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The Variable Gradient Method of Generating Liapunov Functions with Application to Automatic Control Systems

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PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

***The Variable Gradient Method
of Generating Liapunov Functions
with Application to Automatic
Control Systems***

J. E. Gibson, Principal Investigator

D. G. Schultz

Control and Information Systems Laboratory

April, 1962

Lafayette, Indiana



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THE VARIABLE GRADIENT METHOD OF GENERATING LIAPUNOV
FUNCTIONS WITH APPLICATIONS TO AUTOMATIC
CONTROL SYSTEMS

SUPPORTED BY
NATIONAL SCIENCE FOUNDATION
WASHINGTON, D. C.

by

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Lafayette, Indiana

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ABSTRACT

Schultz, Donald Gene, Ph.D., Purdue University, April, 1962. The Variable Gradient Method of Generating Liapunov Functions, with Applications to Automatic Control Systems.

Major Professor: John E. Gibson.

The contribution of this thesis is the introduction and development of the variable gradient method of generating Liapunov functions. A Liapunov function, V , is considered to be generated if the form of V is not known before the generating procedure is applied.

Two previous attempts at the generation of Liapunov functions to prove global asymptotic stability for nonlinear autonomous systems have been made. These attempts are summarized and evaluated in some detail, as they form the basis for the variable gradient approach proposed in this thesis.

It is assumed that the system whose stability is being investigated is represented by n first order, ordinary, nonlinear differential equations in state variable form

$$\dot{\underline{x}} = \underline{X}(\underline{x}) \qquad \underline{X}(\underline{0}) = 0 \qquad (1)$$

The particular state variables used throughout the thesis are the phase variables. This was done for convenience.

The problem of finding a scalar $V(\underline{x})$ to satisfy a particular Liapunov theorem is recast into the problem of

finding a vector function, ∇V , having suitable properties. As the name implies, ∇V is assumed to be a vector of n elements, ∇V_i , each of which has n arbitrary coefficients. These coefficients, designated as a_{ij} , may be constants or functions of the state variables. In its most general form, the variable gradient is assumed to be

$$\nabla V = \begin{pmatrix} a_{11}(\underline{x})x_1 + a_{12}(\underline{x})x_2 + \dots + a_{1n}(\underline{x})x_n \\ a_{21}(\underline{x})x_1 + a_{22}(\underline{x})x_2 + \dots \\ \vdots \\ a_{n1}(\underline{x})x_1 + a_{n2}(\underline{x})x_2 + \dots + a_{nn}(\underline{x})x_n \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} \quad (2)$$

V may be determined as a line integral of ∇V if the following $(n-1)n/2$ partial differential equations are satisfied.

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \quad (3)$$

Here ∇V_i are the elements of the vector ∇V . The equations (3) are referred to as generalized curl equations.

dV/dt may also be determined from ∇V .

$$\frac{dV}{dt} = \nabla V' \dot{\underline{x}} \quad (4)$$

An outline of the procedure by which a suitable V and dV/dt may be determined for a particular problem, starting from the variable gradient of (2) is as follows.

1. Assume a gradient of the form (2).

2. From the variable gradient, determine dV/dt by equation (4).
3. In conjunction with and subject to the requirements of the generalized curl equations (3), constrain dV/dt to be at least negative semi-definite.
4. From the now known ∇V , determine V .
5. Invoke the necessary theorem to establish stability.

Numerous examples are worked to illustrate the procedure outlined above. V functions are generated that involve higher order terms in \underline{x} , integrals, and terms involving three state variables as factors. The problem of determining Hurwitz like criteria for nonlinear systems is considered in some detail.

The last chapter attempts to extend the variable gradient approach to nonautonomous systems. The results of this chapter, though somewhat marginal, are of interest from the point of view of further research.

CHAPTER I

Introduction and Organization of the Thesis

The second method of Liapunov is a general method for determining the stability of autonomous or nonautonomous, linear or nonlinear, ordinary differential equations. The method was advanced in Russia by the mathematician A. M. Liapunov near the end of the nineteenth century and translated into French in 1907. Little use was made of the method until the early 1940's, when the Russians began to realize the value of the Liapunov approach in connection with the analysis of nonlinear, automatic control systems.

The French translation of the original Liapunov manuscript was reproduced by the Princeton University Press in 1947, but publications in English did not begin to appear until around 1955. Since that time interest in Liapunov's second method has steadily increased, and with it the number of English publications, either in the original, or translations from the Russian, French, or German.

The earliest works in English were almost completely mathematical in nature, and without exception included extensive references to original Russian papers. In 1959, two Ph.D. theses in engineering appeared on the subject, and each of these also drew heavily from the Russian. In contrast, this thesis may be considered as something of a "second generation" effort, as the extensive translation of early material from the foreign languages, particularly Rus-

sian, had already been done before this work was started. As a consequence, the majority of references are in English, and, more importantly, these references are readily available.

The historical background of the second method is largely mathematical, and due in part, perhaps, to the communication barrier between mathematicians and engineers, the theoretical capabilities of Liapunov's second method far exceed the present practical applications. In fact, the lack of a systematic means of generating the so-called "V function" of Liapunov to satisfy the existing powerful theorems has been deplored in almost every English publication on the subject.

The purpose of this report is to develop a logical and systematic means of generating Liapunov functions. The means by which this is accomplished is called the variable gradient method of generating Liapunov functions. The method is based upon the introduction of a completely arbitrary vector, the variable gradient, and a number of auxiliary equations, called the generalized curl equations. Procedures are described by which the unknowns in the gradient are determined, and from the gradient, both V and dV/dt can be determined directly. This approach reduces the emphasis on the ingenuity and experience of the investigator that has so long been linked with the engineering application of the Liapunov theorems.

Following this introductory material, Chapter II is a review of the basic Liapunov theorems, with the definitions of necessary terms. The emphasis is on clarity rather than on completeness. Only those theorems that are to be used in the chapters immediately following are presented. Later, as more completeness is needed, additions are made as required.

The variable gradient method is an outgrowth of the work of Ingwerson [1], [2] and Szego [3], [4] described in the third chapter. The author was fortunate enough to see the early work of both of these individuals before it appeared in the journals, and because the two papers were read at essentially the same time, the foundation for the variable gradient approach practically suggested itself. The work of Ingwerson and Szego is dealt with in some detail as a foundation for the variable gradient method.

The variable gradient method is proposed in Chapter IV as applicable to the autonomous system. Here, as in the rest of the thesis, the analysis is restricted to systems containing only single-valued nonlinearities. No special attention is given to linear systems, as they are considered as special cases of the nonlinear type. To demonstrate the capability of the variable gradient method, examples are worked in Chapter V to illustrate the different types of V functions that can be generated. Addi-

tional examples are included that deal with practical servo problems. In each case the starting point is not a set of n first order differential equations, but the block diagram from which these equations are derived. Thus the reader is dealing with problems with which he may be expected to be familiar, so that only the framework within which the problem is considered is different.

In Chapter VI, the variable gradient method is extended to include time-variable-parameter systems and systems with an input. Here the order of magnitude of the problem is increased, and the results might be considered to be somewhat marginal.

Chapter VII is a short summary of the report, with recommendations for further study. The Appendix outlines several methods of determining the closedness of higher order Liapunov functions. The Appendix is considered a vital portion of this report, as the motivation for slightly restricting the form of the variable gradient hinges on the means that are used to show that the higher forms generated actually do represent closed surfaces in n dimensional space.

The contribution of this report is the introduction and development of the Variable Gradient Method of generating Liapunov functions, and the application of this method to different types of problems in the field of automatic control.

CHAPTER II

The Second Method of Liapunov for Autonomous Systems

2.1 Introduction and Organization of the Chapter

The second method of Liapunov is a means of determining the stability of a system of n simultaneous, first-order, ordinary, differential equations. In this chapter the automatic-control system is interpreted in terms of equations of this type. Before the introduction of the actual Liapunov theorems, the concepts of "definiteness" and "closedness" are considered, as is the precise meaning of the term "stability".

The more basic Liapunov theorems and their extensions are presented, however no proofs of the theorems are included, as the theorems have been adequately proved in the literature. Rather, an attempt is made to present the material in such a way that the reader with a knowledge of phase-plane analysis will understand the physical implications involved in the statement of the theorems.

2.2 Notation

The following notation is used throughout. Vectors are designated by underlined quantities, as \underline{x} or \underline{X} . The only exception to this is the gradient of a scalar function, a vector, which is denoted by ∇V . A function of an under-

lined quantity is a function of the elements of the vector. Thus $\underline{X}(\underline{x})$ is a vector function identical to $\underline{X}(x_1, x_2, \dots, x_n)$ and $V(\underline{x})$ is a scalar function equal to $V(x_1, x_2, \dots, x_n)$. The transpose of a vector \underline{x} is designated as \underline{x}' . The capital letters A, B and C are reserved exclusively for square matrices in the theoretical Chapters II, III and IV only. Capital letters other than these refer to scalar quantities.

2.3 System Representation

The application of the second method of Liapunov to the determination of the stability of an autonomous, physical system presupposes that the n-th order dynamic system under consideration is specified by n, simultaneous, first-order, ordinary differential equations of the form

$$\begin{aligned} \dot{x}_1 &= b_{11} x_1 + b_{12} x_2 + \dots + b_{1n} x_n \\ \dot{x}_2 &= b_{21} x_1 + b_{22} x_2 + \dots + b_{2n} x_n \\ &\vdots \\ \dot{x}_n &= b_{n1} x_1 + b_{n2} x_2 + \dots + b_{nn} x_n \end{aligned} \tag{2.1}$$

In the linear system, the b_{ij} 's would be constant, but more generally, the b_{ij} 's may be functions of x_1, x_2, \dots, x_n . For a given nonlinear system, the b_{ij} 's are not

necessarily unique, as a term such as $x_1 x_2$ would serve to indicate.

For convenience, though not necessity, vector notation is used to represent the system of equations (2.1), so that (2.1) may be rewritten as

$$\dot{\underline{x}} = B(\underline{x}) \underline{x} \quad (2.2)$$

or

$$\dot{\underline{x}} = \underline{X}(\underline{x}) \quad (2.3)$$

A further assumption is made that the variables \underline{x} are chosen such that

$$\underline{X}(0) = 0 \quad (2.4)$$

This in no way restricts generality, as a linear change in coordinates can be made to shift the equilibrium point to the origin.

In equations (2.2) and (2.3) the variables x_i are functions of time, and a knowledge of the vector \underline{x} completely describes the state of the system for all time. Hence, the variables \underline{x} are referred to as the state variables of the system. It should be noted in passing that any given system may be represented by an infinite number of equations of the form (2.2) or (2.3), as the state variables are not necessarily the physical variables of the system, but may be any linear combination of these

physical variables Gibson, et al., 5 .

Often, physical systems are not described by equations such as (2.2) or (2.3). A basic assumption of this report is that the system under study is representable in block-diagram form, and that either the block diagram or an nth-order differential equation representing the system is known. If systems with time lag are ignored, the requirement that the block diagram be known is identical to the requirement that an nth-order differential equation be known, as a block diagram is simply a pictorial representation of a differential equation. Systems with time delay will not be considered, as they result in differential difference equations.

In the example problems to follow in later chapters, the problem is always stated first in terms of a block diagram, and this is reduced to the form

$$\frac{d^n x}{dt^n} + a_n \frac{d^{n-1} x}{dt^{n-1}} + \dots a_2 \frac{dx}{dt} + a_1 x = 0 \quad (2.5)$$

This may be written more conveniently as

$$x^{(n)} + a_n x^{(n-1)} + \dots a_2 \dot{x} + a_1 x = 0 \quad (2.6)$$

Equation (2.5) is easily reduced to n simultaneous first order equations by assuming as the state variables, the system output or error and its n - 1 derivatives. Thus, with x_1 equal to x, and this choice of state variable,

(2.6) becomes

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= -a_n x_n - a_{n-1} x_{n-1} - \dots - a_2 x_2 - a_1 x_1\end{aligned}\tag{2.7}$$

This particular choice of state variables is referred to as the phase variables, a name that stems from the coordinates of the usual phase plane on which the behavior of a second-order system of the form of (2.7) is usually depicted. This choice of the phase variables is a natural one for the engineer, as these variables have a ready physical interpretation. In a positional servo, for example; x_1 could be chosen as output position; x_2 , velocity; x_3 , acceleration; etc. The behavior of the system can then be depicted as taking place in an n-dimensional phase space, analogous to the two-dimensional phase plane, with time not explicitly indicated.

Sometimes equations in "normal" or "canonic" form [Cunningham, 6], or in the canonic form of Lur'e [7] are convenient. However, the variable gradient method of generating Liapunov functions to be developed is not dependent upon the representation of the system, as long

as n , first-order, differential equations are given. Phase variables will be used because of their simplicity, although later an example will be worked in an alternate coordinate system (Example 5.6).

2.4 The Concepts of Definiteness and Closedness

The concept of definiteness is utilized in the statement of the theorems of Liapunov, and the following definitions apply. The following definitions follow Malkin [8].

Definition 2.1. [Malkin, 8] Positive (Negative)

Definite

A scalar function $V(\underline{x})$ is positive (negative) definite if for

$$\|\underline{x}\| \leq h \text{ where } \|\underline{x}\|^2 = x_1^2 + x_2^2 + \dots + x_n^2$$

$V(\underline{x}) > 0$ ($V(\underline{x}) < 0$) for all $\underline{x} \neq 0$ and $V(\underline{0}) = 0$.

Definition 2.2. [Malkin, 8] Positive (Negative)

Semidefinite

A scalar function $V(\underline{x})$ is positive (negative) semidefinite if for

$$\|\underline{x}\| \leq h$$

$V(\underline{x}) \geq 0$ ($V(\underline{x}) \leq 0$) for all \underline{x} and $V(\underline{0}) = 0$.

In the above definitions, h may be arbitrarily small,

in which case V would be definite in an arbitrarily small region about the origin. If h is infinite, V is definite in the whole space.

Definition 2.3 [Malkin, 8] Indefinite

A scalar function $V(\underline{x})$ is indefinite if it is neither of the above, and therefore, no matter how small the h , in the region

$$\|\underline{x}\| \leq h$$

$V(\underline{x})$ may assume both positive and negative values.

A few simple examples will clarify the definitions.

The function

$$V = x_1^2 + x_2^2$$

is positive definite if the system under consideration is second order, but it is only semidefinite if the system is third order, since, for $x_1 = x_2 = 0$, V is 0 for arbitrary x_3 . Similarly, for a third order system, the function

$$V = x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$$

is only semidefinite, because for $x_3 = 0$ and $x_1 = -x_2$, $V = 0$. A function such as $V = x_1$ or $V = x_1 - x_2$ is obviously indefinite, no matter what the order of the system.

When V is a quadratic form, expressible as

$$V = \underline{x}' C \underline{x} \quad (2.8)$$

where C is a square matrix with constant coefficients, the usual means of determining the definiteness of the form is through the application of Sylvester's Theorem [LaSalle 9].

Sylvester's Theorem

In order that the quadratic form (2.8) be positive definite, it is necessary and sufficient that the principal minors of its determinate, that is, the magnitudes

$$|c_{11}| > 0, \quad \begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{12} & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ c_{1n} & c_{2n} & \dots & c_{nn} \end{vmatrix} > 0$$

be positive.

Closely allied to the concept of definiteness is the concept of a simple closed curve or surface. A surface is said to be simple if it does not intersect itself and closed if it intersects all paths that lead from the origin to infinity [Letov, 10]. That is, a simple closed surface is topologically equivalent to the surface of an n dimensional sphere. If V is a positive-definite func-

tion, then the equations $V = K$, a constant, represent a set of nested, closed surfaces about the origin in a sufficiently small region. In order to insure that the region extends to infinity, it is necessary to insure that the curve $V = K$ is closed for sufficiently large K . Letov [10] states that the closure of the curves $V = K$ is assured if, in addition to positive definiteness, the Liapunov function approaches infinity as the norm of \underline{x} goes to infinity, that is, if

$$\lim_{\|\underline{x}\| \rightarrow \infty} V(\underline{x}) \rightarrow \infty \quad (2.9)$$

As an example of a curve that is positive definite and yet closed only for values of K less than 1, Letov [10] cites the following example from Barbashin [11].

$$V = x_1^2 + \frac{x_2^2}{1 + x_2^2}$$

A second example of Letov includes an integral in the Liapunov function. If V is given as

$$V = \int_0^{x_1} f(\gamma_1) d\gamma_1 + x_2^2$$

and

$$\lim_{x_1 \rightarrow \infty} \int_0^{x_1} f(\gamma_1) d\gamma_1 = a$$

then the curve $V = K$ is closed only for values of K less than a .

2.5 Definitions of Stability

The concept of stability of a linear system with constant coefficients is basic to control engineering. Such a system is defined to be stable [Bower and Schultheiss, 12] if and only if its output in response to every bounded input remains bounded. A necessary and sufficient condition for the stability of a linear system is that the absolute value of its weighting function, $\omega(t)$, be integrable over the infinite range, i.e.,

$$\int_0^{\infty} |\omega(t)| dt < \infty \quad (2.10)$$

The weighting function of a linear system is simply the inverse Laplace transform of the transfer function of the system.

Not only is the concept of stability clearly defined, but the range of stability is not in question. If a linear system is stable, then it is stable for any input, re-

ardless of size.

This is not at all the case in nonlinear systems, as stability is a local concept and a possible function of the input. Kalman [13] defines eight types of stability, Antosiewicz [14] nine types, and Ingwerson [1] twenty different types. Many of these definitions, however, apply to nonautonomous systems, and many are not of interest in engineering applications. Hence, here only stability in the sense of Liapunov and asymptotic stability will be defined. Definitions applicable to nonautonomous systems are given in Chapter VI.

The definitions here follow LaSalle [15], and assume that the system is expressed as equation (2.3).

Assume that the equilibrium state being investigated is located at the origin, and that $\underline{X}(0) = 0$. Let $\|\underline{x}\|$, the norm of \underline{x} , be the Euclidean length of the vector \underline{x} , where $\|\underline{x}\|^2 = x_1^2 + x_2^2 + \dots + x_n^2$. Let $S(R)$ be a spherical region of radius $R > 0$ around the origin, where $S(R)$ consists of points \underline{x} satisfying $\|\underline{x}\| < R$.

Definition 2.4. Stability in the Sense of Liapunov

The origin is said to be stable in the sense of Liapunov, or, simply stable, if, corresponding to each $S(R)$ there is an $S(r)$ such that solutions starting in $S(r)$ do not leave $S(R)$ as $t \rightarrow \infty$.

Definition 2.5. Asymptotic Stability

If the origin is stable and, in addition, every solution starting in $S(r)$ not only stays within $S(R)$ but approaches the origin as $t \rightarrow \infty$, then the system is called asymptotically stable.

The definitions themselves emphasize the local character of these types of stability for nonlinear systems, as the region $S(r)$, the region of initial conditions, may be arbitrarily small. If the region $S(r)$ includes the entire space, the stability defined by 2.4 and 2.5 above is global.

Note that in the above, the region $S(R)$ is a function of the initial conditions, or more precisely, a function of the region of allowable initial conditions. As a consequence of this fact, a linear system with poles on the $j\omega$ axis is stable in the sense of Liapunov. Hence, as far as automatic controls are concerned, Liapunov stability has only historical importance. The type of stability of interest is asymptotic stability, and more specifically, global asymptotic stability.

The concept of asymptotic stability does have one disadvantage, however. The region $S(R)$ is a function of $S(r)$, but the relationship of the size of $S(R)$ with respect to $S(r)$ is not specified. Hence it is quite conceivable that a system that is asymptotically stable, or

even globally asymptotically stable, might still perform quite badly, as, for example, a linear, second-order system with a damping ratio of .05. More will be said about the region $S(R)$ with respect to various inputs in Chapter VI.

2.6 Liapunov Stability Theorems

A large number of theorems exist which are related to the second method of Liapunov; for example, Donaldson [16] lists 32. Only three theorems of immediate interest are stated below.

The original theorem due to Liapunov, Theorem 2.1, is applicable only to an arbitrarily small region about the origin.

Theorem 2.1 [Malkin, 8]

If it is possible to find a $V(\underline{x})$, definite with respect to sign, whose total derivative with respect to time is also a function of definite sign, opposite in sense to that of V , then equation (2.3) under assumption (2.4) is asymptotically stable.

Modern convention assumes that $V(\underline{x})$ is positive definite. Thus, in a geometric sense, the equations $V = K$, where K is a positive constant, represent a one parameter family of simple closed surfaces nested about the origin

in the space of \underline{x} . However, $V(\underline{x})$ does not necessarily represent a closed surface in the whole space, and only local asymptotic stability may be concluded.

With V assumed to be positive definite, Theorem 2.1 requires that dV/dt be negative definite. This rather severe requirement on dV/dt is overcome by LaSalle [15] in the following theorem.

Theorem 2.2

If there exists a real scalar function $V(\underline{x})$ continuous with continuous first partials, such that $V(\underline{0}) = 0$ and

1. $V(\underline{x}) > 0$ for $\underline{x} \neq 0$
2. $V(\underline{x}) \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$
3. $dV/dt \leq 0$ for $\underline{x} \neq 0$ (At least negative semidefinite)

4. dV/dt not identically zero along a solution of the system other than the origin, then system (2.3), under assumption (2.4), is globally asymptotically stable.

Conditions 1 and 2 insure that V represents a closed surface in the entire space. The requirement of Theorem 2.1 that dV/dt be negative definite to insure asymptotic stability is replaced by the conditions 3 and 4. These conditions require that dV/dt be only negative semidefi-

nite, as long as it is not identically zero along a solution of the system. In order to insure that $dV/dt = 0$ is not a solution of (2.3), it is only necessary to substitute the solution of this equation back into (2.3). In practice this is often a trivial problem.

If condition 2 above is not fulfilled, it is impossible to conclude global asymptotic stability. Often, however, it is possible to conclude stability in a well defined region through the use of the following theorem.

Theorem 2.3 [La Salle, 15]

Let Ω be a bounded, closed (compact) set with the property that every solution of (2.3) under assumption (2.4) which begins in Ω remains for all future time in Ω . Suppose there is also a scalar function $V(\underline{x})$ which has continuous first partials in Ω and is such that $dV/dt \leq 0$ in Ω . Let E be the set of all points in Ω where $dV/dt = 0$. Let M be the largest invariant set* in E . Then every solution starting in Ω approaches M as $t \rightarrow \infty$.

If the set M is the origin, asymptotic stability may be concluded. In order to make use of this theorem, of

*A set M is said to be invariant if each solution starting in M remains in M for all time.

course, it is necessary to define the region Ω and show that all solutions starting in Ω remain in Ω as time goes to infinity. Means by which such an Ω may be determined are discussed in connection with the gradient in Chapter IV.

It should be emphasized that the stability theorems presented above give sufficient, but not necessary, conditions for the stability of equations (2.3). The failure of a particular V function to prove stability in no way implies that the system in question is unstable. Instability can only be established by recourse to theorems directly involving instability.

2.7 Geometric Interpretation of Liapunov's Theorems

It is possible to give a relatively simple geometrical interpretation to the theorems of the previous section. Since Theorem 2.2 is the most useful, interpretation will be made in terms of it. For purposes of illustration, it is assumed that the system in question is second order, so that the system behavior may be interpreted on a plane instead of in n dimensions. Extension to n dimensions follows readily.

It is assumed that V and dV/dt meet the conditions of Theorem 2.2. The equation V equals a constant represents a series of closed curves around the origin, with the size of these curves increasing as the constant is

increased from c_1 to c_2 , etc. as in Fig. 2.1. Because of condition 2 of Theorem 2.2, these closed curves extend over the entire x_1x_2 plane. If coordinates are chosen such that x_2 is the derivative of x_1 , then the state plane of x_1x_2 is the phase plane.

Since dV/dt is negative semidefinite, it is either negative or zero everywhere in the state plane. If dV/dt is zero along a curve that is not a trajectory of the system, then, if at any time the trajectory lies on such a curve, it will not remain on the curve where dV/dt is zero. Rather, the trajectory will move to a region where dV/dt is negative. This negative derivative of V insures that as time increases, V will decrease, and in the limit as time goes to infinity, V decreases to the origin.

But V is a function of the state variables. The condition $V(\underline{0}) = 0$ is only possible if the state variables also go to zero as time goes to infinity. This is the meaning of asymptotic stability.

If dV/dt were to equal zero along a curve that was a solution of (2.3), as, for example, if $dV/dt = 0$ along a limit cycle of (2.3), and if the trajectory were to coincide with this curve at one point, the trajectory would remain forever coincident with the curve $dV/dt = 0$.

While the physical interpretation of the meaning of Theorem 2.2 is not difficult, the determination of a Lia-

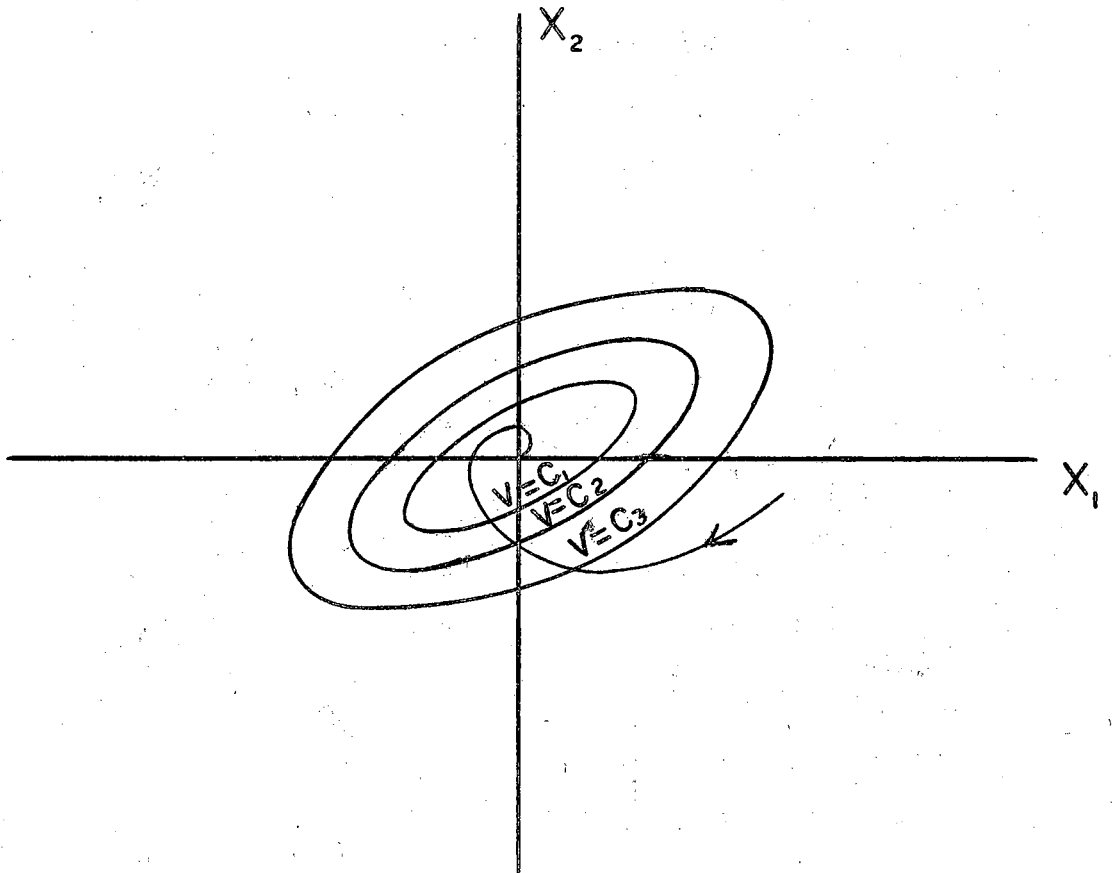


Fig. 2.1. Phase Plane Trajectory Crossing the Curves $V(x, x) = \text{Constant}$ in the Direction of Decreasing V

punov function, $V(\underline{x})$, to satisfy the conditions is indeed a difficult task. The remainder of this report is devoted to means of determining such Liapunov functions.

CHAPTER III

Methods of Generating Liapunov Functions for Autonomous Systems

3.1 Introduction and Organization of the Chapter

The major difficulty in applying the second method of Liapunov to practical problems is the lack of a means of determining a suitable V function. This lack of technique is well recognized and is mentioned in almost every English publication on the subject. The ability to determine the required V function is usually depicted as an art, dependent upon the skill, experience, and even the luck of the investigator. The purpose of this chapter is to explain in detail methods that now exist for generating Liapunov functions. A Liapunov function is said to be generated if the final form of the V function is not known before the generating procedure is applied.

Several of the better known methods of solving non-linear differential equations by the second method are considered briefly. These methods, due mostly to the Russian authors, assume the form of V initially, and thus V is not said to be "generated". The methods of Ingwerson [1], [2] and Szego [3], [4] are treated in detail, because they are actual generating methods within the meaning of the word as here used, and because the Variable Gradient Method, described in later chapters, is based

upon a combination of these two techniques.

3.2 Well Known Techniques Applicable to the Second Method of Liapunov

The work of Lur'e [7], Letov [10], Rekasius [17], Aizerman [18], and Krasovskii [19] is considered briefly in this section. A more comprehensive treatment, aside from the original references, is to be found in the Purdue University's Control and Information Systems Laboratory Report 61-5 [Gibson, et al., 20].

The methods of Lur'e and Letov, and the extensions of these techniques due to Rekasius, consider V 's of a quadratic form or a quadratic form plus an integral, after the system equations have been arranged in a suitable canonic form. The coefficients of the variables in the quadratic form are not assumed but are determined on the basis of a set of stability equations that naturally result. Since the form of V is assumed, this is not considered to be a V function which is generated. The method of Aizerman is similar in the sense that V functions are not generated. Aizerman approximates the nonlinear element of the actual system by a straight line characteristic and then determines the quadratic V function for the approximate linear system. The hope is, of course, that the same V will be successful in proving the stability of the actual nonlinear system.

Krasovskii's method is more of an existence theorem than a working technique. Krasovskii has shown that it is possible to use the phase velocities, not the phase coordinates, as variables in a quadratic form for V . That is, Krasovskii has shown that a suitable V function is

$$V(\underline{X}) = \underline{X}' A \underline{X}$$

Here the \underline{X} 's are the right hand side of equation (2.3).

Krasovskii's method deserves some special mention, however, because even though V is assumed to be a quadratic form in \underline{X} , in the state variables \underline{x} , V will be a function of higher order. Perhaps it is this fact that prompted others to investigate the generation of V functions of higher order form.

3.3 The Method of Ingwerson [1], [2]

3.3.1 Theory and Mechanics of Ingwerson's Method

The method of Ingwerson is a technique for generating Liapunov functions for the general nonlinear system. The method is based upon the successive integration of matrices, and yields sufficient conditions for the stability of nonlinear systems that are always correct for small disturbances. The method is applicable to systems represented by equations (2.2) or (2.3), under assumption (2.4). Phase variables are used exclusively, so that the equations of motion in expanded form are as in (2.7). Thus the

matrix $B(\underline{x})$ of (2.2) becomes

$$B(\underline{x}) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ \vdots & & & & \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \quad (3.1)$$

In the linear autonomous system, the a 's of (3.1) above are all constants. Note also that the matrix B is simply the Jacobian of (2.3), so that the elements of B are $\partial X_i / \partial x_j$, and they are constants.

For the linear autonomous case Ingwerson proceeds in a manner similar to that of Krasovskii and assumes V as a general quadratic form

$$V = \underline{x}' A \underline{x} \quad (3.2)$$

For this V , dV/dt becomes

$$dV/dt = \underline{x}' [B'A + AB] \underline{x} \quad (3.3)$$

dV/dt is constrained to be negative semidefinite, of the form

$$dV/dt = - \underline{x}' C \underline{x} \quad (3.4)$$

where the choice of C is restricted to those matrices which have all elements equal to zero, except one element

of the principal diagonal. This element is set equal to a positive constant.

If the left sides of equations (3.3) and (3.4) are equated, as in (3.5),

$$B'A + AB = - C \quad (3.5)$$

It is then possible to solve this matrix equation for the elements of A in terms of the known elements of the matrices B and C. Obviously, the elements of A are dependent upon the choice of the matrix C. For an nth order system, n possible C matrices exist, and corresponding to each C_i is an A_i matrix. Ingwerson has solved the matrix equations of (3.5) for n up to and including 4. These solutions are tabulated for second and third order systems in Table I and give necessary and sufficient conditions for the stability of linear systems. The results for fourth order systems are not considered significant, since, although dV/dt is constrained to satisfy the usual Hurwitz conditions, these same conditions are violated by the resulting V.

In the linear case, if the matrix A is considered to be the coefficient matrix of a quadratic V function, it is observed that the elements of A are equal to

$$a_{ij} = \frac{1}{2} \frac{\partial^2 V}{\partial x_i \partial x_j} \quad (3.6)$$

Table I

MATRICES FOR THE LIAPUNOV FUNCTIONS

Second Order

$$A_1 = \begin{bmatrix} a_2 & 0 \\ 0 & 1 \end{bmatrix} \quad C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2a_1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_1^2 + a_2 & a_1 \\ a_1 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 2a_1 a_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Third Order

$$A_1 = \begin{bmatrix} a_3^2 & a_2 a_3 & 0 \\ a_2 a_3 & a_1 a_3 + a_2^2 & a_3 \\ 0 & a_3 & a_2 \end{bmatrix} \quad C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(a_1 a_2 - a_3) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_1 a_3 & a_3 & 0 \\ a_3 & a_1^2 + a_2 & a_1 \\ 0 & a_1 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(a_1 a_2 - a_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_1 a_2^2 - a_2 a_3 + a_1^2 a_3 & a_1^2 a_2 & a_1 a_2 - a_3 \\ a_1^2 a_2 & a_1^3 + a_3 & a_1^2 \\ a_1 a_2 - a_3 & a_1^2 & a_1 \end{bmatrix} \quad C_3 = \begin{bmatrix} 2a_3(a_1 a_2 - a_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This suggests a double integration to obtain V directly from A , and it is this idea that is carried over into the nonlinear case.

In the nonlinear case the same problem formulation is assumed. In order to obtain a B matrix of the form (3.1), which is also the Jacobian of the system, (2.3) is differentiated, with the result that

$$\ddot{\underline{x}} = B(\underline{x}) \dot{\underline{x}} \quad (3.7)$$

Now B is no longer a constant matrix but $B(\underline{x})$, as the a 's are, in general, functions of \underline{x} .

In a manner analogous to that used in the linear case, the matrix equation (3.5) is solved for the elements of the matrix A in terms of the chosen C and the known $B(\underline{x})$. (Although this step is not justified, comment will be reserved until the section on analysis of Ingwerson's method.) The resulting A matrix is also a function of \underline{x} , so A is actually $A(\underline{x})$.

Ingwerson points out the conditions that are necessary for the elements of a matrix, such as $A(\underline{x})$, to be the second partial derivative of a scalar, such as V . This is required by (3.6) if integration is to be used to determine a unique V . $A(\underline{x})$ must be symmetrical and the equation

$$\frac{\partial a_{ij}}{\partial x_k} = \frac{\partial a_{ik}}{\partial x_j} \quad (3.8)$$

must be satisfied for the elements of $A(\underline{x})$. In general the elements of $A(\underline{x})$ do not satisfy (3.8) if the system is non-linear. The difficulty is overcome by altering the elements of $A(\underline{x})$ to form a new matrix $A(x_i, x_j)$. This is accomplished by letting all of the variables in each element of $A(\underline{x})$ vanish except x_i and x_j , where i and j are the respective indices of the row and column containing the element. The elements of $A(x_i, x_j)$ now satisfy (3.8).

Once this $A(x_i, x_j)$ is found, a vector, the gradient of a scalar function V , is determined by the integration

$$\nabla V = \int_0^{\underline{x}} A(x_i, x_j) d\underline{x} \quad (3.9)$$

If the components of ∇V in the x_i direction are designated as ∇V_i , V is determined as a line integral of ∇V , as

$$V = \int_0^{\underline{x}} \nabla V' d\underline{x} \quad (3.10a)$$

The upper limit here is not meant to imply that V is a vector quantity, as in (3.9), but rather that the integral is a line integral to an arbitrary point in the phase space located at $\underline{x} = (x_1, x_2, \dots, x_n)$. Because of previous constraints on $A(\underline{x})$, the line integration indicated by (3.10a) is independent of the path of integration. The simplest

path of integration is indicated by the expanded form of (3.10a) to be

$$V = \int_0^{x_1} \nabla V_1(\gamma_1, 0 \dots 0) d\gamma_1 + \int_0^{x_2} \nabla V_2(x_1, \gamma_2, 0 \dots 0) d\gamma_2$$

$$+ \dots + \int_0^{x_n} \nabla V_n(x_1, x_2, \dots x_{n-1}, \gamma_n) d\gamma_n \quad (3.10b)$$

Once V is known, dV/dt may be determined either directly from V or from the gradient, as

$$\frac{dV}{dt} = \nabla V' \dot{\underline{x}} = \nabla V' \underline{\dot{x}} \quad (3.11)$$

The mechanics of this method are best illustrated by a simple example taken from Ingwerson [2]. Consider the undamped, second-order system of Fig. 3.1, which is stabilized by a variable gain. The equations of motion written in the form of (2.7) are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{b_0}{J} x_1 - \frac{b_1}{J} x_1^2 x_2$$

$B(\underline{x})$ is the Jacobian of the above, or

$$B(\underline{x}) = \begin{bmatrix} 0 & 1 \\ \frac{b_0}{J} - \frac{2b_1}{J} x_1 x_2 & -\frac{b_1}{J} x_1^2 \end{bmatrix}$$

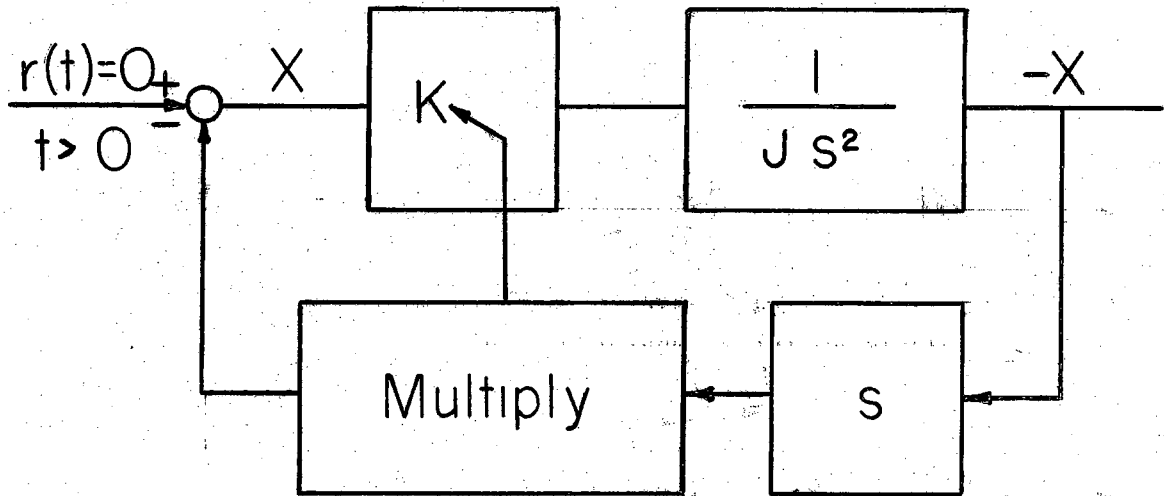
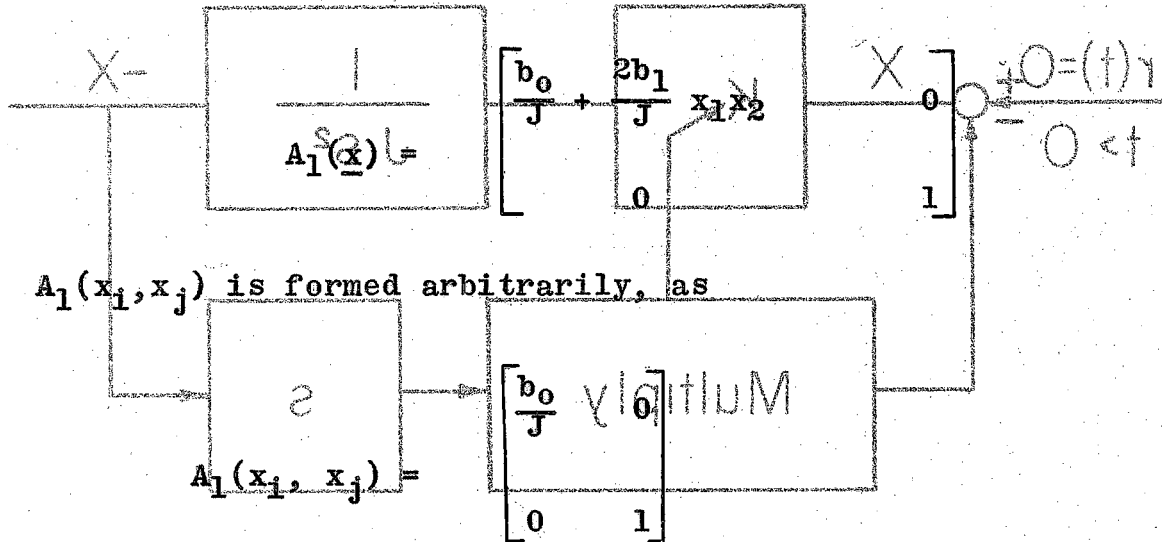


Fig. 3.1. The Ingwerson Example of an Undamped System Compensated by a Nonlinear Compensator

With the matrix C equal to C_1 , $A(\underline{x})$ is equal to $A_1(\underline{x})$, which is given by Table I to be

$$A(\underline{x}) = A_1(\underline{x}) = \begin{bmatrix} a_2 & 0 \\ 0 & 1 \end{bmatrix}$$

Substituting from $B(\underline{x})$, $A_1(\underline{x})$ is found to be



From (3.9) ∇V , a vector, is

$$\nabla V = \begin{bmatrix} \frac{b_0}{J} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d\gamma_1 \\ d\gamma_2 \end{bmatrix} = \begin{bmatrix} \frac{b_0}{J} x_1 \\ x_2 \end{bmatrix}$$

From (3.10), V is determined to be

$$V = \int_0^{x_1} \frac{b_0}{J} \gamma_1 d\gamma_1 + \int_0^{x_2} \gamma_2 d\gamma_2$$

$$V = \frac{b_0}{2J} x_1^2 + \frac{x_2^2}{2}$$

dV/dt follows from (3.11) as

$$\frac{dV}{dt} = - \frac{b_1}{J} x_1^2 x_2^2$$

V and dV/dt meet the conditions of Theorem 2.2, and hence the system of Fig. 3.1 is globally asymptotically stable. If a satisfactory result had not been obtained, the procedure would have been repeated, with C equal to C_2 . If the results were still not satisfactory, a combination of C_1 and C_2 might be tried. Of course, the method is not guaranteed to work in every case, but it often does give good results.

3.3.2 Analysis of the Ingwerson Method

In the development outlined above, two steps were taken quite arbitrarily, such that the resulting procedure is not formally correct in a mathematical sense. Nor does Ingwerson claim that what he has done is rigorous. The justification is purely pragmatic.

A step taken above that might lead one to question the validity of the method is the formation of the matrix $A(x_1, x_j)$ from the matrix $A(\underline{x})$. As mentioned, this is necessary to insure that the integrations subsequently performed will yield a unique scalar function V . However, the V thus determined is in no way assured to satisfy the conditions of any theorem.

The other arbitrary substitution is not as obvious. For the linear case (3.3) is a valid equation, but for the nonlinear case with the system specified by (2.3), dV/dt is found to be

$$\frac{dV}{dt} = \underline{x}' A(\underline{x}) \underline{x} + \underline{x}' \frac{dA(\underline{x})}{dt} \underline{x} + \underline{x}' A(\underline{x}) \underline{x} \quad (3.12)$$

If dV/dt is now constrained as in (3.4), (3.5) does not follow, as

$$\underline{x}' A(\underline{x}) \underline{x} + \underline{x}' \frac{dA(\underline{x})}{dt} \underline{x} + \underline{x}' A(\underline{x}) \underline{x} \neq \underline{x}' [B(\underline{x})' A(\underline{x}) + A(\underline{x}) B(\underline{x})] \underline{x}$$

If the nonlinear system is linearized, however, (3.12) does reduce to (3.3), and valid results are realized in the vicinity of the origin.

The question remains, if the Ingwerson method is not formally correct, why does the method often give good results.

The question can perhaps best be answered by a re-examination of the mechanics of the Ingwerson technique. As an initial step, C_i is chosen arbitrarily. This choice of C_i determines dV/dt , as in (3.4). However, the choice of C_i also uniquely determines $A(\underline{x})$, $A(x_i, x_j)$, ∇V and V itself. In short, the initial arbitrary choice of C_i completely determines both V and dV/dt . Hence the choice of C_i amounts to a rather elaborate means of guessing not only dV/dt , but also V . Since there are always a large

number of V functions capable of proving stability for a given problem, the method often gives results.

Of course it may not be possible to constrain dV/dt to be as required by (3.4). This fact was pointed out by Ingwerson. He indicated that it might be necessary to combine two different C_i matrices, or to even include off diagonal terms in the final C matrix in order to be able to find a suitable V function. However, it seems like an almost hopeless task to try and modify an unsatisfactory V by making an alteration in the matrix C , which is one matrix equation and two integrations removed from V .

Because of the completely mechanical operations required once C has been chosen, solutions exist which are not achievable by the Ingwerson method. Consider the following example as a case in point. The system is represented by the block diagram of Fig. 3.2. The differential equations of motion are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -(x_1 + cx_2)^3 - bx_3$$

and $B(\underline{x})$ is

$$B(\underline{x}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3(x_1 + cx_2)^2 & -3c(x_1 + cx_2)^2 & -b \end{bmatrix}$$

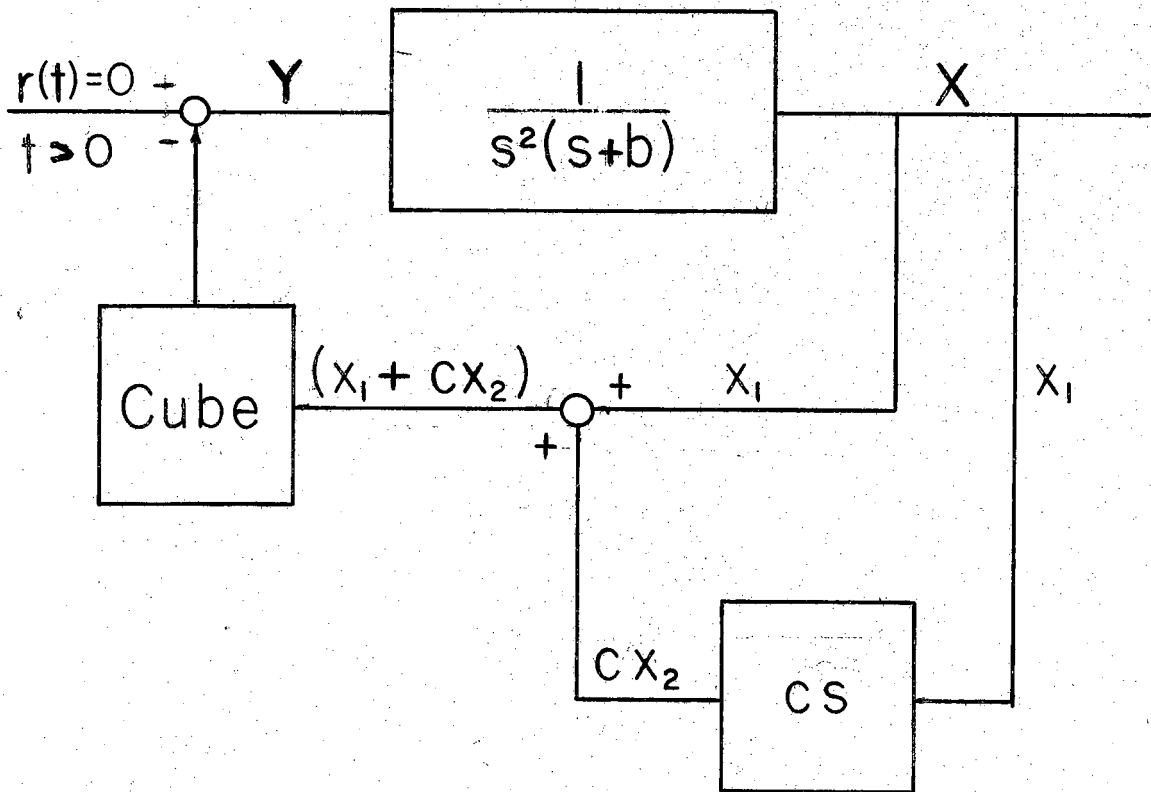


Fig. 3.2. Third Order Example of Ingwerson

If in this problem an attempt is made to constrain dV/dt to be negative semidefinite in x_1^2 or x_3^2 through an initial choice of C_1 or C_3 , a satisfactory result is not obtained. Yet an answer does exist for dV/dt in terms of x_3^2 . Such a dV/dt , along with the corresponding V , is

$$V = bx_2^2 + 2x_2x_3 + cx_3^2 + \frac{1}{2}(x_1 + cx_2)^4$$

and

$$\frac{dV}{dt} = -2x_3^2(bc - 1)$$

Why the method of Ingwerson is unable to produce this result can be seen by considering only the first element of $A_3(\underline{x})$, corresponding to C_3 . The element a_{11} of $A_3(\underline{x})$ is

$$a_{11} = 9bc^2(x_1 + cx_2)^4 + 9c(x_1 + cx_2)^4 + 3b^2(x_1 + cx_2)^2$$

The element a_{11} of $A_3(x_1, x_1)$ is

$$a_{11} = 9bc^2x_1^4 + 9cx_1^4 + 3b^2x_1^2$$

Since a term in x_1^4 appears in this element, a term in x_1^6 will appear in V . This term does not appear in the actual V that proved to be a successful solution to this problem. Hence, a satisfactory solution is not attainable by the

Ingwerson method when dV/dt is constrained to be a function of x_3^2 .

Ingwerson did obtain a solution to this problem in terms of x_2^2 in dV/dt . The point here is not that a problem has been worked which was not solved by Ingwerson, since the problem was solved by him in terms of x_2^2 . The point is to demonstrate the inflexibility of the approach, once C_i has been chosen. In problems where dV/dt necessarily contains terms in $x_i x_j$, the choice of a single C_i or a combination of C_i 's will not produce a solution.

Much of what has been said concerning the Ingwerson method of generating Liapunov functions has been said in a negative sense. Yet Ingwerson's contribution is significant. The idea of integrating a vector ∇V as a line integral to determine the scalar V is original, and this idea offers a new approach to the generation of Liapunov functions, as is explained in the following chapters. Further, the method is applicable to cases in which the nonlinearity is expressed as a polynomial or as a general function of \underline{x} .

3.4 The Method of Szego

3.4.1 Theory and Mechanics of Szego's Method

The Szego method of generating Liapunov functions which is presented here is based on material from refer-

This fact is most easily seen in two dimensions, as in Fig. 3.3. Here the solution of the curve $dV/dt = 0$ is simply two lines in the plane, as shown. It is assumed arbitrarily that in regions M of Fig. 3.3, dV/dt is negative, and that in regions N it is positive, and, of course, dV/dt is 0 on the lines. As the curves are brought closer together, the regions N shrink, so that when the two curves coincide, dV/dt is negative in the whole space.

Although the coefficients $a_{ij}(x_i, x_j)$ are not allowed (to be functions of x), they are considered to be general polynomials in x_i and x_j . An exposition of the method is most easily made in the second-order case.

Consider the system described by the equations

$$\dot{x}_1 = f(x_1, x_2) \quad (3.14)$$

$$\dot{x}_2 = f(x_1, x_2) \quad (3.15)$$

and

$$\dot{V} = 2 \left[a_{11}(x_1) + \frac{1}{2} x_1 \frac{\partial a_{11}(x_1)}{\partial x_1} \right] x_1 x_2 + 2f(x_1, x_2) \left[a_{12}(x_1) x_1 + a_{22} x_2 \right] + 2 \left[a_{12}(x_1) + \frac{1}{2} x_1 \frac{\partial a_{12}(x_1)}{\partial x_1} \right] x_2^2 \quad (3.16)$$

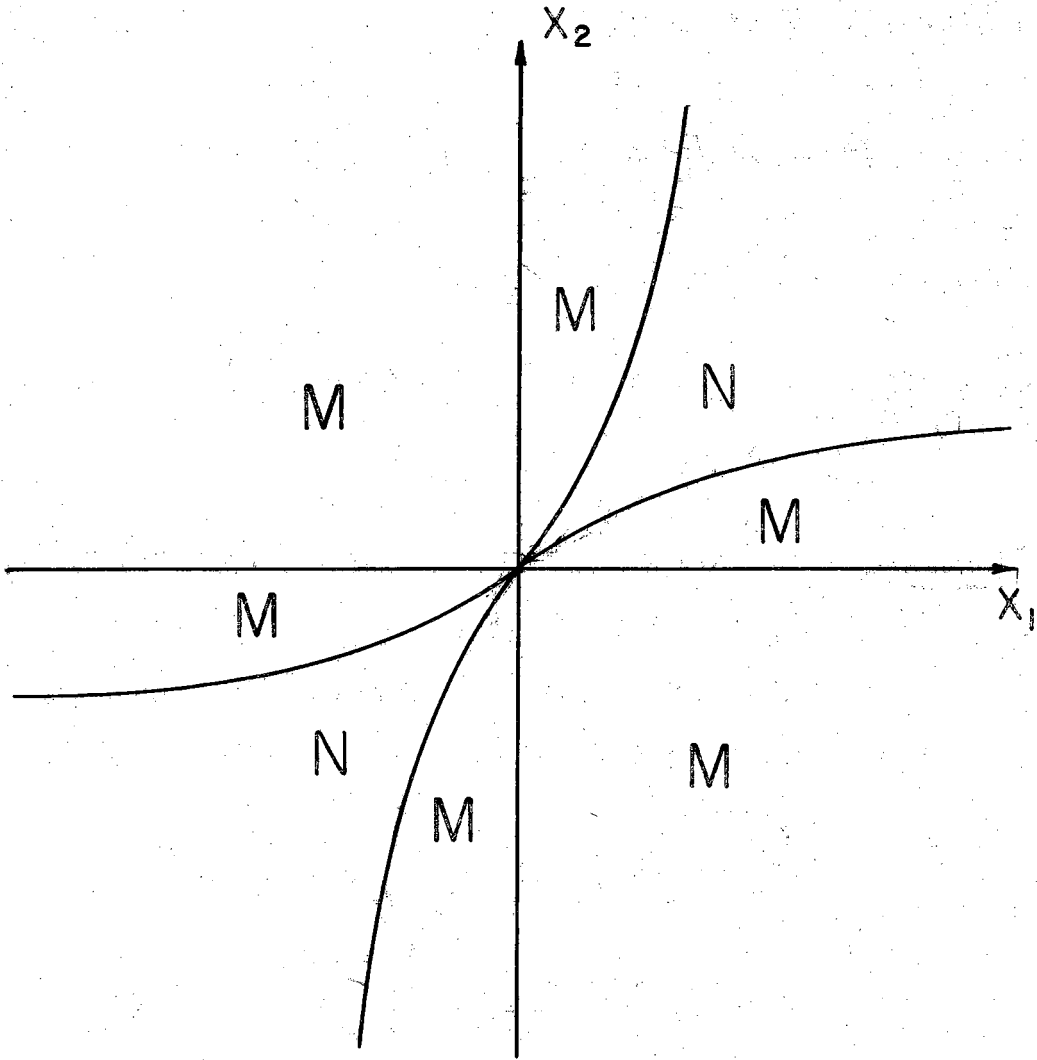


Fig. 3.3. An Illustration of Szego's Method of Constraining dV/dt to be Negative Semidefinite by Forcing Solutions to the Equation $dV/dt = 0$ to Coincide

Note that the form of $a_{11}(x_1)$ and $\frac{1}{2}x_1 \partial a_{11}(x_1) / \partial x_1$ is identical, since $a_{11}(x_1)$ is a polynomial. Hence the bracketed terms above may be replaced by a new coefficient, $a_{ij}'(x_1)$, where

$$a_{ij}'(x_1) = a_{ij}(x_1) + \frac{1}{2} \frac{\partial a_{ij}(x_1)}{\partial x_1} x_1 \quad (3.17)$$

Thus dV/dt becomes

$$\frac{dV}{dt} = 2a_{11}'(x_1)x_1x_2 + 2a_{12}'(x_1)x_2^2 + 2f(x_1, x_2) [a_{12}(x_1)x_1 + a_{22}x_2] \quad (3.18)$$

Note that in the above equation, two sets of coefficients now exist, $a_{ij}(x_i, x_j)$ and $a_{ij}'(x_i, x_j)$. To eliminate the excessive number of arbitrary coefficients, consider an auxiliary equation of the same form as dV/dt , such as

$$\begin{aligned} \psi(\underline{x}) = & 2a_{11}'(x_1)x_1x_2 + 2a_{12}'(x_1)x_2^2 \\ & + 2f(x_1, x_2) [a_{12}'(x_1)x_1 + a_{22}x_2] \quad (3.19) \end{aligned}$$

Now instead of forcing the solutions of the equation $dV/dt = 0$ to coincide, the solutions of the equation $\psi(\underline{x}) = 0$ are forced to coincide. Thus the $a_{ij}'(x_i, x_j)$'s are evaluated, and a V function which produces a proper $\psi(\underline{x})$ is determined.

However, $\psi(\underline{x})$ is not the function of interest. The function of interest is dV/dt , but dV/dt does have the same form as $\psi(\underline{x})$. Hence it is reasonable to expect that a V function of the same form that was used in connection with $\psi(\underline{x})$ might also yield a dV/dt that could be constrained to be at least negative semidefinite, as $\psi(\underline{x})$ was constrained to be at least negative semidefinite. Thus, the problem is started over, this time not with an arbitrary V function, but with a V function of the form determined from the consideration of the auxiliary equation $\psi(\underline{x})$. The coefficients of this new V function are left arbitrary, and they are determined by constraints on dV/dt which make it at least negative semidefinite.

What has been said in general above is clarified by the following example. The block diagram of the system is pictured in Fig. 3.4, and the dynamic equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_1^3$$

Assume V is as in (3.13) or (3.15)

$$V = a_{11}(x_1)x_1^2 + 2a_{12}(x_1)x_1x_2 + x_2^2$$

After differentiation and substitution, dV/dt is found to be

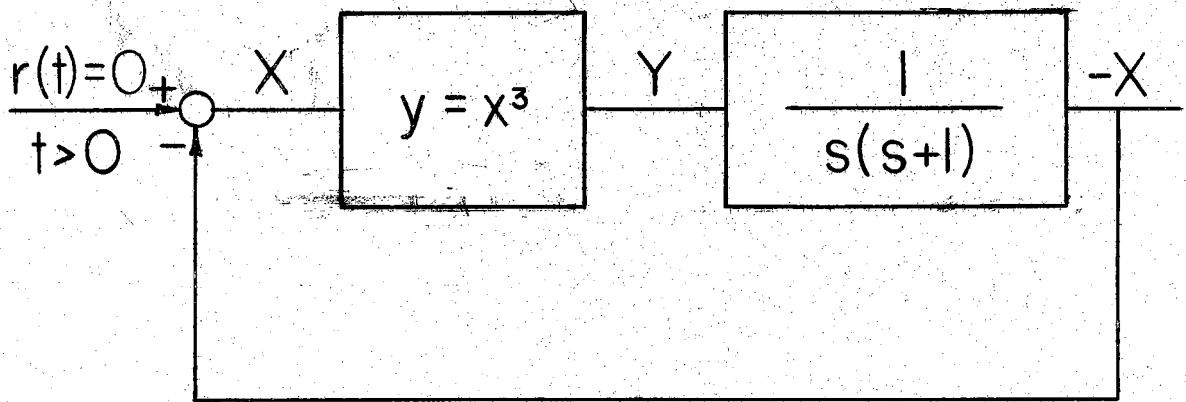


Fig. 3.4. Block Diagram of a Second Order System with a Cubic Nonlinearity

$$\frac{dV}{dt} = x_2^2 \left[2a_{12}'(x_1) - 2 \right] + x_2 \left[2a_{11}'(x_1)x_1 - 2a_{12}(x_1)x_1 - 2x_1^3 \right] - 2a_{12}(x_1)x_1^4$$

and

$$\psi(\underline{x}) = x_2^2 \left[2a_{12}'(x_1) - 2 \right] + x_2 \left[2a_{11}'(x_1)x_1 - 2a_{12}'(x_1)x_1 - 2x_1^3 \right] - 2a_{12}'(x_1)x_1^4$$

Here dV/dt and $\psi(\underline{x})$ are arranged as quadratics in x_2 . The roots can be made to coincide if the radical in the usual quadratic formula is made equal to zero, that is if

$$\beta^2 - 4\alpha\gamma = 0, \text{ where, for } \psi(\underline{x}),$$

$$\alpha = 2a_{12}'(x_1) - 2$$

$$\beta = 2a_{11}'(x_1)x_1 - 2a_{12}'(x_1)x_1 - 2x_1^3$$

$$\gamma = 2a_{12}'(x_1)x_1^4$$

As Szego does in his example problem, Case b of [4],

α and β are constrained to be 0. Thus

$$a_{12}'(x_1) = a_{12}' = 1$$

With this substitution in β ,

$$a_{11}'(x_1) = 1 + x_1^2$$

Thus the V associated with $\psi(\underline{x})$ is known, and the form of V associated with dV/dt is also known. The problem is now started over, under the assumption that V is

$$V = ax_1^4 + bx_1^2 + cx_1x_2 + x_2^2$$

Here a , b , and c are arbitrary constants. For $a = \frac{1}{2}$, $b = 1$, and $c = 2$, dV/dt is

$$\frac{dV}{dt} = -2x_1^4$$

and V is

$$V = x_1^4/2 + x_1^2 + 2x_1x_2 + x_2^2$$

Here V is positive definite* and dV/dt negative semidefinite. Theorem 2.2 applies, since dV/dt is not zero along a trajectory, as $x_1 = 0$ is not a solution of the given equations. Thus the given equations are globally asymptotically stable, or, perhaps more significantly, the system described by these equations is globally asymptotically stable.

In the application of this method to the third order case, difficulties arise that are not apparent in the example above. Consider again the Ingwerson third-order

*Note that the constant portions of V remain identical to those previously determined for the auxiliary equation $\psi(\underline{x})$. This is always true.

system of Fig. 3.2, for which the equations of motion are given as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= - (x_1 + cx_2)^3 - bx_3\end{aligned}$$

In order to appreciate the difficulties that arise, it is necessary to consider this problem in detail. From (3.14) V is set equal to

$$\begin{aligned}V &= a_{11}(x_1)x_1^2 + 2a_{12}(x_1, x_2)x_1x_2 + 2a_{13}(x_1)x_1x_3 \\ &+ a_{22}(x_2)x_2^2 + 2a_{23}(x_2)x_2x_3 + a_{33}x_3^2\end{aligned}$$

$\frac{1}{2} dV/dt$ is found to be

$$\begin{aligned}\frac{1}{2} \frac{dV}{dt} &= \left[a_{11}(x_1) + \frac{1}{2}x_1 \frac{\partial a_{11}(x_1)}{\partial x_1} \right] x_1x_2 \\ &+ \left[a_{12}(x_1, x_2) + x_1 \frac{\partial a_{12}(x_1, x_2)}{\partial x_1} \right] x_2^2 \\ &+ \left[a_{12}(x_1, x_2) + x_2 \frac{\partial a_{12}(x_1, x_2)}{\partial x_2} \right] x_1x_3 \\ &+ \left[a_{13}(x_1) + x_1 \frac{\partial a_{13}(x_1)}{\partial x_1} \right] x_2x_3 \\ &+ \left[a_{22}(x_2) + \frac{1}{2}x_2 \frac{\partial a_{22}(x_2)}{\partial x_2} \right] x_2x_3 + \\ &+ \left[a_{23}(x_2) + x_2 \frac{\partial a_{23}(x_2)}{\partial x_2} \right] x_3^2 \\ &- a_{13}(x_1)x_1(x_1 + cx_2)^3 - ba_{13}(x_1)x_1x_3 \\ &- a_{23}(x_2)x_2(x_1 + cx_2)^3 \\ &- ba_{23}(x_2)x_2x_3 - a_{33}x_3(x_1 + cx_2)^3 - a_{33}bx_3^2\end{aligned}$$

The starred terms above are of the same form, but are not necessarily equal. Hence in substituting the $a_{ij}'(x_i, x_j)$'s into dV/dt , an additional coefficient must be introduced. The double starred term above is assumed to be equal to $b'_{12}(x_1, x_2)$, and $\frac{1}{2}dV/dt$ is ordered as a quadratic in x_3 .

$$\begin{aligned} \frac{1}{2} \frac{dV}{dt} = & - x_3^2 [a_{33}'b - a_{23}'(x_2)] \\ & - x_3 [a_{33}'(x_1 + cx_2)^3 + ba_{23}'(x_2)x_2 + ba_{13}'(x_1)x_1 \\ & - a_{22}'(x_2)x_2 - a_{13}'(x_1)x_2 - b'_{12}(x_1, x_2)x_1] \\ & - a_{23}'(x_2)x_2(x_1 + cx_2)^3 + a_{13}'(x_1)x_1(x_1 + cx_2)^3 \\ & - a_{12}'(x_1, x_2)x_2^2 - a_{11}'(x_1)x_1x_2 \end{aligned}$$

The formation of $\psi(\underline{x})$ is accomplished as before. The $a_{ij}'(x_i, x_j)$ terms are simply substituted for the $a_{ij}(x_i, x_j)$, but here it is also necessary to substitute $a_{12}'(x_1, x_2)$ for $b'_{12}(x_1, x_2)$. The resulting $\psi(\underline{x})$ is therefore

$$\begin{aligned} \psi(\underline{x}) = & - x_3^2 [a_{33}'b - a_{23}'(x_2)] \\ & - x_3 [a_{33}'(x_1 + cx_2)^3 + ba_{23}'(x_2)x_2 + ba_{13}'(x_1)x_1 \\ & - a_{22}'(x_2)x_2 - a_{13}'(x_1)x_2 - a_{12}'(x_1, x_2)x_1] \\ & - a_{23}'(x_2)x_2(x_1 + cx_2)^3 + a_{13}'(x_1)x_1(x_1 + cx_2)^3 \\ & - a_{12}'(x_1, x_2)x_2^2 - a_{11}'(x_1)x_1x_2 \end{aligned}$$

This may be constrained to have the surfaces resulting from the equation $\psi(\underline{x}) = 0$ coincide if the radical of the usual quadratic formula is made equal to zero. In this case β and γ are made zero. In γ a term in x_1^4 results which cannot be cancelled unless $a_{13}'(x_1)$ is zero. Since one coefficient is always arbitrary, set $a_{23}'(x_2) = 1$. Then $\gamma = 0$ results in

$$x_2(x_1 + cx_2)^3 = a_{12}'(x_1, x_2)x_2^2 + a_{11}'(x_1)x_1x_2$$

or

$$x_1^3 + 3cx_1^2x_2 + 3c^2x_1x_2^2 + c^3x_2^3 = a_{12}'(x_1, x_2)x_2 + a_{11}'(x_1)x_1$$

If $a_{11}'(x_1) = x_1^2$, then

$$a_{12}'(x_1, x_2) = 3cx_1^2 + 3cx_1x_2 + c^3x_2^2$$

When these known coefficients are substituted into the equation $\beta = 0$,

$$3cx_1^3 + 3c^2x_1^2x_2 + c^3x_2^2x_1 + a_{22}'(x_2)x_2 =$$

$$bx_2 + a_{33}'x_1^3 + 3a_{33}'cx_1^2x_2 + 3a_{33}'c^2x_1x_2^2 + a_{33}'c^3x_2^3$$

If terms in like powers and like variables are equated, four equations result, as

$$3cx_1^3 = a_{33}'x_1^3 \quad (3.20a)$$

$$3c^2x_1^2x_2 = 3a_{33}'cx_1^2x_2 \quad (3.20b)$$

$$c^3x_1x_2^2 = 3a_{33}'c^2x_1x_2^2 \quad (3.20c)$$

$$a_{22}'(x_2)x_2 = bx_2 + a_{33}'c^3x_2^3 \quad (3.20d)$$

From (3.20d), $a_{22}'(x_2)$ equals

$$a_{22}'(x_2) = b + a_{33}'c^3x_2^2$$

However, if (3.20a) is solved for a_{33}' , the result, $a_{33}' = 3c$, does not satisfy the remaining equations. These are simply not equalities, although in each case it is seen that a_{33}' should be of the form $a_{33}' = Kc$, where K is a constant. Hence the fact that these terms do not cancel in $\psi(\underline{x})$ is overlooked, in hope that the terms will actually cancel when the form of V determined from $\psi(\underline{x})$ is applied to $\frac{1}{2}dV/dt$. Thus the V function with which the problem may be reworked is

$$V = a_1x_1^4 + a_2x_1^3x_2 + a_3x_1^2x_2^2 + a_4x_1x_2^3 + a_5x_2^4 \\ + bx_2^2 + 2x_2x_3 + a_6x_3^2$$

In this case β and γ can be forced to 0, so that

$$\frac{dV}{dt} = -2x_3^2(bc - 1)$$

and, with the coefficients evaluated,

$$V = bx_2^2 + 2x_2x_3 + cx_3^2 + \frac{1}{2}(x_1 + cx_2)^4$$

V is positive definite and dV/dt negative semidefinite in such a manner that Theorem 2.2 applies. Thus the system is globally asymptotically stable.

3.4.2 Analysis of the Szego Method

As a consequence of this last example, it is clear that the success of the Szego method of generating Liapunov functions depends completely upon the similarity in form of the undetermined coefficients and of dV/dt and $\psi(\underline{x})$. It is true that the form determined for V above was successful in solving the problem in question, even though $\psi(\underline{x})$ could not be constrained as desired. However, in a problem picked at random the opposite might well be true; that is, it may be possible to constrain $\psi(\underline{x})$ as desired, but not dV/dt . Then, of course, no result would be obtained. Hence the Szego method, like the Ingwerson method, is not guaranteed to work.

On another point, the method of constraining $\psi(\underline{x})$ or dV/dt is unnecessarily restrictive. The idea of forcing the two surfaces that result from the equation $\psi(\underline{x}) = 0$ to coincide is conceptually appealing as it was described with reference to Fig. 3.3. Yet the meaning is not always

clear, as in the second-order example cited above. The two values of x_2 were forced to be identical by letting α and β be zero, where α and β are defined in (3.21). Yet if α is allowed to be zero, x_2 becomes unbounded, as α also appears in the denominator of the quadratic formula. Thus the graphical or pictorial significance is lost. Actually, as long as β is forced to be zero, $a_{12}'(x_1)$ can take on any value from 0 to 2 inclusive, and the resulting $\psi(\underline{x})$ is still at least negative semidefinite. This problem is worked as an illustrative example in the chapter to follow, and this point is discussed further.

The last adverse criticism of the Szego method is based upon an initial assumption of the problem statement, namely that the nonlinearity in question can be represented in polynomial form. This objection stems from the usual complaint that any power series of finite number of terms either goes to plus or minus infinity for large x . This behavior is not typical of the nonlinearities of physical systems, and it may very well be that it is impossible to prove global asymptotic stability for a system which is, in fact, globally asymptotically stable, simply because the assumption of the nonlinearity in polynomial form produces an unbounded output for large x .

In defense of the Szego method, it should be stressed that the method is easy to apply and often does give re-

sults. Also, many nonlinear differential equations of classical interest do have a polynomial representation of the nonlinearity, such as the Van der Pol equation for example. In [3] and [4] Szego brackets the limit cycle of the Van der Pol equation by forcing the equation $\psi(\underline{x}) = 0$ to represent a closed and bounded surface. The reader is referred to the above references for further treatment of this excellent example. Formally speaking, the mechanics of application are as described here.

The idea of assuming the unknown coefficients to be polynomials of the state variables is made use of in the following chapter.

CHAPTER IV

The Variable Gradient Method of Generating Liapunov Functions for Autonomous Systems

4.1 Introduction and Organization of the Chapter

This chapter is devoted to the development and application of the variable gradient method of generating Liapunov functions. The method is mathematically sound and is characterized by its ability to handle systems containing multiple nonlinearities in which the nonlinearity is known as a definite function of the state variables, or simply as a general function of \underline{x} . The method overcomes the theoretical and practical limitations of the two methods described in the previous sections.

Two main sections follow this brief introduction. The first of these is devoted to the theoretical considerations upon which the variable gradient method is based. This is followed by a detailed explanation as to how these theoretical considerations can be implemented. Example problems are treated separately in the following chapter.

4.2 Theoretical Considerations

It is assumed here, as in the previous chapters, that the physical system under consideration is represented by (2.3), under assumption (2.4).

$$\dot{\underline{x}} = \underline{X}(\underline{x}) \quad (2.3)$$

$$\underline{X}(\underline{0}) = \underline{0} \quad (2.4)$$

The following theorem is due to Massera [21, p. 200]. A preferred form of the theorem is quoted from Kalman [13] for autonomous systems.

Theorem 4.1 [Kalman, 13, p. 397]

If the system described by (2.3) under assumption (2.4) is Lipschitzian,* and if the equilibrium state, $\underline{x}_e = 0$ is globally asymptotically stable, then there exists a $V(\underline{x})$ which is infinitely differentiable with respect to \underline{x} that is capable of proving global asymptotic stability.

The Lipschitz condition implies continuity of \underline{X} in \underline{x} . Hence all physical systems that are globally asymptotically stable and whose nonlinearities satisfying the Lipschitz condition satisfy the conditions of Theorem 4.1. The theorem could be reworded to say that if a physical system with a continuous nonlinearity whose derivative exists and is bounded everywhere is globally asymptotically stable, then an infinitely differentiable $V(\underline{x})$ exists which is capable of proving this type of stability via Liapunov's second method.

Theorem 2.2 requires that $V(\underline{x})$ be continuous with continuous first partials. If the scalar $V(\underline{x})$ has first par-

* $\underline{X}(\underline{x})$ satisfies the Lipschitz condition in a region R if the following condition is satisfied

$$\| \underline{X}(\underline{\gamma}) - \underline{X}(\underline{\delta}) \| \leq K \| \underline{\gamma} - \underline{\delta} \|$$

tials with respect to \underline{x} , this is equivalent to saying that the gradient of $V(\underline{x})$ exists. This ∇V is a unique n dimensional vector with n components ∇V_i in the x_i direction. Thus if a physical system with continuous nonlinearities is globally asymptotically stable, at least one ∇V exists which can be determined from a $V(\underline{x})$ capable of proving such stability.

Instead of assuming a knowledge of V , from which ∇V may be determined, assume that ∇V is known. It is shown in standard texts on vector calculus [Lass, 22, pp. 297-301] that for a scalar function V to be obtained uniquely from a line integral of a vector function, ∇V , the following $(n - 1) n/2$ equations must be satisfied.

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \quad i, j = 1, 2, \dots, n \quad (4.1)$$

Equations (4.1) are necessary and sufficient conditions that the scalar function V be independent of the path of the line integration. In the three dimensional case, the above equations are identical to those obtained from setting the curl of a vector equal to zero. This form of Stokes theorem is familiar to electrical engineers from field theory. Equations (4.1) are thus an n dimensional representation of Stokes theorem, and these equations will be referred to hereafter as curl equations.

A ∇V determined from a $V(\underline{x})$ capable of proving glo-

bal asymptotic stability necessarily meets the conditions of (4.1). This is seen as follows. Theorem 4.1 guarantees that

$$\frac{\partial^2 V(\underline{x})}{\partial x_i \partial x_j} \quad \text{and} \quad \frac{\partial^2 V(\underline{x})}{\partial x_j \partial x_i} \quad (4.2)$$

exist and are continuous, as V is infinitely differentiable. A theorem from advanced calculus [Taylor, 23, p. 220] states that if expressions (4.2) are continuous in the whole region, then, in the whole region,

$$\frac{\partial^2 V(\underline{x})}{\partial x_i \partial x_j} = \frac{\partial^2 V(\underline{x})}{\partial x_j \partial x_i} \quad (4.3)$$

This is simply a restatement of (4.1). Hence a knowledge of either $V(\underline{x})$ or ∇V uniquely defines the other. The conclusion from the above is stated as a theorem.

Theorem 4.2

If the system described by (2.3) under assumption (2.4) is Lipschitzian, and if the equilibrium state, $\underline{x}_e = 0$, is globally asymptotically stable, then a ∇V exists, from which $V(\underline{x})$ may be obtained by line integration, and the $V(\underline{x})$ so obtained is capable of establishing global asymptotic stability.

This is rather powerful existence theorem. If a given system has nonlinearities that can be represented by continuous functions, and if that system is globally asymp-

totically stable, then a gradient capable of establishing this stability exists.

Since the knowledge of either V or ∇V uniquely determines the other, Theorem 2.2 may be restated in terms of the gradient function.

Theorem 4.3

If for the equations (2.3) under assumption (2.4) there exists a real vector function ∇V with elements ∇V_i , such that

$$1. \quad \frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}$$

2. $\nabla V' \underline{X}(\underline{x}) \leq 0$, but not identically zero on a solution of (2.3) other than the origin and such that the scalar function $V(\underline{x})$ formed by a line integration of ∇V is continuous with continuous first partials, and

$$3. \quad V(\underline{x}) > 0 \text{ for } \underline{x} \neq 0$$

$$4. \quad V(\underline{x}) \rightarrow \infty \text{ as } \|\underline{x}\| \rightarrow \infty$$

then (2.3) is globally asymptotically stable.

This theorem is not new in the sense that it is an extension or a generalization of an existing theorem. However, in this restatement of Theorem 2.2, the role of the gradient function is emphasized.

If condition 4 above is not satisfied or if condition 2 is not satisfied in the whole space, it is impossible to

conclude global asymptotic stability, and Theorem 2.3 may be used to prove stability in a smaller region. As implied by Ingwerson [1], a possible means of defining the region Ω exists if V is positive definite and

1. One of the surfaces, $V = \text{a constant}$, bounds the region.
2. The gradient of V , ∇V , is not zero anywhere in the region except at the equilibrium position.
3. dV/dt is negative or zero inside the region.

Proof of the fact that the region Ω can be defined in such a way is quite simple. If V is positive definite, $V(\underline{0}) = 0$, and in a neighborhood of the origin, ∇V is such that every point, movement along the gradient is movement toward a higher value of V . The requirement that all of the elements of ∇V not be zero except at the origin insures that V has no relative maximum between 0 and the curve $V = K$ which bounds the region. Since dV/dt is always negative or zero inside $V = K$, solutions starting within $V = K$ remain within Ω .

Notice that here again the gradient is important. The following section is devoted to discussion of a method of generating Liapunov V functions, starting with a variable gradient.

4.3 Implementation of Theorem 4.3

A comparison of Theorems 2.2 and 4.3 clearly indicates a shift in emphasis. The problem of determining a V function which satisfies Liapunov's theorem is transformed into the problem of finding a ∇V such that the n dimensional curl of this gradient is equal to zero, or, in other words, (4.1) is satisfied. Further, the V and dV/dt determined from ∇V must be sufficient to prove stability, according to either theorem, as the theorems are equivalent. On the surface it may appear as though the problem is actually being made more difficult, although the opposite is true. The existence of the auxiliary curl equations is the device that enables a solution of the stability problem, starting with ∇V .

As the name "variable gradient" implies, the task of implementing Theorem 4.3 is accomplished by the assumption of a vector, ∇V , with n undetermined components. In order to make this vector general enough to embrace all possible solutions, each of the n undetermined components of the gradient is further assumed to be made up of n elements of the form $\alpha_{ij} x_j$. The α 's are assumed to be general functions of \underline{x} or polynomials with an unspecified number of terms, such that ∇V is equal to

$$\nabla V = \begin{pmatrix} \alpha_{11} x_1 + \alpha_{12} x_2 + \dots + \alpha_{1n} x_n \\ \alpha_{21} x_1 + \alpha_{22} x_2 + \dots \\ \vdots \\ \alpha_{n1} x_1 + \dots + \alpha_{nn} x_n \end{pmatrix} = \begin{pmatrix} \nabla V_1 \\ \nabla V_2 \\ \vdots \\ \nabla V_n \end{pmatrix} \quad (4.4)$$

The α 's are assumed to be made up of a constant portion α_{ijk} , and a variable portion α_{ijv} . The variable portion is a function of the state variables, so that

$$\alpha_{ij} = \alpha_{ijk} + \alpha_{ijv}(\underline{x}) \quad (4.5)$$

and

$$\nabla V = \begin{pmatrix} \alpha_{11k} + \alpha_{11v}(\underline{x}) x_1 + \alpha_{12k} + \alpha_{12v}(\underline{x}) x_2 + \dots + \alpha_{1nk} + \alpha_{1nv}(\underline{x}) x_n \\ \alpha_{21k} + \alpha_{21v}(\underline{x}) x_1 + \dots \\ \vdots \\ \alpha_{nk} + \alpha_{nv}(\underline{x}) x_1 + \dots + \alpha_{nnk} + \alpha_{nnv}(\underline{x}) x_n \end{pmatrix} \quad (4.6)$$

Several interesting facts are apparent from an examination of the i th element of the gradient,

$$\nabla V_i = \alpha_{i1k} + \alpha_{i1v}(\underline{x}) x_1 + \dots + \alpha_{iik} + \alpha_{iiv}(\underline{x}) x_i + \dots + \alpha_{ink} + \alpha_{inv}(\underline{x}) x_n$$

The solution of a given problem may require that ∇V_i contain terms that have more than one state variable as factors. It is evident that such terms may be determined

from terms such as $\alpha_{ij}(\underline{x})x_j$, such that $\alpha_{iiv}(\underline{x})$ need only be $\alpha_{iiv}(x_i)$.

V is to be determined as a line integral of ∇V , according to equation (3.10).

$$\begin{aligned}
 V = & \int_{\text{line}} \nabla V' \cdot \underline{dx} = \int_0^{x_1} \nabla V_1(\gamma_1, 0 \dots 0) d\gamma_1 \\
 & + \int_0^{x_2} \nabla V_2(x_1, \gamma_2, 0 \dots 0) d\gamma_2 + \dots \\
 & + \int_0^{x_n} \nabla V_n(x_1, x_2, \dots, x_{n-1}, \gamma_n) d\gamma_n \quad (3.10)
 \end{aligned}$$

Note that the α_{ii} coefficients give rise to terms such as

$$\frac{\alpha_{iik} x_i^2}{2} \quad \text{and} \quad \int_0^{x_i} \alpha_{iiv}(\gamma_i) \gamma_i d\gamma_i$$

Here it has been assumed that $\alpha_{iiv}(\underline{x})$ has been set equal to $\alpha_{iiv}(x_i)$, as mentioned above. For V to be positive definite in the neighborhood of the origin, α_{iik} must be always positive. For V to represent a closed surface in the whole space, or for V to be always positive, $\alpha_{iiv}(x_i)$ must be an even function of x_i and > 0 for large x_i . Also, if $\alpha_{iik} = 0$, $\alpha_{iiv}(x_i)$ must be even and greater than zero for all x_i .

What has been said above in regard to the α_{ij} 's has been said in view of requirements that have to be met by the resulting V function if Theorem 4.3 is to apply. This line of thinking is pursued further in the following paragraphs.

Since the α_{ijv} are allowed to be functions of the state variables, it is expected that V may well contain higher order terms in the state variables. Since this is the case, the question of the positive definiteness of the resulting V becomes important.

The term positive definiteness is usually used in reference to quadratic forms, although the concept does have meaning for a form of arbitrary order. Geometric means of insuring that a scalar function, as $V(\underline{x})$, represents a closed surface are discussed in the appendix. The geometric method used requires that one of the state variables in V be raised to the second order, and no higher. This is accomplished by forcing one α_{ij} to be equal to a constant, and by forcing the remaining α_{ijv} not to be functions of x_i .

These restrictions were originally made so that dV/dt could be constrained by letting the solutions to the equation $dV/dt = 0$ coincide. As mentioned in Section 3.4.1, this technique is unnecessarily restrictive, as will be made clear in Example 5.1. However, the assumptions that

one of the α_{ii} is a constant and that the remaining α_{ijv} are not a function of x_i do insure that the V finally produced from ∇V will be a quadratic in x_i , as is necessary for the geometric considerations of the Appendix.

In problems involving automatic control systems, the x_n term frequently appears linearly in the n first order equations that describe the motion of the system. For this reason, the assumptions of the previous paragraph are applied to the x_n variable. Specifically, α_{nn} is set equal to 2. This seemingly arbitrary choice of α_{nn} in the gradient is equivalent to the assumption of an arbitrary constant, or scale factor, in V . The choice of $\alpha_{nn} = 2$ insures that V will contain a term in x_n^2 .

In view of the above discussion, ∇V is now

$$\nabla V = \left\{ \begin{array}{l} \alpha_{11k} + \alpha_{11v}(x_1) x_1 + \alpha_{12k} + \alpha_{12v}(x_1, x_2, \dots, x_{n-1}) x_2 \\ \quad + \dots + \alpha_{1nk} + \alpha_{1nv}(x_1, x_2, \dots, x_{n-1}) x_n \\ \alpha_{21k} + \alpha_{21v}(x_1, x_2, \dots, x_{n-1}) x_1 + \alpha_{22k} + \alpha_{22v}(x_2) x_2 + \dots \\ \quad \vdots \\ \quad \vdots \\ \alpha_{nk} + \alpha_{nv}(x_1, x_2, \dots, x_{n-1}) x_1 + \dots + 2x_n \end{array} \right\} \quad (4.7)$$

Through an examination of the requirements on V , the most general gradient of (4.6) has been somewhat simplified in form to that of (4.7). Without loss in generality, the

α_{ii} have been constrained to be functions of x_i alone. With slight loss of generality, one of the α_{ii} , here α_{nn} , has been set equal to an arbitrary constant, and the α_{ijv} have been constrained to be $\alpha_{ijv}(x_1, x_2, \dots, x_{n-1})$. This has been accomplished in view of the future requirements of V . Further knowledge of the unknown coefficients in ∇V is obtainable from an examination of the generalized curl equations, (4.1).

Consider the expanded form of equation (4.1),

$$\begin{aligned} \frac{\partial \nabla V_i}{\partial x_j} = & \frac{\partial \alpha_{ilv}(x_1, x_2, \dots, x_{n-1})x_l}{\partial x_j} + \dots \\ & \frac{\partial \alpha_{ijk}x_j}{\partial x_j} + \frac{\partial \alpha_{ijv}(x_1, x_2, \dots, x_{n-1})x_j}{\partial x_j} + \dots \\ & \frac{\partial \alpha_{inv}(x_1, x_2, \dots, x_{n-1})x_n}{\partial x_j} \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \frac{\partial \nabla V_j}{\partial x_i} = & \frac{\partial \alpha_{jlv}(x_1, x_2, \dots, x_{n-1})x_l}{\partial x_i} + \dots \\ & \frac{\partial \alpha_{jik}x_i}{\partial x_i} + \frac{\partial \alpha_{jiv}(x_1, x_2, \dots, x_{n-1})x_i}{\partial x_i} + \dots \\ & \frac{\partial \alpha_{inv}(x_1, x_2, \dots, x_{n-1})x_n}{\partial x_i} \end{aligned}$$

Here $\frac{\partial \alpha_{jik}x_i}{\partial x_i}$ and $\frac{\partial \alpha_{ijk}x_j}{\partial x_j}$ result in constant terms. If constant terms on either side of the equal sign are equated, it is seen that

$$\alpha_{ijk} = \alpha_{jik}$$

Thus further knowledge of the variable gradient is provided, this time from the curl equations. A knowledge of the necessary values of the remaining unknowns in ∇V can be acquired from a joint consideration of the generalized curl equations and dV/dt .

dV/dt is determined from the variable gradient by means of equation (3.11). In order to satisfy either Theorem 2.2 or 4.3, dV/dt must necessarily be constrained to be at least negative semidefinite. In general, an attempt is made to make dV/dt negative semidefinite in as simple a way as possible. This is accomplished if

$$\frac{dV}{dt} = -K x_i^2 \quad (K > 0) \quad (4.9)$$

where K is initially assumed to be a constant. If dV/dt is constrained as in (4.9), the remaining terms in dV/dt must be forced to cancel. This is accomplished by grouping terms of similar state variables and choosing the α_{ij} 's to force cancellation. The α_{ij} 's are assumed constants, unless cancellation or the generalized curl equations require a more complicated form.

Grouping of terms is guided by the restrictions on the α_{ij} 's stated above. For example, if in a third order system, dV/dt contains the terms $\alpha_{11}x_1x_2$, $\alpha_{12}x_2^2$ and $-x_1x_2^3$, the indefinite term, $-x_1x_2^3$, could not be grouped with

$\alpha_{11}x_1x_2$, as α_{11} can only be a function of x_1 . However, if $-x_1x_2^3$ were grouped with $\alpha_{12}x_2^2$, it could be eliminated by letting $\alpha_{12} = x_1x_2$.

The choice of the α_{ij} 's to force cancellation is not arbitrary, as the generalized curl equations must be satisfied. In fact, if one coefficient is chosen through necessity to eliminate undesirable terms in dV/dt , information concerning the required value of one or more of the unknown coefficients is often supplied directly from the generalized curl equations. Thus dV/dt is constrained to be at least negative semidefinite in conjunction with and subject to the requirements of the generalized curl equations, (4.1).

If it proves to be impossible to constrain dV/dt as in (4.8), it is necessary to attempt to constrain dV/dt to be negative semidefinite in terms of two state variables, then three, etc., until the final attempt is made to force dV/dt to be negative definite. If no solution is yet available, it may be necessary to revert to the more general gradient function of (4.5), or an attempt at a proof of instability may be in order. In problems that have been treated to date, these latter alternatives have not been necessary.

In summary of what has been said in this section, the following outline for the formal application of the variable gradient method is included.

1. Assume a gradient of the form (4.6).

2. From the variable gradient, form dV/dt , as

$$\frac{dV}{dt} = \nabla V' \dot{\underline{x}}, \quad (3.11).$$

3. In conjunction with and subject to the requirements of the generalized curl equations, (4.1), constrain dV/dt to be at least negative semi-definite.
4. From the known gradient, determine V and the region of closedness of V .
5. Invoke the necessary theorem to establish stability.

This procedure is illustrated with examples in the chapter to follow.

4.4 Discussion of the Variable Gradient Method of Generating Liapunov Functions for Autonomous Systems

This chapter has discussed the theoretical considerations upon which the variable gradient approach is based. Whether or not the method as outlined is applicable to problems of interest in automatic control remains to be shown in the following chapter of illustrative examples.

It has been shown here that for all globally asymptotically stable systems whose nonlinearities satisfy the Lipschitz condition, a vector, ∇V , exists from which a scalar V may be determined uniquely by line integration. This scalar V function is capable of proving such stability via

the second method of Liapunov. This conclusion is stated as an existence theorem, Theorem 4.2.

Existence theorems are reassuring, but rarely helpful in solving engineering problems. To say a solution exists does not necessarily imply that it can be found. However, in order to emphasize the possible role of the variable gradient in solving the stability problem, Theorem 2.2 is restated as Theorem 4.3. Here it is emphasized that the gradient enjoys a somewhat unique position, in that both V and dV/dt may be determined directly from ∇V . Furthermore, if V is to be unique, the generalized curl equations (4.1) must be satisfied. Thus, through the introduction of the variable gradient, $(n-1)n/2$ additional equations are also introduced. It is the existence and use of these curl equations that facilitates the search for a suitable V and dV/dt to satisfy Liapunov's theorems.

Initially a gradient function of sufficient generality to embrace all solutions was assumed. However, in view of the future requirements on V , the generality of the gradient was decreased to insure that the resulting V is a quadratic in one of the state variables. Obviously this excludes the generation of V functions such as $V = x_1^4 + x_2^4$, which might well be a suitable solution to a particular problem.

It is difficult to assess exactly how much generality has been lost, particularly in view of the fact that often

an infinite number of V functions exist which are capable of proving stability in any given case. For the types of problems treated in the following chapter, the assumption is apparently not a prohibitive one. For other classes of problems, perhaps different initial assumptions concerning the variable gradient may be in order. However, it is felt that the existence of the curl equations and the ability to determine both V and dV/dt directly from the gradient are significant advantages in attempting to find a suitable Liapunov function.

CHAPTER V

Examples Using the Variable Gradient Method

5.1 Introduction and Organization of the Chapter

The variable gradient approach outlined in the previous chapter is a method for generating Liapunov functions. The ultimate criteria of any method of obtaining problem solutions is not the elegance or generality of the formulation, but rather the applicability of the technique to the class of problems under consideration. This chapter includes examples of increasing complexity to illustrate both the use of the method and the results that are obtainable.

The first four examples serve to illustrate the mechanics of the method and the types of V functions which have been generated.

Example 5.1 is a simple illustrative problem. Example 5.2 considers the Ingwerson third order example that has been discussed in connection with the methods of Ingwerson and Szego. The V functions generated in each of these first two cases includes higher order terms in the state variables. The ease with which integrals appear in the generated V function is illustrated in Example 5.3, and a V function which includes three state variables as factors is produced in Example 5.4.

The remaining examples illustrate the results that are available from the application of the variable gradient

method to several of the more interesting types of problems. Example 5.5 considers two systems, each of which has more than one singularity. A system with a limit cycle is discussed in Example 5.6. The last example is a rather extensive discussion of the so-called "generalized Routh-Hurwitz conditions" for nonlinear systems.

5.2 Examples

Example 5.1

Assume the system is given by the block diagram of Fig. 5.1, such that the equations of motion written in state variable form become, with $x_1 = x$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_1^3$$

Step 1

$$\nabla V = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} \nabla V_1 \\ \nabla V_2 \end{pmatrix}$$

Step 2

$$\begin{aligned} \frac{dV}{dt} &= \nabla V' \dot{\underline{x}} \\ &= x_1 x_2 (a_{11} - a_{21} - 2x_1^2) + x_2^2 (a_{12} - 2) - a_{21} x_1^4 \end{aligned}$$

Step 3

If the given system is stable, there are a large

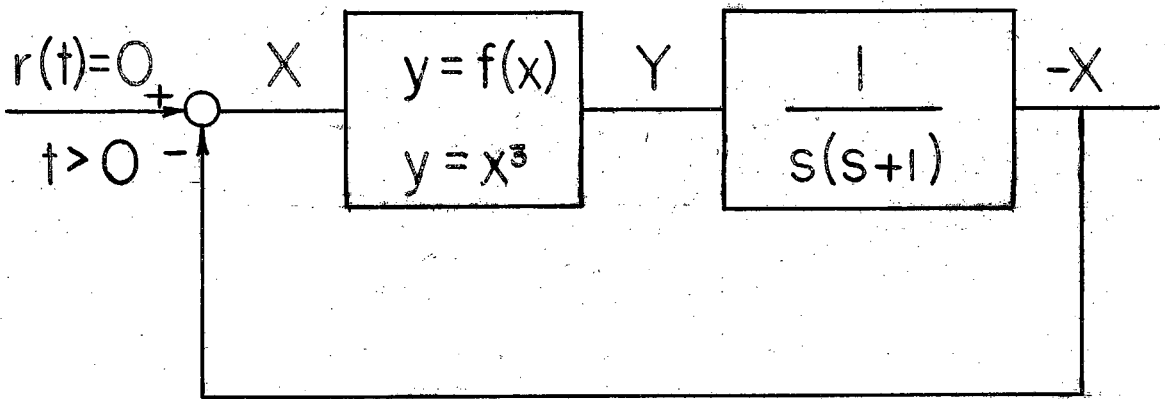


Fig. 5.1. Block Diagram of the Control System of Example 5.1

or even infinite, number of V functions, with a corresponding number of dV/dt 's, which will show the system to be stable. In fact, it is the existence of this large number of suitable Liapunov functions as opposed to the one unique solution of the initial nonlinear differential equation that gives the Liapunov method an advantage over classical methods in the determination of stability.

Here there are a large number of ways in which dV/dt might be constrained in order to prove stability. However, in order to be able to conclude anything about stability, dV/dt must be at least negative semidefinite. In Example 5.1, this can be accomplished by setting the coefficient of x_1x_2 equal to zero and by assuring that x_2^2 and x_1^4 have zero or negative coefficients. The latter can be accomplished if α_{12} is any positive number from 0 to 2, and if α_{21} is any positive number whatever. This is less restrictive than forcing the solutions of the equation $dV/dt = 0$ to coincide, as discussed in Section 3.3. Hence, α_{12} is assumed to be a constant between 0 and 2, and since it is constant, $\alpha_{21} = \alpha_{12}$. With the coefficient of x_1x_2 set equal to zero, dV/dt becomes

$$\frac{dV}{dt} = -x_2^2(2 - \alpha_{12}) - \alpha_{12}x_1^4$$

The requirement that the coefficient of x_1x_2 be zero is satisfied if

$$a_{11} = a_{12} + 2x_1^2$$

Therefore, with these substitutions, ∇V becomes

$$\nabla V = \begin{pmatrix} a_{12}x_1 + 2x_1^3 + a_{12}x_2 \\ a_{12}x_1 + 2x_2 \end{pmatrix}, \quad 0 \leq a_{12} \leq 2$$

Step 4

V is determined from (3.10) to be the line integral

$$V = \int_0^{\underline{x}} \nabla V' d\underline{x} = \int_0^{x_1} (a_{12}\gamma_1 + 2\gamma_1^3) d\gamma_1 + \int_0^{x_2} (a_{12}x_1 + 2\gamma_2) d\gamma_2$$

$$V = \frac{x_1^4}{2} + \frac{a_{12}x_1^2}{2} + a_{12}x_1x_2 + x_2^2, \quad 0 \leq a_{12} \leq 2$$

Step 5

Here V is positive definite and $\lim V \rightarrow \infty$ as the $\|\underline{x}\| \rightarrow \infty$, such that V represents a closed surface in the whole space. Since dV/dt is also at least negative semidefinite in the whole space, by either Theorem 2.2 or 4.3, the system of Fig. 5.1 is globally asymptotically stable.

Example 5.2

This is the third-order example of Ingwerson, the block diagram of which is given in Fig. 3.2. Ingwerson was unable to obtain a solution to this problem when dV/dt was

constrained to be a function of x_3^2 , and the solution achieved by Szego was achieved only through a rather special set of fortunate circumstances, as shown in Section 3.3.2. The equations of motion are repeated here for convenience.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -(x_1 + cx_2)^3 - bx_3\end{aligned}$$

From (4.6), the gradient is written as

$$\nabla V = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + 2x_3 \end{pmatrix}$$

From (3.11), dV/dt in ordered form becomes

$$\begin{aligned}\frac{dV}{dt} &= x_1x_2(a_{11} - a_{32}x_1^2 - 3a_{31}cx_1^2) \\ &+ x_2^2(a_{12} - 3a_{32}cx_1^2 - 3a_{32}c^2x_1x_2 - 3a_{31}c^2x_1^2 - a_{31}c^3x_1x_2 - a_{32}c^3x_2^2) \\ &+ x_1x_3(a_{21} - ba_{31} - 2x_1^2 - \underline{6cx_1x_2} - \underline{6c^2x_2^2}) \\ &+ x_2x_3(a_{22} + a_{13} - a_{32}b - 2c^3x_2^2) \\ &+ x_3^2(a_{23} - 2b) - a_{31}x_1^4\end{aligned}$$

Since the solution is being attempted in terms of x_3^2 , a_{31} is set equal to zero to eliminate the x_1^4 term, and thus a_{13} is also zero. If a_{32} is set equal to zero in order to

force the x_2^4 term to be zero, both a_{12} and a_{21} would have to be zero, and this is not possible. Hence, a_{32} is left undetermined for the moment. Note the two underlined terms above. When removed from the parentheses in which they are now enclosed, these terms contain the three state variables as factors. One might at first wonder exactly how these terms should be grouped, whether they should be with the x_1x_2 terms, the x_2x_3 terms, or with the x_1x_3 terms as they are now located. Under the restrictions placed on the a 's by equation (4.7), only the present location is allowed.

For terms in x_1x_3 to vanish, a_{21} must be equal to

$$a_{21} = 2x_1^2 + 6cx_1x_2 + 6c^2x_2^2$$

Similarly, for terms in x_2x_3 and x_1x_2 to vanish,

$$a_{22} = ba_{32} + 2c^3x_2^2$$

and

$$a_{11} = a_{32}x_1^2$$

Thus far, ∇V has been determined to be

$$\nabla V = \left\{ \begin{array}{l} a_{32}x_1^3 + a_{12}x_2 \\ 2x_1^3 + 6cx_1^2x_2 + 6c^2x_1x_2^2 + ba_{32}x_2 + 2c^3x_2^3 + a_{23}x_3 \\ a_{32}x_2 + 2x_3 \end{array} \right\}$$

and dV/dt has been found to be

$$\frac{dV}{dt} = x_2^2 (\alpha_{12} - 3\alpha_{32}cx_1^2 - 3\alpha_{32}c^2x_1x_2 - \alpha_{32}c^3x_2^2) + x_3^2 (\alpha_{23} - 2b)$$

By means of the curl equation relating ∇V_1 and ∇V_2 , the coefficient α_{12} may be determined. In solving for this coefficient, information regarding α_{32} is automatically obtained. First, both sides of the equation

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

are determined to be

$$\frac{\partial \nabla V_1}{\partial x_2} = x_1^3 \frac{\partial \alpha_{32V}}{\partial x_2} + \alpha_{12K} + \alpha_{12V} + x_2 \frac{\partial \alpha_{12V}}{\partial x_2}$$

$$\frac{\partial \nabla V_2}{\partial x_1} = 6x_1^2 + 12cx_1x_2 + 6c^2x_2^2 + bx_2 \frac{\partial \alpha_{32V}}{\partial x_1} + x_3 \frac{\partial \alpha_{23V}}{\partial x_1}$$

If terms in equal powers are equated, the first result is

$$x_3 \frac{\partial \alpha_{23V}}{\partial x_1} = 0 \quad ; \quad \frac{\partial \alpha_{23V}}{\partial x_1} = 0$$

A possible combination of terms is

$$x_1^3 \frac{\partial \alpha_{32V}}{\partial x_2} = bx_2 \frac{\partial \alpha_{32V}}{\partial x_1}$$

This equation has the solution $\alpha_{32V} = 0$ or $\alpha_{32V} = x_1^4 - 2bx_2^2$.

If the simplest solution is chosen, $\alpha_{32V} = 0$ and

$$\alpha_{23} = \alpha_{32} = \alpha_{32K} + 0$$

Two equations remain, namely

$$a_{12K} = 0$$

$$a_{12V} + x_2 \frac{\partial a_{12V}}{\partial x_2} = 6x_1^2 + 12cx_1x_2 + 6c^2x_2^2$$

Thus the form of a_{12} is known immediately as

$$a_{12} = 0 + a_{12V} = \beta_1 x_1^2 + \beta_2 cx_1x_2 + \beta_3 c^2x_2^2$$

Simple manipulations with the two equations above determine that

$$a_{12V} = 6x_1^2 + 6cx_1x_2 + 2c^2x_2^2$$

The only remaining coefficient to be determined is

$a_{23K} = a_{32K}$. The required value is obtained if a_{12V} above is substituted into dV/dt .

$$\begin{aligned} \frac{dV}{dt} = & x_2^2(6x_1^2 + 6cx_1x_2 + 2c^2x_2^2 - 3a_{32K}cx_1^2 - 3a_{32K}c^2x_1x_2 - a_{32K}c^3x_2^2) \\ & + x_3^2(a_{23K} - 2b) \end{aligned}$$

By equating terms of equal powers, a_{32K} is soon determined to be $2/c$. Now dV/dt is completely known, as is ∇V .

$$\frac{dV}{dt} = - \frac{2x_3^2}{c} (bc - 1)$$

and

$$\nabla V = \left\{ \begin{array}{l} 2/c x_1^3 + 6x_1^2x_2 + 6cx_1x_2^2 + 2c^2x_2^3 \\ 2x_1^3 + 6cx_1^2x_2 + 6c^2x_1x_2^2 + 2b/c x_2 + 2c^3x_2^3 + 2/c x_3 \\ + 2/c x_2 + 2x_3 \end{array} \right\}$$

dV/dt has been constrained to be negative semidefinite and the generalized curl equations are satisfied for ∇V ; in fact, they are the means by which ∇V is determined. All that remains to be done is to determine V and the region for which V represents a closed surface. From (3.12), V is found to be

$$V = \int_0^{x_1} \frac{2}{c} \gamma_1^3 d\gamma_1 + \int_0^{x_2} (2x_1^3 + 6cx_1^2\gamma_2 + 6c^2x_1\gamma_2^2 + \frac{2b}{c}\gamma_2 + 2c^3\gamma_2^3) d\gamma_2$$

$$+ \int_0^{x_3} (\frac{2}{c}x_2 + 2\gamma_3) d\gamma_3$$

After grouping terms, V is

$$V = \frac{b}{c} x_2^2 + \frac{2}{c} x_2 x_3 + x_3^2 + \frac{1}{2c} (x_1 + cx_2)^4$$

The fractions in both V and dV/dt may be removed by multiplying each by the constant c . As a final result

$$cV = V^* = bx_2^2 + 2x_2x_3 + cx_3^2 + \frac{1}{2}(x_1 + cx_2)^4$$

$$cdV/dt = dV/dt^* = -2x_3^2(bc - 1)$$

dV/dt is negative semidefinite and not equal to zero on a solution of the system if $(bc - 1) > 0$, and V is positive definite under the same conditions. V also satisfies the limiting condition as the norm of x goes to in-

finiteness, and hence V represents a closed surface in the whole space. According to Theorem 4.3, the given system is globally asymptotically stable if both b and c are positive and if $(bc - 1) > 0$.

The solution to this problem is lengthy perhaps, but nowhere was the procedure vague or difficult. In the evaluation of α_{12} , it might have been assumed that α_{32} was a constant. Or this fact might have been guessed as in ∇V_2 the coefficient of x_2 was $b\alpha_{32}$, and another term in x_2^3 already exists. This would have reduced the length of the solution, but in no way would have changed the results.

Example 5.3

The two previous examples considered systems in which the nonlinearity was expressed as a polynomial in \underline{x} , and the resulting V functions contained higher order terms in \underline{x} , as opposed to the usual quadratic form for V . This example differs from the first two in that the nonlinearity is not known as a definite function of \underline{x} , and further, the linear portion of the system contains a zero located at an arbitrary point β .

The problem of example three is illustrated by the block diagram of Fig. 5.2. In this synthesis problem it is desired to know the restrictions on the nonlinearity and on β for which the system will be globally asymptotically stable. The problem is considered significant because of

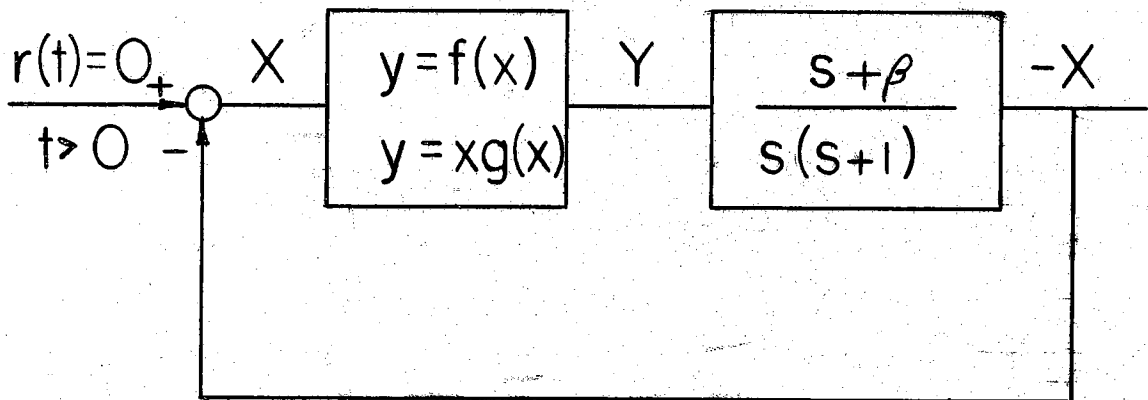


Fig. 5.2. Block Diagram of the Control System of Example 5.3

the integrals that appear naturally in the Liapunov function which is generated.

For $x_1 = x$, the equations of motion are

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \frac{\partial y}{\partial x_1} x_2 - \beta g(x_1) x_1\end{aligned}$$

As before, let

$$\nabla V = \begin{cases} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + 2x_2 \end{cases}$$

so that

$$\begin{aligned}\frac{dV}{dt} &= x_1 x_2 \left[a_{11} - a_{21} \frac{\partial y}{\partial x_1} - a_{21} - 2\beta g(x_1) \right] \\ &\quad - x_2^2 \left[2 + 2 \frac{\partial y}{\partial x_1} - a_{12} \right] - a_{21} \beta g(x_1) x_1^2\end{aligned}$$

If the coefficient of the $x_1 x_2$ term is forced to vanish,

$$a_{11} = a_{21} + a_{21} \frac{\partial y}{\partial x_1} + 2\beta g(x_1)$$

and

$$\nabla V = \begin{cases} a_{21}x_1 + a_{21} \frac{\partial y}{\partial x_1} x_1 + 2\beta g(x_1)x_1 + a_{12}x_2 \\ a_{21}x_1 + 2x_2 \end{cases}$$

The optimum choice of $a_{12} = a_{21}$, a constant, is best seen from a joint examination of V and dV/dt . As before, from (3.10), V is

$$V = \frac{\alpha_{21}}{2} x_1^2 + \alpha_{21} x_1 x_2 + x_2^2 + \alpha_{21} \int_0^{x_1} \frac{\partial y}{\partial \gamma_1} \gamma_1 d\gamma_1 + 2\beta \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1$$

dV/dt has not changed. It is seen from dV/dt that if α_{12} is 0, then dy/dx , the slope of the nonlinearity, may take on a maximum negative slope of unity before the x_2^2 term changes its sign. Hence it might be decided to let α_{12} be just that. However, if this is done, V becomes

$$V = x_2^2 + 2\beta \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1$$

V is positive definite if the integral is always positive, and V represents a closed surface in the whole plane if the integral goes to infinity as the upper limit goes to infinity. To remove this latter restriction on closedness, α_{12} might be chosen as the arbitrarily small number ϵ . Then the allowable minimum slope of the nonlinearity, as determined in dV/dt , is not changed significantly, yet V is closed in the whole space independent of the integrals, as long as they are positive. Since the nonlinearity was specified as $y = xg(x)$, $g(x)$ is always positive if the nonlinearity lies in the first and third quadrant, and, under these conditions, the integral involving $g(x_1)$ is always positive.

The final form of V and dV/dt is then

$$V = \frac{\epsilon}{2} x_1^2 + \epsilon x_1 x_2 + x_2^2 + \epsilon \int_0^{x_1} \frac{\partial y}{\partial \gamma_1} \gamma_1 d\gamma_1 + 2\beta \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1$$

$$\frac{dV}{dt} = -2x_2^2 \left(1 + \frac{\partial y}{\partial x_1} - \frac{\epsilon}{2}\right) - \epsilon \beta g(x_1) x_1^2$$

It is seen that as long as β is positive, or the zero is in the LHP, the value of β is not important. As mentioned, the problem is included as an example to illustrate the ease with which integrals are introduced into V without having to guess their existence beforehand.

Example 5.4

Example 5.4 is artificial in the sense that the block diagram, Fig. 5.3, which corresponds to the dynamic equations of motion of the system, contains five loops and is not a system that might be expected to be encountered in practice. However, the system does contain more than one nonlinear element, and it is particularly interesting because the linearized first approximation of the system, as determined by dropping all higher order terms, has poles on the $j\omega$ axis of the s plane. Hence, the linearized first approximation of the system yields no information concerning the stability of the actual nonlinear system. It is shown in this example that the exact nonlinear system is asymptotically stable in the entire state space; that is, it is globally asymptotically stable.

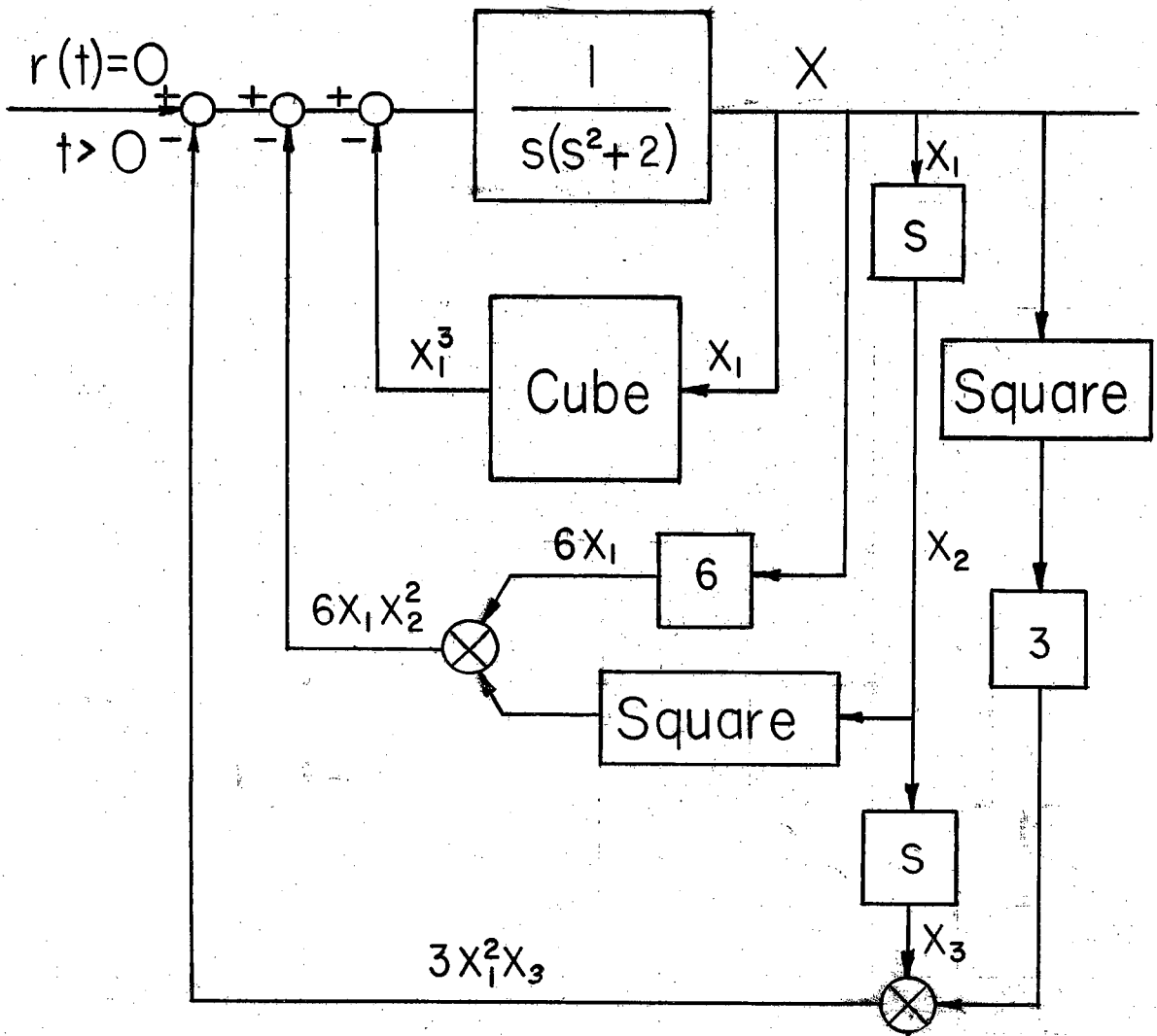


Fig. 5.3. Block Diagram of the Control System of Example 5.4

The problem is interesting from another point of view. The V function that proves asymptotic stability contains a term $6x_1^2 x_2 x_3$. However, the presence of 3 state variables as factors does not alter the procedure that has been previously established. The problem is solved in exactly the same way.

The equations of motion corresponding to the system in Fig. 5.3 are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -3x_1^3 x_3 - 2x_2 - 6x_1 x_2^2 - x_1^3$$

Here the large number of negative terms in dV/dt is reduced by allowing one of the α_{3j} 's to be zero. In this case α_{31} is set equal to zero, and as the ultimate objective, dV/dt is constrained to be a function of x_2^2 . Therefore, the negative term in $x_1^2 x_3^2$ is cancelled by setting α_{23} equal to $6x_1^2$, and one curl equation is used to determine that α_{32} is also $6x_1^2$. A second of the curl equations determines that α_{13} is $12x_1 x_2$, and with these substitutions, dV/dt is found to be

$$\begin{aligned} \frac{dV}{dt} = & x_1 x_2 (\alpha_{11} - 6x_1^4) + x_2^2 (\alpha_{12} - 36x_1^3 x_2) - \\ & - 12x_1^2 x_2^2 + x_2 x_3 (\alpha_{22} - 4) \\ & + x_1 x_3 (\alpha_{21} - 2x_1^2 - 18x_1^3 x_2) \end{aligned}$$

Terms in x_1x_2 , x_2x_3 and x_1x_3 can be eliminated by setting

$$\alpha_{11} = 6x_1^4$$

$$\alpha_{22} = 4$$

$$\alpha_{21} = 2x_1^2 + 18x_1^3x_2$$

The term in $x_1^3x_2^3$ can be forced to vanish by causing α_{12} to equal $36x_1^3x_2$. Hence the attempt here has been to force dV/dt to be

$$\frac{dV}{dt} = -12x_1^2x_2^2$$

by using the gradient function

$$\nabla V = \left\{ \begin{array}{l} 6x_1^5 + 36x_1^3x_2^2 + 12x_1x_2x_3 \\ 2x_1^3 + 18x_1^4x_2 + 4x_2 + 6x_1^2x_3 \\ 6x_1^2x_2 + 2x_3 \end{array} \right\}$$

That this is not a satisfactory gradient function can be seen by applying the remaining curl equation

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

$$\frac{\partial \nabla V_1}{\partial x_2} = 72x_1^3x_2 + 12x_1x_3$$

$$\frac{\partial \nabla V_2}{\partial x_1} = 6x_1^2 + 72x_1^3x_2 + 12x_1x_3$$

These are not equal, and it is seen that α_{12V} must contain

more than one term, as did a_{21v} . The second term is determined from the equation directly above as $6x_1^2$. Hence, the final value of ∇V is

$$\nabla V = \begin{pmatrix} 6x_1^5 + 36x_1^3x_2^2 + 6x_1^2x_2 + 12x_1x_2x_3 \\ 2x_1^3 + 18x_1^4x_2 + 4x_2 + 6x_1^2x_3 \\ 6x_1^2x_2 + 2x_3 \end{pmatrix}$$

From the ∇V , dV/dt and V are determined in the same manner as before.

$$V = x_1^6 + 2x_1^3x_2^2 + 9x_1^4x_2^2 + 2x_2^2 + 6x_1^2x_2x_3 + x_3^2$$

$$\frac{dV}{dt} = -6x_1^2x_2^2$$

Using geometric considerations, (see Appendix) it is possible to show that V satisfies the conditions of Theorem 2.2. dV/dt is negative semidefinite, and the system is globally asymptotically stable.

Example 5.5

The block diagram of Fig. 5.4 pictures a non-minimum phase control system whose dynamic equations of motion are, with $K = 0$, $\delta = 1$, $\gamma = -1$ and $\beta = 2$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + 2x_1 - x_1^3$$

The configuration of the given system is such that the des-

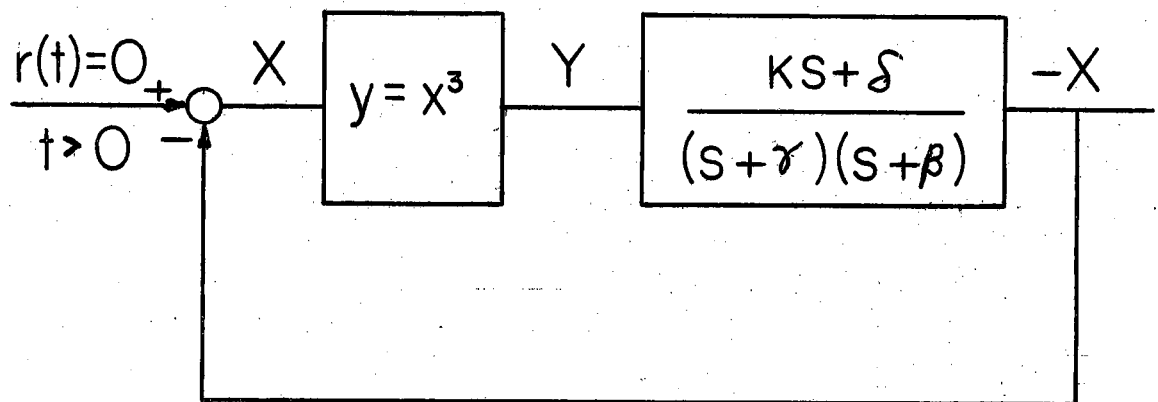


Fig. 5.4. Block Diagram of the Control System of Example 5.5

cribing equations above contain singularities not only at the origin, but at $\pm \sqrt{2}$, and the linearized first approximation indicates that the solution is unstable in the neighborhood of the origin. This information need not be known in advance, as it is included in the rather interesting solution of this problem. With ∇V as in (4.7), dV/dt is found to be

$$\begin{aligned} \frac{dV}{dt} = & x_1 x_2 (\alpha_{11} - \alpha_{21} + 4 - 2x_1^2) \\ & + x_2^2 (\alpha_{12} - 2) + 2\alpha_{21} x_1^2 - \alpha_{21} x_1^4 \end{aligned}$$

If an attempt is made to constrain dV/dt in terms of x_1 , no choice of α_{21} is possible, such that dV/dt will be at least semidefinite in the whole plane. However, if α_{21} is allowed to be 0, and

$$\alpha_{11} = 2x_1^2 - 4$$

then

$$\frac{dV}{dt} = -2x_2^2$$

and

$$\nabla V = \begin{pmatrix} 2x_1^3 - 4x_1 \\ 2x_2 \end{pmatrix}$$

By integrating in the usual manner, the resulting V is

$$V = \frac{x_1^4}{2} - 2x_1^2 + x_2^2$$

For very small values of x_1 , the fourth-power term above is negligible compared to the second-power term, and may be neglected. The remaining quadratic form is not a definite function, and hence does not represent a family of closed curves about the origin, no matter how small the neighborhood. Geometric considerations, however, indicate that the curve is indeed closed, though not around the origin, and a family of these V curves is plotted in Fig. 5.5. The curve $V = 0$ bounds the region Ω of Theorem 2.3. Since dV/dt is negative in the whole plane, any solution starting within the curve $V = 0$ will proceed to the enclosed singularity as time runs to infinity. It is impossible to say whether a solution starting outside of the curve $V = 0$ will terminate at the singularity located at $+\sqrt{2}$ or $-\sqrt{2}$. It will definitely not terminate at the origin, since the equations of first approximation determine the origin to be unstable.

Thus for the choice of constants that was initially made, a complete analysis of the system requires an evaluation including negative values of V .

If the constants in Fig. 5.4 are chosen so that $K = 1$, $\delta = -1$, $\gamma = 1$, $\beta = 2$, the equations of motion of the system are

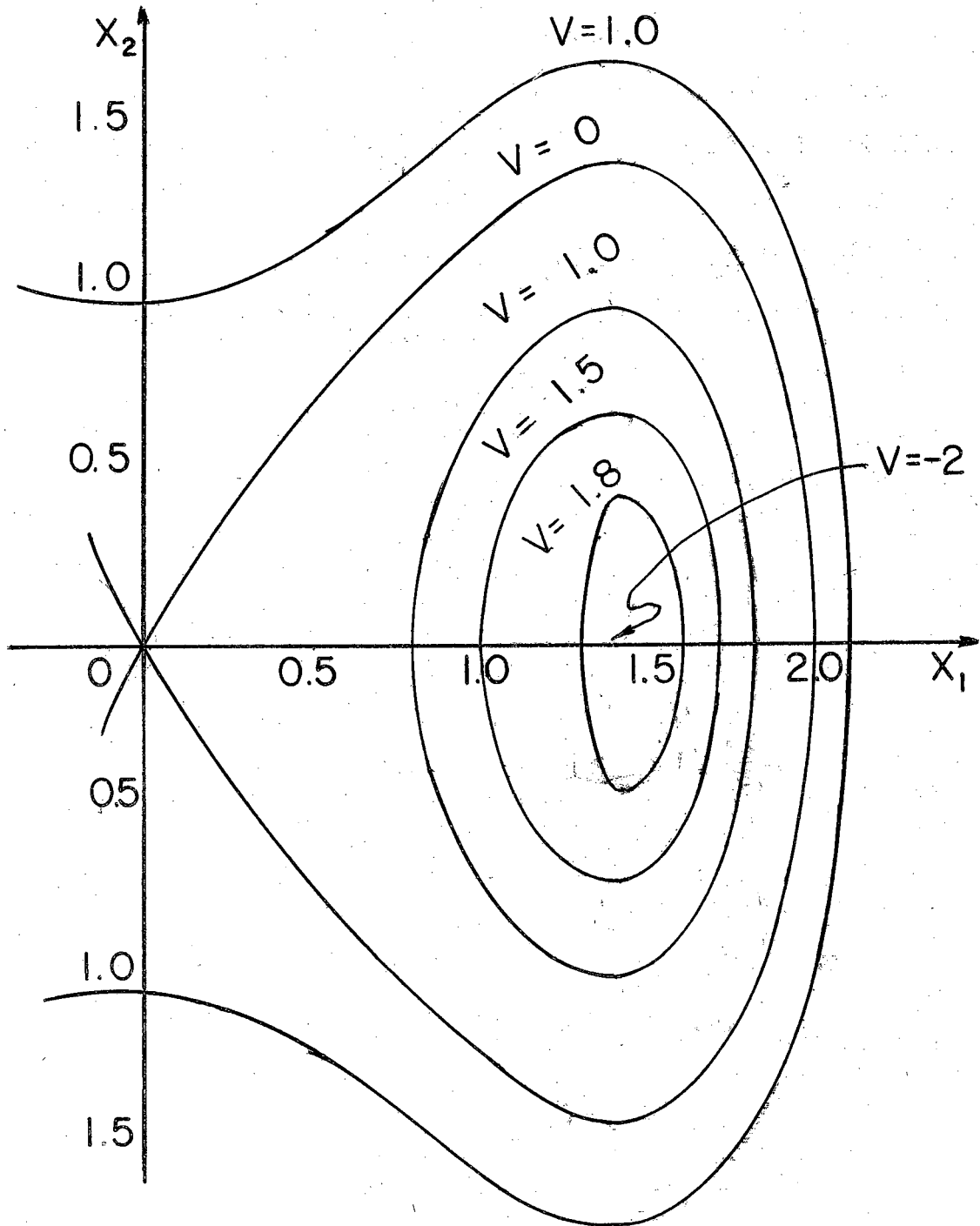


Fig. 5.5. V Curves of Example 5.5

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -3(x_1^2 + 1)x_2 - 2x_1 + x_1^3$$

This time the origin is stable and the two nodes at $\pm\sqrt{2}$ are unstable. An analysis almost identical to that above results in a V and dV/dt of

$$V = 2x_1^2 - \frac{x_1^4}{2} + x_2^2$$

$$\frac{dV}{dt} = -3x_2^2(x_1^2 + 1)$$

Again the plot of V thus determined is quite unusual. For all values of V from $V = 0$ to $V = 2$, the equations actually represent three disconnected curves, as may be seen from Fig. 5.6. In this case the region Ω of Theorem 2.3 is bounded by the curve $V = 2$ for $|x_1| < \sqrt{2}$. All trajectories that enter this region approach the origin at $t \rightarrow \infty$. Several typical trajectories, as determined by the isocline method, are superimposed on the plot of the V curves in Fig. 5.6. The behavior of these trajectories agrees with the interpretation that results from viewing the V curves alone.

Example 5.6

The application of Liapunov's second method to second-order systems with limit cycles has been considered in papers by Szego [4], Ingwerson [2], LaSalle [15], and in

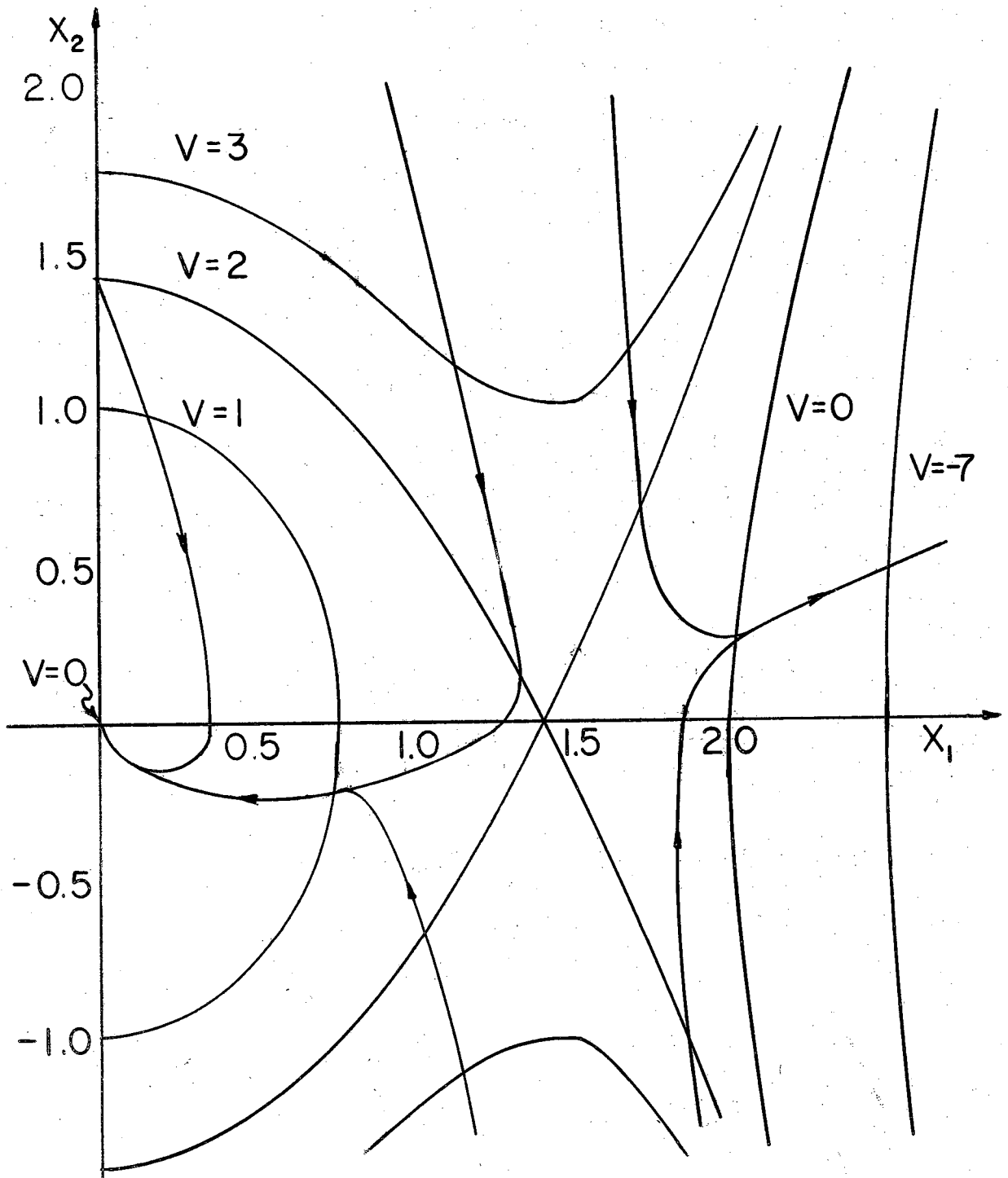


Fig. 5.6. V Curves and Trajectories of the Alternate Example of Example 5.5

the recent book by [Graham and McRuer 24]. The first two authors make use of the phase variables, which have been used exclusively in this report thus far, while the remaining authors use a more general state variable. The application of the variable gradient method is independent of the coordinate system, as is demonstrated in this example of the Lewis servomechanism [Graham, 24, p. 360] in which the solution is obtained in both coordinate systems.

A possible block diagram of the Lewis servomechanism is given in Fig. 5.7, and the equation governing the dynamics of the system is

$$\ddot{x} + 2 \int (1 - a|x|) \dot{x} + x = 0$$

As Graham points out, this is a special case of the Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

for which La Salle [15, p. 23] has recommended the change in variable

$$y = \dot{x} + \int_0^x f(x) dx$$

With this substitution, the two first-order equations of motion become

$$\begin{aligned} \dot{x} &= y - \int_0^x f(x) dx \\ \dot{y} &= -g(x) \end{aligned}$$

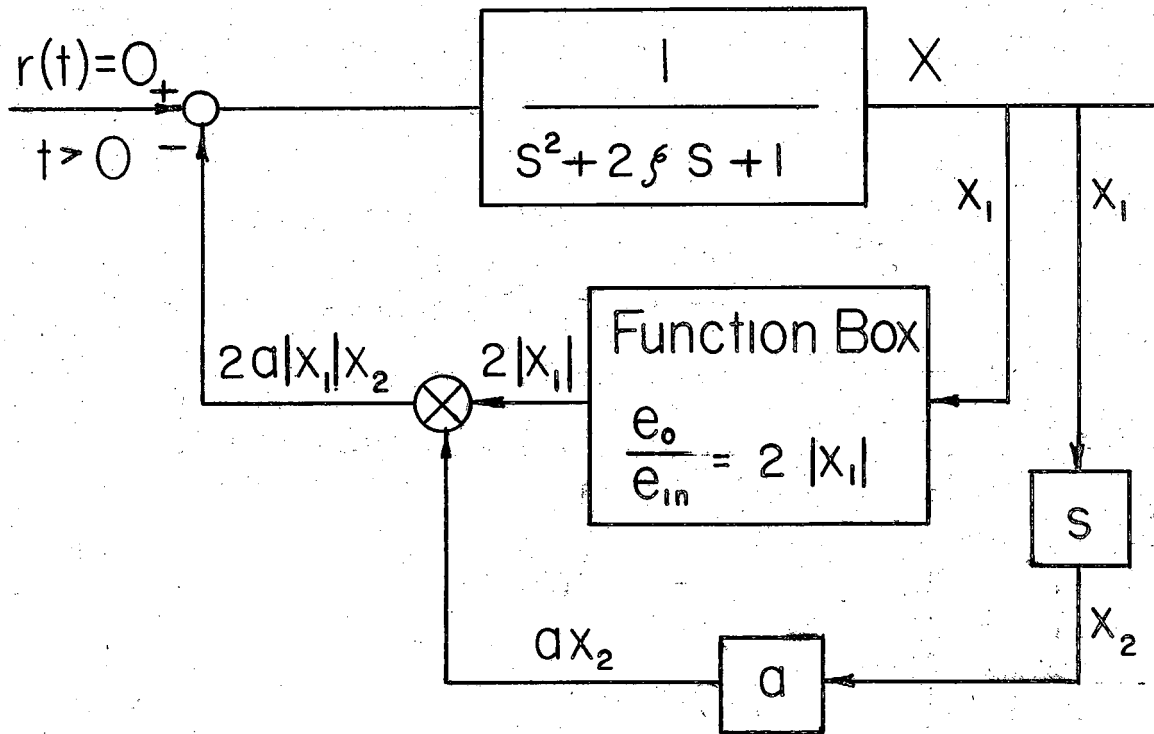


Fig. 5.7. Block Diagram of the Control System of Example 5.6

Here the variable y is no longer the velocity, but the velocity plus an integral involving the nonlinearity. Hence the x, y plane no longer represents the phase plane. For the specific problem under consideration, the two first-order equations are

$$\dot{x} = y - x + \frac{x^2}{2}$$

$$\dot{y} = -x$$

Here the equations are normalized with $2\} = a = 1.0$. From the variable gradient, dV/dt is found to be

$$\begin{aligned} \frac{dV}{dt} = & xy(\alpha_{11} - 2 - \alpha_{12}) - x^2(\alpha_{11} + \alpha_{21}) \\ & + \alpha_{11} \frac{x^3}{2} + \alpha_{12} y^2 + \frac{\alpha_{12} x^2 y}{2} \end{aligned}$$

If dV/dt is to be negative semidefinite in any region, α_{12} must be set equal to zero. With $\alpha_{12} = 0$, $\alpha_{21} = 0$, and if α_{11} is 2, dV/dt becomes

$$\frac{dV}{dt} = -x^2(2 - x)$$

∇V is simply

$$\nabla V = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

and V is found by line integration to be

$$V = x^2 + y^2$$

V is the equation of a circle in the x, y plane, and the

given physical system is asymptotically stable within the radius 2 of a circle in the x, y plane. Any limit cycle must lie outside of this circle.

A similar solution is obtained through the use of phase coordinates which, with x_1 equal to x , describe the system as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + |x_1| x_2 - x_1$$

Proceeding as above from the variable gradient, dV/dt is

$$\begin{aligned} \frac{dV}{dt} = & x_1 x_2 (\alpha_{11} - \alpha_{21} + \alpha_{21} |x_1| - 2) \\ & + x_2^2 (\alpha_{12} - 2 + 2|x_1|) - \alpha_{21} x_1^2 \end{aligned}$$

A decision to constrain dV/dt to be negative semidefinite in terms of x_2^2 results in a dV/dt which is negative only within the range $-1 < x_1 < 1$, an answer that agrees with the results obtained from the application of Bendixson's first theorem [Graham, 24, p. 350]. A better solution is obtained if dV/dt is constrained in terms of x_1^2 . Toward this end, α_{12} is set equal to

$$\alpha_{12} = 2 - 2|x_1|$$

The application of the only curl equation that applies in this second order case determines that α_{21} is

$$\alpha_{21} = 2 - |x_1|$$

With these substitutions, the x_1x_2 term in dV/dt is cancelled by allowing α_{11} to be

$$\alpha_{11} = 4 - 3|x_1| + x_1^2$$

and thus dV/dt is constrained to be

$$\frac{dV}{dt} = -x_1^2(2 - |x_1|)$$

The coefficients in the gradient whose values were initially unknown have now been determined, and the gradient is

$$\nabla V = \begin{pmatrix} 4x_1 - 3|x_1|x_1 + x_1^3 + 2x_2 - 2|x_1|x_2 \\ 2x_1 - |x_1|x_1 + 2x_2 \end{pmatrix}$$

V is determined from the usual line integration to be

$$V = 2x_1^2 - x_1^3 + \frac{x_1^4}{4} + 2x_1x_2 - |x_1|x_1x_2 + x_2^2$$

V is a closed curve within the range for which dV/dt is negative semidefinite. This curve, $V = 4$, is identical with that obtained using the coordinates recommended by LaSalle, if the indicated change of variables is made. The results are indicated in Fig. 5.8, which was taken directly from Graham and McRuer [24, p. 351]. It is seen that the curve $V = 4$ closely resembles the limit cycle, while the conclusion based on Bendixson's theorem indicates that no limit cycle exists between $x_1 = \pm 1$. This latter conclusion, while true, gives little information.

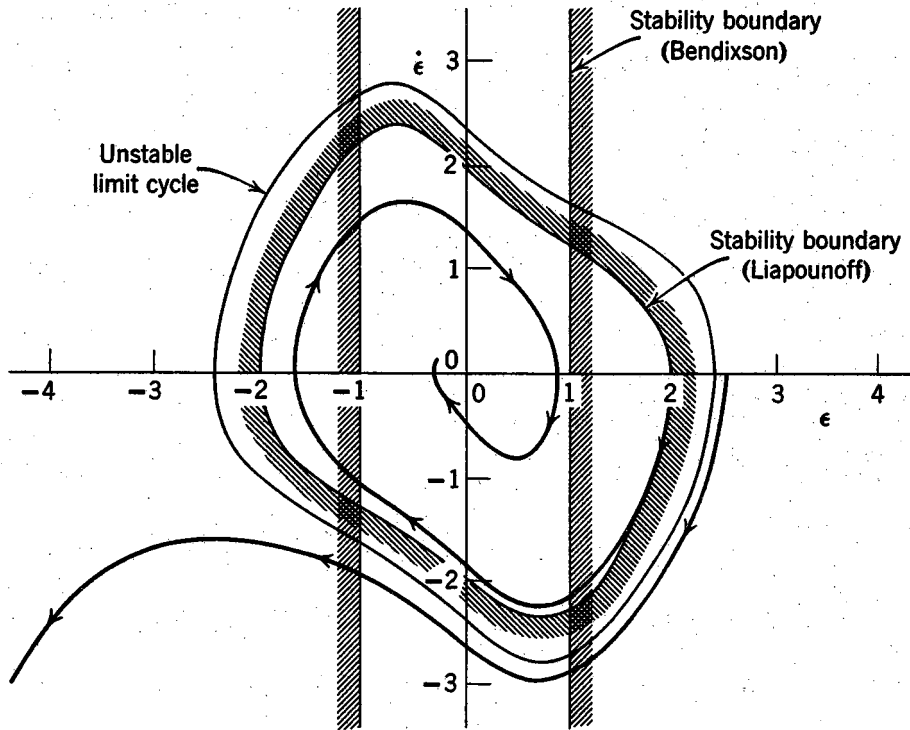


Fig. 5.8. Estimates of the Region of Stability Given by Bendixson's First Theorem and by the Second Method of Liapounoff (From [24, p. 351])

In some cases, as, for example, in the van der Pol equation, it is possible to find a surface over which dV/dt is zero. In such cases the limit cycle can be bracketed by V curves tangent inside and outside to the $dV/dt = 0$ curve [Szego, 4].

Example 5.7

The last of the examples to be included in this section on autonomous systems is the so-called "Aizerman problem." Simply stated, the problem is to determine a "generalized Hurwitz" criteria for nth order nonlinear systems of the form

$$x^n + a_n(\underline{x})x^{n-1} + a_{n-1}(\underline{x})x^{n-2} + \dots + a_1(\underline{x})x = 0$$

where the coefficients are not constants but functions of the state variables. This problem has been considered by Aizerman [25] and by Hahn [26], and solutions to different phases of the problem have been contributed by Ingwerson [2], LaSalle [15], and Barbashin [11]. The discussion here is restricted to second and third order systems.

Consider the rather general second order nonlinear differential equation

$$\ddot{x} + A(x, \dot{x})\dot{x} + B(x)x = 0$$

In terms of the phase variables, the given second-order equation is equivalent to the following two, first-order equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -A(x_1, x_2)x_2 - B(x_1)x_1$$

Starting from the variable gradient, (4.7), dV/dt is determined to be

$$\begin{aligned} \frac{dV}{dt} = & x_1 x_2 \left[\alpha_{11} - \alpha_{21} A(x_1, x_2) - 2B(x_1) \right] \\ & + x_2^2 \left[\alpha_{12} - A(x_1, x_2) \right] - \alpha_{21} B(x_1) x_1^2 \end{aligned}$$

The most general result is achieved when $\alpha_{12} = \alpha_{21} = 0$, and if the $x_1 x_2$ term is caused to vanish, dV/dt becomes

$$\frac{dV}{dt} = -2A(x_1, x_2)x_2^2$$

and ∇V is

$$\nabla V = \left\{ \begin{array}{l} 2B(x_1)x_1 \\ 2x_2 \end{array} \right\}$$

V is once again determined by a line integration, and the result is

$$V = 2 \int_0^{x_1} B(\gamma_1)\gamma_1 d\gamma_1 + x_2^2$$

and

$$\frac{dV}{dt} = -2A(x_1, x_2)x_2^2$$

If the coefficients $A(x_1, x_2)$ and $B(x_1)$ were constants, the Routh-Hurwitz condition for stability of the given dif-

ferential equation would be that the two coefficients be positive. If the two coefficients, now a function of \underline{x} , are positive for all \underline{x} , the V and dV/dt determined above are positive definite and negative semidefinite respectively. Thus the system described by the given differential equation is asymptotically stable in a region about the origin. If the integral in V goes to infinity as the norm of \underline{x} goes to infinity, then V represents a closed surface in the whole space, and the system is globally asymptotically stable.

In a sense, the condition imposed on the integral is an additional requirement to the usual Routh-Hurwitz condition that the coefficients be positive. In another sense, it may appear less restrictive, as here $B(x_1)$ seemingly need not be always positive, as long as the integral is positive for all x_1 . The system of Fig. 5.9 is such a system. The differential equation describing the system is

$$\ddot{x} + \dot{x} + x(1 - x^2 + \frac{x^4}{4.5}) = 0$$

Here $A(x_1, x_2)$ is simply unity and $B(x_1)$ is

$$B(x_1) = 1 - x_1^2 + \frac{x_1^4}{4.5}$$

A plot of $B(x_1)$ is pictured in Fig. 5.10, and in the range from 1.24 to $\sqrt{3}$, $B(x_1)$ is actually negative. However, the integral

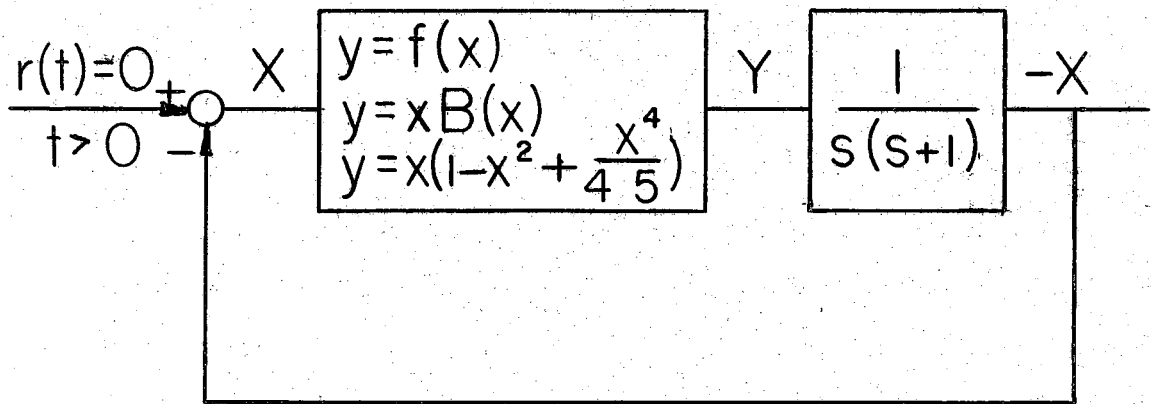


Fig. 5.9. A Nonlinear System Which Apparently Violates the So-Called "Generalized Hurwitz Criteria"

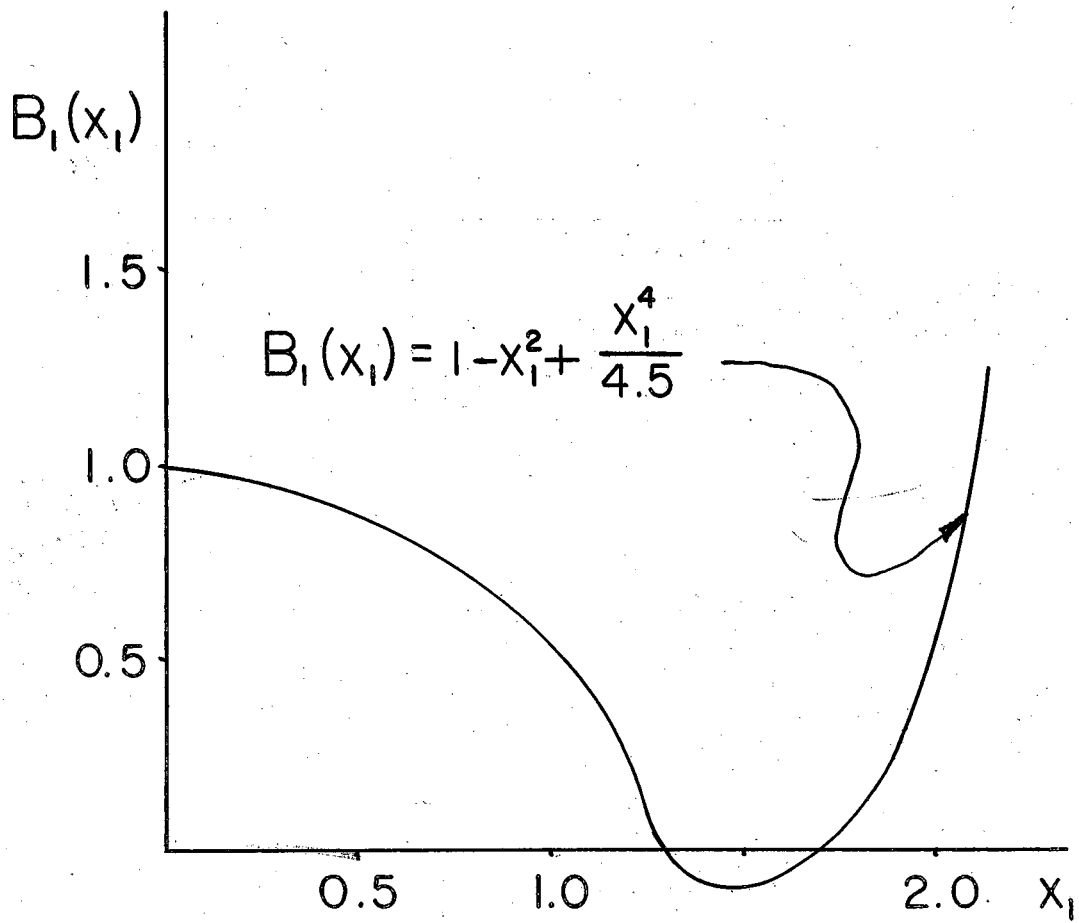


Fig. 5.10. Graph of the Nonlinearity of Fig. 5.9

$$\int_0^{x_1} B(\gamma_1) \gamma_1 d\gamma_1$$

is positive for all x_1 , the least value of the integral being .25 at $x_1 = \sqrt{3}$. Here V is always greater than zero for $\underline{x} \neq 0$, and V also goes to infinity as $\|\underline{x}\| \rightarrow \infty$. Under the assumption that $A(x_1, x_2)$ is always greater than zero, dV/dt is negative semidefinite. The conditions of Theorem 2.2 are apparently satisfied, and one is tempted to conclude global asymptotic stability. If this were true, the usual Routh-Hurwitz conditions that A and B be greater than zero would be violated. In this case global asymptotic stability may not be concluded, as the given equation has four additional singularities in addition to the equilibrium point at the origin. Theorem 2.2 does not apply. In general, if $B(x_1)$ ever becomes negative, the system will have more than one stable or unstable equilibrium point.

For the third order linear system of the form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -Ax_3 - Bx_2 - Cx_1$$

the Routh-Hurwitz criteria requires that for stability, $AB - C > 0$. If the coefficients A , B , and C are not

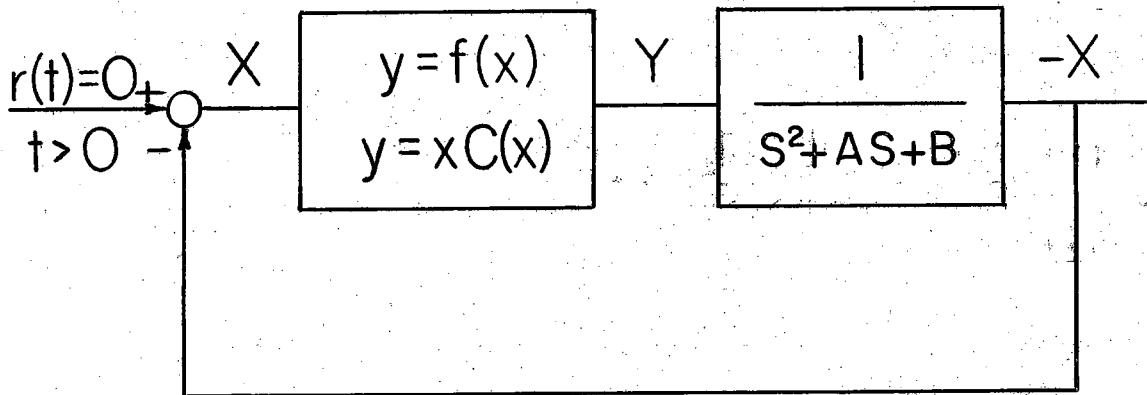
constants, but are functions of the state variables, the question arises, as in the second-order case, if the Routh-Hurwitz conditions are satisfied for all \underline{x} , is the system stable?

This question has been considered by several investigators, and the following information pertaining to their results is presented on the following pages.

1. The block diagram of the system.
2. The differential equation of the system.
3. The V function which proved the system asymptotically stable.
4. The dV/dt determined from the given Liapunov function V .
5. The reference.

In each of the cases cited on these pages, the results were presented by the various authors with only slight justification for the assumptions made in forming the Liapunov function, V . Through the use of the variable gradient, it becomes evident why it is possible to obtain the results above, and further, how these results may be extended.

The basis of the discussion to follow is the general derivative as determined from the variable gradient for the third order system above. In this general derivative, the coefficients of the differential equation are written



1. Block Diagram of the Ingwerson System

2. Differential Equation, with $x_1 = x$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

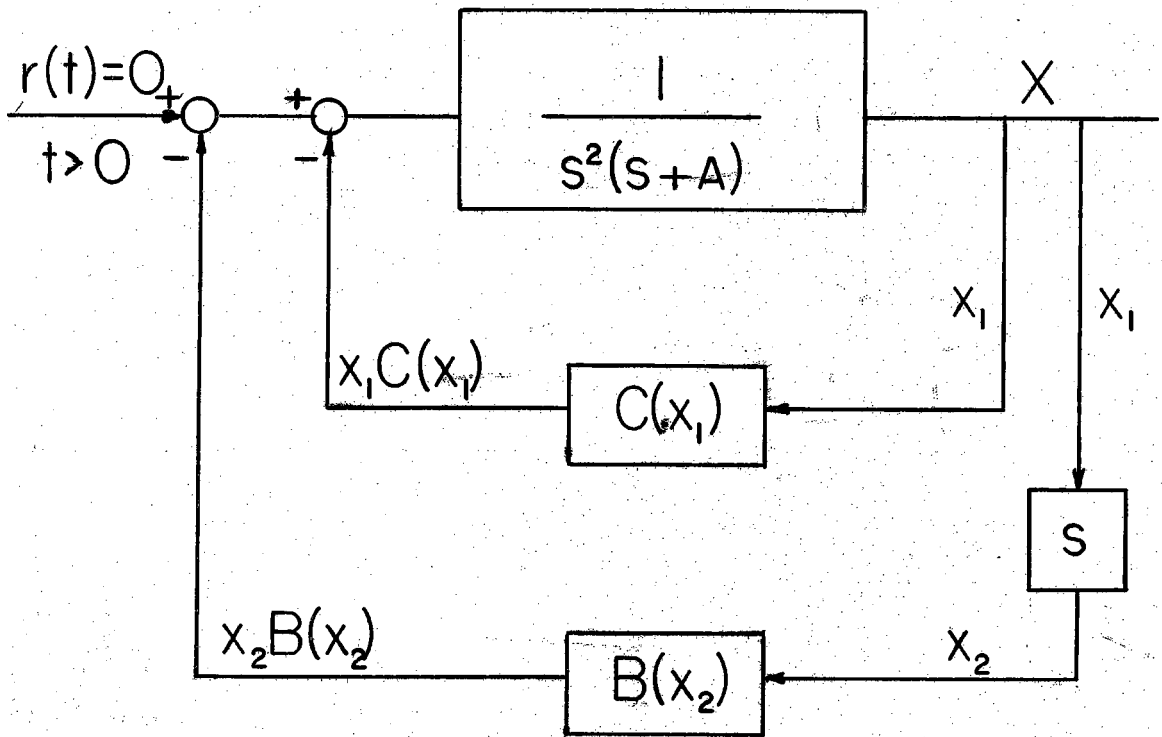
$$\dot{x}_3 = -Ax_3 - Bx_2 - C(x_1)x_1$$

$$3. V = A \int_0^{x_1} C(\gamma_1)\gamma_1 d\gamma_1 + \frac{B^2}{2} x_1^2 + ABx_1x_2 + \frac{A^2 + B}{2} x_2^2 + Bx_1x_3 + Ax_2x_3 + x_3^2$$

$$4. dV/dt = -BC(x_1)x_1^2 + 2C(x_1)x_1x_3 + Ax_3^2$$

5. Reference, Ingwerson, 1, 2

Fig. 5.11. The Ingwerson Example



1. Block diagram of the Barbashin System
2. Differential Equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

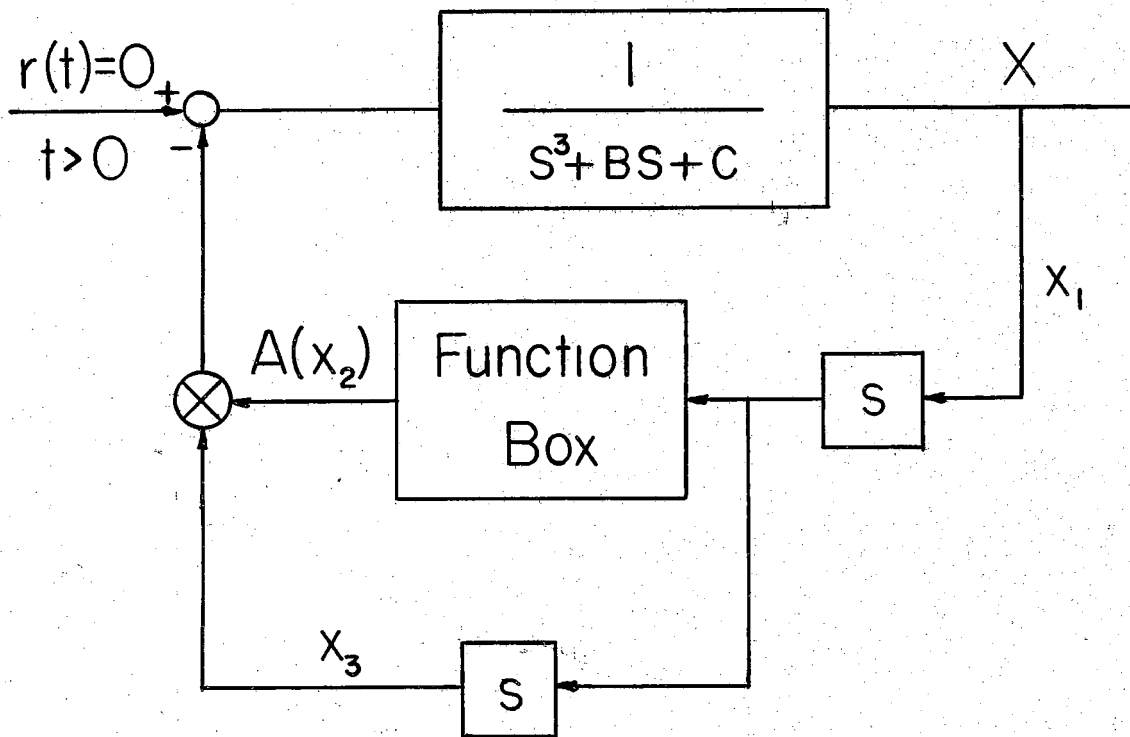
$$\dot{x}_3 = -Ax_3 - B(x_2)x_2 - C(x_1)x_1$$

$$3. \quad V = 2A \int_0^{x_1} C(\gamma_1)\gamma_1 d\gamma_1 + 2C(x_1)x_1x_2 + [A^2 + B(x_2)]x_2^2 + 2Ax_2x_3 + x_3^2$$

$$4. \quad dV/dt = -2x_2^2 [AB(x_2) - C(x_1)] + 2x_1 \frac{\partial C(x_1)}{\partial x_1} x_2^2$$

5. Reference, [Kalman, 13, p. 384]

Fig. 5.12. The Example of Barbashin



1. Block Diagram of the Example of La Salle
2. Differential Equation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -A(x_2)x_3 - Bx_2 - Cx_1$$

$$3. \quad V = \frac{C^2}{B} x_1^2 + 2Cx_1x_2 + Bx_2^2 + \frac{2C}{B} x_2x_3$$

$$+ x_3^2 + \frac{2C}{B} \int_0^{x_2} A(\gamma_2)\gamma_2 d\gamma_2$$

$$4. \quad dV/dt = \frac{-2x_3^2}{B} [A(x_2) B - C]$$

5. Reference La Salle 15

Fig. 5.13. The Example of La Salle

as though they are constants. In the discussion to follow, one or more of these coefficients will be allowed to be functions of the state variables.

The general dV/dt is

$$\begin{aligned} \frac{dV}{dt} = & x_1 x_2 (\alpha_{11} - B\alpha_{31} - C\alpha_{32}) \\ & + x_2 x_3 (\alpha_{13} + \alpha_{22} - A\alpha_{32} - 2B) \\ & + x_1 x_3 (\alpha_{21} - A\alpha_{31} - 2C) \\ & - C\alpha_{31} x_1^2 + x_2^2 (\alpha_{12} - B\alpha_{32}) - x_3^2 (2A - \alpha_{23}) \end{aligned}$$

Consider the Ingwerson example, which corresponds to a rather practical automatic control system configuration. The significant feature of the solution of this problem is not the V function itself, but rather the manner in which it was possible to constrain dV/dt . dV/dt is constrained in terms of x_1 and x_3 . Why this is possible is evident from careful consideration of the general derivative above, where C is now considered to be a function of x_1 , or $C = C(x_1)$. $C(x_1)$ appears in the $x_1 x_2$ term along with α_{11} , which may be a function of x_1 . Hence the $x_1 x_2$ term may be caused to vanish by letting

$$\alpha_{11} = B\alpha_{31} + C(x_1)\alpha_{32}$$

Consequently, an integral appears in V , since $C(x_1)$ is not known explicitly. $C(x_1)$ also appears in the $x_1 x_3$ term, and

this coefficient may be retained as long as terms in x_1^2 and x_3^2 are also retained. Thus in the x_1x_3 term, α_{21} is allowed to be $A\alpha_{31}$. If α_{31} is not allowed to be zero, the x_1^2 term does not vanish, and the remaining constants are determined rather mechanically to obtain Ingwerson's result, as indicated in Fig. 5.11.

This result was possible for two reasons. $C(x_1)$ appeared as a coefficient of the same term as α_{11} , and hence could be cancelled by α_{11} . $C(x_1)$ did not appear as a coefficient of a term which also had α_{22} as a coefficient. If this had been the case, no cancellation would be possible, as α_{22} can not be a function of x_1 . These points are emphasized in the following paragraphs.

An alternate solution is possible for this problem. The $C(x_1)$ term in x_1x_3 may be forced to vanish by letting α_{21} be equal to $2C(x_1)$, with $\alpha_{31} = 0$, and thus the system may be constrained in terms of x_2^2 alone. Here it is interesting to note that since α_{21} is a function of x_1 , the curl equation

$$\frac{\partial \nabla v_2}{\partial x_1} = \frac{\partial \nabla v_1}{\partial x_2}$$

requires that α_{12} be

$$\alpha_{12} = 2C(x_1) + 2x_1 \frac{\partial C(x_1)}{\partial x_1}$$

Thus dV/dt , as determined from the gradient containing this additional partial derivative term, is

$$\frac{dV}{dt} = - 2x_2^2 [AB - C(x_1)] + 2x_2^2 x_1 \frac{\partial C(x_1)}{\partial x_1}$$

In systems where the nonlinearity is of the saturating type, as, for example, $y = \arctan x_1$ or $\arctan x_1$ plus some kx_1 , the last term is always negative. This alternate solution, of course, is less general than the Ingwerson result. The point is that from an examination of the general derivative, as determined from the variable gradient, more than one means of attacking the problem is evident.

Further examination of the general dV/dt reveals that the term B, if allowed to be $B(x_2)$, enjoys the same unique situation as $C(x_1)$ did above, if at the same time α_{13} is set equal to 0. With $\alpha_{31} = 0$, α_{11} will not contain a term in x_2 from $B(x_2)$. Then $B(x_2)$ in the x_2x_3 term may be cancelled with the α_{22} coefficient, which is allowed to be a function of x_2 , and dV/dt may be constrained in terms of x_2^2 . This result is contained in the Barbashin result quoted by Kalman.

The coefficient A, if allowed to be $A(x_2)$, is in an identical situation as $B(x_2)$ above, if α_{31} is allowed to be zero once again. Then $A(x_2)$ in the x_2x_3 term may be cancelled by the α_{22} . This time it is necessary to cons-

train dV/dt in terms of x_3^2 to avoid the appearance of $A(x_2)$ in α_{23} , as this $A(x_2)$ would appear as a coefficient of x_1x_2 and could not be cancelled by α_{11} . If dV/dt is constrained in terms of x_3^2 , the solution of LaSalle results, as in Fig. 5.13.

The thought immediately arises that if $A(x_2)$ and $B(x_2)$ have the same position, why not let each of them be functions of x_2 at the same time. This proves to be impossible. If an attempt is made to constrain dV/dt to be negative semidefinite in terms of any one or two state variables, in each case α_{11} ultimately proves to be a function of either $A(x_2)$ or $B(x_2)$. Similar difficulty arises in other cases in which two variable coefficients are considered, as $A(x_i) B(x_j)$, $A(x_i) C(x_j)$ or $B(x_i) C(x_j)$, $i, j = 1, 2, 3$, except for the Barbashin problem, Fig. 5.12.

In the Barbashin example, the nonlinearities are described as $B = B(x_2)$ and $C = C(x_1)$. dV/dt can be constrained in terms of x_2^2 , with $\alpha_{31} = 0$ and $\alpha_{23} = \alpha_{32} = 2A$, a constant and not a function of x_2 . When this is done, $\alpha_{12} = 2C(x_1)$ and, as in the alternate solution of the Ingwerson example, a partial derivative is introduced in the derivative.

It may be somewhat disconcerting to learn that more solutions are not available from the variable gradient method for those cases in which more than one variable

coefficient is considered. An examination of the block diagram of such systems indicates why this is the case. If two coefficients are functions of the state variables, the linear portion of the system contains only two terms in s . In the Barbashin example, Fig. 5.12, the linear portion of the system is $1/s^2(s + A)$, and it is indeed surprising that the system is stable at all. For the case when $A = A(x_1)$ and $C = C(x_1)$, as in Fig. 5.14, the linear portion of the system has three poles on the $j\omega$ axis. It comes as no great shock that global asymptotic stability cannot be proved in this case.

The difficulty lies in the differential equation representation that is being considered. The cases of Fig. 5.12 and 5.14 represent configurations that are seldom met in automatic control practice. In fact, it is the author's opinion that such cases are of little more than academic interest.

A case of practical interest is that of the third-order system with one zero, as pictured in Fig. 5.15. A second-order servo motor compensated with a lead lag network can be considered to be of this configuration, as can a third-order model of the motor compensated with tach feedback.

A solution is not possible, or at least not obvious, by using the differential equation representation that has

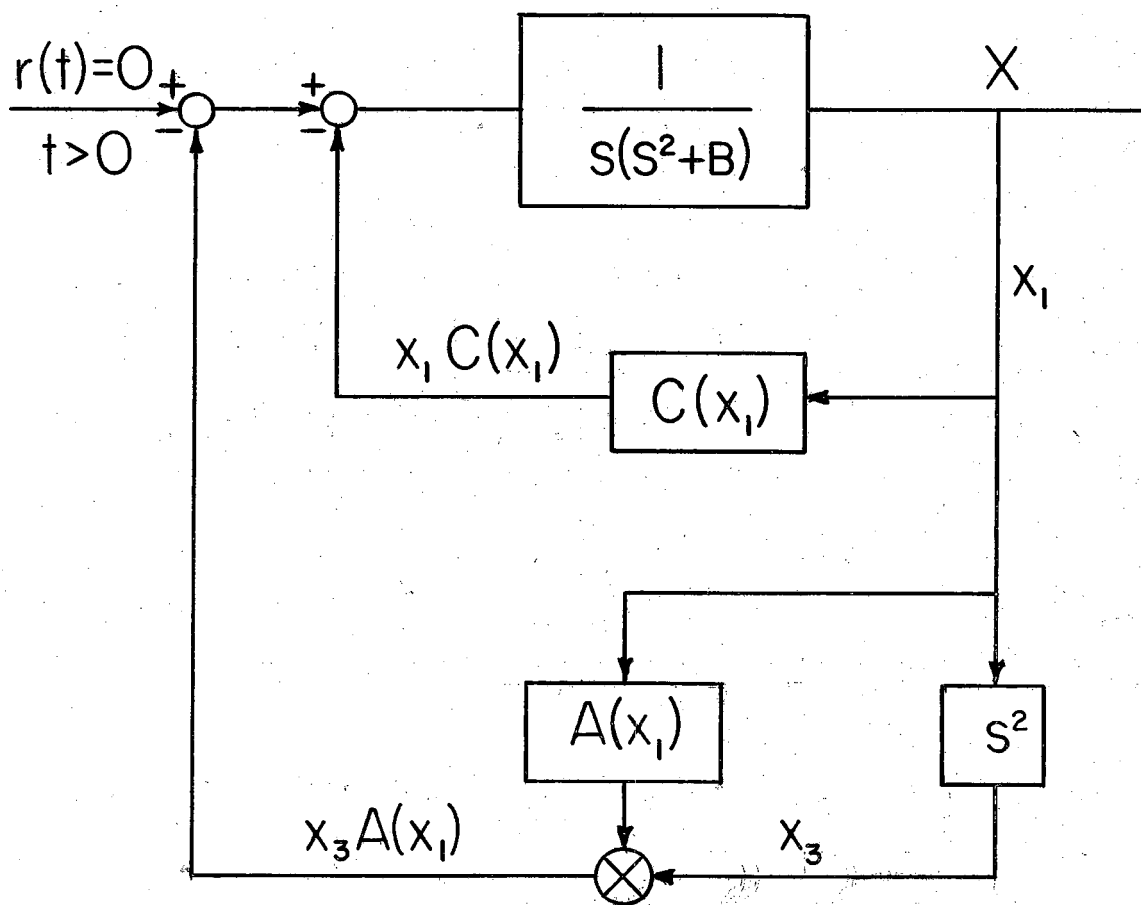


Fig. 5.14. Block Diagram of an Hypothetical Control System

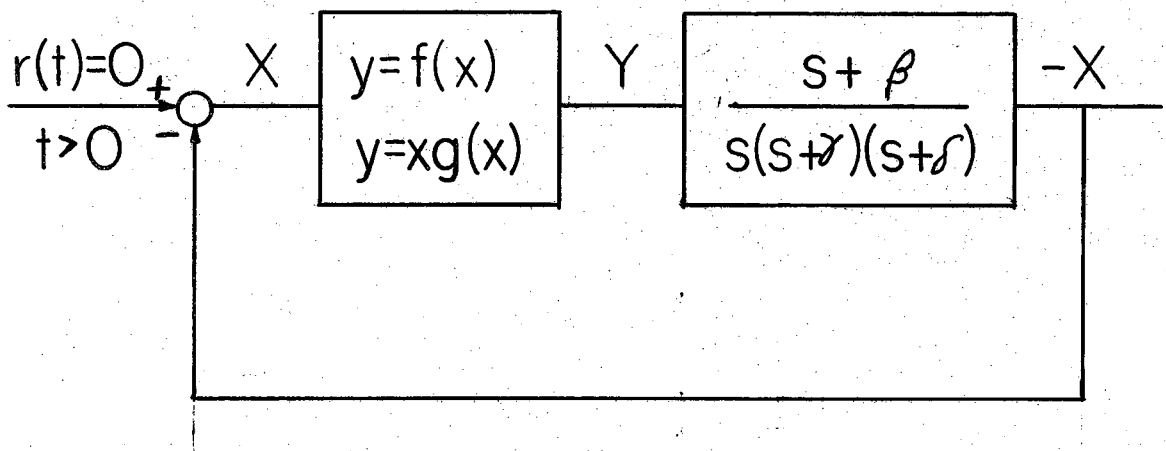


Fig. 5.15. Block Diagram of a Practical Control System

been the subject of this example thus far. Such a representation would cause both C and B to become functions of x_1 . However, if the system differential equations are determined directly from the block diagram and used in that form, the method applies directly and the results are of immediate interest. Based upon the block diagram of Fig. 5.15, the three, first-order differential equations describing the system dynamics are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -(\gamma + \delta)x_3 - \gamma\delta x_2 - \frac{\partial f(x_1)}{\partial x_1} x_2 - \beta g(x_1)x_1$$

Letting $(\gamma + \delta) = M$ and $\gamma\delta = N$, these equations become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -M x_3 - N x_2 - \frac{\partial f(x_1)}{\partial x_1} x_2 - \beta g(x_1)x_1$$

Through the use of the variable gradient, as in (4.7), dV/dt in ordered form is found to be

$$\begin{aligned} \dot{V} = & x_1 x_2 \left[a_{11} - a_{31} N - a_{31} \frac{\partial f(x_1)}{\partial x_1} - a_{32} \beta g(x_1) \right] \\ & + x_2 x_3 \left[a_{13} + a_{22} - a_{32} M - 2N - \frac{2 \partial f(x_1)}{\partial x_1} \right] \\ & + x_1 x_3 \left[a_{21} - a_{31} M - 2\beta g(x_1) \right] \\ & + x_2^2 \left[a_{12} - a_{32} N - a_{32} \frac{\partial f(x_1)}{\partial x_1} \right] \\ & + x_3^2 (a_{23} - 2M) - a_{31} \beta g(x_1) x_1^2 \end{aligned}$$

Stability of this system is definitely a function of β . β appears in dV/dt as a coefficient of the $x_1 x_2$ term and of the $x_1 x_3$ term. The β dependent portion of $x_1 x_2$ is easily cancelled by a suitable choice of a_{11} . Hence, dV/dt may be most easily constrained in terms of x_1 and x_3 . With this in mind, the x_2^2 terms are forced to vanish if

$$a_{12} = a_{32} N + a_{32} \frac{\partial f(x_1)}{\partial x_1}$$

From the $x_2 x_3$ term, since a_{22} cannot be a function of x_1 , a_{13} must have the form

$$a_{13} = a_{13k} + \frac{2 \partial f(x_1)}{\partial x_1}$$

Thus, ∇V_1 is known to be

$$\nabla V_1 = a_{11} x_1 + \left[a_{32} N + a_{32} \frac{\partial f(x_1)}{\partial x_1} \right] x_2 + \left[a_{13k} + \frac{2 \partial f(x_1)}{\partial x_1} \right] x_3$$

Using the first curl equation, it is seen that

$$\frac{\partial \nabla V_1}{\partial x_2} = \alpha_{32} N + \alpha_{32} \frac{\partial f(x_1)}{\partial x_1} = \frac{\partial \nabla V_2}{\partial x_1}$$

Here it is recalled that $f(x_1) = x_1 g(x_1)$. Hence if α_{21} is allowed to be

$$\alpha_{21} = \alpha_{32} N + \alpha_{32} g(x_1)$$

∇V_2 becomes

$$\nabla V_2 = \alpha_{32} N x_1 + \alpha_{32} g(x_1) x_1 + \alpha_{22} (x_2) x_2 + \alpha_{23} x_3$$

and

$$\frac{\partial \nabla V_2}{\partial x_1} = \alpha_{32} N + \alpha_{32} \frac{\partial f(x_1)}{\partial x_1}$$

if α_{23} is assumed to be a constant. The first curl equation is satisfied.

Consider a second curl equation

$$\frac{\partial \nabla V_1}{\partial x_3} = \alpha_{13k} + \frac{2 \partial f(x_1)}{\partial x_1} = \frac{\partial \nabla V_3}{\partial x_1}$$

With the relationship between $f(x_1)$ and $g(x_1)$ in mind, α_{31} is allowed to be

$$\alpha_{31} = \alpha_{13k} + 2g(x_1)$$

Since α_{23} was assumed constant, $\alpha_{32} = \alpha_{23}$, and ∇V_3 is now

$$\nabla V_3 = \alpha_{13k} x_1 + 2g(x_1) x_1 + \alpha_{23} x_2 + 2x_3$$

The second curl equation is satisfied. Because no coeffi-

icients in ∇V_2 of ∇V_3 are functions of x_2 or x_3 , the last curl equation has already been satisfied by setting

$$a_{23} = a_{32}, \text{ a constant.}$$

If the $x_1 x_2$ term in dV/dt is eliminated, dV/dt will have been constrained in terms of x_1 and x_3 . The $x_1 x_2$ term is eliminated if a_{11} is set equal to

$$\begin{aligned} a_{11} &= a_{31}N + a_{31} \frac{\partial f(x_1)}{\partial x_1} + a_{32}\beta g(x_1) \\ &= a_{13k}N + 2Ng(x_1) + a_{13k} \frac{\partial f(x_1)}{\partial x_1} + 2g(x_1) \frac{\partial f(x_1)}{\partial x_1} \\ &\quad + a_{23}\beta g(x_1) \end{aligned}$$

dV/dt is now

$$\begin{aligned} \frac{dV}{dt} &= - a_{31}\beta g(x_1)x_1^2 - x_1 x_3 \left[a_{31}M + 2\beta g(x_1) - a_{21} \right] \\ &\quad - x_3^2 (2M - a_{23}) \end{aligned}$$

Substituting for a_{31} and a_{21} , dV/dt becomes

$$\begin{aligned} \frac{dV}{dt} &= - \left[a_{13k} + 2g(x_1) \right] \beta g(x_1)x_1^2 \\ &\quad - x_1 x_3 \left[a_{13k}M - a_{23}N + 2g(x_1)M + 2\beta g(x_1) - a_{23}g(x_1) \right] \\ &\quad - x_3^2 (2M - a_{23}) \end{aligned}$$

The constant portion of $x_1 x_3$ above is removed if

$$a_{13k} = \frac{a_{23}N}{M}$$

Since $a_{22k} + a_{13k} = a_{23}^M + 2N$, $a_{22} = a_{23}^M + 2N - \frac{a_{23}^N}{M}$

Now all of the elements of ∇V are known within a constant to be

$$\nabla V_1 = \left[\frac{a_{23}^N}{M} + (2N + a_{23}^\beta)g(x_1) + \frac{a_{23}^N}{M} \frac{\partial f(x_1)}{\partial x_1} + 2g(x_1) \frac{\partial f(x_1)}{\partial x_1} \right] x_1$$

$$+ \left[a_{23}^N + a_{23} \frac{\partial f(x_1)}{\partial x_1} \right] x_2 + \left[\frac{a_{23}^N}{M} + \frac{2 \partial f(x_1)}{\partial x_1} \right] x_3$$

$$\nabla V_2 = a_{23}^N x_1 + a_{23} g(x_1) x_1 + \left[a_{23}^M + 2N - \frac{a_{23}^N}{M} \right] x_2 + a_{23} x_3$$

$$\nabla V_3 = \frac{a_{23}^N}{M} x_1 + 2g(x_1) x_1 + a_{23} x_2 + 2x_3$$

and

$$\frac{dV}{dt} = \left\{ - 2\beta g(x_1)^2 x_1^2 - x_1 x_3 g(x_1) [2\beta + 2M - a_{23}] - x_3^3 (2M - a_{23}) \right\}$$

$$- \frac{a_{23}^N}{M} \beta g(x_1) x_1^3$$

Note that the x_3^2 term remains negative for $a_{23} < 2M$.

If in $\left\{ \right\}$ above, the following substitutions are made

$$2M - a_{23} = 2\beta \quad \text{or} \quad a_{23} = 2(M - \beta)$$

$$x_1 g(x_1) = z_1$$

$$x_3 = z_3$$

dV/dt becomes

$$\frac{dV}{dt} = - \left\{ 2\beta z_1^2 + 4\beta z_1 z_3 + 2\beta z_3^2 \right\} \\ - \frac{2(M - \beta)\beta N}{M} g(x_1)x_1^2$$

- $\left\{ \right\}$ is negative semidefinite, and the remaining term in dV/dt is also not positive for positive N , M , β , and $g(x_1)$, as long as $\beta \leq M$.

If the poles of the linear portion of the original system are in the LHP, M and N are both positive. $g(x_1)$ is positive if the nonlinearity $y = f(x_1) = x_1 g(x_1)$ lies in the first and third quadrant. $(M - \beta) \geq 0$ if β is less than or equal to the sum of the open loop poles of the linear portion of the given system. It is interesting to note that this is exactly the condition required for the root locus of the linear portion of the system to remain in the LHP for all values of gain from 0 to ∞ .

If it is assumed that the solution of the equation $dV/dt = 0$ does not satisfy the given system equations, then dV/dt satisfies the requirements of either Theorem 2.2 or 4.3. However, before any decision concerning stability can be made, the closedness of V must be established. V is determined by a line integration of the gradient to be

$$\begin{aligned}
 V = & (2N + \alpha_{23}\beta) \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1 + \frac{\alpha_{23}N}{M} \int_0^{x_1} \frac{\partial f(\gamma_1)}{\partial \gamma_1} \gamma_1 d\gamma_1 \\
 & + 2 \int_0^{x_1} \gamma_1 g(\gamma_1) \frac{\partial f(\gamma_1)}{\partial \gamma_1} d\gamma_1^* + \frac{\alpha_{23}N^2}{2M} x_1^2 + \alpha_{23}N x_1 x_2 \\
 & + \alpha_{23}g(x_1)x_1 x_2 + \left[\alpha_{23}M + 2N - \frac{\alpha_{23}N}{M} \right] \frac{x_2^2}{2} + \frac{\alpha_{23}N}{M} x_1 x_3 \\
 & + 2g(x_1)x_1 x_3 + \alpha_{23}x_2 x_3 + x_3^2
 \end{aligned}$$

The integral with asterisk above can be evaluated, since $x_1 g(x_1) = f(x_1)$. The integral becomes

$$2 \int_0^{f(x_1)} f(\gamma_1) df(\gamma_1) = f(x_1)^2 = g(x_1)^2 x_1^2$$

The V determined above is quite complicated. It is difficult to draw any conclusion for an arbitrary non-linearity and an arbitrary zero location, β . However, it is possible to select a zero location that will prove global asymptotic stability for a large class of nonlinearities. If $\beta = M$, $\alpha_{23} = 0$, then V becomes

$$\begin{aligned}
 V = & 2N \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1 + N x_2^2 \\
 & + \left[g(x_1)^2 x_1^2 + 2g(x_1)x_1 x_3 + x_3^2 \right]
 \end{aligned}$$

If in [], $g(x_1)x_1$ is set equal to z_1 , and x_3 set equal to z_3 , V becomes

$$V = 2N \int_0^{x_1} g(\gamma_1)\gamma_1 d\gamma_1 + Nx_2^2 + [z_1^2 + 2z_1z_3 + z_3^2]$$

V is positive definite if N is positive and if the integral is greater than zero for all x_1 . If the integral also goes to ∞ as $x_1 \rightarrow \infty$, V is sufficient to satisfy the conditions of global asymptotic stability.

dV/dt , corresponding to the V above is

$$\frac{dV}{dt} = -2\beta(z_1 + z_3)^2$$

If dV/dt is not identically zero on a solution of the system, which would be rarely true for such a complex dV/dt , global asymptotic stability of the given system is assured.

The class of nonlinearities for which V is positive definite is very large. The nonlinearity need not be an odd function, although it must lie in the first and third quadrant enough of the time so that the integral stays positive. If the nonlinearity saturates at any finite value, the integral will go to ∞ as $x_1 \rightarrow \infty$. The slope of the nonlinearity is not important.

In a sense, the solution to the above problem is dis-

appointing to a control engineer. The solution required that $\beta = M$, the sum of the open loop poles. Practically speaking, this is impossible. A better answer would be a range of β for which global asymptotic stability could be concluded. Another alternative requirement might be the size of the region of global asymptotic stability for a given range of β . Such questions can be answered, but not until the nonlinearity is specified.

5.3 Discussion of the Application of the Variable Gradient Method to Specific Problems

Chapter IV included a general discussion of the variable gradient method of generating Liapunov functions. This chapter has applied the method to specific problems of engineering interest. As a consequence of this application to a large range of stability problems, the following conclusions are reached:

1. As concerns nonlinearities, the method is applicable to single-valued, continuous nonlinearities where the nonlinearity is known as a polynomial, as a specific function of \underline{x} , as a general function of \underline{x} , or as a curve determined from experimental results.
2. As concerns coordinate systems, the method is applicable independent of the particular state variable formulation used. In the examples,

the phase variables were used almost exclusively. This was done for convenience, and because it is possible to treat in the same way systems that have one or more integrations, multiple poles, poles or zeros in the RHP, etc.

3. As concerns V functions, the method generates V functions to suit the problem at hand. This fact was illustrated in Examples 5.1 to 5.4, where V function with higher order terms, integrals, and terms involving three state variables as factors were generated.

The question may be asked as to why this method of assuming a general gradient is better than a method assuming a general V . The answer is clear in terms of the examples of this chapter. If a V general enough to include the solutions of all of the examples had been selected as a starting point for each problem, the number of terms resulting in dV/dt would have been completely prohibitive.

CHAPTER VI

The Application of the Variable Gradient Method to Nonautonomous Systems

6.1 Introduction and Organization of the Chapter

The term nonautonomous system refers to all systems which are either forced or nonstationary, or both, independent of linearity or nonlinearity. The form of the differential equations arising from time-varying-parameter (TVP) systems and from driven stationary systems is different, thus it is convenient to treat these two types of systems in separate sections.

The first type of system to be considered is the nonstationary type, as this is more closely allied to the work that has been presented in the previous section. The definitions and modifications necessary to take care of the explicit time variations in the system differential equations are made, and this is followed by a discussion of several adaptations of the variable gradient that make it possible to take into account this new condition. Examples indicate the application to both linear and nonlinear, time-variable-parameter systems.

Forced systems cannot be said to be stable in the sense that they seek an equilibrium point. Hence a discussion of stability of this type of system is not appli-

cable, and is replaced by a discussion of ultimate boundedness. A theorem on boundedness is cited and specific examples are given to indicate the means that are available through the variable gradient approach for determining the region of ultimate boundedness.

6.2 Time-Variable-Parameter Systems

The pattern of the section devoted to time-variable-parameter (TVP) systems is similar to the pattern established in the consideration of autonomous systems. After the necessary definitions are presented, the Liapunov theorem applicable is stated, and means of implementing this theorem along the lines of the variable gradient are considered.

6.2.1 Definitions and Applicable Theorem

The purpose of this section on TVP system is to determine the stability of a set of n , first-order, ordinary, differential equations of the form

$$\dot{\underline{x}} = \underline{X}(\underline{x}, t), \quad \text{where} \quad \underline{X}(\underline{0}, t) = \underline{0} \quad (6.1)$$

Because of the explicit time dependence of the right hand side of equations (6.1), it is necessary at the outset to define the exact meaning of the term stability in this nonstationary case.

The following definitions are made under the assumption that the equilibrium state being investigated is the origin

and that $\underline{X}(0, t) = \underline{0}$. The definitions are compatible with the usual definitions, as, for instance, those of Kalman [13] or Szego [27]. However, as in Section 2.4 on autonomous systems, the definitions are stated in terms of the regions $S(r)$ and $S(R)$, rather than in terms of ε and $\eta(\varepsilon)$.

Definition 6.1 Stability in the Sense of Liapunov

The origin is said to be stable with respect to the coordinates x_i and the initial time t_0 , if, corresponding to each $S(R)$ there is an $S(r)$ such that every solution starting in $S(r)$ does not leave $S(R)$ for all $t > t_0$.

Definition 6.2 Uniform Stability

The origin is said to be uniformly stable with respect to the coordinates x_i if, independent of the initial time t_0 , corresponding to each $S(R)$ there is an $S(r)$ such that every solution starting in $S(r)$ does not leave $S(R)$ as $t \rightarrow \infty$.

Definition 6.3 Asymptotic Stability

The origin is said to be asymptotically stable with respect to the coordinates x_i and the initial time t_0 if, corresponding to each $S(R)$ there is an $S(r)$ such that every solution starting in $S(r)$ not only stays within $S(R)$ but approaches the origin as $t_0 < t \rightarrow \infty$.

Definition 6.4 Uniform Asymptotic Stability

The origin is said to be uniformly asymptotically stable with respect to the coordinates x_i if, independent of the initial time t_0 , corresponding to each $S(R)$ there is an $S(r)$ such that every solution starting in $S(r)$ not only stays within $S(R)$ but approaches the origin as $t \rightarrow \infty$.

In each case above the type of stability defined is local. If the region $S(r)$ includes the entire space, each type of stability defined above is global. As before, interest is principally in global stability, and because, in general, an automatic control system must function independent of some arbitrary time t_0 , the principal interest is in global uniform asymptotic stability.

Since equation (6.1) above is an explicit function of time, it might be expected that the Liapunov function required to prove stability may likewise be a function of time. This is true, and the basic theorem applicable to the nonautonomous case is as follows.

Theorem 6.1 [Kalman, 13, p. 379]

If for the system of equations (6.1) there exists a scalar function $V(\underline{x}, t)$ with continuous first partials with respect to \underline{x} and t such that $V(\underline{0}, t) = 0$ and

1. $V(\underline{x}, t)$ is positive definite; that is, there

exists a continuous, non-decreasing, scalar function α such that $\alpha(\underline{0}) = 0$ and, for all t and $\underline{x} \neq 0$

$$0 < \alpha(\|\underline{x}\|) \leq V(\underline{x}, t)$$

2. There exists a scalar function γ such that $\gamma(\underline{0}) = 0$, and dV/dt along the motion starting at t, \underline{x} satisfies for all t and $\underline{x} \neq 0$,

$$\frac{dV}{dt} \leq \gamma(\|\underline{x}\|) < 0$$

3. There exists a continuous, non-decreasing scalar function β such that $\beta(\underline{0}) = 0$ and, for all t ,

$$V(\underline{x}, t) \leq \beta(\|\underline{x}\|)$$

4. $\alpha(\|\underline{x}\|) \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$

THEN the equilibrium state $\underline{x}_e = 0$ is globally, uniformly, asymptotically stable for $t \geq 0$.

Note that Theorem 6.1 requires a new definition for positive definiteness in the nonautonomous case. For $V(\underline{x}, t)$ to be positive definite, $V(\underline{x}, t)$ must be greater than or equal to another positive definite function, which is independent of time, and this inequality must hold for all time. Kalman points out in a footnote that the requirement on dV/dt is less than the requirement of negative definiteness, as γ is not required to be a non-decreasing

function. However, here, in attempting to apply the theorem, a negative definite dV/dt is always sought.

Conditions 1 and 4 above insure that at any instant of time, $V(\underline{x}, t_1)$ represent a family of nested, closed surfaces about the origin in the entire space. Because V is an explicit function of time, conditions 1 and 3 are necessary to insure that the variations of this family of surfaces with time are not such that stability cannot be concluded.

Consider, for example, the family of surfaces in two dimensions

$$V = e^{-t} x_1^2 + e^{-t} x_2^2 \quad (6.2)$$

If both sides of the above equation are divided by the exponential, it is seen that as time increases, this family of circles has an increasing radius. Even though dV/dt may be negative, V may be increasing at such a rate that the net movement of the trajectories may be away from the origin. This is an intuitive explanation of the necessity of requirement 1 of Theorem 6.1.

Condition 3 is sometimes stated as a requirement that $V(\underline{x}, t)$ have an infinitely small upper bound [Kalman, 13], meaning that V must be bounded in all of its coordinates for all time. Szego [27] states this requirement differentially, as

$$\lim_{\|\underline{x}\| \rightarrow 0} V(\underline{x}, t) = 0 \text{ uniformly on } t, \text{ for } t \geq t_0$$

The problem of determining a $V(\underline{x}, t)$ to fit the conditions of the theorem for a given problem is necessarily more difficult than in the autonomous case. The conditions of the theorem are more restrictive, and dV/dt must be determined not only from the gradient but from the gradient and another partial derivative with respect to t . Methods of determining $V(\underline{x}, t)$ are the subject of the following pages.

6.2.2 Methods of Generating Liapunov Functions for Non-Stationary Systems

Three methods are proposed in this section for the solution of TVP problems via the second method of Liapunov. These methods rely heavily upon the variable gradient techniques which have been developed in previous chapters.

Method I

The first method is based upon the fact that the constants of a physical system are never actually constant, but are always changing, due to aging, and environmental changes. In the analysis of physical systems, these time variations are ignored, and yet the results of the theoretical analysis often agree quite well with physical reality. The first method suggested for the generation of Liapunov functions for TVP systems is a procedure identical

to that in which the parameters are assumed to be constant. Time variations are ignored completely, and the system is treated as a fixed parameter system.

At first glance, it seems that this approach has little chance of success, until it is realized that dV/dt will almost surely contain a derivative with respect to time of the time varying coefficient. Only in the exceptional case could this time derivative be expected to cancel. To ignore the time variation in forming V simply amounts to the acceptance of a time varying term in dV/dt before the problem is started.

This procedure is satisfactory if it is possible to limit the appearance of the time varying coefficient in V to be the coefficient of a definite term. This assures that in dV/dt the term arising from $\partial V/\partial t$ will be a definite term in one of the state variables. It is possible that this term may be over ridden by other negative definite terms in the same state variable in dV/dt . This matter is clarified in the following example.

Example 6.1

Consider the second order differential equation,

$$\ddot{x} + A\dot{x} + B(x,t)x = 0$$

which, in phase variable form becomes

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Ax_2 - B(x_1, t)x_1$$

As before, let

$$\nabla V = \begin{cases} \alpha_{11}x_1 + \alpha_{12}x_2 \\ \alpha_{21}x_1 + 2x_2 \end{cases}$$

dV/dt becomes

$$\begin{aligned} \frac{dV}{dt} &= x_1x_2[\alpha_{11} - A\alpha_{21} - 2B(x_1, t)] \\ &+ x_2^2(\alpha_{12} - 2A) \\ &- \alpha_{21}B(x_1, t)x_1^2 + dV/dt \end{aligned}$$

If $B(x_1, t)$ is considered for a moment only as a function of x_1 , α_{11} becomes

$$\alpha_{11} = A\alpha_{21} + 2B(x_1, t)$$

and, with α_{21} still unspecified, ∇V is

$$\nabla V = \begin{cases} A\alpha_{21}x_1 + 2B(x_1, t)x_1 + \alpha_{12}x_2 \\ \alpha_{21}x_1 + 2x_2 \end{cases}$$

As in previous chapters, V is produced by a line integration of ∇V , and is found to be

$$V = \frac{A\alpha_{21}}{2} x_1^2 + 2 \int_0^{x_1} B(x_1, t)x_1 dx_1 + \alpha_{21}x_1x_2 + x_2^2$$

With V completely known, dV/dt is also completely known.

$$\frac{dV}{dt} = -x_2^2(2A - \alpha_{12}) - \alpha_{21}B(x_1, t)x_1^2 + 2 \int_0^{x_1} \frac{\partial}{\partial t} B(x_1, t)x_1 dx_1$$

Quite obviously α_{21} should be made as large as possible, or $\alpha_{21} = 2A - \epsilon$, and the resulting V and dV/dt are

$$V = (A - \epsilon)A x_1^2 + 2(A - \epsilon)x_1x_2 + x_2^2 + 2 \int_0^{x_1} B(x_1, t)x_1 dx_1$$

and

$$\frac{dV}{dt} = -2(A - \epsilon) B(x_1, t)x_1^2 - \epsilon x_2^2 + 2 \int_0^{x_1} \frac{\partial B(x_1, t)}{\partial t} x_1 dx_1$$

If the integral in V is always greater than zero, V is always greater than the time independent positive definite function

$$W_1(\underline{x}) = \frac{1}{2} \left[A(A - \epsilon)x_1^2 + 2(A - \epsilon)x_1x_2 + x_2^2 \right]$$

Thus V is positive definite.

In order for $-dV/dt$ to be greater than a time independent positive definite function, $B(x_1, t)$ must be always positive and must contain a linear term of arbitrarily small magnitude. That is, $B(x_1, t)$ must be able to be written as

$$B(x_1, t) = B_k + B_v(x_1, t)$$

Here B_k may be arbitrarily small. Then $-dV/dt$ is

$$- \frac{dV}{dt} = 2B_k(A - \epsilon)x_1^2 + \epsilon x_2^2 + 2(A - \epsilon) B_v(x_1, t)x_1^2 - 2 \int_0^{x_1} \frac{\partial B_v(\gamma_1, t)}{\partial t} \gamma_1 d\gamma_1$$

If

$$(A - \epsilon) B_v(x_1, t)x_1^2 > \int_0^{x_1} \frac{B_v(\gamma_1, t)}{t} \gamma_1 d\gamma_1 \quad (6.3)$$

for all x_1 and t , then $-dV/dt$ is greater than the time independent positive definite function

$$W_2(\underline{x}) = \frac{1}{2} \left[2B_k(A - \epsilon)x_1^2 + \epsilon x_2^2 \right]$$

V and dV/dt meet conditions 1 and 2 of Theorem 6.1. Unless the nonlinearity is specified, it is not possible to guarantee that conditions 3 and 4 are realized. For in-

stance, if $B(x_1, t) = B_k + e^t x_1^2$, V would not be bounded in its x_1 coordinate, and condition 3 is violated.

The differential equation under discussion corresponds to the block diagram of Fig. 6.1. A nonlinearity with a small linear element and which lies in the first and third quadrant for all t would have $B(x_1, t)$ always positive as required. This is also the type of nonlinearity of interest in the automatic control area.

Two further observations can be made. If the partial of $B_V(x_1, t)$ with respect to t is negative, the inequality (6.3) is always valid. If the partial is not negative, then $B(x_1, t)$ must contain a significant constant portion. It is the minimum value of $B(x_1, t)$ that is of importance in (6.3). Secondly, the amplitude of variations in $B_V(x_1, t)$ are not of importance, but rather the rate of variation is the critical item.

It is possible to construct many nonlinearities for which inequality (6.3) is valid; for example, a linear term plus an odd function of x_1 multiplied by e^{-t} would be sufficient, since the partial with respect to t would be negative. This might be expected, since the forward gain would be decreasing as time increased in this case. A nonlinearity for which the gain is increasing would be

$$y = k_1 x_1 + k_2 x_1^2 + k_3 x_1 t e^{-\delta t} \quad \text{valid for } Ak_1 > \frac{k_3}{\delta}$$

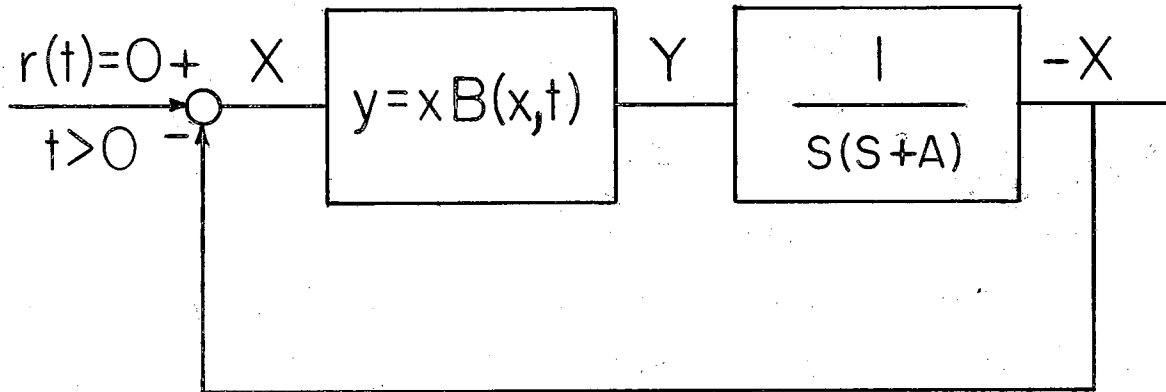


Fig. 6.1. Block Diagram of the Control System of Example 6.1

Here the δ in the exponential may be arbitrarily small, so that the gain actually increases as t for any range, but the exponential must be included to satisfy condition 3 of Theorem 5.1.

Method II

The second method of generating Liapunov functions for TVP systems is based upon the realization that the additional constraints on V in Theorem 6.1 appear because V is an explicit function of time. Hence, Method II simply requires that dV/dt be constrained in such a way that time does not occur in V , that is, V is simply $V(\underline{x})$. This method compliments Method I rather nicely, as no derivatives with respect to time appear in dV/dt , and hence the quantity of interest here is magnitude rather than rate of variation. Again the method is illustrated by the rather general example of Fig. 6.2, for which the equations of motion are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\delta(t)x_2 - g(x_1)x_1$$

This example corresponds to a nonlinear system with a time varying load or damping. From the usual gradient, dV/dt is determined to be

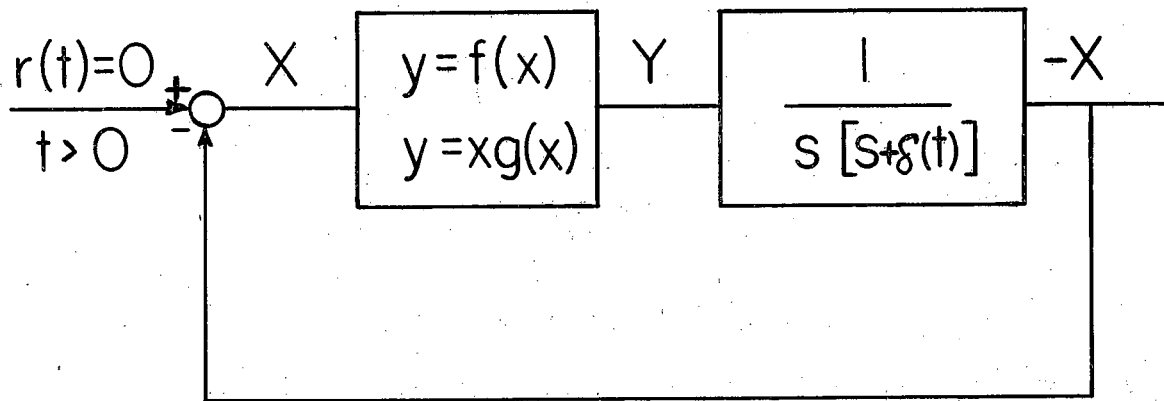


Fig. 6.2. Block Diagram of the System Discussed Under Method II

$$\frac{dV}{dt} = x_1 x_2 [a_{11} - a_{21} \delta(t) - 2g(x_1)] + x_2^2 [a_{12} - 2\delta(t)] - a_{21} g(x_1) x_1^2$$

Since it is postulated that V could not be an explicit function of time, dV/dt is constrained to be

$$\frac{dV}{dt} = -a_{21} g(x_1) x_1^2 - a_{21} \delta(t) x_1 x_2 - x_2^2 (2\delta(t) - a_{12})$$

by use of the gradient

$$\nabla V = \begin{pmatrix} 2g(x_1)x_1 + a_{12}x_2 \\ a_{21}x_1 + 2x_2 \end{pmatrix}$$

Certain assumptions must be made in regard to $g(x_1)$ and $\delta(t)$ to insure that $-dV/dt$ is positive definite. First it is assumed that $\delta(t)$ is always greater than some arbitrarily small number, ϵ . This is not illogical in terms of the system, as this simply requires the "pole" of the linear portion of the system to remain in the LHP. If $\delta(t)$ has a least value ϵ , the x_2^2 term in dV/dt requires that $a_{12} = a_{21} < 2\delta(t)$, or a_{12} is also ϵ . To insure that $-dV/dt$ is greater than another positive definite function, $W(\underline{x})$, $g(x_1)$ must also have at least a small constant portion, so that $-dV/dt$ may be written as

$$-\frac{dV}{dt} = \epsilon g_k x_1^2 + \epsilon \delta(t) x_1 x_2 + x_2^2 (2\delta(t) - \epsilon) + \epsilon g_v(x_1) x_2^2$$

Then

$$-\frac{dV}{dt} > \frac{\epsilon}{2} (g_k x_1^2 + \delta(t) x_1 x_2 + x_2^2)$$

and $-dV/dt$ is positive definite. As in previous problems, the nonlinearity in the system has been assumed to lie in the first and third quadrants.

V is simply determined from the gradient to be

$$V = 2 \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1 + \epsilon x_1 x_2 + x_2^2$$

or

$$V = g_k x_1^2 + \epsilon x_1 x_2 + x_2^2 + 2 \int_0^{x_1} g_v(\gamma_1) \gamma_1 d\gamma_1$$

Now, as desired, V is independent of time, and V is also positive definite and goes to ∞ as $\|\underline{x}\| \rightarrow \infty$. The conditions of Theorem 6.1 are satisfied, and the system is uniformly, globally asymptotically stable.

To be specific, in the example above, assume that $\delta(t)$ is $A_1 + A_2 \sin \omega t$, and let y be a nonlinearity of the form $y = K_1 x_1 + K_2 x_1 g(x_1)$, where K_1 may be arbitrarily small. The system of Fig. 6.2 is globally, uniformly,

asymptotically stable as long as $\delta(t)$ is always positive, or if $A_1 > A_2$. In contrast to the solution of the previous problem, there is no restriction here on the rate of variation. ω may be any number whatever. In addition, it should be noted here that the variation is large, so that no artificial restriction need be made that variations be slow and/or small.

Method III

At the outset of the investigation of TVP systems, it was felt that this third method would prove to be the most successful in solving the stability problem. However, the results attainable by this approach prove to be less general than those mentioned above, and as a consequence, this last method of generating V functions will only be mentioned as a subject for further consideration.

In consideration of equations of the form (6.1), in the most general case, V might be expected to be a function of both \underline{x} and t . If time were considered as simply another coordinate, say x_{n+1} , dV/dt could still be considered as $dV/dt = \nabla V' \underline{\dot{x}}$. Thus instead of V being $V(\underline{x}, t)$, V becomes $V(x_1, x_2, \dots, x_{n+1})$, and it is possible now to treat the system as though it were a constant parameter system. The idea of increasing the order of the system by considering time as an additional variable was suggested by Rozonoer [30] in connection with Pontryagin's maximum

this section of nonautonomous systems is included for completeness, and because some of the results previously obtained apply directly to the driven system.

In a discussion of forced systems, the concepts of stability and ultimate stability are replaced by those of boundedness and ultimate boundedness, as defined below.

Definition 6.5 [Rekasius, 28] Boundedness

The system of equations (6.1) is said to be bounded if for every bounded region $S(r)$ there exists another bounded region $S(R)$ such that every solution starting in $S(r)$ remains in $S(R)$ for all time $t > 0$.

This type of boundedness is often referred to as stability in the sense of Lagrange. It differs from stability in the sense of Liapunov in that the region $S(r)$ must be chosen first. A system with a limit cycle is stable in the sense of Lagrange as long as $S(R)$ is chosen large enough to enclose the limit cycle. Such a system is not stable in the sense of Liapunov.

Definition 6.6 [Rekasius, 28] Ultimate Boundedness

The system of equations (6.1) is said to be ultimately bounded if it is bounded, and if there exists a bounded region Ω in the state space such that every solution starting in the complement of Ω , Ω^* , will approach Ω asymptotically as $t \rightarrow \infty$.

The basic theorem relative to ultimate boundedness is due to Yoshizawa [29]. A statement of the theorem due to Rekasius [28] is given below.

Theorem 6.2 [Rekasius, 28]

Let Ω be a bounded region of the equilibrium state $\underline{x}_e = 0$ of the system of equations (6.1) and let Ω^* be its complement. Then (6.1) is ultimately bounded to Ω if there exists a scalar function $V(\underline{x})$ such that

- 1) $V(\underline{x}) > 0$ for all \underline{x} in Ω^*
- 2) $V(\underline{x})$ is locally Lipschitzian
- 3) $\lim_{\|\underline{x}\| \rightarrow \infty} V(\underline{x}) = \infty$
- 4) $dV/dt \leq 0$ for all \underline{x} in Ω^* .

If the region Ω is simply the origin, this theorem corresponds to a Liapunov stability theorem. Conditions 1, 2, and 3 above require that $V(\underline{x})$ represent a one parameter family of nested closed surfaces about the region Ω .

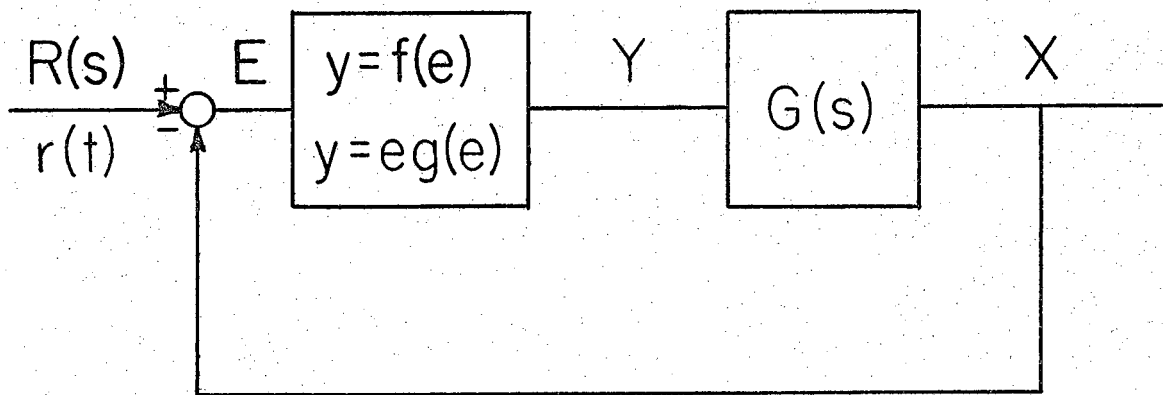
The problem is to determine the size of the region Ω , the region to which the solution is ultimately bounded. Of course it would be more desirable if one were to be able to establish a region in which the solution always remained, but this capability is not afforded by the above theorem.

For simplicity, consider first the usual block diagram representation of a unity-ratio automatic control sys-

tem in which the input, $r(t)$, is no longer zero. The configuration is that of Fig. 6.3. Here once again no special attention is given the linear system. It is treated as a special case of the nonlinear system, in which $g(e)$ is the forward gain, K . In general, the nonlinear differential equations describing the system of Fig. 6.3 are

$$\begin{aligned}
 \dot{e}_1 &= e_2 \\
 \dot{e}_2 &= e_3 \\
 &\vdots \\
 \dot{e}_n &= -a_n e_n - a_{n-1} e_{n-1} \cdots a_2 e_2 - a_1 e_1 \\
 &\quad - f(e)^m - b_m f(e)^{n-1} \cdots b_2 \frac{df(e)}{dt} - b_1 f(e) \\
 &\quad + r^n + a_n r^{n-1} + \cdots a_2 \frac{dr}{dt} + a_1 r
 \end{aligned}
 \tag{6.4}$$

Here the equations are written in terms of error, e , rather than in terms of the output, x , and the use of the phase coordinates is retained, as elsewhere in this work. Notice that the above formulation requires that as many derivatives of the input exist as the order of the system. Also, the nonlinearity must possess as many derivatives as there are zeros in the system. These are definitely limitations on the type of system that can be handled in the phase coordinates by the approach being described. In



$$G(s) = \frac{s^m + b_m s^{m-1} + \dots + b_2 s + b_1}{s^n + a_n s^{n-1} + \dots + a_2 s + a_1}$$

Fig. 6.3. Block Diagram of a Conventional Control System with an Input

systems with no zeros, it is possible to write the equations in terms of \underline{x} , and these limitations no longer exist.

Because of the form of equation (6.4) and because of the similarity of the theorem on ultimate boundedness to that of the theorem for asymptotic stability, many of the results of the previous sections are directly applicable. The procedure is similar to that used in Example 5.6 in connection with limit cycles. Here, however, it is necessary to find the region outside of which dV/dt is always negative, and then to choose the smallest V curve to circumscribe that region.

In the examples that follow, the input and its derivatives, as in (6.4) are replaced by M . M is the maximum value of

$$\left[r^n + a_n r^{n-1} + \dots + a_2 \dot{r} + a_1 r \right] \max = M$$

As a first example, consider the block diagram of Fig. 6.4. In terms of error, the equations of motion become

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = -Ae_2 - \frac{df(e_1)}{de_1} e_2 - e_1 g(e_1) + \dot{r} + Ar$$

or

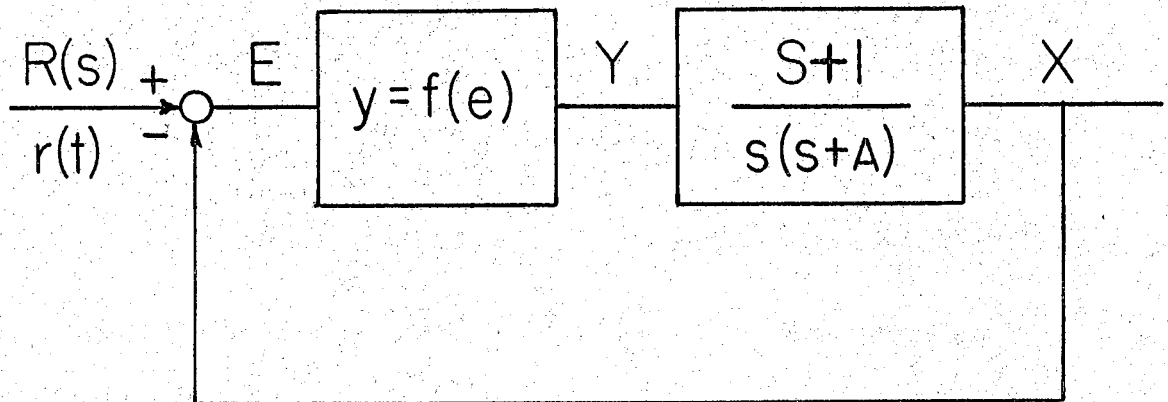


Fig. 6.4. Block Diagram of a More Specific Example of a Forced System

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = -Ae_2 - \frac{df(e_1)}{de_1} e_2 - e_1 g(e_1) + M$$

From the gradient, dV/dt is found to be

$$\begin{aligned} \frac{dV}{dt} = & e_1 e_2 \left[a_{11} - Aa_{21} - a_{21} \frac{df(e_1)}{de_1} - 2g(e_1) \right] \\ & + e_2^2 \left[a_{12} - 2A - 2 \frac{df(e_1)}{de_1} \right] - a_{21} g(e_1) e_1^2 \\ & + a_{21} M e_1 + 2M e_2 \end{aligned}$$

dV/dt must be constrained so that the region in which dV/dt is not negative is a minimum. In this case, the bound of e_1 is independent of a_{21} , and hence a_{21} is chosen to be the arbitrarily small number ϵ , so that the extent of e_2 might be minimized. The coefficient of the $e_1 e_2$ term is forced to be zero through an obvious choice of a_{11} , and dV/dt is thus constrained to be

$$\begin{aligned} \frac{dV}{dt} = & -e_2^2 \left[2A + \frac{2df(e_1)}{de_1} - \epsilon \right] + 2M e_2 \\ & - \epsilon \left[e_1^2 g(e_1) - M e_1 \right] \end{aligned}$$

and, from the gradient, V is

$$\begin{aligned} V = & \frac{\epsilon A}{2} e_1^2 + \epsilon e_1 e_2 + e_2^2 + \epsilon \int_0^{x_1} \frac{df(\gamma_1)}{d\gamma_1} d\gamma_1 \\ & + 2 \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1 \end{aligned}$$

For $-dV/dt$ to be positive definite, the magnitude of e_1 must be

$$|e_1| > \frac{M}{g(e_1)}$$

To be specific, consider the linear system for which the input is a ramp. Then M is A and $g(e_1)$ is K , the forward gain, and e_1 is bounded by

$$|e_1| > \frac{A}{K}$$

Here in order to make the region a minimum, A should be reduced to as small as number as possible, and K should be increased to as large as possible. This is completely reasonable for the given system with a ramp input. Thus the second method of Liapunov begins to look like a design tool when applied to systems where the form of the input is known.

In the case of a nonlinear system, the minimum value of $g(e_1)$ must be considered in determining the size of the bound. Thus in the case of a nonlinearity such as $y = \arctan e_1$, which represents saturation, $g(e_1)$ is unity for $e_1 = 0$, but it goes to 0 as e_1 goes to infinity. Hence, for this nonlinearity, the size of the bounded region would be infinite. If it were possible to approximate the given nonlinearity with $y = \arctan e_1 + k_1 e_1$, the bounded region would be a function of k_1 , and the method

would give a result. The Liapunov method here suggests that the designer not let his components "saturate completely".

It should be noted in passing that although apparently little use was made of the variable gradient approach in the solution of this problem, actually the resulting V contains two integrals.

If the system of Fig. 6.4 had a unity numerator, the equations of motion may be written in terms of \underline{x} as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Ax_2 - f(e) = -Ax_2 - (r - x)g(e)$$

From the usual gradient, dV/dt is formed, and it is seen that only a portion of the coupling term, x_1x_2 , may be removed by letting α_{11} be $\alpha_{12}A$. The remaining dV/dt is

$$\frac{dV}{dt} = -\alpha_{21}g(e)x_1^2 - 2g(e)x_1x_2 - x_2^2[2A - \alpha_{12}]$$

$$+ \alpha_{21}g(e)rx_1 + 2g(e)rx_2$$

The term in x_1x_2 could not be removed because $g(e)$ is a function of r as well as x_1 . Hence the problem of determining the region of boundedness is somewhat more complicated. Essentially a portion of the x_1^2 and x_2^2 terms must be allocated to take care of the coupling term in x_1x_2 , and the remainder of these terms is used to determine the

boundary. This becomes more evident in the following development. Let

$$\begin{aligned} \alpha_{21}g(e)x_1^2 &= g(e) \left[\alpha_{21} - K_1 \right] x_1^2 + g(e) K_1 x_1^2 \\ (2A - \alpha_{12})x_2^2 &= \left[2A - \alpha_{12} - K_2 \right] x_2^2 + K_2 x_2^2 \end{aligned}$$

With this substitution, dV/dt becomes

$$\begin{aligned} \frac{dV}{dt} &= - \left[K_1 g(e) x_1^2 + 2g(e) x_1 x_2 + K_2 x_2^2 \right] \\ &\quad - g(e) \left[\alpha_{21} - K_1 \right] x_1^2 + \alpha_{21} g(e) r x_1 \\ &\quad - \left[2A - \alpha_{12} - K_2 \right] x_2^2 + 2g(e) r x_2 \end{aligned}$$

The large square bracket is made definite from geometric considerations by forcing

$$K_1 K_2 > g(e) \max$$

The bounds of x_1 and x_2 are then determined from the remaining terms to be

$$\begin{aligned} |x_1| &> \frac{r}{\alpha_{21} - K_1} \\ |x_2| &> \frac{2g(e)_{\max} r}{2A - \alpha_{21} - K_2} \end{aligned}$$

Here the result depends upon the magnitude of the input and not upon any of its derivatives. In a linear system, $g(e)$ would correspond to K , the forward gain. For

high-gain systems, A must be large to keep the region of x_2 small, which again agrees completely with the usual linear design. Now, however, it is possible to draw conclusions of a similar nature for the nonlinear system.

6.4 Analysis of the Variable Gradient Method as Applied to Nonautonomous Systems

In this chapter the variable gradient method was applied to solution of differential equations representing systems with time-varying coefficients or with forcing terms. The examples presented as expositions of the method were all based on second order systems, which in itself indicates the degree of achievement or flexibility that has thus far been achieved in dealing with these more difficult systems. It is felt, however, that the fact that anything at all was achieved is significant.

Of particular interest in both types of problems that were considered is the fact that linear and nonlinear systems received the same treatment. Hence if a design or synthesis procedure could be worked out for the linear system, the results would be directly applicable to the nonlinear case. If the second method of Liapunov is all that Letov [10] claims in the introduction to his book, a linear system design and synthesis procedure should be forthcoming, and with it the nonlinear technique.

Perhaps it is superfluous, but it seems that this is an interesting area for further research.

CHAPTER VII

Summary and Conclusions

7.1 Summary

The second or direct method of Liapunov is a powerful tool for the analysis of the stability of ordinary differential equations. Although originally conceived and developed by the Russian mathematician Liapunov in the late 19th century, the method has received considerable attention from other competent mathematicians only in recent years. As a consequence, the theoretical sophistication involved in the development and proof of the original and supplementary Liapunov theorems far exceeds the applications to which these theorems can be applied.

The difficulty in applying Liapunov's theorems lies in the determination of a V function which meets the conditions of the given theorem. In the past, the determination of a suitable V function for a given differential equation has been a task that has relied heavily upon the ingenuity and experience of the investigator. This work presents a systematic approach to the determination of a Liapunov's V function that in some measure overcomes this problem. The new method is known as the variable gradient method of generating Liapunov functions.

The precise meaning of the term "generating" is de-

defined in the sense that it is used in this work, and the two principal means of generating Liapunov functions that have been proposed to date are examined in some detail. The desirable and undesirable features of each of these methods are emphasized, and the more desirable features of each are incorporated into a new technique. The variable gradient method that results is based upon the assumption of a variable gradient that is thought to be of a sufficiently general nature to include all possible gradients within its structure. This gradient is assumed to be a vector of n components where n corresponds to the order of the differential equation in question. Each component of the gradient is further assumed to be made up of n terms, each of which has an unspecified coefficient. These coefficients are determined from constraints on dV/dt , with the aid of $(n - 1)n/2$ additional curl equations that must be satisfied if the V function determined from the resulting gradient is to be unique. Once the elements of the gradient are known, both V and dV/dt are determined directly from the gradient. Because of the general nature of the gradient, if solution to a physical problem with continuous, single valued, nonlinearity exists, in theory, the solution exists within the framework outlined.

The variable gradient method of generating V func-

tions is characterized by its ability to handle systems containing multiple nonlinearities in which the nonlinearity is known as a definite function of the state variables or simply as a general function of \underline{x} . Systems with one or more integrations, multiple poles, or complex conjugate poles are treated in the same way. As opposed to the more usual quadratic form for V , with this method it is possible to generate V functions which include state variables raised to higher powers than 2, depending upon the actual representation of the nonlinearity. Also, V functions which include one or more integrals are derived quite naturally, as are V 's containing terms that involve, not two, but three state variables as factors.

The capability of generating this broader class of Liapunov functions that is described above is demonstrated through simple examples, through the reproduction and extension of the results of other investigators, and through the solution of original problems. The last chapter of this report is devoted to extensions of the variable gradient method to nonautonomous systems.

7.2 Recommendations for Further Study

The variable gradient method developed above is a general technique for the generation of the Liapunov V function. In this report the example problems considered include only single-valued nonlinearities, and the coor-

dinate system used is almost exclusively that of the phase variables. An obvious extension would include the consideration of multiple valued nonlinearities, or a coordinate system in canonic or other special form. For example, if the basis of system description is to be a set of nonlinear equations, these may be generated by the use of Lagrange's equations. The resulting equations include variables that are intrinsic to the physical system in question, the so-called generalized coordinates. It is quite conceivable that the resulting set of second-order differential equations might result in a set of n , first-order, differential equations that would be more meaningful and easier to handle than the phase variables considered here.

As developed in Chapter IV, the variable gradient method is applicable to the n th order system, yet only second and third order systems are considered as examples. Obviously it is desirable to apply the method to higher order systems. Limit cycles were considered for only second-order systems, yet it is known from experience and from describing function analysis that higher-order systems also exhibit periodic behavior. In short, this work proposes a method of generating Liapunov functions, and this method is used to solve as many different types of problems as possible, in order to show the generality of the method.

No particular attempt is made at a deep penetration of any one particular class of problems, other than the consideration of the Aizerman problem. In this sense it might be said that this report suggests more problems than it actually solves.

The greatest area of interest lies in the furthering of the work in the last chapter on nonautonomous systems. There it was observed that linear and nonlinear systems were treated in the same manner, at least for the second order systems. For example, in the discussion of forced systems, it was noticed that the region of dV/dt had to be a closed region, such that the V curve might circumscribe it. Yet in the previous solutions of Chapter V, advantage was taken of the fact that dV/dt was not required to be definite, as long as it was not zero on a solution of the system. It is conceivable that a determination of stability in one coordinate system for the autonomous case and the region of ultimate boundedness for the driven system in another coordinate system might be in order.

The discussion of ultimate boundedness in itself is a compromise. What is actually of interest is the maximum value of the response for a given input, or better yet, the maximum deviation from a given or desired response. There is no theorem as yet to aid in this pursuit.

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APPENDIX*

Since no general analytic method is known for proving the definiteness of V functions other than quadratic forms, the purpose of this appendix is to provide a geometrical basis for the establishment of the definiteness, or closedness, of higher order V functions, such as those generated by the examples above.

That Sylvester's inequalities are not adequate in the case where V is not a quadratic form can be seen from a consideration of the Liapunov function that follows.

$$V = x_1^6 + x_1^4 + 2x_1^3x_3 + 2x_2^2 + x_3^2$$

Here it is possible to arrange V in what could be considered a quadratic form with variable coefficients. However, the arrangement is not unique as is indicated by the two configurations below.

$$\text{Case I} - V_1 = (x_1^4 + x_1^2 + 2x_1x_3)x_1^2 + 2x_2^2 + x_3^2$$

$$\text{Case II} - V_2 = (x_1^4 + x_1^2)x_1^2 + (2x_1^2)x_1x_3 + 2x_2^2 + x_3^2$$

The coefficient matrix to which Sylvester's inequalities apply indicates that in Case I the V function is indefinite, while in Case II the function is definite -- clearly a contradiction.

*This method was originally suggested by Dr. G. P. Szego.

In order to determine the definiteness of higher order V functions, it is possible to employ basic geometrical considerations. Consider, for example, a second-order case where

$$V = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

According to Sylvester, if $a_{11} a_{22} - a_{12}^2 > 0$, then V is a positive definite function, or a closed function. In fact, if the above condition on the a 's holds true, the closed curves representing different values of V are simply a family of nested ellipses in the x_1x_2 plane.

Suppose that in this example x_2 is determined as a function of V and x_1

$$x_2 = \frac{-a_{12}x_1 \pm \sqrt{x_1^2(a_{12}^2 - a_{11}a_{22}) + a_{22}V}}{2a_{22}}$$

Assuming that the a_{ii} are positive, for any constant value of V , x_2 has two values for small values of x_1 . If the coefficient of x_1^2 under the radical is negative, as x_1 is increased, a value of x_1 is reached for which the two values of x_2 are identical. Beyond this value of x_1 , the two values of x_2 are no longer real. Of course, the condition that insures this closure of the curve is identical to Sylvester's conditions, namely that

$$a_{11}, a_{22} > 0$$

$$[a_{11} a_{22} - a_{12}^2] > 0$$

The idea of closeness depending upon the two values of a variable becoming imaginary is the concept that is used in determining the definiteness of higher order V functions. For this reason, in all examples a_{nn} was assumed to be 2 and the a_{ij} 's were not allowed to be functions of x_n . Thus the resulting V is always a quadratic in x_n , and the quadratic formula can be used to solve for the two values of x_n .

In the third order systems, of course, it is necessary to show that V represents a closed surface rather than a closed curve. This procedure can be reduced to the examination of a closed curve by considering one of the state variables a constant. Thus a three dimensional closed surface is cut by a plane, and to insure closeness, each curve of intersection must be a closed curve. As the plane of intersection is moved along its axis, the curve of intersection must eventually vanish.

This procedure can be illustrated by the V function of which was cited above. Here, since x_1 appears in the most complicated form, let x_1 be a constant, k , so that V becomes

$$V = k^6 + k^4 + 2k^3x_3 + 2x_2^2 + x_3^2$$

The term in x_3 alone can be eliminated by a linear change in variables to produce a form amenable to Sylvester's theorem. Let

$$x_2 = z_2 + \alpha \quad \text{and} \quad x_3 = z_3 + \beta$$

The constants are found to be

$$\alpha = 0 \quad , \quad \beta = -k^3$$

such that

$$V - k^4 = 2z_2^2 + z_3^2$$

For a particular value of $V = c$ and x_1 for which k^4 is less than V , this is the equation of an ellipse in the z_2z_3 plane. As x_1 is increased till $x_1^4 = V$, the ellipse finally vanishes and closeness of the surface is demonstrated.

The V function chosen in this example was a simple one for expository purposes. The V function resulting from Example 5.4 is considerably more complicated, yet closeness can be demonstrated in exactly the same way. In the case of a fourth order system, a geometrical interpretation is not possible to visualize, yet the procedure is the same. For each x_1 a constant, it is necessary to show

that the resulting surface was closed, and that this sphere finally vanished as the value of x_i is increased. Although the concept is not difficult, the work involved increases rapidly as the order of the system is increased.