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Distribution Functions for Outputs of Certain Linear Filters for Random Square-wave Inputs

J. A. McFadden, Principal Investigator

A. R. Cohen

August 1, 1961

Lafayette, Indiana

for
DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
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FOR

DEPARTMENT OF THE NAVY

OFFICE OF NAVAL RESEARCH

WASHINGTON, D. C.

by

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

a_k	value of $x(t)$ in the interval $t_o-(k+1)\tau_o < t \leq t_o-k\tau_o$.
b	a constant, $0 < b < 1$.
$E\{A\}$	expectation of A.
$E\{A\|B\}$	expectation of A given B.
$F(z)$	characteristic function of the output at an axis-crossing of the input.
$F_+(z), F_-(z)$	conditional characteristic function of the output at an axis crossing of the input.
$G(z)$	characteristic function of the output at a sample point.
$G_+(z), G_-(z)$	conditional characteristic function of the output at a sample point.
i	$\sqrt{-1}$
I^+_{jk}, I^-_{jk}	a closed interval.
J^+_{jk}, J^-_{jk}	an open interval.
ℓ	time from an arbitrary point to last axis crossing.
m_k	kth moment of the output at a sample point or at an axis crossing of the input.
p	probability of no axis crossing occurring at a sample point.
$P(y)$	distribution function of the output at an arbitrary point.
$P_+(y), P_-(y)$	conditional distribution function of the output at an arbitrary point.
$P_o(\tau)$	distribution function of the lengths of axis-crossing intervals.
$Pr\{A\}$	probability of A.
$Pr\{A\|B\}$	probability of A given B.
q	probability of an axis-crossing occurring at a sample point.
$Q(y)$	distribution function of the output at an axis-crossing of the input.

LIST OF SYMBOLS (continued)

$Q_+(y), Q_-(y)$	conditional distribution function of the output at an axis-crossing of the input.
$Q_o(\ell)$	distribution function of the time from an arbitrary point to the last axis-crossing.
r	a random variable, $0 \leq r \leq 1$.
$R(y)$	distribution function of the output at a sample point.
$R_+(y), R_-(y)$	conditional distribution function of the output at a sample point.
t	time.
t_o	a sample point.
t_k	an axis crossing.
T	time constant of an RC filter.
T_o	integration time of an ideal finite-time integrator.
$U(z)$	real part of $G_+(z)$ or $F_+(z)$.
$V(z)$	imaginary part of $G_+(z)$ or $F_+(z)$.
$W(t)$	weighting function of a filter.
$x(t)$	the input.
$y(t)$	the output.
ϵ	"is a member of", i.e. a A indicates that a is a member of A.
ξ	dummy variable of integration.
ρ	reflection operator.
τ	length of an axis-crossing interval
τ_o	elementary pulse width of a coin-toss square wave.

ABSTRACT

This report considers the problem of the calculation of the distribution function of the output of a linear filter with a random square-wave input. The systems considered are the finite-time integrator, the RC low-pass filter, and certain restricted higher-order filters. The inputs are square-waves in which the lengths of axis-crossing intervals are random, but statistically independent.

For the finite-time integrator with a coin-toss square-wave input, a difference equation for the characteristic function of the output is derived and solved.

The continuity and differentiability properties of the distribution function of the output of an RC low-pass filter are discussed.

Under specified conditions on an RC low-pass filter with a coin-toss square-wave input, the distribution function of the output is constructed. For the same problem, a functional equation is derived for the characteristic function of the output, and a recurrence relation is obtained for certain moments of the output.

For a general square-wave input, an integral equation is derived for the characteristic function of the output of an RC low-pass filter at an axis-crossing of the input. From this equation a second recurrence relation for the moments of the output is obtained.

For the coin-toss square-wave input, certain higher-order systems are also considered. In particular, when a second-order system is tuned to the clocking rate of the input, the problem is reduced to an equivalent first-

order problem, and the distribution function for the output at an arbitrary instant is expressed in terms of a related function for the RC filter.

CHAPTER I

INTRODUCTION

The problem of finding the distribution function of the output of a linear filter for a random square-wave input will be stated here, and a brief historical background will be given.

The various types of filters and inputs with which we shall be concerned will be defined, and some of the basic properties of the output distribution functions will be discussed.

1. Statement of the Problem

Consider a general linear, time-invariant filter with a weighting function $W(t)$. If the input to the filter is $x(t)$, the output can be written as

$$y(t) = \int_{-\infty}^{t} W(t-\xi)x(\xi)d\xi \quad . \tag{1.1.1}$$

Now, if the multivariate distribution function of the input to this system is Gaussian, the output will also be Gaussian, and the first-order distribution function of the output will be specified by knowledge of the first and second moments of the output.

However, if the input is non-Gaussian, the output, in turn, will be non-Gaussian. For this case the first and second moments of the output, which are relatively easy to determine, do not specify the first-order output distribution function as they do in the Gaussian case.

This, then, is the problem which is to be studied: to obtain the necessary information to specify the first-order distribution function of the outputs of several types of linear, time-invariant filters for a certain restricted class of non-Gaussian inputs. This class of inputs will consist of binary random square waves (see Fig. I-1) in which the lengths of the intervals between axis crossings are independent; the lengths of these intervals will be further restricted so that they all have the same distribution function. Finally, we shall require that $|x(t)| = 1$.

2. Historical Background

This problem has been considered directly by a number of investigators. The solution of the problem for a random telegraphic wave[1] into an RC low-pass filter has been obtained by McFadden [14], [15], [16], and Wonham and Fuller [19]. Results concerning a random telegraphic wave into an ideal, finite-time integrator and other more general systems have been obtained by McFadden [14], [15], and for an RC low-pass filter results concerning a special class of inputs have also been obtained [16]. Some of the methods used below are based on the work by Darling and Siegert [3] on Markov processes.

It will be shown in Chapter III that the random variable defined by

$$\sum_{k=1}^{\infty} a_k b^k, \quad 0 < b < 1,$$

where $a_k = \pm 1$ with equal probability and where a_k is independent of a_j $(j \neq k)$, is directly related to the output of an RC low-pass filter for certain types of random square-wave inputs. A certain amount of information about the distribution function of this random variable has been obtained by Jessen and

---

1 A random telegraphic wave is a square wave in which the axis crossings are Poisson-distributed in time.

Winter [9], Erdös [5], Rice [18], and several other investigators.

3. Definitions and General Remarks

Because of the complexity of the problem, even with the restricted class of inputs which we shall consider, we shall limit ourselves almost entirely to two types of filters. The first of these will be the finite-time integrator (Chapter II) with the weighting function,

$$W(t) = \begin{cases} \frac{1}{T_o}, & 0 < t \leq T_o, \\ 0, & \text{elsewhere}, \end{cases} \tag{1.3.1}$$

and which is shown graphically in Fig. I-2a. The second will be the RC low-pass filter (Chapters III-V) which has the weighting function (see Fig. I-2b),

$$W(t) = \begin{cases} 0, & t \leq 0, \\ \frac{1}{T} e^{-t/T}, & t > 0, \end{cases} \tag{1.3.2}$$

where T is the time constant of the filter. In addition we shall consider certain more general, but restricted, filters in Chapter VI.

For the random square waves which we shall consider, the interval between any two adjacent axis crossings will be called an axis-crossing interval. We shall let τ be the length of an arbitrary axis-crossing interval, and $P_o(\tau)$, where $P_o(\tau)=0$ for $\tau \leq 0$, its distribution function.

Contained in the class of inputs under consideration are the square waves which we shall refer to as coin-toss square waves. These square waves have the property that axis-crossings can occur only on a set of sample points which are equally spaced in time. At each sample point we shall let p be the probability that no axis crossing occurs, and $q=1-p$ the probability that one does occur. The distance between sample points will be called the elementary pulse width and will be denoted by τ_o. For this waveform, therefore, the length of an axis-crossing interval is $\tau = n\tau_0$, $n = 1,2,\ldots$, and $P_0(\tau)$ is given by

$$P_0(\tau) = \begin{cases} 0, & \tau \le \tau_0 ; \\ q \sum_{k=0}^{n-1} p^k & n\tau_0 < \tau \le (n+1)\tau_0, \quad n=1,2,\ldots \end{cases} \tag{1.3.3}$$

A coin-toss square wave with its associated distribution function is depicted in Fig. I-3. When $p \neq q$ we shall refer to this square wave as a generalized coin-toss square wave.[2]

The distribution function of the output at an axis crossing will be denoted by $Q(y)$, and the conditional distribution function of the output, given that the input was positive or negative just to the left of this point, will be denoted by $Q_+(y)$ or $Q_-(y)$, respectively. In the case of a coin-toss square wave the distribution function of the output at a sample point will be denoted by $R(y)$, with definitions similar to the above for $R_+(y)$ and $R_-(y)$. At an arbitrary point the distribution function of the output will be denoted by $P(y)$, and we shall define $P_+(y)$ and $P_-(y)$ as being the distribution functions conditional on whether the input is positive or negative at this point. The characteristic functions corresponding to $Q(y)$ and $R(y)$ will be denoted by $F(z)$ and $G(z)$, respectively, i.e.,

$$\left.\begin{aligned} F(z) &= \int_{-\infty}^{\infty} e^{izy}\, dQ(y); \\ G(z) &= \int_{-\infty}^{\infty} e^{izy}\, dR(y). \end{aligned}\right\} \tag{1.3.4}$$

---

2 Cf. Feller [6] p. 341, example (a).

Similarly, $F_+(z)$, $F_-(z)$, $G_+(z)$, and $G_-(z)$ will be defined as the characteristic functions corresponding to $Q_+(y)$, $Q_-(y)$, $R_+(y)$, and $R_-(y)$, respectively.

Since all the axis-crossing interval lengths have the same distribution function, the input has the property that

$$\Pr\{x(t)=+1\} = \Pr\{x(t)=-1\} = \tfrac{1}{2}, \qquad (1.3.5)$$

for all t, where $\Pr\{A\}$ means "probability of A". Since we are considering only linear filters, this symmetric property will appear at the output in the following way:

$$\left.\begin{aligned} Q(y) &= 1 - Q(-y); \\ Q_+(y) &= 1 - Q_-(-y); \\ R(y) &= 1 - R(-y); \\ R_+(y) &= 1 - R_-(-y); \\ P(y) &= 1 - P(-y); \\ P_+(y) &= 1 - P_-(-y). \end{aligned}\right\} \qquad (1.3.6)$$

It follows from (1.3.6) that F(z) and G(z) are even functions, and

$$\left.\begin{aligned} F_+(z) &= F_-^*(z); \\ G_+(z) &= G_-^*(z), \end{aligned}\right\} \qquad (1.3.7)$$

where the star indicates complex conjugate. And finally, because of (1.3.5), we have that

$$\left.\begin{aligned} Q(y) &= \tfrac{1}{2}\; Q_+(y)+Q_-(y) \;; \\ R(y) &= \tfrac{1}{2}\; R_+(y)+R_-(y) \;; \\ P(y) &= \tfrac{1}{2}\; P_+(y)+P_-(y) \;; \\ F(z) &= \tfrac{1}{2}\; F_+(z)+F_-(z) \;; \\ G(z) &= \tfrac{1}{2}\; G_+(z)+G_-(z) \;. \end{aligned}\right\} \qquad (1.3.8)$$

We shall consider, in Chapter II, the output of a finite-time integrator with a generalized coin-toss square-wave input. We shall first derive a difference equation involving the characteristic function at a sample point. This equation will then be solved and the solution will be extended to an arbitrary point.

In Chapter III we shall investigate the properties of the distribution of the output of an RC low-pass filter, and for that purpose we shall derive an integral equation involving $Q_+(y)$ and $Q_-(y)$.

We shall then specialize to the case of a generalized coin-toss square wave into an RC low-pass filter in Chapter IV. Under a given restriction, we shall construct the distribution function of the output at a sample point. A functional equation involving $G_+(z)$ and $G_-(z)$ will be derived from which we shall obtain a recurrence relation for the moments of the output at a sample point.

We shall derive, in Chapter V, an integral equation involving $F_+(z)$ and $F_-(z)$ for the output of an RC low-pass filter with a general square-wave input. From this equation we shall obtain a second recurrence relation for the moments of the output at an axis-crossing of the input.

Conditions are derived in Chapter VI for which the results of the case of an RC low-pass filter with a generalized coin-toss square-wave input can be extended to certain higher-order systems with the same input.

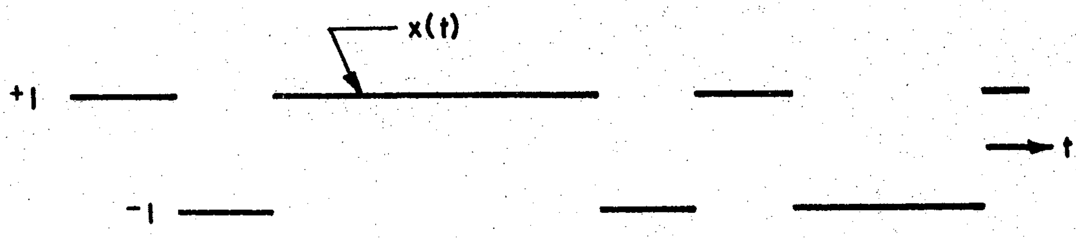
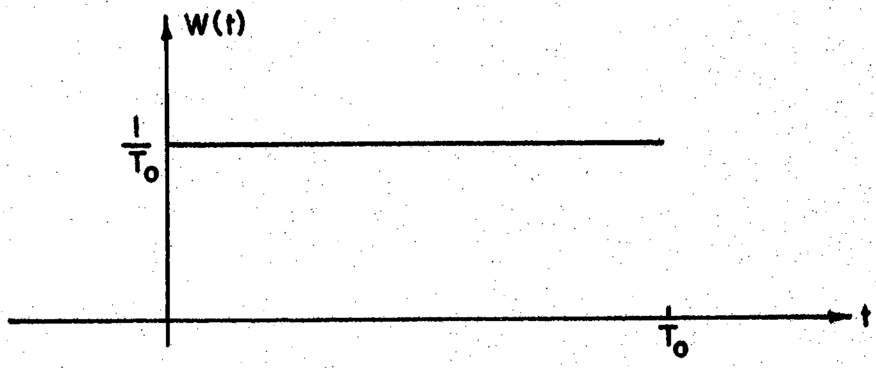
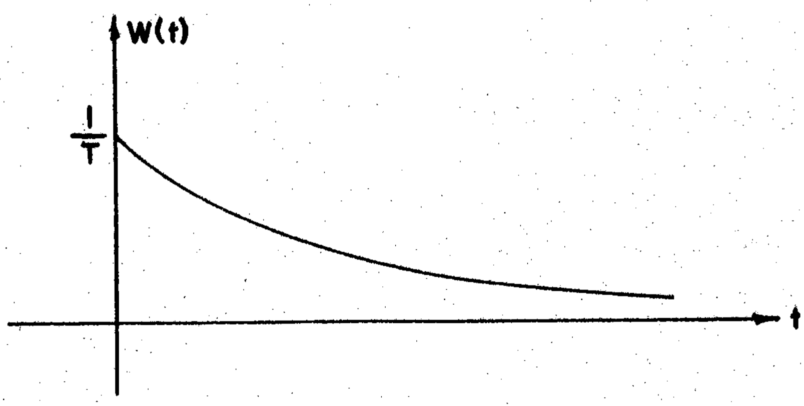


Fig. 1-1

A Binary Random Square Wave



(a) Finite-Time Integrator



(b) RC Low-Pass Filter

Fig. 1-2

Weighting Functions of the Finite-Time Integrator

and the RC Low-Pass Filter

CHAPTER II

THE FINITE-TIME INTEGRATOR WITH GENERALIZED COIN-TOSS SQUARE-WAVE INPUT

We shall now investigate the distribution function of the output of a finite-time integrator when the input is a generalized coin-toss square wave. We shall approach this problem by first calculating the characteristic function of the output at a sample point, then using the inverse Fourier transform to obtain the distribution function, and finally extending the solution to an arbitrary point.

The characteristic function at a sample point will be obtained through the solution of a difference equation which will be derived below. This solution will be obtained in a form which is easily transformed.

The distribution function at an arbitrary point will then be found under the assumption that this point is uniformly distributed between two adjacent sample points.

1. The Characteristic Function of the Output at a Sample Point

We shall now consider the problem of finding, at a sample point, the characteristic function of the output of a finite-time integrator for a coin-toss square wave input, and we shall use a Darling-Siegert approach for discrete time (see Darling and Siegert [3]).

Let t_o be an arbitrary sample point. Then from the definition of the generalized coin-toss square wave, we can characterize the input[1] as

---

[1] Note that, in this characterization, as we increase j, we move backwards in time.

$$x(t) = \sum_j a_j x_j(t), \tag{2.1.1}$$

where

$$x_j(t) = \begin{cases} 1, & t_o-(j+1)\tau_o < t \leq t_o - j\,\tau_o; \\ 0, & \text{elsewhere}, \end{cases} \tag{2.1.2}$$

and $a_j = \pm 1$ with the associated transition probabilities,[2]

$$\begin{aligned} &Pr\,\{a_j=1 \mid a_{j+1}=1\} = Pr\,\{a_j=-1 \mid a_{j+1}=-1\} = p; \\ &Pr\,\{a_j=1 \mid a_{j+1}=-1\} = Pr\,\{a_j=-1 \mid a_{j+1}=\ 1\} = q. \end{aligned} \tag{2.1.3}$$

Moreover, since the occurrence of an axis-crossing, or the absence of an axis-crossing, at any sample point is independent of what took place at all previous sample points, we have that

$$Pr\,\{a_j \mid a_{j+1}, a_{j+2}, \ldots\} = Pr\,\{a_j \mid a_{j+1}\}; \tag{2.1.4}$$

therefore the sequence $\{a_j\}$ forms a simple Markov chain.

Now let us define the functional,

$$m(t_o, \tau_1) = \int_{t_o-\tau_1}^{t_o} W(t_o-\xi)x(\xi)\,d\xi\,,$$

which, for the finite-time integrator, becomes

$$m(t_o, \tau_1) = \frac{1}{T_o}\int_{t_o-\;{}_1}^{t_o} x(\xi)d\xi\,, \qquad 0 \leq \tau_1 \leq T_o. \tag{2.1.5}$$

When $\tau_1 = T_o$, equation (2.1.5) becomes the output $y(t_o)$. Since the input is stationary, the distribution function of this functional does not depend on t_o. Therefore let us write

---

2 $Pr\ \{A|B\}$ denotes "probability of A given B".

$$m(t_o, \tau_1) \equiv m(\tau_1), \tag{2.1.6}$$

bearing in mind that the functional itself does depend on time. With $\tau_1 = n\tau_o$, the application of (2.1.1) and (2.1.2) to (2.1.5) yields

$$m(n\tau_o) = \frac{\tau_o}{T_o} \sum_{j=0}^{n-1} a_j \tag{2.1.7}$$

and

$$m\left[(n+1)\tau_o\right] = \frac{\tau_o}{T_o} \sum_{j=0}^{n} a_j , \tag{2.1.8}$$

so that

$$m\left[(n+1)\tau_o\right] = m(n\tau_o) + \frac{\tau_o}{T_o} a_n. \tag{2.1.9}$$

At this point, let us define[3]

$$H_{a_o,a_{n-1}}(n\tau_o;z) = E\left\{\exp\left[izm(n\tau_o)\right] \mid a_o, a_{n-1}\right\} \Pr\left\{a_o \mid a_{n-1}\right\} \tag{2.1.10}$$

and

$$H_{a_o,a_n}\left[(n+1)\tau_o;z\right] = E\left\{\exp\left[izm\left((n+1)\tau_o\right)\right] \mid a_o, a_n\right\} \Pr\left\{a_o \mid a_n\right\}, \tag{2.1.11}$$

and proceed with the method of Darling and Siegert. We shall derive difference equations for the conditional characteristic functions in (2.1.10).

Using (2.1.9) we obtain

$$H_{a_o,a_n}\left[(n+1)\tau_o;z\right] =$$

$$\exp\left(iza_n\frac{\tau_o}{T_o}\right)E\left\{\exp\left[izm(n\tau_o)\right] \mid a_o, a_n\right\} \Pr\left\{a_o \mid a_n\right\}. \tag{2.1.12}$$

We can write that

---

[3] $E\{A|B\}$ means "expectation of A given the condition B".

$$E\{\exp[izm(n\tau_o)]\mid a_o,a_n\}$$

$$=\sum_{a_{n-1}} E\{\exp[izm(n\tau_o)]\mid a_o,a_{n-1},a_n\}\Pr\{a_{n-1}\mid a_o,a_n\} \qquad (2.1.13)$$

where the summation is taken over the two possible values of a_{n-1}, and moreover, by a Markov property,

$$\Pr\{a_{n-1}\mid a_o,a_n\} = \frac{\Pr\{a_{n-1}\mid a_n\}\Pr\{a_o\mid a_{n-1}\}}{\Pr\{a_o\mid a_n\}}\,. \qquad (2.1.14)$$

Since $m(n\tau_o)$ depends only on $a_o,a_1,\ldots,a_{n-1}$, and not on a_n, and because of the Markov property[4] of the sequence $\{a_j\}$, we have that

$$E\{\exp[izm(n\tau_o)]\mid a_o,a_{n-1},a_n\}$$
$$= E\{\exp[izm(n\tau_o)]\mid a_o,a_{n-1}\}\,. \qquad (2.1.15)$$

Therefore, after combining the last three equations and substituting the results into (2.1.12), we obtain

$$H_{a_o,a_n}[(n+1)\tau_o;z]$$
$$= \exp\left(iza_n\frac{\tau_o}{T_o}\right)\sum_{a_{n-1}} E\{\exp[izm(n\tau_o)]\mid a_o,a_{n-1}\}\Pr\{a_o\mid a_{n-1}\}\Pr\{a_{n-1}\mid a_n\}. \qquad (2.1.16)$$

Because of (2.1.10), this becomes

$$H_{a_o,a_n}[(n+1)\tau_o;z]$$
$$= \exp\left(iza_n\frac{\tau_o}{T_o}\right)\sum_{a_{n-1}} H_{a_o,a_{n-1}}(n\tau_o;z)\Pr\{a_{n-1}\mid a_n\}\,. \qquad (2.1.17)$$

To eliminate the condition on a_o, we introduce new conditional characteristic functions having a condition only on a_k. Let

$$H_{a_k}[(k+1)\tau_o;z] = H_{a_o,a_k}[(k+1)\tau_o;z] + H_{-a_o,a_k}[(k+1)\tau_o;z]. \qquad (2.1.18)$$

---

4 See Doob [4], p. 81, equation (6.3).

Substitution of (2.1.17) into (2.1.18) yields

$$H_{a_n}[(n+1)\tau_o;z] = \exp(iza_n\frac{\tau_o}{T_o}) \sum_{a_{n-1}} H_{a_{n-1}}(n\tau_o;z)\Pr\{a_{n-1}|a_n\}. \qquad (2.1.19)$$

Since $a_n = \pm 1$, $a_{n-1} = \pm 1$, and referring back to (2.1.3), we can write (2.1.19) as

$$H_+[(n+1)\tau_o;z] = \exp(iz\frac{\tau_o}{T_o})[qH_-(n\tau_o;z) + pH_+(n\tau_o;z)] \qquad (2.1.20)$$

and

$$H_-[(n+1)\tau_o;z] = \exp(-iz\frac{\tau_o}{T_o})[pH_-(n\tau_o;z)+qH_+(n\tau_o;z)]. \qquad (2.1.21)$$

Going back to the definitions, it is easy to show

$$H_+(k\tau_o;z) = H_-^*(k\tau_o;z). \qquad (2.1.22)$$

Consequently the two unknown functions (2.1.20) and (2.1.21) appear to be complex conjugates of each other and therefore we need consider only one. Thus we may rewrite (2.1.20) as

$$H_+[(n+1)\tau_o;z] = \exp(iz\frac{\tau_o}{T_o})[qH_+^*(n\tau_o;z) + pH_+(n\tau_o;z)]. \qquad (2.1.23)$$

Let us now define the marginal characteristic function of the function $m(n\tau_o)$,

$$H(n\tau_o;z) = \tfrac{1}{2}[H_+^*(n\tau_o;z) + H_+(n\tau_o;z)], \qquad (2.1.24)$$

thus eliminating the condition on a_{n-1}.

Let us now consider the output $y(t_o)$. From (2.1.5) we can see that

$$m(T_o) = y(t_o)$$

from which it follows that, for $T_o = N\tau_o$,

$$G(z) = H(N\tau_o;z). \qquad (2.1.25)$$

Suppose, however, that the integration time T_o is not an integral number of pulse widths. For $T_o = N\tau_o + \delta$, $0 < \delta \leq \tau_o$, we have, by an argument similar to that used in deriving (2.1.23),

$$H_+(N\tau_o+\delta;z) = \exp(iz\frac{\delta}{T_o})[qH_+^*(N\tau_o;z) + pH_+(N\tau_o;z)] \qquad (2.1.26)$$

and

$$G(z) = H(N\tau_0+\delta\,;z) = \tfrac{1}{2}\left[H_+^*(N\tau_0+\delta\,;z) + H_+(N\tau_0+\delta\,;z)\right] . \qquad (2.1.27)$$

Therefore the characteristic function $G(z)$ of the output is obtained by solving the difference equation (2.1.23), and then by using (2.1.25) or (2.1.27); the initial condition,

$$H_+(0;z) = 1, \qquad (2.1.28)$$

is obtained by setting $\tau_1 = 0$ in (2.1.5).

2. Solution of the Difference Equation

There are several methods by which we can solve (2.1.23); however, we shall use a scheme which will yield a function $G(z)$ that is easily Fourier-transformable, so as to simplify the calculation of the distribution function from the characteristic function.

It can easily be shown that the solution is of the form,

$$H_+(n\tau_0;z) = \sum_{k=0}^{n-1} c_{n,k} \exp\left[i(n-2k)z\frac{\tau_0}{T_0}\right],\ n\geq 1, \qquad (2.2.1)$$

where the coefficients are real and are given by

$$c_{n,k} = \frac{1}{2\pi}\int_0^{2\pi} H_+(n\tau_0;z)\exp\left[-i(n-2k)z\frac{\tau_0}{T_0}\right] dz,\quad 0\leq k\leq n-1, \qquad (2.2.2)$$

Since $H_+(n\tau_0;0) = 1$, we have the property

$$\sum_{k=0}^{n-1} c_{n,k} = 1. \qquad (2.2.3)$$

Substitution of (2.2.1) into the difference equation (2.1.23) yields

$$\sum_{k=0}^{n} c_{n+1,k} \exp\left[i(n+1-2k)z\frac{\tau_o}{T_o}\right]$$

$$= q \sum_{j=0}^{n-1} c_{n,j} \exp\left[-i(n-1-2j)z\frac{\tau_o}{T_o}\right] \tag{2.2.4}$$

$$+ p \sum_{j=0}^{n-1} c_{n,j} \exp\left[i(n+1-2j)z\frac{\tau_o}{T_o}\right],$$

and the comparison of the coefficients of like terms yields

$$c_{n+1,0} = pc_{n,0}; \tag{2.2.5a}$$

$$c_{n+1,k} = qc_{n,n-k}+pc_{n,k}, \quad 1 \le k \le n-1; \tag{2.2.5b}$$

$$c_{n+1,n} = qc_{n,0}. \tag{2.2.5c}$$

From (2.2.1) and (2.2.3) we have the initial conditions

$$c_{1,0} = 1,$$

$$c_{1,k} = 0, \quad k \neq 1,$$

so that we can write immediately

$$\left.\begin{array}{l} c_{n+1,0} = p^n, \quad n \ge 0 \; ; \\ c_{n+1,n} = qp^{n-1}, \quad n \ge 1. \end{array}\right\} \tag{2.2.6}$$

To obtain the rest of the coefficients we must solve the partial difference equation (2.2.5b). We can solve this equation by induction, and we find, in general, that

$$c_{n+1,k} = p^{n-1}q + \sum_{j=1}^{[\frac{n}{2}]} \frac{(k-j+1)_{j-1}}{(j-1)!} \frac{(n-k-j+1)_j}{j!} p^{n-2j} q^{2j}$$

$$+ \sum_{j=1}^{[\frac{n-1}{2}]} \frac{(k-j)_j}{j!} \frac{(n-k-j+1)_j}{j!} p^{n-2j-1} q^{2j+1}, \tag{2.2.7}$$

$$1 \le k \le n-1, \quad n \ge 2,$$

where

$$(r)_j = r(r+1)(r+2)\ldots(r+j-1) \tag{2.2.8}$$

and where $[s]$ is the largest integer which is less than or equal to s. Through the use of suitable manipulations, (2.2.7) can be put into more compact form involving the hypergeometric function ${}_2F_1(a,b;c;x)$:

$$\begin{aligned} c_{n+1,k} &= (n-k)p^{n-2}q^2\ {}_2F_1(1-k,k-n+1;2;\frac{q^2}{p^2}) \\ &\quad + p^{n-1}q\,{}_2F_1(1-k,k-n;1;\frac{q^2}{p^2}). \end{aligned} \tag{2.2.9}$$

For $p=q=\frac{1}{2}$, this reduces to

$$\begin{aligned} c_{n+1,k} &= (\tfrac{1}{2})^n \left[(n-k)\,{}_2F_1(1-k,k-n+1;2;1) + {}_2F_1(1-k,k-n;1;1)\right] \\ &= (\tfrac{1}{2})^n \binom{n}{k}, \end{aligned}$$

where $\binom{n}{k}$ denotes a binomial coefficient.

We have now solved for the distribution function of the output at a sample point since, from (2.2.1), we can see that $R_+(y)$ will be a step function with jumps having the magnitudes given by the appropriate set of coefficients $c_{n,k}$.

3. The Distribution Function at an Arbitrary Point

Since we now have the distribution function at a sample point, we are ready to extend the solution to an arbitrary point.

Let t_o be a sample point, and t be an arbitrary point, such that $t_o < t \leq t_o + \mathcal{T}_o$.

Let us assume further that

$$x(t) = 1, \quad t_o < t \leq t_o + \mathcal{T}_o.$$

Then we have

$$y(t) = \begin{cases} y(t_o) + \dfrac{t-t_o}{T_o}, & y(t_o) + \dfrac{t-t_o}{T_o} < 1; \\ 1, & y(t_o) + \dfrac{t-t_o}{T_o} \geq 1. \end{cases} \tag{2.3.1}$$

Since $x(t')$ for $t_o - \tau_o < t' \leq t_o$ can still take on either a positive or negative value, we have, for a given value of $s = t-t_o$,

$$\begin{aligned} &\Pr\{y(t) \leq Y \mid x(t) = +1, s\} \\ &\quad = p \Pr\{y(t_o) \leq Y - \frac{s}{T_o} \mid x(t-\tau_o) = +1, s\} \\ &\quad + q \Pr\{y(t_o) \leq Y - \frac{s}{T_o} \mid x(t-\tau_o) = -1, s\}, \end{aligned} \tag{2.3.2}$$

The variable $s = t-t_o$ is independent of the input; then, if $P_s(s)$ is the distribution function of s, $0 < s \leq \tau_o$, we obtain

$$P_+(y) = \begin{cases} 0, & y \leq -1; \\ \displaystyle\int_0^{\tau_o} \left[pR_+(y-\frac{s}{T_o}) + qR_-(y-\frac{s}{T_o})\right] dP_s(s), & -1 < y \leq 1; \\ 1, & 1 < y, \end{cases} \tag{2.3.3}$$

as the conditional distribution function of the output at an arbitrary point t, given that $x(t) = +1$. If we assume that s is distributed uniformly over the interval $0 < s \leq \tau_o$, we obtain finally

$$P_+(y) = \begin{cases} 0, & y \leq -1; \\ \dfrac{1}{\tau_o}\displaystyle\int_0^{\tau_o} \left[pR_+(y-\frac{s}{T_o}) + qR_-(y-\frac{s}{T_o})\right] ds, & -1 < y \leq 1; \\ 1, & 1 < y. \end{cases} \tag{2.3.4}$$

In section 2 of this chapter, we saw that $R_+(y)$ and $R_-(y)$ are step functions. Therefore, the integral in (2.3.4) is relatively easy to evaluate, and, once we have solved for $P_+(y)$, we can find $P_-(y)$ and $P(y)$ through the use of (1.3.6) and (1.3.8).

CHAPTER III

THE RC LOW-PASS FILTER:

GENERAL REMARKS

In the following paragraphs we shall investigate some of the continuity and differentiability properties of the distribution function of the output of an RC low-pass filter for a random square-wave input. To aid in this investigation we shall derive an integral equation in terms of $Q_+(y)$ and $Q_-(y)$, and an integral establishing the relationship between $P_+(y)$ and $Q_-(y)$.

For practical purposes, since an axis-crossing interval of zero length would produce the same output as if those two axis-crossings had not occurred, we shall require $P_o(\tau)$ to be continuous at $\tau=0$.

1. Derivation of the Basic Equations

In order to facilitate the study of the continuity and differentiability properties of the distribution function of the output, we shall now derive an integral involving $Q_+(y)$ and $Q_-(y)$, and an integral which gives $P_+(y)$ in terms of $Q_-(y)$.

Let $t_o < t_1$ be two consecutive axis-crossings of the input, and assume that

$$x(t) = +1, \; t_o < t \leq t_1. \tag{3.1.1}$$

Let $y_o = y(_o)$ and $y_1 = y(t_1)$ be the values of the output at these points. Since the system is an RC low-pass filter, y_o and y_1 are related by

$$y_1 = 1-(1-y_o)e^{-\tau/T}, \tag{3.1.2}$$

where $\tau = t_1 - t_o$. Then, for a given value of τ, we have

$$\Pr\{y_1 \leq y | \tau\} = \Pr\{y_o \leq 1-(1-y)e^{\tau/T} | \tau\}. \tag{3.1.3}$$

If we bear in mind that (3.1.1) still holds, and integrate (3.1.3) over τ, we obtain the integral equation,

$$Q_+(y) = \int_0^{T \log \frac{2}{1-y}} Q_-\left[1-(1-y)e^{\tau/T}\right] dP_o(\tau), \tag{3.1.4}$$

where the lower limit is obtained by setting the argument of Q_- equal to y (its maximum value), and the upper limit by setting this argument equal to -1 (its minimum value). We can put (3.1.4) into better form by letting $\xi = 1-(1-y)e^{\tau/T}$ and integrating by parts. Noting that $P_o(0) = 0$ and $Q_-(-1) = 0$, we then obtain

$$Q_+(y) = \int_{-1}^{y} P_o(T \log \frac{1-\xi}{1-y}) dQ_-(\xi) \tag{3.1.5}$$

as the second form of the integral equation. From (1.3.6) and (1.3.8) we see that solution of the integral equation for $Q_+(y)$ or $Q_-(y)$ enables us to calculate $Q(y)$. Thus if (3.1.5) can be solved, we can obtain the distribution functions for the output at an axis-crossing of the input.

Now let $y'=y(t)$ be the output at an arbitrary point t, $t_o < t \leq t_1$, and assume that (3.1.1) is still satisfied. Let $\ell = t - t_o$ so that we have

$$y'=1-(1-y_o)e^{-\ell/T} \tag{3.1.6}$$

If we denote the distribution function of ℓ as $Q_o(\ell)$, we have[1]

$$Q_o(\ell) = \beta \int_0^{\ell} \left[1-P_o(\tau)\right] d\tau, \tag{3.1.7}$$

where β is the expected number of axis-crossings per unit time in the input, i.e.

$$\frac{1}{\beta} = \int_0^{\infty} \tau \, dP_o(\tau). \tag{3.1.8}$$

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1 See McFadden [16], p. 175.

In the manner identical to that used in the derivation of (3.1.5), we find

$$P_+(y) = \int_{-1}^{y} Q_o(T \log \frac{1-\xi}{1-y}) \, d\, Q_-(\xi). \tag{3.1.9}$$

Thus, once we have solved (3.1.5), we can find $P_+(y)$, from which we can calculate $P(y)$, the distribution function of the output at an arbitrary instant.

2. Continuity and Differentiability Properties of the Distribution Function of the Output

We shall now investigate some of the continuity and differentiability properties of the distribution function of the output.

First let us consider the distribution function at an axis-crossing of the input. Since $P_o(\tau) = 0$ for $\tau \leq 0$, we can rewrite (3.1.5) as

$$Q_+(y) = \int_{-1}^{1} P_o(T \log \frac{1-\xi}{1-y}) \, dQ_-(\xi). \tag{3.2.1}$$

We can now state the following:

Lemma 1. If $P_o(\tau)$ is continuous on $0 \leq \tau < \infty$, then $Q(y)$ is continuous on $-1 < y < 1$.

Proof: If $P_o(\tau)$ is continuous on $0 \leq \tau < \infty$, then $P_o(T \log \frac{1-\xi}{1-y})$ is a continuous function of y on the interval $-1 < y < 1$; moreover $P_o(\tau)$ is bounded by 1. Hence $Q_+(y)$ is continuous on $-1 < y < 1$, and by (1.3.6) and (1.3.8), $Q_-(y)$ and $Q(y)$ are continuous on the same interval.

We also have the following:

Lemma 2. If $P_o(\tau)$ is Lipschitz on $0 \leq \tau < \infty$ and $p_o(\tau)$ is its density function, then $Q(y)$ is absolutely continuous on $-1 < y < 1$ and, if $q_+(y)$ and $q_-(y)$ are the density functions corresponding to $Q_+(y)$ and $Q_-(y)$ respectively, then

$$q_+(y) = \frac{T}{1-y} \int_{-1}^{1} p_o(T \log \frac{1-\xi}{1-y}) \, q_-(\xi) \, d\xi \,, \; -1 < y < 1. \tag{3.2.2}$$

<u>Proof:</u> Let M be the Lipschitz constant, and let I_j: $\alpha_j < y < \beta_j$, $j-1,2,\ldots,N$, be non-overlapping subintervals of $-1<y<1$ such that

$$\delta = \sum_{j=1}^{N} (\beta_j - \alpha_j) > 0.$$

Then

$$\sum_{j=1}^{N} \left| Q_+(\beta_j) - Q_+(\alpha_j) \right| \leq \sum_{j=1}^{N} \int_{-1}^{1} \left| P_o(T \log \frac{1-\xi}{1-\beta_j}) - P_o(T \log \frac{1-\xi}{1-\alpha_j}) \right| dQ_-(\xi)$$

$$\leq MT \sum_{j=1}^{N} \int_{-1}^{1} \left| \log \frac{1-\xi}{1-\beta_j} - \log \frac{1-\xi}{1-\alpha_j} \right| dQ_-(\xi)$$

Using the mean value theorem, we can rewrite the above equation as

$$\sum_{j=1}^{N} \left| Q_+(\beta_j) - Q_+(\alpha_j) \right| \leq MT \sum_{j=1}^{N} \left| \frac{\beta_j - \alpha_j}{1-\beta_j + \theta(\beta_j - \alpha_j)} \right| , \quad 0<\theta<1,$$

$$< MT \sum_{j=1}^{N} \left| \frac{\beta_j - \alpha_j}{1-\beta_{max}} \right| = \frac{MT}{1-\beta_{max}} \delta \quad = \varepsilon ,$$

where $\beta_{max} = \max[\beta_1, \beta_2, \ldots, \beta_N]$. Therefore $Q_+(y)$ is absolutely continuous on $-1<y<1$, and by (1.3.6) and (1.3.8), $Q_-(y)$ and $Q(y)$ are also absolutely continuous on $-1<y<1$. Now we can write (3.2.1) as

$$Q_+(y) = \int_{-1}^{1} P_o(T \log \frac{1-\xi}{1-y}) q_-(\xi)\, d\xi .$$

Let $\{y_n\}$ be a sequence such that $y_n \to y$. Then using the mean value theorem we can write

$$\frac{Q_+(y_n)-Q_+(y)}{y_n-y} = \int_{-1}^{1} \frac{P_o(T\log\frac{1-\xi}{1-y_n})-P_o(T\log\frac{1-\xi}{1-y})}{y_n-y}\, q_-(\xi)d\xi$$

$$= \frac{T}{1-y-(1-\theta)(y_n-y)} \int_{-1}^{1} \frac{P_o(T\log\frac{1-\xi}{1-y_n})-P_o(T\log\frac{1-\xi}{1-y})}{T\left[\log\frac{1-\xi}{1-y_n} - \log\frac{1-\xi}{1-y}\right]}\, q_-(\xi)\, d\xi ,$$

$$0<\theta<1, \quad -1<y<1.$$

As $n \to \infty$ we have dominated convergence, and, in the limit, we obtain (3.2.2).

Now let us consider the distribution function at an arbitrary point. Equation (3.1.9) can be rewritten in the form

$$P_+(y) = \int_{-1}^{1} Q_o(T\log\frac{1-\xi}{1-y})\, dQ_-(\xi), \tag{3.2.3}$$

and we can state the following:

<u>Lemma 3.</u> $P(y)$ is absolutely continuous and, if $p_+(y)$ is the density function of $P_+(y)$ and $q_o(\ell)$ is the density function of $Q_o(\ell)$, then

$$p_+(y) = \frac{T}{1-y}\int_{-1}^{1} q_o(T\log\frac{1-\xi}{1-y})\, dQ_-(\xi), \quad -1<y<1. \tag{3.2.4}$$

<u>Proof:</u> From (3.1.7) we see that $Q_o(\ell)$ is Lipschitz, with Lipschitz constant equal to unity, on $0\leq \ell < \infty$. Then, following the same procedure used in the proof of Lemma 2, we can show that $P_+(y), P_-(y)$, and $P(y)$ are all absolutely continuous on $-1<y<1$, and (3.2.4) is valid. Moreover, from (3.1.7), we see that

$$q_o(\ell) = \beta\left[1-P_o(\ell)\right].$$

<u>3. The Special Case of a Coin-Toss Square Wave Input</u>

We shall now consider the case in which the input is a coin-toss square wave, and we shall investigate the properties of the distribution function of the output at a sample point. We note that, in this case, $P_o(\tau)$ is not continuous at every point in $0 \le \tau < \infty$, but has a denumerably infinite number of discontinuities.

Let us characterize the input in the same manner as in Chapter II, Section 1. Using this characterization in (1.1.1), we obtain for the output, after interchange of summation and integration,

$$y(t_o) = \sum_{k=0}^{\infty} a_k \int_{t_o-(k+1)\tau_o}^{t_o-k\tau_o} W(t_o-\xi)d\xi ,$$

where t_o is a sample point. Since $W(t) = \frac{1}{T}\exp(-\frac{t}{T})$, $t>0$, we find that the output at a sample point can be represented as

$$y(t_o) = (1-b)\sum_{k=0}^{\infty} a_k b^k, \tag{3.3.1}$$

where

$$b=e^{-\tau_o/T}. \tag{3.3.2}$$

We note that $0<b<1$, so that the sum in (3.3.1) is convergent. We can now make some rather general statements about the distribution function of $y(t_o)$ if $p=q=\frac{1}{2}$.

When $p=q=\frac{1}{2}$, we see that a_k becomes independent of a_j, $j\neq k$. It has been shown by Jessen and Wintner [9] that the distribution function of the random variable defined in (3.3.1), when the coefficients a_o, $a_1, \ldots$ are independent, has the property that it is either an absolutely continuous function or a singular function[2] on the interval $-1 \le y(t_o) \le 1$. Moreover, it has also been

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2 A singular function is a function which is continuous and not equal to a constant over its range of definition, but which has a zero derivative almost everywhere.

shown[3] that for $0 < b < \frac{1}{2}$ this distribution function is singular, for $b=(\frac{1}{2})^{1/k}$, $k=1,2,\ldots$, it is absolutely continuous, and for $\frac{1}{2} < b < 1$ it can have either property (never both).

If $p \neq q$, the a_k are no longer independent and the above statements do not apply. It will be shown in Chapter IV, however, that even if $p \neq q$, the distribution function of $y(t_0)$ is definitely not absolutely continuous for $0 < b < \frac{1}{2}$, but consists of a singular part (definitely) and a discontinuous part (possibly).

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[3] See Jessen and Wintner [9], and Kershner and Wintner [11].

CHAPTER IV

THE RC-LOW PASS FILTER WITH GENERALIZED COIN-TOSS SQUARE-WAVE INPUT

We shall present a method by which the distribution function of the output of an RC low-pass filter, with a generalized coin-toss square-wave input, can be constructed when $b < \frac{1}{2}$. This will be done by first constructing the set of values in $-1 \leq y \leq 1$ which the output at a sample point is not allowed to assume. Then, using the integral equation (3.1.4), we shall construct the distribution function.

A functional equation, involving the characteristic function of the output at a sample point, will be derived. From this equation we shall derive a recurrence relation for the moments of the output at a sample point.

Finally, we shall consider the problem of extending the distribution function at a sample point to an arbitrary point.

1. Construction of the Set of Unallowed Values of the Output at a Sample Point for $b < \frac{1}{2}$.

We shall now examine the particular values which the output at a sample point can assume when $b < \frac{1}{2}$, i.e., when the time constant T is less than $\tau_o(\log 2)^{-1}$. This information will then enable us to construct the set of values which the output at a sample point cannot assume.

First of all, we note that, because of the symmetry involved, if y_* is an allowed (or unallowed) value of the output y_o, then $-y_*$ is also an allowed (or unallowed) value of y_o. Therefore we shall focus most of our attention on the interval $0 \leq y_o \leq 1$, which includes all the possible values of the output which are positive.

From Chapter III we have that y_o, the output at a sample point, can be written as

$$y_o = (1-b) \sum_{k=0}^{\infty} a_k b^k. \tag{4.1.1}$$

Since $b < \frac{1}{2}$, we see that the closed interval, in which the end points, are defined by $a_k = 1$, $k = 0,1,2,\ldots$ and $a_o = -1$, $k = 1,2,\ldots$, contains all the allowed positive values of y_o; i.e.,

$$(1-b)(1-b-b^2-\ldots) \leq y_o \leq (1-b)(1+b+b^2+\ldots),$$

or

$$I_1^+: \quad 1-2b \leq y_o \leq 1. \tag{4.1.2}$$

Because of the above mentioned symmetric property, the allowed negative values of y_o are contained in the reflection of I_1^+ about the point $y_o = 0$. This reflection operator will be denoted by

$$I^- = \rho I^+: \quad -\alpha_1 \leq y_o \leq -\alpha_2, \tag{4.1.3}$$

where

$$I^+: \quad \alpha_2 \leq y_o \leq \alpha_1.$$

Thus

$$I_1^- = \rho I_1^+: \quad -1 \leq y_o \leq -1+2b. \tag{4.1.4}$$

The interval I_1^+ can then be broken down into two closed intervals which contain all the allowed positive values of y_o. The first of these intervals has end-points defined by $a_o = 1$, $a_k = -1$, $k = 1,2,\ldots$, and $a_1 = -1$, $a_k = 1$, $k = 0,2,3,\ldots$; and in the second they are defined by $a_o = a_1 = 1$, $a_k = -1$, $k = 2,3,\ldots$, and $a_k = 1$, $k = 0,1,2,\ldots$; i.e.,

$$1-2b \leq y_o \leq (1-b)(1-b+b^2+b^3+\ldots);$$
$$(1-b)(1+b-b^2-b^3-\ldots) \leq y_o \leq 1,$$

or

$$\left.\begin{array}{l} I_{21}^+: \quad 1-2b \leq y_o \leq 1-2b+2b^2; \\ I_{22}^+: \quad 1-2b^2 \leq y_o \leq 1, \end{array}\right\} \tag{4.1.5}$$

and the corresponding allowed negative values are contained in

$$I_{21}^{-} = \rho I_{21}^{+};\ I_{22}^{-} = \rho I_{22}^{+}. \tag{4.1.6}$$

Each of the intervals in (4.1.5) can be broken down further into two closed intervals containing the allowed positive values of y_o:

$$1-2b \leq y_o \leq (1-b)(1-b-b^2+b^3+b^4+\ldots);$$

$$(1-b)(1-b+b^2-b^3-b^4-\ldots) \leq y_o \leq 1-2b+2b^2;$$

$$1-2b^2 \leq y_o \leq (1-b)(1+b-b^2+b^3+b^4+\ldots);$$

$$(1-b)(1+b+b^2-b^3-b^4-\ldots) \leq y_o \leq 1,$$

or

$$\left.\begin{array}{ll} I_{31}^{+}: & 1-2b \leq y_o \leq 1-2b+2b^3\ ; \\ I_{32}^{+}: & 1-2b+2b^2-2b^3 \leq y_o \leq 1-2b+2b^2\ ; \\ I_{33}^{+}: & 1-2b^2 \leq y_o \leq 1-2b^2+2b^3; \\ I_{34}^{+}: & 1-2b^3 \leq y_o \leq 1\ ; \end{array}\right\} \tag{4.1.7}$$

and the allowed negative values are given by

$$I_{3k}^{-} = \rho I_{3k}^{+},\ k = 1,2,3,4. \tag{4.1.8}$$

This process can be continued, but we shall, at this point, focus our attention on the intervals of unallowed values of y_o.

The first interval of unallowed values of y_o occurs as the open interval contained between I_1^+ and I_1^-:

$$J_1:\ -1+2b < y_o < 1-2b. \tag{4.1.9}$$

Between I_{21}^+ and I_{21}^- we obtain another open interval containing unallowed positive values of y_o:

$$J_{21}^{+}:\ 1-2b+2b^2 < y_o < 1-2b^2. \tag{4.1.10}$$

Using the reflection operator, we obtain the corresponding open interval containing unallowed negative values:

$$J_{21}^{-} = \rho J_{21}^{+}:\ -1+2b^2 < y_o < -1+2b-2b^2. \tag{4.1.11}$$

Similarly, we obtain two more such intervals, one between I_{31}^{+} and I_{32}^{+}, and the other between I_{33}^{+} and I_{34}^{+}:

$$\left.\begin{array}{ll} J_{31}^{+}: & 1-2b+2b^3 < y_o < 1-2b+2b^2-2b^3; \\ J_{32}^{+}: & 1-2b^2+2b^3 < y_o < 1-2b^3, \end{array}\right\} \qquad (4.1.12)$$

with corresponding intervals for unallowed negative values. This process can be continued also.

We can now calculate the lengths of these intervals of unallowed values of y_o:

$$|J_1| = 2(1-2b);$$

$$|J_{21}^{+}| = |J_{21}^{-}| = 2b(1-2b);$$

$$|J_{31}^{+}| = |J_{31}^{-}| = |J_{32}^{+}| = |J_{32}^{-}| = 2b^2(1-2b),$$

and in general we find that

$$|J_{nk}^{+}| = |J_{nk}^{-}| = 2b^{n-1}(1-2b), \quad k=1,2,\ldots,2^{n-2}. \qquad (4.1.13)$$

Summation of the lengths of all the intervals of unallowed values of y_o yields

$$|J_1| + \sum_{n=2}^{\infty} \sum_{k=1}^{2^{n-2}} |J_{nk}^{+}| + |J_{nk}^{-}|$$

$$= 2(1-2b) + \sum_{n=2}^{\infty} \sum_{k=1}^{2^{n-2}} 4b^{n-1}(1-2b)$$

$$= 2. \qquad (4.1.14)$$

Therefore the Lebesgue measure of the set of unallowed values of y_o is 2, and since all the possible values of y_o are contained in $-1 \leq y_o \leq 1$, we find that the set of allowed values of y_o is of measure zero. Moreover, since we can make a one-to-one mapping of the values of the expression in (4.1.1) onto the binary expansion of numbers in $0 \leq y \leq 1$, the set of allowed values of the output at a sample point is of cardinality c.

From the above argument we can now present a simpler construction of the sets of allowed and unallowed values of the output.

We note, first of all, that J_1 is the middle open interval of length $2(1-2b)$ in $-1 \leq y_0 \leq 1$. Let us remove this interval, leaving two closed intervals. Then J^+_{21} and J^-_{21} are the middle open intervals of length $2b(1-2b)$ in these remaining closed intervals. Let us remove these open intervals, leaving four closed intervals. Then, from each of these closed intervals, the middle open interval of length $2b^2(1-2b)$ is removed. Continuing this process ad infinitum yields the desired sets. It can now be seen that this process is similar to the construction of the Cantor ternary set, and for $b = \frac{1}{3}$, the set of allowed values of the output at a sample point becomes identical to the Cantor ternary set constructed on the interval $-1 \leq y \leq 1$. This construction is illustrated in Fig. IV-1.

2. Construction of the Distribution Function of the Output at an Axis Crossing of the Input when $b < \frac{1}{2}$.

Noting that the results of the previous section apply to an axis crossing[1] of the input, we can now construct the distribution function of the output at an axis crossing of the input under the condition that $b < \frac{1}{2}$.

For convenience we repeat the relations necessary for the construction of this distribution function. From Chapter I the symmetric relation is

$$Q_-(y) = 1-Q_+(-y), \tag{4.2.1}$$

where y denotes the output at an axis crossing of the input. From Chapter III we have the integral equation,

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[1] An axis crossing of the input can occur only at a sample point.

$$Q_+(y) = \int_0^{T \log \frac{2}{1-y}} Q_-\left[1-(1-y)e^{\tau/T}\right] dP_o(\tau), \tag{4.2.2}$$

where, for the generalized coin-toss square wave, we have from Chapter I that

$$P_o(\tau) = \begin{cases} 0 \quad , & \tau \le 0 \ ; \\ 1-p^n, & n\tau_o < \tau \le (n+1)\tau_o, \quad n=0,1,2,\dots \ . \end{cases} \tag{4.2.3}$$

Substitution of (4.2.3) into (4.2.2) yields

$$Q_+(y) = \begin{cases} 0 \quad , & y \le 1-2b; \\ q \sum_{n=1}^{N} p^{n-1} Q_-(1- \frac{1-y}{b^n}), & 1-2b^N < y \le 1-2b^{N+1}, \end{cases} \tag{4.2.4}$$

where $N = 1,2,\dots$. Let

$$w_n(y) = 1 - \frac{1-y}{b^n}, \tag{4.2.5}$$

so that (4.2.4) can be rewritten as

$$Q_+(y) = \begin{cases} 0 \quad , & y \le 1-2b; \\ q \sum_{n=1}^{N} p^{n-1} Q_-[w_n(y)], & 1-2b^N < y \le 1-2b^{N+1}, \ N \ge 1. \end{cases} \tag{4.2.6}$$

Noting that the distribution function must be constant over intervals containing only unallowed values of y, we can start the construction.

From (4.2.6) and (4.2.1) we have

$$\left.\begin{aligned} Q_+(y) &= 0 \\ Q_-(y) &= 1 \end{aligned}\right\} \quad y \in J_1. \tag{4.2.7}$$

Then (4.2.6) with N = 1 gives

$$Q_+(y) = qQ_-[w_1(y)], \qquad y \in J_{21}^+,$$

and from (4.2.5) we find that $w_1(y) \in J_1$ for $y \in J_{21}^+$. Therefore, from (4.2.7) we find that

$$\left.\begin{aligned} Q_+(y) &= q \\ Q_-(y) &= 1 \end{aligned}\right\} \quad y \in J_{21}^+, \tag{4.2.8}$$

and using (4.2.1) we have

$$\left.\begin{aligned} Q_+(y) &= 0 \\ Q_-(y) &= 1-q \end{aligned}\right\} \quad y \in J_{21}^- \tag{4.2.9}$$

Further, we find that

$$\left.\begin{aligned} Q_+(y) &= qQ_-[w_1(y)] \\ w_1(y) &\in J_{21}^- \end{aligned}\right\} \quad y \in J_{31}^+,$$

and

$$\left.\begin{aligned} Q_+(y) &= qQ_-[w_1(y)] + pq\, Q\, [w_2(y)] \\ w_1(y) &\in J_{21}^+ \\ w_1(y) &\in J_1 \end{aligned}\right\} \quad y \in J_{32}^+$$

which then gives

$$\left.\begin{aligned} Q_+(y) &= q(1-q) \\ Q_-(y) &= 1 \end{aligned}\right\} \quad y \in J_{31}^+, \tag{4.2.10}$$

and

$$\left.\begin{aligned} Q_+(y) &= q(1+p) \\ Q_-(y) &= 1 \end{aligned}\right\} \quad y \in J_{32}^+, \tag{4.2.11}$$

and, correspondingly, we obtain from (4.2.1)

$$\left.\begin{aligned} Q_+(y) &= 0 \\ Q_-(y) &= 1 \end{aligned}\right\} \quad y \in J_{31}^-, \tag{4.2.12}$$

and

$$\left.\begin{aligned} Q_+(y) &= 0 \\ Q_-(y) &= 1-q(1+p) \end{aligned}\right\} \quad y \in J_{32}^-. \tag{4.2.13}$$

In this manner the distribution functions $Q_+(y)$ and $Q_-(y)$ can be constructed to any degree of accuracy.

Using the relation

$$Q(y) = \tfrac{1}{2}[Q_+(y)+Q_-(y)] \tag{4.2.14}$$

from Chapter I, we can calculate the values of the marginal distribution of y over the intervals of constancy. We obtain from the above results

$$Q(y) = \tfrac{1}{2}, \qquad y \in J_1; \tag{4.2.15a}$$

$$\left.\begin{aligned} Q(y) &= \tfrac{1}{2}(1+q), & y &\in J_{21}^{+}; \\ &= \tfrac{1}{2}(1-q), & y &\in J_{21}^{-}; \end{aligned}\right\} \tag{4.2.15b}$$

$$\left.\begin{aligned} Q(y) &= \tfrac{1}{2}(1+q-q^2), & y &\in J_{31}^{+}; \\ &= \tfrac{1}{2}(1+2q-q^2), & y &\in J_{32}^{+}; \\ &= \tfrac{1}{2}(1-q+q^2), & y &\in J_{31}^{-}; \\ &= \tfrac{1}{2}(1-2q+q^2), & y &\in J_{32}^{-}. \end{aligned}\right\} \tag{4.2.15c}$$

For illustrative purposes consider the case where $p=q=\frac{1}{2}$. The above equations become

$$\begin{aligned} Q(y) &= \tfrac{1}{2}, & y &\in J_1; \\ &= \tfrac{3}{4}, & y &\in J_{21}^{+}; \\ &= \tfrac{1}{4}, & y &\in J_{21}^{-}; \\ &= \tfrac{5}{8}, & y &\in J_{31}^{+}; \\ &= \tfrac{7}{8}, & y &\in J_{32}^{+}; \\ &= \tfrac{3}{8}, & y &\in J_{31}^{-}; \\ &= \tfrac{1}{8}, & y &\in J_{32}^{-}. \end{aligned}$$

From these values we can see the general trend for this case. In Fig. IV-2 the distribution functions for $p=q=\frac{1}{2}$ and $p=\frac{1}{4}$, $q=\frac{3}{4}$, both for $b=\frac{1}{3}$, are shown.

Now we can see that, for the generalized coin-toss square wave with $b < \frac{1}{2}$, the distribution function of the output at an axis crossing of the input is not absolutely continuous. For $p=q=\frac{1}{2}$ it is easy to show that this distribution function is continuous and therefore must be singular; this fact agrees with the statement in Chapter III. We can also see that the derivative of the distribution function is zero almost everywhere, so that the function does all its rising on the set of allowed values of y, which is of measure zero. Since this set was shown to be of cardinality c, the distribution function for $p \neq q$ must consist of a singular part and possibly a discontinuous part.

For the special case $b=\frac{1}{2}$, we see, from the previous section that each interval of unallowed values of the output reduces to an open, degenerate interval. Therefore the set of allowed values of the output is the interval $-1 \leq y \leq 1$. Moreover, if $p=q=\frac{1}{2}$ and we choose a sequence b_n such that $b_n \to \frac{1}{2}$, $b_n < \frac{1}{2}$, the above procedure, in the limit, yields the uniform distribution defined over the interval $-1 \leq y \leq 1$. That this is the correct solution can be verified by direct substitution into the integral equation, noting that $Q_+(y)$ and $Q_-(y)$ are the uniform distributions defined over the intervals $0 \leq y \leq 1$ and $-1 \leq y \leq 0$, respectively. This special case appears in Jessen and Wintner [9].

3. The Characteristic Function of the Output at a Sample Point.

We shall now derive a functional equation involving the characteristic functions of the output at a sample point. From this equation we shall then derive a recurrence relation for the moments of the output at a sample point.

In this problem we shall use a method similar to the one used in Chapter II. Let $t_o > t_1 > t_2 > \dots$ be the sample points of the input. Define the following functional:

$$m(t,s) = \int_s^t W(t-\xi)x(\xi)d\xi\,, \quad s<t. \qquad (4.3.1)$$

Since $W(t) = \frac{1}{T}\exp(-\frac{t}{T})$, $t>0$, we obtain, using the characterization of $x(t)$ from Chapter II,

$$m(t_o,s) = bm(t_1,s)+a_o(1-b). \qquad (4.3.2)$$

Now with $s=t_o-(n+1)\tau_o$, let us define

$$H_{a_1,a_n}(t_1,s;z)=E\{\exp[izm(t_1,s)]\ |a_1,a_n\}\Pr\{a_1|a_n\}, \tag{4.3.3}$$

and

$$H_{a_o,a_n}(t_o,s;z)=E\{\exp[izm(t_o,s)]\,|\,a_o,a_n\}\Pr\{a_o|a_n\}. \tag{4.3.4}$$

Using (4.3.2), equation (4.3.4) becomes

$$H_{a_o,a_n}(t_o,s;z)$$
$$= \exp[iza_o(1-b)]\ E\{\exp[izbm(t_1,s)]\,|a_o,a_n\}\Pr\{a_o|a_n\},$$

which, by virtue of the argument used in Chapter II, can then be written as

$$H_{a_o,a_n}(t_o,s;z)$$
$$= \exp[iza_o(1-b)]\sum_{a_1} E\{\exp[izbm(t_1,s)]\ |a_o,a_1,a_n\}\Pr\{a_o|a_1\}\Pr\{a_1|a_n\}, \tag{4.3.5}$$

where the summation is taken over the two values of a_1. By an argument similar to that in Chapter II, we can omit the condition on a_o in (4.3.5). Then comparing (4.3.5) with (4.3.3), we see that (4.3.5) can be written as

$$H_{a_o,a_n}(t_o,s;z)$$
$$= \exp[iza_o(1-b)]\sum_{a_1} H_{a_1,a_n}(t_1,s;bz)\ \Pr\{a_o|a_1\}\ . \tag{4.3.6}$$

From (4.3.1) we see that

$$\lim_{s\to-\infty} m(t,s) = y(t).$$

Now, remembering that $s=t_o-(n+1)\tau_o$, define[2]

$$\lim_{s\to-\infty} H_{a_j,a_n}(t_j,s;z) = H_{a_j}(t_j;z). \tag{4.3.7}$$

---

[2] Note that $s\to-\infty$ implies $n\to\infty$.

Since the input and the output are stationary, $H_{a_j}(t_j;z)$ is independent of t_j. Therefore,

$$G_+(z) = H_+(t_j;z)\ ;\ G_-(z) = H_-(t_j;z). \tag{4.3.8}$$

If we put $a_o = +1$, (4.3.6) becomes

$$G_+(z) = \exp[iz(1-b)][qG_-(bz)+pG_+(bz)]. \tag{4.3.9}$$

Solution of (4.3.9), together with the conjugate relation (1.3.7), then yields the characteristic function of the output at a sample point.

Although equations (4.3.9) and (1.3.7) are not easily solved, we can obtain from them a recurrence relation for the moments of the output at a sample point.

Put

$$\left.\begin{aligned} G_+(z) &= U(z)+iV(z); \\ G_-(z) &= U(z)-iV(z), \end{aligned}\right\} \tag{4.3.10}$$

where $U(z)$ is an even function and $V(z)$ is an odd function when z is real. Substitution of (4.3.10) into (4.3.9) and separation of real and imaginary parts yields

$$\left.\begin{aligned} U(z) &= U(bz)\cos(1-b)z-(p-q)V(bz)\sin(1-b)z; \\ V(z) &= U(bz)\sin(1-b)z+(p-q)V(bz)\sin(1-b)z. \end{aligned}\right\} \tag{4.3.11}$$

The marginal characteristic function at a sample point is then

$$G(z) = \tfrac{1}{2}[G_+(z)+G_-(z)] = U(z). \tag{4.3.12}$$

If $p = q = \frac{1}{2}$, the first equation in (4.3.11) reduces to

$$U(z) = U(bz)\cos(1-b)z, \tag{4.3.13}$$

the solution of which is

$$U(z) = G(z) = \prod_{j=0}^{\infty} \cos[b^j(1-b)z]. \tag{4.3.14}$$

This solution has been obtained previously by Jessen and Wintner [8] and Wonham [20]. For $b=\frac{1}{2}$ equation (4.3.14) yields

$$G(z) = \prod_{j=0}^{\infty} \cos(\tfrac{1}{2})^{j+1} z = \frac{\sin z}{z}$$

which is the Fourier transform of the uniform distribution on $-1 \leq y \leq 1$. This special case was discussed at the end of section 2.

Now for arbitrary p and q let

$$\left.\begin{aligned} U(z) &= \sum_{j=0}^{\infty} \frac{m_{2j}}{(2j)!} (iz)^{2j} ; \\ V(z) &= -i \sum_{j=0}^{\infty} \frac{m_{2j+1}}{(2j+1)!} (iz)^{2j+1}, \end{aligned}\right\} \qquad (4.3.15)$$

where m_k is the kth moment of the output at a sample point, given that the input was positive just previous to this sample point, i.e.,

$$m_k = \int_{-1}^{1} y^k \, dR_+(y). \qquad (4.3.16)$$

It is easy to show that m_k exists for all k. We can also represent these moments as

$$\left.\begin{aligned} m_{2j} &= (-1)^j \left.\frac{d^{2j}U(z)}{dz^{2j}}\right|_{z=0} ; \\ m_{2j+1} &= (-1)^j \left.\frac{d^{2j+1}V(z)}{dz^{2j+1}}\right|_{z=0} \end{aligned}\right\} \qquad (4.3.17)$$

in addition to (4.3.17) we have that

$$\left.\begin{aligned} \left.\frac{d^{2j+1}U(z)}{dz^{2j+1}}\right|_{z=0} &= 0 ; \\ \left.\frac{d^{2j}V(z)}{dz^{2j}}\right|_{z=0} &= 0 . \end{aligned}\right\} \qquad (4.3.18)$$

Application of (4.3.17) and (4.3.18) to (4.3.11) yields two recurrence relations, one for the even moments and one for the odd moments:

$$m_{2n} = (1-b)^{2n}\left[\sum_{j=0}^{n}\binom{2n}{2j}\left(\frac{b}{1-b}\right)^{2j}m_{2j}+(p-q)\sum_{j=0}^{n-1}\binom{2n}{2j+1}\left(\frac{b}{1-b}\right)^{2j+1}m_{2j+1}\right], \tag{4.3.19}$$

and

$$m_{2n+1} = (1-b)^{2n+1}\left[\sum_{j=0}^{n}\binom{2n+1}{2j}\left(\frac{b}{1-b}\right)^{2j}m_{2j}+(p-q)\sum_{j=0}^{n}\binom{2n+1}{2j+1}\left(\frac{b}{1-b}\right)^{2j+1}m_{2j+1}\right]. \tag{4.3.20}$$

Equations (4.3.19) and (4.3.20) can then be combined to give

$$m_k = (1-b)^k\left[\sum_{j=0}^{[\frac{k}{2}]}\binom{k}{2j}\left(\frac{b}{1-b}\right)^{2j}m_{2j}\right. \tag{4.3.21}$$

$$\left. +(p-q)\sum_{j=0}^{[\frac{k-1}{2}]}\binom{k}{2j+1}\left(\frac{b}{1-b}\right)^{2j+1}m_{2j+1}\right],\quad k=0,1,2,\ldots$$

For $p=q=\frac{1}{2}$, this recurrence relation reduces to

$$m_k = (1-b)^k\sum_{j=0}^{[\frac{k}{2}]}\binom{n}{2j}\left(\frac{b}{1-b}\right)^{2j}m_{2j}\quad ; \tag{4.3.22}$$

This expression has been obtained previously by Wonham [21].

From (4.3.16), (1.3.6), and (1.3.8) we see that the $2k^{th}$ moment of the output at a sample point, with no conditions on the input, is the same as m_{2k}, but the $(2k+1)^{st}$ moment of the output is zero, i.e.,

$$m_{2k} = \int_{-1}^{1} y^{2k}\, dR(y),\quad k=0,1,2,\ldots, \tag{4.3.23}$$

and

$$\int_{-1}^{1} y^{2k+1}\, dR(y) = 0, \quad k=0,1,2,\ldots \quad (4.3.24)$$

It can be seen from (4.3.21) that the calculation of these moments will be extremely tedious, if not impossible, for arbitrary b and arbitrary p and q. However, if b, p, and q are specified, any number of moments can be computed by machine. Then, using the even moments, a polynomial approximation of G(z) can be made, which, in turn gives an approximation for R(y).

4. The Distribution Function of the Output at an Arbitrary Point

Let us assume that the distribution function of the output at a sample point, or at an axis crossing of the input, is known. We shall now extend this distribution function to an arbitrary point.

We should note here that, in general, the distribution function at a sample point is different from the distribution at an axis crossing of the input. If the distribution at an axis crossing of the input is known, then (3.1.9) can be used to find the distribution function of the output at an arbitrary point. On the other hand, if the distribution function at a sample point is known, equation (3.1.9) is not applicable and a different method must be used to make the extension to an arbitrary point. We shall use a method similar to the one used in Chapter II.

Let t_o be a sample point, and let t be an arbitrary point such that $t_o < t \leq t_o + \tau_o$. Let us assume that

$$x(t) = +1, \quad t_o < t \leq t_o + \tau_o.$$

Then the outputs at t and at t_o are related by

$$y(t) = 1 - [1 - y(t_o)] \exp\left[-\frac{t-t_o}{T}\right] \quad (4.4.1)$$

Then, for a given value of $s = t-t_o$, we can write

$$\Pr\{y(t)\leq Y \mid x(t) = +1,s\}$$
$$= p\Pr\{y(t_o)\leq 1-(1-Y)e^{s/T} \mid x(t-\tau_o) = +1,s\}$$
$$+q\Pr\{y(t_o)\leq 1-(1-Y)e^{s/T} \mid x(t-\tau_o) = -1,s\}\ , \quad -1 < Y \leq 1.$$

Assuming that s is uniformly distributed over the interval $0 < s \leq \tau_o$, we obtain finally

$$P_+(y) = \begin{cases} 0\ , & y \leq -1; \\ \dfrac{1}{\tau_o}\displaystyle\int_0^{\tau_o} \left[pR_+(1-(1-y)e^{s/T}) \right. & \\ \qquad \left. +qR_-(1-(1-y)e^{s/T})\right] ds, & -1 < y \leq 1; \\ 1\ , & 1 < y. \end{cases} \tag{4.4.2}$$

The distribution function $P_-(y)$ can be calculated in a similar manner, or through the use of (1.3.6), after which (1.3.8) can be used to find the marginal distribution function at an arbitrary point.

Therefore, we see that, if $Q_+(y)$ or $R_+(y)$ is known, we can calculate the distribution function at an arbitrary point.

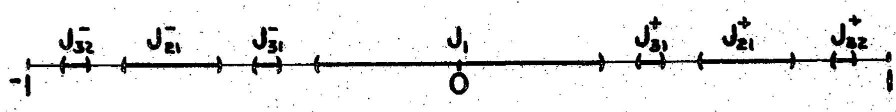

Fig. IV-1

Construction of the Set of Unallowed Values of the Output

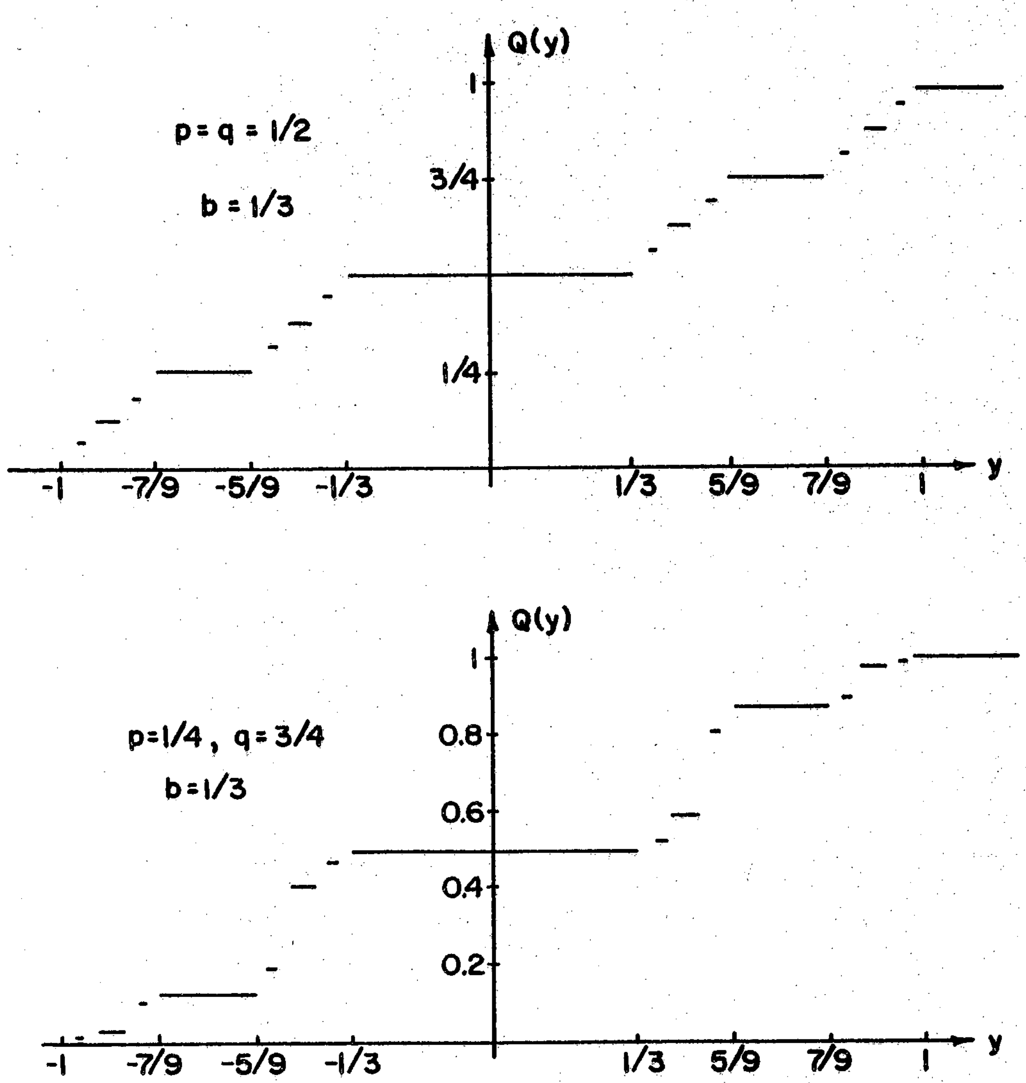

Fig. IV-2

The Distribution Function of the Output at a Sample Point for $p=q=1/2$ and $p=1/4$, $q=3/4$ when $b=1/3$.

CHAPTER V

THE RC LOW-PASS FILTER WITH GENERAL SQUARE-WAVE INPUT

An integral equation, analogous to the equation in Chapter III, which involves the characteristic functions of the output of an RC low-pass filter at an axis-crossing of the square-wave input, will be derived. No assumptions will be made concerning the input, other than the restriction that the axis-crossing intervals of the input be independent.

From this integral equation, we shall obtain a recurrence relation for the moments of the output at an axis-crossing of the input.

Finally, it will be shown, for certain types of inputs, that the integral equation can be reduced to a differential equation.

1. Derivation of the Integral Equation

We shall now derive an integral equation involving the characteristic functions of the output at an axis-crossing of the input.

Let $t_1 > t_2 > t_3 > \dots$ be the successive axis-crossings of the input. Let

$$x(t) = a_1, \; t_2 < t \leq t_1,$$

where $a = \pm 1$ with equal probability, then

$$x(t) = a_2 = -a_1, \; t_3 < t \leq t_2. \tag{5.1.1}$$

The output at t_1 is related to the output at t_2 by

$$y(t_1) = e^{-\tau/T}\, y(t_2) + a_1(1-e^{-\tau/T}), \tag{5.1.2}$$

where $\tau = t_1 - t_2$. Now we define

$$H_{a_1}(t_1;z) = E\left\{\exp\left[izy(t_1)\right] \mid a_1\right\} \tag{5.1.3}$$

and

$$H_{a_2}(t_2;z) = E\left\{\exp\left[izy(t_2)\right] \mid a_2\right\}. \tag{5.1.4}$$

Because of (5.1.1) we see that (5.1.4) can be rewritten as

$$H_{a_2}(t_2;z) = E\left\{\exp\left[izy(t_2)\right] \middle| a_1\right\}. \tag{5.1.5}$$

Substituting (5.1.2) into (5.1.3) and comparing the resulting equation with (5.1.5), we obtain

$$H_{a_2}(t_1;r;z) = \exp\left[iza_1(1-r)\right] H_{a_2}(t_2;rz), \tag{5.1.6}$$

where[1]

$$r = e^{-\tau/T}. \tag{5.1.7}$$

We note that (5.1.6) depends on r, and to remove this dependence we average over r. From (5.1.7) we have

$$\Pr\left\{r \le r_o\right\} = \Pr\left\{e^{-\tau/T} \le r_o\right\} = \Pr\left\{\tau \le -T \log r\right\}$$

and, denoting the distribution function of r by $P_1(r)$, we obtain

$$P_1(r) = \begin{cases} 0, & r \le 0; \\ 1-P_o(-T\log r), & 0 < r \le 1; \\ 1, & 1 < r. \end{cases} \tag{5.1.8}$$

Now performing the averaging operation on (5.1.6), we find

$$H_{a_1}(t_1;z) = \int_0^1 H_{a_1}(t_1;r;z)\,dP_1(r)$$

$$= \int_0^1 \exp\left[iza_1(1-r)\right] H_{a_2}(t_2;rz)\,dP_1(r). \tag{5.1.0}$$

Because of stationarity, we can eliminate the time dependence and write

$$F_+(z) = H_+(t_j;z);$$
$$F_-(z) = H_-(t_j;z).$$

---

1 This r is a random variable; whereas b, as defined in (3.3.2), is a known constant.

If we put $a_1 = +1$, so that $a_2 = -1$, equation (5.1.9) becomes finally

$$F_+(z) = \int_0^1 e^{iz(1-r)} F_-(rz)\, dP_1(r). \tag{5.1.10}$$

Therefore a solution of the integral equation (5.1.10), together with the conjugate relation (1.3.7), will give us the characteristic function of the output at an axis-crossing of the input.

2. A Recurrence Relation for the Moments of the Output at an Axis-Crossing of the Input

Using the integral equation (5.1.10), we shall now derive a recurrence relation for the moments of the output at an axis-crossing of the input.

We note first of all that the moments,

$$m_k = \int_{-1}^{1} y^k dQ_+(y), \quad k=0,1,2,\ldots, \tag{5.2.1}$$

exist and are finite. Therefore we can write

$$\left.\begin{aligned} F_+(z) &= \sum_{k=0}^{\infty} m_k \frac{(iz)^k}{k!} \;; \\ F_-(z) &= \sum_{k=0}^{\infty} m_k \frac{(-iz)^k}{k!} . \end{aligned}\right\} \tag{5.2.2}$$

Then m_k is given by

$$m_k = (-i)^k \left.\frac{d^k F_+(z)}{dz^k}\right|_{z=0}, \quad k=0,1,2,\ldots, \tag{5.2.3}$$

or

$$m_k = i^k \left.\frac{d^k F_-(z)}{dz^k}\right|_{z=0}, \quad k=0,1,2,\ldots, \tag{5.2.4}$$

Differentiating (5.1.10) n times, setting z=0, and applying (5.2.3) and (5.2.4), we obtain two recurrence relations for the moments, one for the even moments, and one for the odd moments:

$$m_{2n} = \sum_{j=0}^{2n} \binom{2n}{j} m_j E\left\{ r^j (1-r)^{2n-j} \right\}, \qquad n=0,1,2,\ldots, \qquad (5.2.5)$$

and

$$m_{2n+1} = \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} m_j E\left\{ r^j (1-r)^{2n-j+1} \right\}, \ n=0,1,2,\ldots, \qquad (5.2.6)$$

where

$$E\left\{ r^j (1-4)^{k-j} \right\} = \int_0^1 r^j (1-r)^{k-j}\, dP_1(r). \qquad (5.2.7)$$

Finally, these two recurrence relations can be combined to give a single relation:

$$m_k = \sum_{j=0}^{k} (-1)^{kj} \binom{k}{j} m_j E\left\{ r^j (1-r)^{k-j} \right\}, \quad k=0,1,2,\ldots \ . \qquad (5.2.8)$$

Again we see that, as in Chapter IV, the calculation of the moments using (5.2.8) will be very tedious, but using machine computation, we can calculate as many moments as we choose. Then since

$$\int_{-1}^{1} y^k dQ(y) = \begin{cases} m_k, & k \text{ even;} \\ 0\ , & k \text{ odd,} \end{cases}$$

we can use the even moments to make a polynomial approximation to $F(z)$.

3. Reduction of the Integral Equation to a Differential Equation for Special Types of Inputs

We shall show that, under certain conditions, it is possible to reduce the integral equation (5.1.10) to a differential equation.

In equation (5.1.10) let us assume that $P_1(r)$ is differentiable everywhere, and let

$$p_1(r) = \frac{dP_1(r)}{dr} \text{ .} \tag{5.3.1}$$

Then, if we make the variable transformation $\xi = rz$, the integral equation becomes

$$F_+(z) = \frac{e^{iz}}{z} \int_0^z e^{-i\xi} \; F_-(\xi) \; p_1(\tfrac{\xi}{z}) \, d\xi \text{ .} \tag{5.3.2}$$

From (5.1.8) we see that, when $P_1(r)$ is differentiable, $P_o(\tau)$ is differentiable, and if

$$p_o(\tau) = \frac{dP_o(\tau)}{d\tau} \text{ ,} \tag{5.3.3}$$

we find

$$p_1(r) = \frac{T}{r} \, p_o(-T \log r), \quad 0 < r \leq 1. \tag{5.3.4}$$

As the first example, consider the case where

$$p_o(\;\;) = \begin{cases} 0 \text{ ,} & \tau \leq 0; \\ \frac{1}{T} \sum_{k=1}^{N} A_k \exp(-\alpha_k \frac{\tau}{T}), & \tau > 0; \end{cases} \tag{5.3.5}$$

where we must have

$$\int_0^{\infty} p_o(\tau) \, d\tau = 1.$$

Then from (5.3.4) we have

$$p_1(r) = \sum_{k=1}^{N} A_k r^{\alpha_k - 1} \tag{5.3.6}$$

and the integral equation becomes

$$F_+(z) = e^{iz} \sum_{k=1}^{N} A_k z^{-\alpha_k} \int_0^z e^{-i\xi} F_-(\xi) \xi^{\alpha_k - 1} \, d\xi \text{ .} \tag{5.3.7}$$

We note that, if we multiply both sides of (5.3.7) by $e^{-iz}z^{\alpha_j}$, $1 \le j \le N$, and differentiate with respect to z, we can eliminate the integral term for k=j. Repeating this process until all the integral terms disappear will yield an Nth-order differential equation involving derivatives of $F_+(z)$ and $F_-(z)$. The solution of this differential equation, together with the conjugate relation (1.3.7), will give us the characteristic function of the output at an axis-crossing of this input.

As a special case, let N=1. This is a random telegraph-wave input, and the resulting differential equation is

$$zF_+'(z)+(\alpha_1-iz)F_+(z)-\alpha_1F_-(z) = 0, \tag{5.3.8}$$

where the prime denotes differentiation with respect to z. Letting

$$\left.\begin{aligned} F_+(z) &= U(z)+iV(z); \\ F_-(z) &= U(z)-iV(z), \end{aligned}\right\} \tag{5.3.9}$$

where U(z) and V(z) are real for real z, we obtain two equations, one for the real part, and one for the imaginary part. Elimination of V(z) yields the equation

$$zU''(z)+2\alpha_1U'(z)+zU(z) = 0,$$

the solution of which is

$$U(z) = F(z) = \Gamma(\alpha_1+\tfrac{1}{2})\left(\frac{2}{z}\right)^{\alpha_1-\frac{1}{2}} J_{\alpha_1-\frac{1}{2}}(z), \tag{5.3.10}$$

where $J_\lambda(z)$ is the Bessel function of the first kind, of order λ. This solution has been obtained previously by McFadden [15], and by Wonham and Fuller [19].

For N=2, the differential equation is

$$\begin{aligned} z^2F_+''(z)+(c_1z-2iz^2)F_+'(z)+(c_2-z^2-ic_1z)F_+(z) \\ = c_3zF_-'(z)+(c_2-ic_3z)F_-(z), \end{aligned} \tag{5.3.11}$$

where

$$c_1 = \alpha_1 + \alpha_2 + 1;$$
$$c_2 = \alpha_1 \alpha_2; \qquad (5.3.12)$$
$$c_3 = A_1 + A_2.$$

Equation (5.3.11) does not lend itself to solution as readily as in the case for N=1, and, at present, no solution is known. However, application of (5.2.3) and (5.2.4) will yield a three-term recurrence relation for the moments of the output at an axis-crossing of the input. It should be noted that the same recurrence relation can be obtained from (5.2.8).

As the second example, consider the case where

$$p_0(\tau) = a^2 \frac{\tau}{T} \exp(-\frac{a\tau}{T}) \quad , \qquad \tau \geq 0. \qquad (5.3.13)$$

For this case we have

$$p_1(r) = a^2 r^{a-1} \log r, \qquad (5.3.14)$$

and the integral equation becomes

$$F_+(z) = a^2 \frac{e^{iz}}{z^a} \int_0^z e^{-i\xi} F_-(\xi) \xi^{a-1} \log \frac{z}{\xi} \, d\xi . \qquad (5.3.15)$$

If we multiply both sides of (5.3.15) by z^a and differentiate, we obtain

$$z^a F_+'(z) + (\frac{a}{z} - i) z^a F_+(z) = a^2 \frac{e^{iz}}{z} \int_0^z e^{i\xi} F_-(\xi) \xi^{a-1} d\xi .$$

Note that we have eliminated the log term. Multiplying both sides by z, and differentiating a second time, we find

$$z^2 F_+''(z) + (2a+1 - 2iz) z F_+'(z) + a^2 - z^2 - i(2a+1) z \, F_+(z) = a^2 F_-(z). \qquad (5.3.16)$$

Using the conjugate relation, we can obtain a fourth-order differential eua- tion involving only one unknown function. However, no solution of this equa- tion is known, but as in the case of equation (5.3.11), we can obtain a three-term recurrence relation for the moments of the output at a sample point of the input.

Therefore we see that, for the cases considered in this section, we can reduce the integral equation (5.1.10) to a differential equation.

CHAPTER VI

EXTENSIONS TO CERTAIN SECOND-ORDER SYSTEMS WITH COIN-TOSS SQUARE-WAVE INPUTS

In this chapter conditions will be derived under which the distribution function of the output of a general filter at a sample point becomes identical to that of an RC low-pass filter with the same coin-toss input and suitable choice of T.

For a second-order system which is tuned to the clocking rate of the input we shall calculate the distribution function of the output at an arbitrary point, assuming that the corresponding problem for the RC los-pass filter has been solved.

1. The Output Random Variable at a Sample Point for Certain Higher-Order Systems

We shall now derive the restrictions under which the distribution function of the output of a general filter at a sample point becomes identical to that of an RC low-pass filter, when the input to both filters is a coin-toss square wave.

Let W(t) be the weighting function of the general filter, and assume that it satisfies the conditions,

$$\left.\begin{aligned} &W(t) = 0, \qquad\qquad t < 0; \\ &\int_0^{\infty} W(t)dt = 1. \end{aligned}\right\} \tag{6.1.1}$$

The output at a sample point t_o is then given by

$$y(t_o) = \int_{-\infty}^{t_o} W(t_o - \xi)x(\xi)d\xi . \tag{6.1.2}$$

If we use the characterization of the coin-toss square wave from Chapter II, the output becomes

$$y(t_o) = \sum_{j=0}^{\infty} a_j \phi_j , \tag{6.1.3}$$

where

$$\phi_j = \int_{t_o-(j+1)\tau_o}^{t_o-j\tau_o} W(t-\xi)d\xi = \int_{j\tau_o}^{(j+1)\tau_o} W(\xi)\, d\xi . \tag{6.1.4}$$

Now let us restrict the weighting function to satisfy the condition

$$\phi_{j+1} = b\, \phi_j, \quad j = 0,1,2,\dots , \qquad 0 < b < 1. \tag{6.1.5}$$

Essentially, this restriction means that the area under the curve representing the weighting function between $(j+1)\tau_o$ and $(j+2)\tau_o$ is equal to b times that area between $j\tau_o$ and $(j+1)\tau_o$ for all $j=0,1,2,\dots$. Moreover, if (6.1.5) is satisfied for a given input with an elementary pulse width τ_o, then a similar expression will be satisfied by the system only for inputs with elementary pulse widths which are integral multiples of τ_o. Then, if (6.1.5) is satisfied, the output at a sample point is

$$y(t_o) = \phi_o \sum_{j=0}^{\infty} a_j b^j. \tag{6.1.6}$$

Furthermore, from (6.1.1) and (6.1.5) we find that

$$\int_0^{\infty} W(\xi)d\xi = \sum_{j=0}^{\infty} \phi_j = \frac{\phi_o}{1-b} = 1$$

or

$$\phi_o = 1-b, \tag{6.1.7}$$

so that the output becomes

$$y(t_o) = (1-b) \sum_{j=0}^{\infty} a_j b^j. \qquad (6.1.8)$$

We note that this expression is identical to equation (3.3.1), which represents the output of an RC low-pass filter at a sample point for a coin-toss square-wave input. Therefore we can conclude, if (6.1.5) is satisfied, that the distribution function of the output of the general filter at a sample point is the same as the distribution function for the output of the RC filter at a sample point, providing, of course, that the values of b are the same for the two cases. We must bear in mind, however, that at an arbitrary point the distribution functions for the outputs of the general filter and of the RC filter will, in general, be quite different; it is only at a sample point that they are the same.

2. The Distribution Function at an Arbitrary Point for Certain Second-Order Systems

In the following we shall derive the conditions which a second-order system, with a coin-toss square-wave input, must satisfy in order that the distribution function of the output at an arbitrary point can be calculated, using the results of the preceding section.

Let us assume that equation (6.1.5) is satisfied and that the distribution function at a sample point is known. Since we are considering a second-order system, we must first investigate the derivative of the output at a sample point. Let the weighting function of the system be W(t), and let us assume that the system is characterized by the differential equation,

$$A\ddot{y}(t) + B\dot{y}(t)+Cy(t)=x(t) \qquad (6.2.1)$$

where the dots denote differentiation with respect to time.

The output at time t is given by (1.1.1). Differentiation with respect to t yields

$$\dot{y}(t) = \int_{-\infty}^{t} \frac{dW(t-\xi)}{d(t-\xi)}\, x(\xi)\, d\xi + W(0)x(t) \qquad (6.2.2)$$

where we have assumed that $W(t)$ is differentiable on $0<t<\infty$. Equation (6.2.2) can be put into the form,

$$\dot{y}(t) = -\int_{\xi=-\infty}^{\xi=t} x(\xi)\, dW(t-\xi) + W(0)x(t) \qquad (6.2.3)$$

Using the same characterization for $x(t)$ as above, we find, at the sample point $t=t_o$,

$$\dot{y}(t_o) = W(0)x(t_o) - \sum_{j=0}^{\infty} a_j \int_{\xi=t_o-(j+1)\tau_o}^{\xi=t_o-j\tau_o} dW(t_o-\xi)$$

$$= W(0)x(t\ ) - \sum_{j=0}^{\infty} a_j \left\{ W(j\tau_o) - W\left[(j+1)\tau_o\right] \right\}.$$

Now if $W(t)$ satisfies the condition[1]

$$W\left[(j+1)\tau_o\right] = bW(j\tau_o), \qquad j=0,1,2,\ldots, \qquad 0<b<1, \qquad (6.2.4)$$

then the derivative of the output at a sample point becomes

$$y(t_o) = a_o W(0) - (1-b)W(0) \sum_{j=0}^{\infty} a_j b^j,$$

and, comparing this result with (6.1.6), we find

$$y(t_o) = W(0)\left[a_o - y(t_o)\right]. \qquad (6.2.5)$$

Therefore, if (6.2.4) is satisfied, then the first derivative of the output

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1. For a second-order system (6.2.4) implies (6.1.5). Equation (6.2.4) is the assumption that the system is tuned to the clocking rate of the input.

at a sample point is directly related to the output at the same sample point. On the other hand, if (6.2.4) were not satisfied, we should have to know the joint distribution function for $y(t_o)$ and $\dot{y}(t_o)$ in order to calculate the distribution function at an arbitrary point. This added difficulty would arise because then the output at an arbitrary point would depend on both the output and its first derivative at the preceding sample point.

Now let us find an expression for the output at an arbitrary point. In order for the weighting function of a second-order system to satisfy (6.2.4) it must be of the form,

$$W(t) = \begin{cases} 0, & t \leq 0; \\ e^{-\alpha t}\left[K_1 \cos \frac{2\pi nt}{\tau_o} + K_2 \sin \frac{2\pi nt}{\tau_o}\right], & t > 0, \end{cases} \tag{6.2.6}$$

where K_1 and K_2 are such that (6.1.1) is satisfied, and $n=1,2,3,\ldots$. We note, with this weighting function, that equation (6.2.4) and therefore (6.1.5) is satisfied.

For simplicity let $t_o=0$, and put $y(t_o)=y_o$ and $\dot{y}(t_o)=\dot{y}_o$. Then (6.2.5) becomes

$$\dot{y}_o = K_1(a_o-y_o). \tag{6.2.7}$$

The output at time t, $0 < t \leq \tau_o$, is then given by

$$y(t) = e^{-\alpha t}\left[A_1 \cos \frac{2\pi nt}{\tau_o} + A_2 \sin \frac{2\pi nt}{\tau_o}\right] + \frac{a_{-1}}{C},$$

where $a_{-1} = x(t)$, $0 < t \leq \tau_o$, C is the constant from (6.2.1), and where the constants,

$$\left.\begin{aligned} A_1 &= A_1(y_o,\dot{y}_o) \quad , \\ A_2 &= A_2(y_o,\dot{y}_o) \quad , \end{aligned}\right\} \tag{6.2.8}$$

are obtained in the standard fashion. From (6.2.7) we see that (6.2.8) can be rewritten as

$$A_1 = A_1' \ (y_o; a_o, a_{-1});$$

$$A_2 = A_2' \ (y_o; a_o, a_{-1}).$$

The output at time t can then be put into the form,

$$y(t) = y_o f(t; a_o, a_{-1}) + g(t; a_o, a_{-1}), \tag{6.2.9}$$

where

$$\left.\begin{aligned} f(t;a_o,a_{-1}) &= e^{-\alpha t}\left[\cos\frac{2\pi nt}{\tau_o} + \frac{(\alpha - K_1)\tau_o}{2\pi n}\sin\frac{2\pi nt}{\tau_o}\right]; \\ g(t;a_o,a_{-1}) &= \frac{a_{-1}}{C} - e^{-\alpha t}\left[\frac{a_{-1}}{C}\cos\frac{2\pi nt}{\tau_o}\right. \\ &\qquad \left. + \frac{\tau_o(\alpha a_{-1} - K_1 a_o C)}{2\pi nC}\sin\frac{2\pi nt}{\tau_o}\right]. \end{aligned}\right\} \tag{6.2.10}$$

Then for a given value of t we have

$$\begin{aligned} &\Pr\{y \le Y \mid a_{-1} = +1, t\} \\ &\quad = p \Pr\left\{y_o \le \frac{Y - g(t;a_o,a_{-1})}{f(t;a_o,a_{-1})} \;\middle|\; a_o = +1,\ a_{-1} = +1, t\right\} \\ &\quad + q \Pr\left\{y_o \le \frac{Y - g(t;a_o,a_{-1})}{f(t;a_o,a_{-1})} \;\middle|\; a_o = -1,\ a_{-1} = +1, t\right\}. \end{aligned}$$

Assuming that t is uniformly distributed over the interval $0 < t \le \tau_o$, we obtain finally

$$\begin{aligned} P_+(y) &= \frac{p}{\tau_o}\int_0^{\tau_o} R_+\left[\frac{y - g(t;a_o=+1,a_{-1}=+1)}{f(t;a_o=+1,a_{-1}=+1)}\right]dt \\ &\quad + \frac{q}{\tau_o}\int_0^{\tau_o} R_-\left[\frac{y - g(t;a_o=-1,a_{-1}=+1)}{f(t;a_o=-1,a_{-1}=+1)}\right]dt \quad . \end{aligned} \tag{6.2.11}$$

Therefore we see that for a tuned second-order system as described above, we can calculate the distribution function of the output at an arbitrary point if, at a sample point, we know the distribution function of the output of the corresponding RC low-pass filter with the same input. And we see also, for the particular cases in which the above restrictions are met, that we have essentially reduced a second-order system to an equivalent first-order system. In making the extension from a sample point to an arbitrary point, the distinction between the corresponding second- and first-order systems is that we must use (6.2.11) instead of (4.4.2). In addition, since we have not lost the symmetry present in the output of a first-order system, equations (1.3.6) and (1.3.8) are still valid.

Finally, noting that a numerical integration may be necessary in (6.2.11), we may still conclude that this integration is considerably simpler than the corresponding double integration which would be required if it were not for the reduction in order. Furthermore, the required joint distribution would be much more elusive than our $R_+(y)$ and $R_-(y)$, but the reduction in order has made it unnecessary.

CHAPTER VII

CONCLUSIONS

In this final chapter we shall outline some of the problems which need further study.

1. The Finite-Time Integrator

Although we have solved the problem of the distribution function of the output of a finite-time integrator with a generalized coin-toss square-wave input, very little is known concerning this type of filter with a general square-wave input. The only other inputs for which solutions exist are the (Poisson) random telegraphic wave and the alternate-Poisson squave wave[1] (McFadden [14]).

In considering the coin-toss square wave with $p \neq q$, we have effectively introduced a simple Markov dependence between the value of the input at time t and its value of $t-\tau_0$. By defining a higher-order Markov dependence between the values of the input at t and at $t-\tau_0$, $t-2\tau_0, \ldots$, $t-N\tau_0$, we could approximate more general types of square waves; this approach would involve a step-function approximation to the distribution function of the lengths of axis-crossing intervals. Whether or not this type of approximation would yield any useful results is a problem for future study, but, in itself, this general class of coin-toss square waves is worthy of investigation.

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1. The axis crossings of an alternate-Poisson square wave occur at alternate points in a Poisson-distributed sequence of points.

2. The RC Filter

It was mentioned in Chapter III that at a sample point, the distribution function of the output of an RC low-pass filter with a coin-toss square-wave input is either absolutely continuous or singular when $p=q=\frac{1}{2}$. Whether or not this statement is valid for $p \neq q$ is a question that should be investigated. In particular, for $b < \frac{1}{2}$ and $p \neq q$, we should determine whether the resulting distribution function, which we are able to construct, has a discontinuous part or is purely singular. Another point for future study is the determination of the conditions under which a distribution function $P_0(\tau)$, having a discontinuous part more general than that of a coin-toss square-wave, implies continuity in the distribution function of the output at a sample point.

Still another problem which needs investigation is the determination of the output distribution when the input is the more general type of coin-toss square-wave mentioned in section 1 of this Chapter. Again, one possible approach would be to use general coin-toss square waves to approximate more general types of square-waves.

It was pointed out in Chapters IV and V, using the recurrence relations (5.2.8) and (4.3.21), that polynomial approximations to the characteristic function of the output at a sample point, or an axis crossing, can be made. These approximations should be investigated to determine if they yeild useful results; this investigation would include the study of the Fourier transforms of these polynomial approximations to determine whether or not they lead to acceptable approximations for the corresponding distribution functions.

3. General Filters

It was shown in Chapter VI, when certain restriction are placed on the weighting function of a filter with a coin-toss square-wave input, that we can essentially reduce this system to a first-order system as far as its output distribution function at a sample point is concerned. And, in the second-order case with still further restrictions, we were able to obtain an expression for the distribution function of the output at an arbitrary point.

Except for these special cases, there is very little known about the distribution function of the output of a general filter for a random square-wave input. Consequently, even though we have restricted our inputs to be square waves, there exists an extremely large class of problems which has remained essentially untouched. Entirely new methods may be necessary for these problems.

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