the one used on Eq. (6.4-2). This is equivalent to replacing the "equals" sign in Eq. (6.4-3) through (6.4-5) and (6.4-7) through (6.4-9) with a "greater than" sign. Thus it is evident that the region given by

$$- \sum_{i=1}^{n} \sum_{j=1}^{\infty} \alpha_{ij} X_i^j > 0 \qquad (6.4\text{-}9a)$$

is identical to the region given by Eq. (6.4-3), i.e., the "mark" region. It is evident that any point $\vec{X}$ satisfying inequality (6.4-9a) which is subjected to the non-linear transformation will map into a point $\vec{Y}$ satisfying

$$- \sum_{i=1}^{n} \beta_i Y_i > 0 \qquad (6.4\text{-}12a)$$

Therefore, the "mark" region given in Eq. (6.4-2a) is mapped into the semi-infinite region given by inequality (6.4-12a) which is all the region on one side of the surface given by (6.4-12).

A similar analysis shows that the "space" region given by inequality (6.4-2b) is mapped into the region

$$- \sum_{i=1}^{n} \beta_i Y_i < 0$$

which is the region on the other side of the surface given by (6.4-12).

From the foregoing it is clear that the optimum detection can now be achieved through the use of a synchronous receiver having a zero threshold and a local carrier as specified by Eq. (6.4-13). Two possible configurations of such a receiver are shown in Fig. 6.4-1 and 6.4-2.

In practice the operations specified in Eq. (6.4-11) would be approximated with series of finite length.

Suppose the series of Eq. (6.4-10) converges sufficiently rapidly that α_{ij} can be approximated with one term.

$$\alpha_{ij} = (j + 1)\, a_{i,(j+1)}\, M_i \tag{6.4-14}$$

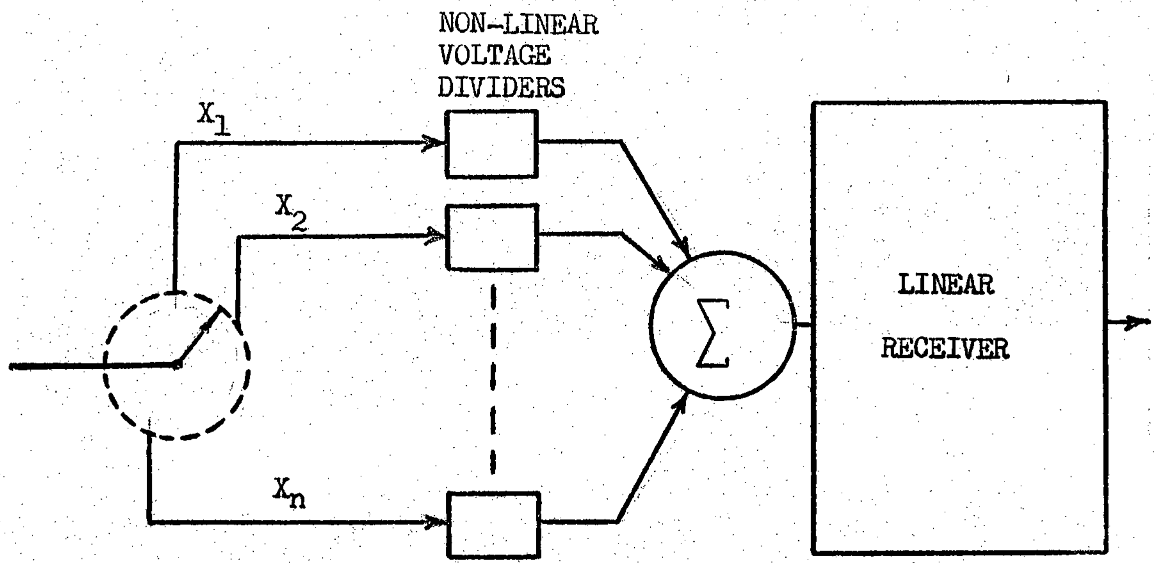
and

$$Y_i = \frac{1}{\beta_i} \sum_{j=0}^{\infty} (j + 1)\, a_{i,(j+1)}\, M_i\, X_i^j \tag{6.4-15}$$

Here it is convenient to choose $\beta_i = M_i$ and

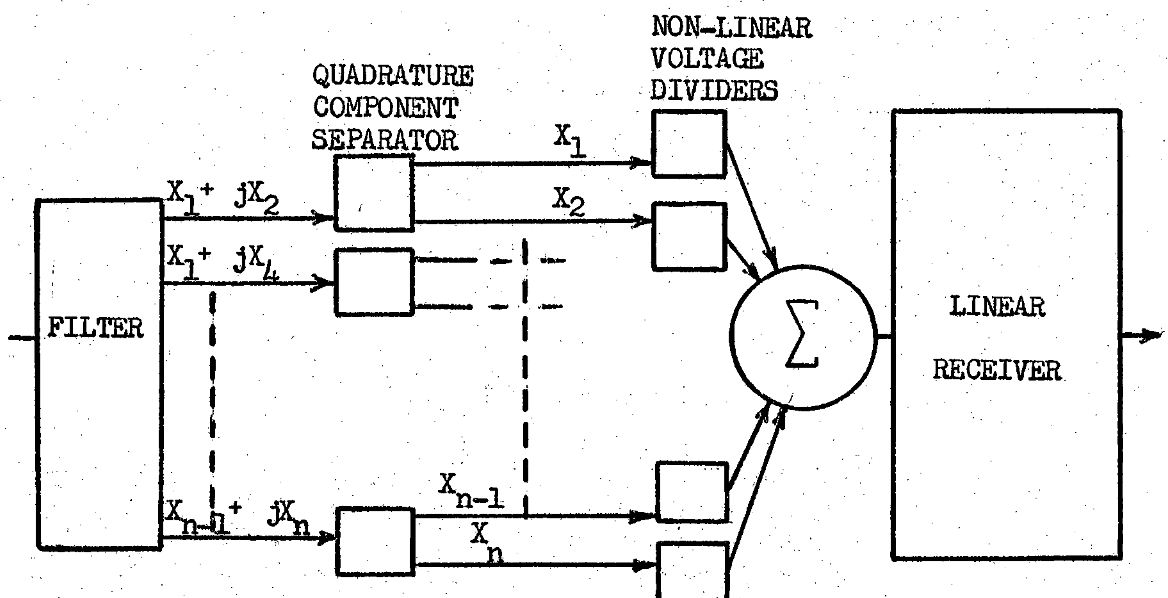
$$Y_i = \sum_{j=0}^{\infty} (j + 1)\, a_{i,(j+1)}\, X_i^j \tag{6.4-16}$$

The form given by Eq. (6.4-16) is especially useful since $\vec{Y}$ does not depend on $\vec{M}$ but only on $\vec{X}$. This means that reception can be accomplished without previous knowledge of the magnitude of $\vec{M}$.



A NON-LINEAR RECEIVER WHICH SAMPLES IN TIME

FIGURE 6.4-1



A NON-LINEAR RECEIVER WHICH SAMPLES INFREQUENCY

FIGURE 6.4-2

Whether the approximation given by Eq. (6.4-14) is appropriate in a given situation can only be determined by examination of Eq. (6.4-10). In general, however, we may anticipate that for a given situation the approximation will be valid for sufficiently small signal-to-noise ratios.

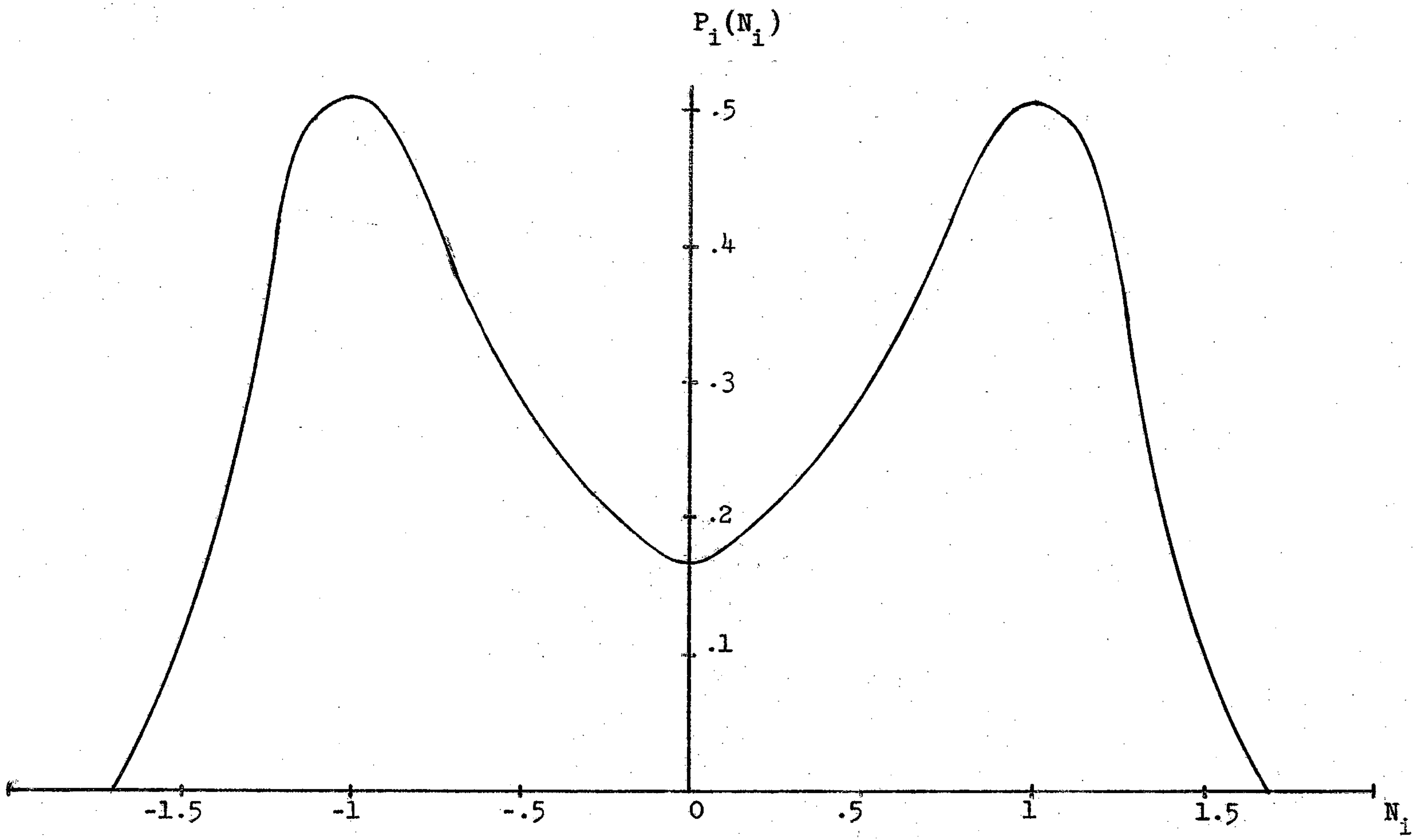
The following example will be used to illustrate the method which was derived above. Suppose that it has been determined that the noise when sampled in time at intervals of 1/2W can be represented as 2WT independent random variables having a common probability distribution whose density functions are symmetrical about the origin. Suppose further that the logarithms of these density functions can be represented with reasonable accuracy by the following Maclurin's series.

$$\log p_i\ (N_i) = a_0 + 2N_i^2 - N_i^4 \tag{6.4-17}$$

A graph of $p_i(N_i) = .1865 \exp (2N_i^{\ 2} - N_i^{\ 4})$ is shown in Fig. 6.4-3. We shall also assume that the signal $\vec{M}$ is a psuedo-random binary noise sequence such that $M_i^{\ 2} = M^2$ where $2WM^2$ is the average signal power.

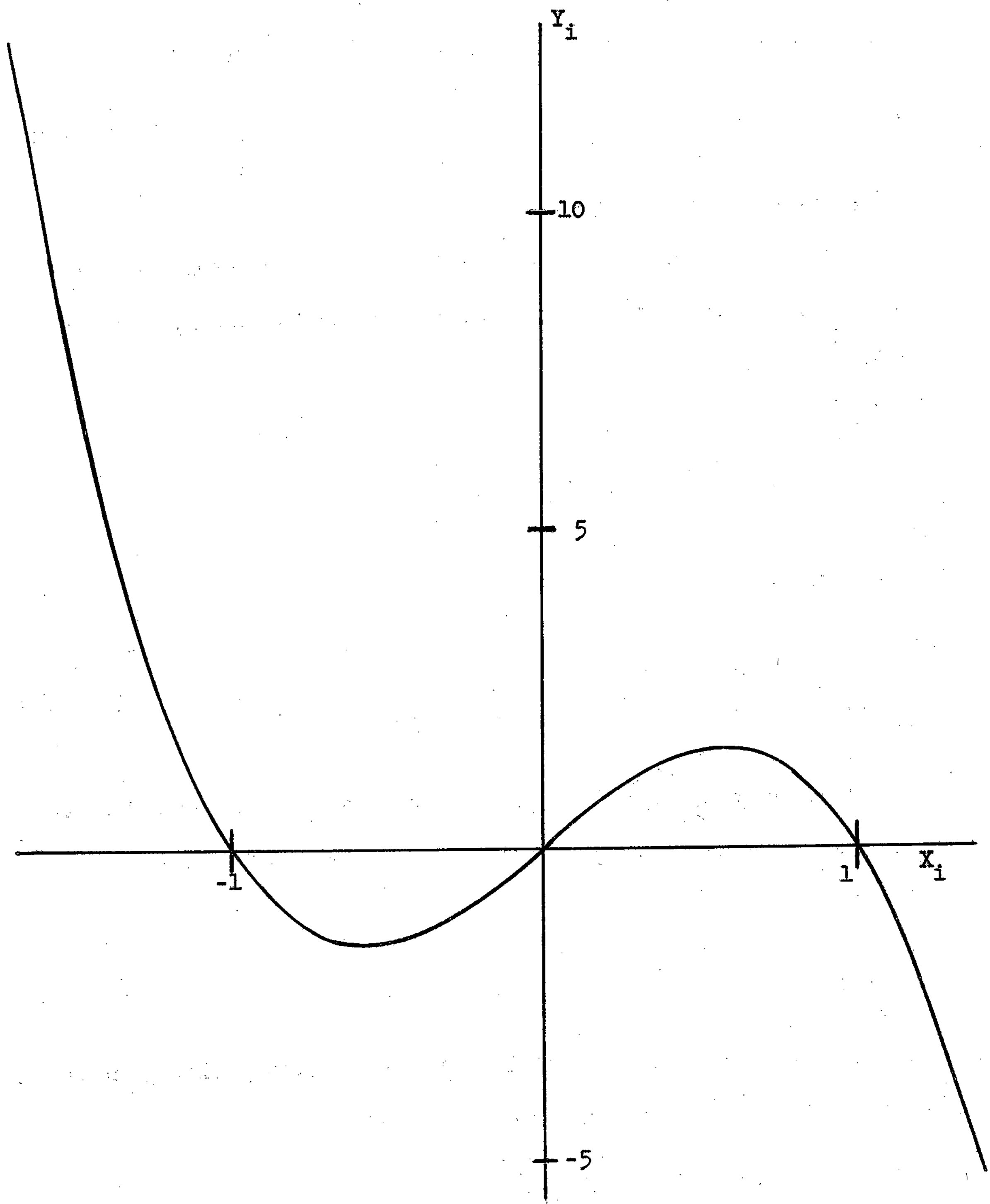
With the aid of Eq. (6.4-10) the following values are determined

$$\begin{aligned} \alpha_{io} &= 0 \\ \alpha_{i1} &= 4M_i - 4M_i^3 \\ \alpha_{i2} &= 0 \\ \alpha_{i3} &= -4M_i \\ \alpha_{ij} &= 0 \text{ for } j > 3 \end{aligned} \tag{6.4-18}$$



THE PROBABILITY DENSITY FUNCTION OF THE NOISE

FIGURE 6.4-3



THE REQUIRED NON-LINEAR TRANSFORM

FIGURE 6.4-4

The exact expression for Y_i as given by Eq. (6.4-11) is

$$Y_i = (4 - 4 M_i^2) X_i - 4 X_i^3 \tag{6.4-19}$$

where β_i has been set equal to M_i.

Through the use of Eq. (6.4-9) it is possible to determine the equation of the optimum decision surface prior to the non-linear transformation

$$\sum_{i=1}^{n} (4 M_i - 4 M_i^3) X_i - 4 M_i X_i^3 = 0 \tag{6.4-20}$$

which reduces to

$$\sum_{i=1}^{n} \left[(1 - M_i^2) - X_i^2 \right] X_i = 0 \tag{6.4-21}$$

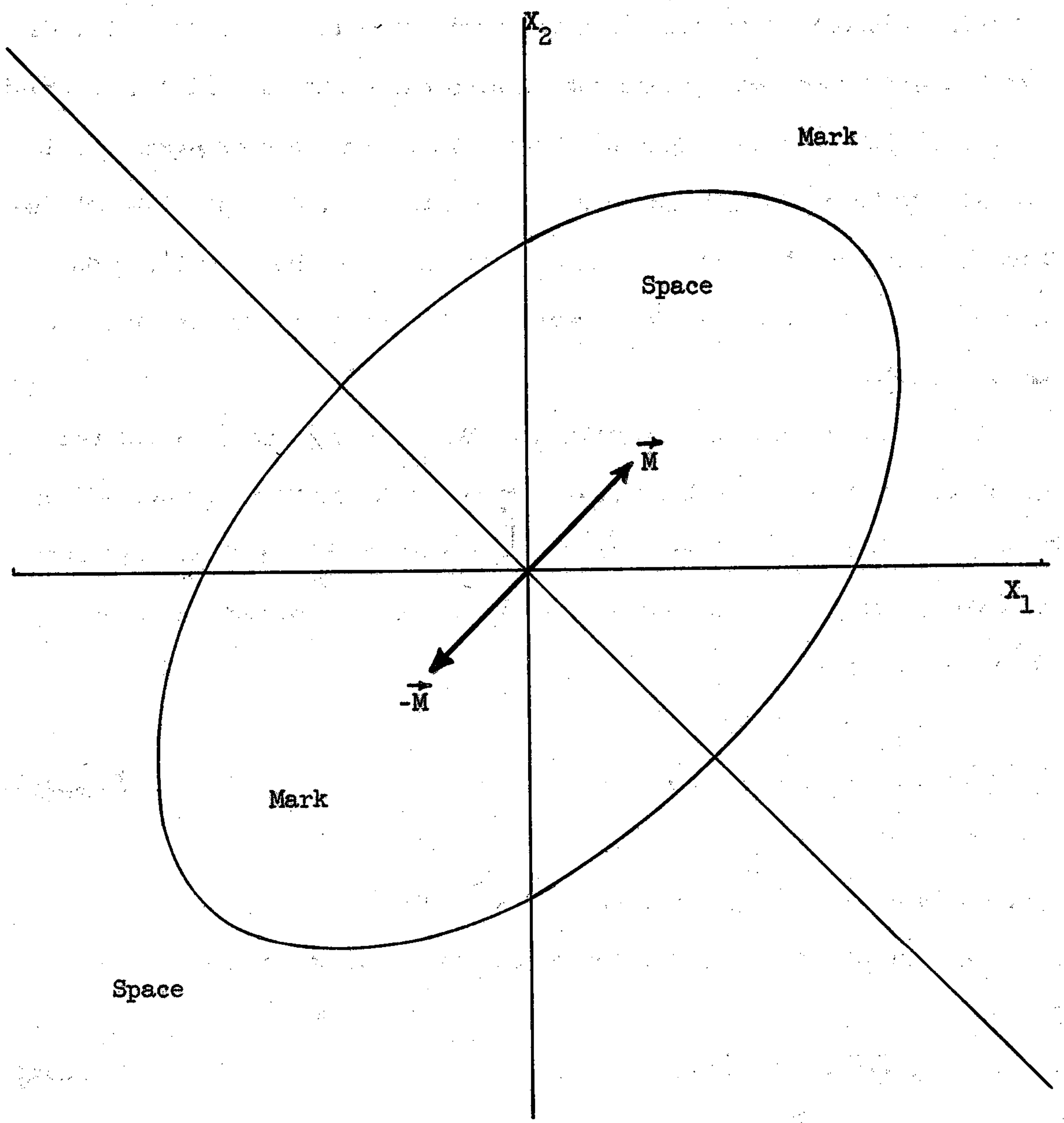
It is of interest to examine the decision surface generated by Eq. (6.4-21) for the case of a two dimensional signal space. Clearly a solution is

$$X_1 + X_2 = 0 \tag{6.4-22}$$

The root corresponding to this solution can be removed yielding an easily solved quadratic having the following solutions

$$X_2 = \frac{X_1 \pm \sqrt{X_1^2 + 4(1 - M_1^2 - X_1^2)}}{2} \tag{6.4-23}$$

The curve given by Eq. (6.4-22) is clearly a straight line and it is possible to show that the curve of Eq. (6.4-23) is an ellipse whose major axis makes a 45° angle with the X_1 axis. A graph of these



THE DECISION SURFACE OF THE SIGNAL SPACE AT THE

INPUT TO THE RECEIVER

FIGURE 6.4-5

curves is shown in Fig. 6.4-5 where $M_i^2 = 0.1$. In the figure the "mark" and "space" signal vectors are indicated by $\vec{M}$ and $-\vec{M}$ respectively. The optimum decisions associated with the various regions bounded by the decision curves have been determined in accordance with the likelihood ratio and are indicated on the figure by the words "mark" and "space". It is interesting to note that the "mark" signal point lies in the "space" decision region and the "space" signal point lies in the "mark" region. Clearly no linear receiver is capable of optimum reception under these circumstances.

It is clear from an examination of Eq. (6.4-19) and (6.4-20) that the transformation given by (6.4-19) produces a decision surface which is a hyperplane and optimum reception can now be obtained with a linear receiver having a zero threshold and the following local carrier as given by Eq. (6.4-13)

$$k\vec{B} = \sum_{i=1}^{n} k\, M_i\, \vec{I}_i = k\, \vec{M}. \tag{6.4-24}$$

where k is an arbitrary constant.

The output of the integrator of the linear receiver is

$$V_d = k\, \vec{B}\cdot\vec{Y} = \sum_{i=1}^{n} k\, M_i\, Y_i \tag{6.4-25}$$

With the aid of Eq. (6.4-19) and the fact that $X_i = N_i + M_i$ the elements of the sum are found to be equal to

$$M_i\ Y_i = M_i\ (4 - 4M_i^2)\ \ (N_i + M_i) - 4(N_i + M_i)^3$$

$$= 4 - M_i\ N_i^3 - 3M_i^2\ N_i^2 + (1 - 4M_i^2)\ M_i\ N_i + (1 - 2M_i^2)\ M_i^2 \qquad (6.4\text{-}26)$$

That each of the M_iY_i are independent random variables with common distributions can be shown by examination of Eq. (6.4-26) and by recalling the properties assumed for M_i and N_i (namely that the N_i are independent random variables whose densitites are common and symmetrical about the origin and that $2W\overline{M_i^2} = 2W\overline{M^2}$ which is the average signal power). From this we can infer that V_d is a random variable with a gaussian distribution. The probability of error P_e is

$$P_e = \frac{1}{2} - \frac{1}{2}\,\text{erf}\left\{\frac{n\,E\,(M_i\ Y_i)}{\sqrt{2n\ \ E(M_i\ Y_i)^2 - E^2\ (M_i\ Y_i)}}\right\} \qquad (6.4\text{-}27)$$

where $E(M_iY_i)$ and $E(M_iY_i)^2$ are the mean value and mean squared value of the random variable M_iY_i. For a signal power of $2WM^2 = .2W$ these quantities are found to be

$$E(M_i\ Y_i) = \int_{-\infty}^{\infty} M_i\ Y_i\ p_n(N_i)\ \ dN_i = -\ 2.15M \qquad (6.4\text{-}28)$$

$$E(M_i\ Y_i)^2 = \int_{-\infty}^{\infty} \left[M_i\ Y_i\right]^2 p_n(N_i)\ dN_i = 21.6M^2 \qquad (6.4\text{-}29)$$

The output signal-to-noise ratio is

$$(SNR) = \frac{n\,E^2(M_i\,Y_i)}{E(M_iY_i)^2 - E^2(M_iY_i)} = 0.22n \qquad (6.4\text{-}30)$$

If instead of the non-linear receiver described above a linear receiver is used which has a zero threshold and the same local carrier then the signal-to-noise ratio is found to be

$$SNR = 0.12n \qquad (6.4\text{-}31)$$

which indicates that the non-linear transformation gives an improvement in the signal-to-noise ratio of 1.83 (2.62db).

If, for example, n = 100 the probability of error of the output of the non-linear receiver is 1.32×10^{-6} while the probability of error for the linear receiver is 2.65×10^{-4}.

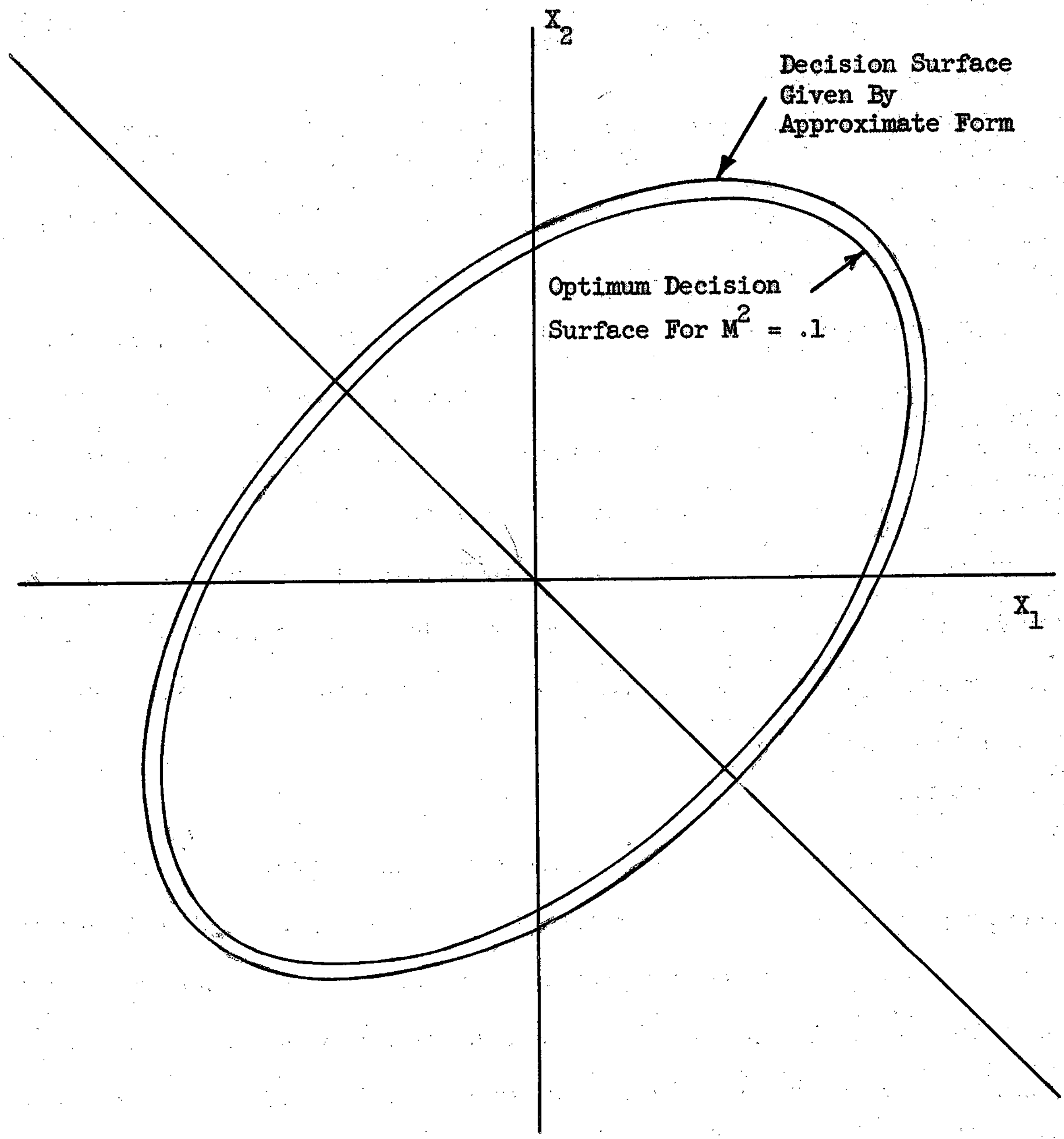
From Eq. (6.4-18) it appears that the approximate form given by Eq. (6.4-16) would be useful. In this case the non-linear transformation is

$$Y_i = 4X_i - 4X_i^3 \qquad (6.4\text{-}32)$$

It is evident that Eq. (6.4-32) is equivalent to equation (6.4-19) with M_i set equal to zero and thus the decision surface in two dimensions corresponding to Eq. (6.4-32) can be obtained from Eq. (6.4-22) and (6.4-23) by setting M_i equal to zero. Therefore the decision surface is given by

$$X_1 + X_2 = 0 \qquad (6.4\text{-}33)$$

and



A COMPARISON OF THE DECISION SURFACES GENERATED

FIGURE 6.4-6

$$X_2 = \frac{X_1 \pm \sqrt{4 - 3X_1^2}}{2} \qquad (6.4\text{-}34)$$

and the difference between the curves given by Eq. (6.4-33) and (6.4-34) and Eq. (6.4-22) and (6.4-23) with M_1^2 set equal to 0.1 is a difference in the length of the major and minor axes of the elipses.

The effectiveness of the transformation given by Eq. (6.4-32) can be determined by comparing the decision surface which it generates with the optimum surface previously determined. This comparison is made in two dimensions in Fig. 6.4-6. As can be seen the smaller the signal-to-noise ratio the better the approximation becomes which in general can be considered a desirable property since excellence of performance is of greater importance at low signal-to-noise ratios that it is at high signal-to-noise ratios.

6.5 Summary

Three methods of obtaining optimum reception through the use of non-linear techniques have been discussed in this chapter. The first two have the serious disadvantage that the parameters of the receiver must be determined in accordance with the strength of the received signal. An approximate form of the third method which is valid for small signal-to-noise ratios is independent of the strength of the received signal, however. For this reason, this method is considered to have the greatest practical significance.

CHAPTER VII

APPLICATIONS OF NON-LINEAR RECEIVERS

7.1 Introduction

In chapter VI, three methods of non-linear reception were described. An important design parameter of two of the systems is the received signal strength. An approximate form of the third method is independent of signal strength and for this reason this method appears to be the most useful. In this chapter the possible applications of this method are discussed.

7.2 Interference consisting of many narrowband stations

In Section 4.3, the nature of the interference which results from many similar narrowband stations was analyzed. It was pointed out that to a good approximation, the signal represented by a WT-dimensional vector (as represented by WT complex numbers), is perturbed by WT random variables and that in many cases these random variables can be considered independent. In fact, it is apparant from the analyses of Section 4.3 that independence can be insured by choosing the baud length T of the wideband system such that it is equal to 1/kB, where B is the bandwidth of the narrowband stations and k is an integer. Thus, if the communication system is symmetrical then the hypothesis for the development of the non-linear receiver described in Section 6.4 is satisfied and optimum reception using this technique is possible.

For example, if the narrowband interference consists of WT adjacent AM stations, each with a bandwidth of 1/T then the density functions of the noise for each of the coordinates can be obtained with the aid of Eq. (4.3-18) which can be written

$$p_i(N_i) = \int_{|N_i|}^{\infty} \frac{p_{ui}(u_i)du_i}{\pi\sqrt{u_i^2 - N_i^2}} \tag{7.2-1}$$

where $p_{ui}(u_i)$ is the probability density function of the amplitude u_i of the i-th narrowband station and $p_i(N_i)$ is the probability density function of the i-th coordinate value of the noise. The parameters of the optimum non-linear receiver can then readily be determined from Chapter VI.

If the interfering stations transmit CW carriers such that

$$p_{ui}(u_i) = \delta(u_i - C_i) \tag{7.2-2}$$

then

$$p_i(N_i) = \begin{cases} \dfrac{1}{\pi\sqrt{C_i^2 - N^2}} ; & N_i^2 \leq C_i^2 \\ 0 ; & N_i^2 > C_i^2 \end{cases} \tag{7.2-3}$$

In order to obtain the desired Maclurin's series it is convenient to approximate Eq. (7.2-3) with the following expression

$$p_i(N_i) = \left[\frac{1}{\pi\sqrt{C_i^2 - N_i^2}}\right] \left[\frac{1}{1 + N_i^m}\right] \tag{7.2-4}$$

where the closeness of the approximation depends on the exponent m.

The coefficients of the first m terms of the Maclurin's series as determined from Eq. (7.2-4) are

$$a_{i0} = -\log \pi C_i \tag{7.2-5}$$

$$a_{ir} = \begin{cases} (r - 1)! \;/\; C^r; & r \text{ even and } 0 < r < m \\ 0; & r \text{ odd} \end{cases} \qquad (7.2\text{-}6)$$

$$a_{im} = \left[-m! + (m - 1)! \right] / C^m \; ; \quad m \text{ even} \qquad (7.2\text{-}7)$$

Suppose that it is decided to terminate the series with a_{i6}. According to Eq. (7.2-6) and Eq. (7.2-7)

$$a_{i2} = 1/C_i^2 \qquad (7.2\text{-}8)$$

$$a_{i4} = 6/C_i^4 \qquad (7.2\text{-}9)$$

$$a_{i6} = -600/C_i^2 \qquad (7.2\text{-}10)$$

The coefficients of the non-linear transformation can now be determined with the aid of Eq. (6.4-10)

$$\frac{\alpha_{i1}}{M_i} = \frac{2}{C_i^2} \left[1 + 12M_i^2 \;/\; C_i^2 - 1800M_i^4 \;/\; C_i^4 \right] \qquad (7.2\text{-}11)$$

$$\frac{\alpha_{i3}}{M_i} = \frac{24}{C_i^4} \left[1 - 250\, M_i^2 \;/\; C_i^2 \right] \qquad (7.2\text{-}12)$$

$$\frac{\alpha_{i5}}{M_i} = -\frac{3600}{C_i^6} \qquad (7.2\text{-}13)$$

Since C_i^2 is twice the variance of the i-th noise component σ_i^2 the quantity M_i^2/C_i^2 is related to the coordinate signal-to-noise ratio by

$$\frac{M_i^2}{C_i^2} = \frac{1}{2}\frac{M_i^2}{\sigma_i^2} \tag{7.2-14}$$

and therefore

$$\frac{\alpha_{i1}}{M_i} = \frac{1}{\sigma_i^2}\left[1 + 6\, M_i^2/\sigma_i^2 - 450\, M_i^4/\sigma_i^4\right] \tag{7.2-15}$$

$$\frac{\alpha_{i3}}{M_i} = \frac{6}{\sigma_i^4}\left[1 - 125\, M_i^2/\sigma_i^2\right] \tag{7.2-16}$$

$$\frac{\alpha_{i5}}{M_i} = -\frac{450}{\sigma_i^6} \tag{7.2-17}$$

In order to obtain optimum reception the following non-linear transformation which is a result of substituting Eq. (7.2-15), (7.2-16) and (7.2-17) into Eq. (6.4-11) may be used

$$Y_i = \frac{M_i X_i}{\beta_i \sigma_i^2}\left\{\left[1 + 6M_i^2/\sigma_i^2 - 450M_i^4/\sigma_i^4\right] + 6\left[1 - 125M_i^2/\sigma_i^2\right] X_i^2/\sigma_i^2 - 450X_i^4/\sigma_i^4\right\} \tag{7.2-18}$$

The equation for the non-linear transformation can be greatly simplified if the value of M_i^2/σ_i^2 is sufficiently small. If $M_i^2/\sigma_i^2 < 100$ the first term of the sum in the braces in Eq. (7.2-18), viz.

$$1 + 6\, M_i^2/\sigma_i^2 - 450\, M_i^4/\sigma_i^4$$

is, for all practical purposes, unity. The range of values of the coefficient of the second term is

$$-4.5 < 6 \left[1 - 125\, M_i^2 / \sigma_i^2 \right] < 6$$

Thus, it is evident that the second term makes a neglibable contribution and can be disregarded. Therefore, the following non-linear transformation can be used.

$$Y_i = \frac{M_i X_i}{\beta_i \,{}_i^2} \left[1 - 450\, X_i^4 / \sigma_i^4 \right] \qquad (7.2\text{-}19)$$

The non-linear transformation given by Eq. (7.2-19) as characterized by the quantity in the brackets does not depend on the strength of the signal. Similar results are obtained by using larger values for m in that all terms excepting the first and the last becomes negligable for small signal-to-noise ratios.

It should be noted that the shape of the density function of the noise in the above example is quite extreme in that its value becomes infinite at the ends of the range of the random variable. In practice it is reasonable to expect the density functions to be better behaved since any amplitude modulation will cause the curve to be smoothed out. In these cases the performance of the approximate form of the non-linear transformation should be satisfactory over much larger signal-to-noise ratios.

7.3 Methods of determining the coordinate probability densities

In order to determine the parameters of the non-linear receiver it is necessary to know the probability density functions of the coordinate values of the noise. There are two methods by which these densities can be determined (1) the analytical method and (2) the experimental method.

In order to accurately determine the probability density functions $p_i(N_i)$ analytically, it is necessary to have an exact knowledge of the characteristics of the interfering waveforms. It may be anticipated that in most cases the designer will not have the necessary knowledge of the nature of the interference and the results of an analytical approach will be of questionable value. For this reason the experimental method will probably be the most useful especially since it should not be difficult to obtain log $p_i(N_i)$ directly. The values of the α_{ij}'s can then be computed and receiver can be adjusted to give optimum performance. Because of the ease in which the receiver can be changed to fit the statistics of the interference and in view of the fact that these statistics may be expected to change from time to time, it is evident that the non-linear receiver will be highly useful when certain adaptive techniques are incorporated.

7.4 Summary

In Section 6.4 a non-linear technique of obtaining optimum reception in the presence of non-gaussian noise was dereived. The application of this technique to the case where the interference is from narrowband stations has been discussed in this chapter and results are obtained for stations transmitting CW carriers.

CHAPTER VIII

TIME COMPRESSION METHODS

8.1 Introduction

Linear and non-linear receivers have been analyzed in the preceding chapters in some detail and in both cases it was generally necessary to perform an appropriate filtering operation subsequent to the further processing of the signal. In the case of the linear receiver, the purpose of the filtering is to whiten the noise spectrum at the input to the synchronous demodulator, whereas, in the case of the non-linear receiver, there is the additional purpose of separating the various signal-plus-noise coordinates so that each may be subjected to the desired non-linear transformation.

If adaptive operations are incorporated in the receiver then certain signal analysis processes must be performed and in general each part of the signal must be analyzed in real tune.

The filtering processes, and in many cases the signal analysis processes referred to above, must simultaneously be carried out on many different parts of the signal. For example, the filtering process must divide the signal-plus-noise waveform at the input of the receiver into the WT component waveforms, each having a bandwidth 1/T.

In general, there are two methods by which such processing can be accomplished: (1) the parallel method (2) the serial method. The parallel method, which is the most direct, employs m duplicate sets of apparatus to perform the m operations simultaneously. In the case of the filtering process, the operation is accomplished through the use of WT individual similar filters, each having a bandwidth of 1/T and each tuned to its appropriate center frequency.

Although in theory this process is feasible, in practice it is impractical if the number of operations is large. Consider, for example, the equipment complexity of the filtering process if WT was one hundred.

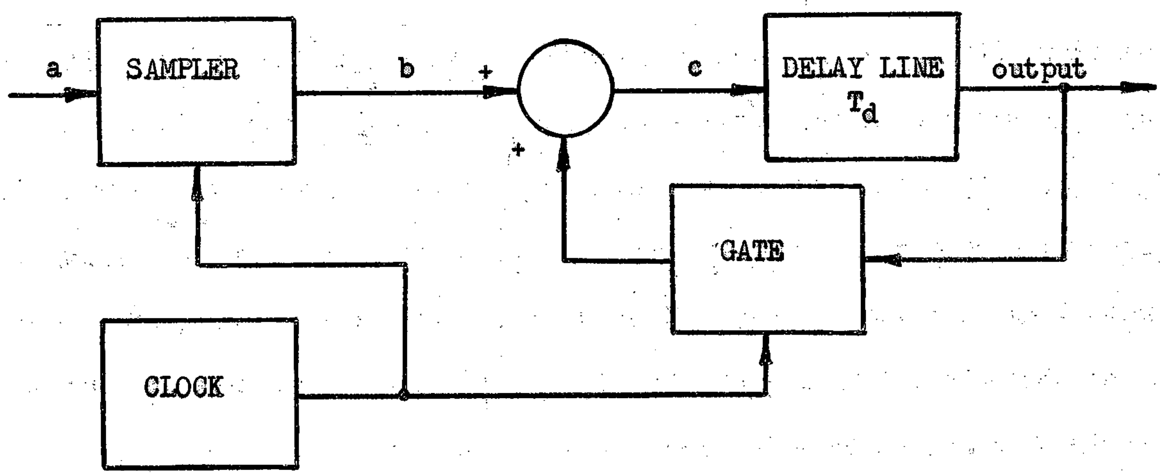
The serial method employs a single set of apparatus which is used over and over m times to perform the m operations in a serial fashion. An example of such a method is the ordinary spectrum analyzer in which the various parts of the signal are heterodyned to the center frequency of a filter. In order to completely process all parts of the signal in real time, however, it is necessary to precede this operation with a time compressor.

8.2 Time Compression

Time compression is accomplished by recording the signal at a certain rate and then playing it back at a much higher rate. In this manner the waveform is speeded up, or compressed in time and can be played over many times in a period of time equal to that which was required for the recording process. A different step in the serial processing of the signal can then be performed each time the signal is played back and in this manner the processing can be performed serially in real time.

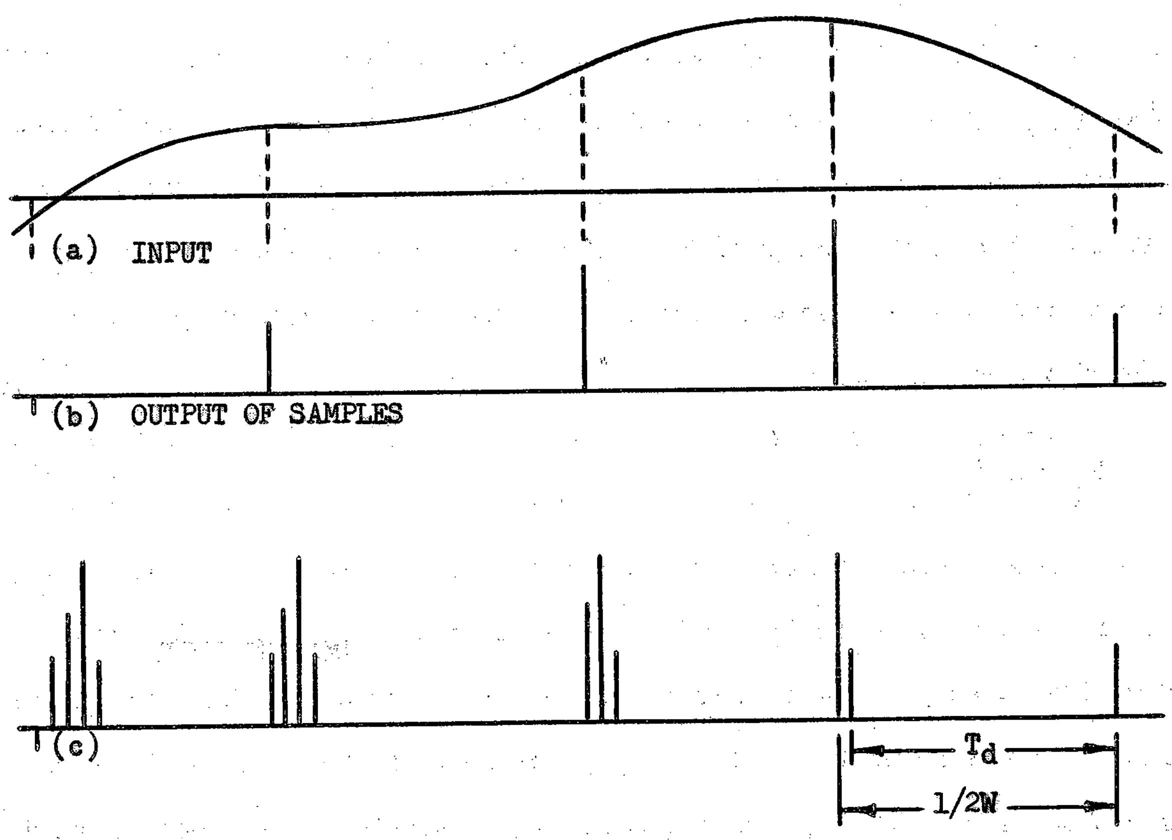
The time compression technique has increasingly been used in the past few years in the field of signal analysis particularly in spectrum analyzers and correlation analyzers[10,11,12]. The method which is usually used to obtain time compression is the deltic method, or delay line time compression method.

Fig. 8.2-1 is the block diagram of a typical deltic system. The in-



A DELTIC SYSTEM

FIGURE 8.2-1



INITIAL WAVEFORMS IN A DELTIC SYSTEM

FIGURE 8.2-2

put is a low pass signal and is bandlimited to W cycles per second. This input is sampled by very narrow clock pulses having a period 1/2W. The time delay T_d of the delay line is made slightly shorter than the sampling period and the delayed samples are fed through the gate and back into the delay line (see Fig. 8.2-2). Since this process reduces the period between the pulses from 1/2W to $1/2W - T_d$ the waveform has been speeded up, or time compressed by a factor

$$k_c = \frac{1/2W}{1/2W - T_d} = \frac{1}{1 - 2WT_d} \qquad (8.2\text{-}1)$$

In order to simplify the timing in the system, $2WT_d$ is chosen so that k_c is an integer.

At the end of the k_d-th sample the delay line will be filled. If the sampling process was discontinued at this time, the pulses would continue to recirculate in the delay line and at the output would be produced a periodic waveform, each period of which would be a time compressed replica of the input. The time duration of the input waveform corresponding to one of the output periods is

$$T_s = k_c T_d \qquad (8.2.2)$$

If, on the other hand, the sampling of the input waveform is continued after the line is filled then it becomes necessary to drop off a recirculating sample each time a new sample is inserted at the delay line input. This is accomplished by closing the gate in the feedback loop when a new sample is taken. The output is a quasi-periodic waveform in which each period differs from the preceeding in that an old pulse has been dropped at the beginning of the period and a new one added on the end.

In both of the cases described above the bandwidth of the output is

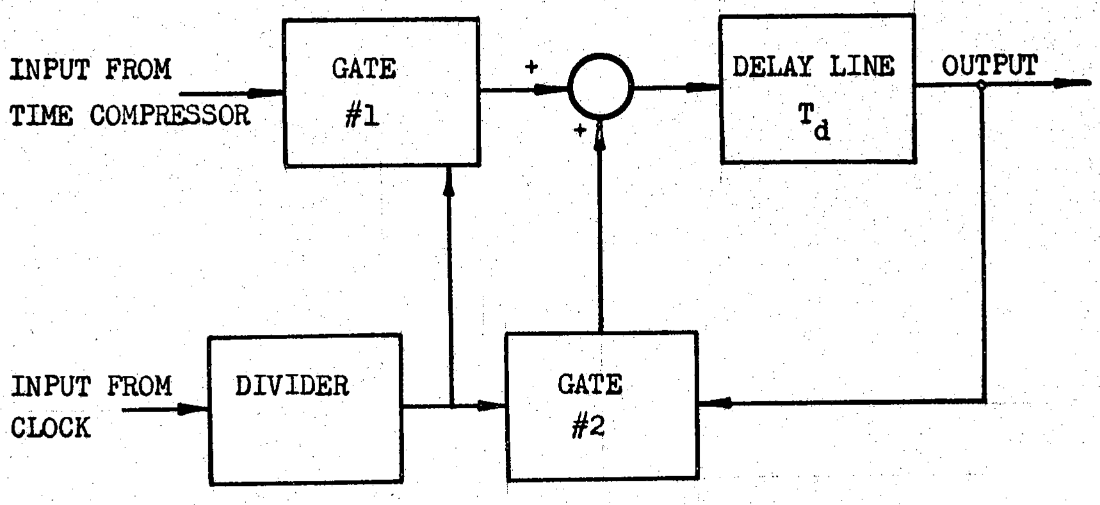
$$W_c = k_c W \tag{8.2-3}$$

In some applications it is required that the output waveform repeat a certain segment of the input waveform k_c times and then repeat the next segment of the input k_c times, etc. This can be accomplished by following the time compressor shown in Fig. 8.2-1 with a buffer storage unit shown in Fig. 8.2-3.

At the end of each T_s seconds the delay line of the time compressor contains a completely new set of pulses and each of these sets corresponds to successive segments of the input. The purpose of the buffer storage unit is to recirculate each successive set of pulses in turn for intervals of T_s seconds. To accomplish this Gate #1 is opened and Gate #2 is closed for a time duration of T_d seconds at the end of each series of k_c clock pulses such that the output signal of the time compressor is fed into the delay line of the buffer storage unit. During the remaining time Gate #1 is closed and Gate #2 is open and the pulse train recirculates around the loop.

Certain practical considerations arise in the design of time compressors. These considerations can be divided into the following areas: (1) bandwidth and switching speeds (2) loop gain and (3) timing.

The bandwidth of the output signal and therefore the minimum bandwidth of the delay line is W_c as given by Eq. (8.2-3). That the required switching speeds are also proportional to k_c is evident from the fact that the maximum allowable width of the sampling pulses is $1/(2WK_c)$. Thus, from a practical standpoint, the feasibility of the time compression technique



BUFFER STORAGE UNIT

FIGURE 8.2-3

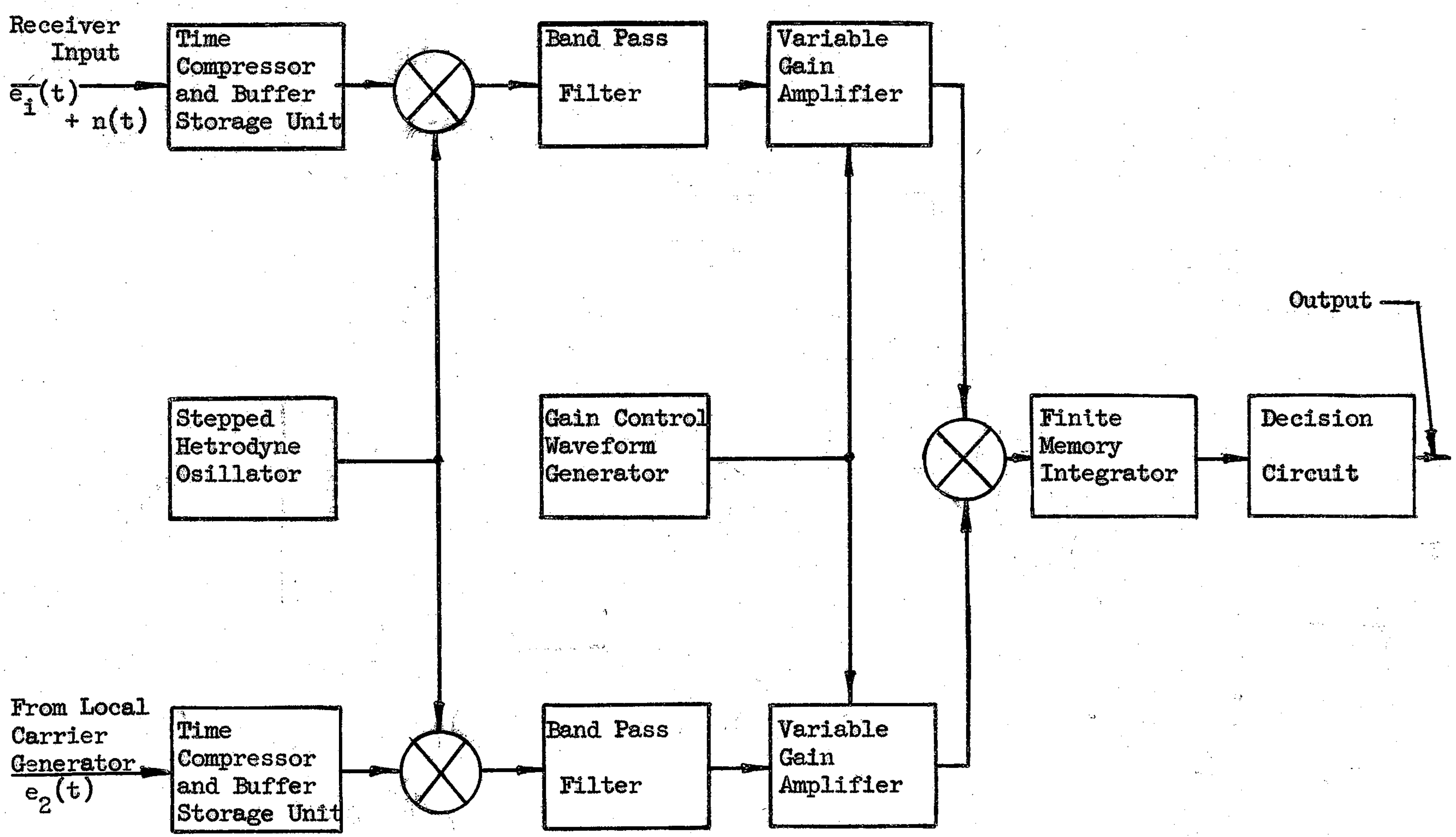
in a given application depends on the signal bandwidth W and the compression factor k_c.

The requirements that the loop gain must be held exactly at unity and that the timing of the sampling be extremely accurate are obvious. These problems are usually solved by infinitely clipping the input signal and using pulse regeneration techniques in the delay line feedback loop. It appears, however, that a more sophisticated approach will solve these problems and at the same time retain the amplitude information of the input.

8.3 Receivers with time compression filters

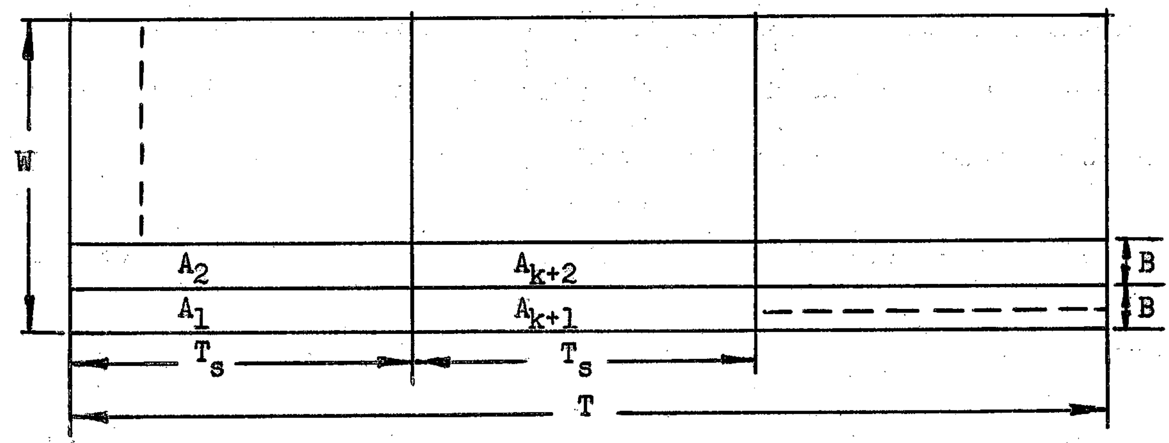
As has been shown in the previous chapters, a rather complex filtering process is required to obtain optimum or even sub-optimum reception. It is apparent from the high dimensionality of the wideband signals being considered that the direct approach of parallel processing with WT filters is prohibitive in terms of the equipment complexity which is required. The use of the time compression technique followed by serial processing will result in a considerable reduction of the equipment required to perform the filtering operation and, thereby, make the desired filtering process feasible.

The block diagram of a wideband linear receiver using time compression filtering is shown in Fig. 8.3-1. The purpose of the filtering is to cause the noise at point "A" to have a white spectrum. The receiver input e (t)+n(t) and the input from the local carrier generator $e_2(t)$ are both individually time compressed. The time-bandwidth diagrams of the input and output of the time compressor are shown in Fig. 8.3-2 where the interfering signals are assumed to have equal bandwidth B and lie adjacent to

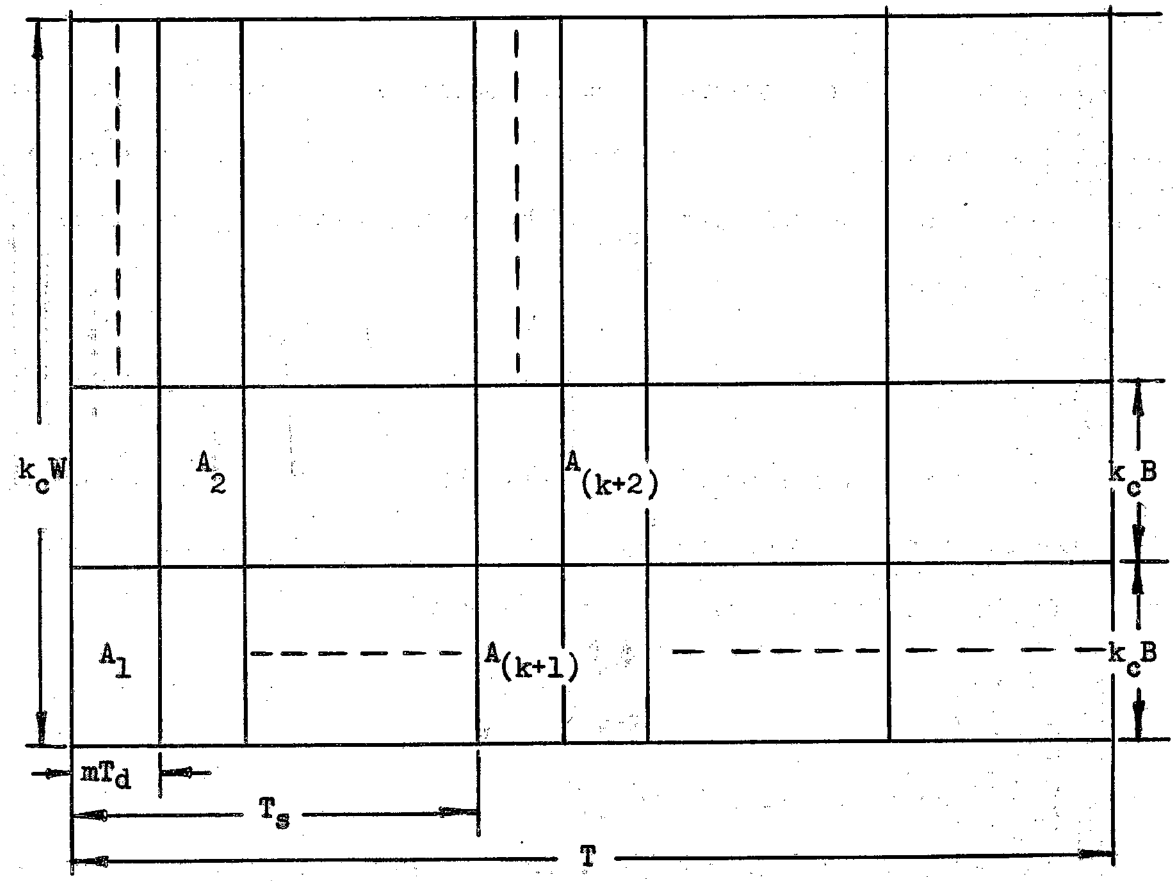


A WIDEBAND LINEAR RECEIVER USING TIME COMPRESSION FILTERING

FIGURE 8.3-1



(a) THE INPUT TO THE TIME COMPRESSION



(b) THE OUTPUT OF THE TIME COMPRESSOR

TIME–BANDWIDTH DIAGRAMS

FIGURE 8.3–2

one another. It is convenient to divide the region occupied by the baud into WT cells of bandwidth B and duration T_s where it will be assumed that the system has been designed so that

$$W/B = k_b = \text{integer} \tag{8.3-1}$$

and

$$T/T_s = k_t = WT/k_b = \text{integer} \tag{8.3-2}$$

These WT cells correspond to WT complex numbers and completely specify the waveform of the baud.

In order to whiten the noise it is necessary to separate these complex numbers and to apply to each an appropriate gain factor. It is evident from the diagram that this process can be done sequentially if the factor k_b is made equal to

$$k_b = \frac{T_s}{mT_d} \tag{8.3-3}$$

If the time compressor of the signal channel is designed so that T_s is the real time length of the input sample which is stored in the time compressor and T_d is the length in compressed time then

$$mk_b = k_c \tag{8.3-4}$$

where k_c is the compression factor.

In order for the filter to separate the waveforms corresponding to the cells of the input it is necessary that mT_d be long enough so that the waveforms of the output cells will contain the complete replicas of the corresponding input cells. Since T_d is the length of this replica then m is the number of times the waveform is repeated and therefore

$$m \geqslant 1. \tag{8.3-5}$$

After the input has been time compressed, the WT components of the waveform are sequentially obtained by a heterodyning and filtering process. The output of the heterodyne oscillator for the duration of a particular baud is

$$e_n(t) = \sum_{j=1}^{k_t} \sum_{i=1}^{k_b} E \left\{ u \left[t - (i - 1)\, mT_d - (j - 1)T_s \right] - u \left[t - imT_d - jT_s \right] \right\} \cos \left\{ 2\pi \left[f_0 + \frac{(2i - 1)}{2} B \right] t + \phi_{(i + j)} \right\} \tag{8.3-6}$$

where f_o is the center frequency of the bandpass filter.

From Fig. 8.3-2, it is evident that the required bandwidth of the filter is $k_c B$.

In order to eliminate filter transients, the factor m can be made somewhat larger than unity and only the last T_d seconds of each output cell used.

As the frequency of the heterodyne oscillator is stepped from one frequency to another to obtain the various frequency components, the gain of the variable gain amplifier is stepped to appropriate gain values so that the noise at point "A" has a constant variance.

The same operations are simultaneously applied to the local carrier generator output $e_2(t)$.

Since the waveforms which appear at the outputs of the variable gain amplifiers correspond to the orthogonal components of the waveforms at the inputs to the time compressors they do not need to be recomposed before being processed further but may be multiplied together directly. After multiplication the usual integration and decision process is performed.

The linear receiver shown in Fig. 8.3-1 can be converted to a non-linear receiver by sequentially inserting the appropriate non-linear voltage dividers at point "A".

8.4 Summary

In the previous chapters various methods of reception have been analyzed. Most of these methods require a rather sophisticated filtering process. To accomplish this process through the use of a bank of filters operating in parallel would result in a prohibitive equipment complexity because of the excessively large number of filters required.

In this chapter a method has been described by which the processing of either the signal or the local carrier can be accomplished sequentially with one filter. Time compression of the signal previous to filtering allows all of the signal to be processed without loss of information. This method offers a relatively simple way of realizing the required filtering process and avoids the inherent disadvantages of the parallel approach.

CHAPTER IX

ADAPTIVE RECEIVERS

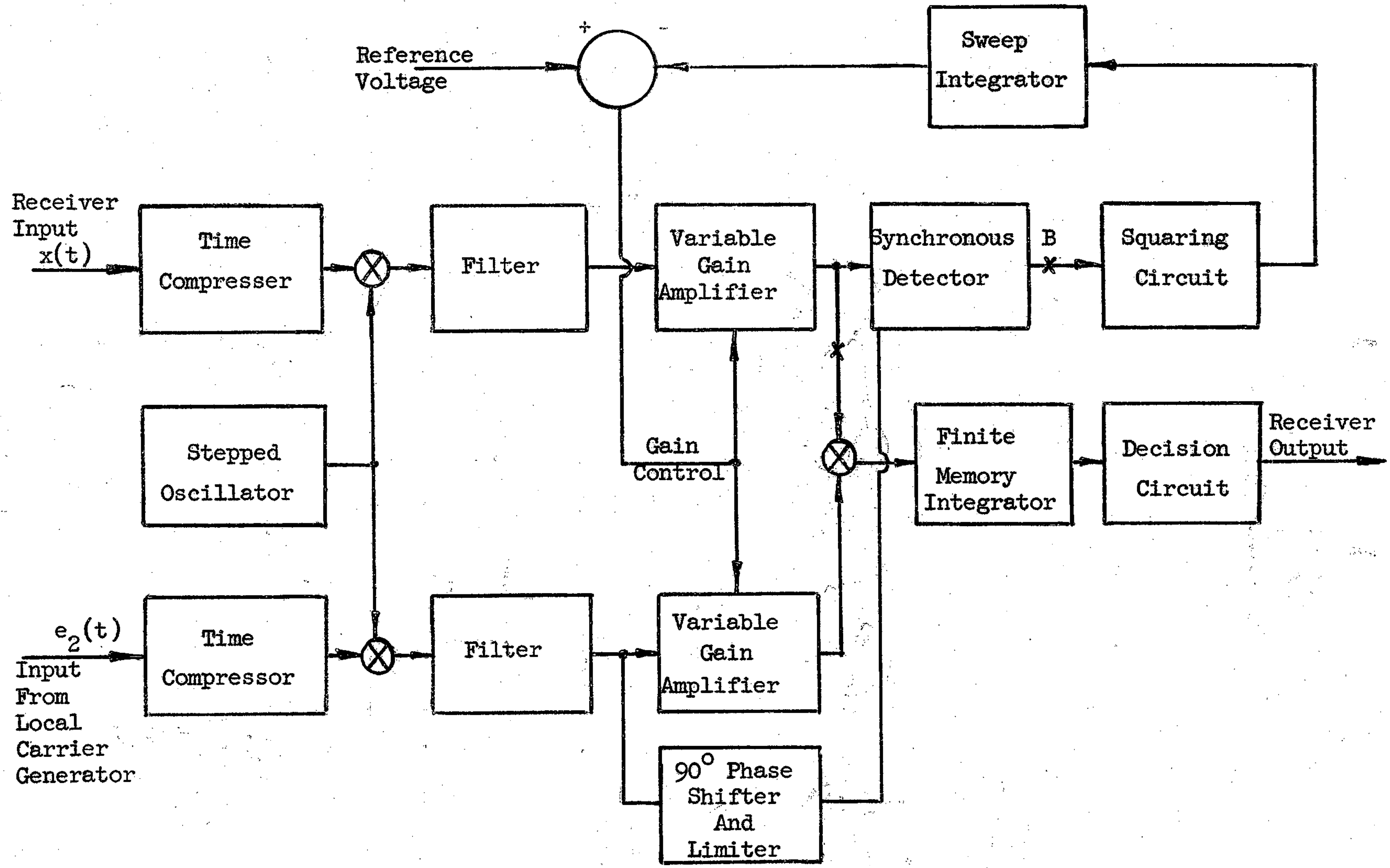
9.1 Introduction

In a great number of applications of wideband systems it will be impossible for the designer to know the shape of the spectrum of the interference or the statistical laws which it follows. In fact, the nature of the interference can be expected to vary from time to time. It is evident that in such instances optimum, or, for that matter, sub-optimum reception can only be obtained from an adaptive receiver which measures the spectrum and the statistics and adapts itself accordingly. In this chapter, several adaptive receivers which incorporate the techniques described in the previous chapters will be outlined.

9.2 Adaptive linear receivers

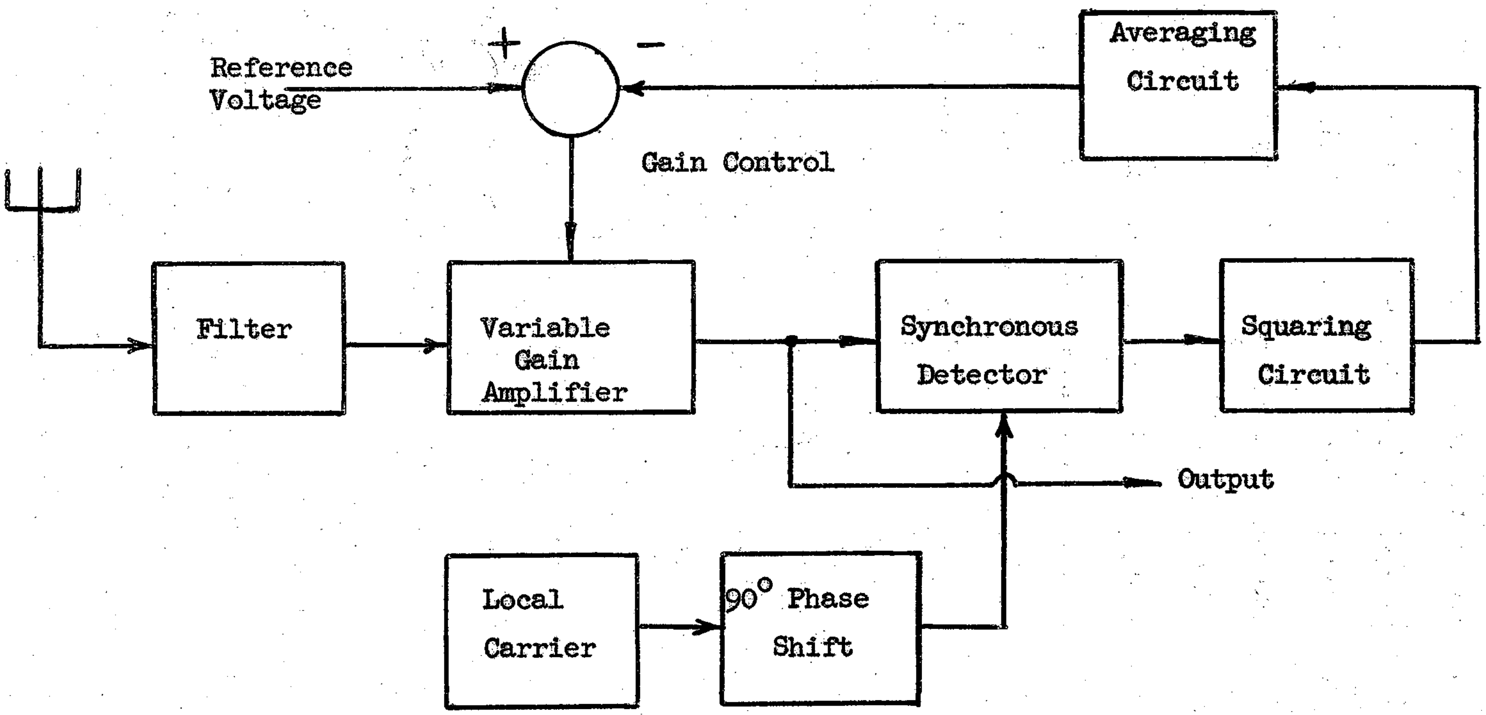
The type of adaptive linear receivers which will be considered in this section analyzes the input spectrum and varies the input filter characteristic accordingly. The block diagram of such a receiver is shown in Fig. 9.2-1. As can be seen, the receiver utilizes time compression filters of the type described in the previous chapter.

In order for the filters to perform their function satisfactorily, the gain control voltage must vary as the filters select the different coordinate values such that the average power of the output noise remains constant across the spectrum. If the process were non-sequential then it could be performed by the method shown in Fig. 9.2-2. The quadrature component of the noise is obtained by demodulating the signal plus noise with a synchronous detector which is driven by the local carrier shifted



AN ADAPTIVE LINEAR RECEIVER

FIGURE 9.2-1



A METHOD OF OBTAINING CONSTANT OUTPUT NOISE POWER.

FIGURE 9.2-2

by 90°. Since the phase difference between the signal and the noise is random with a uniform distribution over the range 0 to 2π, the probability distribution of the quadrature component of the noise will be identical to the probability distribution of the in phase component. Thus, a voltage which is proportional to the average power of the noise can be obtained by squaring and averaging the output of the synchronous detector. If this voltage is subtracted from a reference voltage and the resultant used as a gain control voltage and if the loop gain of the gain control circuit is made large then the output noise power will be held essentially constant. This method can be incorporated into a sequential filtering process by replaing the simple averaging circuit by a sweep integrator as shown in Fig. 9.2-1. The period of the sweep integrator is made equal to the period of the sweep of the stepped oscillator and in this manner the required averaging operation is performed sequentially on the various coordinate values of the noise.

Several variations of the method which has been described are possible. For example, in order to insure that the variable gain amplifier of the local generator channel has the same gain as the one in the signal channel, a single amplifier can be either frequency shared or time shared between the two channels.

9.3 Adaptive non-linear receivers

In this section two types of adaptive non-linear receivers will be outlined. The first adapts to the noise spectrum but has a fixed non-linear characteristic. This type is useful when the narrow band interfering stations are all of the same type and where the statistical law of their amplitudes is known to the designer. The adaptive linear receiver shown in Fig. 9.2-1 can be converted into this type of adaptive non-linear receiver by inserting a non-linear transformation circuit at point "A". A suitable non-linear transformation circuit is one whose input-output relationship satisfies Eq. (6.4-16), viz.,

$$Y_i = \sum_j (j + 1)a_{i,(j+1)} X_i^j \tag{9.3-1}$$

where it has been assumed that it is valid to use the approximate expression for α as given by Eq. (6.4-14). Since the variances have been made equal through the action of the variable gain amplifier of the signal channel and since the various noise coordinate values are assumed to follow the same statistical law, the values of the various $a_{i,(j+1)}$ are independent of the subscript "i".

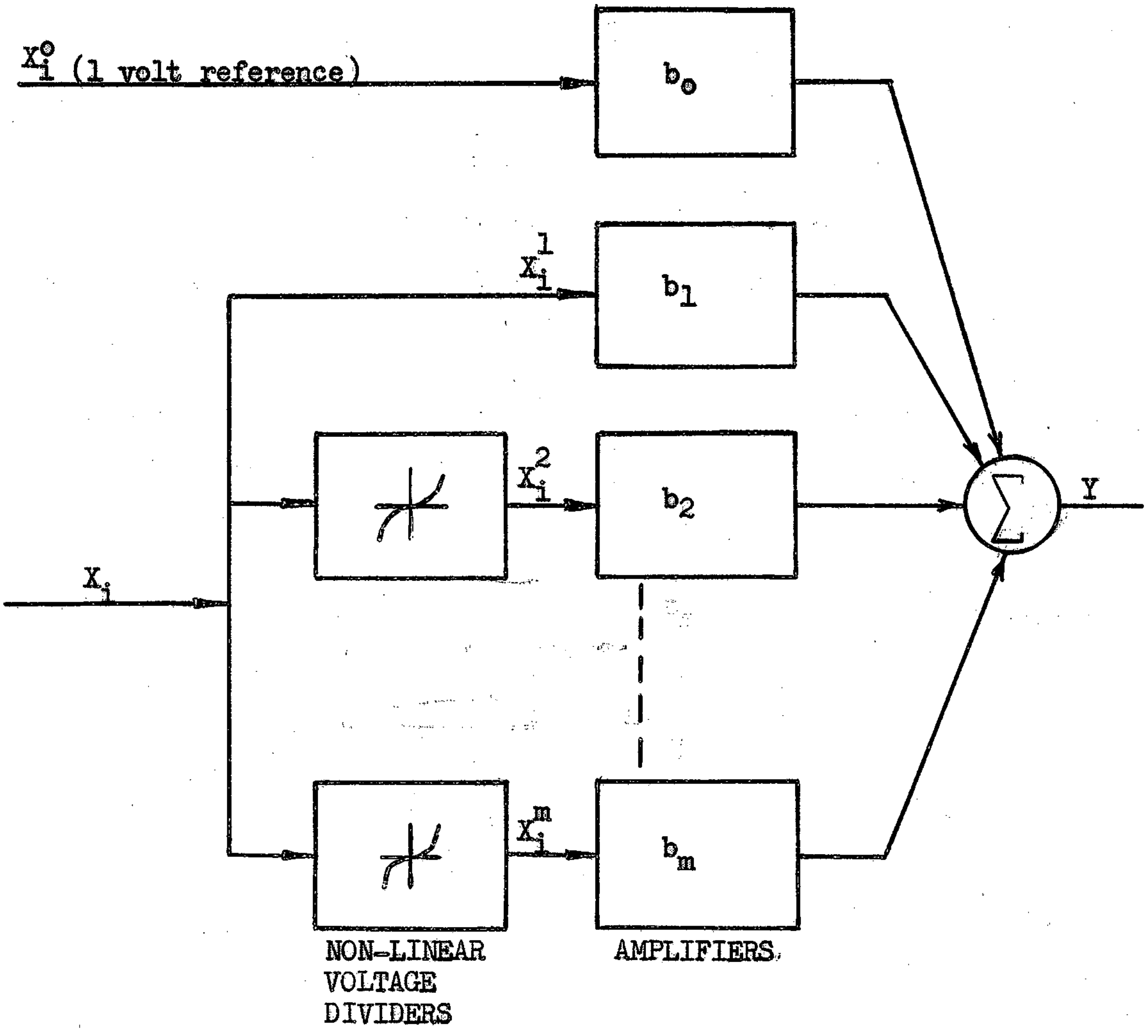
Fig. 9.3-1 shows a convenient way of realizing the non-linear transformation. The various powers of the signal plus noise components (i.e. the X_i's) are obtained through the use of non-linear voltage dividers and their coefficients are determined by the associated amplifier gains. The values of the gains of the amplifiers are given by

$$b_j = (j + 1)\, a_{i,(j+1)} \tag{9.3-2}$$

This form has the advantage that the non-linear characteristic can easily be adjusted to fit any desired noise statistics by an appropriate adjustment of the amplifier gains.

If the transmission is a band pass signal then the probability densities of the coordinate components of the noise will be even functions. In this case the values of b_j will be zero when j is even.

The second type of adaptive non-linear receiver adapts to the noise



A NON-LINEAR TRANSFORMATION CIRCUIT

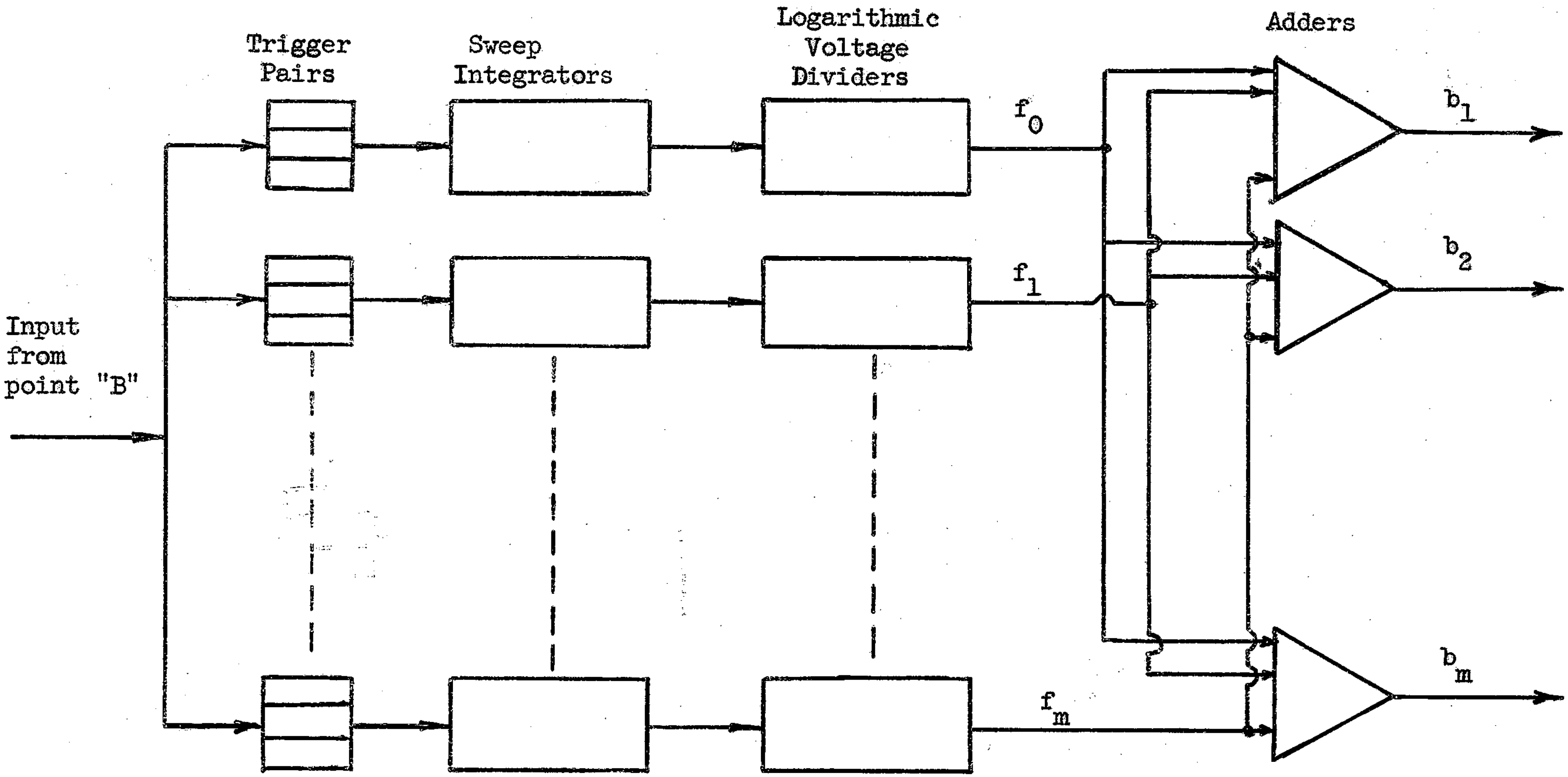
FIGURE 9.3-1

statistics as well as to the spectrum of the noise. This type uses a non-linear transformation circuit similar to the one shown in Fig. 9.3-1. The gains of the amplifiers are controlled electronically by gain control signals which are obtained from an analyzer unit. The function of the analyzer is to obtain points on the probability density curve from the incoming noise waveforms and from these points compute and supply the required amplifier gain control voltages. Such an analyzer is shown in Fig. 9.3-2. Points on the probability density curves are obtained through the use of the triggers and sweep integrators. Each trigger is designed to put out a pulse if its input voltage is within a certain range and each trigger is set so that the range where it triggers a pulse is centered at a noise value where the corresponding probability density is to be determined. For convenience the noise values at which the triggers are set to operate are equally spaced by an amount h. Points on the probability density curves are then obtained by averaging the occurance of the pulses through the use of the sweep integrators. Thus, the probability density function of each of the noise coordinate values as defined by these points appears in sequence at the outputs of the sweep integrators.

Since it is necessary to determine the coefficients of the series

$$f_i(N_i) = \log \left[p_i(N_i) \right] = a_{io} + a_{i1} N_i + a_{i2} N^2 + \cdots \qquad (9.3\text{-}3)$$

it is convenient to convert the data defining the curves $p_i(N_i)$ to data defining the curves $\log p_i(N_i)$. This is done by passing the outputs of the sweep integrators through non-linear voltage dividers having loga-



A NOISE STATISTICS ANALYZER

FIGURE 9.3-2

rithmic input-output characteristics. This data can then be combined with the aid of the Gregory-Newton formula to obtain the required gain values as given by Eq. (9.3-2). The Gregory-Newton formula can be written in the following form.[14]

$$f(x_0 + rh) = f_0 + r\,Df_0 + \frac{r(r-1)}{2!}D^2f_0 + \frac{r(r-1)(r-2)}{3!}D^3f_0 + \cdots \tag{9.3-4}$$

where

$$f_0 = f(x_0); \quad f_1 = f(x_0 + h); \quad f_2 = f(x + 2h); \quad \text{etc.}$$

and

$$Df_0 = f_1 - f_0$$

$$D^2f_0 = f_2 - 2f_1 + f_0$$

$$D^3f_0 = f_3 - 3f_2 + 3f_1 - f_0$$

etc.

Since $x = x_o + rf$, Eq. (9.3-4) is equivalent to equation (9.3-3) and the values of the $a_{i,(j+1)}$ can be determined by a suitable manipulation.

For example, if five points are taken at -2h, -h, 0, h, and 2h then

$$r = \frac{x + 2h}{h}$$

and

$$f(x) = f_0 + \frac{(x + 2h)}{h} Df_0 + \frac{(x + 2h)(x + h)}{h} D^2f_0$$

$$+ \frac{(x + 2h)(x + h)\,x}{h^2} D^3f_0 + \frac{(x + 2h)(x + h)\,x\,(x - h)}{h^3} D^4f$$

(9.3-5)

which becomes

$$f(x) = f_2 + x\,\frac{(f_c - 8f_1 + 8f_3 - f_4)}{12h}$$

$$+ x^2\,\frac{(-f_0 + 16f_1 - 18f_2 + 16f_3 - f_4)}{24h^2}$$

$$+ x^3\,\frac{(-f_0 + 2f_1 - 2f_3 + f_4)}{12h^3}$$

$$+ x^4\,\frac{(f_0 - 4f_1 + 6f_2 - 4f_3 + f_4)}{24h^4} \qquad (9.3\text{-}6)$$

If the transmission was a bandpass signal then $f(x)$ is an even function and

$$f(x) = f_2 + x^2\,\frac{(-f_0 + 16f_1 - 9f_2)}{12h^2}$$

$$+ x^4\,\frac{(f_0 - 4f_1 + 3f_2)}{12h^4} \qquad (9.3\text{-}7)$$

The relationships which determine the various amplifier gains can be gotten by combining the results given by Eq. (9.3-2), Eq. (9.3-3) and Eq. (9.3-7).

$$
\begin{aligned}
b_0 &= 0 \\
b_1 &= \frac{-f_0 + 16f_1 - 9f_2}{6h^2} \\
b_2 &= 0 \\
b_3 &= \frac{f_0 - 4f_1 + 3f_2}{3h^4}
\end{aligned}
\qquad (9.3\text{-}8)
$$

9.4 Summary

The designer of a wideband communications system will often not know the shape of the spectrum or the statistical laws of the interfering noise. In this chapter three types of receivers which adapt to the nature of the interference are outlined. The first is a linear receiver which has an adaptive prewhitening filter at its input. The second is similar to the first with the exception that a preset non-linear transformation of the type discussed in section 6.4 is incorporated. The third adapts to the statistics of the noise as well as to the noise spectrum. All of the receivers which are discussed use the time compression technique which was analyzed in the last chapter making possible the minimization of equipment complexity through the use of serial processing.

CHAPTER X

CONCLUSION

In the Introduction, it was pointed out that in the application of wideband communications systems one of the most important sources of noise will be from interfering narrowband stations. In this report the effect of interference from narrowband stations on the information efficiency of wideband communications systems has been analyzed. A study has also been made of the effects of impulse noise.

Linear systems which can be made to give optimum performance if the noise is gaussian are found to perform relatively poorly when the interference is from narrowband stations. The example worked out in Section 6.4 demonstrates this fact since the probability density function of the noise which is used may be assumed to be typical of interference from narrowband stations. For the example given, the output probability of error which is obtained for the linear receiver is 2.65×10^{-4} as compared with 1.32×10^{-6} which is the probability of error which would be obtained from an optimum receiver. The reason that the linear receiver does not perform better is that the decision surface that it generates is a plane whereas the optimum decision surface is curved and, in the example, the decision regions are not even simply connected.

It is possible to obtain optimum reception in such cases by the use of certain non-linear techniques. In essence, these techniques, make it possible to generate the desired decision surface. For the most part, however, these techniques have serious practical disadvantages. One technique, however, which was developed in the course of this research appears to have appreciable practical significance. One approximate form of this technique which is valid for low signal-to-noise ratios has the advantage that no knowledge of the received signal strength is required in the determination of its parameters. The performance of receivers using this technique approaches optimum as the signal-to-noise ratio is decreased. In comparison, the performance of a linear receiver will diverge away from optimum for decreasing signal-to-noise ratios for this type of noise.

Since the noise from the narrowband interfering stations will be highly colored, a complex filtering process to prewhiten the signal will be required if the best results are to be obtained from a linear receiver. Filtering of a similar nature is also required in the case of non-linear reception. The direct approach of using a bank of parallel filters is not feasible since the amount of equipment involved is prohibitive. It appears, however, that time compression of the signal followed by serial processing will make the desired prewhitening and non-linear techniques practical. At the present it appears to be within the state of the art to realize systems having bandwidths of the order of 1 mc and time compression factors of 100. This might correspond to a binary link having a data rate of 1000 bits/sec in an environment of narrowband interference which has 10 kc bandwidths. Since the state of the art is advancing quite rapidly in the area of high speed switching and data handling it appears that the bandwidths and time compression factors can be increased in the future.

The results of the theoretical investigation which is being reported on are sufficiently promising to warrant an experimental phase. In particular, experimental models of the linear and non-linear time compression receivers described in Chapters VIII and IX should be constructed.

In order to simplify the practical problems, frequency and time scaling should be used in the construction of the first model.

Concurrent with the construction of the experimental receivers, the statistics of the interference from various types of narrowband stations should be measured. Based on this data, further analysis of the performance of the non-linear receivers using the techniques discussed in Section 6.4 and Chapter VII can be made providing important specific design information which will be useful in the experimental phase.

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APPENDIX

ANALYSIS OF THE CONTINUOUS SYSTEM

The block diagram of a continuous communications system is shown in Fig. A-1.

We shall assume that m(t), c(t) and n(t) are band limited but are not necessarily white such that the signal and noise spectral densities in the channel are zero for $f < f_o$ and $f > f_o + W$ and that the input signal spectral density is zero for $f > B$. Furthermore we shall assume that m(t), c(t) and n(t) are independent of each other.

The following representations of the waveforms are convenient:

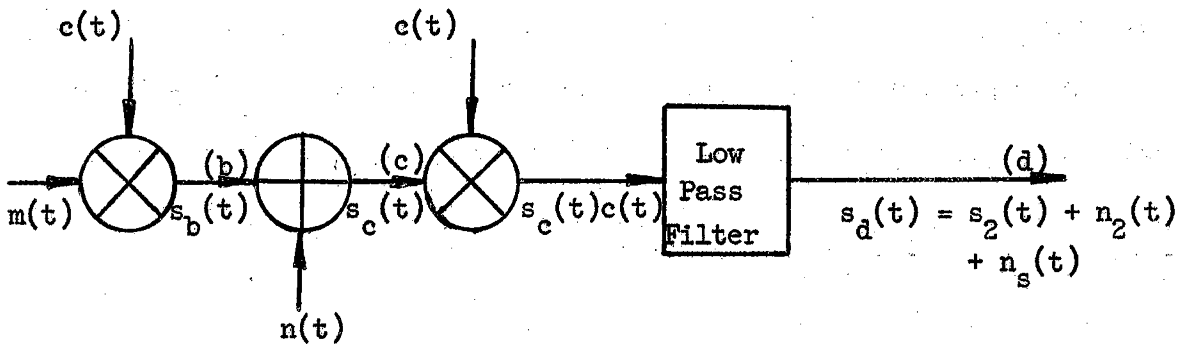
$$c(t) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i \cos(\omega_i t + \phi_i) \qquad \text{(A-1)}$$

$$m(t) = \lim_{p \to \infty} \sum_{k=1}^{p} \beta_k \cos(\omega_k t + \phi_k) \qquad \text{(A-2)}$$

$$n(t) = \lim_{q \to \infty} \sum_{b=1}^{q} \delta_b \cos(\omega_b + \Psi_b) \qquad \text{(A-3)}$$

The average powers of these waveforms are clearly

$$C = \int_0^{f_0 + W} 2C(f)\, df = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\alpha^2}{2} \qquad \text{(A-4)}$$



A CONTINUOUS SYSTEM

FIGURE A-1

$$M = \int_0^B 2M(f)\, df = \lim_{p \to \infty} \sum_{k=1}^{p} \frac{\beta_k^2}{2} \tag{A-5}$$

$$N = \int_{f_0}^{f_0 + W} 2N_0(f)\, df = \lim_{q \to \infty} \sum_{b=1}^{q} \frac{\delta_b^2}{2} \tag{A-6}$$

The waveform of the output of the modulator $s_b(t)$ is obtained by taking the product of the two input waveforms m(t) amd c(t)

$$s(t) = c(t)\, m(t)$$

$$= \lim_{\substack{n \to \infty \\ p \to \infty}} \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\alpha_i \beta_k}{2} \left\{ \cos\left[(\omega_i + \omega_k)\, t + \phi_i + \theta_k\right] \right.$$

$$\left. + \cos\left[(\omega_i - \omega_k)\, t + \phi_i - \theta_k \right] \right\} \tag{A-7}$$

Thus the average power at the output of the modulator is

$$S = \lim_{\substack{n \to \infty \\ p \to \infty}} 2 \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\left[\frac{\alpha_i \beta_k}{2}\right]^2}{2}$$

$$S = CM \tag{A-8}$$

This represents the signal power in the channel.

Noise is added to the signal in the channel and this combination forms the input to the receiver

$$s_c(t) = c(t)\, m(t) + n(t)$$

This input is multiplied by c(t) producing at the output of the multiplier

$$s_c(t)\, c(t) = c(t)\, m(t)\, c(t) + n(t)\, c(t)$$

The multiplier output component due to the presence of the signal is

$$c(t)\, m(t)\, c(t)$$

$$= m(t) \lim_{n \to \infty} \left\{ \sum_{i=1}^{n} \frac{\alpha_i^2}{2} \left[1 + \cos 2\, (\omega_i t + \phi_i) \right] \right.$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{i-1} \alpha_i \alpha_j \left[\cos\ (\omega_i - \omega_j)\, t + \phi_i - \phi_j \right.$$

$$\left. \left. + \cos\ (\omega_i + \omega_j)\, t + \phi_i + \phi_j \right] \right\} \qquad \text{(A-9)}$$

That part of c(t)m(t)c(t) which has a waveform similar to m(t) will be called $s_2(t)$ and will pass with no change through the low pass filter; the remaining part will be called self noise

$$s_2(t) = m(t) \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\alpha_i^2}{2} = m(t)\ C$$

$$S_2 = MC^2 = SC \tag{A-10}$$

The self noise $n_s(t)$ in the output is equal to that part of the self noise in $c(t)m(t)c(t)$ which passes through the low pass filter, i.e. that part where $\omega_i - \omega_j < 2\pi B$.

The self noise waveform at the output of the filter is

$$n_s(t) = m(t) \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=i-m}^{i-1} \alpha_i \alpha_j \cos\left[(\omega_i - \omega_j)\, t + \phi_i - \phi_j\right] \tag{A-11}$$

We desire to evaluate the average power in $n_s(t)$. First it is necessary to determine the voltage at each frequency $\omega_i - \omega_j$. Thus the component of $n_s(t)$ in the interval at ω_k with width Δf is

$$n_s(t)\Big|_{\omega_k} = m(t) \lim_{n\to\infty} \sum_{b=3}^{n+1} \alpha_i \alpha_j \cos\left(\omega_k t + \phi_i - \phi_j\right) \tag{A-12}$$

where

$$i = \frac{b + k}{2}$$

$$j = \frac{b - k}{2}$$

The self noise power is the sum of the powers of each of these components.

Clearly, if the self noise is to be zero the voltage components at each frequency must sum to zero.

$$n_s(t)\Big|_{\omega_k} = 0, \quad k = 1, 2, - - m$$

This is simply another way of prescribing that the power of the carrier is constant.

The worst case will occur when $\phi_i - \phi_j$ equals a constant over the range of b at each value of k. Here all terms in Eq. (A-12) are in phase and the power at ω_k is

$$\Delta N_{sk} = \frac{M}{2}\left[\lim_{n\to\infty}\sum_{b=3}^{n+1} \alpha_i\alpha_j\right]^2$$

$$= \frac{M}{2}\left[2\int_{f_0}^{f_0 + W} \sqrt{2C(f)}\ \sqrt{2C(f - k\Delta f)}\ df\right]^2$$

and the total power is

$$N_s = \lim_{m\to\infty}\sum_{k=1}^{m} \Delta N_{sk}$$

$$= 4M\int_0^B \left[\int_{f_0}^{f_0 + W} \sqrt{C(f)\ C(f - \eta)}\ df\right]^2 d\eta$$

For the case of the flat spectrum where C(f) - C/2W in the range $f_o < f < f_o + W$ we have

$$N_s = 4M \int_0^B \left[\int_{f_0}^{f_0 + W} \frac{c}{W} \, df \right]^2 d\eta$$

$$= MC^2 B = S_2 B$$

or

$$\frac{S_2}{N_s} = \frac{1}{B} \qquad \text{(A-13)}$$

Here, the phase relationship of the noise carrier voltage spectrum is such that the carrier is a single sin x/x shaped pulse. Such pathological cases, however, should be easy to avoid in practice.

Of much more interest is the case where the noise carrier is a random sample of gaussian noise. Here, we do not have a definite relationship for the phase and amplitude distribution; we only know the probability densities of the gaussian noise. Since the phases have a probability density which is uniform in the range 0 to 2π we can conclude that (ϕ_i - ϕ_j) has the same distribution. In other words, the components of $n_s(t)$ which have the same frequency have phases with a uniform random distribution. As is well known the expected total average power delivered by these components is simply the sum of the average powers of the individual components. Thus

$$N_s = M \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=i-m}^{i-1} \frac{\alpha_i^2 \alpha_j^2}{2}$$

$$= 2M \int_{f_0}^{f_0 + W} \int_{f - B}^{f} \left[2C(f) \right] \left[2C(\eta) \right] d\eta \, df$$

Furthermore, if $C(f) = c/W;\ f_0 \leqslant f \leqslant f_0 + W$

$$N_s = 2MC^2 \frac{B}{W}$$

$$\frac{S_2}{N_s} = \frac{1}{2} \frac{W}{B} \tag{A-14}$$

We see that for $W \gg B$ the self noise will be negligable.

Another source of noise is the noise which is added in the channel. The spectral density of the noise at the output of the filter can be obtained by convolution

$$N_2(f) = 2 \int_{f_0}^{f_0 + W} N_0(x)\ C(f - x)\ dx \tag{A-15}$$

or conversely

$$N_2(f) = 2 \int_{f_0}^{f_0 + W} N_0(f - x)\ C(x)\ dx \tag{A-16}$$

If either the carrier or the channel noise has a white spectrum then it is evident from the above equations that the noise spectrum of the filter output will be white and that the output noise power will be

$$N_2 = \frac{CNB}{W} . \qquad (A-17)$$