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Two Dimensional Signal Representation Using Prolate Spheroidal Functions

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PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

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ABSTRACT

The most widely used methods of signal representation are the time function and the frequency function or spectrum representations. This work is concerned with the development of a representation which is a combination of these two.

Two previous attempts at defining this type of signal representation, which is referred to as two dimensional representation, have been made and a summary and evaluation of these attempts is presented.

The primary objective of the work reported here was to develop a practical two dimensional representation which has the desired two dimensional conceptual properties as well as mathematical convenience. The representations defined are based on the angular prolate spheroidal functions. These functions have a number of desirable properties among which are the following: they are orthogonal over both a finite and the infinite interval, they are bandlimited, and they have certain properties concerning their maximal proximity to being timelimited.

The procedure used in defining the first two dimensional representation is to make an orthogonal expansion,

using the prolate spheroidal functions, of each timelimited portion of each bandlimited portion of the signal to be represented. The second two dimensional representation is defined from an orthogonal expansion of each bandlimited portion of each timelimited portion of the signal to be represented. For both of these, the summation over all time intervals and all frequency intervals results in the complete representation of the signal.

It is seen from this that since it is not possible to timelimit and bandlimit simultaneously, these limiting processes have been carried out serially. Due to the peculiar properties of the prolate spheroidal functions, as the number of orthogonal function terms is increased, the representation of a timelimited function converges first in a certain bandwidth, and the representation of a bandlimited function converges first in a certain time interval.

It is demonstrated that both series representations will converge to either a timelimited or a bandlimited portion of the represented signal upon inclusion of the proper terms. Following this several applications of the representations are presented. First, it is shown that the result of the convolution of 2 two dimensionally represented functions may be determined at discrete values of time from the expansion coefficients alone. The spectrum of the product of two functions may be determined in a similar manner at discrete values of frequency. As a result it is possible to determine the contribution made to the output of a linear system at any time due to the portion of the input in any time and frequency interval.

A technique is also developed for the solution of this same problem for the more general time variable linear system with the output being determined in continuous form rather than only at discrete values. It is somewhat more difficult to calculate the coefficients in this case, however.

Another application demonstrated is a method by which the value of Woodward's ambiguity function may be calculated for discrete values of the time and frequency variables.

The two dimensional nature of the representation is demonstrated by two numerical examples using very elementary time functions. A further numerical example is provided for the case of the determination of the output of a linear system at discrete values of time.

This work is concluded by a brief listing of further problems which seem amenable to solution as a result of this type of analysis. This list includes such problems as biological system signal analysis, signal design, and random process representation.

CHAPTER 1

Introduction

In the first portion of this introductory chapter the place of signal theory is established in the larger area of system theory. This is followed by a statement of the problem investigated and a brief outline of the contents of this thesis including the results obtained. The chapter is concluded with a statement of some conventions and notation adopted for this work.

1.1 Signal Theory and its Relation to System Theory

Signal theory is an area of study which has only recently begun to be defined clearly and in a manner distinct from other closely related areas such as information theory and circuit theory. The name is applied to a class of problems which, as the general technology advances, increasingly demand solution.

In the preface to an issue of the IRE Transactions on Circuit Theory devoted entirely to signal theory, Ham (9) defined signal theory as "being concerned with the developing of useful representations for dynamical observables in physical systems and with characterizing the transforming effects of system components on the significant attributes of observables". Since frequently the purpose and significance of a signal is the conveyance of information, the term "significant attributes" may refer to information and the term "useful representations" may refer to representations which cause the information conveyed to be readily accessible conceptually and/or mathematically. Then by the use of the term "signal", it is possible, if desired, to make a distinction in the case of the output of a power amplifier, for example, between the information the output conveys and the energy or capacity of that output to do work. The point is, the terms "information" as here used and "energy" are not necessarily synonymous. They are, however, unquestionably related, for in the case of an electrical signal, for example, the manner in which the energy is arranged must convey the information.

It is, of course, evident that signals are only of interest as a result of their relation to an associated system and the transformations the system performs. It is, therefore, of interest to establish the relationship between signal theory and system theory. Huggins (11) suggested that many of the subjects and techniques presently lumped under the heading of circuit theory might better be called system theory and that this in turn might further be broken down into three parts--that part dealing with the representation of physical elements, that part dealing with the representation of signals, and that part dealing with relations and transformations between the elements and the signals. For the first he suggested the name "circuit theory," for the second the name "signal theory," and for the third "operator theory." This definition of signal theory and its relation to system theory is wholly in accord with the philosophy behind this thesis.

As a result of these definitions, it is apparent that signal representations are needed for two purposes, as Huggins points out. One is for the purpose of studying systems and their transmission properties. The other is for the purpose of revealing the "information bearing attributes" of a signal.

1.2 Discussion of the Thesis

Of all of the various representations of signals available, the time function representation and the Fourier representation have found the widest application by far. This work is concerned with a representation which is a combination of the two. This representation may be thought of graphically (or rather, pictorially) as a plot of the signal to be represented on a two dimensional plane which has been divided into small rectangles. The plane will be considered to have time along the abscissa and frequency along the ordinate. A certain "amount" of signal will be assigned to each rectangle.

From this it follows immediately that the representation demands that in some fashion the spectrum of the function be determined as a function of time. If the Fourier Transform is defined to be the spectrum in question, strictly speaking, the spectrum as a function of time is not defined. The Fourier Transform is defined over the infinite time interval and is not, and cannot by itself, be a function of time. In the words of Gabor (8), "if the term frequency is used in the strict mathematical sense. . . a 'changing frequency' becomes a contradiction in terms as it is a statement involving both time and frequency." The time structure of the spectrum, however, is frequently very important as is evident, for example, in the simple situation of listening to a piano selection. It is the purpose of the proposed representation to give to the spectrum its time structure.

Gabor receives the credit for having originated the idea of the two dimensional representation in 1946 and a summary of his methods and results are presented in Chapter 2. Also in Chapter 2 is contained a brief summary of the results of Lerner (19) who was the first to suggest that Gabor's two dimensional representation be used for signal analysis rather than information theory as was Gabor's intention.

Chapter 3 contains a summary of some of the properties of the angular prolate spheroidal functions, which have been reported in the literature, and which will form the basis of the two dimensional representations to be developed. It will be seen that the properties of these functions are such that the resulting representations provide both the mathematical aspects and the intuitive type of conceptual aspects desired.

The representations, themselves, are developed in Chapter 4. The development of both two dimensional representations presented is carried out in a manner so as to place in evidence as clearly as possible the two dimensional nature of each term and what portion of the signal it represents. The errors due to truncation are also calculated. Following these developments it is shown that the series representations will converge to either a timelimited or a bandlimited portion of the represented signal upon inclusion of the proper terms.

The application of these representations to the determination of the output of a linear system given a two dimensionally represented input is also demonstrated by two different techniques. In the first it is found that the output, at sampled instants of time, which results from the portion of the input signal occupying any arbitrary portion of the time-frequency plane may be calculated easily. This technique is demonstrated only for the linear, time invariant system.

The second technique may be used in the more general linear, time variable system, and the output is given in continuous rather than sampled form. It is also possible with this technique to determine the output due to any two dimensional portion of the input, but this technique is somewhat more difficult to handle mathematically.

Chapter 5 contains numerical examples of some of the representations and operations developed in Chapter 4. Two time functions were chosen for presentation by the same two dimensional representation. The functions were chosen for their simple but fundamentally different time-frequency structure so that it is possible to compare the results with one another and with what intuition predicts. Numerical results are also given for a time invariant linear system problem.

Chapter 6 contains a summary followed by brief descriptions of a number of problems which seem amenable to solution as a result of or in terms of the two dimensional representations presented.

The most significant original contribution of this work is the definition of two dimensional signal representations which are more general than those previously defined and which are in a form such that they may be immediately applied to practical problems. A further contribution is the method of application of these two dimensional representations to the general linear system problem and to the determination of the ambiguity function of a signal.

1.3 Notation and Conventions

In order to specify the class of functions or signals to be treated herein, the following convention will be adopted: throughout this thesis the term "arbitrary function of time" refers to any time function which is of integrable-square on the whole line, i.e.,

$$\int_{-\infty}^{\infty} \left| f(t) \right|^2 dt < \infty \qquad (1\text{-}1)$$

and whose Fourier Transform exists. This restricts the class of functions in a manner similar to Gabor, Lerner, Slepian, and Pollak, and thus makes the class under consideration compatible with the primary references of Chapters 2 and 3.

In order to avoid some confusion which exists in the literature, the definition of the Fourier Transform will be taken as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \varepsilon^{-j\omega t} \, dt \qquad (1\text{-}2)$$

As a result the inverse transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \varepsilon^{j\omega t} \, d\omega = \int_{-\infty}^{\infty} F(f) \varepsilon^{j2\pi f t} \, df \qquad (1\text{-}3)$$

where $\omega = 2\pi f$. Generally, lower case letters are used for time functions and upper case letters for frequency functions.

Further, the terms "Fourier Transform" and "spectrum" are herein considered synonymous as are the terms "mean square value" and "energy."

CHAPTER 2

Review of Two Dimensional Representations

A brief summary of the history of two dimensional representations is presented. The concept of the analytic signal is introduced and applied in the summary of the methods and principle results obtained in the original definition of the two dimensional representation of Gabor. This is followed by a summary of the modifications and generalizations of this definition given by Lerner, and the chapter is concluded by a discussion of these results.

2.1 Introduction

It is readily apparent that one cannot simultaneously limit the lengths of the non-zero portion of both a function and its Fourier Transform. But the fact that the "essentially zero" portion of both a function and its transform cannot simultaneously be too severely limited is not so apparent and dawned slowly on communication engineers during the 1920's and 30's. At about this same time the formulation of wave mechanics was taking shape. One of the results of this formulation, the Heisenberg Uncertainty Principle, states that the product of the variances, or uncertainties in measurement, of position and of momentum is always greater than a certain positive constant. The fact that position and momentum are related functionally by Fourier Transformation and, thus, this uncertainty relation might have application for signals and their spectrums went apparently unnoticed for some time.

It was not until 1946 that Gabor (8) noticed this connection and attempted a mathematical formulation to Hartley's (10) statement of 1928 that the amount of information which can be transmitted over a communication channel is proportional to the product of the bandwidth available and the time available.* Hartley had not given a mathematical formulation to back up his statement.

Gabor was interested in determining the amount of information transmitted by a communication channel. He noted that if he would substitute a signal as a function of time in place of the position probability amplitude function of wave mechanics, then the spectrum of the signal would be analogous to the momentum probability amplitude function, the uncertainty in the measurement of position would become one possible definition for the time duration of the signal, and the uncertainty in the measurement of momentum would become one possible definition for bandwidth. Continuing, the analogy to the Heisenberg uncertainty rela-

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*Landau and Pollak (18) state that L. A. MacColl obtained such a signal uncertainty relation about 1940, but it went unpublished. See the Appendix.

tion then states that the product of the bandwidth and time duration of a signal cannot be less than a certain positive constant.

Thinking of this in another manner, if one considers a two dimensional representation of the signal--time along one axis and frequency along the other--then one sees that there is a certain minimum area that any signal can occupy. It was on this minimum area that Gabor chose to base his definition of information content of signals.

The study of this uncertainty relation has been the subject of considerable work and a brief summary of some of the results obtained is given in the appendix. The study of the two dimensional representation of a signal has not been so extensive.

2.2 Analytic Signals

Before proceeding further it is necessary to introduce the concept of what has become known as the analytic signal. Gabor was apparently the first to present this concept although many have used it since. See, for example, Dugundji (4), Kay and Silverman (12), Ville (31), (32), (33), and Woodward (34). The analytic signal may be defined as follows: For any signal, s(t), and its spectrum, S(ω), the analytic signal is given by

$$s_{+}(t) = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\varepsilon^{j\omega t}\, d\omega \qquad (2\text{-}1)$$

From this it is apparent that the only difference between $s_+(t)$ and $s(t)$ is that in $s_+(t)$ the negative frequency portion of the spectrum has been suppressed and the positive frequency portion is multiplied by a fixed constant. There is, then, somewhat of an analogy between this operation for a nonperiodic function and the expression of a periodic function in the complex form of the Fourier Series with all terms for which the summation index is negative discarded. It is important to note that given either the real or the analytic signal, the other can be found uniquely. Of course, if $s(t)$ is a real function of time, $s_+(t)$ is necessarily complex valued.

The analytic signal, $s_+(t)$, can be found from $s(t)$ entirely in the time domain using the Hilbert Transform of $s(t)$, denoted $H\{s(t)\}$ or $\hat{s}(t)$. This transform is defined by

$$s(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau} \, d\tau \qquad (2\text{-}2)$$

where P denotes a principle value at $t = \tau$. A derivation of this transform is given by Oswald (24). Some of the properties of the transform are given by Titchmarch (30), Stewart (28), and Dugundji (4).

Actually, some intuition may be generated for this transform by noting that under a much broader but still consistent definition of Fourier Transform, Lighthill (20) has shown that the Fourier Transform of $f(t) = \frac{1}{t}$ is given by $-\pi\ j\ \text{sgn}\ f$.* Then if one views Eq. (2-2) as a convolution integral and recalls that the convolution of two time functions is equivalent to multiplication of their spectrums, it is apparent that the Fourier Transform of $\hat{s}(t)$ is given by

$$\mathcal{F}\left\{\hat{s}(t)\right\} = -j\ \text{sgn}\ f\ \mathcal{F}\left\{s(t)\right\} \tag{2-3}$$

and the analytic signal may be defined equivalently by

$$s_+(t) = s(t) + j\ \hat{s}(t) \tag{2-4}$$

The advantage in using $s_+(t)$ rather than $s(t)$ in signal representation and analysis problems will become more apparent shortly.

2.3 Principle Results of Gabor

Gabor chose to base his definitions of time duration and bandwidth on the first and second normalized moments of $s_+(t)$ and its transform, $S_+(f)$. These normalized moments may be defined as follows

$$\overline{t^n} = \frac{\int_{-\infty}^{\infty} s_+^*(t)\ t^n\ s_+(t)\ dt}{\int_{-\infty}^{\infty} s_+^*(t)\ s_+(t)\ dt} \tag{2-5}$$

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* Sgn stands for signum and sgn x, a notation commonly used in this branch of mathematics, is defined as being +1 for $x > 0$ and -1 for $x < 0$.

$$\overline{f^n} = \frac{\int_{-\infty}^{\infty} s_+^*(f)\ f^n\ s_+(f)\ df}{\int_{-\infty}^{\infty} s_+^*(f)\ s_+(f)\ df} \tag{2-6}$$

The effective duration, Δt, and the effective bandwidth, Δf, may then be defined using Eqs. (2-5) and (2-6) as

$$\Delta t = \left[2\pi\overline{(t - \overline{t})^2}\right]^{\frac{1}{2}} \tag{2-7}$$

$$\Delta f = \left[2\pi\overline{(f - \overline{f})^2}\right]^{\frac{1}{2}} \tag{2-8}$$

Thus, duration and bandwidth are defined as being proportional to the root of the second moment, or rms deviation, about the mean epoch with a corresponding definition for bandwidth.

It may now be seen, too, that had s(t) been used instead of $s_+(t)$, all the odd ordered moments of Eq. (2-6) for a real time signal would have been zero.

With the above definitions Gabor shows, using a form of the Schwarz inequality, that

$$\Delta t\ \Delta f \geq \frac{1}{2} \tag{2-9}$$

Eq. (2-9) is one of the principle results of Gabor's presentation. It is his thesis that, due to this result, the information content of a signal may be measured in terms of its time-bandwidth product.

Specifically, Gabor shows that the signal for which the equality in Eq. (2-9) is satisfied is given by

$$s_+(t) = \exp\left[-\alpha^2(t - t_o)^2 + j(2\pi f_o t + \phi)\right] \quad (2\text{-}10)^*$$

and its spectrum

$$S_+(f) = \frac{\sqrt{\pi}}{\alpha} \exp\left\{-\left(\frac{\pi}{\alpha}\right)^2 (f - f_o)^2 - j\left[2\pi t_o(f - f_o) - \phi\right]\right\} \quad (2\text{-}11)^{**}$$

where α controls the sharpness of this pulse, t_o is its epoch in time, f_o is the epoch in frequency, and ϕ is a phase constant. For $s_+(t)$ as given by Eq. (2-10) the duration and bandwidth are given by Gabor as

$$\Delta t = \sqrt{\frac{\pi}{2}}\,\frac{1}{\alpha} \quad (2\text{-}12)$$

and

$$\Delta f = \frac{\alpha}{\sqrt{2\pi}} \quad (2\text{-}13)$$

As a result of the preceeding Gabor decided to repre-

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* It should be noted that even though Gabor used the notation for the analytic signal here, the function given by Eq. (2-10) is not an analytic signal. That is, the spectrum of this function is not zero for negative f as may be seen from Eq. (2-11).

** Gabor apparently erred here as the $\frac{\sqrt{\pi}}{\alpha}$ factor did not appear in the paper. It has been correctly added to Eq. (2-11).

sent any arbitrary signal on a plane which has time along one axis and frequency along the other. Since the minimum area which any signal could occupy was shown to be one half, he proposed to divide up the time-frequency plane into rectangles of length Δt and width Δf centered at the point (t_o, f_o) and to associate with each such rectangle one quantum of information.

He further stated that any signal can be expanded into a presumably infinite series, each term of which is a time and frequency shifted version of the elementary signal given in Eq. (2-10). The time shifting was to be by a distance $n\Delta t$ and the frequency shifting by a distance $\frac{k}{\Delta t}$ where n and k are integers, and in this manner the whole area of the time-frequency plane could be represented. That is, if it is assumed that the time and frequency shifted elementary signals exist only in their rectangle of area 1/2, then such an expansion will cover the whole time-frequency plane completely but without overlap.

One possible form put forth for such a representation of an arbitrary signal, s(t), was given as

$$s(t) = \sum^{n} \sum^{k} c_{nk} \exp\left[-\pi \frac{(t - n\Delta t)^2}{2\ (\Delta t)^2}\right] \exp\left[j2\pi k \frac{t}{\Delta t}\right] \qquad (2\text{-}14)$$

where the c_{nk} are complex constants. Sine-cosine forms were also presented, but they are not of additional interest here. Note that in Eq. (2-14) the terms are centered at a distance $\frac{1}{\Delta t}$ in the frequency direction so that each term must cover two elementary areas. However, the c_{nk} are complex and Gabor's argument was that the real and the imaginary parts, therefore, actually assign two numbers to this double sized elementary area.

2.4 Extensions of Gabor's Representation by Lerner

Although there were a number of further investigations of some of Gabor's original ideas, principally those of the analytic signal and the uncertainty relation, little additional effort was apparently expended on two dimensional representations until 1959 when Lerner (19) proposed some generalizations to this representation. Gabor had placed the emphasis of his work on the information content of the signal; Lerner presented this representation as a signal analysis tool. It was still his intention to make an expansion in terms of an elementary function. Each term was still to be a time and frequency shifted version of the elementary function. But for the purpose of a signal analysis tool he suggested that choosing the elementary function of the expansion as the one which has minimum time-bandwidth product is unnecessarily restrictive. Any function might be used as the elementary signal depending on the application. It is not even necessary that the elementary signal chosen be essentially confined to a given time duration or a given bandwidth.

Lerner further suggested that the shape of the fundamental areas in the time-frequency plane may be chosen with arbitrary length-to-width ratio. Gabor had already shown this to be possible with his representation by simply varying the constant, α, of Eq. (2-10).

A third suggestion was that from the elementary signal chosen, an orthonormal set could be formed. More specifically, call the elementary function v(t). From it define

$$v_{mn}(t) = v(t - n\Theta)\ \varepsilon^{j2\pi mt/\Theta} \qquad (2\text{-}15)$$

Then a set of functions, $u_{mn}(t)$, called unit functions, may be defined as linear combinations of the $v_{mn}(t)$ such that the $u_{mn}(t)$ form a complete orthonormal set. A matrix method was given by which the coefficients of these linear combinations may be obtained.

2.5 Discussion

Gabor put forth a number of new and interesting ideas. He was apparently the first to use the Hilbert Transform operation for problems of this type. Certainly the concept of the two dimensional representation of a signal was interesting to say the least. To express a function of time in such a manner that its structure is apparent in both the time and frequency domain rather than the time or the frequency domain was an interesting and useful thought.

It, perhaps, should be mentioned here that, in one sense, the term "two dimensional representation" is misleading. Notice that what is suggested is to take a function of time, a one dimensional representation, and express it in two dimensions, i.e., an additional dimension without requiring additional information. The point is, the two dimensions are not unrelated. In spacial dimensions, where the coordinates are unrelated, this is roughly analogous to measuring the width of this page and from this alone determining what the length of the page is.

Gabor's whole mathematical analysis is heavily based on the mathematics of quantum mechanics, even to the point of the notation he used. For example, the familiar operator notation of quantum mechanics was defined, and the notation used for the analytic time function was $\psi(t)$.* with its spectrum, $\phi(f)$, both symbols commonly used in quantum mechanics. Gabor certainly cannot be criticized for noting the analogy between the mathematics of these two disciplines. Indeed, some of the greatest advances have been made by noting such analogies. But following the analogy too far did cause him to fail in his attempt to define information mathematically. For, although he was able to define the minimum area occupied by a signal, he was not able to use this concept to show how to determine a number which specifies the information content of any given message in a unique manner. In general, there

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* In this discussion it was necessary to change the notation from $\psi(t)$ to $s_t(t)$ since the notation, $\psi(t)$, would conflict with that to be introduced in the next chapter.

is some non-zero number associated with every rectangle of the diagram for an arbitrary signal. Thus, in general, every signal contains infinite information in this sense. No particular significance was attached to the size of these numbers with regard to information content. And really none can except that they show in what portion of the time-frequency plane the signal is concentrated. This demonstrates again the point that while the energy and information of a signal are related, they are not identical.

In reality the uncertainty relation only reflects in two dimensions, for example, what the sampling theorem does in time domain analysis, i.e., that it takes at least 2 Δf numbers per second to represent a time function which has a bandwidth, Δf, or, as a second example, what the Fourier series gives in the frequency domain for periodic functions, i.e., that it takes 2 Δf Δt numbers to represent the periodic function where in this case Δt is the period (duration) and Δf is the bandwidth of the representation ($\frac{n}{\Delta t}$ in this case). Even this reflection is not as strong as it may seem. For it is seen that in the definition of duration and bandwidth, Eqs. (2-7) and (2-8), it was necessary to insert a factor of 2π arbitrarily in order to make the time-bandwidth product equal one half. The uncertainty relation, therefore, compares with the Fourier series and the sampling theorem only to within an order of magnitude.

Perhaps part of the confusion for Gabor arose because the energy of his elementary signal depends on the length-to-width ratio of the elementary rectangle and not on the area of it as might be expected. The attention to energy considerations might have caused him to notice his error in calculating the Fourier Transform of $s_+(t)$ before making the erroneous statement that the elementary signal as given by Eqs. (2-10) and (2-11) "passes into a 'delta function'" as $\alpha \rightarrow \infty$.

Perhaps the biggest question left unanswered by both Gabor and Lerner is the question of convergence of their representations. Gabor does not mention this question at all, and Lerner merely states without proof or reference that his functions $v_{mn}(t)$ "are complete in the sense that we may write almost any 'well-behaved' signal, s(t), as $s(t) = \Sigma\, a_{mn}\, v_{mn}(t)$" and that the $u_{mn}(t)$ are an "exhaustive listing" of the functions obtained from them. Intuition may lead one to the same conclusion, but, to say the least, it would be reassuring to have a more concise discussion of the circumstances and type of convergence. Neither has given any reason for a fundamental area of 0.50 rather than, for example, 0.49 so far as series representation and convergence are concerned.

Lerner's greatest contributions, then, were the realization that this representation was better suited for signal analysis than information theory and the method of orthogonalization of the elementary functions. His suggestion that the elementary signals need not be largely confined to one rectangle of the time-frequency plane, while perhaps true, is not nearly so helpful when it is realized that the binomial expansion in his matrix orthogonalization process will not converge unless the elementary signals are largely confined to their own rectangle only. And, from a practical standpoint, the amount of work involved in determining the coefficients for either Gabor's or Lerner's representation would be considerable. This is further evidenced by the fact that neither author published any specific numerical examples.

CHAPTER 3

The Spheroidal Functions and the ψ Functions

The origin of the angular prolate spheroidal functions is given and a list of a number of their properties is presented. After definition of the notation to be used for bandlimitation and timelimitation is given, the ψ functions are defined and their pertinent properties are developed.

3.1 Introduction

The angular prolate spheroidal functions arise from the separation of the scalar wave equation expressed in prolate spheroidal coordinates and have been applied to the solution of a number of boundary value problems, principally in the areas of electromagnetic theory, acoustics, and quantum mechanics. Books by Flammer (6), Stratton, et al. (29), Morse and Feshbach (23), and Meixner and Schäfke (22) contain considerable information concerning the properties of these functions for application to such problems. Flammer and Meixner and Schäfke give extensive lists of further references.

However, recently Slepian, Pollak, and Landau (18), (25), and (27) in a series of 3 papers have pointed out a number of properties of these functions which make them valuable in certain signal representation problems without reference to a coordinate system.

It should be noted that the various authors do not agree on the notation for the solutions to the separated wave equation nor on the method of their normalization.* For the presentation in section 3.2 the notation of Flammer will be used since it seems the most widely accepted and the most useful. The numerical data used in the calculations of constants presented was taken from the tables given in Stratton, et al., since it is the most extensive available to date. As a result the numerical values of u_n given in Table 3-1 assume the normalization of Little and Corbato in Stratton, et al. The notation used in the definition of the ψ functions follows that of Slepian and Pollak (27).

3.2 Solutions to the Angle Function Differential Equation

Upon separation of the scalar wave equation expressed in prolate spheroidal coordinates it is found that the angle functions must satisfy the differential equation

$$\frac{d}{dt}\left[(1 - t^2)\frac{dS_{mn}(c,t)}{dt}\right] + \left[\chi_{mn} - c^2t^2 - \frac{m^2}{1 - t^2}\right]S_{mn}(c,t) = 0 \tag{3-1}$$

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*The problem is so acute that in Stratton, et al. both the notation and normalization of Stratton and Chu in the theoretical portion of this volume differ from that used in the numerical tables prepared by Little and Corabato, the theoretical portion having been written a number of years prior to the preparation of the tables.

where c is a real, positive parameter or zero. It is found that continuous solutions in the closed t interval (-1,1) exist only for certain discrete, real, positive values of χ. Since this equation reduces to the associated Legendre equation when c = 0, the solutions to this equation, known as prolate spheroidal angle functions of the first kind, of order m and degree n, are frequently represented in terms of an infinite sum of associated Legendre functions as

$$S_{mn}(c,t) = \sum_{r=0,1}^{\infty}{}' \; d_r^{mn}(c) \; P_{m+r}^{m}(t) \qquad (3\text{-}2)$$

where the prime over the summation sign indicates that the sum is taken over only even values of r when n - m is even and over only odd values of r when n - m is odd. By substitution of Eq. (3-2) into Eq. (3-1), after some manipulation, it is possible to generate a recursion formula for the coefficients $d_r^{mn}(c)$. The numerical values of these coefficients are available in Stratton, et al. for m: 0(1)8, n: m(1)8 and c: 0(0.1)1.0(0.2)8.0. Only the zeroth order angle functions are of interest here and therefore, henceforth m = 0.

The solution of the second portion of the separated scalar wave equation results in the radial functions, the first kind of which is designated $R_{mn}^{(1)}(c,t)$. Here, too, only the zeroth order function is of interest, so m = 0.

Listed below are the pertinent properties given in the literature, most of which were given by Meixner and Schäfke and were quoted by Slepian and Pollak.

The angle functions, $S_{0n}(c,t)$:

1. Are continuous in the t interval (-1,1) and real for real t.
2. May be extended to be entire functions of the complex variable t.
3. Are continuous functions of c for $c \geqslant 0$.
4. Reduce uniformly to the Legendre Polynomials, $P_n(t)$, in (-1,1) as $c \to 0$.
5. Are orthogonal in the t interval (-1,1).
6. Are complete in (-1,1) over the class of functions integrable in absolute square in that interval.
7. Are orthogonal in $(-\infty, \infty)$.
8. Are complete in $(-\infty, \infty)$ over the class of band-limited functions.
9. Have exactly n zeroes in (-1,1).
10. Are even for n even or odd for n odd.

11(a) Are normalized by Flammer, and Stratton and Chu such that $S_{0n}(c,0) = P_n(0)$ n even

and $S'_{0n}(c,0) = P'_n(0)$ n odd

11(b) Are normalized by Little and Corbato, and Morse and Feshbach such that $S_{0n}(c,1) = P_n(1) = 1$.

11(c) Are normalized by Meixner and Schäfke such that

$$\int_{-1}^{1} \left[S_{0n}(c,t)\right]^2 dt = \frac{2}{2n+1}$$

12. Are related to the radial functions by the equation $R_{0n}^{(1)}(c,t) = K_n(c)\ S_{0n}(c,t)$ where $K_n(c)$ is real.

13. Satisfy the following relation for all t, real or complex,

$$\frac{2c}{\pi}\left[R_{0n}^{(1)}(c,1)\right]^2 S_{0n}(c,t) = \int_{-1}^{1} \frac{\sin c(t-s)}{\pi\ (t-s)}\ S_{0n}(c,s)ds \tag{3-3}$$

14. Satisfy the following relation for all t, real or complex,

$$2j^n\ R_{0n}^{(1)}(c,1)S_{0n}(c,t) = \int_{-1}^{1} \varepsilon^{jcts}\ S_{0n}(c,s)ds \tag{3-4}$$

It was the orthogonality over two different points sets, properties 5 and 7, together with the integral relation, Eq. (3-3), that interested Slepian, Landau and Pollak. In addition to these, the Eq. (3-4) will be of particular interest here.

3.3 Bandlimiting and Timelimiting Notation

Before proceeding, the following notation will be defined. Let D_ℓ be an operator which timelimits an arbitrary time function, f(t), to an interval of length T seconds centered at $t = \ell T$.

$$D_\ell f(t) = \begin{cases} f(t) & (\ell - \frac{1}{2})T < t < (\ell + \frac{1}{2})T \\ 0 & t < (\ell - \frac{1}{2})T, \qquad t > (\ell + \frac{1}{2})T \end{cases} \tag{3-5}$$

D without subscript is taken to mean $\ell = 0$.

Similarly, define B_p to be an operator which bandlimits an arbitrary function, $f(t)$, whose spectrum is $F(\omega)$, to a band of length 2Ω radians per second centered at $2p\Omega$.

$$B_p F(\omega) = \begin{cases} F(\omega) & (2p - 1)\Omega < \omega < (2p + 1)\Omega \\ 0 & \omega < (2p - 1)\Omega, \qquad \omega > (2p + 1)\Omega \end{cases} \tag{3-6}$$

Here, too, B without subscript is taken to mean p = 0.

It may easily be shown that the bandlimiting operation of Eq. (3-6) may be carried out on an arbitrary function of time, $f(t)$, entirely in the time domain by the relation

$$B_p f(t) = \int_{-\infty}^{\infty} f(\tau)\varepsilon^{jp\Omega(t - \tau)} \frac{\sin \Omega(t - \tau)}{\pi(t - \tau)} d\tau \tag{3-7}$$

In an analogous manner, the timelimiting operation may be carried out entirely in the frequency domain by

$$D_\ell F(\omega) = \int_{-\infty}^{\infty} F(s)\varepsilon^{-j\ T(\omega - s)} \frac{\sin(\omega - s)\frac{T}{2}}{\pi(\omega - s)} ds \tag{3-8}$$

3.4 The ψ Functions

As defined and normalized in the manner for which numerical data is available, the angle functions have varying amounts of energy in $(-\infty,\infty)$. It is convenient for the present purposes to scale and renormalize these functions in the manner used by Slepian and Pollak. To this end define

$$\lambda_n(c) = \frac{2c}{\pi}\left[R_{0n}^{(1)}(c,1)\right]^2 \qquad n = 0,1,2,\ldots \tag{3-9}$$

and also

$$\left[u_n(c)\right]^2 = \int_{-1}^{1}\left[S_{0n}(c,t)\right]^2 dt \tag{3-10}$$

Finally, define

$$\psi_n(c,t) = \sqrt{\frac{2\,\lambda_n(c)}{T\,u_n^2(c)}}\;S_{0n}(c,\tfrac{2t}{T}) \tag{3-11}*$$

It is seen that the constants $u_n(c)$ are the rms values of the prolate spheroidal functions in the t interval (-1,1). The numerical values for some of these constants are given in Table 3-1. The significance of the constant $\lambda_n(c)$ will become evident presently.

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* It is necessary here to digress slightly from the notation of Slepian and Pollak to correct an error. Their definition of the ψ functions did not contain the $\sqrt{2/T}$ factor. If it were included, the remainder of their results would be correct. Without it, they are not.

n	u_n (c = 0.5)	u_n (c = 1.0)	u_n (c = 2.0)	u_n (c = 3.0)	u_n (c = 4.0)
0	1.45422	1.58317	2.24546	3.97191	8.15470
1	0.824757	0.850710	0.974434	1.27105	1.95566
2	0.632890	0.635089	0.655957	0.725044	0.885229
3	0.534594	0.534890	0.537622	0.550082	0.589321
4	0.471426	0.471509	0.472134	0.474531	0.482425
5	0.426411	0.426443	0.426670	0.427433	0.429512
6	0.392240	0.392254	0.392356	0.392687	0.393516
7	0.365161	0.365168	0.365219	0.365386	0.365794
8	0.343022	0.343026	0.343054	0.343145	0.343367

Table 3-1.

Table of u_n Normalization Coefficients.

Henceforth, when no confusion will result from so doing, the parameter c will be omitted from the notation for these constants and functions, i.e. $u_n(c)$ will be designated simply u_n.

It is easy to show from property 14, Eq. (3-4), of section 3.2 that the ψ functions are bandlimited.

$$2j^n R_{0n}^{(1)}(c,1)\; S_{0n}(c,\tau) = \int_{-1}^{1} \varepsilon^{jc\tau s} S_{0n}(c,s)ds \qquad (3\text{-}12)$$

Make the substitutions $\tau = \frac{2t}{T}$, $s = \frac{\omega}{\Omega}$, and, here and in the sequel, $c = \frac{\Omega T}{2}$. Eq. (3-12) then becomes

$$2j^n R_{0n}^{(1)}(c,1)\; S_{0n}(c,\tfrac{2t}{T}) = \frac{1}{\Omega}\int_{-\Omega}^{\Omega} S_{0n}(c,\tfrac{\omega}{\Omega})\varepsilon^{j\omega t}d\omega \qquad (3\text{-}13)$$

Multiplying both sides by $\frac{\Omega}{2\pi}\sqrt{\frac{2\ \lambda_n}{T\ u_n^2}}$ and substituting Eq. (3-11), the definition of $\psi_n(t)$, this becomes

$$\frac{\Omega}{\pi}\, j^n R_{0n}(c,1)\; \psi_n(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \psi_n(\tfrac{T\omega}{2\Omega})\varepsilon^{j\omega t}d\omega \qquad (3\text{-}14)$$

an expression given by Slepian and Pollak. Applying the definition of λ_n, Eq. (3-9), it follows that

$$j^n \sqrt{\frac{\Omega}{T\pi}}\sqrt{\lambda_n}\;\; \psi_n(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \psi_n(\tfrac{T\omega}{2\Omega})\varepsilon^{j\omega t}d\omega \qquad (3\text{-}15)$$

which shows not only that the ψ functions are bandlimited but that the functional form of the time functions in the t interval $(-\frac{T}{2},\frac{T}{2})$ and that of their spectrums in the ω interval $(-\Omega,\Omega)$ are the same apart from a constant.

Further, rewrite Eq. (3-3), property 13 of section 3.2, as

$$\frac{2c}{\pi}\left[R_{0n}^{(1)}(c,1)\right]^2 S_{0n}(c,\tau) = \int_{-1}^{1} \frac{\sin c(\tau - s)}{\pi(\tau - s)} S_{0n}(c,s)ds \tag{3-16}$$

Now make the substitutions $\tau = \frac{2t}{T}$ and $s = \frac{2\zeta}{T}$, and multiply both sides by $\sqrt{\frac{2}{T}\frac{\lambda_n}{u_n^2}}$. If the proper identifications using Eqs. (3-9) and (3-11) are made, Eq. (3-16) becomes

$$\lambda_n \psi_n(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{\sin \Omega(t - \zeta)}{\pi(t - \zeta)} \psi_n(\zeta)\, d\zeta \tag{3-17}$$

Slepian and Pollak, as had others before, recognized that the ψ functions satisfied this eigenvalue problem, and they proved that the eigenvalues are real, positive and ordered as $\lambda_0 > \lambda_1 > \lambda_2 \dots$. As a result of this they showed that the bandlimited function which loses the least portion of its energy upon being timelimited is $\psi_0(t)$, and the timelimited function which loses the least portion of its energy upon being bandlimited is $D\psi_0(t)$. Franz (7) had attacked this problem earlier. His conclusion was that one cannot concentrate the energy of a timelimited signal into a given bandwidth "essentially more" than in the case of the rectangular signal. This conclusion seems to be confirmed since the spectrum of a rectangular signal, which is of the form $\frac{\sin x}{x}$, and the spectrum of $D\psi_0(t)$, which is of the form $\psi_0(\omega)$, are quite similar in appearance. The portion of the energy of $\psi_n(t)$ in the t interval $(-\frac{T}{2},\frac{T}{2})$ may be found by

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\psi_n(t)\right]^2 dt = \frac{2\lambda_n}{T u_n^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[S_{0n}(\tfrac{2t}{T})\right]^2 dt \qquad (3\text{-}18)$$

Let $\tau = \frac{2t}{T}$. Then

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\psi_n(t)\right]^2 dt = \frac{2\lambda_n}{T u_n^2}\int_{-1}^{1} \left[S_{0n}(\tau)\right]^2 \frac{T}{2}d\tau = \lambda_n \qquad (3\text{-}19)$$

It is obvious from Eq. (3-19) and properties five and six of section 3.2 that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \psi_m(t)\ \psi_n(t)dt = \lambda_n\ \delta_{mn} \qquad (3\text{-}20)$$

where δ_{mn} is the Kronecker delta, and that the set is complete over this interval. Further, Slepian and Pollak have shown that

$$\int_{-\infty}^{\infty} \psi_m(t)\ \psi_n(t)dt = \delta_{mn} \tag{3-21}$$

by using Eqs. (3-17) and (3-20) and the identity

$$\int_{-\infty}^{\infty} \frac{\sin \Omega(t-u)}{\pi(t-u)} \frac{\sin \Omega(u-s)}{\pi(u-s)} du = \frac{\sin \Omega(t-s)}{\pi(t-s)} \tag{3-22}$$

Thus the ψ functions are orthogonal over the t interval $(-\frac{T}{2},\frac{T}{2})$ and orthonormal over the t interval $(-\infty,\infty)$.

In addition to the above, Eq. (3-17) may be rewritten

$$\lambda_n \psi_n(t) = \int_{-\infty}^{\infty} \frac{\sin \Omega(t-\zeta)}{\pi(t-\zeta)} D\, \psi_n(\zeta)\, d\zeta \tag{3-23}$$

By comparing this with Eq. (3-7), it is seen that

$$BD\, \psi_n(t) = \lambda_n\, \psi_n(t) \tag{3-24}$$

The role of the eigenvalues, λ_n, is now more apparent. Eq. (3-21) shows that each ψ function does have unit total energy and Eq. (3-20) shows that the portion of this total energy which is in the interval $(-\frac{T}{2},\frac{T}{2})$ is given by λ_n.

Or, from another viewpoint, when $\psi_n(t)$ is timelimited, the amount of energy remaining is λ_n. By Eq. (3-24) if, after being timelimited, the function is subsequently bandlimited, the amount of energy remaining is λ_n^2, and so on through subsequent timelimitations and bandlimitations.

There are other properties of these eigenvalues which are not so apparent. Shown in Table 3.2 are some numerical values for the λ_n. Slepian and Pollak pointed out that λ_n remains relatively constant for increasing n until n becomes greater than $\frac{2c}{\pi}$ and then it decreases very rapidly. The ramifications of this property aid in determining some relationships between the error in a series representation of a signal and the number of terms included in this representation, and the discussion of this will be postponed until after the series representations of Chapter 4 have been introduced.

It was shown in Eq. (3-15) that the Fourier Transform of $\psi_n(t)$ is given by

$$\mathcal{F}\{\psi_n(t)\} = \begin{cases} \frac{j^{-n}}{\sqrt{\lambda_n}} \sqrt{\frac{\pi T}{\Omega}} \; \psi_n\left(\frac{T\omega}{2\Omega}\right) & |\omega| < \Omega \\ 0 & |\omega| > \Omega \end{cases} \qquad (3\text{-}25)$$

The Fourier Transform of $D\psi_n(t)$ will now be derived. Begin again with Eq. (3-4).

$$2j^n R_{0n}^{(1)}(c,1)\; S_{0n}(c,\tau) = \int_{-1}^{1} \varepsilon^{jc\tau s} S_{0n}(c,s)ds \qquad (3\text{-}26)$$

Let $s = \frac{2t}{T}$ and $\tau = \frac{-\omega}{\Omega}$. With $c = \frac{\Omega T}{2}$ Eq. (3-26) becomes

$$2j^n R_{0n}^{(1)}(c,1)\; S_{0n}\left(c,\frac{-\omega}{\Omega}\right) = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} S_{0n}\left(c,\frac{2t}{T}\right)\varepsilon^{-j\omega t}dt \qquad (3\text{-}27)$$

c = 0.5

n	L_n	h
0	3.09690	-01
1	8.58107	-03
2	3.91745	-05
3	7.21139	-08
4	7.27142	-11
5	4.63777	-14
6	2.04135	-17
7	6.57662	-21
8	1.61828	-24

c = 1.0

n	L_n	h
0	5.72582	-01
1	6.27913	-02
2	1.23748	-03
3	9.20098	-06
4	3.71793	-08
5	9.49144	-11
6	1.67157	-13
7	2.15445	-16
8	2.12072	-19

c = 2.0

n	L_n	h
0	8.80560	-01
1	3.55641	-01
2	3.58676	-02
3	1.15223	-03
4	1.88816	-05
5	1.93585	-07
6	1.36606	-09
7	7.04888	-12
8	2.77679	-14

c = 3.0

n	L_n	h
0	9.75829	-01
1	7.09963	-01
2	2.05139	-01
3	1.82038	-02
4	7.08147	-04
5	1.65512	-05
6	2.64101	-07
7	3.07374	-09
8	2.72813	-11

c = 4.0

n	L_n	h
0	9.95886	-01
1	9.12107	-01
2	5.19055	-01
3	1.10211	-01
4	8.82788	-03
5	3.81292	-04
6	1.09509	-05
7	2.27864	-07
8	3.60655	-09

Table 3-2.

Table of Eigenvalues $\lambda_n = L_n \times 10^h$.

Multiply both sides by $\sqrt{\frac{2\lambda_n}{T u_n^2}}$ and multiply the left side by $\sqrt{\frac{2c}{\pi}}\sqrt{\frac{\pi}{2c}}$. Eq. (3-27) then becomes

$$Tj^n \sqrt{\frac{\pi}{2c}}\sqrt{\frac{2c}{\pi}}\, R_{0n}^{(1)}(c,1)\, \psi_n\left(\frac{-T\omega}{2\Omega}\right) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \psi_n(t)\varepsilon^{-j\omega t}dt \qquad (3\text{-}28)$$

The right hand side of this expression is seen to be the Fourier Transform of $D\psi_n(t)$. Since the ψ functions are even or odd depending on whether n is even or odd and, identifying λ_n on the left, this expression may be written, finally, as

$$\mathcal{F}\left[D\psi_n(t)\right] = j^{-n}\sqrt{\frac{T\pi}{\Omega}}\sqrt{\lambda_n}\,\psi_n\left(\frac{T\omega}{2\Omega}\right) \qquad (3\text{-}29)$$

Comparing Eqs. (3-29) and (3-25) it is interesting to note that

$$_n(t) = \begin{cases} \frac{1}{\lambda_n}\,\mathcal{F}\{D\,\psi_n(t)\} & |\omega| < \Omega \\ 0 & |\omega| > \Omega \end{cases} \qquad (3\text{-}30)$$

There remains now to be demonstrated the orthonormality of the frequency shifted ψ functions. That is, it will be shown that

$$\int_{-\infty}^{\infty} \psi_m(t)\varepsilon^{j2p\Omega t}\psi_n(t)\varepsilon^{-j2q\Omega t}dt = \delta_{mn}\,\delta_{pq} \qquad (3\text{-}31)$$

where p and q are any positive or negative integers or zero. This may be shown very simply by defining $r = p - q$ and rewriting Eq. (3-31) as

$$\int_{-\infty}^{\infty} \psi_m(t)\, \psi_n(t) \varepsilon^{j2r\Omega t} dt = \delta_{mn}\, \delta_{pq} \qquad (3\text{-}32)$$

Applying Parseval's theorem, this becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_m(\omega)\phi_n(-\omega + 2r\Omega)d\omega = \delta_{mn}\, \delta_{pq} \qquad (3\text{-}33)$$

where the Fourier Transform of $\psi_i(t)$ is denoted by $\phi_i(\omega)$. But $\phi_i(\omega)$ has been shown to be bandlimited to the region for which the magnitude of the argument is less than Ω and, thus, the two terms of the integrand are concurrently non-zero only if $r = 0$ or $p = q$. But if $r = 0$ Eq. (3-32) reduces to the infinite orthogonality relation, Eq. (3-21), and, therefore, Eq. (3-31) is established.

CHAPTER 4

Two Dimensional Representations Using Prolate Spheroidal Functions

In this chapter the properties of the ψ functions developed previously will be applied to obtain two new two dimensional representations in which the time-frequency structure and the convergence properties are clearly evident. It will be seen that the fundamental area in the time-frequency plane to be represented, in addition to having arbitrary width-to-heighth ratio, is not of fixed but largely arbitrary size. Attention is given to errors arising from truncation of the series. Several useful properties and applications of these representations are presented.

4.1 Introduction

The decomposition of a function of time (or a function of frequency) into a form in which both the time and frequency structure are apparent can be extremely confusing. In order to reduce the complexity to at least some extent, the two representations to be presented will be formed in two steps. In section 4.2 an intermediate representation is generated by approximation in the time domain. In section 4.3 this representation is further broken down, the result being one of the desired two dimensional representations. The second two dimensional representation is generated in a similar manner in sections 4.4 and 4.5. This approach has been purposely chosen to place in evidence the time-frequency structure and to make apparent the error which arises due to truncation and the location of this error in the time-frequency plane.

In order to prevent the notation from becoming more cumbersome than necessary, the following convention is adopted: Unless specifically stated otherwise, a summation sign, Σ, with a summation index p,q,k, or ℓ associated with it will signify a summation over terms for which the summation index takes on all integer values, both positive and negative, and zero; a summation sign with a summation index, m or n associated with it will signify a summation over terms for which the index takes on only positive integer values and zero. The two dimensional plane is defined to have time as the abscissa and frequency as the ordinate, and p and q will always be used as indices in the vertical or frequency direction. The indexes k or ℓ will be used as indices in the horizontal or time direction, while n or m will refer to indexing over the set of orthogonal functions.

4.2 A Representation by Approximation in the Time Domain

Assume that an arbitrary function of time, f(t), or its spectrum, $F(\omega)$, has been given. Certainly $f(t)$ may be written as

$$f(t) = \ldots \frac{1}{2\pi}\int_{-3\Omega}^{-\Omega} F(\omega)\varepsilon^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)\varepsilon^{j\omega t}d\omega \ldots +$$

$$\frac{1}{2\pi}\int_{(2p-1)\Omega}^{(2p+1)\Omega} F(\omega)\varepsilon^{j\omega t}d\omega \ldots \qquad (4\text{-}1)$$

$$f(t) = \sum^{p} B_p f(t) \qquad (4\text{-}2)$$

Now consider only the p^{th} term of Eq. (4-2), $B_p f(t)$. This single term, being a bandlimited function, may certainly be represented in terms of the ψ functions as

$$B_p f(t) = \sum^{n} a_{np\ell}\, \psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \qquad (4\text{-}3)^*$$

---

* As pointed out by Courant and Hilbert (2) even though the set of orthogonal functions on the right is a complete set, this equation may not be a true equality in the strict sense of the word. However, the convergence on the right is in the mean square sense

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left| B_p f(t) - \sum_{n=0}^{N} a_{np\ell}\, \psi_n(t - \ell T)\varepsilon^{j2p\Omega t}\right|^2 dt = 0$$

(cont. on following page)

Clearly the set of orthogonal functions on the right of Eq. (4-3) is complete since by multiplying both sides by $\varepsilon^{-j2p\Omega t}$ and translating a distance ℓT, this expression becomes

$$\varepsilon^{-j2p\Omega(t + \ell T)} B_p f(t + \ell T) = \sum^{n} a_{np\ell} \psi_n(t) \qquad (4\text{-}4)$$

The function on the left of Eq. (4-4) is now bandlimited to $-\Omega < \omega < \Omega$ and certainly of finite energy, and $\{\psi_n(t)\}$ are known to be complete over this class of functions.

The coefficients $a_{np\ell}$ may be determined by multiplying Eq. (4-3) by $\psi_m(t - \ell T)\varepsilon^{-j2p\Omega t}$, integrating and applying the orthonormality relation, Eq. (3-21), as follows

$$\int_{-\infty}^{\infty} \{B_p f(t)\} \psi_m(t - \ell T)\varepsilon^{-j2p\Omega t} dt =$$

$$\sum^{n} a_{np\ell} \int_{-\infty}^{\infty} \psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \psi_m(t - \ell T)\varepsilon^{-j2p\Omega t} dt = a_{mp\ell} \qquad (4\text{-}5)$$

---

* (cont. from previous page)
and it is within this broader definition of equality, where only the mean square difference between the two sides is zero, that Eq. (4-3) and a number of similar equations throughout this chapter is written as an equality.

Therefore

$$a_{np\ell} = \int_{-\infty}^{\infty} \left\{B_p f(t)\right\} \psi_n(t - \ell T)\varepsilon^{-j2p\Omega t} dt \qquad (4\text{-}6)$$

The ease of evaluation of Eq. (4-6) may be increased by applying Parseval's Theorem and writing it as

$$a_{np\ell} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{B_p F(-\omega)\right\} \phi_n(\omega + 2p\Omega)\varepsilon^{-j(\omega + 2p\Omega)\ell T} d\omega \qquad (4\text{-}7)$$

where $\phi_n(\omega)$ is the Fourier Transform of $\psi_n(t)$. Now both of the first two terms of the integrand are non-zero only for $-(2p + 1)\Omega < \omega < -(2p - 1)\Omega$, and, thus, the equation is unchanged by replacing $B_p F(\omega)$ by simply $F(\omega)$. Therefore Eq. (4-6) may be written as

$$a_{np\ell} = \int_{-\infty}^{\infty} f(t) \psi_n(t - \ell T)\varepsilon^{-j2p\Omega t} dt \qquad (4\text{-}8)$$

Alternately, the dual orthogonality property of the ψ functions allows the $a_{np\ell}$ coefficients to be calculated by an integration over a finite range, for, because

$\left[B_p f(t)\right]\varepsilon^{-j2p\Omega t}$ is bandlimited to $|\omega| < \Omega$, the finite orthogonality relation, Eq. (3-20) may be applied. In this case Eq. (4-5) would have been

$$\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left\{B_p f(t)\right\} \psi_m(t - \ell T)\varepsilon^{-j2p\Omega t} dt \;=$$

$$\sum^{n} a_{np\ell} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \psi_n(t - \ell T)\varepsilon^{j2p\Omega t}\psi_m(t - \ell T)\varepsilon^{-j2p\Omega t} dt \tag{4-9}$$

and therefore

$$a_{np\ell} = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left\{B_p f(t)\right\} \psi_n(t - \ell T)\varepsilon^{-j2p\Omega t} dt \tag{4-10}$$

The fact that the coefficients can be calculated by integration over a finite range is of considerable importance in the numerical evaluation of these coefficients.

This orthogonal series, Eq. (4-3), may be substituted into Eq. (4-2) to obtain a representation of the arbitrary function, f(t), as follows

$$f(t) = \sum^{p} \sum^{n} a_{np\ell} \psi_n(t - \ell T)\varepsilon^{j2p\Omega t}$$

where the $\{a_{np\ell}\}$ are calculated by Eqs. (4-6), or (4-10).

The structure of this representation in both the time and the frequency domain may be readily seen. In the frequency domain every term is restricted to a band of width, 2Ω, centered at $2p\Omega$, and for a given p the inclusion of each additional term in n decreases, in an optimum least square sense, the mean square difference between the spectrum of f(t) in this band and the spectrum of the series representation of f(t).

Concerning the time domain structure, it is known from the finite orthogonality relation, Eq. (3-20), and the previously mentioned properties of the eigenvalues, λ_n, that, at least for the first few terms in the summation over n, the ψ functions exist largely in the t interval $(-\frac{T}{2},\frac{T}{2})$. Thus, this kind of representation lends itself to the decomposition of a signal which exists largely only in a time interval T seconds, centered at $t = \ell T$. Since both T and Ω may be picked arbitrarily, the representation is quite flexible. However, $c = \frac{\Omega T}{2}$, and the magnitude of the λ_n, for a given n, depends on c; therefore, the number of terms necessary in the summation over n surely depends on c as might be expected.

The mean square error due to truncation of the series representation of $B_p f(t)$ is easily obtainable. Assume that for a given p the summation over n is truncated after N_p terms. The mean square error is given by

$$M_{p\infty} = \int_{-\infty}^{\infty} \left| B_p f(t) - \sum_{n=0}^{N_p} a_{np\ell} \psi_n(t - \ell T) \varepsilon^{j2p\Omega t} \right|^2 dt =$$

$$\int_{-\infty}^{\infty} \left| \sum_{n=N_p+1}^{\infty} a_{np\ell} \psi_n(t - \ell T) \varepsilon^{j2p\Omega t} \right|^2 dt = \sum_{n=N_p+1}^{\infty} \left| a_{np\ell} \right|^2$$

(4-11)

Further, it may easily be shown, by carrying out the operation on the left above that the size of this error may be calculated by

$$M_{p\infty} = \int_{-\infty}^{\infty} \left| B_p f(t) \right|^2 dt - \sum_{n=0}^{N_p} \left| a_{np\ell} \right|^2 \qquad (4\text{-}12)$$

In a similar manner it may be shown that the mean square error in the time interval $(\ell - \frac{1}{2})T<t<(\ell + \frac{1}{2})T$ in the representation of $B_p f(t)$ by the truncated series is given by

$$M_{pT} = \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left|B_p f(t)\right|^2 dt - \sum_{n=0}^{N_p} \left|a_{np\ell}\right|^2 \lambda_n \tag{4-13}$$

Since, after the first few terms in n, λ_n becomes small very fast, it may be expected that in the frequency interval $(2p - 1)\Omega<\omega<(2p + 1)\Omega$ the fit in the time interval $(\ell - \frac{1}{2})T<t<(\ell + \frac{1}{2})T$ will be good after only a few terms. Since λ_n depends on c, the number of terms necessary also depends on c.

The mean square error of the representation of f(t), Eq. (4-12), may be obtained in a manner similar to the above. If only the terms $-P\leq p\leq P$ and $0\leq n\leq N_p$ are included, the error would be

$$\int_{-\infty}^{\infty} \left| f(t) - \sum_{p=-P}^{P} \sum_{n=0}^{N_p} a_{np\ell}\psi_n(t - \ell T)\varepsilon^{j2p\Omega t}\right|^2 dt =$$

$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt - \sum_{p=-P}^{P} \sum_{n=0}^{N_p} a^*_{np\ell} \int_{-\infty}^{\infty} f(t)\psi_n(t - \ell T)\varepsilon^{-j2p\Omega t} dt$$

$$- \sum_{p=-P}^{P} \sum_{n=0}^{N_p} a_{np\ell} \int_{-\infty}^{\infty} f(t)\psi_n(t - \ell T)\varepsilon^{j2p\Omega t} dt$$

$$+\sum_{p=-P}^{P}\sum_{q=-P}^{P}\sum_{n=0}^{Np}\sum_{m=0}^{N_q} a_{np\ell}a^*_{np\ell}\int_{-\infty}^{\infty}\psi_n(t-\ell T)\varepsilon^{-j2p\Omega t}\psi_m(t-\ell T)\varepsilon^{-j2q\Omega t}dt \tag{4-14}$$

Now from Eq. (4-8) the integral in the second term on the right is $a_{np\ell}$ and the integral in the third term is $a^*_{np\ell}$. Due to Eq. (3-31) the integral in the last term is equal to the product of the Kronecker deltas, $\delta_{mn}\delta_{pq}$. Therefore, Eq. (4-14) reduces to

$$\int_{-\infty}^{\infty}|f(t)|^2dt - \sum_{p=-P}^{P}\sum_{n=0}^{N_p}|a_{np\ell}|^2 \tag{4-15}$$

That is, the error is simply the difference between the mean square value of f(t) and that of its representation, and this is true even though N_p may be different for each value of p. Of course, completeness insures that this error may be made as small as desired by including enough terms. It is of considerable interest to note here, too, that the two dimensional structure of the error is also apparent.

4.3 A Further Decomposition to Two Dimensions

The representation of a signal developed in section 4.2, while quite satisfactory for a certain class of signal representation problems, does not have the general two dimensional structure desired. It cannot deal satisfactorily with a signal if it is desired to break up the time duration of the signal into more than one interval of length T seconds.

It is possible to obtain the desired two dimensional structure by carrying out a further decomposition of the representation of section 4.2. Begin once more with a representation of the arbitrary function, f(t), as in Eq. (4.2).

$$f(t) = \sum_{p} B_p f(t) \tag{4-16}$$

Again consider only the p^{th} term of this series, $B_p f(t)$. The term may be represented as

$$B_p f(t) = \sum_{\ell} \sum_{n} b_{np\ell} D_\ell \psi_n(t - \ell T) \varepsilon^{j2p\Omega t} \tag{4-17}$$

In this case, for each ℓ, each term is zero outside of the time interval $(\ell - \frac{1}{2})T < t < (\ell + \frac{1}{2})T$, and, although all terms are not confined entirely to the frequency interval $(2p - 1)\Omega < \omega < (2p + 1)\Omega$, they are to as great an extent as is possible in view of the appropriate properties of the ψ functions pointed out in section 3.4. Once again the orthogonal set on the right side of Eq. (4-17) is complete since $\{\psi_n(t)\}$ is complete in the time interval $(-\frac{T}{2}, \frac{T}{2})$.

The coefficients $b_{np\ell}$ may be calculated by multiplying Eq. (4-17) by $D_k \psi_m(t - kT)\varepsilon^{-j2p\Omega t}$ and integrating.

$$\int_{-\infty}^{\infty} \left\{B_p f(t)\right\} D_k \psi_m(t - kT) \varepsilon^{-j2p\Omega t} dt$$

$$= \sum^{\ell} \sum^{n} b_{np\ell} \int_{-\infty}^{\infty} D_k \psi_m(t - kT) \varepsilon^{-j2p\Omega t} D_\ell \psi_n(t - \ell T) \varepsilon^{j2p\Omega t} dt \tag{4-18}$$

and therefore

$$b_{np\ell} = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left\{B_p f(t)\right\} \psi_n(t - \ell T) \varepsilon^{-j2p\Omega t} dt \tag{4-19}$$

An expression for $b_{np\ell}$ may be obtained for which $f(t)$, and not $B_p f(t)$, is required. To obtain this expression substitute Eq. (3-7) into Eq. (4-19).

$$b_{np\ell} = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \int_{-\infty}^{\infty} f(\tau) \varepsilon^{j2p\Omega(t-\tau)} \frac{\sin \Omega(t-\tau)}{\pi(t-\tau)} d\tau \, \psi_n(t-\ell T) \varepsilon^{-j2p\Omega t} dt \tag{4-20}$$

Reversing the order of integration, this becomes

$$b_{np\ell} = \frac{1}{\lambda_n} \int_{-\infty}^{\infty} f(\tau)\varepsilon^{-j2p\Omega\tau} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \frac{\sin \Omega(t-\tau)}{\pi(t-\tau)} \psi_n(t-\ell T)\,dt\,d\tau \tag{4-21}$$

If now the substitution $s = (t - \ell T)$ is made and then Eq. (3-17) is applied, this reduces to

$$b_{np\ell} = \int_{-\infty}^{\infty} f(\tau)\psi_n(\tau - \ell T)\varepsilon^{-j2p\Omega\tau}d\tau \tag{4-22}$$

A representation of the complete arbitrary function, f(t) may be obtained by substituting Eq. (4-17) into Eq. (4-16).

$$f(t) = \sum_p \sum_\ell \sum_n b_{np\ell} D_\ell \psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \tag{4-23}$$

where the $\{b_{np\ell}\}$ are given by Eqs. (4-19) or (4-22).

The mean square error due to truncation of the series representation of the bandlimited function as in Eq. (4-17) may be calculated as before. For this purpose assume that only terms for $-L \leq \ell \leq L$ and $0 \leq n \leq N_\ell$ have been included. Then the mean square difference between $B_p f(t)$ and the series representation of it is given by

$$\int_{-\infty}^{\infty} \left| B_p f(t) - \sum_{\ell=-L}^{L} \sum_{n=0}^{N_\ell} b_{np\ell} \, D_\ell \psi_n(t - \ell T) \varepsilon^{-j2p\Omega t} \right|^2 dt =$$

$$\int_{-\infty}^{\infty} \left| B_p f(t) \right|^2 dt - \sum_{\ell=-L}^{L} \sum_{n=0}^{N_\ell} \left| b_{np\ell} \right|^2 \lambda_n \qquad (4\text{-}24)$$

It may be mentioned in passing that the mean square error involved in truncating the complete series representation of f(t), Eq. (4-23), is not simply the sum of the mean square errors of the representations of each $B_p f(t)$ terms. However, such a sum would give a good estimate of the total error, particularly in the case of large values of c. Basically this results from the fact that the time and frequency shifted ψ functions are not orthogonal over a finite time interval. That is, if the limits of integration of Eq. (3-31) are changed to $-\frac{T}{2}$ and $\frac{T}{2}$, the value of the integral would not be zero for m = n and $p \neq q$. This value would be expected to be small compared to the result for p = q, however, since in the frequency domain most of the energy of the first term of the integrand would be near $p\Omega$ and most of that of the second would be near $q\Omega$.

It is of interest here again to examine the two dimensional structure of this representation as given in Eq. (4-23). First, recall that for a given ℓ and p, the sum of the terms over n represents a finite piece in the time domain of a band limited function or $D_\ell B_p f(t)$. Certainly, every term in Eq. (4-23) is absolutely time limited and (the extent depending on c) nearly bandlimited when n is small. However, for a given p and ℓ, each term contains information only from an absolutely bandlimited portion of f(t) and largely from a timelimited portion. This can be seen most clearly by applying Parseval's Theorem to Eq. (4-22) and from Eq. (4-22) itself, respectively.

4.4 A Representation by Approximation in the Frequency Domain

A representation analogous to that presented in section 4.2 may be constructed by the same process used there, but beginning in the frequency domain rather than the time domain. The strength of this analogy would be more evident if the development was carried out in the frequency domain. However, since the analogy is not as important as the properties of the representations, themselves, and since calculation would usually be carried out in the time domain, the development will be done in the time domain.

To carry out this development begin with the arbitrary function, f(t), or its spectrum, $F(\omega)$, expressed as

$$F(\omega) = \ldots + \int_{-\frac{3T}{2}}^{\frac{T}{2}} f(t)\varepsilon^{-j\omega t}dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\varepsilon^{-j\omega t}dt + \ldots + \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} f(t)\varepsilon^{-j\omega t}dt + \ldots \quad (4-25)$$

$$f(t) = \sum^{\ell} D_{\ell} f(t) \quad (4-26)$$

The ℓ^{th} term, $D_{\ell}f(t)$, of Eq. (4-26) may be represented in terms of an orthogonal expansion of the ψ functions as

$$D_{\ell} f(t) = \sum^{n} c_{np\ell} D_{\ell} \psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \quad (4-27)$$

Once again, the set of functions on the right is a complete set since the ψ functions are complete over the time interval $(-\frac{T}{2},\frac{T}{2})$.

If Eq. (4-27) is substituted into Eq. (4-26) a representation valid for f(t) is obtained.

$$f(t) = \sum^{\ell}\sum^{n} c_{np\ell} D_{\ell}\psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \quad (4-28)$$

The coefficients, $c_{np\ell}$, may be calculated by multiplying Eq. (4-28) by $D_k\psi_m(t - kT)\varepsilon^{-j2p\Omega t}$ and integrating as follows

$$\int_{-\infty}^{\infty} f(t)D_k\psi_m(t - kT)\varepsilon^{-j2p\Omega t}dt =$$

$$\sum^{\ell}\sum^{n} c_{np\ell} \int_{-\infty}^{\infty} D_{\ell}\psi_n(t - \ell T)\varepsilon^{j2p\Omega t} D_k\psi_m(t - kT)\varepsilon^{-j2p\Omega t}dt = c_{mpk}\lambda_n \tag{4-29}$$

Therefore

$$c_{np\ell} = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} f(t)\psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \tag{4-30}$$

In order to examine the time-frequency structure of this representation, take the Fourier Transform of both sides of Eq. (4-28). The result is

$$F(\omega) = \sum^{\ell}\sum^{n} c_{np\ell}\hat{\phi}_n(\omega - 2p\Omega)\varepsilon^{-j(\omega - 2p\Omega)\ell T} \tag{4-31}$$

where $\hat{\phi}_n(\omega)$ is the Fourier Transform of $D\psi_n(t)$. Since $\hat{\phi}_n(\omega)$ differs from $\psi_n(\omega)$ (or rather $\psi_n(\frac{T\omega}{2\Omega})$) by only a constant, the above representation provides, for each ℓ, a least squares fit to the entire spectrum of a timelimited function but in such a way that, as the number of n terms is increased from zero, the fit will be best first near $\omega = 2p\Omega$.

The time domain approximation is apparent from Eq. (4-28) itself. The sum of the terms over n for each ℓ simply gives a least squares fit to a timelimited portion of f(t).

The mean square error in any time interval due to truncation of the summation over n is available from Eq. (4-27) as follows:

$$\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left| f(t) - \sum_{n=0}^{N_\ell} c_{np\ell} \psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \right|^2 dt =$$

$$\int_{(\ell+\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left| f(t) \right|^2 dt - \sum_{n=0}^{N_\ell} \left| c_{np\ell} \right|^2 \lambda_n \qquad (4\text{-}32)$$

where it was assumed that the series was terminated after $N_\ell + 1$ terms.

The mean square error due to truncation of Eq. (4-27) in the interval $(2p - 1)\Omega < \omega < (2p + 1)$ may also be obtained. To do so form the mean square value of the Fourier Transform of the difference between $D_\ell f(t)$ and the truncated series representation of it in that region.

$$M_{pT} = \frac{1}{2\pi} \int_{(2p-1)\Omega}^{(2p+1)\Omega} \left| D_\ell F(\omega) - \sum_{n=0}^{N_\ell} c_{np\ell} \phi_n(\omega - 2p\Omega)\varepsilon^{-j(\omega-2p\Omega)\ell T} \right|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| B_p D_\ell F(\omega) - \sum_{n=0}^{N_\ell} c_{np\ell} B_p \hat{\phi}_n(\omega - 2p\Omega)\varepsilon^{-j(\omega-2p\Omega)\ell T} \right|^2 d\omega \tag{4-33}$$

where again $\hat{\phi}_n(\omega)$ is the Fourier Transform of $D_\ell \psi_n(t)$. Returning to the time domain, this is

$$M_{pT} = \int_{-\infty}^{\infty} \left| B_p D_\ell f(t) - \sum_{n=0}^{N_\ell} c_{np\ell} B_p \left[D_\ell \psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \right] \right|^2 dt \tag{4-34}$$

Multiplying this out and carrying out the indicated operations recalling that $BD\psi_n(t) = \lambda_n \psi_n(t)$ from Eq. (3-24) this becomes

$$M_{pT} = \int_{-\infty}^{\infty} \left| B_p D_\ell f(t) \right|^2 dt - \sum_{n=0}^{N_\ell} \left| c_{np\ell} \right|^2 \lambda_n^2 \tag{4-35}$$

To calculate the total error due to truncation of the complete representation of f(t), Eq. (4-28), it is immediately apparent from Eq. (4-32) that it is only necessary to sum Eq. (4-32) over all ℓ. Here, too, a different number of n terms may be used for each ℓ so as to adjust the goodness of fit in any time interval as desired.

4.5 A Further Decomposition to Two Dimensions

As with the representation of section 4.2, the representation of section 4.4 does not have a general two dimensional structure. It could deal adequately with a signal which exists largely in the frequency interval $(2p - 1)\Omega < \omega < (2p + 1)\Omega$, but it would not be very satisfactory for more general signals. Also as before, this representation may be further decomposed so as to obtain the desired two dimensional structure.

To carry out the decomposition, begin with the arbitrary function, $f(t)$, again represented as in Eq. (4-26)

$$f(t) = \sum^{\ell} D_\ell f(t) \qquad (4\text{-}36)$$

The ℓ^{th} term of this series may be represented as

$$D_\ell f(t) = \sum^{p} \sum^{n} d_{np\ell} \left[BD_\ell \psi_n(t - \ell T)\right] \varepsilon^{j2p\Omega t} \qquad (4\text{-}37)$$

Taking the Fourier Transform of this expression, noting again from Eq. (3-21) that $BD\psi_n(t) = \lambda_n \psi_n(t)$, yields

$$D_\ell F(\omega) = \sum^{p} \sum^{n} d_{np\ell} \lambda_n \phi_n(\omega - 2p\Omega) \varepsilon^{-j(\omega - 2p\Omega)\ell T} \qquad (4\text{-}38)$$

where again $\phi_n(\omega)$ is the Fourier Transform of $\psi_n(t)$. Since $\phi_n(\omega)$ is zero when the magnitude of its argument is greater than Ω, the orthogonal series here approximates a bandlimited section of the spectrum of a timelimited function or $B_p D_\ell f(t)$. Here, too, the ψ functions are complete over this finite interval and so the mean square error of this representation may be reduced to any arbitrarily small amount.

The coefficients, $d_{np\ell}$, may be obtained from Eq. (4-37) in the usual manner. Multiply both sides by $\psi_m(t-\ell T)\varepsilon^{-j2q\Omega t}$ and integrate over the appropriate limits.

$$\int_{-\infty}^{\infty} D_\ell f(t)\psi_m(t - \ell T)\varepsilon^{-j2q\Omega t}dt \quad =$$

$$\sum^{p}\sum^{n} d_{np\ell} \int_{-\infty}^{\infty} \lambda_n\psi_n(t - \ell T)\varepsilon^{j2p\Omega t}\psi_m(t - \ell T)\varepsilon^{-j2q\Omega t}dt \tag{4-39}$$

After a change of variables in the integral on the right it becomes Eq. (3-31) which has been shown to be equal to the product of the Kronecker deltas, $\delta_{mn}\delta_{pq}$. Therefore

$$d_{np\ell} \;=\; \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} f(t)\psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \tag{4-40}$$

Of course, by substituting Eq. (4-37) into Eq. (4-36) which uses these coefficients, a representation for f(t) is obtained as follows

$$f(t) = \sum^{\ell}\sum^{p}\sum^{n} d_{np\ell}\left[BD_\ell\psi_n(t - \ell T)\right]\varepsilon^{j2p\Omega t} \tag{4-41}$$

If the expansion, Eq. (4-37), of a timelimited portion of f(t) is truncated such that only terms for $-P \leq p \leq P$ and $0 \leq n \leq N_p$ are included, it can be shown in a manner similar to previous such calculations that the mean square error involved is given by

$$\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} |f(t)|^2 dt - \sum_{p=-P}^{P} \sum_{n=0}^{N_p} |d_{np\ell}|^2 \lambda_n^2 \qquad (4\text{-}42)$$

It may be noted that, as with the representation of section 4.3, the mean square error involved in truncating the complete series representation of f(t), Eq. (4-41), is not simply the sum over all ℓ of the errors in representing the D_ℓ f(t) terms as given by Eq. (4-39). For as pointed out before, the mean square value of the sum of two time functions is not equal to the sum of their mean square values.

The two dimensional structure of this representation is very analogous to that of section 4.3. It can be seen from Eq. (4-41) that every term is bandlimited to $(2p - 1)\Omega < \omega < (2p + 1)\Omega$ and, due to the properties of the ψ functions, largely time limited to $(\ell - \frac{1}{2})T < t < (\ell + \frac{1}{2})T$. But it can be seen from Eq. (4-40) that each term contains no information at all about f(t) from outside $(\ell - \frac{1}{2})T < t < (\ell + \frac{1}{2})T$ and relatively little (for the n terms of primary interest) from outside of $(2p - 1)\Omega < \omega < (2p + 1)\Omega$.

4.6 The Bandlimited or Timelimited Representation of a Function

So that the four representations presented may be compared, they are repeated here in order of their original presentation

$$f(t) = \sum^{p}\sum^{n} a_{np\ell}\Psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \quad (4\text{-}43)$$

$$a_{np\ell} = \int_{-\infty}^{\infty} f(t)\Psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \quad (4\text{-}44)$$

$$f(t) = \sum^{p}\sum^{\ell}\sum^{n} b_{np\ell} D_\ell \Psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \quad (4\text{-}45)$$

$$b_{np\ell} = \int_{-\infty}^{\infty} f(t)\Psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \quad (4\text{-}46)$$

$$f(t) = \sum^{\ell}\sum^{n} c_{np\ell} D_\ell \Psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \quad (4\text{-}47)$$

$$c_{np\ell} = \frac{1}{\lambda_n}\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} f(t)\Psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \quad (4\text{-}48)$$

$$f(t) = \sum^{\ell}\sum^{p}\sum^{n} d_{np\ell}\left[BD_\ell \Psi_n(t - \ell T)\right]\varepsilon^{j2p\Omega t} \quad (4\text{-}49)$$

$$d_{np\ell} = \frac{1}{\lambda_n}\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} f(t)\Psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \quad (4\text{-}50)$$

First of all, it is immediately apparent from Eqs. (4-41) and (4-43) that for a given f(t), the $a_{np\ell}$ and $b_{np\ell}$ are identical. The same is true of $c_{np\ell}$ and $d_{np\ell}$. Also, it was apparent from the derivation of the second representation, Eq. (4-45), that for this representation, for fixed p and ℓ, the $\sum^{n}$ yields the function $D_\ell B_p f(t)$ and for fixed p, the $\sum^{\ell}\sum^{n}$ yields the function $B_p f(t)$. It is also true that for fixed ℓ, the $\sum^{p}\sum^{n}$ yields the function $D_\ell f(t)$. This may be seen by comparing the first and second representations, Eqs. (4-43) and (4-45), and noting that if only one value of ℓ in the second is used, the two representations are identical except that every term in the second is the timelimited version of the corresponding term of the first. Therefore, since the first representation yields the function f(t), the second yields $D_\ell f(t)$.

A corresponding result may be obtained for the third and fourth representations. It is apparent from the derivation of the fourth representation, Eq. (4-49), that for fixed ℓ and p the $\sum^{n}$ results in $B_p D_\ell f(t)$ and for fixed ℓ the $\sum^{p}\sum^{n}$ results in $D_\ell f(t)$. It is also true that for fixed p the $\sum^{\ell}\sum^{n}$ results in $B_p f(t)$. This may be seen again by noting that if only one value of p is chosen in the fourth representation, the only difference between the two is that every term in the fourth is the bandlimited version of the corresponding one in the third, and, therefore, since $\sum^{\ell}\sum^{n}$ yields f(t) in the third, the $\sum^{\ell}\sum^{n}$ yields $B_p f(t)$ in the fourth.

Thus, if one has the expansion coefficients of a function for either the representation of Eq. (4-45) or that of Eq. (4-49), not only is the portion of f(t) which occupies, in the sense of the ψ functions, any finite area of the time-frequency plane available, but also that portion which is in any vertical or horizontal strip of infinite length or any set of such strips.

4.7 Multiplication of Two Dimensional Representations or of Their Spectrums

A further group of properties of these representations may be demonstrated. If it is desired to combine two functions expressed in two dimensional form either by forming their product or by forming the product of their spectrums, it is possible to obtain a sampled form of the result from the appropriate expansion coefficients alone.

For example, suppose it is desired to determine the output, f(t), which results from applying an input, e(t), to a linear system whose impulse response is g(t). Then certainly

$$F(\omega) = E(\omega)G(\omega) \tag{4-51}$$

where $E(\omega)$, $F(\omega)$, and $G(\omega)$ are the Fourier Tranforms of e(t), f(t), and g(t), respectively. An alternate method of calculation is the use of convolution in the time domain and so

$$f(t) = \int_{-\infty}^{\infty} e(\tau)g(t - \tau)d\tau \tag{4-52}$$

If, then, e(t) and g(t) are expressed in terms of Eq. (4-47), f(t) may be written as

$$f(t) =$$

$$\sum^{k}\sum^{\ell}\sum^{m}\sum^{n} c_{np\ell}\,\hat{c}_{mpk}\int_{-\infty}^{\infty} D_{\ell}\,\Psi_n(\tau-\ell T)\varepsilon^{j2p\Omega\tau} D_{\frac{t}{T}-k}\,\Psi_m(t-\tau-kT)\varepsilon^{j2p\Omega(t-\tau)}dt$$

$$= \sum^{k}\sum^{\ell}\sum^{m}\sum^{n} c_{np\ell}\,\hat{c}_{mpk}\varepsilon^{j2p\Omega t}\int_{-\infty}^{\infty} D_{\ell}\,\Psi_n(\tau-\ell T) D_{\frac{t}{T}-k}\,\Psi_m(t-\tau-kT)d\tau \tag{4-53}$$

where $c_{np\ell}$ are the coefficients for e(t), and $\hat{c}_{mpk}$ are the coefficients for g(t). Then for t = rT where r is an integer, due to the finite orthogonality property, Eq. (3-20), and property ten of section 3.2, this reduces to

$$f(rT) = \sum^{\ell}\sum^{n} c_{np\ell}\,\hat{c}_{np(r-\ell)}(-1)^n\lambda_n\varepsilon^{j2p\Omega rT} \tag{4-54}$$

Thus the output at sampled instants can be obtained from only the products of the coefficients, a very simple operation to carry out.

A similar result may be demonstrated for the first of the two dimensional representations, Eq. (4-45). First, notice that if the transforms of the representations are substituted into Eq. (4-51) the result is

$$F(\omega) = \sum^{p}\sum^{\ell}\sum^{n} b_{np\ell}\hat{\phi}_n(\omega - 2p\Omega)\varepsilon^{-j(\omega - 2p\Omega)\ell T} \text{ X}$$

$$\sum^{q}\sum^{k}\sum^{m} \hat{b}_{mqk}\hat{\phi}_m(\omega - 2q\Omega)\varepsilon^{-j(\omega - 2q\Omega)kT} \qquad (4\text{-}55)$$

where $b_{np\ell}$ are the coefficients for $E(\omega)$, $\hat{b}_{mqk}$ are the coefficients for $G(\omega)$ and $\phi_i(\omega)$ is the transform of $D\psi_i(t)$. Recall, however, from the derivation of this representation in section 4.3 that for a given p and q the $\sum^{\ell}\sum^{n}$ and the $\sum^{k}\sum^{m}$, respectively, are bandlimited functions, and thus the product of $\sum^{p}\sum^{\ell}\sum^{n}$ and $\sum^{q}\sum^{k}\sum^{m}$ would be zero unless $p = q$. Then expressing Eq. (4-55) in the time domain by convolution, this would be

$$f(t) = \sum^{p}\sum^{\ell}\sum^{k}\sum^{m}\sum^{n} b_{np\ell}\,\hat{b}_{mpk} \text{ X}$$

$$\int_{-\infty}^{\infty} D_\ell\psi_n(\tau - \ell T)\varepsilon^{j2p\Omega\tau} D_{\frac{t}{T}-k}\psi_m(t-\tau-kT)\varepsilon^{j2p\Omega(t-\tau)}d\tau \qquad (4\text{-}56)$$

But the integral in Eq. (4-56) is identical to that in Eq. (4-53) and so, by following the same steps as before, it can be shown that

$$f(rT) = \sum^{p} \sum^{\ell} \sum^{n} b_{np\ell} b_{np(r-\ell)} (-1)^n \lambda_n \varepsilon^{j2p\Omega rT} \tag{4-57}$$

and one may see, for example, how the output time function is affected by varying the bandwidth of the input.

The other two representations yield a similar type of result in the frequency domain for the multiplication of the representations of two signals in the time domain. Assume that the time functions, e(t) and f(t), are given and the product g(t) = e(t)f(t) is to be formed. This may be computed in the frequency domain as

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\lambda) F(\omega - \lambda) d\lambda \tag{4-58}$$

Then if both $E(\omega)$ and $F(\omega)$ are expressed in terms of the transform of the representation of Eq. (4-43) and these substituted into Eq. (4-58) the result would be

$$G(\omega) = \frac{1}{2\pi} \sum^{p} \sum^{q} \sum^{m} \sum^{n} a_{np\ell} \hat{a}_{mq\ell} \quad X$$

$$\int_{-\infty}^{\infty} \phi_n(\lambda - 2p\Omega) \varepsilon^{-j(\lambda - 2p\Omega)\ell T} \phi_m(\omega - \lambda - 2q\Omega) \varepsilon^{-j(\omega - \lambda - 2q\Omega)\ell T} d\lambda \tag{4-59}$$

where $a_{np\ell}$ are the coefficients for $E(\omega)$, $\hat{a}_{mq\ell}$ are the coefficients for $F(\omega)$ and $\phi_i(\omega)$ is the transform of $\psi_i(t)$.

If $\zeta = (\lambda - 2p\Omega)$ this becomes

$$G(\omega) = \frac{1}{2\pi} \sum^{p} \sum^{q} \sum^{m} \sum^{n} a_{np\ell} \hat{a}_{mq\ell} \quad X$$

$$\int_{-\infty}^{\infty} \phi_n(\zeta)\phi_m(\omega-\zeta-2(p+q)\Omega)\varepsilon^{-j[\omega-2(p+q)\Omega]\ell T} d\zeta \qquad (4\text{-}60)$$

In this case at $\omega = 2r\Omega$ where r is an integer, due to the fact that $\phi_n(\omega)$ is bandlimited to $|\omega| < \Omega$, the integrand is zero unless $q = r - p$ and Eq. (4-60) becomes

$$G(2r\Omega) = \frac{1}{2\pi} \sum^{p} \sum^{m} \sum^{n} a_{np\ell} a_{m(r-p)\ell} \int_{-\infty}^{\infty} \phi_n(\zeta)\phi_m(-\zeta)d\zeta \qquad (4\text{-}61)$$

Applying Parseval's Theorem and the orthonormality condition, this reduces to

$$G(2r\Omega) = \sum^{p} \sum^{n} a_{np\ell} \hat{a}_{n(r-p)\ell} \qquad (4\text{-}62)$$

To derive the similar result for the second two dimensional representation, Eq. (4-49), begin first in the time domain with

$$g(t) = e(t)f(t) =$$

$$\sum^{\ell} \sum^{p} \sum^{n} d_{np\ell} \lambda_n \psi_n(t-\ell T)\varepsilon^{j2p\Omega t} \sum^{k} \sum^{q} \sum^{m} \hat{d}_{mqk} \lambda_m \psi_m(t-kT)\varepsilon^{j2q\Omega t} \qquad (4\text{-}63)$$

Recall here from the derivation of this representation in section 4.5 that for a given ℓ and k the summation $\sum^{p}\sum^{n}$ and the summation $\sum^{q}\sum^{m}$, respectively, are timelimited functions and thus the product of $\sum^{\ell}\sum^{p}\sum^{n}$ and $\sum^{k}\sum^{q}\sum^{m}$ is zero except when $\ell = k$. If now Eq. (4-63) is written in terms of the frequency domain using Eq. (4-58) the result is

$$G(\omega) = \frac{1}{2\pi} \sum^{\ell}\sum^{p}\sum^{q}\sum^{m}\sum^{n} d_{np\ell}\, \hat{d}_{mq\ell} \quad X$$

$$\int_{-\infty}^{\infty} \phi_n(\lambda - 2p\Omega)\varepsilon^{-j(\lambda-2p\Omega)\ell T}\phi_m(\omega-\lambda-2q\Omega)\varepsilon^{-j(\omega-\lambda-2q\Omega)\ell T} d\lambda \tag{4-64}$$

But the integral on the right side is the same as that of Eq. (4-59) and since the value of this integral was shown there to be 2π for m = n and p = q and zero otherwise, assuming $\omega = 2r\Omega$, Eq. (4-64) reduces to

$$G(2r\Omega) = \sum^{\ell}\sum^{p}\sum^{n} d_{np\ell}\, \hat{d}_{n(r-p)\ell}\, \lambda_n^2 \tag{4-65}$$

Thus by very simple calculations, one can, for example, see how the various parts of the energy spectrum of a signal change with time.

4.8 A Technique for Time Variable Systems

By a slightly different technique it is possible to obtain the output in continuous form for the more general time-variable system in terms of the two representations.

Assume once again that the input to a system is given by e(t) and the resulting output is f(t). Time variable systems are also frequently specified in terms of their impulse response, which in this case is a function of two variables, $g(t,\tau)$. This function is interpreted as the output at time t due to an impulse occurring at the input at time τ. Then it is known (see, for example, Ref. 1, page 96) that the output of the system is given by

$$f(t) = \int_{-\infty}^{t} e(\tau)g(t,\tau)d\tau \tag{4-66}$$

Since for a physically realizable system $g(t,\tau) = 0$ for $\tau > t$, the upper limit could have just as well been infinite.

Assume first that the input, e(t), is represented in terms of Eq. (4-47)

$$e(t) = \sum^{\ell} \sum^{n} \hat{c}_{np\ell} D_{\ell} \Psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \tag{4-67}$$

where

$$\hat{c}_{np\ell} = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} f(t)\Psi_n(t - \ell T)\varepsilon^{-j2p\Omega t}dt \tag{4-68}$$

The impulse response, $g(t,\tau)$, may be represented as

$$g(t,\tau) = \sum^{\ell} \sum^{n} c_{np\ell}(t) D_{\ell} \psi_n(\tau - \ell T)\varepsilon^{j2p\Omega\tau} \qquad (4\text{-}69)$$

where

$$c_{np\ell}(t) = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} g(t,\tau)\psi_n(\tau - \ell T)\varepsilon^{-j2p\Omega\tau} d\tau \qquad (4\text{-}70)$$

The system output results from the substitution of Eqs. (4-69) and (4-67) into Eq. (4-66)

$f(t) =$

$$\sum^{\ell} \sum^{k} \sum^{m} \sum^{n} \hat{c}_{np\ell}\, c_{mpk}(t) \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \psi_n(\tau-\ell T)\varepsilon^{j2p\Omega\tau} D_k \psi_m(\tau-kT)\varepsilon^{j2p\Omega\tau} d\tau \qquad (4\text{-}71)$$

Upon a change of variable

$f(t) =$

$$\sum^{\ell} \sum^{m} \sum^{n} c_{np\ell}\, c_{mp\ell}(t) \int_{-\frac{T}{2}}^{\frac{T}{2}} \psi_n(\lambda)\varepsilon^{j2p\Omega(\lambda+\ell T)} \psi_m(\lambda)\varepsilon^{j2p\Omega(\lambda+\ell T)} d\lambda \qquad (4\text{-}72)$$

Factoring out the constant portion of the exponent and applying Parseval's Theorem to the integral that remains,

$$f(t) =$$

$$\sum^{\ell}\sum^{m}\sum^{n} \hat{c}_{np\ell}\, c_{mp\ell}(t)\, \frac{\varepsilon^{j4p\Omega\ell T}}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}_n(\omega-2p\Omega)\, \hat{\phi}_m(-\omega+2p\Omega)\, d\omega \tag{4-73}$$

Upon a change of variable and substitution for $\phi_n(\zeta)$ from Eq. (3-29)

$$f(t) =$$

$$\sum^{\ell}\sum^{m}\sum^{n} \hat{c}_{np\ell}\, c_{mp\ell}(t)\, \frac{\varepsilon^{j4p\Omega\ell T}}{2\pi}\, j^{-(m+n)}\, \frac{T\pi}{\Omega}\sqrt{\lambda_m\lambda_n} \int_{-\infty}^{\infty} \psi_n\left(\frac{T\zeta}{2\Omega}\right)\psi_m\left(-\frac{T\zeta}{2\Omega}\right) d\zeta \tag{4-74}$$

It is seen that this reduces to

$$f(t) = \sum^{\ell}\sum^{n} \hat{c}_{np\ell}\, c_{np\ell}(t)\, \lambda_n \varepsilon^{j4p\Omega\ell T} \tag{4-75}$$

A similar result may be obtained for the two dimensional representation of Eq. (4-45). In this case the representation of $g(t,\tau)$ would be

$$g(t,\tau) = \sum^{p}\sum^{\ell}\sum^{n} b_{np\ell}(t) D_\ell \psi_n(\tau - \ell T)\varepsilon^{j2p\Omega t} \tag{4-76}$$

where

$$b_{np\ell}(t) = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left[B_p g(t,\tau)\right] \psi_n(\tau - \ell T)\varepsilon^{j2p\Omega\tau} d\tau \tag{4-77}$$

The representation for e(t) would be similar except that the coefficients would be constants rather than functions of t. Then to find the system output, f(t), both of these representations may be substituted into Eq. (4-66). Upon so doing it may be recalled again that for this type of representation for each p the $\sum^{\ell}\sum^{n}$ is a bandlimited function, and, therefore, as before, the integral of the product of all terms of unlike p will be zero. The output may then be written as

$$f(t) = \sum^{p}\sum^{\ell}\sum^{k}\sum^{m}\sum^{n} \hat{b}_{np\ell}\, b_{mpk}(t) \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \psi_n(\tau-\ell T)\,\varepsilon^{j2p\Omega\tau} D_k \psi_m(\tau-kT)\,\varepsilon^{j2p\Omega\tau}\, d\tau \qquad (4\text{-}78)$$

However, this integral is seen to be identical with that in Eq. (4-71) and, therefore, the output becomes

$$f(t) = \sum^{p}\sum^{\ell}\sum^{n} \hat{b}_{np\ell}\, b_{np\ell}(t)\,\lambda_n \varepsilon^{j4p\Omega\ell T} \qquad (4\text{-}79)$$

With these results it is seen to be possible to determine, as before, the effect of any portion of the input signal or of the mathematical description of the system on the output signal. The output is given as a continuous time function rather than as values at discrete times as with the approach of section 4.7. The disadvantage here is that the coefficients for the representation of g(t,τ) are more difficult to calculate.

4.9 Ambiguity Functions

Frequently in the study of radar signals and similar problems, application is made of a concept introduced by Woodward (34) which is known as the ambiguity function. Klauder et.al. (16) and Klauder (15) made use of this concept in the design of chirp radar. The ambiguity function is defined (using Woodward's notation) by

$$\chi(\tau,\phi) = \int_{-\infty}^{\infty} u(t)u^*(t + \tau)\varepsilon^{-2\pi j\phi t}dt \qquad (4\text{-}80)$$

This function is seen to be a correlation function for a combined time and frequency shift. The variable, τ, is the time shift and in the radar problem is associated with the range of the target. The variable, ϕ, is considered as the Doppler shift associated with the velocity of the radar target. The details of the application of this function will not be discussed here, but it will be shown that the ambiguity function of a given signal may be evaluated at discrete values of τ and ϕ by a technique very similar to that employed in section 4.7 for the linear system problem.

Assume that the signal, u(t), is represented in terms of the two dimensional representation of Eq. (4-45). Then substituting into Eq. (4-80), this gives

$$\chi(\tau,\phi) = \int_{-\infty}^{\infty} \left\{ \sum^{p} \sum^{\ell} \sum^{n} b_{np\ell} D_{\ell} \psi_n(t - \ell T) \varepsilon^{j2p\Omega t} \right. \times$$

$$\sum^{q} \sum^{k} \sum^{m} b^{*}_{qkm} D_{k-\frac{\tau}{T}} \psi_m(t+\tau-kT) \varepsilon^{-j2q\Omega t} \varepsilon^{-j2\pi\phi t} \, dt \qquad (4\text{-}81)$$

Assume also that the values of $\chi(\tau,\phi)$ are desired for $\tau = rT$ and $2\pi\phi = 2s\Omega$, r and s both being integers either positive, negative or zero.

Recall, as before, for the first portion of the integrand of Eq. (4-81), the portion representing u(t), that for a given p the $\sum^{\ell} \sum^{n}$ is bandlimited to $(2p - 1)\Omega < \omega < (2p + 1)\Omega$. Similarly, for the second portion of the integrand, the portion representing $u^{*}(t + \tau)$, for a given q the $\sum^{k} \sum^{m}$ is the conjugate of a function bandlimited to $(2q - 1)\Omega < \omega < (2q + 1)\Omega$. If the exponential involving ϕ is included as part of the second portion of the integrand and if $2\pi\phi = 2s\Omega$, then for a given q the $\sum^{k} \sum^{m}$ is the conjugate of a function bandlimited to $(2q + 2s - 1)\Omega < \omega < (2q + 2s + 1)\Omega$. Therefore, due to Parseval's Theorem, the integral of the product of these two portions will be zero unless $p = q + s$ or $q = p - s$. Eq. (4-81), assuming also $\tau = rT$, then reduces to

$$\chi(rT,\tfrac{s\Omega}{\pi}) =$$

$$\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \sum^{p} \sum^{\ell} \sum^{k} \sum^{m} \sum^{n} b_{np\ell}\, b^{*}_{m(p-s)k} \psi_n(t-\ell T) D_{k-r} \psi_m(t+rT-kT)\,dt$$

$$= \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \sum^{p} \sum^{\ell} \sum^{m} \sum^{n} b_{np\ell}\, b^{*}_{m(p-s)(r+\ell)} \psi_n(t-\ell T) \psi_m(t-\ell T)\,dt$$

$$= \sum^{p} \sum^{\ell} \sum^{n} b_{np\ell}\, b^{*}_{n(p-s)(r+\ell)} \lambda_n \qquad (4\text{-}82)$$

Again, this is a very simple operation to perform.

CHAPTER 5

Examples of Two Dimensional Representations

In this chapter numerical examples of one of the two dimensional representations are presented for two different time functions. The expansion coefficients for the larger terms are given together with the representation of the function in both tabular and graphical form. Following this the result of the convolution of the representations of the two functions is given.

5.1 The Representation of $f(t) = u(t)\varepsilon^{-t}$

As a first example the function $f(t) = u(t)\varepsilon^{-t}$, where $u(t)$ is the unit step function, will be presented in terms of the representation of section 4.3. This representation was given by Eqs. (4-23) and (4-19) which are repeated here for convenience.

$$f(t) = \sum_{p} \sum_{\ell} \sum_{n} b_{np\ell} D_{\ell} \Psi_n(t - \ell T)\varepsilon^{j2p\Omega t} \tag{5-1}$$

where

$$b_{np\ell} = \frac{1}{\lambda_n} \int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \left\{B_p f(t)\right\} \Psi_n(t - \ell T)\varepsilon^{-j2p\Omega t} dt \tag{5-2}$$

For this representation, the following values were arbitrarily chosen: $c = 0.5$, $T = 0.2$, and therefore $\Omega = 5.0$. The coefficients, $b_{np\ell}$, were calculated for $0 \leq n \leq 1$, $0 \leq p \leq 1$, and $0 \leq \ell \leq 8$. The results of these calculations are presented in Table 5-1 as a magnitude and an angle in radians in a format such that each rectangle of the table may be thought of as an area in the time-frequency plane. It is seen that terms for indexing in the ℓ or time direction proceed to the right while terms for $p = 1$ lie above those for $p = 0$ as they would in the time-frequency plane. Of course, so long as $f(t)$ is a real function of time, $b_{n(-p)\ell}$ is the complex conjugate of $b_{np\ell}$; thus the values of the coefficients for $p = -1$ are also available and could have been placed in a row of rectangles below the row containing the terms for $p = 0$. No additional information is conveyed by so doing, however.

The number in parentheses in each rectangle of Table 5-1 is $\sum_{n=0}^{1} |b_{np\ell}|^2 \lambda_n$ for the coefficients of that rectangle. Recall that for a given p and ℓ the $\sum_{n}$ of the terms in Eq. (5-1) results in $D_\ell B_p f(t)$ and $\sum_{n} |b_{np\ell}|^2 \lambda_n$ is the mean square value of $D_\ell B_p f(t)$. Hence the parenthesized number indicates the degree of approximation of $D_\ell B_p f(t)$ and indicates the portion of $f(t)$ assigned to each rectangular area by this representation.

By considering the trends of these parenthesized numbers from rectangle to rectangle it is seen that the representation illustrates many of the trends which intuition

p	n \ ℓ	0	1	2
1	0	0.13724/4.833	0.11871/4.646	0.06954/4.352
	1	0.04237/3.378	0.07963/2.092	0.10979/1.717
		(0.005849)	(0.004420)	(0.001601)
0	0	0.34978/0.0	0.50223/0.0	0.56130/0.0
	1	0.31841/0.0	0.19587/0.0	0.00617/0.0
		(0.03876)	(0.07844)	(0.09757)

p	n \ ℓ	3	4	5
1	0	0.02577/3.366	0.03235/1.953	0.03118/1.442
	1	0.09079/1.424	0.04298/0.822	0.03475/5.468
		(0.0002764)	(0.0003425)	(0.0003116)
0	0	0.51523/0.0	0.40215/0.0	0.28293/0.0
	1	0.15622/π	0.22073/π	0.18126/π
		(0.08242)	(0.05426)	(0.02507)

p	n \ ℓ	6	7	8
1	0	0.01502/0.601	0.01685/5.295	0.02011/4.708
	1	0.04683/4.805	0.03096/4.264	0.01882/2.729
		(0.0000887)	(0.0000983)	(0.000128)
0	0	0.20387/0.0	0.17367/0.0	0.16883/0.0
	1	0.09117/π	0.01980/π	0.00500/π
		(0.01295)	(0.00935)	(0.00883)

Table 5-1. Coefficients, $b_{np\ell}$, for $f(t) = u(t)\varepsilon^{-t}$.

might predict. Considering the horizontal strip in which $p = 0$ it is seen that the general time domain structure is present, and, as ℓ increases, after reaching a maximum, the representation tends toward zero. In the horizontal strip for $p = 1$ it is seen that the representation is largest at the origin and decreases quickly with increasing ℓ. In addition to the gross time domain structure, this may be attributed to the jump discontinuity of $f(t)$ at the origin.

It may be shown easily that the total energy of $f(t) = u(t)\varepsilon^{-t}$ is 0.500. The total energy of this function in the time interval of representation, $-0.10 < t < 1.70$ sec., is 0.483. Further, it is shown easily that for $f(t) = u(t)\varepsilon^{-t}$

$$F(\omega) = \frac{1 - j\omega}{1 + \omega^2} \qquad (5\text{-}3)$$

From this the energy of $Bf(t)$, i.e. the energy of $F(\omega)$ in $-5 < \omega < 5$, is found to be 0.4372; the energy in $5 < \omega < 15$ is 0.0208. From Table 5-1 it may be shown that $\sum_{\ell=0}^{8} \sum_{n=0}^{1} |b_{n0\ell}|^2 \lambda_n$ is 0.4039 and therefore from Eq. (4-24) the mean square error in representing $Bf(t)$ is 0.0333. The $\sum_{\ell=0}^{8} \sum_{n=0}^{1} |b_{n1\ell}|^2 \lambda_n$ is 0.0131 and, therefore, the mean square error in representing $B_1 f(t)$ is 0.0077.

The results of the substitution of the values of Table 5-1 into Eq. (5-1) at 19 values of t are shown in Table 5-2 and plotted in Figs. 5-1 and 5-2. The tabular results are given both for $f_o(t)$, the summation using only terms

t	$f_o(t)$ p = 0 only	$f_1(t)$ p = -1, 0 and 1	$f(t) = u(t)\varepsilon^{-t}$
-0.1	0.3102	0.0759	0.0
0.0	0.4412	0.4827	1.0
0.1	0.5373	0.8256	0.9048
0.2	0.6335	0.9136	0.8187
0.3	0.6772	0.7576	0.7408
0.4	0.7081	0.6242	0.6703
0.5	0.6802	0.5844	0.6065
0.6	0.6499	0.5850	0.5488
0.7	0.5665	0.4953	0.4966
0.8	0.5073	0.4368	0.4493
0.9	0.4075	0.3758	0.4066
1.0	0.3569	0.3909	0.3679
1.1	0.2785	0.3336	0.3329
1.2	0.2572	0.2950	0.3012
1.3	0.2158	0.2511	0.2725
1.4	0.2191	0.2579	0.2466
1.5	0.2046	0.2303	0.2231
1.6	0.2130	0.1986	0.2019
1.7	0.2025	0.1643	0.1827

Table 5-2. Values of $f_o(t)$ and $f_1(t)$ for $f(t) = u(t)\varepsilon^{-t}$.

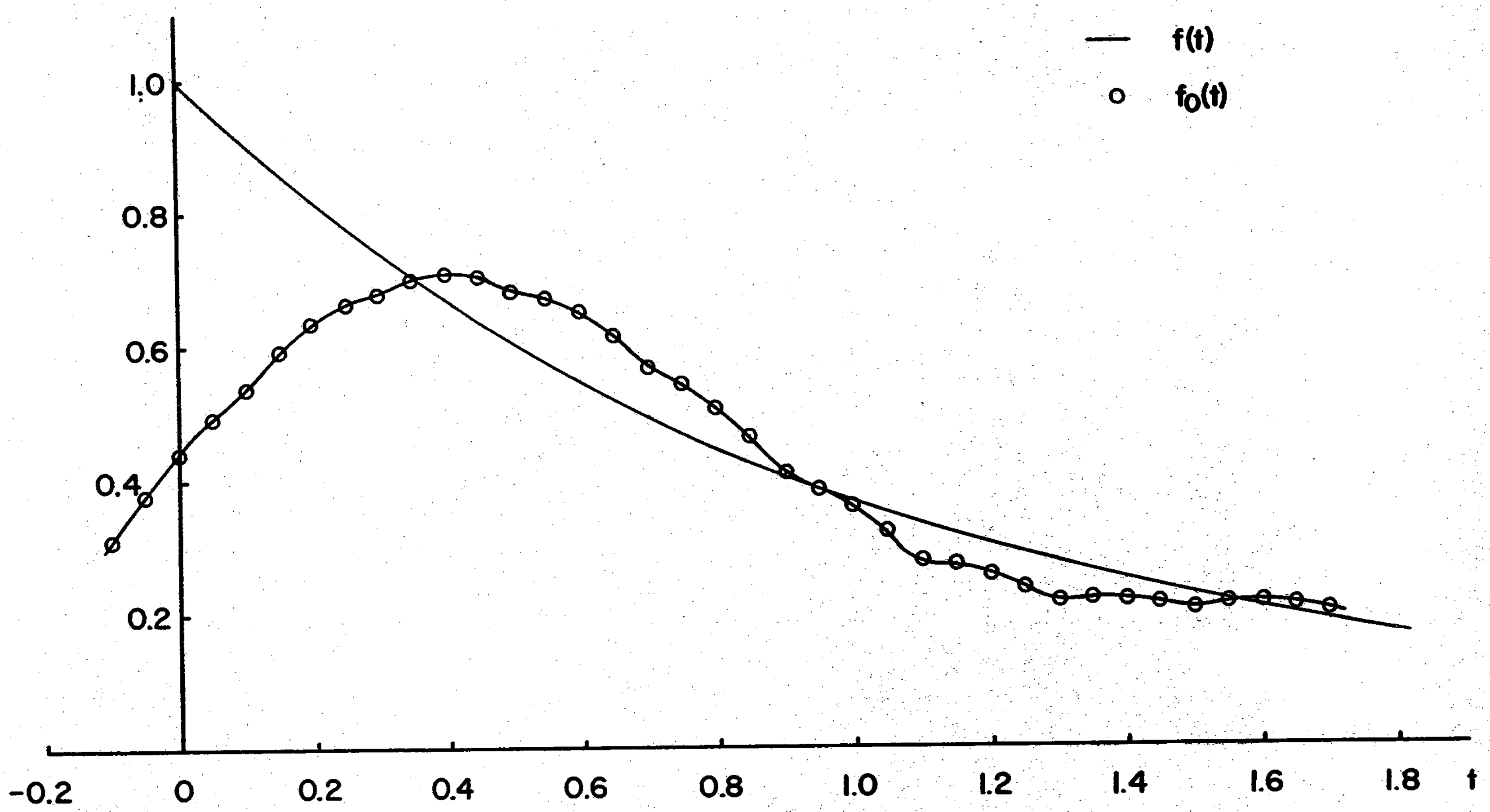

FIG. 5-1. THE FUNCTION $f_0(t)$ FOR $f(t) = u(t)\,\varepsilon^{-t}$

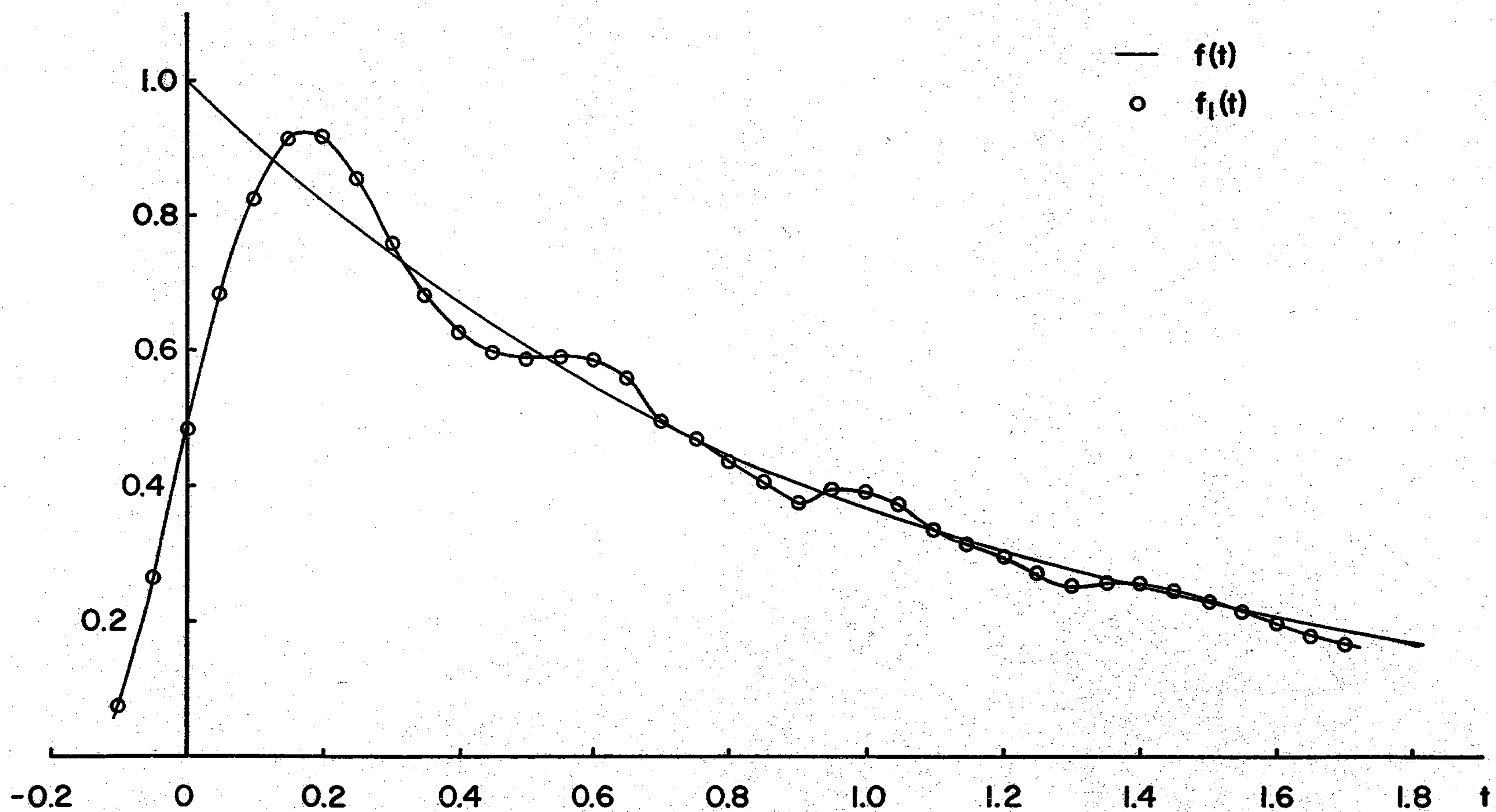

FIG. 5–2. THE FUNCTION $f_1(t)$ FOR $f(t) = u(t)\,\varepsilon^{-t}$

for which p = 0, and $f_1(t)$, the summation using terms for p = -1, 0, and +1, in Table 5-2. The function, $f_o(t)$, is shown in Fig. 5-1; $f_1(t)$ is shown in Fig. 5-2. In each case, f(t) is also given for comparison. It is even more apparent here that the inclusion of $B_1f(t)$ and $B_{-1}f(t)$ has the greatest effect near the time origin. It is seen, however, that the greatest error also still occurs near the time origin. Hence, if it were desired to reduce the overall error still further, more terms for which ℓ = 0 or 1 should be included. This might have also been suspected from Table 5-1 even if the exact functional form were not known. That is, in Table 5-1 when comparing coefficients in vertical strips, it is seen that, as p is increased, the coefficients decrease the slowest when ℓ is small; therefore, additional terms in this region are likely to make the greatest contribution. For example, the coefficient b_{020} is $0.06432\underline{/4.765}$, which is still almost half as large in magnitude as b_{010}.

5.2 The Representation of $f(t) = u(t)\varepsilon^{-t} \sin 3t$

The example given in section 5.1 was relatively severe in that it contained a jump discontinuity, thus causing the spectrum of f(t) to decrease only as $\frac{1}{\omega}$ as ω becomes large. For a second example the function $f(t) = u(t)\varepsilon^{-t} \sin 3t$ is chosen. It is seen that this function has a jump discontinuity in its first derivative, and, therefore, its spectrum falls off as $\frac{1}{\omega^2}$ for large ω. It is shown easily that for this function

$$F(\omega) = \frac{3}{(1 + j\omega)^2 + 9} \qquad (5\text{-}4)$$

Once again the representation is given in terms of Eqs. (5-1) and (5-2). The appropriate coefficients are given in Table 5-3 in the same format as before. Again the general time structure is apparent from the parenthesized numbers, $\sum_{n=0}^{1} |b_{np\ell}|^2 \lambda_n$. By looking at those in the horizontal strip for which p = 0, it is seen that in the time interval represented, 1.8 sec., the sinusoidal function nearly completes one oscillation. It might further be expected that most of the oscillatory information would be contained in this horizontal strip since for this strip, $-5 < \omega < 5$, and the sinusoidal variation has frequency $\omega = 3$. This, too, is seen to be the case since the strip for which p = 1 shows little tendency to oscillate. Again this strip seems more influenced by the discontinuity in the derivative as would be expected.

It may be shown in this case that the total energy of the function $f(t) = u(t)\varepsilon^{-t}\sin 3t$ is 0.225. The total energy of f(t) in $-0.10 < t < 1.70$ is 0.2127. From $F(\omega)$, the energy in $-5 < \omega < 5$ may be evaluated as 0.2135. The energy in $5 < \omega < 15$ may be evaluated as 0.0056. From Table 5-3 it is seen that $\sum_{\ell=0}^{8} \sum_{n=0}^{1} |b_{n0\ell}|^2 \lambda_n$ is 0.2030, and, therefore, the mean square error in representing Bf(t) is 0.0105. The $\sum_{\ell=0}^{8} \sum_{n=0}^{1} |b_{n1\ell}|^2 \lambda_n$ is 0.0033, and, therefore, the error in

p	n \ ℓ	0	1	2
1	0	0.05860/3.481	0.05439/3.077	0.03896/2.540
	1	0.04064/2.068	0.04645/1.062	0.05512/0.392
		(0.001077)	(0.000935)	(0.000496)
0	0	0.16494/ 0.0	0.32277/ 0.0	0.42384/ 0.0
	1	0.27974/ 0.0	0.24878/ 0.0	0.08715/ 0.0
		(0.009096)	(0.03279)	(0.05569)

p	n \ ℓ	3	4	5
1	0	0.02561/1.652	0.02324/0.640	0.02084/4.761
	1	0.04968/6.103	0.03554/5.330	0.02998/4.271
		(0.000225)	(0.000178)	(0.000142)
0	0	0.41002/ 0.0	0.27714/ 0.0	0.07975/ 0.0
	1	0.13614/ π	0.31120/ π	0.35272/ π
		(0.05223)	(0.02462)	(0.00304)

p	n \ ℓ	6	7	8
1	0	0.01556/5.217	0.01417/4.126	0.01391/3.254
	1	0.03023/3.401	0.02479/2.589	0.02018/1.525
		(0.000083)	(0.000067)	(0.000064)
0	0	0.09882/ π	0.19217/ π	0.18338/ π
	1	0.25020/ π	0.06963/ π	0.09152/ 0.0
		(0.00356)	(0.01148)	(0.01049)

Table 5-3. Coefficients, $b_{np\ell}$, for $f(t) = u(t)\varepsilon^{-t}\sin 3t$.

representing $B_1 f(t)$ is 0.0023.

The results of the substitution of the values of Table 5-3 into Eq. (5-1) at 19 different values of t are given in Table 5-4 and plotted in Figs. 5-3 and 5-4. Again these results are presented for both $f_o(t)$, the summation using only terms for which $p = 0$, and $f_1(t)$, the summation using terms for which $p = -1, 0,$ and $+1$. The function $f(t) = u(t)\varepsilon^{-t}\sin 3t$ is also presented for comparison.

5.3 A Linear System Example

In order to illustrate one of the properties of section 4.6, assume a signal $e(t) = u(t)\varepsilon^{-t}$ is applied to the input of a linear system whose impulse response is $g(t) = u(t)\varepsilon^{-t}\sin 3t$. It is evident that the resulting output, $f(t)$, is

$$f(t) = \int_0^t \varepsilon^{-(t-\tau)}\, \varepsilon^{-\tau}\sin 3\tau d\tau = \frac{\varepsilon^{-t}}{3}(1 - \cos 3t) \qquad (5\text{-}5)$$

If both $e(t)$ and $f(t)$ are represented as in Eq. (5-1), using the same values of c and T for both, then from Eq. (4-56) it is evident that $f(t)$ is given by

$$f(t) = \sum^{p}\sum^{\ell}\sum^{k}\sum^{m}\sum^{n} b_{np\ell}\hat{b}_{mpk}\varepsilon^{j2p\Omega t}\int_{(\ell-\frac{1}{2})T}^{(\ell+\frac{1}{2})T} \psi_n(\tau-\ell T)D_{\frac{t}{T}-k}\psi_m(t-\tau-kT)d\tau \qquad (5\text{-}6)$$

where $b_{np\ell}$ are the coefficients of $e(t)$ and $\hat{b}_{mpk}$ are the

t	$f_o(t)$ p = 0 only	$f_1(t)$ p = -1, 0 and 1	$f(t) =$ $u(t)\varepsilon^{-t}\sin 3t$
-0.1	0.1002	-0.0256	0.0
0.0	0.2081	0.0687	0.0
0.1	0.3005	0.2386	0.2674
0.2	0.4072	0.4561	0.4623
0.3	0.4804	0.5885	0.5803
0.4	0.5346	0.6297	0.6248
0.5	0.5442	0.5981	0.6050
0.6	0.5172	0.5302	0.5345
0.7	0.4468	0.4335	0.4287
0.8	0.3496	0.3081	0.3035
0.9	0.2232	0.1642	0.1738
1.0	0.1006	0.0527	0.0519
1.1	-0.0298	-0.0568	-0.0525
1.2	-0.1247	-0.1271	-0.1333
1.3	-0.2082	-0.1951	-0.1874
1.4	-0.2424	-0.2156	-0.2149
1.5	-0.2558	-0.2178	-0.2181
1.6	-0.2313	-0.1990	-0.2011
1.7	-0.1894	-0.1702	-0.1691

Table 5-4. Values of $f_o(t)$ and $f_1(t)$ for $f(t) = u(t)\varepsilon^{-t}\sin 3t$.

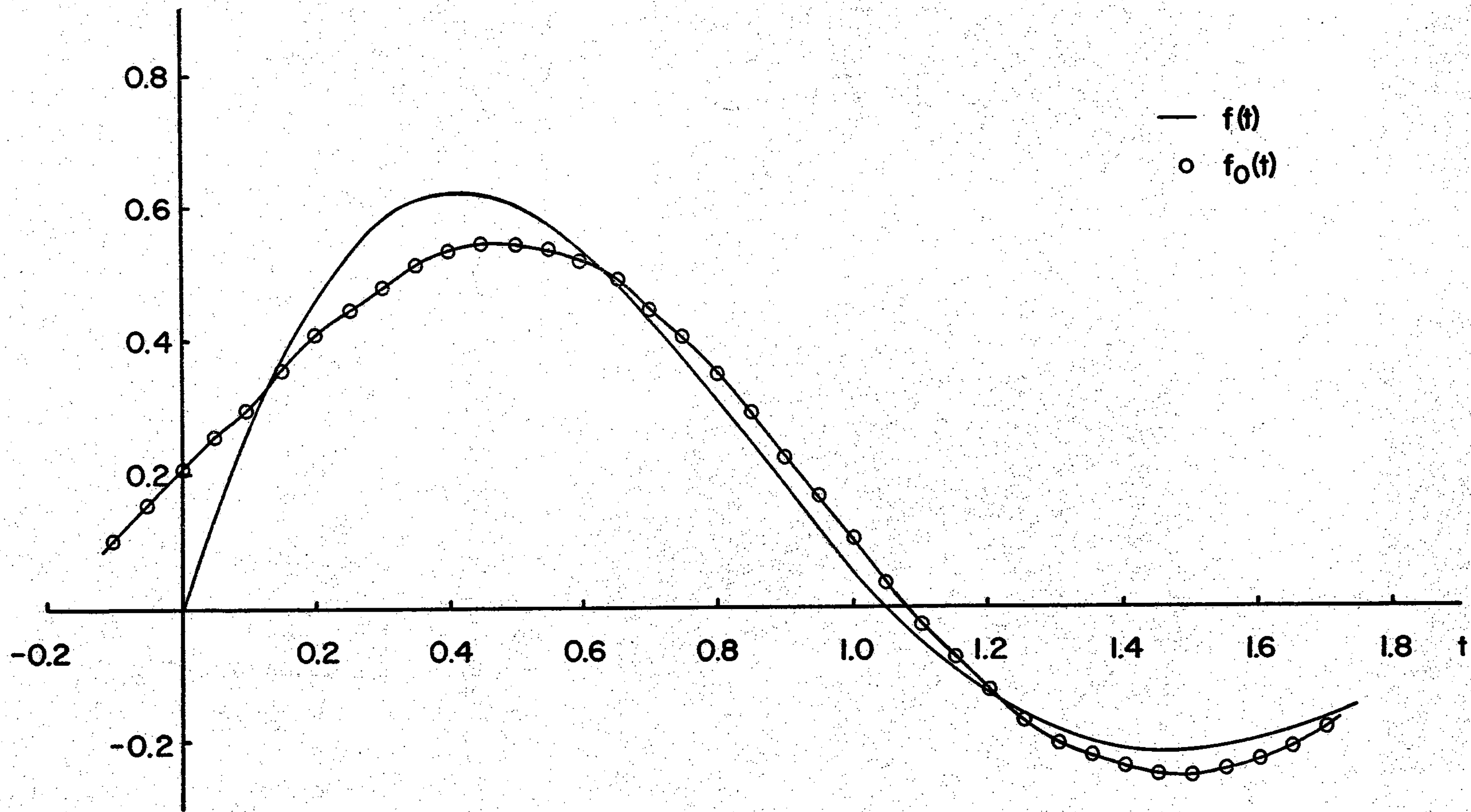

FIG. 5-3. THE FUNCTION $f_0(t)$ FOR $f(t) = u(t)\,\varepsilon^{-t}$ SIN 3t.

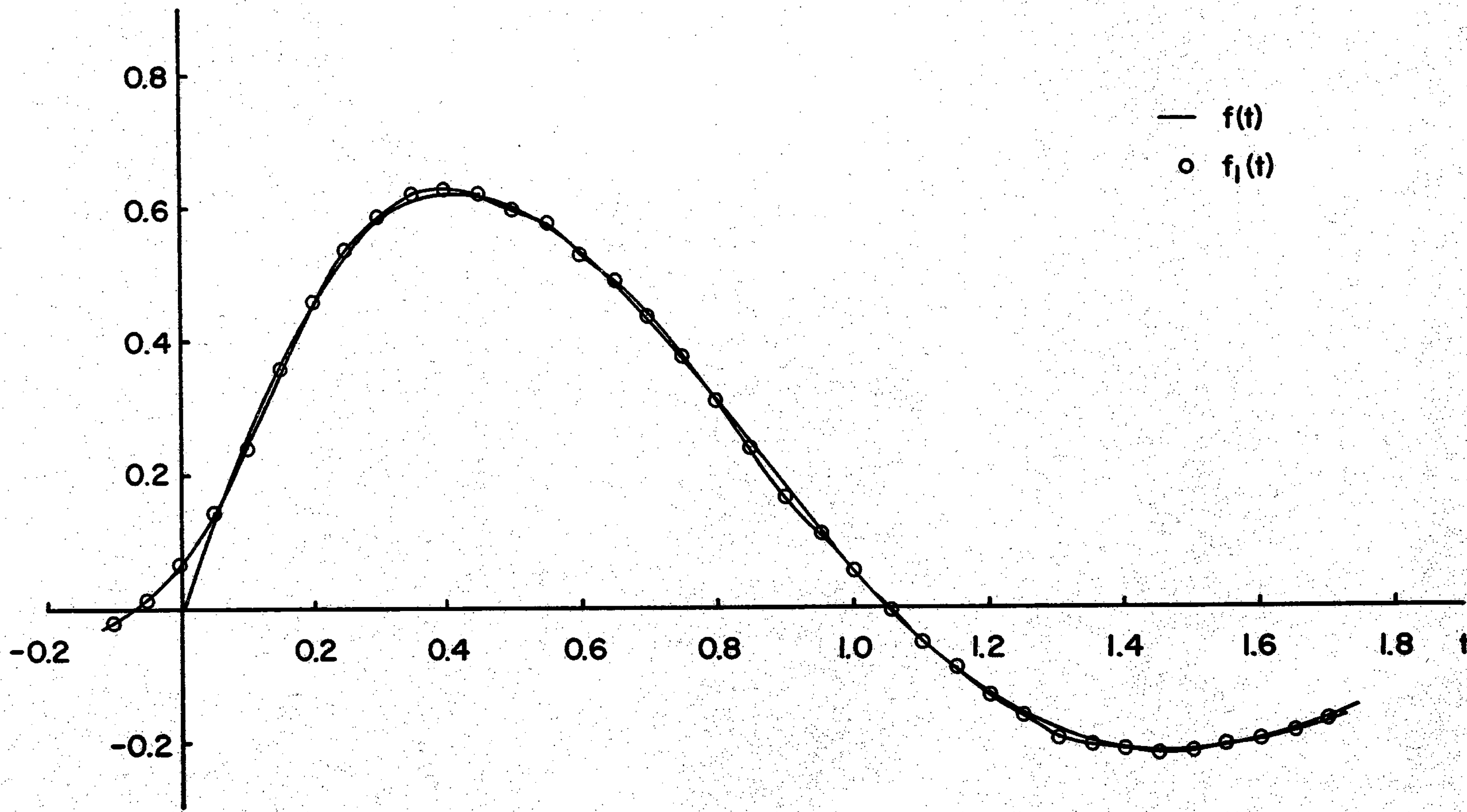

FIG. 5-4. THE FUNCTION $f_1(t)$ FOR $f(t) = u(t)\,\varepsilon^{-t}$ SIN 3t.

coefficients of g(t). As stated in Eq. (4-57), f(t) evaluated at values of t which are integer multiples of T is given by

$$f(rT) = \sum^{p} \sum^{\ell} \sum^{n} b_{np\ell} \, \hat{b}_{np(r-\ell)} \varepsilon^{j2p\Omega rT} (-1)^n \lambda_n \qquad (5\text{-}7)$$

where r is any integer, positive, negative, or zero. If the coefficients of Tables 5-1 and 5-3 are substituted into Eq. (5-7) the values given in Table 5-5 and plotted in Fig. 5-5 are obtained. Again the results are presented for $f_0(t)$, using only the terms for which $p = 0$, and for $f_1(t)$, using terms for which $p = -1$, 0, and +1. The function, f(t), is again shown for comparison.

The triple summation of Eq. (5-7) is further broken down in Table 5-6 so that the contribution of each term is evident. Here it can be seen that the terms for which $n = 1$ are about two orders of magnitude smaller than the corresponding ones for which $n = 0$. This is due at least in part to the fact that λ_1 is almost two orders of magnitude smaller than λ_0. Due to the rapid decrease in size of λ_n as n increases through values greater than $\frac{2c}{\pi}$, the contribution of terms involving these values of n is expected to be small.

r	t	$f_o(rT)$ p = 0 only	$f_1(rT)$ p = -1, 0 and 1	$f(t) = \frac{\varepsilon^{-t}}{3}(1 - \cos 3t)$
0	0.0	0.01710	0.01487	0.0
1	0.2	0.05946	0.05200	0.04767
2	0.4	0.1241	0.1298	0.1425
3	0.6	0.1933	0.2005	0.2245
4	0.8	0.2414	0.2419	0.2602
5	1.0	0.2478	0.2419	0.2440
6	1.2	0.2087	0.2038	0.1904
7	1.4	0.1397	0.1406	0.1225
8	1.6	0.06771	0.07251	0.06141

Table 5-5. Representation of Linear System Output.

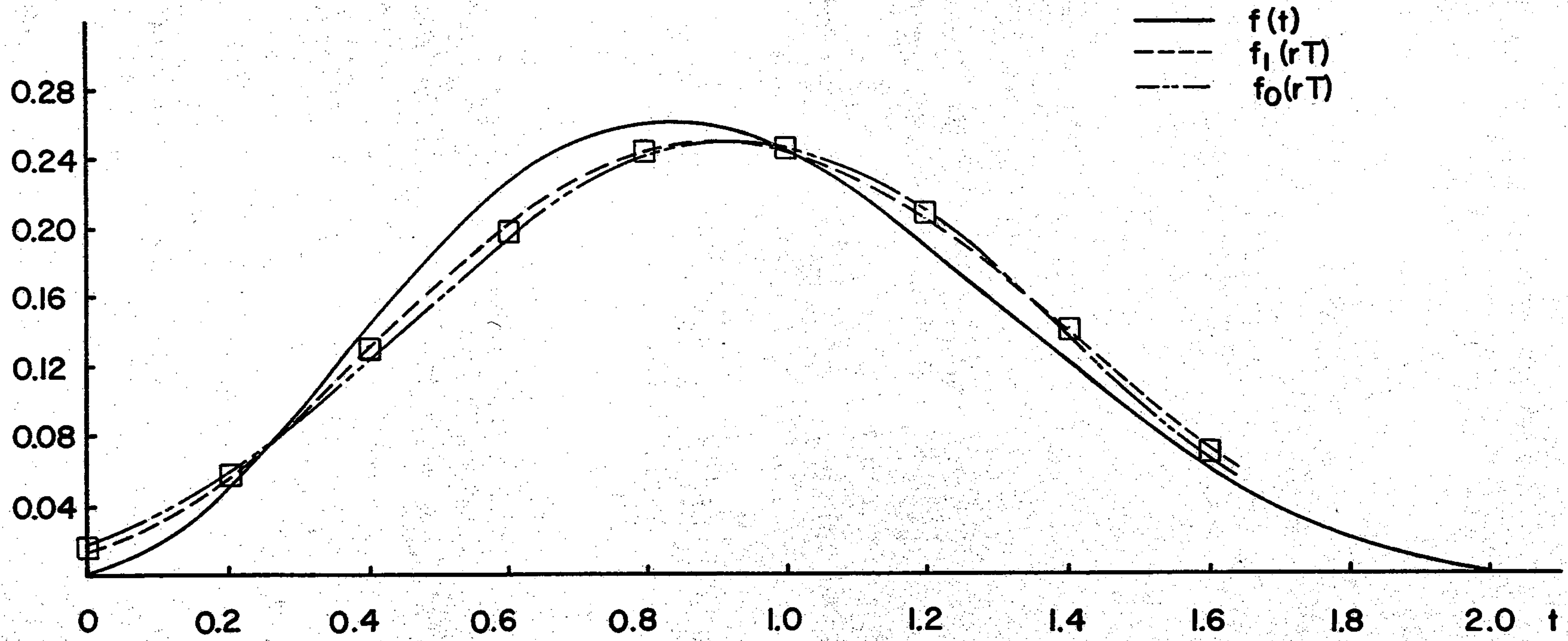

FIG. 5-5. REPRESENTATION OF LINEAR SYSTEM OUTPUT.

		p = 0				p = 1			
		n = 0		n = 1		n = 0		n = 1	
r	ℓ	M	h	M	h	M	h	M	h
0	0	1.787	-02	-7.643	-04	-2.209	-03	-1.978	-05
1	0	3.496	-02	-6.797	-04	-4.092	-03	-3.336	-05
1	1	2.565	-02	-4.702	-04	-3.289	-03	-5.510	-05
2	0	4.591	-02	-2.381	-04	1.219	-03	-3.379	-06
2	1	5.020	-02	-4.182	-04	2.661	-03	-4.088	-05
2	2	2.867	-02	-1.477	-05	1.875	-03	-5.237	-06
3	0	4.441	-02	3.720	-04	2.171	-03	3.519	-05
3	1	6.592	-02	-1.465	-04	2.332	-03	4.434	-05
3	2	5.610	-02	-1.313	-05	1.524	-03	6.993	-05
3	3	2.632	-02	3.750	-04	8.986	-04	6.317	-05
4	0	3.002	-02	8.503	-04	1.217	-03	1.396	-05
4	1	6.377	-02	2.288	-04	-3.021	-04	6.002	-05
4	2	7.367	-02	-4.601	-06	-1.150	-03	8.047	-05
4	3	5.150	-02	3.335	-04	-2.614	-04	3.526	-05
4	4	2.054	-02	5.298	-04	7.591	-04	3.161	-06
5	0	8.637	-03	9.638	-04	-9.150	-04	-7.889	-06
5	1	4.311	-02	5.231	-04	-1.560	-03	-6.943	-06
5	2	7.127	-02	7.187	-06	-1.056	-03	-4.820	-05
5	3	6.763	-02	1.168	-04	-6.096	-04	-6.281	-05
5	4	4.020	-02	4.712	-04	-8.498	-04	-2.658	-05
5	5	1.445	-02	4.352	-04	-8.003	-04	-6.186	-06

Table 5-6. Table of Terms for Eq. (5-7)

Given as $M \times 10^h$.

		p = 0				p = 1			
		n = 0		n = 1		n = 0		n = 1	
r		M	h	M	h	M	h	M	h
6	0	-1.070	-02	6.836	-04	-1.320	-03	-2.193	-05
6	1	1.240	-02	5.929	-04	-1.082	-03	-3.622	-05
6	2	4.817	-02	1.643	-05	-2.822	-04	-6.566	-05
6	3	6.543	-02	-1.825	-04	-1.055	-04	-6.033	-05
6	4	5.278	-02	1.651	-04	-5.521	-04	-3.243	-05
6	5	2.828	-02	3.870	-04	-7.241	-04	-2.632	-05
6	6	1.041	-02	2.189	-04	-5.075	-04	-3.265	-05
7	0	-2.082	-02	1.903	-04	-6.835	-04	-7.893	-06
7	1	-1.537	-02	4.206	-04	3.395	-04	-3.307	-05
7	2	1.386	-02	1.862	-05	7.144	-04	-2.366	-05
7	3	4.422	-02	-4.172	-04	2.468	-04	1.817	-05
7	4	5.106	-02	-2.579	-04	1.649	-04	1.768	-05
7	5	3.714	-02	1.356	-04	4.863	-04	-1.747	-05
7	6	2.038	-02	1.946	-04	1.967	-04	-1.962	-05
7	7	8.871	-03	4.749	-05	-4.377	-04	-1.904	-06
8	0	-1.986	-02	-2.501	-04	5.925	-04	6.809	-06
8	1	-2.989	-02	1.171	-04	9.754	-04	8.731	-06
8	2	-1.718	-02	1.321	-05	6.073	-04	3.660	-05
8	3	1.272	-02	-4.729	-04	3.106	-04	4.470	-05
8	4	3.451	-02	-5.894	-04	4.507	-04	2.587	-05
8	5	3.593	-02	-2.118	-04	4.803	-04	2.261	-05
8	6	2.676	-02	6.818	-05	3.472	-04	3.102	-05
8	7	1.736	-02	4.223	-05	4.164	-04	1.943	-05
8	8	8.623	-03	1.202	-05	4.284	-04	4.829	-06

Table 5-6 (continued). Table of Terms for Eq. (5-7) Given as $M \times 10^h$.

CHAPTER 6

Summary of Results, Conclusions and Extensions

In this final chapter summaries of the evolution of two dimensional representations and of the work reported in the previous chapters are presented including a few remarks concerning the choice of the prolate spheroidal functions. The chapter is concluded by brief descriptions of some of the problems which seem amenable to solution as a result of this type of analysis.

6.1 The Evolution of the Two Dimensional Representations and Summary

When Gabor first proposed the concept of a two dimensional description of a signal he was attempting to learn about information and how it is conveyed by a signal. Although he failed in his attempt to define information mathematically, he did suggest so many new ideas that his paper is still referenced in new studies fifteen years after it appeared. Because of his desire to define information, he chose to fix both the size and the shape of the smallest area of interest in the time-frequency plane. If the two dimensional representation is to be applied to signal analysis problems rather than information theory, the reasons for those restrictions are foreign to the application and undesirable.

Lerner's contributions were to suggest the use of the two dimensional representation for signal analysis, to discard the restriction on the shape of the fundamental area of the time-frequency plane, i.e., the length-to-width ratio, and to point out a method of orthogonal expansion in terms of a somewhat arbitrary function called a basis function. He chose to retain the restriction on the size of this fundamental area. And while he gave some hints as to the reasons for choosing any given function as the basis function, he gave no hint as to how the choice is made. Also the reason for choosing a fundamental area of one half was not specifically stated. Gabor's reason for this choice was that both the sampling theorem for bandlimited functions and the Fourier series for timelimited functions dictate this value. In addition, if these definitions of bandwidth and time duration are used, the "uncertainty relation" also suggests the value of one half.

For the two dimensional representations defined in Chapter 4 it is seen that this restriction on fundamental area size has also been discarded, and, indeed, this results in one of the fundamental differences.

It was pointed out in Chapter 3 that, roughly speaking, the number of the eigenvalues, λ_n, and thus the number of ψ functions which are significant in the fundamental interval is about $\frac{2c}{\pi} + 1$. Then it might be supposed that the number of terms necessary here to achieve about the same accuracy as the sampling theorem would yield is the same as the sampling theorem suggests. This is confirmed mathematically by Pollak (23) in an as yet unpublished manuscript. As a result of this, the concept of the dimensionality or degrees of freedom of a signal is confirmed by these two dimensional representations of a signal, and it is possible to vary the accuracy with which the signal is representated in any given area of the time-frequency plane by varying the number of terms used. Thus, the accuracy of this representation is not a matter of whether a certain region is represented or not, but to what extent is it represented, and, hence, a two dimensional representation of the type suggested by Lerner is, to this extent, simply a special case of this.

In the final analysis any two dimensional representation of a signal in which time is one dimension and frequency is the other must, in one sense, be arbitrary. That is, any such two dimensional representation depends on the functions on which it is based. To infer otherwise is to infer that a function can be simultaneously time- and bandlimited. It is, therefore, important to choose the functions to be used as wisely as possible. In this case the term "wise" must refer to at least two groups of properties: those of intuition and conceptual aid and those of mathematical convenience.

In particular, the bandlimited property is convenient both conceptually and mathematically. And, given the requirement of bandlimitation, the prolate spheroidal functions are particularly well-suited for this purpose. Their properties concerning the maximally close approximation to a function which is simultaneously time- and bandlimited appeal to the intuition by affording maintenance of the two dimensional structure as strictly as possible. Their orthogonality properties, among others, are convenient mathematically. And certainly properties such as symmetry and the number of zeroes in the fundamental interval are useful as aids to visualization.

In summary, then, it is seen that two very similar two dimensional representations have been defined which have the property of finality of coefficients and which have no theoretical limitation on either the size or shape of the fundamental area to be represented. It is also possible to adjust the accuracy with which various parts of the signal are represented.

Very important among the properties of these representations is the fact that the integrations necessary to calculate the coefficients are not difficult to carry out numerically. All of the calculations reported here were carried out on a Royal McBee RPC-4000 digital computer which is a very small, relatively slow digital computer. It is not necessary at any point to calculate a Hilbert Transform or any other improper integral.

Another point to be noted is that, aside from the choice of c and T, very little preliminary work is necessary when applying the representations to a particular problem. While Lerner's work is certainly significant, it did have the rather severe limitation, so far as application is concerned, that it required a considerable amount of preliminary work before useful results could be obtained.

It was pointed out in section 1.1 that signal representations are needed for two purposes. These purposes were given as the revelation of the information-bearing attributes of a signal and the study of systems and their transmission properties. The results for a numerical example of the application of a two dimensional representation to a problem in each of these categories have been presented. In the first it was demonstrated that the representation has the two dimensional properties expected for the examples chosen. In the second one method proposed for the solution of the linear system problem was demonstrated. Further, a method for solving a more general linear system problem was presented as well as a method for signal analysis in terms of the ambiguity function. Other possible applications are indicated in section 6.2.

6.2 Extensions and Further Applications

Perhaps one of the ultimate objectives of signal theory, so far as can be foreseen at this time, is in the area of signal design. Frequently problems arise in which the general characteristics of a system are largely known and some specific information about the system is desired from the output. The question then arises as to what input signal should be used in order to best obtain this information from the output. The radar system is an example of such a problem; another is the system identification problem.

The question raised, then, is one of how to design a signal with certain properties, perhaps subject to constraints in the time and/or frequency domain, much in the manner by which modern techniques allow the design of systems. It seems reasonable to assume that practical two dimensional signal representations are a fairly long first step toward the solution of the problem and that further research in this direction would be profitable.

A second problem arises upon solution of this first one. For the system problem mentioned, once the optimum signal has been designed and applied, how is the resulting output signal best analyzed to obtain the desired information about the system. This, then, is the identification problem and it seems that the results of Chapter 4 may be able to shed new light upon existing solutions to this problem.

In the area of signal analysis there are a number of problems which could yield very valuable results if the available methods could be extended and improved. Examples of such problems are especially numerous in the area of biological systems. The analysis of such signals as electrocardiograms and electroencephalograms is apparently still in its infancy. Lowenberg (19) states in the case of the study of electroencephalograms by frequency domain techniques, that in many cases it appears that electroencephalographic signals may be abnormal only for brief intervals of time, and, therefore, it seems desirable to use a unique method for representing short intervals which are known to be abnormal. Thus the two dimensional representation seems particularly well-suited for this purpose.

From the study of the two dimensional representations presented it would seem that a logical extension would be to a continuous spectrum-time function. It is certainly possible to force a continuous function from the integral expressions for any of the four coefficients defined in Chapter 4 by simply allowing $p\Omega$ and ℓT to become continuous variables in a manner similar to that employed by Lerner for his two dimensional representations. The problem is a bit more complex than this, however. Most of the previous attempts to define a spectrum-time function (see for example Kharkevich (14), Fano (5), or Rothschild (24)) have either purposely or inadvertently assumed that the unknown or irrelevant parts of the time function to be so represented are zero. This introduces a fictitious discontinuity in the function, and, thus, grossly affects the frequency function. Although quite natural, for the above purpose, this assumption is not necessarily the most logical nor the most desirable. It seems possible that further study of the spectrum-time functions in light of the prolate spheroidal two dimensional representations may also yield useful results in this direction.

It would be desirable, for this application, to be aware of any further properties of the prolate spheroidal functions which are readily available, in particular, properties of the spheroidal functions as a function of c.

One further possible application of the two dimensional representations will be mentioned here. There exists a method of orthogonal series representation of random processes, known as the Karhunen-Loève representation, in which the expansion coefficients become the random variables. It may be shown (see, for example, Davenport and Root (3)) that in order for the expansion coefficients to be uncorrelated, the orthogonal functions must satisfy the following integral relation

$$\int_a^b R(t,s)\, \phi_n(s)\, ds = \lambda_n\, \phi_n(t) \tag{6-1}$$

where $\phi_n(s)$ is the orthogonal function, $R(t,s)$ is the correlation function of the random process, and (a,b) is the region of orthogonality. Slepian and Pollak pointed out that in the case of stationary, bandlimited, white noise, Eq. (6-1) and Eq. (3-17) are identical, and, thus, the ψ functions are the required orthogonal set. In this case the generalization to two dimensions does not immediately follow. The similarities are strong enough, however, to make a further investigation seem justifiable.

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APPENDIX

Some Results Concerning the Time-Bandwidth Uncertainty Principle

Over the years a considerable amount of work has been expended in determining how severely the time duration and the bandwidth of a function can be simultaneously restricted. This problem has become known as the uncertainty relation problem, by analogy to the Heisenberg uncertainty principle of quantum mechanics. The results obtained vary over a considerable range and depend heavily on the definition of bandwidth and time duration used.

Listed below are some of the results which have appeared in the literature together with the associated definitions of bandwidth and time duration. They are listed in the order in which the work was completed.

1. MacColl, L. A., 1940 as reported by Landau and Pollak (18)

If $$\frac{\int_{t_o}^{t_o+T} |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt} = \alpha_1 \qquad \text{(A-1)}$$

and $$\frac{\int_{\omega_o}^{\omega_o+\Omega} \left|F(\omega)\right| d\omega}{\int_{-\infty}^{\infty} \left|F(\omega)\right| d\omega} = \alpha_2 \qquad \text{(A-2)}$$

then $$\Omega T > 2\pi\alpha_1\alpha_2^2 \qquad \text{(A-3)}$$

2. Gabor (8), 1946

These results were presented in Chapter 2 and are repeated here for comparison. See Eqs. (2-7), (2-8) and (2-9)

If $$\Delta t = \left[2\pi\overline{(t - \overline{t})^2}\right]^{\frac{1}{2}} \qquad \text{(A-4)}$$

and $$\Delta f = \left[2\pi\overline{(f - \overline{f})^2}\right]^{\frac{1}{2}} \qquad \text{(A-5)}$$

then $$\Delta t \Delta f \geq \frac{1}{2}$$

3. Lampard (17), 1956

If $$\Delta\tau = \frac{\frac{1}{2}\int_{-\infty}^{\infty} \Psi(\tau)d\tau}{\Psi(0)} \qquad \text{(A-7)}$$

where $\Psi(\tau)$ is the autocorrelation function of $f(t)$

and if $$\Delta f = \frac{2\int_{-\infty}^{\infty} \omega(f)df}{\omega(0)} \qquad \text{(A-8)}$$

where $\omega(f)$ is the power spectrum of $f(t)$, then

$$\Delta\tau \; \Delta f = 1 \qquad \text{(A-9)}$$

4. Kay and Silverman (12), 1957

Define the first moment

$$\langle t \rangle = \int_{-\infty}^{\infty} t \left| f\ (t) \right|^2 dt \tag{A-10}$$

and the time duration, Δt,

$$(\Delta t)^2 = \int_{-\infty}^{\infty} (t - \langle t \rangle)^2 \left| f(t) \right|^2 dt \tag{A-11}$$

Define the transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\varepsilon^{-j\omega t} d\omega \tag{A-12}$$

and the moment

$$\langle \omega \rangle_+ = \int_{0}^{\infty} \omega \left| F(\omega) \right|^2 d\omega \tag{A-13}$$

also the bandwidth, $\Delta\omega$, by

$$(\Delta\omega_+)^2 = 2\int_{0}^{\infty} (\omega - \langle \omega \rangle_+)^2 \left| F(\omega) \right|^2 d\omega \tag{A-14}$$

Then $$\Delta t \Delta\omega_+ \geqslant \alpha \tag{A-15}$$

where α is $\frac{1}{2}$ if $F(0) = 0$. But α may be smaller than $\frac{1}{2}$ for other signals and examples are given for which $\alpha \approx 0.3$. The problem of finding the greatest lower bound for α was left unsolved. In a post script added to this paper in 1959 (13) they stated that a Russian author solved this problem in 1934 and found that $\alpha = \frac{1}{2\sqrt{3}}$.

5. Zakai, (35) 1960

Define the bandwidth as

$$\Delta f_p = \frac{1}{2\pi} \left[\frac{\left(\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \right)^{\frac{1}{2}}}{\left(\int_{-\infty}^{\infty} |F(\omega)|^p d\omega \right)^{\frac{1}{p}}} \right]^{\frac{2p}{p-2}} \qquad \text{(A-16)}$$

where $F(\omega)$ is defined as in Eq. (A-12).

Define the time duration Δt_q in a manner similar to this substituting $f(t)$ for $F(\omega)$ and q for p.

Then the uncertainty relation is

$$\Delta t_q \ \Delta f_p \geqslant 1 \qquad \text{(A-17)}$$

for

$$\frac{1}{p} + \frac{1}{q} \geqslant 1 \qquad \begin{matrix} p \geqslant 0 \\ q \geqslant 0 \end{matrix} \qquad \text{(A-18)}$$

For the case

$$\frac{1}{p} + \frac{1}{q} < 1 \qquad \begin{matrix} p > 0 \\ q > 0 \end{matrix} \qquad \text{(A-19)}$$

the greatest lower bound, g.l.b., for the uncertainty product is

$$\text{g.l.g.}\left[\Delta t_q \ \Delta f_p\right] = 0 \qquad \text{(A-20)}$$

6. Landau and Pollak (18), 1961

Landau and Pollak chose to state their results in a different form. For an arbitrary function, $f(t)$, with spectrum, $F(\omega)$, define the following

$$\frac{\int_{t_o - \frac{T}{2}}^{t_o + \frac{T}{2}} \left| f(t) \right|^2 dt}{\int_{-\infty}^{\infty} \left| f(t) \right|^2 dt} = \alpha^2 \tag{A-21}$$

and

$$\frac{\int_{-\Omega}^{\Omega} \left| F(\omega) \right|^2 d\omega}{\int_{-\infty}^{\infty} \left| F(\omega) \right|^2 d\omega} = \beta^2 \tag{A-22}$$

Their statement was that there is a function satisfying Eqs. (A-21) and (A-22) under the following conditions and only under the following conditions:

1. If $\alpha = 0$ when $0 \leq \beta < 1$
2. If $0 < \alpha < \sqrt{\lambda_o}$ when $0 \leq \beta \leq 1$
3. If $\sqrt{\lambda_o} \leq \alpha < 1$ when $\cos^{-1}\alpha + \cos^{-1}\beta \geq \cos^{-1}\sqrt{\lambda_o}$
4. If $\alpha = 1$ when $0 < \beta \leq \sqrt{\lambda_o}$

One may conclude from these six results and others which have appeared in the literature that this is a well studied problem and the time-bandwidth product is of the order of one, and that a clear definition of bandwidth or duration which is sufficiently general for most problems and yet simple to use has not yet been given.

ERRATA

TECHNICAL REPORT EE62-2

Page	Line	Should read	Instead of
12	Eq. (2-2)	$\hat{s}(t) = \ldots$	$s(t) = \ldots$
16	last line	The phrase "Δt in the time direction and a distance" was omitted immediately preceeding the last line.	
28	Eq. (3-7)	$\ldots \varepsilon^{j2p\Omega(t-\tau)} \ldots$	$\ldots \varepsilon^{jp\Omega(t-\tau)} \ldots$
28	Eq. (3-8)	$\ldots \varepsilon^{-j\ell T(\omega-s)} \ldots$	$\ldots \varepsilon^{-j\ T(\omega-s)} \ldots$
37	Eq. (3-30)	$\mathcal{F}\{\psi_n(t)\} = \ldots$	$_n(t) = \ldots$
42	footnote, last line	... are written as equalities.	... is written as an equality.
47	Eq. (4-14) third term on right	$\ldots f^*(t) \ldots$	$\ldots f(t) \ldots$
48	Eq. (4-14) last term	$\ldots \varepsilon^{j2p\Omega t} \ldots$	$\ldots \varepsilon^{-j2p\Omega t} \ldots$
54	Eq. (4-25)	$\ldots + \int_{-\frac{3T}{2}}^{-\frac{T}{2}} \ldots$	$\ldots + \int_{-\frac{3T}{2}}^{\frac{T}{2}} \ldots$
64	Eq. (4-53)	$\ldots \varepsilon^{j2p\Omega(t-\tau)} d\tau$	$\ldots \varepsilon^{j2p\Omega(t-\tau)} dt$
68	Eq. (4-64)	The right side should contain the multiplicative factor, $\lambda_m \lambda_n$.	
74	Eq. (4-81)	b^*_{mqk}	b^*_{qkm}
74	Eq. (4-81)	$\ldots\ \varepsilon^{-j2\pi\phi t}\} dt$	$\ldots\ \varepsilon^{-j2\pi\phi t}\ dt$
75	Eq. (4-82)	Both integral signs should occur immediately preceeding $\psi_n(t-\ell T)$.	
99	21	In the second, one method...	In the second one method...
104	Ref. 7	Fränz, K., "Über Signale...	Franz, K., "Uber Signale...