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J.E. Gibson

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PURDUE UNIVERSITY
SCHOOL OF ELECTRICAL ENGINEERING

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Control and Information Systems Laboratory

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PREFACE

This report was prepared by Purdue University, School of Electrical Engineering, Prof. J. E. Gibson acting as Principal Investigator, under USAF Contract No. AF 29(600)-1933. This contract is administered under the direction of the Guidance and Control Division, Air Force Missile Development Center, Holloman Air Force Base, New Mexico by Mr. J. H. Gengelbach, the initiator of the study.

FOREWORD

This is Volume III of a three volume final report for Air Force Research Project Number AF 29(600)-1933. Volume I, issued October 1960, is titled SPECIFICATION AND DATA PRESENTATION IN LINEAR CONTROL SYSTEMS. Volume II is a continuation of Volume I and deals with Indices of Performance for Control Systems, Time Variable Parameter Systems and Specifications for Sampled Data Systems. Volume II is being printed along with this volume.

The reader is directed to the FOREWORD of Volume I for the general approach of Purdue to the specifications of control systems. It was pointed out there that the present volume would be more introductory and tutorial than the others because of the nature of the material. There is no satisfactory treatment in English at the present time of the Engineering applications of the Second Method of Liapunov. This fact is widely recognized, and a number of authors are rushing to meet this deficiency. For the present, however, the field is virgin.

The only direct attempt to specify control systems by means of the second method comes in the discussion of the Aizerman index of performance which utilizes the concept of the V function. The discussion of Aizerman's original work is included in Volume II. Further work on this approach has been completed at Purdue separate from this project, and a paper is shortly to appear concerning it.

Although the Second Method applies directly to Nonlinear Systems, this report is not to be viewed as an adequate statement of the state of the art in the Specifications of Nonlinear Automatic Control Systems. It is background, tutorial reading on an important tool in the analysis and synthesis of Nonlinear Systems. Such background material is not necessary, for example, on the Describing Function or Phase Plane Analysis since they have been widely treated in English. Much of the material included here was available only in Russian when the work started and some is the fruit of original research. Our further work in the Specification of Nonlinear Automatic Control Systems will lean heavily on this volume.

At the present time the Purdue group is drawing up an interim report on the State of the Art of Specifications for Nonlinear Systems. Industry will be invited to comment on this work so that, if further work is done, the final Specifications will reflect a wide spectrum of industrial thinking, rather than simply that of a small academic group. This general area is, of course, most difficult, and it appears impossible to the definitive at this time. It should be recognized, however, that the possibility exists even today of designing and building nonlinear systems that are lighter, simpler, cheaper and more reliable than the linear systems they are to replace. Thus it seems imperative that, as the state of the art advances, these advances be included in Air Force control system specifications. It will be in the best interests of the Air Force to include the possibility of Nonlinear Systems in their procurement specifications for aerospace systems as soon as such trustworthy and inclusive specifications can be developed.

Since Volume I on Linear Systems was delivered, the conclusions have been extracted and a paper containing them presented at the Winter General Meeting of the AIEE with the cognizance of AFMDC. The paper will appear in the Transactions of AIEE and thus will afford a wide circulation of the concepts even though supplies of the original report have been exhausted.

ABSTRACT

This report investigates the stability of autonomous closed-loop control systems containing nonlinear elements. An n-th order nonlinear autonomous system is described by a set of n first order differential equations of the type

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots x_n) \qquad i = 1, 2, \ldots n.$$

Liapunov's second (direct) method is used in the stability analysis of such systems. This method enables one to prove that a system is stable (or unstable) if a function

$$V = V(x_1, x_2, \ldots x_n)$$

can be found which, together with its time derivative, satisfies the requirements of Liapunov's stability (or instability) theorems. At the present time there are no generally applicable straightforward procedures available for constructing these Liapunov's functions. Several Liapunov's functions, applicable to systems described in the canonic form of differential equations, have been reported in the literature. In this report it is shown that any autonomous closed-loop system containing a single nonlinear element can be described by canonic differential equations.

The stability criteria derived from the Liapunov's functions for canonic systems give sufficient and not necessary conditions for stability. It is known that these criteria reject many systems which are actually stable.

The reasons why stable systems are sometimes rejected by these simplified stability criteria are investigated in the report. It is found that a closed-loop system will always be rejected by these simplified stability criteria if the root locus of the transfer function G(s),

representing the linear portion of the system, is not confined to the left-half of the s-plane for all positive values of the loop gain.

A pole-shifting technique and a zero-shifting technique, extending the applicability of the simplified stability criteria to systems that are stable for sufficiently high and/or sufficiently low values of the loop gain, are proposed in this report. New simplified stability criteria have been developed which incorporate the changes in the canonic form of differential equations caused by the application of the zero-shifting technique.

Other methods of constructing Liapunov's functions for nonlinear control systems are presented in Chapter III. These include the work of Pliss, Aizerman and Krasovski. Numerous other procedures, which have been reported in literature, apply to only very special cases of automatic control systems. No attempt has been made to account for all of these special cases and the presentation of methods of constructing Liapunov's functions is limited to only those which are more generally applicable.

A pseudo-canonic transformation has been developed which enables one to find stability criteria of canonic systems without the use of complex variables.

The results of this research indicate that the second method of Liapunov is a very powerful tool of exact stability analysis of nonlinear systems. Additional research, especially in the direction of the methods of construction of Liapunov's functions, will not only yield new analysis and synthesis procedures but also will aid in arriving at a set of meaningful performance specifications for nonlinear control systems.

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CHAPTER I

INTRODUCTION

The concept of stability of linear automatic control systems is not only very useful in the analysis and qualitative evaluation of these systems but also has yielded several very useful synthesis procedures, such as the Nyquist Diagram, root-locus, etc. The conventional methods of analysis that are applicable to nonlinear systems, such as the describing function, phase space analysis, etc., are more complicated, less generally applicable, and cannot always be used as synthesis procedures. The basic difficulty of the application of such methods is due to the fact that they represent an attempt to find the solutions of the nonlinear differential equations describing the system. In autonomous linear systems one may prove that a system is stable (or unstable) without the need to find the response of the system, i.e., without the need to solve the differential equations describing the system. It would be very desirable to extend the applicability of such methods of linear system stability analysis to closed-loop systems that contain one or more nonlinear elements.

The importance of such an extension cannot be overemphasized, since any control system that can be considered linear in its normal mode of operation will inevitably become nonlinear for either sufficiently large or sufficiently small values of its response or the initial disturbances. Unfortunately, the concepts of linear system stability cannot be extended to nonlinear systems without considerable modification in both the definition and the meaning of system stability.

A linear system is defined as stable (see e.g., Bower and Schultheiss [1], p. 104) if and only if its output in response to every bounded input remains bounded.

This stability definition has a very precise meaning in linear systems. If a linear system is stable, it automatically meets all of the following requirements:

a) its driven response is bounded for all bounded driving functions;

b) its disturbed response (i.e., its response due to initial disturbances in the absence of driving functions) approaches an equilibrium state, at which the response and all of its time derivatives are zero, asymptotically with time $t \rightarrow \infty$;

c) stability is independent of the magnitude of either the initial disturbances or the continuously acting driving functions.

In nonlinear systems stability may or may not imply that all or any of the above requirements are satisfied in the entire phase space of system response variable (i.e., in the space of the system response and its first n-1 time derivatives, where n is the order of the system). It is possible, for example, to have a bounded disturbed response and unbounded driven response, or stability may depend upon the magnitudes of initial disturbance and/or continuously acting driving function. The response of a system which is stable in some region A of the phase space, containing the origin, may either remain bounded or become unbounded outside this region.

To differentiate between these possible modes of nonlinear system behavior it becomes necessary to define different types of system stability. This is done in Section 1.1 of this Chapter.

Once a suitable definition of stability, applicable to both linear and nonlinear systems, has been agreed upon, it becomes necessary to find methods which can be used to prove stability or instability of actual closed-loop systems. The methods of stability analysis used in this report are based upon the theory of physical system stability developed at the turn of the century by the Russian mathematician A. M. Liapunov [2].

Liapunov divides all the methods of solution of the stability problem into two groups ([3], p. 13). The first approach consists of consideration of the disturbed response, i.e., of finding general or special solutions of the corresponding differential equation. These solutions are usually found in the form of series (finite or infinite). Liapunov refers to the entire group of all such approaches as the first method. Hence, there is not a single procedure for attacking nonlinear problems that could be referred to as Liapunov's first method, but rather Liapunov refers to all approaches that attempt to find the solution of the differential equations describing the system as the first method. Thus, for example, the Krylov and Bogoliubov transform (describing function) technique would fall under Liapunov's first method.

The second method is, in Liapunov's terminology, the sum of all the techniques and approaches whereby the system stability (or instability) is established by considering some special functions of the response variable and its time derivatives. From the characteristics of these functions, together with the system differential equation, conclusions can be drawn about system stability. Since the second method deals with procedures which enable one to decide upon system stability directly from the system differential equation and some arbitrary functions without finding the solution of the differential equation, it is sometimes referred to as Liapunov's direct method.

The difficulties in the application of Liapunov's method of stability analysis to practical control systems are due to the fact that it is necessary to construct a certain function of the system variables which satisfies the requirements of Liapunov's theorems of stability or instability. These functions are not unique, and an infinite number of such functions may exist for a single system. However, no general methods of constructing such functions are known and in most cases it is very difficult to find a function satisfying the requirements of Liapunov's stability (or instability) theorems. The object of this report is to find methods of constructing Liapunov's functions for autonomous closed-loop systems which contain a single nonlinear element.

Several methods of constructing Liapunov's functions for various types of physical systems have been reported by Lur'e [4], Letov [5], Yakubovich [6], Aizerman [7], Krasovsky [8], Chetaev [9], Barbashin [10] and Hahn [11]. A disadvantage common to all these methods is that they are applicable only to either low order or to special types of physical systems. Lur'e [4] and Letov [5] proposed several Liapunov's functions (i.e., functions that satisfy the requirements of Liapunov's stability or instability theorems) for two special groups of control systems that can be described mathematically by the so-called canonic forms of system differential equations. They have shown that two special groups of minor-loop systems, referred in the Russian literature as the "direct control" and the "indirect control" systems can be transformed into one of the two canonic forms. The transformation of system differential equations into the first canonic form is generalized in Chapter II of this report. It is shown that any closed-loop system containing only one nonlinear element can be transformed into the first canonic form. The formulae are developed for this transformation, enabling one to use Liapunov's functions applicable to the first canonic form of system differential equations.

The transformation of some actual control systems into the second canonic form is presented in Section 2.3. A critical evaluation of this transformation reveals that it is applicable only to a very small number of closed-loop systems.

A summary of the simplified stability criteria based on the first canonic form of system differential equations is presented in Section 3.1. This summary includes the latest simplified stability criteria reported in the current periodicals and some simplifications that result from the generalization of the first canonic transformation developed in Section 2.2 of this report.

It is shown in Section 3.2 that a plot of the root-locus may be used to predict which systems will be rejected by these simplified stability criteria. The application of the root-locus concept reveals that these simplified stability criteria select as stable only those systems that are stable for all positive values of the open-loop gain and reject all systems that may be actually stable for intermediate values of gain, but are unstable for sufficiently low and/or sufficiently high values of gain. To avoid this difficulty, a pole-shifting technique and a zero-shifting technique are proposed in Chapter III. A modification of the first canonic form of system differential equations is proposed for use in connection with the zero-shifting technique. Simplified stability criteria based upon this modified canonic form of differential equations are developed in Section 3.4.

An alternate approach for achieving the same results as those obtained by the first canonic transformation without the need to introduce complex variables is presented in Chapter III. Liapunov's functions, to be used with this new canonic transformation, are also developed.

While the theory underlying the methods of stability analysis presented in the report is applicable to both time varying and time invariant nonlinear control systems, the methods of stability analysis developed in this report are applicable directly (i.e., without modifications) only to autonomous nonlinear systems, i.e., to systems that can be described mathematically by one or a set of differential equations, the coefficients of which do not vary with time. The analysis of time varying parameter nonlinear systems falls outside the scope of this report.

1.1 Definitions of Stability

An autonomous physical system may be described mathematically by a set of simultaneous first-order differential equations of the form

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots x_n) \qquad i = 1, 2, \ldots n . \qquad (1.1)$$

Such a set of differential equations corresponds to a linear system, if the functions X_i are linear. If the functions X_i are nonlinear, the system is said to be nonlinear.

If the set of differential equations represents a physically realizable system, the functions X_i must be defined in some fixed region G of the space of the variables $x_1, x_2, \ldots x_n$. This space will be referred to as the state space of the variables x_i.

The equilibrium states of the system (also referred to in the literature as singularities or singular points) are given by the real roots of the equations

$$X_i(x_1, x_2, \ldots x_n) = 0 \qquad i = 1, 2, \ldots n . \qquad (1.2)$$

These so-called null solutions of Eq. 1.2

$$x_i = \bar{x}_i \qquad i = 1, 2, \ldots n \qquad (1.3)$$

describe the statics of the control system. One of the important questions of the theory of automatic control is the question whether the equilibrium states (1.3) represent physically realizable operating conditions of the system. If the system is brought sufficiently close to an equilibrium state $\bar{x}_i$ and remains close to this state in the absence of external disturbances, then the equilibrium state is stable. If the system response moves away from an equilibrium state without the action of any external disturbances, then the equilibrium state is unstable. In the case of an unstable equilibrium state, the system response may either stay within some region G' of the state space, containing the singularity, or it may increase without bounds.

In physically realizable, linear, autonomous systems there is only one equilibrium state, and that is at the origin of the normalized coordinate system of the state space. If this equilibrium state is stable, the entire state space represents a stable region of the system response, and vice versa. Either of these two statements may or may not be true in a nonlinear control system. A nonlinear control system may have both stable and unstable equilibrium states (singularities) and also both stable and unstable regions of response. Consequently, stability in nonlinear systems is a local concept (Kalman [12], p. 5). Stability makes sense only when associated with some region of the state space containing a singularity. For the purpose of a stability investigation, it is more convenient to normalize the differential equations describing the system, i.e., to place the singularity, at which stability is investigated, at the origin of the state space. This is accomplished by the change in variable

$$y_i = x_i - \bar{x}_i \qquad i = 1, 2, \ldots n. \qquad (1.4)$$

The number of normal forms of the system differential equations

$$\frac{dy_i}{dt} = Y_i(y_1, y_2, \ldots y_n) \qquad i = 1, 2, \ldots n \qquad (1.5)$$

is equal to the number of equilibrium states (singularities) of the undisturbed system. Equation (1.4) represents a linear transformation which translates the origin of the coordinate system to an equilibrium state of the control system. The null solutions of equation (1.5)

$$\bar{y}_i = 0 \qquad i = 1, 2, \ldots n \tag{1.6}$$

are referred to, according to Liapunov's terminology ([5], p. 14) as the undisturbed response of the control system. At time $t = 0$, let the response of the control system have initial values, $y_{10}, y_{20}, \ldots y_{no}$, at least one of which is not equal to zero. For this type of given initial disturbance there exist unique and real solutions

$$y_i = y_i(y_{10}, y_{20}, \ldots y_{no}, t) \qquad i = 1, 2, \ldots n; \quad t \geqslant 0 \tag{1.7}$$

referred to as the disturbed response of the control system. According to this terminology, stability of nonlinear systems can be formulated in the following way (Letov [5], p. 15), (Malkin [3], p. 5):

Definition 1: The undisturbed response (1.6) of the control system is stable if, for any given arbitrarily small real positive number ϵ, there can be found another positive number η (ϵ) such that, for all initial disturbances y_{io} in the region G', defined by the inequality

$$0 \leqslant |y_{io}| \leqslant \eta \qquad i = 1, 2, \ldots n, \tag{1.8}$$

the disturbed response (1.7) will satisfy the inequality

$$0 \leqslant |y_i(t)| \leqslant \epsilon \qquad i = 1, 2, \ldots n \tag{1.9}$$

for any time $t > 0$.

This definition can be interpreted geometrically in the following way: for all initial disturbances contained within the hypersphere λ(A) of the n-dimensional state space, defined by the inequality

$$0 \leqslant \sum_{i=1}^{n} y_{io}^2 \leqslant \lambda, \tag{1.10}$$

the disturbed response for any time $t > 0$ after removal of the initial disturbances is contained within another hypersphere, defined as

$$\sum_{i=1}^{n} y_i^2(t) < A \qquad i = 1, 2, \dots n, \qquad (1.11)$$

provided λ is chosen sufficiently small.

The above definition of stability does not guarantee that the system response will be bounded if the system is subjected to continuously acting disturbances that are bounded in magnitude. Obviously the stability question is physically meaningless for unbounded driving functions (the term "driving function" will be used here to designate continuously acting bounded disturbances, and the response to such disturbances will be referred to as "driven response").

In the presence of driving functions, a system can be described by a set of simultaneous first-order differential equations, which, by means of the linear transformation (1.4), can be brought into the normal form

$$y_s = Y_s(y_1, y_2, \dots y_n) + R_s(t, y_1, y_2, \dots y_n)$$

$$s = 1, 2, \dots n. \qquad (1.12)$$

The functions R_s represent the driving functions and are assumed to be bounded. The stability definition has to be modified for driven systems in the following way ([5], p. 30), ([3], p. 10):

Definition 2: The driven response (1.12) is stable if, for any given arbitrary small real positive number ϵ, there can be found two other positive numbers, $\eta_1(\epsilon)$ and $\eta_2(\epsilon)$, such that for all initial disturbances y_{s0} in the region

$$0 \leqslant |y_{s0}| < \eta_1, \qquad s = 1, 2, \dots n \qquad (1.13)$$

the response (1.12) will satisfy the inequality

$$0 \leqslant |y_s| < \epsilon \qquad s = 1, 2, \dots n \qquad (1.14)$$

provided that at any time $t > 0$ the functions R_s satisfy the following inequality:

$$\left|R_s(t, y_1, y_2, \dots y_n)\right| < \eta_2 \qquad s = 1, 2, \dots n \,. \qquad (1.15)$$

The response of a system is considered to be unstable if it does not satisfy either one of the above two definitions. To differentiate between control systems that exhibit sustained oscillations and systems that approach an equilibrium state asymptotically in the absence of driving functions, a new definition becomes necessary.

Definition 3: If the undisturbed response of a system is stable in some region G' according to Definition 1, and if, in addition, its response approaches the equilibrium point $y_1 = y_2 = \dots y_n = 0$ asymptotically, i.e.,

$$\lim_{t \to \infty} y_i(t) = 0 \qquad i = 1, 2, \dots n \,, \qquad (1.16)$$

then the undisturbed response (1.6) of the system is said to be asymptotically stable.

The above three definitions of stability are mathematically sound and rigorous. They are, however, unsatisfactory from the engineering viewpoint since they describe the behavior only in a sufficiently small neighborhood of a singularity (equilibrium state). While an equilibrium state (i.e., the undisturbed response) of a nonlinear system may be stable, sufficiently large driving functions or initial disturbances may bring the system response outside the region of stability in the neighborhood of an equilibrium state and cause considerable oscillations or even self-destruction of the system.

While in linear systems stability of the equilibrium state (i.e., stability of the undisturbed response) implies the stability of the system in the entire phase space of its response variable (Kalman [12], p. 6), in nonlinear systems such an inference is invalid. The important factor in the qualitative evaluation of a nonlinear system is not the stability of an equilibrium state, but rather the size of the region of stability around an equilibrium state. Consequently, the above three definitions of stability of an equilibrium state will not be used directly in this report, but rather they will serve the purpose of defining different types of stability of the system. The following terminology will be used to describe the stability of time-invariant (autonomous) nonlinear systems in the report:

a) if a disturbed control system satisfies the stability Definition 1 in the entire state space, it is called globally stable;

b) if stability conditions are satisfied in some limited region A of the state space enclosing the origin, the system is said to be locally stable;

c) if asymptotic stability conditions are satisfied in the entire state space the system is globally asymptotically stable;

d) if asymptotic stability exists in some limited region A around the origin, the system is referred to as locally asymptotically stable;

e) a system may be stable in some region A of the state space around the origin and unstable outside this region, in which case the system should be referred to as both locally stable (or locally asymptotically stable if such is the case) and globally unstable.

If the system exhibits stability under every bounded continuously acting disturbing function (i.e., if it is stable according to Definition 2), it is referred to as totally stable (Massera [13], pp. 182-184).

It should be noted that the stability concept is meaningful only with respect to a given set of variables. Hence, the stability specifications and investigation should be based on the actual variables of the physical system, or such transformations of these variables which do not change the quality of stability information. An example of a nonlinear transformation will be used to illustrate this point.

Example 1.1

The undisturbed response of the system represented by the equation

$$\frac{d^2x}{dt^2} + (2 - 3x^2)\frac{dx}{dt} - 2x^3 = 0$$

is unstable with respect to its output variable x and its time derivatives. It has a first integral of the form

$$\frac{dx}{dt} - x^3 = K\epsilon^{-2t} ,$$

from which it is easily shown that the output x increases without bounds. However, through the substitution

$$\frac{dx}{dt} - x^3 = y$$

the differential equation is changed to

$$\frac{dy}{dt} + 2y = 0 .$$

Thus the system appears to be stable with respect to the variable y and its time derivatives, even though it is unstable in terms of its actual response.

The above example illustrates the danger of arriving at erroneous conclusions regarding the stability of a system by the use of variables that do not appear in the system.

Definitions of stability that are different from the definitions of this report have also been used in literature. For example, Ku and Wolf ([14], p. 144) use the following definition:

> "A nonlinear system is said to be stable if, to every bounded-decaying driving function or input and for all initial conditions, the response x(t) approaches zero as time increases to infinity."

According to this definition, stability is no longer a local concept. Such a definition will classify as unstable all systems that fail to meet the requirements for global asymptotic stability. The results of a stability analysis may, obviously, differ if different definitions are used for stability. The reasons for selecting the definitions of stability to be used in this report are the following:

a) these definitions are the most widely used in the literature;

b) they define stability as a property of the system which does not depend upon the type of input (driving functions) applied to the system;

c) they are applicable to all continuous, autonomous nonlinear systems;

d) a large amount of theoretical work, known as Liapunov's second (direct) method of stability analysis, is based on the preceding definitions of stability. Liapunov's stability theory will be used in the report to develop simplified stability criteria and methods of stability analysis for nonlinear systems.

All the preceding stability definitions consider the stability of the equilibrium state of the system in some bounded or unbounded region of the state space. In systems which exhibit periodic self-sustained oscillations (limit cycles) these definitions would only consider what happens around the equilibrium state inside the limit cycle. It thus becomes necessary to introduce a new definition for the stability of limit cycles.

Let $\overline{y}_1(t)$ represent the periodic response of the system (1.5). The minimum distance, in the state space, between the actual and the periodic response of the disturbed system is given by

$$\rho(y_1, \dots y_n) = \inf \sqrt{\sum_{i=1}^{n} (y_i(t) - \bar{y}_i(t))^2} \qquad (1.17)$$

Definition 4: (Zubov [15], p. 207) The limit cycle (= periodic response) of the system (1.5) is asymptotically stable if for any given real positive number ϵ there can be found another real positive number $\eta(\epsilon)$ such that if

$$\rho\,(y_1(0), \dots y_n(0) < \eta$$

then

$$\rho\,(y_1(t), \dots y_n(t) < \epsilon$$

for any time $t > 0$ and

$$\lim_{t \to +\infty} \rho(y_1(t), \dots y_n(t) = 0 .$$

The limit cycle is globally asymptotically stable if the above conditions are satisfied in the entire state space (i.e., in the entire space of the variables $y_1, \dots y_n$).

1.2 Liapunov's Direct (Second) Method of Stability Analysis

Liapunov has shown that the stability of a physical system, described by a set of first order differential equations

$$\frac{dy_i}{dt} = Y_i(y_1, y_2, \dots y_n) \qquad i = 1, 2, \dots n \qquad (1.5)$$

can be determined analytically if it is possible to establish the so-called V-function of the variables y_i, such that these functions and their time derivatives possess certain characteristic properties. A necessary condition is that the system be continuous, i.e., the functions Y_i are continuous with respect to all the variables y_i. The continuity restriction is primarily of theoretical importance, since any changes in practical physical systems will take a finite, even though possibly a very small, amount of time.

The results of Liapunov's Stability Theory are expressed in his theorems on stability and instability. These theorems, together with a few additional theorems by other researchers, that will be used in this report, are presented in this section. The proofs of these theorems are contained in numerous references and will not be repeated here.

Theorem 1.1: If there exists a real-valued function $V(y_1, y_2, \dots y_n)$ with the following properties:

a) $V(y_1, y_2, \dots y_n)$ is continuous through first partial derivatives;

b) V is positive definite, i.e.,

$$V(y_1, y_2, \dots y_n) > 0 \quad \text{for all } |y_i| > 0,$$

$$V(0) = 0;$$

c)* $\lim V(y_1, y_2, \dots y_n) = \infty$ for all y_i,

$$|y_i| \longrightarrow \infty$$

then

1) the system (1.5) is stable with respect to the variable y_i if there is some region G, defined by $0 < |y_i| < L$, where L is some real positive constant, such that in this region the derivative $\frac{dV}{dt}$ is negative semidefinite, i.e.,

*Some authors (Malkin, etc.) do not include condition (c) in the stability theorems and, hence, do not eliminate the possibility of $V \longrightarrow 0$ as $|y_i| \longrightarrow \infty$. For an example where this may lead to erroneous results see Letov ([5], p. 21).

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial y_i} \cdot \frac{dy_i}{dt} \leqslant 0$$

for all $y_i \neq 0$, $t > 0$;

2) the system (1.5) is asymptotically stable with respect to the variable y_i if in the above region

$\frac{dV}{dt} < 0$;

or

$\frac{dV}{dt} \leqslant 0$ and the curve $\frac{dV}{dt} = 0$ is not a trajectory (solution) of the system (1.5).

3) the system (1.5) is globally asymptotically stable with respect to the variable y_i if condition 2) above is satisfied in the entire state space of the variable y_i.

This theorem is proved in [12]. Its application is illustrated here by an example.

Example 1.2:

Consider the system described by the differential equation

$$\frac{d^2y}{dt^2} + 0.2\left[1 + (\frac{dy}{dt})^2\right]\frac{dy}{dt} + y = 0 \text{ .}$$

To transform this differential equation into a set of two simultaneous first-order differential equations let

$y = y_1$,

and

$\frac{dy}{dt} = y_2$.

Then

$$\frac{dy_1}{dt} = y_2 ,$$

$$\frac{dy_2}{dt} = -.2(y_2^2 + 1)y_2 - y_1 .$$

Select as the V-function the quadratic form

$$V = \frac{1}{2} (y_1^2 + y_2^2) .$$

The time derivative of this V-function is

$$\frac{dV}{dt} = -.2(y_2^2 + 1)y_2^2 ,$$

which is negative semidefinite everywhere (i.e., $\frac{dV}{dt} \leqslant 0$) and non-zero along any trajectory. Consequently, according to Theorem 1.1, the system is globally asymptotically stable. The experimental phase plane solution of this system is shown in Fig. 1.1. Fig. 1.1 also illustrates the geometrical interpretation of Liapunov's stability theorem. A positive definite V-function represents a family of closed surfaces, represented in Fig. 1.1 by the dotted lines. A negative semidefinite time derivative of such a V-function implies that the trajectories of the response will either stay on a closed surface or intersect these closed surfaces in an inward direction.

As illustrated by the preceding example, a quadratic form, defined by

$$V = \sum_{i=1}^{n} \sum_{k=i}^{n} a_{ik} y_i y_k \qquad \text{with } a_{ik} = a_{ki} \tag{1.18}$$

is requently used as the V-function for the system. It has been shown* that a quadratic form can always be used as a Liapunov's function for

*See, e.g., Malkin ([3], p. 57).

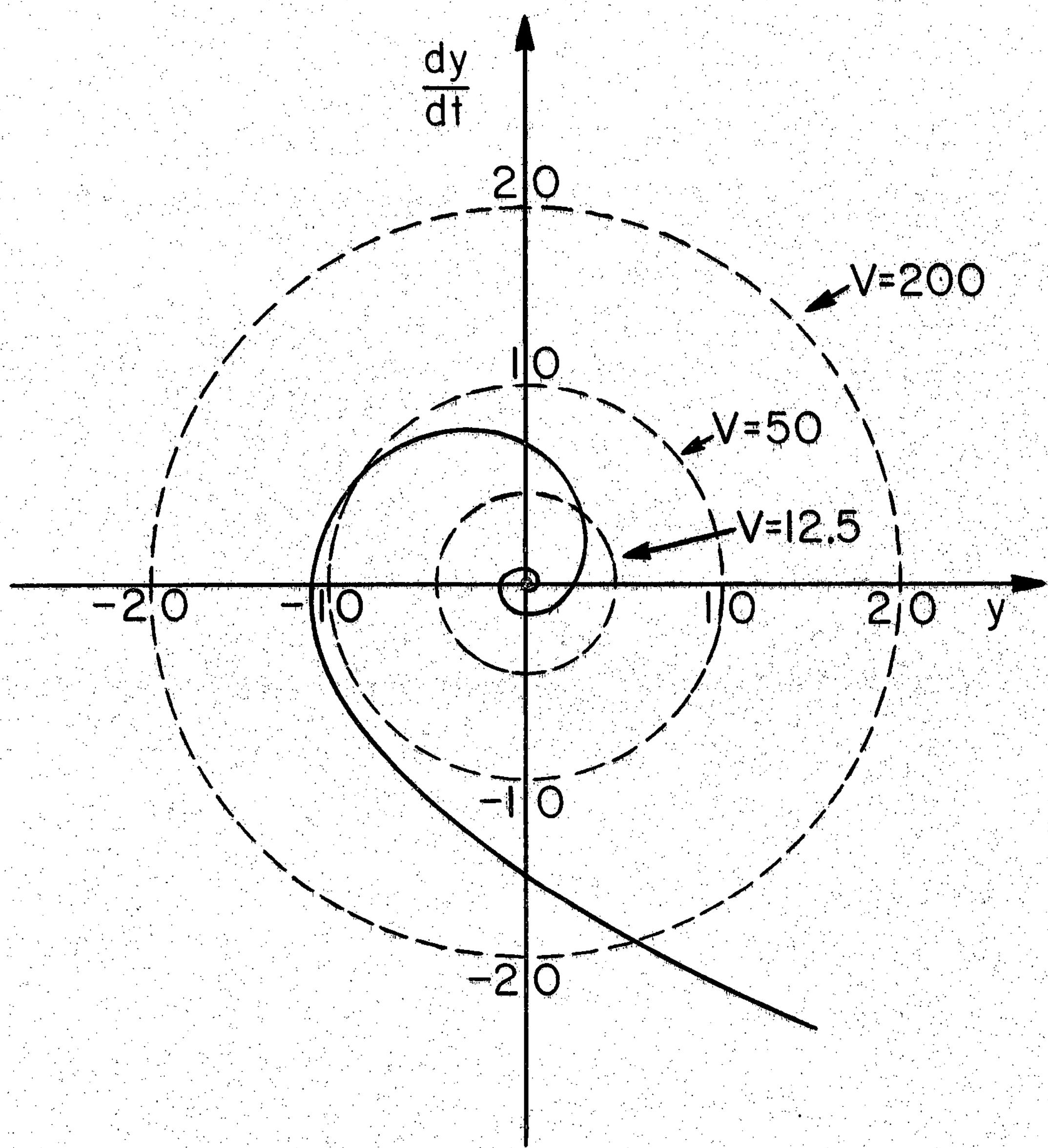

Figure 1.1

Phase Plane Portrait of the System of Example 1.2.

linear autonomous systems. It is easy to prove the definiteness of sign of a quadratic form. A quadratic form is positive definite if and only if all the determinants

$$\left|a_{11}\right|, \quad \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \qquad \text{etc.}$$

are positive. This theorem, known as the Sylvester theorem, (see, e.g., Lefschetz ([16], p. 113), can be used not only to select a positive definite V-function, but also to check its time derivative for values of y_i for which it is negative definite.

For unstable systems a region of instability can be established by means of the following theorem.

Theorem 1.2*: If there exists a real-valued function $V(y_1, y_2, \ldots y_n)$ with the following properties:

a) $V(y_1, y_2, \ldots y_n)$ is continuous;

b) the time derivative of V is negative definite, i.e.,

$$\frac{dV}{dt} = W(y_1, \ldots y_n) < 0, \text{ for all}$$

$$y_i \neq 0, \quad W(0) = 0;$$

c) $$\lim_{|y_i| \to \infty} W(y_1, \ldots y_n) = -\infty$$

for all y_i,

---

*The proof of this theorem is given by Zubov ([15], p. 48).

then

1) the system (1.5) is unstable with respect to the variable y_i in the region G in which $V(y_1, \ldots y_n)$ is not positive semidefinite;

2) the system (1.5) is globally unstable with respect to the variable y_i if $V(y_1, \ldots y_n)$ is not positive semidefinite in the entire state space.

The application of Theorem 1.2 will be illustrated by the following example.

Example 1.3:

Consider a third-order closed-loop system with a saturating amplifier, as shown in Fig. 1.2 and Fig. 1.3. Let

$$G_m(s) = \frac{1}{s^2} ,$$

$$G_e(s) = \frac{100}{1 + 0.1s}$$

and the saturation characteristics of the amplifier be described by (see Fig. 1.3)

$$y = f(x) = xg(x)$$

where the function g(x) satisfies the inequality

$$0.001 \leqslant g(x) \leqslant 1 .$$

Then, if the disturbance is removed at time $t = 0$,

$$r(t) = 0 \quad \text{for all } t > 0 ,$$

and

$$e(t) = -z(t) .$$

The differential equation describing this system is

$$0.1 \frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} = -100xg(x) .$$

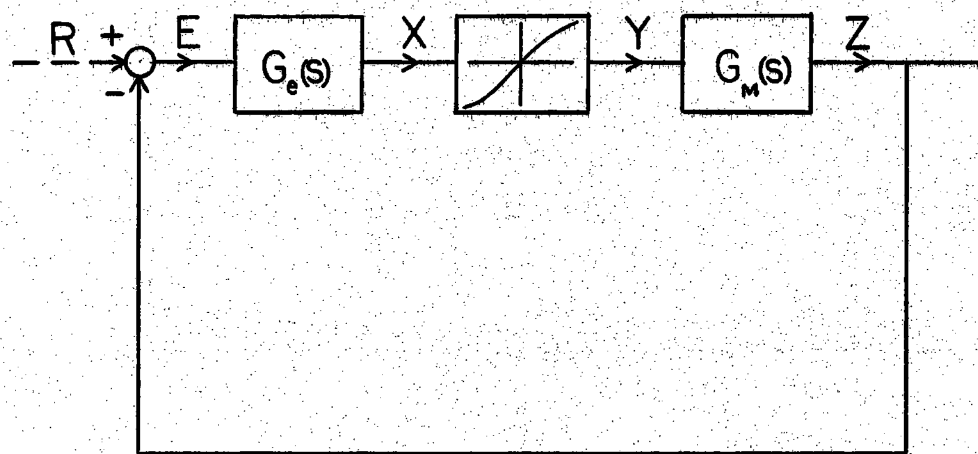

Figure 1.2

Block Diagram of the System of Example 1.3.

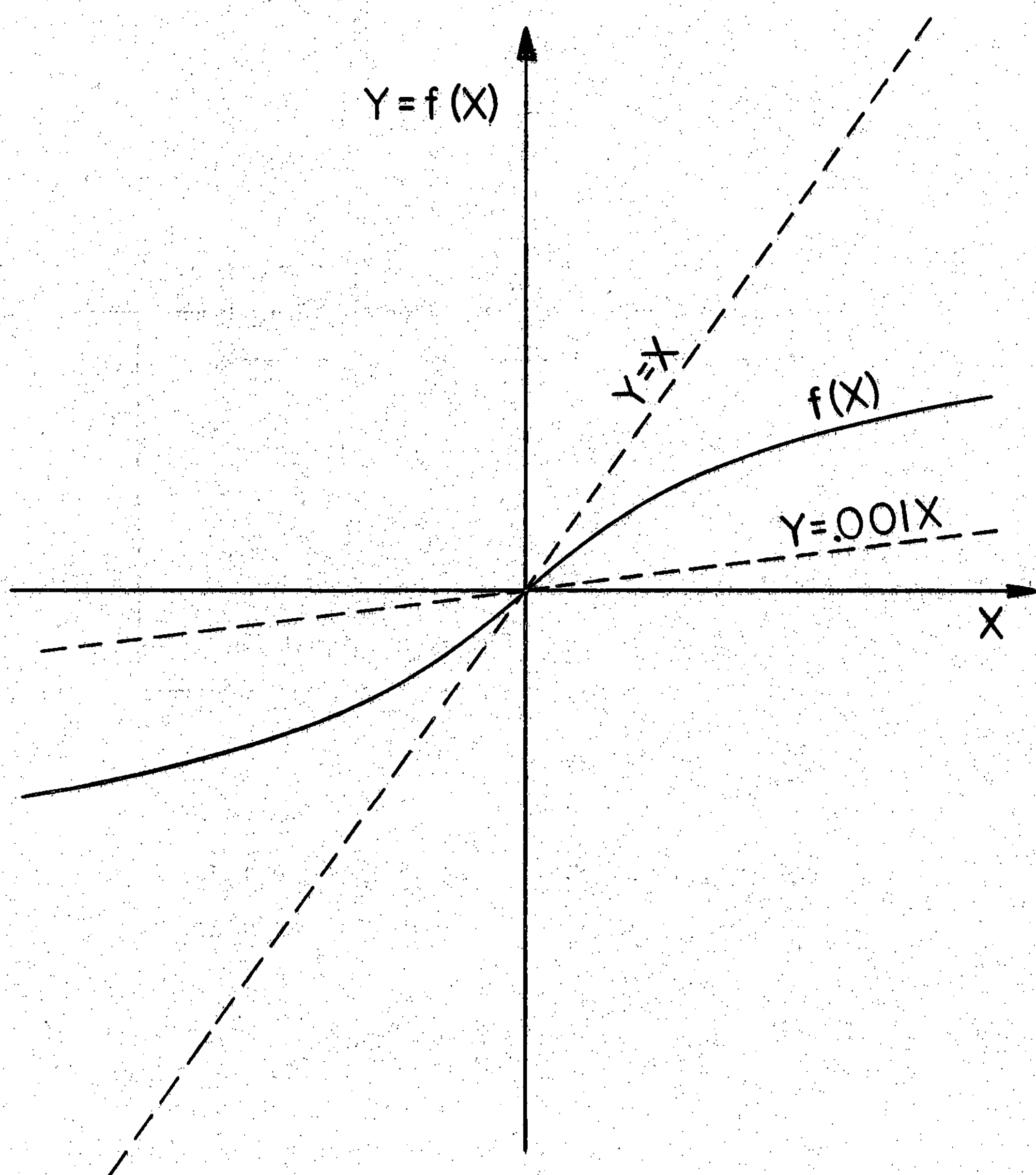

Figure 1.3

Amplifier Saturation Characteristics

of the System of Example 1.3.

Let

$$x = x_1 ,$$

$$\frac{dx}{dt} = x_2 ,$$

$$\frac{d^2x}{dt^2} = x_3 .$$

Then a set of simultaneous first order differential equations describing this system is

$$\frac{dx_1}{dt} = x_2 ,$$

$$\frac{dx_2}{dt} = x_3 ,$$

and

$$\frac{dx_3}{dt} = -100x_3 - 10{,}000x_1 g(x_1) .$$

If

$$V(x_1,x_2,x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{33}x_3^2 + a_4\int_0^{x_1} x\, g(x)\, dx$$

is selected as a Liapunov's function for this system, then

$$\frac{dV}{dt} = \left[-10{,}000g(x_1)a_{13}\right] x_1^2 + \left[2a_{11} - 10{,}000g(x_1)a_{23} + g(x_1)a_{23} + g(x_1)a_4\right] x_1x_2 + \left[a_{12} - 100a_{13} - 20{,}000g(x_1)a_{33}\right] x_1x_3 + (a_{12})x_2^2 + (a_{13} + 2a_{22} - 100a_{23})x_2x_3 + (a_{23} - 200a_{33})x_3^2 .$$

In order to make $\frac{dV}{dt}$ negative definite, let

$$a_{11} = 0 ,$$

$$a_{33} = 0 ,$$

$a_{13} = -0.0001$,

$a_{12} = 0.01$,

$a_{23} = 1,000$,

$a_{22} = 50,000.00005$,

$a_4 = 10,000,000$.

Then

$$V(x_1,x_2,x_3) = 0.01x_1x_2 - 0.0001x_1x_3 + 50,000.00005x_2^2 + 1,000x_2x_3 + 10,000,000\int_0^{x_1} x\, g(x)\, dx$$

and

$$\frac{dV}{dt} = g(x_1)\, x_1^2 + 0.02x_1x_3 + 0.01x_2^2 + 1,000x_3^2 .$$

Consequently, applying the Sylvester theorem, one finds

$g(x_1) > 0$,

$0.01g(x_1) - 0 > 0$

and

$10g(x_1) - 0.000004 > 0$.

Hence $\frac{dV}{dt}$ is positive definite while V is not negative, which proves that the system is globally unstable.

The preceding example illustrates the procedure of finding Liapunov's functions developed by Aizerman [7].*

It should be noted that Theorem 1.2 is much more powerful than Theorem 1.1, since it is always possible to select a negative definite function W. Then, if the conditions of Theorem 1.2 are not satisfied,

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*This procedure is discussed in detail in Chapter III.

the W-function automatically meets the requirements of Theorem 1.1 and shows that the system is stable, while the converse is not true (i.e., violation of requirements of Theorem 1.1 does not necessarily imply instability). Unfortunately, in most cases it is much more difficult to find the V-function for a given $W = \frac{dV}{dt}$ than to find W for a given V-function. In view of this difficulty, it sometimes is more convenient to use a modification of Theorem 1.2 which is given here as Theorem 1.3.

Theorem 1.3*: If there exist two real valued functions $V(y_1, \ldots y_n)$ and $W(y_1, \ldots y_n)$ such that

a) the function V is continuous;

b) $\frac{dV}{dt} = \lambda V + W$,

where λ is a positive constant;

c) the function W is negative semidefinite ($W \leqslant 0$);

d) $\lim W(y_1, \ldots y_n) = -\infty$ as $|y_i| \longrightarrow \infty$ for all y_i,

then

1) the system (1.5) is unstable with respect to the variable y_i in the region G in which $V(y_1, \ldots y_n)$ is not positive definite

2) the system (1.5) is globally unstable with respect to the variable y_i if $V(y_1, \ldots y_n)$ is not positive definite in the entire state space.

The above three theorems can be used to provide an answer to the question of the stability of a control system provided that a V-function satisfying the requirements of any one of the above three Liapunov's

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*For the proof of this theorem see, e.g., Zubov ([15], p. 46).

theorems (henceforth referred to as Liapunov's function) can be found. A trial and error procedure was used to find suitable Liapunov's functions in Examples 1.2 and 1.3. Such a procedure is very difficult even in low order systems ($n < 3$) and cannot in general be applied to higher order systems where a quadratic form (Eq. 1.18) cannot be used as a Liapunov's function for the system. This illustrates the need of a systematic approach to find Liapunov's functions that would be applicable to large groups of control systems of a particular type. Such an approach, applicable to a large group of practical control systems, is developed in Chapter II of this report.

The ultimate goal in the stability analysis of nonlinear control systems is the estimate (or even an exact determination) of the region of either asymptotic stability or instability.* This will be accomplished if a Liapunov's function for the system is found. In view of the difficulties involved in finding a Liapunov's function for the system, even when systematic approaches to find these functions are available, it is very desirable to know in advance whether to look for a Liapunov's function satisfying the requirements of the stability theorem or a Liapunov's function satisfying the requirements of an instability theorem. Fortunately, in many practical systems, this question can be answered easily by the methods of linear system stability analysis (such as Routh-Hurwitz table) to the equations of the first approximation of the system. This question is discussed in more detail in Section 1.3.

The preceding three theorems are applicable to systems which are described by a set of simultaneous differential equations of the type of

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*A control system that exhibits a range of stability and not asymptotic stability around an equilibrium state will act as an oscillator and can hardly be considered as satisfactory.

Eq. 1.5. Since Y_1 in Eq. 1.5 is not an explicit function of time, Eq. 1.5 can represent a closed-loop control system only if the driving function r(t) is (see Fig. 1.2) a constant, or if it is removed at time $t = 0$. Consequently, the Theorems 1.1 through 1.3 deal with stability of control systems in the absence of a driving function. A much more important question than the question of stability in the absence of a driving function is that of the response of the system in the presence of a bounded driving function. It is obvious that a direct analysis of the stability of a closed-loop system in the presence of a bounded driving function (i.e., total stability) would require the use of Liapunov's functions which should be explicit functions of time. The available systematic methods of construction of Liapunov's functions, discussed in Chapter II of this report, would no longer be applicable if the system were subjected to the continuously acting input (driving function). Furthermore, in automatic control systems the exact nature of the input (driving function) is usually unknown. Only the maximum value of the magnitude of the input to which the system will be subjected during its operation can be estimated in many cases. Fortunately, it is possible to prove total stability (i.e., stability in the presence of continuously acting inputs, as defined by Definition 3 of Section 1.1) for systems that are globally asymptotically stable in the absence of inputs (driving functions). The following theorem, due to Malkin, can be used to prove total stability of systems that satisfy the requirements of Theorem 1.1 for global asymptotic stability.

Theorem 1.4: The system (1.12) is totally stable (i.e., it is stable in the presence of continuously acting bounded inputs, according to Definition 3 of Section 1.1) if all of the following hold:

a) it is globally asymptotically stable in the absence of an input (driving function);

b) the terms R_s (Eq. 1.12), representing the input (driving function), are bounded;

c) the terms R_s (Eq. 1.12), representing the input (driving function) can be separated from the terms Y_s, representing the system in the absence of the input (driving function);

d) the terms Y_s (Eq. 1.12), representing the system in the absence of input (driving function, possess continuous partial derivatives with respect to the variable y_i ($i = 1, 2, \ldots n$).

The proof of Theorem 1.4 is given by Malkin ([3], pp. 304-318). The above theorem enables one to use Theorem 1.1 to prove not only global asymptotic stability but also total stability of systems in which it is possible to separate the Y_s and R_s terms in Eq. 1.12. It eliminates the need to use Liapunov's functions which are explicit functions of time to prove total stability of such systems. The systematic methods of Chapter II for finding Liapunov's functions satisfying the requirements of Theorem 1.1 can thus be applied to prove total stability of many non-linear autonomous closed-loop systems.

1.3 Stability Investigation From Equations of First Approximation

In the majority of the problems of control theory, the functions y_i of Eq. 1.5 can be expanded into power series, converging in some region about the origin of the coordinate system, e.g.,

$$\sum_{i=1}^{n} |y_i| < H \tag{1.19}$$

provided the constant H is sufficiently small.* In such cases, the equations describing the nonlinear system can always be rearranged in the form

$$\frac{dy_i}{dt} = a_{i1}y_1 + \dots + a_{in}y_n + f_i(y_1 \dots y_n) \quad i = 1, 2, \dots n \qquad (1.20)$$

where $a_{ik}(k, i = 1, \dots n)$ are the constants of the linear portion of the expansion, and the functions F_i do not contain terms of lower than second degree.

To decide on stability of the equilibrium state, often only the so-called equations of the first approximation

$$\frac{dy_i}{dt} = a_{i1}y_1 + \dots + a_{in}y_n \quad i = 1, \dots n \qquad (1.21)$$

need be investigated.

Since the equations of first approximation represent a set of linear homogeneous differential equations, the problem of stability of the equilibrium state is reduced to the problem of stability of a linear system. Consequently, it becomes sufficient to investigate the characteristic equation

$$(s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n) = 0 \qquad (1.22)$$

of the linearized system, where the λ's are the roots of this characteristic equation. If the system represents a so-called noncritical case, i.e., if none of the roots of the characteristic equation of its first approximation lie on the imaginary axis of the s-plane while all other roots have negative real parts, then the following two Liapunov's theorems (Malkin [3], pp. 61-63) can be used:

Theorem 1.5: If the real parts of all the roots λ_i of the characteristic equation (Eq. 1.21) of the first approximation

*This restriction represents the essential difference between the method of first approximation and the above outlined method of direct stability investigation.

are negative, then the equilibrium state is asymptotically stable, independent of the terms of F_i higher than the first degree.

Theorem 1.6: If among the roots λ_i of the characteristic equation (Eq. 1.21) there is at least one root with positive real part, then the equilibrium state is unstable, independent of terms of higher than the first degree.

The concept of structural stability is closely related to the above discussed critical cases. Structural stability is defined (Cunningham [17], p. 282) as "the property of a physical system such that the qualitative nature of its operation remains unchanged if parameters of the system are subject to small variations". In structurally unstable systems an equilibrium state represents the critical case. Hence, the preceding theorems are applicable only to structurally stable systems. In structurally unstable systems stability (instability) of the undisturbed response is determined by the function F_i of the nonlinear form, and it then becomes necessary to investigate Eq. 1.20 in its original form.

In regard to the stability investigation from the equations of first approximation, the fact that stability of the equilibrium state is a local concept must be re-emphasized. No conclusion about system stability outside the region defined by Eq. 1.19 may be drawn from the equations of first approximation. Nevertheless, this approach enables one to decide the type of V-function that may be applicable to a particular problem (i.e., whether the V-function should satisfy Laipunov's stability theorem or instability theorems).

Although it is easy to find suitable V-functions for the linear equation of first approximation, a much faster procedure is to apply the Routh-Hurwitz criterion to the linearized system of first approximation.

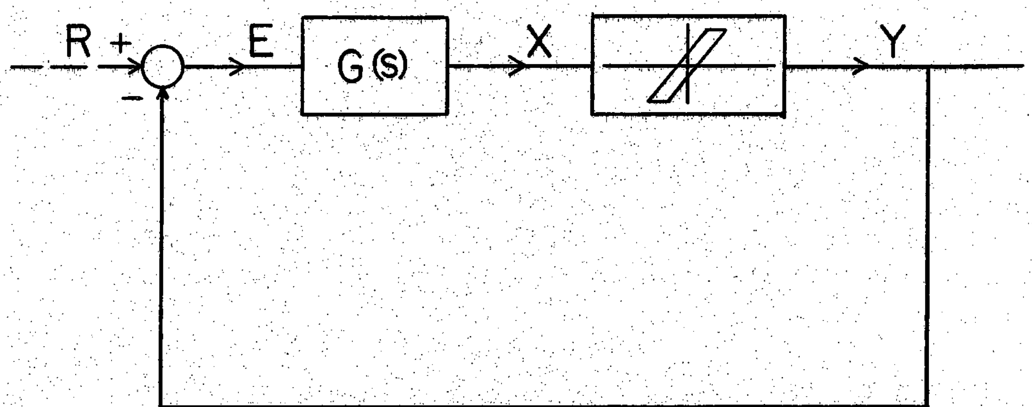

Figure 1.4

Block Diagram of the System of Example 1.4.

Example 1.4:

Consider a third order system with backlash, as shown in Fig. 1.4. Let

$$\frac{X(s)}{E(s)} = \frac{10\ (1 + 0.2s)}{s^2\ (1 + 0.1s)}$$

or

$$0.1\ \frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} = 2\ \frac{de}{dt} + 10e\ ,$$

Removal of the input r(t) yields

$$e(t) = -\ y(t)$$

and

$$\frac{d^3x}{dt^3} + 10\ \frac{d^2x}{dt^2} + 2\ \frac{dy}{dt} + 10y = 0\ .$$

The gear backlash characteristics can be expressed as

$$y(t) = 0.1x(t) - 0.01\ \frac{\frac{dx}{dt}}{\left|\frac{dx}{dt}\right|}\ .$$

In order to overcome the difficulties due to the discontinuity of $\frac{1}{\left|\frac{dx}{dt}\right|}$ at $\frac{dx}{dt} = 0$, let the function $\frac{1}{\left|\frac{dx}{dt}\right|}$ be approximated (as shown in Fig. 1.5), for small values of $\frac{dx}{dt}$, by

$$\frac{1}{\left|\frac{dx}{dt}\right|} \approx 100e^{-4.6\left(\frac{dx}{dt}\right)^2} = 100 - 460\left(\frac{dx}{dt}\right)^2 + \ldots\ .$$

Then the equation of the system first approximation becomes

$$\frac{d^3x}{dt^3} + 8\ \frac{d^2x}{dt^2} - 9.8\ \frac{dx}{dt} + x = 0\ .$$

Obviously, the corresponding characteristic equation

$$s^3 + 8s^2 - 9.8s + 1 = 0$$

has roots in the right-half of the s-plane and the equilibrium

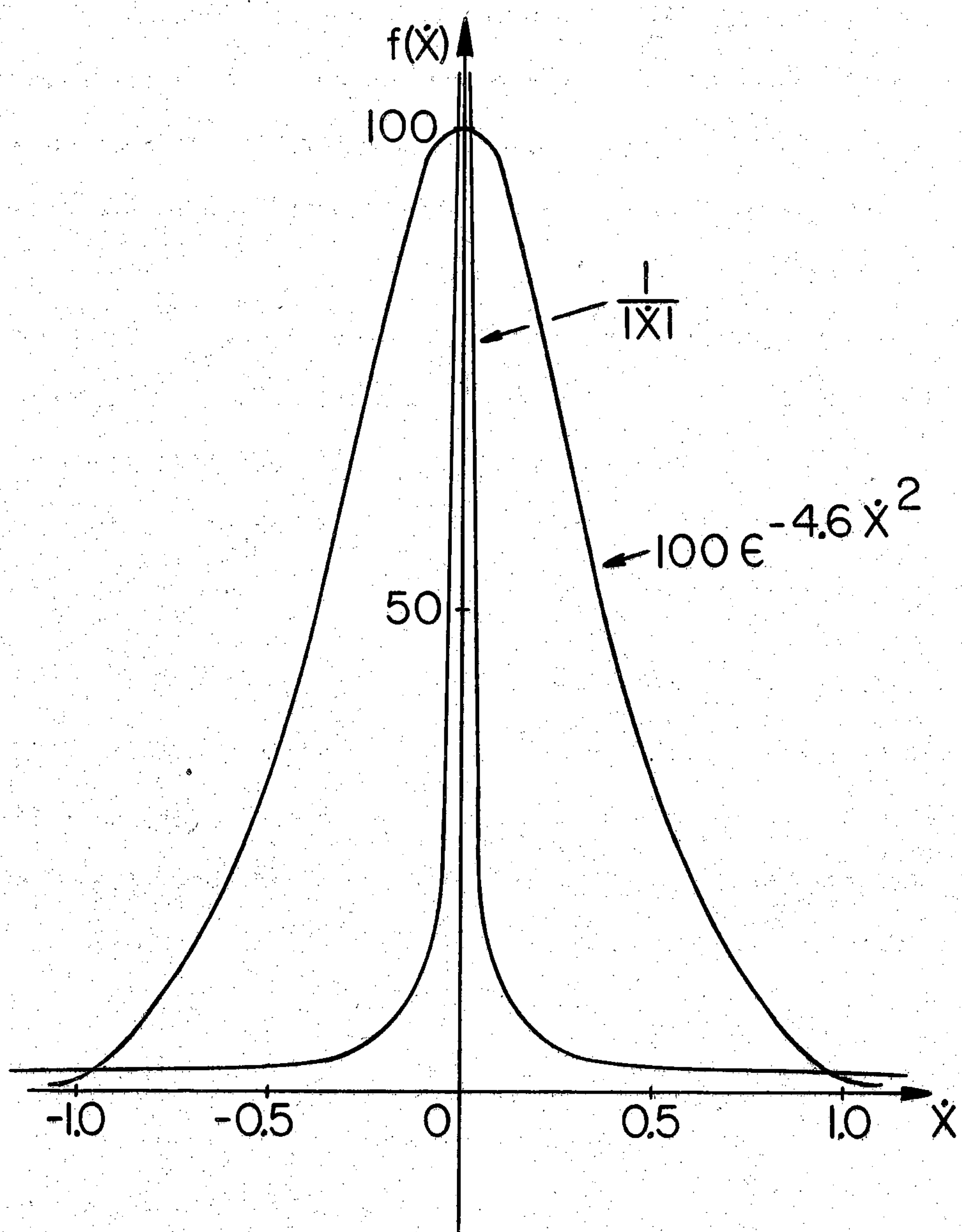

Figure 1.5

Approximation Used in Connection with Backlash Characteristics of the System of Example 1.4.

state $y = x = 0$ is unstable, according to Theorem 1.6. If it were desirable to estimate the region in the phase space in which this system is unstable, one could try to find a Liapunov's function satisfying the requirements of Theorem 1.2 rather than the requirements of Theorem 1.1.

Thus, the preceding example illustrates the use of the theorems on stability from equations of first approximation in deciding whether to look for Liapunov's functions satisfying the stability or the instability theorems for the purpose of estimating the region of stability or instability in the phase space.

1.4 Stability of Limit Cycles

The analysis of limit cycle stability represents an extension of Liapunov's Second Method to systems for which only local asymptotic stability or instability could be proved by direct application of the "second method". The following theorem, due to Zubov ([15], p. 208), can be used to prove that a system has a stable limit cycle.

Theorem 1.7: In order that the limit cycle (periodic solution) of the system (1.5) be asymptotically stable, it is necessary and sufficient that there exist two functions V and W satisfying the following conditions:

1) the function $V(y_1, \ldots y_n)$ is defined and continuous in some region of the state space containing the limit cycle; the function $W(y_1, \ldots y_n)$ is defined in the entire state space;

2) the function $V(y_1, \ldots y_n)$ is negative everywhere in the above region except on the limit cycle (periodic solution) of (1.5); the function

$W(y_1, \ldots y_n)$ is positive everywhere except on the limit cycle of (1.5);

3) the functions V and W are equal to zero at every point on the limit cycle of (1.5);

4) $$\frac{dV}{dt} = W\sqrt{1 + \sum_{i=1}^{n} Y_i^2 (y_1, \ldots y_n)} \qquad (1.23)$$

5) $$\lim_{y_i \to y_i^*} V(y_1, \ldots y_n) = -\infty \quad (i = 1, 2, \ldots n)$$

where y_i^* represent any point on the boundary of stability region.

6) $$\lim_{|y_i| \to \infty} V(y_1, \ldots y_n) = \lim_{|y_i| \to 0} V(y_1, \ldots y_n) = -\infty$$

must be satisfied for global asymptotic stability of the limit cycle (periodic solution) of (1.5).

This theorem represents one of the first attempts to extend the ideas of Liapunov's Second Method to systems with self-sustained oscillations. Its application, however, runs into considerable difficulty even where the equations of the limit cycle are known.

Example 1.5:

Consider the system described by the following equations

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = (1 - y_1^2 - y_2^2)y_2 - y_1 \ .$$

Chose as a tentative V-function

$$V = -\frac{\left[1 - y_1^2 - y_2^2\right]^2}{y_1^2 + y_2^2} \ .$$

Differentiating the above equation with respect to time and substituting the differential equations of the system one finds

$$\frac{dV}{dt} = - \frac{2y_2^2 (1 + y_1^2 + y_2^2)}{y_1^2 + y_2^2} V$$

Note that $\frac{dV}{dt}$ can take on zero value at points which are not on the limit cycle.

$$y_1^2 + y_2^2 = 1 .$$

It appears that the requirement for positiveness of $\frac{dV}{dt}$ in Theorem 1.5 could be relaxed, allowing

$$\frac{dV}{dt} \geqslant 0$$

as long as the curve $\frac{dV}{dt} = 0$ is not a trajectory of the system off the limit cycle. The mathematical proof of such weaker theorem is, however, not available at the present time. This limits considerably the application of Theorem 1.5 to practical (physically realizable) systems.

CHAPTER II

STABILITY OF CANONIC SYSTEMS

2.1 Introduction

The major difficulty in applying Liapunov's "Second Method" to the analysis of practical control systems is due to the lack of a straightforward procedure of finding a Liapunov's function (i.e., a function of the system variables satisfying Liapunov's stability or instability theorems). However, several Liapunov's functions have been developed that apply to a large group of control systems that can be described by the so-called "first canonic form" or the "second canonic form" of system differential equations. The transformations which change the form of system differential equations into a set of canonic differential equations are called the canonic transformations. Hence, canonic transformations represent a systematic appraoch for finding Liaponuv's functions for a large number of nonlinear control systems.

2.2 The First Canonic Transformation

The Russian automatic control literature, in particular the books by Lur'e [4] and Letov [5], contains detailed discussions of the application of canonic transformation for "direct control" and "indirect control" systems. There is no equivalent English terminology to differentiate between direct and indirect control, while the literal translation of Russian terms does not convey much information. In either of the two cases, however, the system may be represented by the block diagram shown in Fig. 2.1. In either case, the nonlinear element is retained in the forward path of the minor loop. Consequently, it is possible to combine the linear feedback paths of both loops into an equivalent single-loop system, as shown in Fig. 2.2 It is not always possible, however, to

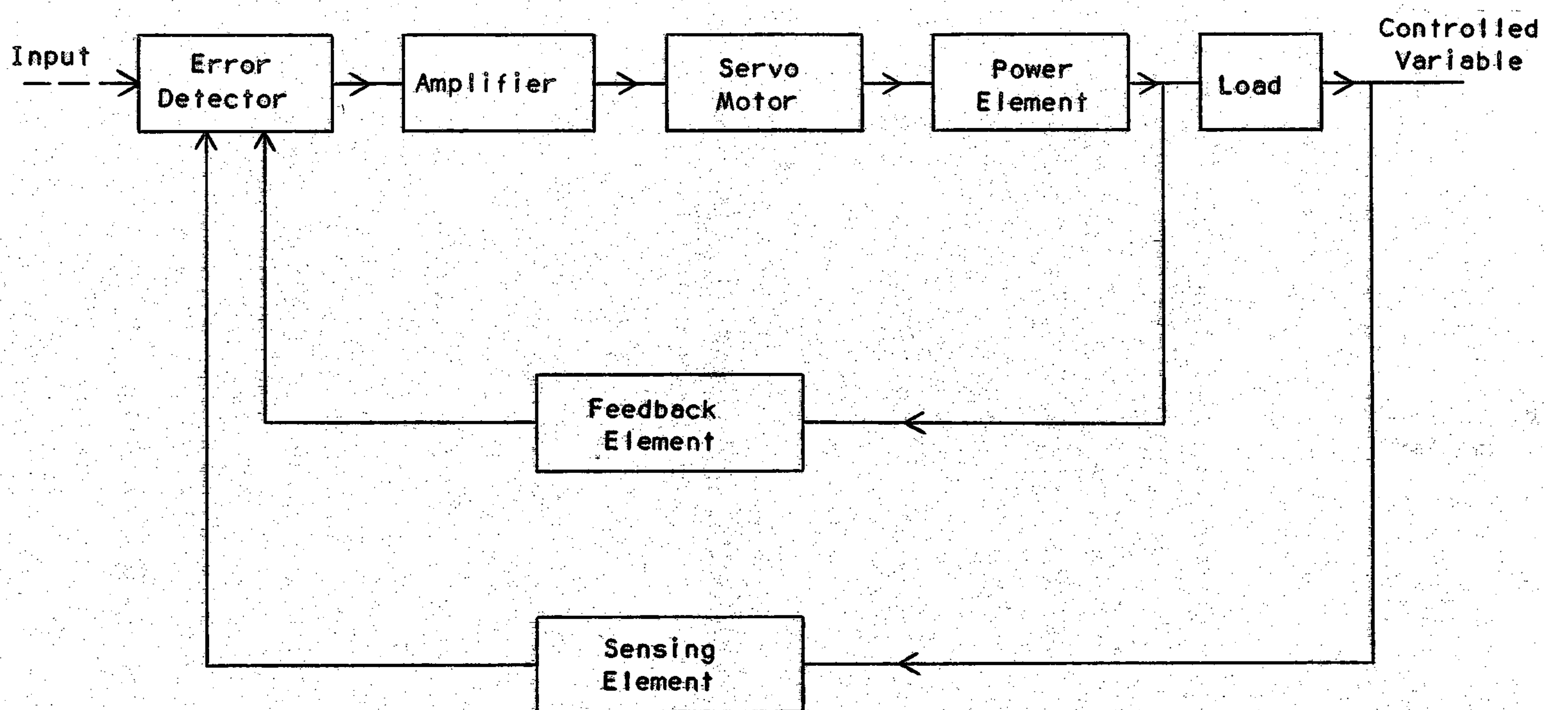

Figure 2.1

Schematic Diagram of an Indirect Control System.

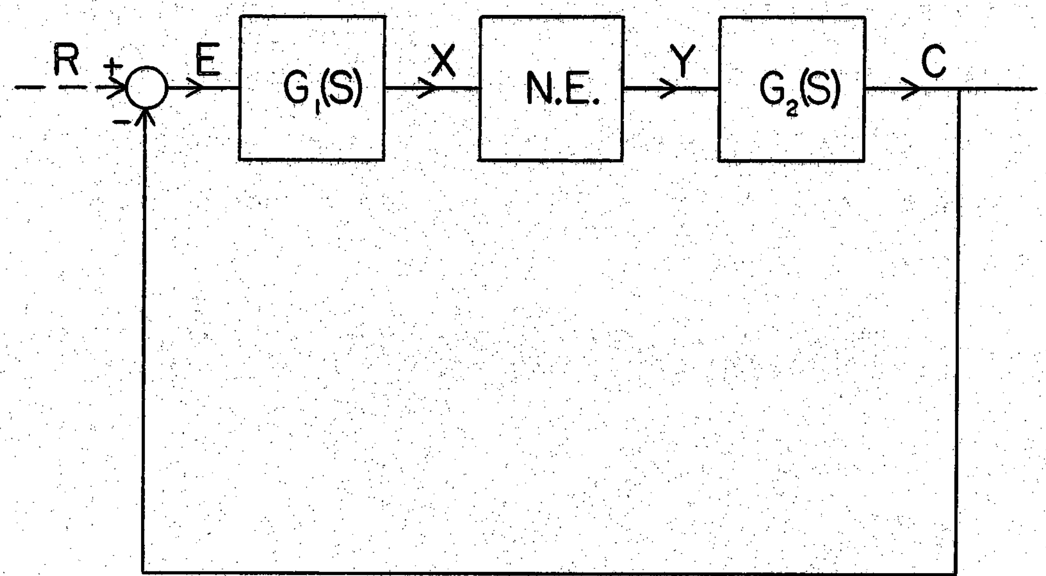

Figure 2.2

Block Diagram of a Closed-loop System

with a Single Nonlinear Element.

transform a single-loop system with a single nonlinear element, as shown in Fig. 2.2 into either an equivalent direct control or indirect control system which is physically realizable (i.e., which is described by differential equations with real coefficients). Consequently, the canonic transformation of either direct control or indirect control systems represents only a special case of canonic transformations of single-loop systems with a single nonlinear element, as shown in Fig. 2.2. Since the first canonic transformation is applicable to the more general case of systems with a single nonlinear element, there is little, if any, justification to discuss the special cases of direct and indirect control systems.

The systems to which the procedure of stability analysis presented in this chapter is applicable can be represented by the block diagram shown in Fig. 2.2.

It is assumed that the input into the system, $r(t)$, is removed at time $t = 0$, i.e.,

$$r(t) = 0 \qquad \text{for all} \quad t > 0.$$

Under the above assumption the block diagram of the system can be simplified as shown in Fig. 2.3.

It will also be assumed that the input-ourput characteristics of the nonlinear gain element can be described by a continuous function

$$y = f(x); \qquad f(o) = 0,$$

where x is the input into and y the output of the nonlinear element. The function $f(x)$ is assumed to be single-valued and analytical in a sufficiently small neighborhood of the point $x = 0$.

The following equations (Lur'e [18], p. 1357) represent the first canonic form of system differential equations:

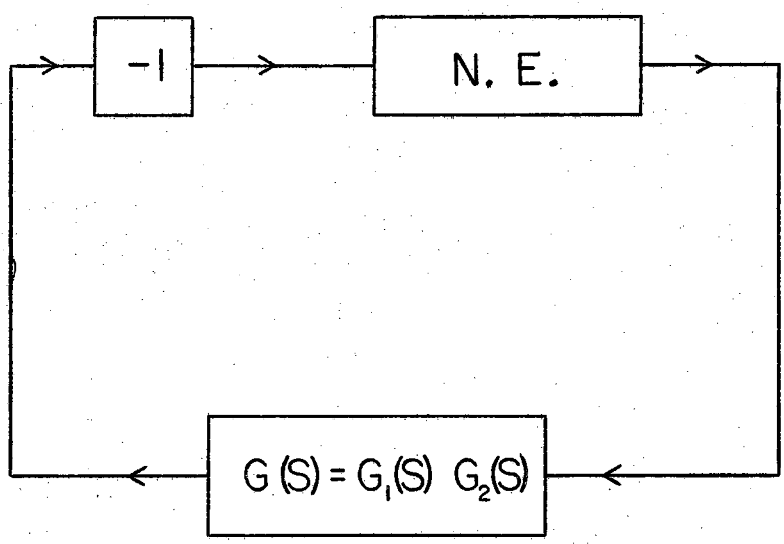

Figure 2.3

Simplified Block Diagram of a Closed-loop System with a Single Nonlinear Element.

$$\frac{dz_i}{dt} = \lambda_i z_i + f(x) \qquad i = 1, 2, \ldots n \tag{2.1a}$$

and

$$x = \sum_{i=1}^{n} \alpha_i z_i , \tag{2.1b}$$

where λ_i and α_i are constants, x is the variable representing the input into the nonlinear element and z_i's are the variables obtained from the canonic transformations. The variables z_i will be referred to as the canonic variables.

Differentiation of Eq. 2.1b with respect to time, followed by the substitution of Eq. 2.1a yields

$$\frac{dx}{dt} = \sum_{i=1}^{n} \beta_i z_i - rf(x) \tag{2.1c}$$

where

$$\beta_i = \alpha_i \lambda_i \qquad i = 1, 2, \ldots n \tag{2.2}$$

and

$$r = - \sum_{i=1}^{n} \alpha_i \qquad i = 1, 2, \ldots n . \tag{2.3}$$

Eq. 2.1a is called the principal part, while Eq. 2.1b and Eq. 2.1c are called the complementary part of the first canonic form of system differential equations.

To show that Eq. 2.1 actually represents the system of Fig. 2.3, let

$$D = \frac{d}{dt} .$$

Then, from Eq. 2.1a and Eq. 2.1b one finds

$$(D - \lambda_i) z_i = y \qquad i = 1, 2, \ldots n \tag{2.4}$$

and

$$x = \sum_{i=1}^{n} \alpha_i z_i ,$$

where

$$y = f(x)$$

represents the nonlinear element characteristic.

Solving Eq. 2.4 for z_i and substituting into Eq. 2.1b, one obtains

$$\frac{x}{y} = \sum_{i=1}^{n} \frac{\alpha_i}{D - \lambda_i} . \tag{2.5}$$

Note that the loop transfer function of the system of Fig. 2.3 is

$$G(s) = G_1(s)\, G_2(s) = - \frac{X(s)}{Y(s)} . \tag{2.6}$$

Consequently, from Eq. 2.5 and Eq. 2.6 the loop transfer function is

$$G(s) = - \sum_{i=1}^{n} \frac{\alpha_i}{s - \lambda_i} \tag{2.7}$$

Equation 2.7 indicates that the constants λ_i are the poles of the loop transfer function G(s) and the constants α_i are the negative values of the residues of G(s) at the corresponding poles. Thus, the first canonic form of differential equations for the system of Fig. 2.3 (or that of Fig. 2.2 with either constant or zero input) can be obtained from the partial fraction expansion of the loop transfer function G(s). It is also apparent from Eq. 2.7 that the number of poles n and zeros m of the transfer function G(s), representing the linear part of the loop, must satisfy the requirement that $m < n$. (2.8)

The above discussion shows that every set of canonic differential equations represents a closed-loop system with a single nonlinear element. Unfortunately, not all closed-loop systems with a single nonlinear element are transformable into the first canonic form. It has been shown (Rekasius [19]) that the canonic variables z_i are defined by the following equation

$$z_i = \frac{1}{\alpha_i} \frac{\begin{vmatrix} 1, & 1, & \ldots, & 1, & F_1, & 1, & \ldots, & 1 \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_{i-1}, & F_2, & \lambda_{i+1}, & \ldots, & \lambda_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-2}, & \lambda_2^{n-2}, & \ldots, & \lambda_{i-1}^{n-2}, & F_{n-1}, & \lambda_{i+1}^{n-2}, & \ldots, & \lambda_n^{n-2} \\ \lambda_1^{n-1}, & \lambda_2^{n-1}, & \ldots, & \lambda_{i-1}^{n-1}, & F_n, & \lambda_{i+1}^{n-1}, & \ldots, & \lambda_n^{n-1} \end{vmatrix}}{C} \tag{2.9}$$

where the constant C is the so-called Vandermonde determinant

$$C = \prod_{1 \leqslant j < i \leqslant n} (\lambda_i - \lambda_j) \tag{2.10}$$

and

$$F_i = \frac{d^{i-1}x}{dt^{i-1}} + \sum_{k=1}^{i-1} \sum_{j=1}^{n} \lambda_j^{i-1} \alpha_j \frac{d^{i-1-k}y}{dt^{i-1-k}} . \tag{2.11}$$

The above equation indicates that the first canonic form does not exist if the transfer function G(s) has multiple poles.

Fig. 2.4 shows the block diagram interpretation of the first canonic transformation as applied to the system of Fig. 2.3. From this diagram one can readily see that every system with only simple poles in G(s) and with the number of such poles exceeding the number of its zeros can be transformed into the first canonic form.

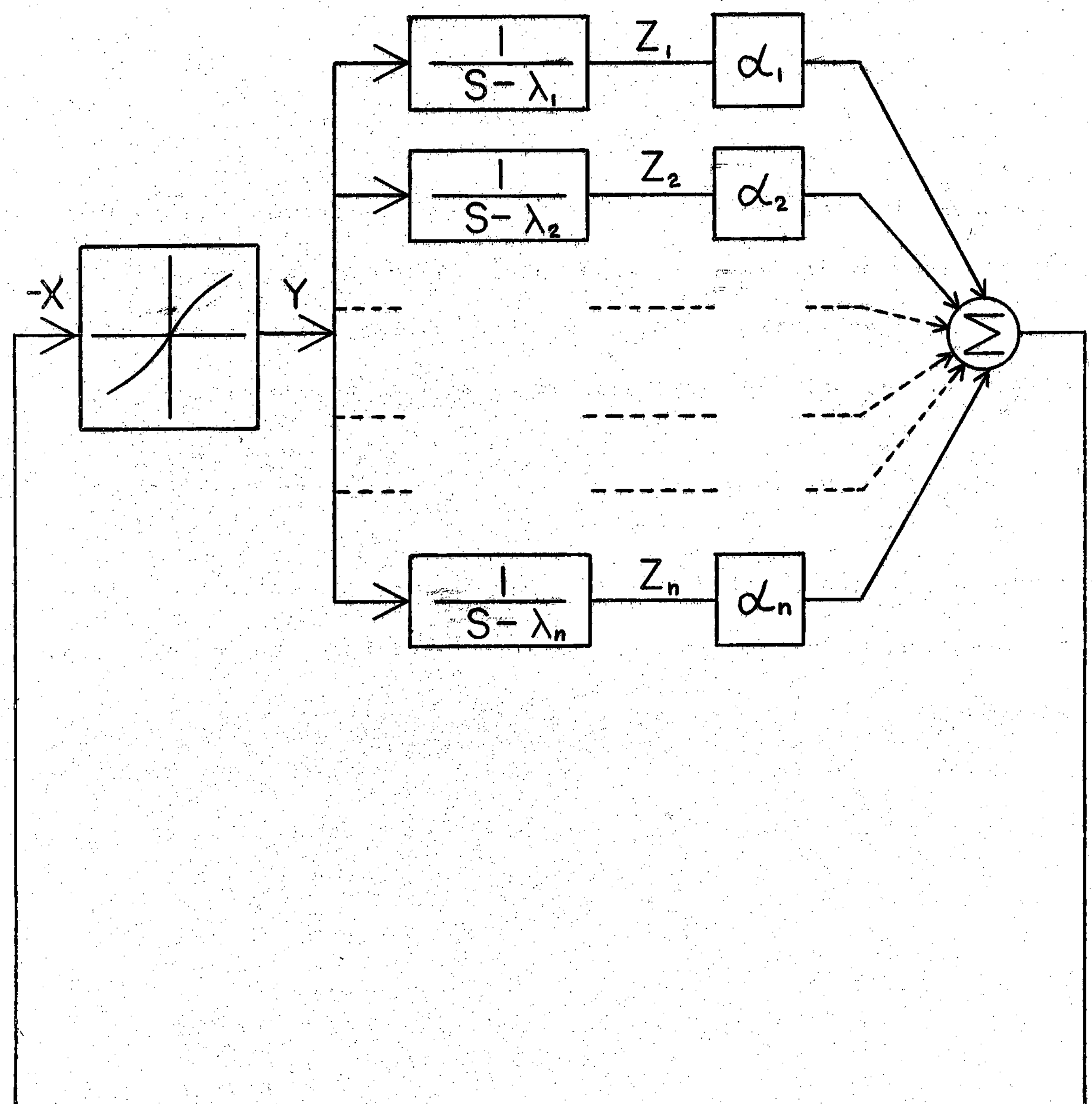

Figure 2.4

Block Diagram Representation of Canonic Transformation.

2.3 Simplified Stability Criteria

Lur'e[18] considered the function

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j z_i z_j}{\lambda_i + \lambda_j} - \int_0^x f(x)dx \qquad (2.12)$$

as a possible Liapunov's function for systems described by the first canonic form of differential equations. It can be shown (Lur'e[18], pp. 46-47) that this function is negative semidefinite if the nonlinear element characteristic satisfies the following inequality:

$$\int_0^x f(a)da \geqslant 0 \; ; \qquad (2.13)$$

provided that the constants a_i are real for corresponding real λ_i's and are in pairs of complex conjugates for corresponding complex conjugate pairs of λ_i's and that Re $\lambda_i < 0$.

The time derivative of this Liapunov's function, in connection with the first canonic form of system differential equations, is

$$\frac{dV}{dt} = rf(x)^2 + \left(\sum_{i=1}^{n} a_i z_i \right)^2$$

$$- f(x) \sum_{i=1}^{n} z_i \left(\beta_i - 2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} \right) . \qquad (2.14)$$

The time derivative of this Liapunov's function (Eq. 2.14) can be made positive semidefinite by letting

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = \beta_i \qquad i = 1, 2, \ldots n \; . \qquad (2.15)$$

Lur'e has also shown that by adding to the Liapunov's function of Eq. 2.12, the term

$$\phi = A_1 z_1^2 + A_2 z_2^2 + \ldots + A_s z_s^2 + C_1\, z_{s+1}\, z_{s+2} + C_3\, z_{s+3}\, z_{s+4} + \ldots C_{s-n-1}\, z_{n-1}\, z_n \qquad (2.16)$$

where the constants A and C are infinitesimally small negative numbers, the time derivative of the Liapunov's function (Eq. 2.14) can be made positive definite. The constants A_i are associated with the real canonic variables z_i ($i = 1, 2, \ldots s$) and the constants C_i are associated with complex canonic variables z_i($i = s, s+1, \ldots n$). Consequently, the application of Liapunov's stability theorem leads to the following stability theorem known as Lur'e's Theorem:

Theorem 2.1: (Lur'e's Theorem) If a system described by Eq. 2.1 satisfies the following conditions:

a) there exists at least one solution of a set of stability equations (Eq. 2.15) such that a_i are real for corresponding real λ_i's and are in pairs of complex conjugates for corresponding complex conjugate pairs of λ_i;

b) $\int_0^x f(a)da \geqslant 0$; $f(o) = 0$;

c) the constant $r \geqslant 0$;

d) Re $\lambda_i < 0$ for all $i = 1, 2, \ldots n$;

then the system is globally asymptotically stable. Local asymptotic stability can also be established by means of Lur'e's Theorem, if there is a range of values of x, containing the equilibrium state, over which Eq. 2.13 is satisfied.

The preceding stability equation (Eq. 2.15) may frequently reject systems that are actually stable, since it puts too many restrictions on the system. Since Lur'e's theorem represents sufficient conditions for asymptotic stability, which may not always be necessary conditions for stability, it is possible to relax the requirements of Lur'e's theorem considerably, thus making it applicable to a greater number of stable systems.

By adding to and subtracting from Eq. 2.14 the quantity

$$2 \sqrt{r}\; f(x) \sum_{i=1}^{n} a_i z_i$$

and then selecting as stability equations

$$2a_i \left(\sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - \sqrt{r} \right) = \beta_i \qquad i = 1, 2, \ldots n \tag{2.17}$$

Lur'e obtained

$$\frac{dV}{dt} = \left[\sqrt{r}\; f(x) + \sum_{i=1}^{n} a_i z_i \right]^2 . \tag{2.18}$$

Consequently, Eq. 2.17 can also be used as a stability equation in Lur'e's Theorem. In other words, the roots a_i of Eq. 2.17 can be used instead of the roots a_i of Eq. 2.15 to prove that a system is stable by the use of Lur'e's Theorem.

Lur'e also considered the function

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i\, a_j}{\lambda_i + \lambda_j} z_i z_j \tag{2.19}$$

as a possible Liapunov's function in connection with the first canonic form of differential equations and obtained the stability equation

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = \alpha_i ; \qquad i = 1, 2, \ldots n . \tag{2.20}$$

A system is asymptotically stable if:

a) the roots a_i of Eq. 2.20 satisfy the requirements of Lur'e's stability theorem,

b) Re $\lambda_i < 0$ for all $i = 1, 2, \ldots n$,

c) the nonlinear element characteristic satisfies the inequality

$$xf(x) \geqslant 0 \text{ for all } |x| > 0; \; f(o) = 0. \tag{2.21}$$

Various other simplified stability criteria (i.e., other stability equations* based on the above two Liapunov's functions as well as other V-functions) have been successfully applied to prove stability of closed-loop systems with a single nonlinear element. The books by Lur'e [4] and Letov [5] contain many examples of such simplified stability criteria. A summary of these simplified stability criteria, applicable to systems expressed in the first canonic form of differential equations, is presented in Table 2.1. Since none of these criteria represent necessary conditions for asymptotic stability, one criterion may succeed where another fails. The system may be stable even if all of these simplified criteria fail. The choice of the criterion to be tried first depends to a great extent upon one's experience and intuition.

The use of the simplified stability criteria described above will be illustrated by the following example.

Example 2.1: Consider the closed-loop system shown in Fig. 2.2.

Let the loop transfer function for this system be

$$G(s) = -\frac{X(s)}{Y(s)} = \frac{s+1}{(s+2)(s+3)(s+5)} \, .$$

---

*The solution of these stability equations for second and third order systems is given in Appendix A.

TABLE 2.1

Simplified Stability Criteria for Systems Described by the First Canonic Form of Differential Equations.

No.	Stability Equation	Conditions for Asymptotic Stability	Reference
1.	$2a_i\left[\sum_{j=1}^{n}\frac{a_j}{\lambda_i+\lambda_j}-\sqrt{r}\right]=\beta_i$ $i = 1, 2, \ldots n$	(a) The roots a_i are real for real λ_i's; a_i's are in pairs of complex conjugates for complex conjugate pairs of λ_i's. (b) Re $\lambda_i < 0$ for all $i = 1, 2, \ldots n$. (c) The constant $r \geqslant 0$. (d) The nonlinear element satisfies the inequality $\int_0^x f(x)dx \geqslant 0$, $f(o) = 0$.	(Lur'e [4], p. 50)
2.	$2a_i\left[\sum_{j=1}^{n}\frac{a_j}{\lambda_i+\lambda_j}-\sqrt{r}\right]=\beta_i+A_i$ $i = 1, 2, \ldots n$	(a), (b), (c) and (d) the same as under No. 1. (e) A_i's are any real positive constants.	(Lur'e [4], p. 50)
3.	$2a_i\sum_{j=1}^{n}\frac{a_j}{\lambda_i+\lambda_j}=\alpha_i$ $i = 1, 2, \ldots n$	(a) and (b) the same as under No. 1. (c) The nonlinear element satisfies the inequality $x\, f(x) \geqslant 0$; $f(o) = 0$.	(Lur'e [4], p. 52)

TABLE 2.1 - Continued

No.	Stability Equation	Conditions for Asymptotic Stability	Reference
4.	$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = A_i + \alpha_i$ $i = 1, 2, \dots n$	(a), (b) and (c) same as under No. 3. (d) A_i's are any real positive constants.	(Lur'e [4], p. 52)
5.	$2b_i \sum_{j=1}^{n} \frac{b_j}{\lambda_i + \lambda_j} = -\frac{\alpha_i}{\lambda_i}$ $i = 1, 2, \dots n$	(a) The roots b_i are real for real λ_i's and are in pairs of complex conjugates for complex conjugate pairs of λ_i's. (b), (c) and (d) the same as under No. 1. This criterion yields identical results as criterion No. 1.	(Rekasius [19], p. 80)
6.	$2b_i \sum_{j=1}^{n} \frac{b_j}{\lambda_i + \lambda_j} = -\frac{\beta_i + A_i}{\lambda_i^2}$ $i = 1, 2, \dots n$	(a) Same as under No. 5. (b), (c), (d) and (e) same as under No. 2. This criterion yields the same results as criterion No. 2.	This criterion can be derived from criterion No. 1 following exactly the same procedure as in the derivation of criterion No. 5.
7.	$\frac{r^2}{\sum_{i=1}^{n} \alpha_i \beta_i} > \sum_{i=1}^{n} \frac{1}{\lambda_i}$ $i = 1, 2, \dots n$	(a) The constants λ_i are real negative, i.e., $\lambda_i < 0$ for all $i = 1, 2, \dots n$. (b) The constant r is non-negative $(r > 0)$. (c) The nonlinear element characteristics satisfy the inequality $\int_0^x f(s)\, dx \geqslant 0;\ f(o) = 0.$	(Rekasius [19], p. 89)

TABLE 2.1 – Continued

No.	Stability Equation	Conditions for Asymptotic Stability	Reference
8.	$\frac{a_i^2}{\lambda_i} + \frac{2a_i\ a_{i+1}}{\lambda_i + \lambda_{i+1}} = \beta_i$ $\frac{a_{i+1}^2}{\lambda_{i+1}} + \frac{2a_i\ a_{i+1}}{\lambda_i + \lambda_{i+1}} = \beta_{i+1}$ $i = 1, 3, 5, \ldots n-1$	(a), (b), (c) and (d) same as under No. 1. (e) The order of the system n must be even.	(Letov [5], pp. 173-174)
9.	$\frac{a_i^2}{\lambda_i} + \frac{2a_i\ a_{i+1}}{\lambda_i + \lambda_{i+1}} = \alpha_i$ $\frac{a_{i+1}^2}{\lambda_{i+1}} + \frac{2a_i\ a_{i+1}}{\lambda_i + \lambda_{i+1}} = \alpha_{i+1}$ $i = 1, 3, 5, \ldots n$	Same as under No. 8, except that the nonlinear element must satisfy the inequality $x\ f(x) \geqslant 0,\ f(o) = 0.$	(Letov [5], p. 176)
10.	$\frac{a_i^2}{\lambda_i} + \frac{2a_i\ a_{i+1}}{\lambda_i + \lambda_{i+1}} - 2\sqrt{\frac{2r}{n}}\, a_i = \beta_i$ $\frac{a_{i+1}^2}{\lambda_{i+1}} + \frac{2a_i\ a_{i+1}}{\lambda_i + \lambda_{i+1}} - 2\sqrt{\frac{2r}{n}}\, a_{i+1} = \beta_{i+1}$ $i = 1, 3, 5, \ldots n-1$	Same as under No. 8.	(Letov [5], p. 176)

According to Eq. 2.1, the first canonic form of system differential equations is

$$\frac{dz_1}{dt} = -2z_1 + f(x)$$

$$\frac{dz_2}{dt} = -3z_2 + f(x)$$

$$\frac{dz_3}{dt} = -5z_3 + f(x)$$

and

$$x = 0.333z_1 - z_2 + 0.667z_3$$

$$\frac{dx}{dt} = -0.667z_1 + 3.000z_2 - 3.333z_3 + 0.000f(x) \ .$$

The time derivative of the V-function (Eq. 2.12) is found by substituting the canonic equations into Eq. 2.14:

$$\begin{aligned}\frac{dV}{dt} = {} & (a_1z_1 + a_2z_2 + a_3z_3)^2 + 2A_1z_1^2 \\ & + 3A_2z_2^2 + 5A_3z_3^2 - f(x)\Big[(A_1 - 0.667 \\ & + 0.500a_1^2 + 0.400a_1a_2 + 0.286a_1a_3)z_1 \\ & + (A_2 + 3.000 + 0.400a_1a_2 + 0.333a_2^2 \\ & + 0.250a_2a_3)z_2 + (A_3 - 3.333 + 0.286a_1a_3 \\ & + 0.250a_2a_3 + 0.200a_3^2)z_3\Big] .\end{aligned}$$

It will be observed that $\frac{dV}{dt}$ of the preceding equation can be made positive definite by setting the terms in f(x) equal to zero and by selecting the values of the constants A_1, A_2, and A_3 as sufficiently small positive numbers. Hence let

$$A_1 = A_2 = A_3 = 0.$$

Furthermore, from Eq. 2.17 one obtains

$$0.500a_1^{\;2} + 0.400a_1a_2 + 0.286a_1a_3 = 0.667,$$

$$0.400a_1a_2 + 0.333a_2^{\;2} + 0.250a_2a_3 = -3.000,$$

$$0.286a_1a_3 + 0.250a_2a_3 + 0.200a_3^{\;2} = 3.333.$$

Simultaneous solution of the above equations yield the constants

$$a_1 = +3.333,$$

$$a_2 = -12.000,$$

$$a_3 = +11.667.$$

Thus, from Eq. 2.12, the V-function is

$$V = -\int_0^x f(x)dx - 2.778z_1^{\;2} + 8.000z_1z_2 - 5.555z_1z_3$$

$$+ 24.000z_2^{\;2} + 17.500z_2z_3 - 13.611z_3^{\;2} ,$$

and, from Eq. 2.18,

$$\frac{dV}{dt} = (3.333z_1 - 12.000z_2 + 11.667z_3)^2 .$$

Since V is negative definite and $\frac{dV}{dt}$ is positive semidefinite, the system is globally stable, provided that the saturating amplifier characteristics satisfy Eq. 2.13. Furthermore, the nature of the roots a_i will not change if the constants A_1, A_2 and A_3 are chosen as sufficiently small positive numbers, thus making $\frac{dV}{dt}$ positive definite. Hence, one concludes that this system is stable if the nonlinear gain element characteristic is confined to the first and third quadrants of Fig. 2.5.

The preceding example also illustrates the following important advantages of Liapunov's second method over other methods of nonlinear system analysis:

a) the second method of Liapunov can be applied to higher order systems described by the first canonic form of differential equations, while the phase plane analysis and graphical integration methods are restricted essentially to first and second order systems.

b) Liapunov's functions used in stability analysis of systems described by the first canonic form of differential equations do not require the knowledge of the exact input-output characteristics of the nonlinear element. It is sufficient to show that the nonlinear element characteristic satisfies Eq. 2.13 and is continuous with respect to the input variable x. The describing function analysis, for example, would require a more precise knowledge of the nonlinear element characteristics.

The input-output characteristics of the nonlinear element of a closed-loop system containing a single nonlinear element were subjected to some restriction in all the simplified stability criteria considered in the previous sections. These restrictions were mathematically expressed by Eq. 2.13 or Eq. 2.21. The range of values of x over which these restrictions were satisfied by the nonlinear element determined the region of the state space of the variables z_i in which stability of a system could be proved by the simplified stability criteria. The restriction expressed by Eq. 2.21 is illustrated by the shaded region of Fig. 2.5. Eq. 2.13 imposes somewhat weaker restrictions on the input-output characteristic of the nonlinear element. These are included in the restrictions imposed by Eq. 2.21. Since the simplified stability criteria of the preceding section do not impose any restrictions on the nonlinear element characteristics other than those expressed by either Eq. 2.13 or Eq. 2.21, these simplified stability criteria cannot differentiate between systems that differ only in their nonlinear element characteristics as long as these characteristics fall within the allowable region of Eq. 2.21.

That is, if a simplified stability criterion has proved that a system is stable, then the same criterion will still prove stability if the nonlinear element is replaced by another nonlinear element whose characteristics fall within the unshaded area of Fig. 2.5.

Now it is possible to determine the reasons why such simplified criteria will reject many systems that are actually stable. By replacing the actual nonlinear element of a system by a linear element that still satisfies Eq. 2.21 (i.e., by applying simplified stability criteria to linearized systems) the analysis is not changed. The characteristics of such a linear element

$$y = kx \qquad 0 < k < \infty$$

could fall anywhere in the first and third quadrants of the input-output plane of the actual nonlinear element (Fig. 2.5). Hence, it is apparent that, in the case of a linearized system, the simplified stability criteria would select as stable only those systems that are stable for all positive values of the open-loop gain k. If the root-locus of the loop transfer function G(s) is not confined to the left-half of the s-plane, a linearized system will, for some positive values of gain, be unstable. Hence, the simplified stability criteria will reject all those systems of the type of Fig. 2.6 which have the root-locus of their loop transfer function G(s) crossing the $j\omega$-axis of the s-plane. Consequently, it is possible to predict which systems will be rejected by the simplified stability criteria of this section by inspection of the root-locus of the transfer function of the linear portion of the loop, G(s). It must be emphasized, however, that:

a) the fact that the root-locus of a system with a single nonlinear gain element is confined to the left-half of the s-plane does not imply that the system must be stable;

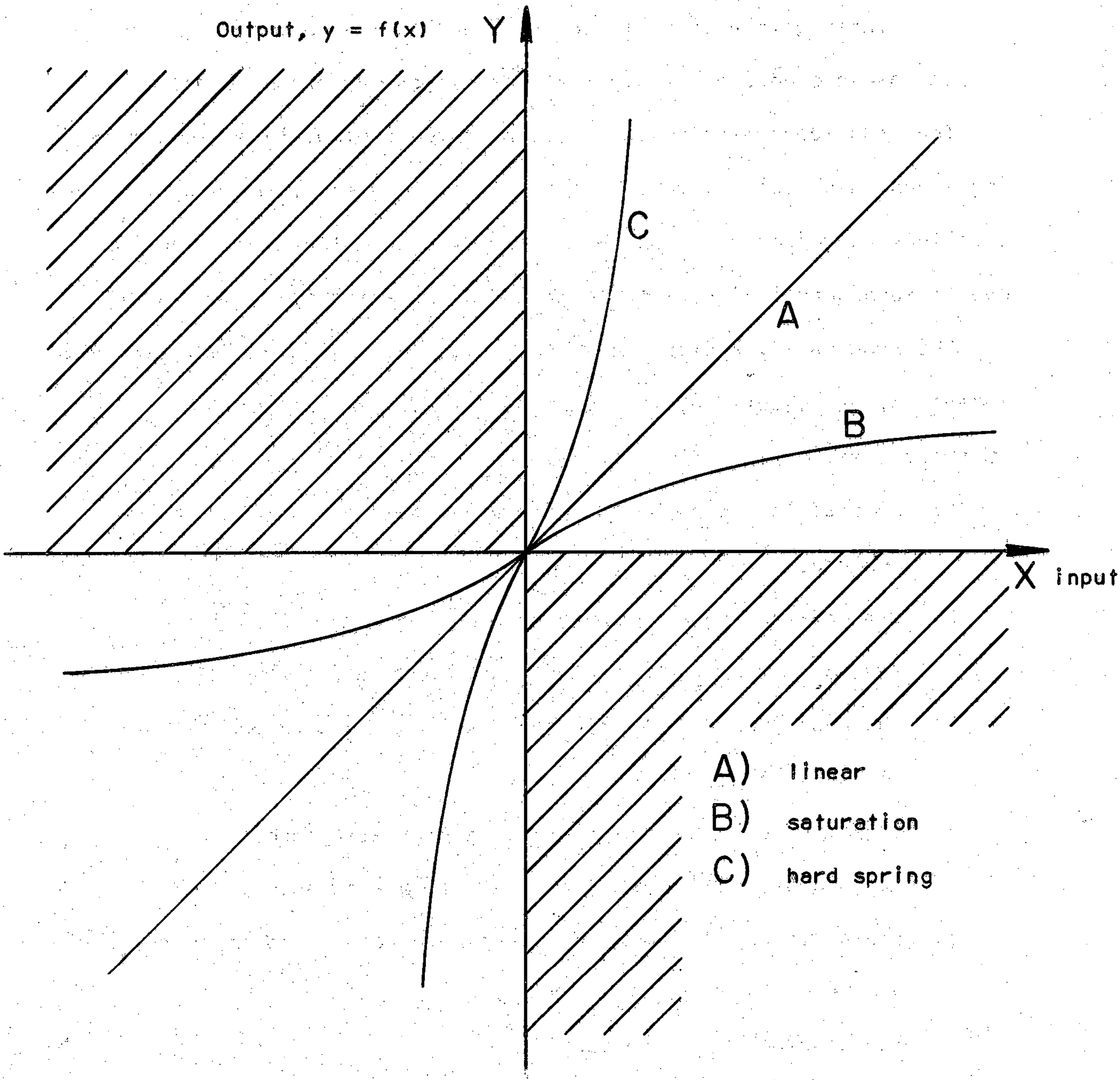

Figure 2.5

Characteristics of a Nonlinear Gain Element.

b) the fact that the root-locus of a system with a single non-linear gain element is not confined to the left-half of the s-plane does not imply that the system may not be stable.

The importance of the root-locus plot (or sketch) in systems with a single nonlinear gain element is its ability to predict which systems will definitely be rejected by the simplified Liapunov stability criteria. Several such practical systems that will be rejected are shown in Fig. 2.6.*

The reasons why a stable system, containing a single nonlinear gain element may be rejected by the simplified stability criteria can be summarized as:

1) Some of the open-loop poles are in the right-half of the s-plane.

2) Some of the open-loop zeros are in the right-half of the s-plane.

3) The root-locus of the system is not confined to the left-half of the s-plane.

4) Open-loop poles are at the origin of the s-plane.

5) Open-loop transfer function has multiple poles.

6) The difference between open-loop poles and zeros is equal or greater than 2 (i.e., $n - m \geqslant 2$).

7) The constant r is non-positive.

The above listed reasons indicate that the majority of stable linear closed-loop systems would be rejected by the simplified stability criteria based on the first canonic form of system differential equations. Hence,

*Other methods of linear system analysis, such as Nyquist Diagram could also be used for this purpose. It is more convenient, however, to use the root-locus in connection with Liapunov's Second Method which analyzes the system behavior in time rather than in frequency domain.

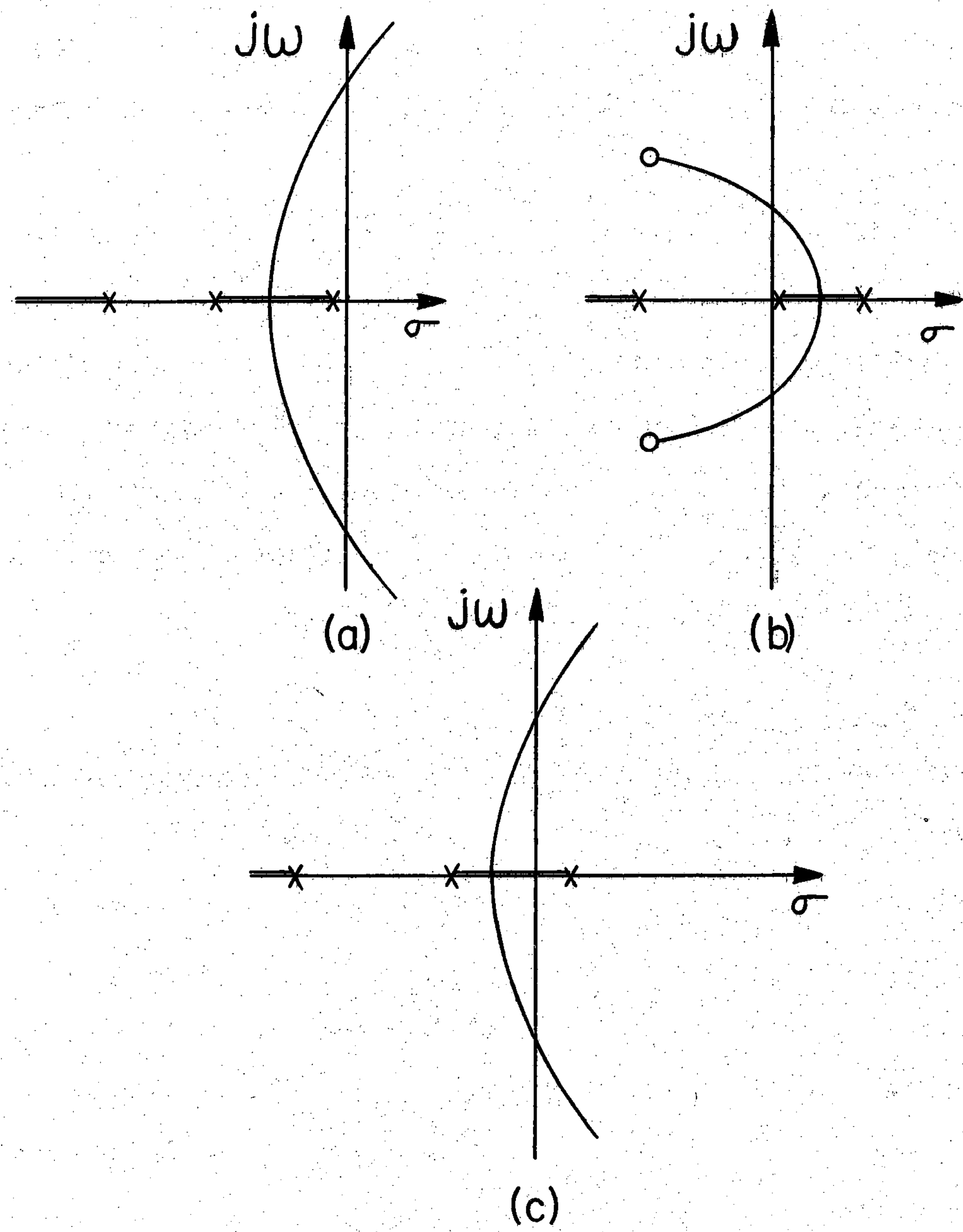

Figure 2.6

Root-loci of Third Order Systems Which Would be Rejected by the Simplified Stability Criteria:

a - Stable for Low Values of Gain

b - Stable for High Values of Gain

c - Stable for Intermediate Values of Gain.

it may be seen that the so-called advantage (b) above might be considered a disadvantage in disguise. It is essential to include a better definition of the nonlinearity than the mere fact that it is included in the first and third quadrants if heretofore excluded stable systems are to be properly identified.

It would be very desirable to modify the simplified stability criteria in order to increase their applicability (i.e., decrease the number of stable systems that these criteria reject for one or several of the reasons listed above). An obvious way to accomplish this is by restricting the gain characteristics of the nonlinear element to only a fraction of the first and third quadrants of the input-output plane of the nonlinear element, as shown in Fig. 2.5 by the dotted lines.

2.4 The Pole Shifting Technique

The purpose of the pole-shifting technique is to put restrictions on the minimum gain of the nonlinear element in order that the simplified stability criteria will no longer reject stable systems whose gain does not fall below such a minimum value. In order to accomplish this, the horizontal (input) axis of the input-output characteristic plane of the nonlinear element is rotated in the counterclockwise direction through an angle ϕ. The rotation of the input (x)-axis is equivalent to the change from the original output variable y to a new variable y', defined as

$$y' = g(x) = y - C_p x \tag{2.22}$$

where C_p is a real constant and determines the angle ϕ of rotation of the horizontal axis in the input-output plane of the nonlinear element. This change in the variable y, representing the output of the nonlinear element, is illustrated graphically in Fig. 2.7 for a positive value of the constant C_p. The angle ϕ is expressed as

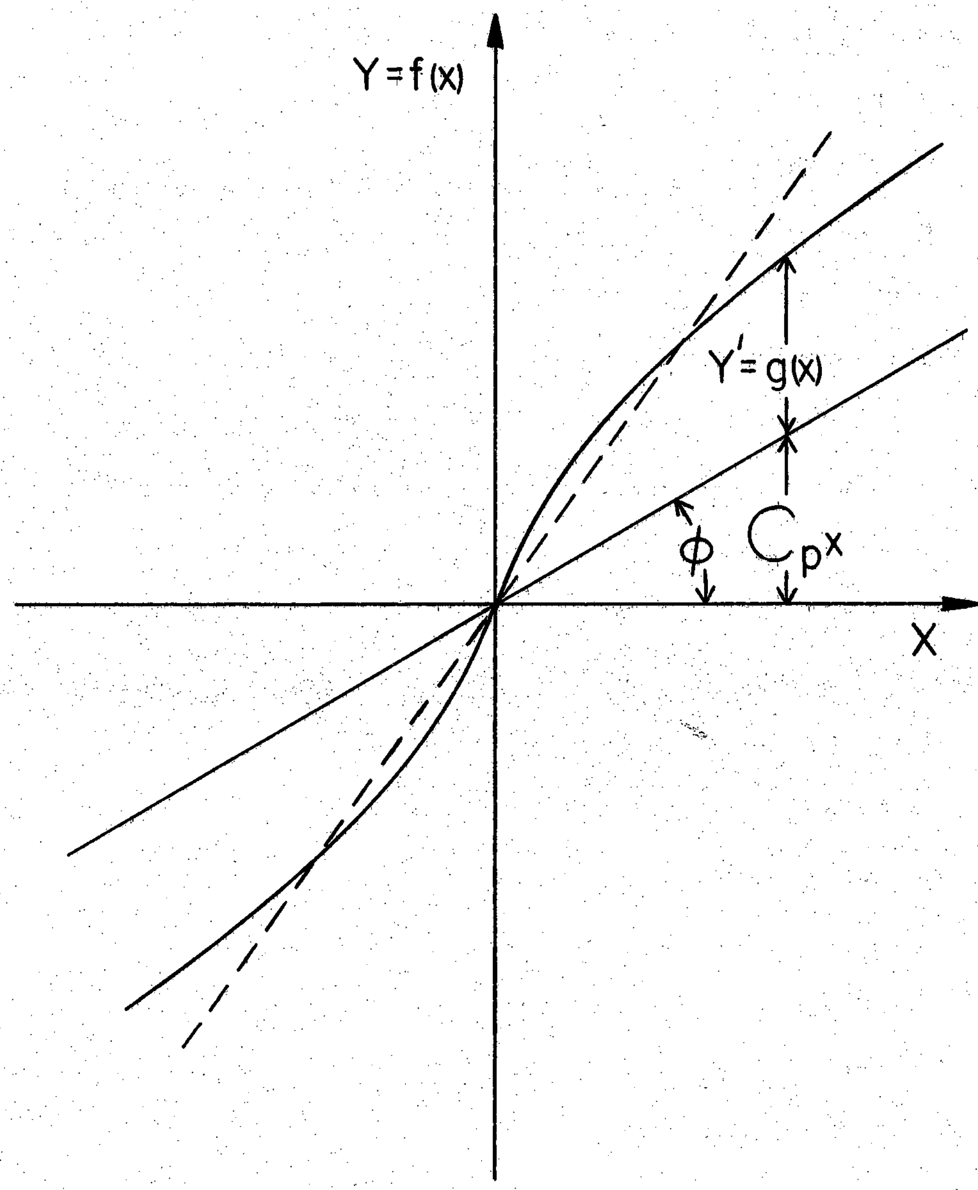

Figure 2.7

Illustration of Pole-Shifting Technique.

$$\phi = \arctan C_p \ . \tag{2.23}$$

The maximum value of the angle ϕ, and consequently, the maximum value of C_p, is determined by the angle through which the horizontal axis of the x-y plane (Fig. 2.7) can be rotated before intersecting the nonlinear element characteristic curve.

The new variable y' will be used in the first canonic form of system differential equations and thus will be contained in the equations of the simplified stability criteria.

To accomplish the purpose of the pole-shifting technique (i.e., to limit the minimum value of nonlinear element gain) this new variable must satisfy the inequality

$$x\ g(x) \geqslant 0; \quad g(0) = 0 \tag{2.24}$$

in the region of the state space in which stability can be proved by the simplified stability criteria of Section 2.3. In the case of global stability, the above inequality must hold in the entire state space of the variables z_i. It is important to note that the original system variable x, representing the input into the nonlinear element, is retained in the new canonic equations resulting from the pole-shifting. The simplified criteria based on the canonic equations prove the existence of a Liapunov's function of the variable x and its time derivatives. Thus, it is obvious that the proofs of stability based upon the new canonic variables z_i after the pole-shifting are still valid as long as Eq. 2.24 is satisfied.

The original transfer function of the linear portion of the loop

$$G(s) = -\frac{X(s)}{Y(s)} = \frac{s^m + a_{m-1}\ s^{m-1} + \dots + a_1\ s + a_o}{s^n + b_{n-1}\ s^{n-1} + \dots + b_1\ s + b_o}\ ; \quad n > m \tag{2.25}$$

is changed, as the result of the change in the variable y caused by the pole-shifting procedure, to

$$G'(s) = -\frac{X(s)}{Y'(s)}$$

$$= \frac{s^m + a_{m-1}\, s^{m-1} + \dots + a_1\, s + a_o}{s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_o + C_p(s^m + a_{m-1} s^{m-1} + \dots + a_o)}\ ; \quad n > m$$

(2.26)

or

$$C_p G'(s) = \frac{C_p G(s)}{1 + C_p G(s)}\ . \tag{2.27}$$

Eq. 2.27 makes it possible to interpret the effect of pole-shifting procedures on the root-locus. It will be observed that a change in the numerical value of the constant C_p does not affect the zeros ω_i of the transfer function G'(s). It does, however, affect the poles λ_i of the transfer function G'(s) in an exactly analogous way as the change in the value of open-loop gain. The poles of the loop transfer function G'(s) move, for increasing positive value of C_p, in the s-plane along the root-locus corresponding to increasing loop gain. The root-locus for negative values of the constant C_p corresponds to root-locus of negative loop-gain (positive feedback).

It is obvious that an arbitrarily small change in the constant C_p will separate any multiple poles of the transfer function. Thus, the pole-shifting technque enables one to perform canonic transformations for systems with multiple open-loop poles. Also, if the system is open-loop unstable (open-loop poles on the jω-axis or in the right-half of the s-plane), it may be possible to move the poles of G'(s) into the left-half of the s-plane by a suitable choice of the constant C_p.

The canonic transformation of the new loop transfer function $G'(s)$ is performed in the same manner as discussed previously for the original loop transfer function $G(s)$. The coefficients λ_i and α_i of the new first canonic form of system differential equations obtained from the new loop transfer function $G(s)$ can be determined from the root-locus of $G'(s)$. The root-locus of $G'(s)$ can be constructed by using the well-known techniques from linear control system theory. In the construction of the root-locus of $G'(s)$, the coefficient C_p is treated as a loop gain in the construction of root-loci for linear systems. It must, however, be emphasized that the root-locus of $G'(s)$ is due entirely to the linear transformation (Eq. 2.22) defining the new variable y' while the actual gain of the system does not vary.

In many cases an arbitrarily small positive value of C_p will violate Eq. 2.23 for sufficiently large absolute values of input variable x. Examples of nonlinear characteristics that may not admit any positive values of C_p due to restriction of Eq. 2.23 are perfect saturation, negative resistance characteristics of vacuum tubes, etc. In such cases a small negative value of C_p may be used to separate multiple poles, provided none of the poles are close to the $j\omega$-axis or in the right-half of the s-plane. The application of pole-shifting technique to prove stability of systems that would be rejected by the simplified stability criteria of Section 2.2 without the pole-shifting is illustrated by the following example.

Example 2.2: Consider a nonlinear system shown in Fig. 2.3 with

$$G(s) = \frac{s + 3}{6(s^2 + 2s - 1)}$$

and the nonlinear element with the hard-spring characteristics such that

$$|f(x)| > |6x| \quad \text{for all} \quad |x| > 0, \quad f(0) = 0 .$$

This means that the x-axis of the nonlinear element characteristic plane can be rotated by an angle ϕ, where

$$C_p = \arctan \phi \leq 6 ,$$

before intersecting the nonlinear element input-output characteristic curve (see Fig. 2.7). It is obvious that the first canonic transformation of G(s) cannot be performed directly because of the poles of G(s) in the right-half s-plane. Hence, it is necessary to apply the pole-shifting technique to this system before simplified stability criteria can be used to prove stability.

Selecting $C_p = 6$ and substituting into Eq. 2.26 one finds

$$G'(s) = \frac{X(s)}{Y(s)} = \frac{s + 3}{6(s + 1)(s + 2)}$$

Application of the first canonic transformation to G'(s) yields the following canonic equations:

$$\frac{dz_1}{dt} = - z_1 + g(x)$$

$$\frac{dz_2}{dt} = - 2z_2 + g(x)$$

$$x = - 0.333z_1 + 0.167z_2 .$$

Thus

$$\lambda_1 = 1 ; \lambda_2 = 2 ; \alpha_1 = 0.333 ; \alpha_2 = - 0.167 .$$

Applying these values to the simplified stability criterion (Eq. 2.20), one obtains

$$a_1 = 1 ; \quad a_2 = - 1 .$$

Hence the requirements of Lur'e's Theorem are satisfied and, consequently, this system is globally asymptotically stable.

The above example illustrates the procedure of pole-shifting and certain of its advantages. It enabled one to prove stability of a system which contained poles of G(s) with positive real parts and thus was not applicable directly to any one of the simplified stability criteria.

2.5 The Zero-Shifting Technique

A procedure, similar to the pole-shifting technique, is proposed in this section to shift the zeros of the transfer function of the linear portion of the loop, G(s). The purpose of the zero-shifting technique is to put restrictions on the maximum gain of the nonlinear element in order that the simplified stability criteria will no longer reject stable systems whose gain does not exceed the maximum value. In order to accomplish this, the vertical (output) axis of the input-output characteristic plane of the nonlinear element is rotated in the clockwise direction through an angle Θ, as illustrated in Fig. 2.8. This rotation of the output (y)-axis is equivalent to the change from the original input variable x to a new variable x', defined as

$$x' = h(y) = x - C_z y \,, \tag{2.28}$$

where C_z is a real positive constant and determines the angle Θ of rotation of the vertical axis in the input-output plane of nonlinear element. This angle Θ is expressed as

$$\Theta = \arctan C_z \,. \tag{2.29}$$

The maximum value of the angle Θ and, consequently, the maximum value of C_z are determined by the angle through which the vertical axis of the x-y plane (Fig. 2.8) can be rotated before intersecting the nonlinear element characteristic curve. The actual value of the constant C_z must be just less than this maximum value determined above. This will make the new function $f(x')$, which represents the output, y, of the nonlinear element

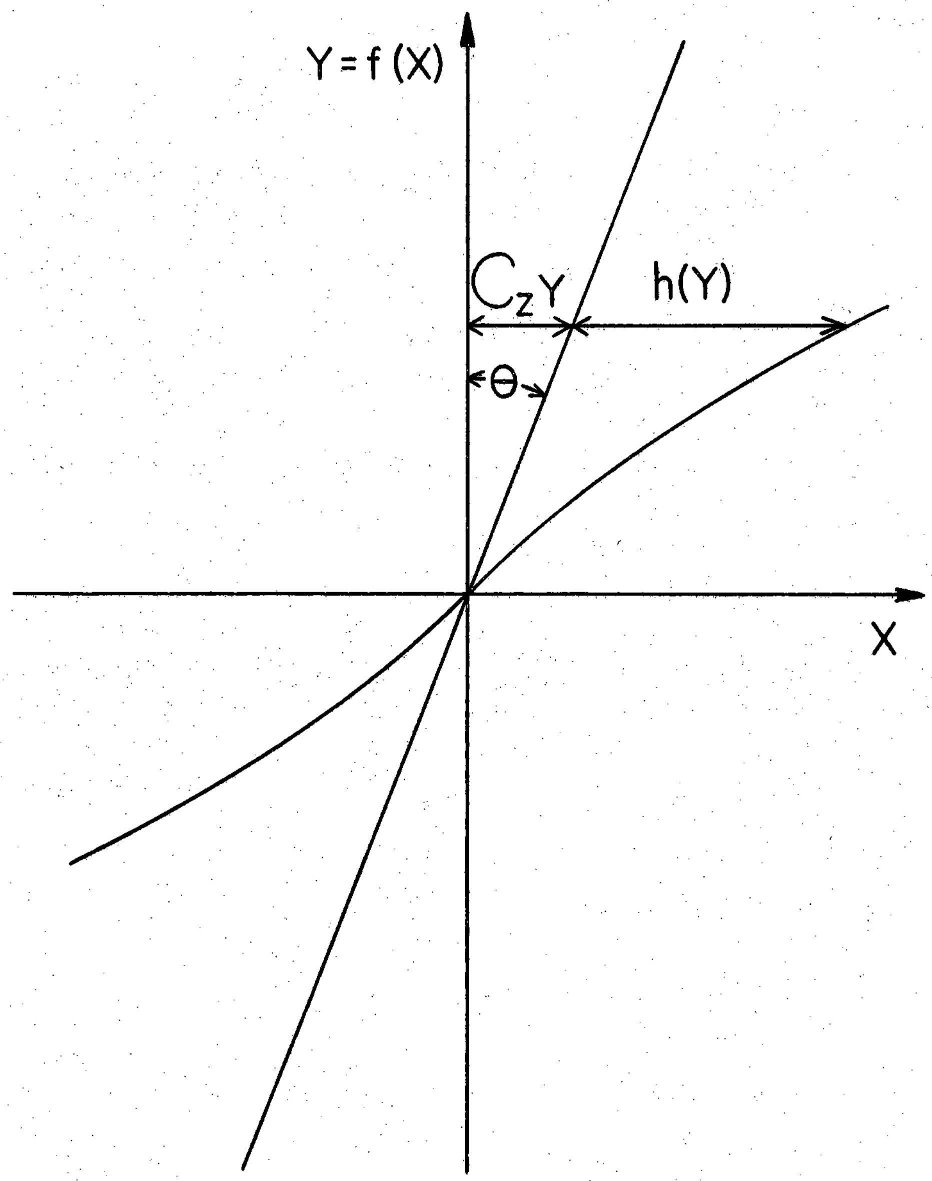

Figure 2.8

Illustration of the Zero-Shifting Technique.

in terms of the new variable x', single-valued for sufficiently small absolute values of the variable x'. Theoretically, the constant C_z could also be a negative number. However, this would result in zeros of the loop transfer function being added in the right-half of the s-plane. Hence, negative values of C_z would yield canonic forms that are unsuitable for stability investigation.

The new variable x' will be used in the first canonic form of the system differential equations and thus will be contained in the stability equations of the simplified stability criteria. Consequently, to accomplish the purpose of the zero-shifting technique (i.e., to limit the maximum value of the nonlinear element gain) this new variable x' must satisfy the inequality

$$x' f(x') \geqslant 0, \quad f(0) = 0 \tag{2.30}$$

in the region of its phase space in which stability can be proved by the simplified stability criteria of this section. In the case of global stability, the above inequality must hold in the entire phase space of the variable x'.

As a result of the change in the variable x, the original transfer function G(s) of the linear portion of the loop (Eq. 2.25) is changed to

$$G'(s) = -\frac{X'(s)}{Y(s)} =$$

$$= \frac{s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_o + C_z(s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o)}{s^n + b_{n-1}\, s^{n-1} + \ldots + b_1\, s + b_o}\ ; \quad n > m\,. \tag{2.31}$$

Eq. 2.31 indicates that the clockwise rotation of the vertical axis in the input-output plane of the nonlinearity introduces additional zeros in the loop transfer function, such that the new transfer function contains an equal number of poles and zeros. It also introduces a scale factor C_z in the new loop transfer function $G'(s)$. Consequently, the new transfer function (Eq. 2.31) cannot be transformed into the first canonic form of system differential equations. The scale factor C_z of this new transfer function could be incorporated in the characteristics of the nonlinear element. However, the transfer function would still violate the requirement that the number of poles of a transfer function be greater than the number of its zeros in order that the transfer function be transformable into the first canonic form of system differential equations. To overcome this difficulty, and hence to retain the advantages of simplified stability criteria based on the first canonic form of system differential equations, requires some modifications in the first canonic form of differential equations. Let the modified first canonic form of system differential equations be

$$\frac{dz_i}{dt} = \lambda_i z_i + f(x') \qquad i = 1, 2, \ldots n \tag{2.32}$$

$$x' = \sum_{i=1}^{n} \alpha_i z_i - r_o f(x') \tag{2.33}$$

and

$$\frac{dx'}{dt} = \sum_{i=1}^{n} \beta_i z_i - r_o \frac{d\, f(x')}{dt} - r_1\, f(x') \tag{2.34}$$

where

$$y = f(x') \tag{2.35}$$

represents the characteristics of the nonlinear element after the zero-shifting. Rewriting the above equations in operational notation and then substituting Eq. 2.32 and Eq. 2.35 into Eq. 2.33, one obtains

$$\frac{X'}{Y} = \sum_{i=1}^{n} \frac{\alpha_i}{D - \lambda_i} - r_o \,. \tag{2.36}$$

Let

$$C_z = r_o \, . \tag{2.37}$$

Then, substitution of Eq. 2.37 and Eq. 2.28 into Eq. 2.36 yields

$$\frac{x}{Y} = \sum_{i=1}^{n} \frac{\alpha_i}{D - \lambda_i} \tag{2.38}$$

or

$$G(s) = - \frac{X(s)}{Y(s)} = - \sum_{i=1}^{n} \frac{\alpha_i}{s - \lambda_i} \, . \tag{2.39}$$

Consequently, the coefficients λ_i, α_i, β_i, and r_1 in the modified canonic form of system differential equations are obtained from the original loop transfer function G(s) and do not change due to zero-shifting. The constant r_o can be computed from Eq. 2.29 as

$$r_o = \tan \Theta \tag{2.40}$$

where Θ is the angle of rotation of the y-axis in the input-output plane of the nonlinear element.

It is necessary to modify the simplified stability criteria if these criteria are to be used with the modified canonic form of system differential equations. The remainder of this section will be devoted to the development of such modified stability criteria.

The function

$$V = - \int_0^{x'} f(x)dx + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i \, a_j \, z_i \, z_j}{\lambda_i + \lambda_j} \tag{2.41}$$

cannot serve as a Liapunov's function in connection with the modified canonic form of system differential equations, since the time derivative of the variable x' depends upon the slope of the nonlinear element characteristic curve. This becomes apparent if Eq. 2.34 is rewritten as

$$\frac{dx'}{dt} = \frac{\sum_{i=1}^{n} \beta_i z_i - r_1 f(x')}{1 + r_o \frac{d f(x')}{dx'}} . \tag{2.42}$$

Substitution of the above equation into the time-derivative of the assumed V-function (Eq. 2.41) yields

$$\frac{dV}{dt} = \frac{r_1 f^2(x')}{1 + r_o \frac{d f(x')}{dx'}} + (\sum_{i=1}^{n} a_i z_i)^2$$

$$- f(x') \left[\sum_{i=1}^{n} z_i (\frac{\beta_i}{1 + r_o \frac{d f(x')}{dx'}} - 2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j}) \right] . \tag{2.43}$$

Previously stability equations were obtained from Liapunov's functions which are analogous to Eq. 2.41 by setting the last term of the time derivative of such Liapunov's functions (which are analogous to the last term of Eq. 2.43) equal to zero. Such a procedure of obtaining stability equations is not applicable in this case, since the last term of Eq. 2.43 is not constant, but rather depends upon the slope of the nonlinear element characteristic curve. The V-function of Eq. 2.41 was considered here only to show that every V-function that contains the variable x' explicitly will be subject to the same weakness and hence cannot be used as a Liapunov's function in connection with the modified canonic form of system differential equations.

Consider next the function*

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i a_j z_i z_j}{\lambda_i + \lambda_j} \tag{2.44}$$

as a possible Liapunov's function to be used in connection with the

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*A suitably small F-term (Eq. 2.16) may be added to this Liapunov's function to prove asymptotic stability of a system.

modified canonic form of system differential equations. Differentiation of Eq. 2.44 with respect to time and substitution of Eq. 2.32 yields

$$\frac{dV}{dt} = (\sum_{i=1}^{n} a_i z_i)^2 + 2f(x') \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i \; a_j \; z_i}{\lambda_i + \lambda_j} . \qquad (2.45)$$

By adding to and subtracting from Eq. 2.45, the quantity

$$f(x') \sum_{i=1}^{n} \alpha_i z_i - r_o f(x') + 2\sqrt{r_o} \sum_{i=1}^{n} a_i z_i$$

one obtains

$$\frac{dV}{dt} = (r_o f(x') + \sum_{i=1}^{n} a_i z_i)^2 + x' f(x')$$

$$+ f(x') \sum_{i=1}^{n} z_i(-2\sqrt{r_o}\; a_i + 2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - \alpha_i) . \qquad (2.46)$$

A set of stability equations may be obtained from Eq. 2.46 by setting its last term equal to zero, i.e.,

$$2a_i \left(\sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} - \sqrt{r_o} \right) = \alpha_i \qquad i = 1, 2, \ldots n . \qquad (2.47)$$

Consider next the function*

$$V = \sum_{i=1}^{n} \frac{a_i^2 z_i^2}{2\lambda_i} + \frac{a_{i+1}^2 z_{i+1}^2}{2\lambda_{i+1}} + \frac{2a_i a_{i+1} z_i z_{i+1}}{\lambda_i + \lambda_{i+1}} ; \quad i = 1, 3, 5, \ldots n-1 \qquad (2.48)$$

as a possible Liapunov's function for systems described by the modified canonic form of differential equations. Its time derivative in connection with the modified canonic form of system differential equations is

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*For systems of even order (n even). If a system is of an odd order, a term $1/2kz_n^2$, where k is a negative constant, may be added to Eq. 2.48.

$$\frac{dV}{dt} = \sum_{i=1}^{n-1} (a_i z_i + a_{i+1} z_{i+1})^2$$

$$+ f(x') \sum_{i=1}^{n-1} \left(\frac{a_i^2}{\lambda_i} + \frac{2a_i a_{i+1}}{\lambda_i + \lambda_{i+1}}\right) z_i$$

$$+ f(x') \sum_{i=1}^{n-1} \left(\frac{a_{i+1}^2}{\lambda_{i+1}} + \frac{2a_i a_{i+1}}{\lambda_i + \lambda_{i+1}}\right) z_{i+1} \qquad i = 1, 3, 5, \ldots n-1 \tag{2.49}$$

By adding to and subtracting from Eq. 2.49 the quantity

$$f(x')\left[\sum_{i=1}^{n-1} \alpha_i z_i + \sum_{i=1}^{n-1} \alpha_{i+1} z_{i+1} - r_o f(x') + 2\sqrt{\frac{2r_o}{n}} \left(\sum_{i=1}^{n-1} a_i z_i + \sum_{i=1}^{n-1} a_{i+1} z_{i+1}\right)\right] \qquad i = 1, 3, 5, \ldots n-1$$

one obtains

$$\frac{dV}{dt} = \sum_{i=1}^{n-1} \left[a_i z_i + a_{i+1} z_{i+1} + \sqrt{\frac{2r_o}{n}}\; f(x')\right]^2 + x' f(x')$$

$$+ f(x') \sum_{i=1}^{n-1} \left(\frac{a_i^2}{\lambda_i} + \frac{2a_i a_{i+1}}{\lambda_i + \lambda_{i+1}} - 2\sqrt{\frac{2r_o}{n}}\; a_i - \alpha_i\right) z_i$$

$$+ f(x') \sum_{i=1}^{n-1} \left(\frac{a_{i+1}^2}{\lambda_{i+1}} + \frac{2a_i a_{i+1}}{\lambda_i + \lambda_{i+1}} - 2\sqrt{\frac{2r_o}{n}}\; a_{i+1} - \alpha_{i+1}\right) z_{i+1}$$

$$i = 1, 3, 5, \ldots n-1 \tag{2.50}$$

A set of stability equations may be obtained from Eq. 2.50 by setting its last two terms equal to zero, i.e.,

$$\frac{a_i^2}{\lambda_i} + \frac{2a_i a_{i+1}}{\lambda_i + \lambda_{i+1}} - 2\sqrt{\frac{2r_o}{n}}\; a_i = \alpha_i \qquad i = 1, 3, 5, \ldots n-1 \tag{2.51}$$

and

$$\frac{a_{i+1}^2}{\lambda_{i+1}} + \frac{2a_i a_{i+1}}{\lambda_i + \lambda_{i+1}} - 2\sqrt{\frac{2r_o}{n}}\; a_{i+1} = \alpha_{i+1} \qquad i = 1, 3, 5, \ldots\ n-1 \tag{2.52}$$

The following simplified stability criterion for systems expressed by the modified canonic form of their differential equations (after performing the zero-shifting procedure) can be formulated from the stability equations developed in this section:

Theorem 2.2: A disturbed system, described by Eq. 2.32 and Eq. 2.33, is asymptotically stable in the region of state space in which the inequality

$$x' \ f(x') \geqslant 0 \ ; \quad f(0) = 0$$

is satisfied, provided that:

a) Re λ_i 0 for all λ_i

b) $r_o \geqslant 0$

c) The roots of a set of stability equations a_i are real for real corresponding λ_i's and are in pairs of complex conjugates for corresponding complex conjugate pairs of λ_i's.

The function $y = f(x')$ in the preceding theorem represents the characteristics of the nonlinear element after the zero-shifting (i.e., after rotation of the vertical axis of Fig. 2.8 by the angle Θ). The stability equations that can be used in the above theorem are either Eq. 2.47 or Eqs. 2.51 and 2.52. In order to prove the above theorem it will be assumed that the conditions a) through c) of the theorem are satisfied by a system described by Eqs. 2.32 and 2.33. Then the V-function from which the particular set of stability equations was derived is a negative definite function. Conditions a) through c) of the theorem also cause the time derivative of the V-function, from which the particular set of stability equations was derived, to be a positive definite function in the region of phase space of the variable in which

$$x' f(x') \geqslant 0 .$$

Consequently, according to Theorem 1.1, the system is asymptotically stable in the above region of the phase space.

The above theorem represents sufficient but not necessary conditions for asymptotic stability. Hence, it still may reject some systems that were previously rejected because the root-locus of the system transfer function G(s) enters the right-half of the s-plane for sufficiently large values of gain. The application of the zero-shifting technique and the associated simplified stability criterion (Theorem 2.2) is illustrated by means of an example.

Example 2.3: Consider a nonlinear system shown in Fig. 2.3 with

$$G(s) = \frac{(s-1)}{(s+1)(s+2)}$$

and the nonlinear element with saturation characteristic such that

$$|f(x)| \leqslant |2x| \quad \text{for all} \quad |x| > 0 \; ; \quad f(0) = 0 .$$

This system is unstable for high values of loop gain. However, after performing the zero-shifting procedure, Eq. 2.31 gives the new transfer function

$$G'(s) = -\frac{X'(s)}{Y(s)} = \frac{C_z(s+1)(s+2) + (s-1)}{(s+1)(s+2)}$$

or

$$G'(s) = \frac{(s-\omega_1)(s-\omega_2)}{(s+1)(s+2)} ,$$

where

$$\omega_{1,2} = -\frac{3}{2} - \frac{1}{2C_z} \pm \sqrt{\left(\frac{3}{2} + \frac{1}{2C_z}\right)^2 - 2 + \frac{1}{C_z}} .$$

Hence, if the constant C_z satisfies the inequality

$$C_z \geqslant 0.5 ,$$

all the poles and zeros of G(s) will be confined to the left-half of the s-plane, and, consequently, a simplified stability criterion may be applicable. This means that stability for this system could be established by means of the simplified criteria only if the non-linear element gain (including the scale factor of the loop transfer function) does not exceed the value of 2, i.e.,

$$|f(x)| \leqslant |2x| .$$

The modified canonic form for this system is obtained from Eqs. 2.32 and 2.33 as

$$\frac{dz_1}{dt} = - z_1 + f(x')$$

$$\frac{dz_2}{dt} = - 2z_2 + f(x')$$

and

$$x' = 2z_1 - 3z_2 - r_o f(x')$$

The stability equation (Eq. 2.47) yields the following roots:

$$a_1 = - \sqrt{r_o} - 2\sqrt{r_o - 0.5} + \left[(\sqrt{r_o} + 2\sqrt{r_o - 0.5})^2 + 6 \right]^{\frac{1}{2}}$$

$$a_2 = + 4\sqrt{r_o} + 2\sqrt{r_o - 0.5} + \left[(4\sqrt{r_o} + 2\sqrt{r_o - 0.5})^2 + 18 \right]^{\frac{1}{2}} .$$

Consequently, this system is stable as long as $r_o = C_z > 0.5$, or as long as the nonlinear element characteristics satisfy the inequality

$$|f(x)| < |2x| .$$

Hence, this system is stable as long as the root-locus of its loop transfer function G(s) is confined to the left-half of the s-plane. In general, however, a system need not be stable even if its root-locus is confined to the left-half of the s-plane and an application of simplified stability criteria may impose more severe restrictions on the equivalent loop gain.

The preceding example illustrates the fact that the zero-shifting technique and the associated simplified stability criteria represent powerful tools for stability analysis. They may be used to prove the stability of systems with a single nonlinear gain element that are stable for low values of gain only. If a system is stable for some intermediate values of equivalent loop gain and unstable for both low and high values of equivalent gain, it may be possible to prove stability by the application of both pole- and zero-shifting techniques. In such cases it is advantageous to apply the pole-shifting technique first, since the zero-shifting technique modifies the canonic form of system differential equations, and, consequently, the formulae used to perform the pole-shifting are no longer applicable after the application of the zero-shifting technique. The simultaneous application of both pole- and zero-shifting techniques is illustrated by the following example.

Example 2.4: Consider a nonlinear system shown in Fig. 2.2 with

$$G(s) = \frac{(s-1)}{(s+1)^2}$$

and the nonlinear gain element whose input-output characteristic satisfies the inequality

$$|0.5x| \leqslant |f(x)| \leqslant |5x| \quad \text{for all} \quad |x| > 0, \quad f(0) = 0 .$$

The simplified stability criteria cannot be applied to this system because G(s) has a double pole. It is necessary to use the pole-shifting technique to separate this double pole. The zero of G(s) in the right-half of the s-plane indicates that a linearized system will be unstable for high values of gain. Thus, pole-shifting must be employed to limit the maximum equivalent nonlinear element gain.

The nonlinear element characteristic applied to Eq. 2.23 yields the maximum allowable value of 0.5 for the constant C_p. With this value of C_p, Eq. 2.26 yields the new transfer function

$$G'(s) = \frac{(s-1)}{(s+2.28)(s+0.22)} .$$

The maximum value of the constant C_z (= r_o) is obtained from Eq. 2.40 as 5.0. Hence the modified canonic form of the system differential equations is obtained from Eqs. 2.32 - 2.33 as

$$\frac{dz_1}{dt} = -2.28z_1 + f(x') ,$$

$$\frac{dz_2}{dt} = -0.22z_1 + f(x') ,$$

and

$$x' = -1.592z_1 + 0.592z_2 - 5.000f(x') .$$

Applying the stability equations (Eq. 2.47) to the above canonic equations, one obtains

$$0.438a_1^2 + 0.800a_1a_2 + 4.470a_1 = 1.592$$

and

$$4.495a_2^2 + 0.800a_1a_2 + 4.470a_2 = -0.592 .$$

Simultaneous solution of these stability equations yields

$$a_1 = 0.400$$

and

$$a_2 = 0.174 .$$

Consequently, according the Theorem 2.2, this system is globally asymptotically stable.

The preceding example illustrates how the stability of a system can be proved by utilizing both the pole- and zero-shifting techniques.

2.6 Analysis by Means of the Second Canonic Form of System Differential Equations

Much less attention has been devoted in the literature to the second canonic form of system differential equations than to the first canonic form. Letov ([5], p. 101) points out that the second canonic form of system differential equations is useful in the stability analysis of systems that contain multiple poles in their loop transfer function G(s), and also in systems which are "inherently unstable" (i.e., in systems which are open-loop unstable since some of their open-loop poles lie in the right-half of the s-plane).

An answer to the question, what systems possess the second canonic form of their differential equations, could not be found in the literature. Hence, an attempt to establish the applicability of the second canonic form to nonlinear closed-loop systems is made in this section.

The second canonic form of differential equations for the disturbed system is

$$\frac{dz_i}{dt} = \omega_i z_i + x \qquad i = 1, 2, \ldots m , \tag{2.53}$$

$$\frac{dx}{dt} = \sum_{i=1}^{m} \gamma_i z_i + \delta x - f(x) , \tag{2.54}$$

where ω_i are the open-loop zeros of system transfer function, γ_i and δ are constants to be defined later and

$$y = f(x)$$

represents the input-output characteristics of the nonlinear element with x representing the input and y the output of the nonlinear element.

Eq. 2.53 and Eq. 2.54 can be rewritten in operational notation as

$$(D - \omega_i) z_i = x \qquad i = 1, 2, \ldots m \tag{2.55}$$

and

$$Dx = \sum_{i=1}^{m} \gamma_i z_i + \delta x - y \ . \tag{2.56}$$

Solution of Eq. 2.55 for z_i and substitution into Eq. 2.56 yields

$$\left[(D - \delta) \prod_{i=1}^{m} (D - \omega_i) - \sum_{i=1}^{m} \gamma_i \prod_{\substack{j=1 \\ j \neq i}}^{m} (D - \omega_j) \right] x$$

$$= - \prod_{i=1}^{m} (D - \omega_i) y \ . \tag{2.57}$$

If the operator D in Eq. 2.57 is replaced by the Laplace transform variable s, the transfer function G(s) of the linear portion of the loop for the system represented by the second canonic form of differential equations is obtained as

$$G(s) = - \frac{X(s)}{Y(s)} = \frac{\prod_{i=1}^{m} (s - \omega_i)}{(s - \delta) \prod_{i=1}^{m} (s - \omega_i) - \sum_{i=1}^{m} \gamma_i \prod_{\substack{j=1 \\ j \neq i}}^{m} (s - \omega_j)} \ . \tag{2.58}$$

Inspection of this transfer function (Eq. 2.58) reveals that the number of its poles m is related to the number of zeros by the equality

$$n = m + 1 \ . \tag{2.59}$$

Eq. 2.59 represents a necessary condition for a closed-loop system with a single nonlinear element, shown in Fig. 2.2, to be transformable into the second canonic form. This restriction to the applicability of the second canonic form of differential equations, imposed by Eq. 2.59 limits the use of second canonic form in stability analysis to only a small fraction of single-loop, single nonlinear element feedback systems. In the case of linear systems, however, Eq. 2.59 indicates that the second canonic form of system differential equations is applicable to those systems which will very likely be stable. It thus appears intuitively that nonlinear systems which possess the second canonic form of differential equations are also very likely to be stable. This is why the discussion of the use of the second canonic form in the stability analysis of nonlinear systems is included in this report even though their applicability is limited to only a small number of control systems.

The constant δ in the second canonic form of system differential equations (i.e., in Eq. 2.54) is obtained by equating the coefficient of the second-highest term of the denominator in Eq. 2.58 to the corresponding term of the denominator of the loop transfer function G(s) of the system shown in Fig. 2.2. Thus

$$\delta = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{m} \omega_i \tag{2.60}$$

where λ_i are the poles and ω_i are the zeros of the loop transfer function G(s).

In order to determine the remaining n-1 coefficients γ_i in the complementary part of the second canonic form of system differential equations, it is more convenient to introduce an auziliary function H(s), defined as

$$H(s) \triangleq \frac{(s-\delta)\,G(s) - 1}{G(s)} \,. \tag{2.61}$$

Since, however,

$$G(s) = \frac{\prod_{i=1}^{m} (s - \omega_i)}{\prod_{i=1}^{n} (s - \lambda_i)} \tag{2.62}$$

(where the scale factor of the loop transfer function G(s) is included in the characteristics of the nonlinear element, i.e., in y = f(x), H(s) can be expressed as

$$H(s) = \frac{(s-\delta)\prod_{i=1}^{m}(s-\omega_i) - \prod_{i=1}^{n}(s-\lambda_i)}{\prod_{i=1}^{m}(s-\omega_i)} \quad . \qquad (2.63)$$

Substitution of Eq. 2.58 into Eq. 2.61 yields

$$H(s) = \frac{\sum_{i=1}^{m} \gamma_i \prod_{\substack{j=1 \\ j\neq i}}^{m}(s-\omega_j)}{\prod_{i=1}^{m}(s-\omega_i)} \qquad (2.64)$$

or

$$H(s) = \sum_{i=1}^{m} \frac{\gamma_i}{s-\omega_i} \quad . \qquad (2.65)$$

Eq. 2.65 shows that the constants γ_i are the residues of the corresponding poles ω_i of H(s). Consequently, the partial fraction expansion of the reciprocal of Eq. 2.62 yields

$$\gamma_i = \frac{-\prod_{j=1}^{n}(\omega_i - \lambda_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{m}(\omega_i - \omega_j)} \qquad i = 1, 2, \ldots m \ . \qquad (2.66)$$

Eqs. 2.60 and 2.66 enable one to calculate the coefficients of the second canonic form of system differential equations from the poles and zeros of the system loop transfer function G(s). These equations also show that the restriction (Eq. 2.59) on the number of poles and zeros of loop transfer function G(s) represent not only a necessary but also a sufficient condition for the equations of a closed-loop system of Fig. 2.2 to be transformable to the second canonic form, since all the coefficients (δ and γ_i) of the canonic equations can be found by means of Eq. 2.60 and Eq. 2.66 as long as the system satisfies Eq. 2.59.

The formulae relating the canonic variables z_i directly to the original system variables x and y and their time derivatives has been derived [19] as

$$z_i = \frac{\begin{vmatrix} 1 & 1 & \cdots & (D-\delta)x + y & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & (D^2 - \delta D - r_1)x + Dy & \cdots & \omega_m \\ \omega_1^2 & \omega_2^2 & \cdots & (D^3 - \delta D^2 - r_1 D - r_2)x + D^2 y & \cdots & \omega_m^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_1^{m-1} & \omega_2^{m-1} & \cdots & (D^{k+1} - \delta D^k - \sum_{j=1}^{k} D^{k-j} r_j)x + D^k y & \cdots & \omega_m^m \end{vmatrix}}{\gamma_i \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & \omega_i & \cdots & \omega_m \\ \omega_1^2 & \omega_2^2 & \cdots & \omega_i^2 & \cdots & \omega_m^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_1^{m-1} & \omega_2^{m-1} & \cdots & \omega_i^{m-1} & \cdots & \omega_m^m \end{vmatrix}} \tag{2.67}$$

where the constants r_j are defined as

$$r_j \triangleq \sum_{i=1}^{m} \omega_i^{j-1} \gamma_i \qquad j = 1, 2, \ldots k \; .$$

The denominator of the above equation contains the Vandermonde determinant which can be written as

$$C = \prod_{1 \leqslant j < i \leqslant m} (\omega_i - \omega_j) \; . \tag{2.68}$$

From Eq. 2.66 one can easily see that the constants γ_i are equal to zero if and only if

$$\omega_i = \omega_j \qquad i,j = 1, 2, \ldots m \; . \tag{2.69}$$

Likewise, the above equation represents the necessary and sufficient conditions for the Vandemonde determinant (Eq. 2.68) to be equal to zero. These results can then be summarized in the following theorem.

Theorem 2.3: A single loop, single nonlinear element system can be described by the second canonic form of differential equations if and only if the following conditions hold:

a) all the zeros ω_i of the transfer function G(s) of the linear portion of the loop are simple;

b) the number of poles n of G(s) is greater by one than the number of its zeros n, i.e., if

$$n = m + 1.$$

The second canonic transformation can be completed by means of Eq. 2.60 and Eq. 2.66 without the need to compute the canonic variables z_i from Eq. 2.67. In systems that are locally and not globally stable Eq. 2.67 may be substituted into the Liapunov's function to find the region of stability in the phase space of the variable x.

The procedure of transforming the mathematical description of a system of Fig. 2.2 from the loop transfer function G(s) into the second canonic form of system differential equations is illustrated by the following example.

Example 2.5: Consider the system of Fig. 2.2 with the loop transfer function

$$G(s) = \frac{(s+1)(s+2)(s+3)}{s^2(s+1+j1)(s+1-j1)} .$$

The poles and zeros of this transfer function are:

$$\lambda_1 = \lambda_2 = 0 ,$$

$$\lambda_3 = -1 - j1 ,$$

$$\lambda_4 = -1 + j1 ,$$

$$\omega_1 = -1 ,$$

$$\omega_2 = -2 ,$$

and

$$\omega_3 = -3 .$$

From Eq. 2.60 one finds

$$\sigma = +4 .$$

From Eq. 2.66 one obtains

$$\gamma_1 = -0.5 ,$$

$$\gamma_2 = +8.0 ,$$

and

$$\gamma_3 = -22.5 .$$

Hence, the second canonic form of differential equations for this system is

$$\frac{dz_1}{dt} = -z_1 + x ,$$

$$\frac{dz_2}{dt} = -2z_2 + x ,$$

$$\frac{dz_3}{dt} = -3z_3 + x ,$$

and

$$\frac{dx}{dt} = -0.5z_1 + 8.0z_2 - 22.5z_3 + 4x - f(x) .$$

Simplified stability criteria of Section 2.6 can now be applied to the above equations to investigate the stability of this system.

It should be noted that the number of equations in the principal part of the second canonic form (Eq. 2.53) is one less than the order of the system which those equations represent. Consequently, the complementary part of the second canonic form (Eq. 2.54) is an independent equation in the set of n independent canonic equations. This means that the variable x is also an independent variable and must be used in stability analysis by means of Liapunov's functions in connection with the second canonic form of system differential equations.

Letov ([5], pp. 192-195) considers the following Liapunov's function which yields useful simplified stability criteria in connection with the second canonic form of system differential equations:

$$V = \sum_{j=1}^{m} \sum_{j=1}^{m} \frac{a_i\, a_j\, z_i\, z_j}{\omega_i + \omega_j} - \frac{B^2}{2} x^2$$

$$- \frac{1}{2}\Big(\sum_{i=1}^{m} A_i z_i\Big)^2 - C_1 z_{s+1} z_{s+2} - C_3 z_{s+3} z_{s+4} - \cdots - C_{m-s-1} z_{m-1} z_m \; ; \qquad (2.70)$$

where A, B, C and a_1, a_2, ... a_s are real constants, and a_{s+1}, a_{s+2}, ... a_m are complex constants appearing in pairs of conjugates. This V-function is negative definite for positive values of the constants A, B and C and for ω_i's with negative real parts only. The time derivative of Eq. 2.70 is, according to Eq. 2.53

$$\frac{dV}{dt} = + \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j z_i z_j + 2x \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{a_i\, a_j\, z_j}{\omega_i + \omega_j}$$

$$- \sum_{i=1}^{s} \omega_i A_i z_i^2 - xB^2 \Big[\sum_{i=1}^{m} \gamma_i z_i + \delta x - f(x) \Big]$$

$$- C_1 (\omega_{s+1} + \omega_{s+2}) z_{s+1} z_{s+2} - \cdots - C_{m-s-1} (\omega_{m-1} + \omega_n) z_{m-1} z_m$$

$$- x \sum_{i=1}^{m} A_i z_i - x \Big[C_1 z_{s+1} z_{s+2} + \cdots + C_{m-s-1} z_{m-1} z_m \Big] . \qquad (2.71)$$

Adding and subtracting the quantity

$$x^2 + 2x \sum_{i=1}^{m} a_i z_i$$

and substituting

$$\sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j z_i z_j = -(\sum_{i=1}^{m} a_i z_i)^2$$

into Eq. 2.71 changes the time derivative of the Liapunov's function (Eq. 2.70) to

$$\frac{dV}{dt} = (\sum_{i=1}^{m} a_i z_i + x)^2 - (\delta B^2 + 1)x^2 + B^2 x\, f(x)$$
$$- \sum_{i=1}^{s} \omega_i A_i z_i^2 + C_1(\omega_{s+1} + \omega_{s+2}) z_{s+1} z_{s+2} + \dots$$
$$+ C_{m-s-1}(\omega_{m-1} + \omega_m) z_{m-1} z_m + x \sum_{i=1}^{s} \Big[2 \sum_{j=1}^{m} \frac{a_i\ a_j}{\omega_i + \omega_j}$$
$$- B^2\ \gamma_{i+s} - 2a_{i+s} - C_{2i-1} \Big] z_i \quad . \tag{2.72}$$

The V-function (Eq. 2.70) will be negative definite even if the terms containing the constants A_i and C_{2i-1} are omitted. Its time derivative, however, could only be positive semidefinite without the terms containing A_i and C_{2i-1}. If the constants A_i and C_{2i-1} are chosen as sufficiently small positive numbers, they will not affect the roots of the stability equations. Hence, if the stability equation is chosen as

$$2 \sum_{j=1}^{m} \frac{a_i\ a_j}{\omega_i + \omega_j} - B^2\ \gamma_i = 0; \qquad i = 1, 2, \dots m , \tag{2.73}$$

Eq. 2.72 becomes

$$\frac{dV}{dt} = (\sum_{i=1}^{m} a_i z_i)^2 + B^2 x\ \big(f(x) - \delta\ x\big) . \tag{2.74}$$

Consequently, the system is asymptotically stable in the region in which

$$x\big(f(x) - \delta\ x\big) > 0 \quad \text{for all} \quad |x| > 0\ ; \quad f(0) = 0 \tag{2.75}$$

is satisfied, provided that:

a) ω_i and γ_i are real for $1 \leqslant i \leqslant s$;

ω_i and γ_i appear in pairs of complex conjugates for $s < i \leqslant m$;

b) Re $\omega_i < 0$ for $i = 1, 2, \ldots m$;

c) The roots of Eq. 2.73, $a_1, a_2, \ldots a_s$ are real and a_{s+1}, $a_{s+2}, \ldots a_{m-1}, a_m$ are in pairs of complex conjugates.

Furthermore, if the stability equation is chosen as

$$2 \sum_{j=1}^{m} \frac{a_i \, a_j}{\omega_i + \omega_j} - B \, \gamma_i - 2a_i = 0 , \tag{2.76}$$

then, from Eq. 2.72 one obtains

$$\frac{dV}{dt} = (\sum_{i=1}^{m} a_i z_i + x)^2 + x\left[x - \sigma B^2 x + B^2 f(x)\right] . \tag{2.77}$$

Consequently, the system is asymptotically stable in the region in which the following inequality holds:

$$x(x - \sigma B^2 x + B^2 f(x)) \geqslant 0 \quad \text{for all } |x| > 0; \quad f(0) = 0 \tag{2.78}$$

provided the following conditions are satisfied:

a) ω_i and γ_i are real for $1 \leqslant i \leqslant s$;

ω_i and γ_i appear in pairs of complex conjugates for $s < i \leqslant m$;

b) Re $\omega_i < 0$ for $i = 1, 2, \ldots m$;

c) The roots of Eq. 2.76, $a_1, a_2, \ldots a_s$ are real and a_{s+1}, $a_{s+2}, \ldots a_{m-1}, a_m$ are in pairs of complex conjugates.

It is also possible to establish asymptotic stability by letting

$$A_i = - B \, \gamma_i \tag{2.79}$$

for all negative γ_i's, so that the corresponding roots a_i are zero. This choice decreases the number of simultaneous stability equations whose solution yields the sufficient stability conditions mentioned above.

It may be observed that both simplified stability criteria restrict the minimum values of the equivalent gain of nonlinear element, as illustrated in Fig. 2.9.

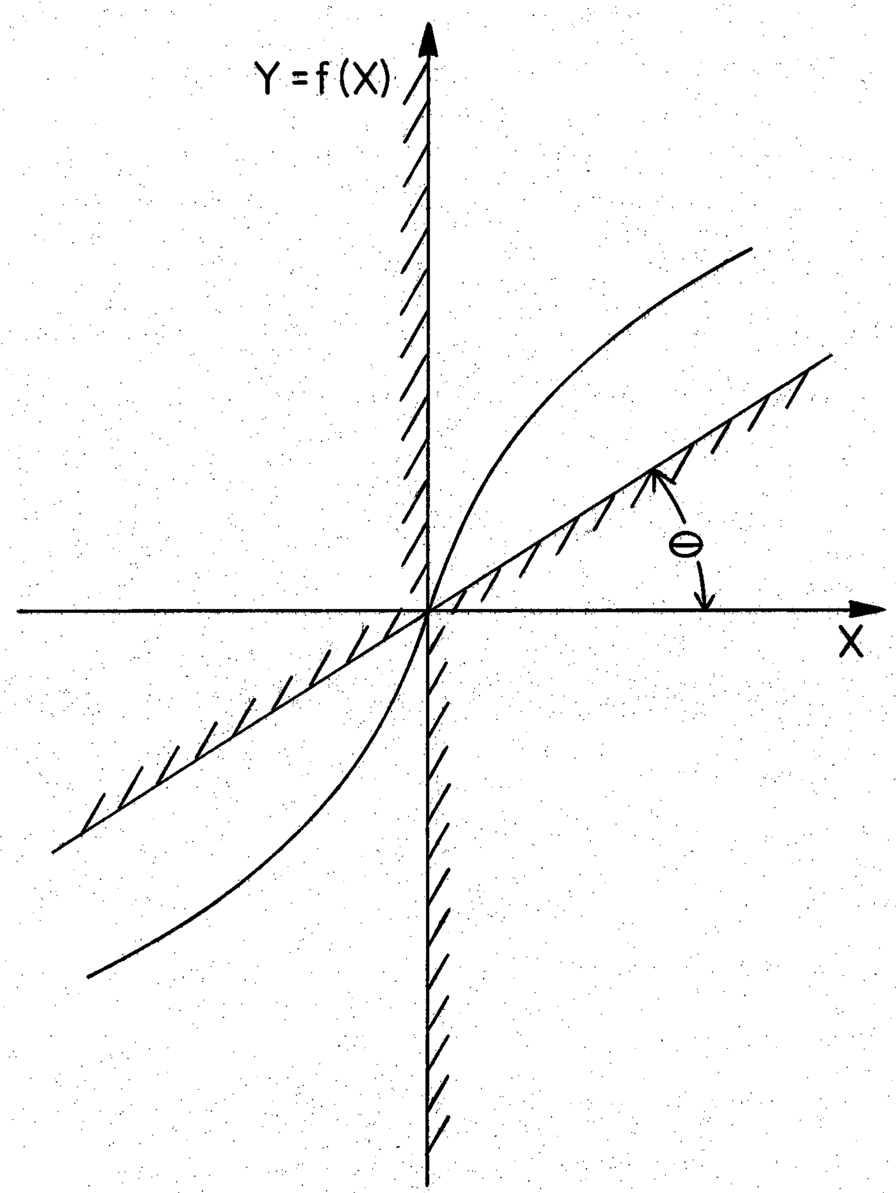

Figure 2.9

Limits on System Gain for Simplified Stability Criteria of the Second Canonic Form.

The angle Θ of clockwise rotation of the input axis of the nonlinear element characteristic plane, which determines the minimum value of nonlinear element equivalent gain, is

$$\Theta = \arctan \delta , \qquad (2.80)$$

if Eq. 2.73 is used as simplified stability criterion, or

$$\Theta = \arctan \frac{(\delta B^2 - 1)}{B^2} \qquad (2.81)$$

if Eq. 2.76 is used as simplified stability criterion. Obviously, there are many stable systems that violate the restrictions of Eq. 2.80 and 2.81, a further limitation of the applicability of the second canonic form in stability analysis. It is possible to avoid these difficulties and extend the applicability of the simplified stability criterion to many more systems that are either rejected by or not applicable to the simplified stability criteria based on the second canonic form by the use of the zero-shifting technique proposed in Section 2.5. The only justification for presenting the second canonic form and the simplified stability criteria associated with the second canonic form is the possibility that in a few systems the simplified stability criteria of this section may yield useful stability information that is not obtainable from other simplified stability criteria. This possibility must be considered in view of the fact that none of the known simplified stability criteria for systems with a single gain nonlinearity represent necessary conditions for stability. It is, however, very unlikely that the approach of the stability analysis presented in this section would yield stability information which is not obtainable from the pole- and zero-shifting techniques of Section 2.4 and Section 2.5.

CHAPTER III

METHODS OF CONSTRUCTING LIAPUNOV'S FUNCTIONS

3.1 Introduction

There is no generally applicable, straightforward procedure of constructing Liapunov's functions for autonomous nonlinear systems. All the known techniques of finding Liapunov's functions for different types of nonlinear control systems are similar to the procedure of finding Liapunov's functions for linear autonomous systems. All these techniques involve the use of a quadratic form as part of the Liapunov's function. Hence, the success in finding a suitable Liapunov's function for a given nonlinear system depends not only upon one's intuition and experience, but also upon thorough knowledge of the methods of finding Liapunov's functions for linear autonomous systems.

The basic difficulty limiting the application of the "second method" in nonlinear system analysis at the present time is the lack of theorems to determine the definiteness (with respect to sign) of higher order forms (i.e., the lack of theorems, similar to Sylvester's Theorem, for higher order forms).

One of the best known procedures of constructing Liapunov's functions has been presented in Chapter II. In cases where canonic transformations either are not applicable or fail to prove stability, one may try several other known techniques of constructing Liapunov's functions. Some of these procedures may also be advantageous in higher order systems in which solution of the stability equations of Chapter II may become difficult and time consuming.

Several other better known methods of finding Liapunov's functions for autonomous nonlinear systems will be presented in this chapter.

These methods are:

The method of Aizerman

The method of Pliss

Krasovski's Theorem

Pseudo-Canonic forms (Purdue)

3.2 Stability of Linear Autonomous Systems

The Routh-Hurwitz criterion provides an easy and convenient way of proving stability or instability of linear autonomous systems. Liapunov's Second Method in turn can be used to prove the Routh-Hurwitz criterion [11]. While the "Second Method" offers no advantages over Routh-Hurwitz in the stability analysis of a particular linear system, there are several reasons for studing the method of constructing Liapunov's functions for linear systems. These are:

a) An infinite number of suitable Liapunov's functions can always be found for a linear autonomous system.

b) Liapunov's functions provide a convenient method of computing the "intergral of error" type performance indices for linear autonomous systems.

c) Liapunov's functions for linear systems can frequently be used to investigate stability of nonlinear autonomous systems.

d) In the case of structurally stable nonlinear autonomous systems, local stability or instability can always be proved by means of Liapunov's functions for linear autonomous systems.

e) The few known methods of constructing Liapunov's functions for nonlinear autonomous systems depend upon the knowledge of Liapunov's functions for linear autonomous systems.

A possible procedure for constructing Liapunov's function for linear autonomous systems is shown in Fig. 3.1. This procedure of constructing Liapunov's functions for linear autonomous systems is by no means the most convenient one. It reveals, however, that any quadratic form $V(x_1, \ldots x_n)$ of the state variables $x_1, \ldots x_n$ will yield the function

$$W(x_1, \ldots x_n) = \frac{dV}{dt}$$

which is also a quadratic form of the state variables $x_1, \ldots x_n$, as long as the system is described by a set of linear autonomous differential equations. Hence, there is no need to solve the differential equations of the system in order to find a suitable Liapunov's function for a linear autonomous system. Liapunov has shown [3] that the following procedure can always be used to construct a Liapunov's function for linear autonomous systems:

a) Assume a general quadratic form, defined as

$$V(x_1, \ldots x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \; ; \quad a_{ij} = a_{ji} \tag{3.1}$$

for the V-function of the state variables $x_1, \ldots x_n$.

b) Differentiate this V-function with respect to time t, i.e., find

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \cdot \frac{dx_i}{dt} \; . \tag{3.2}$$

c) Substitute the system differential equations for $\frac{dx_i}{dt}$ in Eq. 3.2. One may recall that the system is described by a set of first order differential equations

$$\dot{x}_i = X_i(x_1, \ldots x_n) \qquad i = 1, 2, \ldots n \; . \tag{3.3}$$

In the case of linear autonomous systems, these equations are of the form

$$\dot{x}_i = \sum_{j=1}^{n} k_{ij} x_j \qquad i = 1, 2, \ldots n \; . \tag{3.4}$$

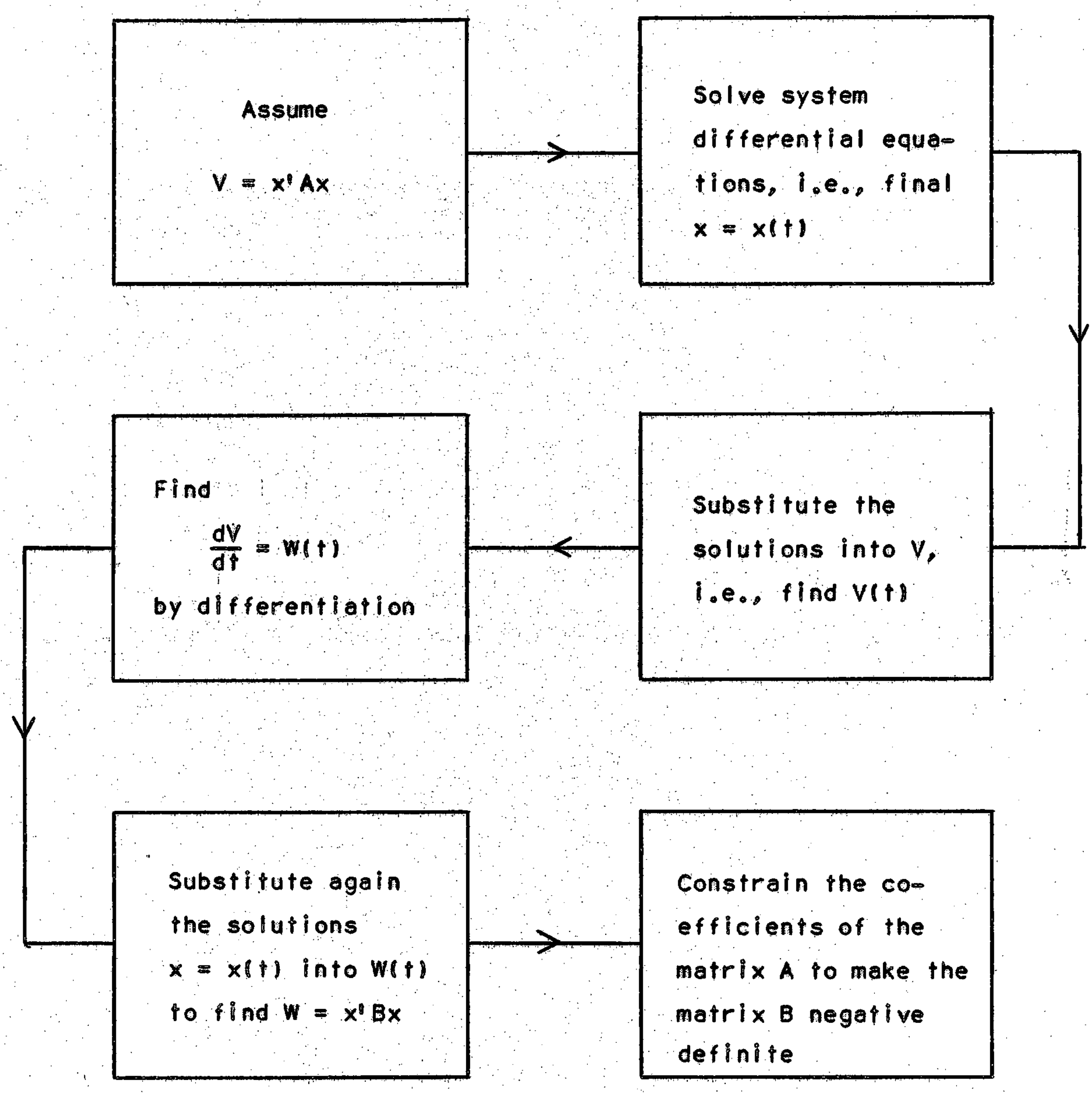

Figure 3.1

A Possible Procedure for Constructing Liapunov's Functions for Linear Autonomous Systems.

d) Substitution of Eq. 3.4 into Eq. 3.2 yields another quadratic form of the state variables $x_i, \ldots x_n$

$$\frac{dV}{dt} = W(x_1, \ldots x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}x_ix_j \; ; \quad b_{ij} = b_{ji} \; . \quad (3.5)$$

e) Constraining now the quadratic form $W(x_1, \ldots x_n)$ to be positive definite (or negative definite) will, in the case of either unstable or asymptotically stable systems, yield the coefficients a_{ij} of Eq. 3.1, such that Eq. 3.1 and Eq. 3.5 will satisfy either the instability or the stability theorems. In the case of stable but not asymptotically stable systems, the above procedure will yield a definite V-function, and the corresponding time derivative $\frac{dV}{dt}$ will be identically equal to zero.

The positive definiteness of V (or W) can be proved by means of Sylvester's Theorem.

Example 3.1:

Consider the linear system, shown in Fig. 3.2. A differential equation describing this system is

$$\ddot{c} + a\dot{c} + bc = ke = - kc \; .$$

Let the state variables be

$$c = x_1 \; ,$$

$$\dot{c} = x_2 \; .$$

Then

$$\dot{x}_1 = x_2$$

and

$$\dot{x}_2 = - ax_2 - (k+b)x_1 \; .$$

From Eq. 3.1 one can write

$$V = a_{11}x_1^{\,2} + 2a_{12}x_1x_2 + a_{22}x_2^{\,2} \; .$$

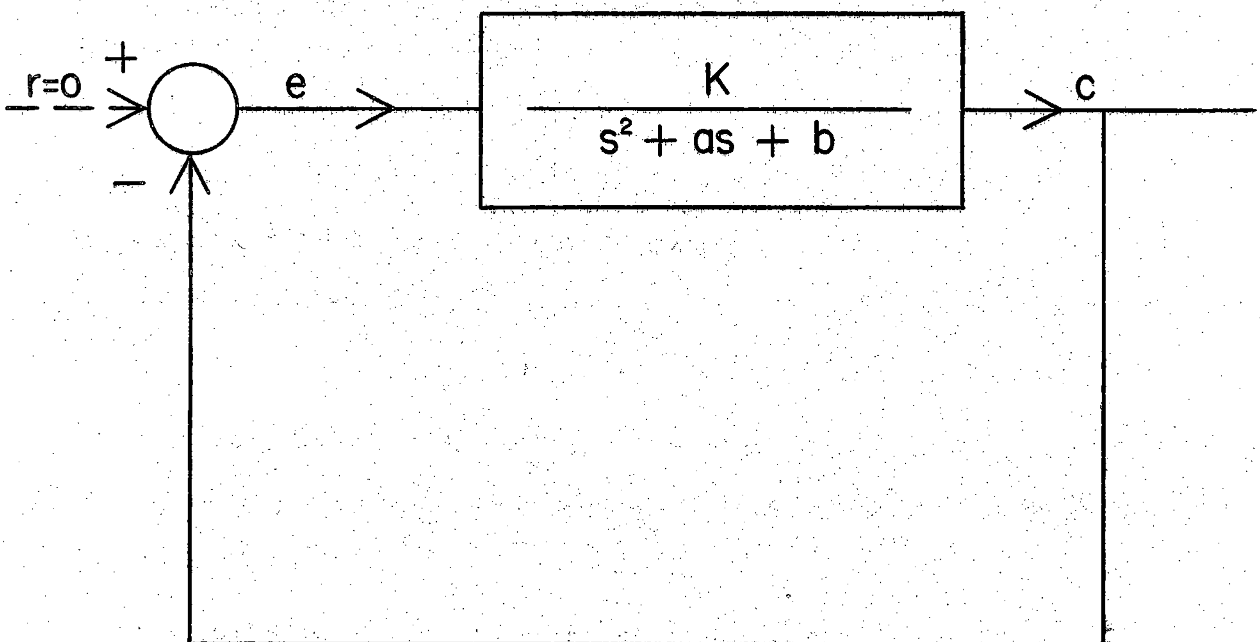

Figure 3.2

Linear Second Order System of Example 3.1.

The differentiation and substitution of x_i's yields

$$\frac{dV}{dt} = W = \left[- 2a_{12}(b+k)\right] x_1^2 + \left[2a_{11} - 2a_{12}a - 2a_{22}(b+k)\right] x_1x_2 + \left[2a_{12} - 2a_{22}a\right] x_2^2 .$$

Constrain Eq. 3.5 to be

$$\frac{dV}{dt} = x_1^2 + x_2^2 .$$

This gives

$$\begin{aligned} b_{11} &= 1 & \qquad & -2a_{12}(b+k) = 1 \\ b_{12} &= 0 & \text{or} \qquad & 2a_{11} - 2a_{12}a - 2a_{22}(b+k) = 0 \\ b_{22} &= 1 & & 2a_{12} - 2a_{22}a = 1 . \end{aligned}$$

Thus the coefficients of the Liapunov's function (Eq. 3.1) are

$$a_{12} = - \frac{1}{2(b+k)}$$

$$a_{22} = - \frac{1+b+k}{2(b+k)a}$$

$$a_{11} = - \frac{a}{2(b+k)} - \frac{1+b+k}{2a} .$$

Hence, from Sylvester's Theorem, the system is asymptotically stable if and only if the following holds:

a) $\frac{1+b+k}{2a} + \frac{a}{2(b+k)} > 0$;

b) $\left[\frac{1+b+k}{2a} + \frac{a}{2(b+k)}\right]\left[\frac{1+b+k}{2a(b+k)}\right] - \left[\frac{1}{2(b+k)}\right]^2 > 0$.

The preceding two inequalities can be simplified to yield conditions for asymptotic stability identical to those of Routh-Hurwitz.

In general, to find a Liapunov's function for an autonomous linear system, one will have to solve

$$n + (n-1) + (n-2) + \dots + 2 + 1$$

linear algebraic equations for the constants a_{ij} of the Liapunov's function.

3.3 Performance Indices -- A Method of Their Computation

The procedure of constructing Liapunov's functions for linear autonomous systems offers a convenient method of computing the "integral of error" type performance indices. For this purpose consider again Eqs. 3.1 and 3.5.

Integrating Eq. 3.5 with respect to time, t, one obtains

$$V(x_1, \dots x_n) - V(x_{10}, \dots x_{n0}) = \int_0^t W(x_1, \dots x_n)dt \qquad (3.6)$$

where x_{10} represents the initial values of the state variables x_i (at time $t = 0$).

As the time $t \longrightarrow \infty$ the above equation becomes

$$V(x_{10}, \dots x_{n0}) - \lim_{t \longrightarrow \infty} V(x_1, \dots x_n) = \int_0^\infty - W(x_1, \dots x_n)dt \qquad (3.7)$$

Since, however, for asymptotically stable systems

$$\lim_{t \longrightarrow \infty} x_i = 0 \qquad i = 1, 2, \dots n \qquad (3.8)$$

and

$$V(0, \dots 0) = 0 , \qquad (3.9)$$

Eq. 3.7 becomes

$$V(x_{10}, \dots x_{n0}) = - \int_0^\infty W(x_1, \dots x_n)dt . \qquad (3.10)$$

If the state variables x_i represent system error and its n-1 time derivatives, and if W (Eq. 3.5) is a positive definite quadratic form, then Eq. 3.10 is an integral of error type performance index for the system, i.e.,

$$PI = \int_0^\infty W(e, \dot{e}, \dots)dt . \qquad (3.11)$$

In general, then, the limit as time $t \rightarrow \infty$ of any Liapunov's function that has a negative definite or negative semidefinite time derivative can be used as an index of performance for the system.

Example 3.2:

Computation of the performance index $PI = \int_0^\infty e^2 dt$.

Consider a unity feedback system (linear) with

$$G(s) = \frac{k}{s(s+a)} .$$

Let

$$R(s) = \frac{1}{s}$$

or

$$r(t) = 1 \quad \text{for } t > 0$$

$$r(t) = 0 \quad \text{for } t \leqslant 0 .$$

Then one may write

$$\ddot{c} + a\dot{c} = ke ;$$

and

$$e = 1 - c \quad (t > o)$$

or

$$\ddot{e} + a\dot{e} + ke = 0 .$$

Let

$$x_1 = e$$

$$x_2 = \dot{e} .$$

Then the system equations become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - ax_2 - kx_1 .$$

Consider a general quadratic form

$$V = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 .$$

This quadratic form yields, in connection with the system equations,

$$\frac{dV}{dt} = W = (-2ka_{12})x_1^{\,2} + (2a_{11} - 2aa_{12} - 2ka_{22})x_1x_2 + (2a_{12} - 2aa_{22})x_2^{\,2}$$

Constrain W to be

$$W = x_1^{\,2} = e^2 \,.$$

This constraint yields the coefficients of the quadratic form V

$$-2ka_{12} = 1$$

$$2a_{11} - 2aa_{12} - 2ka_{22} = 0$$

$$2a_{12} - 2aa_{22} = 0$$

or

$$a_{12} = -\frac{1}{2k}$$

$$a_{22} = -\frac{1}{2ka}$$

$$a_{11} = -\frac{k+a^2}{2ka} \,.$$

Hence,

$$V = -\frac{k+a^2}{2ka}\, x_1^{\,2} - \frac{1}{k}\, x_1x_2 - \frac{1}{2ka}\, x_2^{\,2} \,.$$

The initial values of x_1 and x_2 can be found from the system transfer function

$$E(s) = \frac{R(s)}{1+G(s)} = \frac{s^2 + as}{s(s^2 + as + k)} \,;$$

$$e(o) = \lim_{s \to \infty} \frac{s^2 + as}{s^2 + as + k} = 1 \,.$$

Likewise,

$$\dot{e}(o) = \lim_{s \to \infty} \frac{-sk}{s^2 + as + k} = 0 \,.$$

Hence,

$$x_1(o) = 1 \ ,$$

$$x_2(o) = 0 \ .$$

Then, from Eq. 3.10 and Eq. 3.11 the performance index is

$$PI = \int_o^\infty e^2 dt = \int_o^\infty x_1^2 dt = - V(x_1(o), x_2(o))$$

or

$$PI = \frac{k + a^2}{2ka} \ .$$

The preceding discussion also suggests a convenient and simple procedure for calculating the numerical values of the integral of error type performance indices. All one has to do is to find $V(x_1, \ldots x_n)$ corresponding to the particular W-function of Eq. 3.11 by the procedure outlined in Section 3.2 of this chapter, and then substitute the initial values for $x_i(o)$.

In the case of time-weighted integral of error performance index

$$PI = \int_o^\infty tW(e, \dot{e}, \ldots)dt \qquad (3.12)$$

one can assume a V-function of the type

$$V = V_1(e, \dot{e}, \ldots) + tV_2(e, \dot{e}, \ldots) \ , \qquad (3.13)$$

where both V_1 and V_2 are positive definite or positive semidefinite quadratic forms of the error variable e and its n-1 time derivatives, and use exactly the same procedure as before to compute the numerical values of the performance index PI (Eq. 3.12).

3.4 Aizerman's Method

Aizerman [7] proposed a procedure of constructing Liapunov's functions for nonlinear autonomous systems which is very similar to the procedure of construction of Liapunov's functions for linear autonomous systems. Basically, the method consists of approximating the nonlinear elements of an actual system by straight line characteristics, then finding a Liapunov's function for the resulting linear system of differential equations. The V-function obtained in this way is then applied to the actual (nonlinear) system and the resulting time derivative $\frac{dV}{dt}$ gives a range of deviation of the nonlinear element characteristic from the straight line over which stability (or instability, as the case may be) can be proved by the particular quadratic V-form (i.e., by the V-function obtained from the straight line approximation).

Example 3.3:

As an example of Aizerman's Method consider the system shown in Fig. 3.3. In the absence of input ($r(t) = 0$ for $t > o$) this system can be described by the equations:

$$\ddot{x} + 2\dot{x} - y = 0$$

$$y = f(x) .$$

A possible set of state variables is

$$x_1 = x$$

$$x_2 = \dot{x} .$$

Then the differential equations for this system become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 2x_2 - f(x_1) .$$

A straight-line approximation of the nonlinear element characteristic is shown in Fig. 3.4. This approximation is expressed mathematically as

$$y = f(x_1) \approx 2x_1 .$$

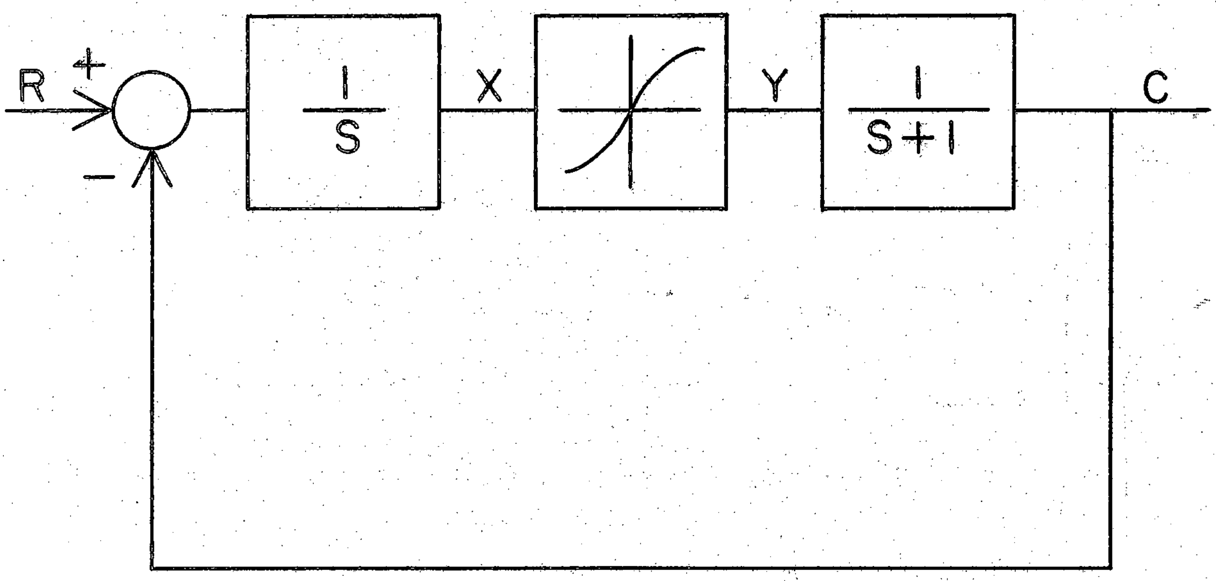

Figure 3.3

Block Diagram of the Nonlinear System of Example 3.3.

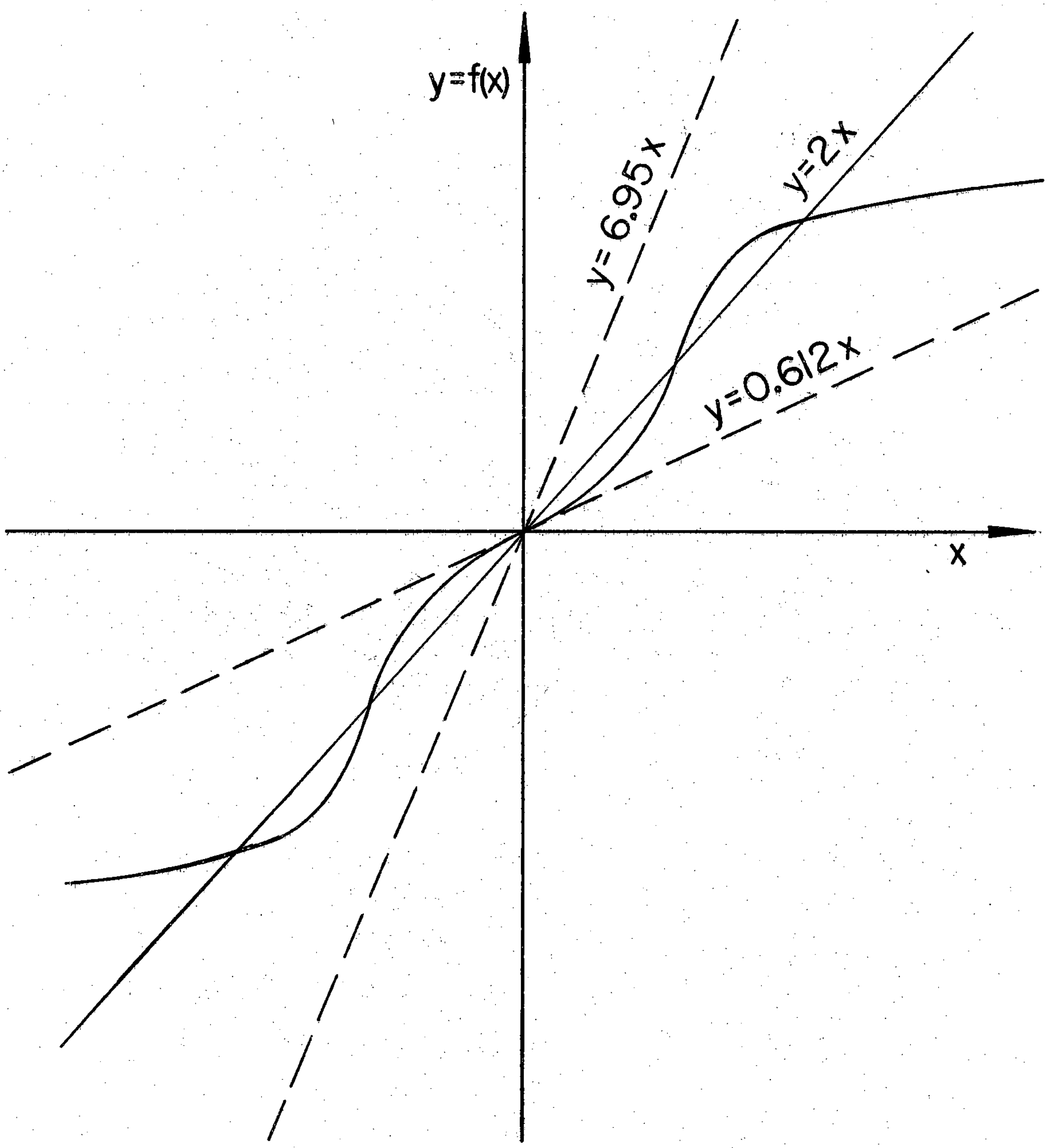

Figure 3.4

Input-Output Characteristics of the Nonlinear Element of Example 3.3.

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_2 - 2x_1 .$$

The V-function for this linearized system is a general quadratic form

$$V(x_1 x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 .$$

This yields

$$\frac{dV}{dt} = W(x_1 x_2) = (-4a_{12})x_1^2 + (2a_{11} - 4a_{12} - 4a_{22})x_1x_2 + (2a_{12} - 4a_{22})x_2^2 .$$

Constrain W to be

$$W = x_1^2 + x_2^2 .$$

Then

$$-4a_{12} = 1$$

$$2a_{12} - 4a_{22} = 1$$

$$2a_{11} - 4a_{12} - 4a_{22} = 0$$

or

$$a_{12} = -\frac{1}{4}$$

$$a_{22} = -\frac{3}{8}$$

$$a_{11} = -\frac{5}{4} .$$

Thus V is a negative definite quadratic form

$$V(x_1, x_2) = -\frac{5}{4}x_1^2 - \frac{1}{2}x_1x_2 - \frac{3}{8}x_2^2 .$$

Applying this V-function to the actual (nonlinear) system differential equation one finds

$$\frac{dV}{dt} = W(x_1 x_2) = \left(\frac{1}{2}\frac{f(x_1)}{x_1}\right)x_1^2 + \left(\frac{3}{4}\frac{f(x_1)}{x_1} - \frac{3}{2}\right)x_1x_2 + x_2^2 .$$

Applying the Sylvester Theorem to this W-function, one finds a set of sufficient conditions for global asymptotic stability:

$$\frac{f(x_1)}{x_1} > 0 \quad \text{or} \quad x_1 f(x_1) > 0$$

and

$$0.612 < \frac{f(x_1)}{x_1} < 6.95 .$$

These restrictions on the characteristic of the nonlinearity are shown in Fig. 3.4.

The advantages of Aizerman's Method are:

1. Its simplicity.
2. Its applicability to systems with more than one nonlinear element.
3. It can be used to justify approximation of a slightly non-linear system by linear differential equations (i.e., it puts bounds on the nonlinearity to assure that, at least stability-wise, the system does not differ appreciably from its linear mathematical model).

The disadvantages of this method are:

1. It is applicable <u>only</u> if the input-output characteristics of the nonlinear elements do not deviate too far from a straight line (i.e., the system may be only slightly nonlinear).
2. If the system contains differentiation (zeros in the transfer function of the linear part of the system), the method puts rather complicated restrictions on nonlinear element characteristics in terms of y, $\frac{dy}{dt}$, etc.

It is important to note that a system may not be globally stable even if its linearized model ($y = kx$) is stable for all values of the equivalent linear gain k. Stability cannot be assumed for granted but must be proved even if the nonlinear element input-output characteristic is confined to a narrow region of the x-y plane, as in Fig. 3.4.

The following two rules are helpful in applying Aizerman's Method:

a) The straight-line approximation $y = kx$ shall be selected in such a way that the input-output characteristic of the non-linear element deviates from this straight line by an equal angular distance in both directions.

b) The W-function $(= \frac{dV}{dt})$ should preferably be constrained to a Euclidian Norm, i.e.,

$$W(x_1, \ldots x_n) = \sum_{i=1}^{n} a_i x_i^2 .$$

This will, in most cases, yield the widest (least severe) restrictions on the input-output characteristic of the non-linear element.

3.5 Krasovski's Theorem

Consider an autonomous nonlinear system described by the equations

$$\dot{x}_i = X_i(x_1, \ldots x_n) \qquad i = 1, 2, \ldots n \tag{3.14}$$

where the right-hand sides X_i are continuous and differentiable functions in the entire state space $-\infty < x_i < \infty$ and the equilibrium state is at the origin of the state space coordinates.

Let us designate by $\frac{\partial X}{\partial x}$ the Jacobian matrix of the function X, i.e.,

$$\frac{\partial X}{\partial x} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{\partial X_n}{\partial x_1} & \cdots & \frac{\partial X_n}{\partial x_n} \end{bmatrix} . \tag{3.15}$$

Theorem 3.1: [8]. In order that the system (Eq. 3.14) be globally asymptotically stable, it is sufficient that there exist a positive symmetric matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad (3.16)$$

with positive eigenvalues and such that the symmetric matrix of the products

$$\frac{1}{2}\left[\left(A \frac{\partial X}{\partial x}\right)_{ik} + \left(A \frac{\partial X}{\partial x}\right)_{ki} \right] \qquad (3.17)$$

has the eigenvalues $\lambda_i(x_1, \ldots x_n)$ which satisfy the inequality

$$\lambda_i < k^2 \qquad i = 1, 2, \ldots n . \qquad (3.18)$$

where k is a real constant.

In order to apply the above theorem to practical systems (i.e., to find the positive matrix A), one may observe that the conditions of the theorem are satisfied by a positive definite quadratic form*

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} X_i X_j , \qquad A_{ij} = A_{ji} \qquad (3.19)$$

of the functions $X_1, \ldots X_n$ (not the state variables $x_1, \ldots x_n$!) whose time derivative

---

*In English literature (Kalman [12]), (Cunningham [20]) it has been stated that the Liapunov function resulting from Krasovski's Theorem will be the square of Euclidian Norm,

$$V = \sum_{i=1}^{n} a_i X_i^{\,2} .$$

This, however, represents only a special case of Krasovski's original theorem and severely limits its applicability. This special case is included in the more general Liapunov's function of Eq. 3.19. (See Appendix B)

$$\frac{dV}{dt} = W(X_1, \ldots X_n) \tag{3.20}$$

is a negative definite function of the functions $X_1, \ldots X_n$ for all real values of the state variables $x_1, \ldots x_n$. The time derivative $\frac{dV}{dt}$ will be of the form

$$\frac{dV}{dt} = W = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}(X_1, \ldots X_n)x_i x_j \; ; \quad B_{ij} = B_{ji} \tag{3.21}$$

where the coefficients B_{ij} are not constant but rather are functions of the state variables x_i. Hence, the procedure of applying Krasovski's Theorem is to assume a general quadratic form (Eq. 3.19), find its time derivative W, and then (if possible) constrain this time derivative W Eq. 3.21) to be a negative definite quadratic form in $X_1, \ldots X_n$ for all real values of the state variables $x_1, \ldots x_n$.

Example 3.4:

Consider again the system of Example 3.3 (Aizerman's Method). The differential equations for this system were found to be

$$\dot{x}_1 = X_1(x_1,x_2) = x_2$$

$$\dot{x}_2 = X_2(x_1,x_2) = - 2x_2 - f(x_1) .$$

Let

$$V = A_{11}X_1^{\,2} + 2A_{12}X_1X_2 + A_{22}X_2^{\,2} .$$

Differentiating the above equation with respect to time t and then substituting the differential equations describing the system one finds

$$W = (-2A_{12}\frac{df}{dx_1})X_1^{\,2} + (2A_{11} - 4A_{12} - 2A_{22}\frac{df}{dx_1})X_1X_2 + (2A_{12} - 4A_{22})X_2^{\,2} .$$

Constraining W to be

$$W = (\tfrac{1}{2}\frac{df}{dx_1})X_1^{\,2} + (\tfrac{3}{4}\frac{df}{dx_1} - \tfrac{3}{2})X_1X_2 + X_2^{\,2}$$

one finds

$$-2A_{12} = \frac{1}{2}$$

$$2A_{12} - 4A_{22} = 1$$

$$2A_{11} - 4A_{12} = -\frac{3}{2}$$

or

$$A_{11} = -\frac{5}{4}$$

$$A_{22} = -\frac{3}{8}$$

$$A_{12} = -\frac{1}{4} \,.$$

Sylvester's Theorem shows that with these values of A_{ij} the V-function is positive definite. Likewise, $\frac{dV}{dt}$ is negative definite if

$$0.573 < \frac{df(x_1)}{dx_1} < 6.98 \,.$$

This inequality represents the sufficient (but not necessary) conditions for global asymptotic stability of the system of this example.

Krasovski's Theorem enjoys the same advantages and disadvantages as the Aizerman's Method. It is possible, however, that a system which fails to meet Aizerman's test may be proved to be globally asymptotically stable by means of Krasovski's Theorem, and vice versa.

3.6 The Work of Pliss

Pliss [21] considered nonlinear autonomous systems with a single nonlinear element described by the set of differential equations

$$\dot{x}_j = \sum_{i=1}^{n} b_{ji}x_k + h_j f(\sigma) \qquad j = 1, 2, \ldots n \qquad (3.22)$$

$$\sigma = \sum_{j=1}^{n} a_j x_j \tag{3.23}$$

where a, b and h are constants and the nonlinear element is defined by the functional relationship

$$f(0) = 0 \ ; \quad c_1 \sigma^2 < \sigma f(\sigma) < c_2 \sigma^2 \ . \tag{3.24}$$

To construct a Liapunov's function for this system, Pliss first analyzes the linearized system, described by the set of linear differential equations

$$\dot{x}_j = \sum_{i=1}^{n} b_{ji} x_i + h_j c \sigma \qquad j = 1, 2, \dots n \tag{3.25}$$

$$\sigma = \sum_{j=1}^{n} a_j x_j \ . \tag{3.26}$$

He then shows that the linearized system may be stable for all values of c in the interval

$$c_1 \leqslant c \leqslant c_2$$

and yet the nonlinear system (Eqs. 3.21 and 3.22) may not be globally stable.

To find sufficient conditions for global asymptotic stability of the nonlinear system (Eqs. 3.21 and 3.22) Pliss uses as the Liapunov's function

$$V_m = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j + \frac{1}{2} c \sigma^m \ . \tag{3.27}$$

By means of this Liapunov's function he arrives at the following results:

The system (Eqs. 3.21 and 3.22) is globally asymptotically stable if

a) for all $c = c_1 + \epsilon$ and $c = c_2 - \epsilon$, where ϵ is an arbitrary small real constant, the linearized system (Eqs. 3.25 and 3.26) is asymptotically stable

b) real numbers

β and $m_{ij} = m_{ji}$, $\quad i, j = 1, 2, \ldots n$

exist such that the quadratic form

$$Q(x_1, \ldots x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} x_i x_j \tag{3.28}$$

is either positive definite or negative definite, where the coefficients r_{ij} are calculated from the equation

$$\begin{aligned} r_{ij} = & \sum_{k=1}^{n} m_{ik} b_{kj} + \sum_{k=1}^{n} m_{kj} b_{kj} \\ & + c(a_j \sum_{k=1}^{n} m_{ki} h_k + a_i \sum_{k=1}^{n} m_{kj} h_k) \\ & + c\beta(a_j \sum_{k=1}^{n} a_k b_{ki} + a_i \sum_{k=1}^{n} a_k b_{kj} \\ & + c^2 a_i a_j \sum_{k=1}^{n} a_k h_k ; \qquad c_1 < c < c_2 . \end{aligned} \tag{3.29}$$

The principal disadvantage of this result is its complexity. It is felt that the same results could be achieved in a simpler manner by means of pole- and zero-shifting in connection with the first canonic transformation (see Chapter II).

3.7 Pseudo-Canonic Transformation

The basic advantage of the first canonic transformation and the associated Liapunov's functions (Chapter II) is the simplicity of restrictions which these Liapunov's functions place upon the input-output characteristic of the nonlinearity for global asymptotic stability. Among the disadvantages of the first canonic form and the associated simplified stability criteria were the necessity to deal with complex variables and the need to solve nonlinear stability equations without an a priori knowledge that these long and tedious computations will yield useful results.

In order to retain the advantages of the canonic transformations and at the same time eliminate some, if not all, of its disadvantages, a pseudo-canonic transformation was developed at Purdue.

Consider the feedback system shown in Fig. 2.2 with a nonlinear element whose characteristic is confined to the first and third quadrants of the input-output plane (Fig. 2.5). Removing the input r(t) at time t = o one may write

$$G(s) = G_1(s)\, G_2(s) = \frac{\prod_{i=1}^{m} (s - w_i)}{\prod_{i=1}^{n} (s - \lambda_i)} \qquad m < n\ . \qquad (3.30)$$

Expansion of the above transfer function into quadratic factors yields

$$G(s) = -\frac{X(s)}{Y(s)} = \sum_{i=1}^{\frac{n}{2}} \frac{(A_i s + B_i)}{(s^2 + a_i s + b_i)}\ . \qquad (3.31)$$

Defining the canonic variables as

$$z_i(s) \triangleq \frac{Y(s)}{s^2 + a_i s + b_i} \qquad i = 1, 3, 5, \ldots n-1 \qquad (3.32)$$

one will obtain the following set of differential equations describing the system:

$$\ddot{z}_i = -a_i \dot{z}_i - b_i z_i + y$$

$$y = f(x) \qquad i = 1, 3, 5, \ldots n-1 \qquad (3.33)$$

$$x = -\sum_{i=1}^{n-1} A_i \dot{z}_i + B_i z_i$$

or

$$\left.\begin{aligned}
\dot{z}_i &= z_{i+1} \\
\dot{z}_{i+1} &= -a_i z_{i+1} - b_i z_i + y \\
y &= f(x) \qquad\qquad\qquad i = 1, 3, 5, \ldots n-1 \\
x &= -\sum_{i=1}^{n-1} A_i z_{i+1} + B_i z_i \\
\dot{x} &= \sum_{i=1}^{n-1} (A_i a_i - B_i) z_{i+1} + A_i b_i z_i - A_i y
\end{aligned}\right\} \quad (3.34)$$

This pseudo-canonic transformation can also be interpreted on the block diagram of the system as shown in Fig. 3.5.

3.8 Construction of Liapunov's Functions Based on Pseudo-Canonic Transformation

Consider as a possible Liapunov's function, the general quadratic form

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} z_i z_j \; ; \qquad c_{ij} = c_{ji} \quad . \qquad (3.35)$$

Differentiating Eq. 3.35 with respect to time, t, and substituting the pseudo-canonic equations (3.34) one gets $\frac{dV}{dt}$ of the form

$$\frac{dV}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} z_i z_j + y \sum_{i=1}^{n} e_i z_i \; ; \quad d_{ij} = d_{ji} \qquad (3.36)$$

where

$$d_{ij} = d_{ij}(a_i, b_i, c_{ij})$$

$$e_i = e_i(a_i, b_i, c_{ij}) \quad .$$

Constrain $\frac{dV}{dt}$ (Eq. 3.36) such that

$$\sum_{i=1}^{n} e_i z_i = x \quad . \qquad (3.37)$$

Then Eq. 3.36 becomes

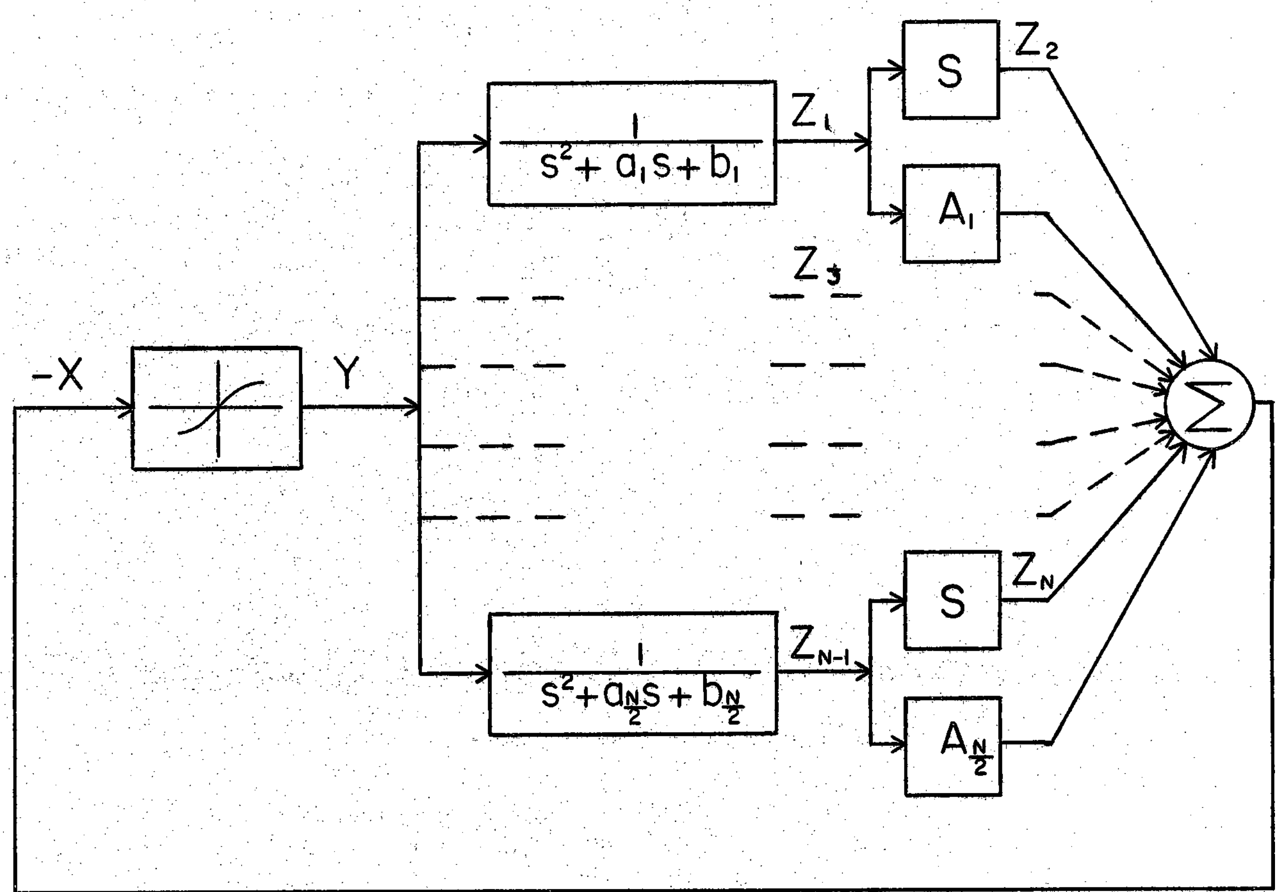

Figure 3.5

Block Diagram Interpretation of the Pseudo-Canonic Form of System Differential Equations.

$$\frac{dV}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} z_i z_j + x\, f(x) \; ; \quad g_{ij} = g_{ji} \tag{3.38}$$

where

$$g_{ij} = g_{ij}\,(a_i,\; b_i,\; c_{ij},\; A_i,\; B_i) \; .$$

If it is then possible to constrain the function W to be positive semi-definite, i.e.,

$$W(z_1, \ldots z_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} z_i z_j \geqslant 0 \; . \tag{3.39}$$

Then, according to Liapunov's Theorems, the system is globally asymptotically stable if V is negative definite, i.e., if

$$V(z_1, \ldots z_n) \leqslant 0 \quad \text{for} \quad \sum_{i=1}^{n} z_i^2 > 0 \; ; \quad V(o) = 0$$

and is unstable if V is not negative definite.

Example 3.5:

Consider the nonlinear system shown in Fig. 2.2 with

$$G(s) = G_1(s)\, G_2(s) = \frac{s + B}{s^2 + S}$$

and the nonlinear element having input-output characteristic of Fig. 2.5 . From Eq. 3.34 one obtains

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = - z_2 + y$$

$$y = f(x)$$

$$x = - (z_2 + Bz_1) \; .$$

The Liapunov's function is, from Eq. 3.35,

$$V = c_{11} z_1^2 + 2c_{12} z_1 z_2 + c_{22} z_2^2 \; .$$

Differentiating and substituting Eq. 3.34 one finds

$$\frac{dV}{dt} = (2c_{11} - 2c_{12})z_1z_2 + (2c_{12} - 2c_{22})z_2^{\,2} + y(2c_{12}z_1 + 2c_{22}z_2) \ .$$

Application of the constraint (Eq. 3.37) yields

$$-2c_{22} = 1 \ ,$$

$$-2c_{12} = B \ ,$$

and

$$\frac{dV}{dt} = (2c_{11} + B)z_1z_2 + (-B + 1)z_2^{\,2} + xy \ .$$

In order to make $\frac{dV}{dt}$ positive semidefinite, let

$$-2c_{11} = B \ ; \qquad B \leqslant 1 \ .$$

Then

$$V = -0.5Bz_1^{\,2} - Bz_1z_2 - 0.5z_2^{\,2}$$

and

$$\frac{dV}{dt} = (1 - B)z_2^{\,2} + xy \ .$$

Consequently, the system is globally asymptotically stable if $0 \leqslant B \leqslant 1$ and globally unstable if $B < 0$. More information about the characteristics of the nonlinearity is necessary to predict global stability or instability for values of $B > 1$. It is interesting to note that, at least for this example, exactly the same stability information is obtained from the first canonic transformation (Chapter II).

The preceding example illustrates the simplicity of construction of Liapunov's functions for pseudo-canonic systems of differential equations. At the same time, the need for stronger restrictions on the nonlinear element characteristic becomes apparent since the particular Liapunov's function (Eq. 3.35) would reject many stable systems. For example, stable linear systems ($y = kx$, $k > 0$) would be rejected if $B > 1$.

To increase the applicability of the method, consider another possible Liapunov's function

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} z_i z_j + k \int_0^x f(x)dx \; ; \qquad c_{ij} = c_{ji} \; . \qquad (3.40)$$

This yields

$$\frac{dV}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} z_i z_j + y(k\dot{x} + \sum_{i=1}^{n} e_i z_i) \; ; \qquad d_{ij} = d_{ji} \; . \qquad (3.41)$$

Substituting Eq. 3.34 for x in the above equation, one finds

$$\frac{dV}{dt} = W = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} z_i z_j + y(\sum_{i=1}^{n} h_i z_i - A_{2i-1} y) \; ;$$

$$g_{ij} = g_{ji} \; . \qquad (3.42)$$

Constrain $\frac{dV}{dt}$ (Eq. 3.42) such that

$$\sum_{i=1}^{n} h_i z_i = x \; . \qquad (3.43)$$

Then $\frac{dV}{dt}$ becomes

$$\frac{dV}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} z_i z_j + f(x)\left[x - \frac{kn}{2} f(x)\right] \; ; \qquad g_{ij} = g_{ji} \; . \qquad (3.44)$$

Fig. 3.6a illustrates the restriction placed by Eq. 3.44 on the input-output characteristic of the nonlinear element. Hence, by constraining

$$W = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} z_i z_j \geqslant 0 \qquad (3.45)$$

one may be able to prove stability of the systems which are unstable for high values of gain. If the system is unstable for low values of gain, then the constraint on $\frac{dV}{dt}$ (Eq. 3.43) may be replaced by

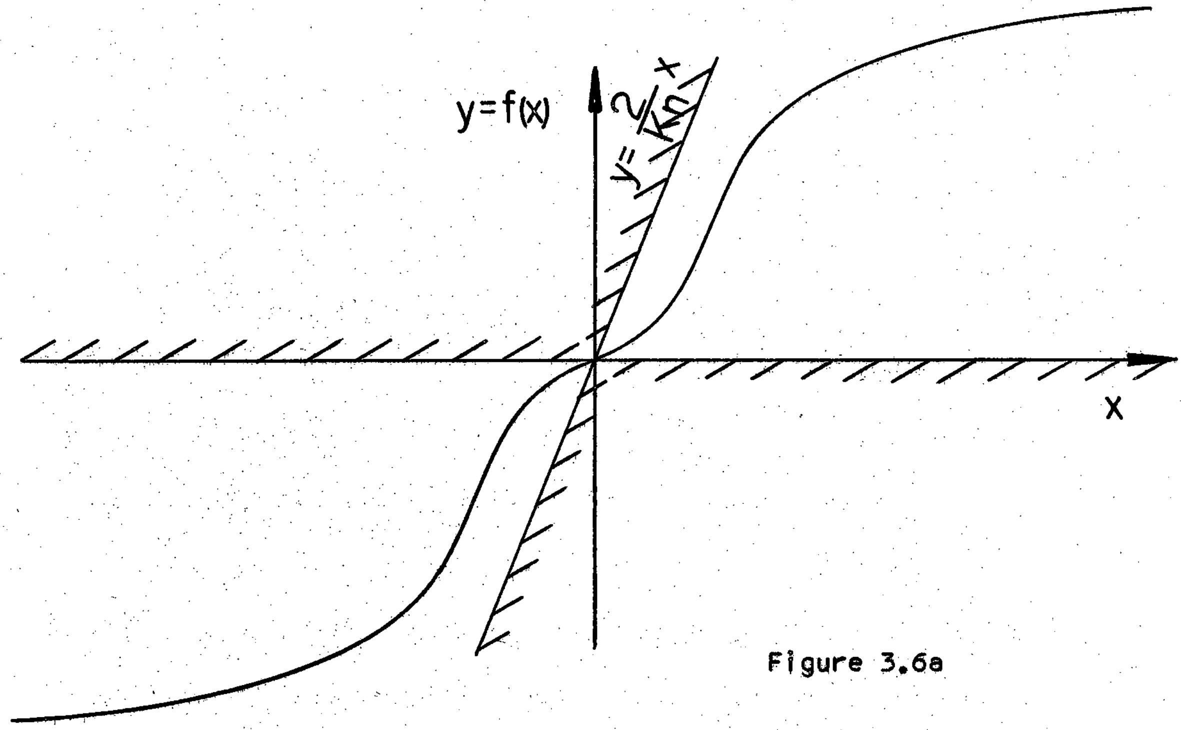

Figure 3.6a

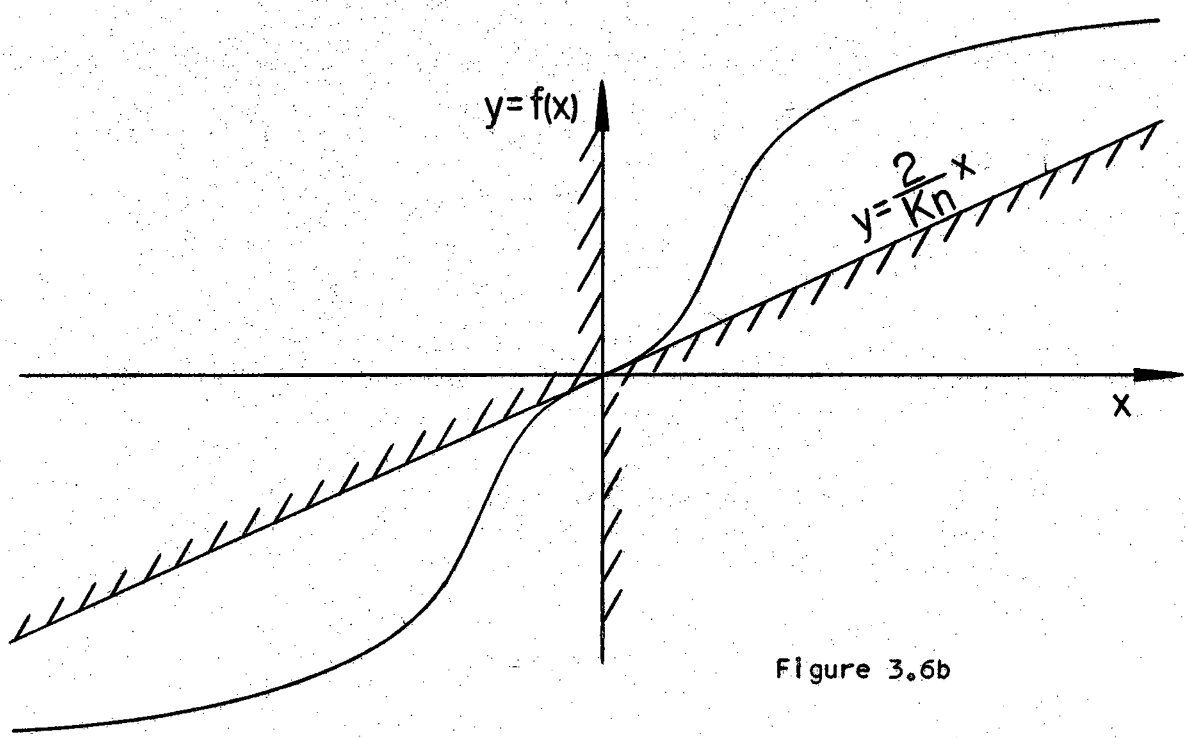

Figure 3.6b

Figure 3.6

Restriction of the Nonlinear Element Characteristic for Systems That Are Unstable for:

a) High Values of Gain

b) Low Values of Gain

$$\sum_{i=1}^{n} h_i z_i = - x \,. \qquad (3.46)$$

This yields

$$\frac{dV}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} z_i z_j + f(x) \left[- \frac{kn}{2} f(x) - x \right] . \qquad (3.47)$$

Fig. 3.6b illustrates the constraint on nonlinear element input-output characteristic by Eq. 3.47.

It is conceivable that other V-functions for pseudo-canonic systems of differential equations could be found, thus extending the applicability of the pseudo-canonic transformation still further. Even at the present time it appears that pseudo-canonic transformation yields useful stability information for the majority of the systems for which stability can be proved by the methods of Chapter II (canonic transformation) and also in some cases in which the canonic transformations are not applicable (e.g., multiple poles of G(s), poles of the origin, etc.). A distinct advantage of the pseudo-canonic transformation over most of the other methods of constructing Liapunov's functions is its ability to predict instability as well as stability. These observations lead one to the conclusion that further research in this direction may yield more useful results.

CHAPTER IV

CONCLUSIONS

Liapunov's "Second Method" of stability analysis is a very powerful tool in the analysis of certain nonlinear control systems. Its applicability is limited at the present time to a relatively small percentage of all practical closed-loop systems. This limitation is due to the lack of straightforward procedures for finding Liapunov's functions that apply to most practical systems. The canonic transformations, developed by Lur'e, enable one to find suitable Liapunov's functions, and, consequently, find sufficient conditions for asymptotic stability of certain practical systems with a single nonlinear gain element. The results of this report extend the applicability of the canonic transformations to all closed-loop systems with a single nonlinear element. This means that the number of systems which may be analyzed for stability by means of known Liapunov's functions for the canonic forms of system differential equations, has been substantially increased.

A critical evaluation of the second canonic form of system differential equations reveals that the applicability of this form of differential equations (and consequently, the associated simplified stability criteria) is limited to a very small percentage of actual control systems, in contrast to the first canonic form of system differential equations which enjoys a much greater applicability. Consequently, the attention has been focused on the first canonic form.

An inherent weakness of all the Liapunov's functions that have been used in the literature in connection with the first canonic form of system differential equations is the fact that these Liapunov's functions yield simplified stability criteria which select as stable only those systems that are actually stable for all positive values of the loop gain. In this report attempts have been made toward developing methods of predicting the conditions under which actually stable systems will be rejected by the simplified stability criteria and also attempts have been made toward eliminating these undesirable rejections. It is found that the root-locus of the linear portion of the loop transfer function G(s) is a useful tool in predicting which systems will definitely be rejected by the simplified stability criteria, as based on the first canonic form of differential equations. The root-locus also enables the designer to design an equalizer, by means of linear system design techniques, which will make the available simplified stability criteria applicable in proving the stability of many closed-loop systems. Needless to say, this approach will in many cases yield systems that are complex, costly, and difficult to build.

A somewhat more significant advance is the generalization of the pole-shifting technique which enables one to prove stability by means of the known simplified stability criteria for systems, the loop gain of which never falls below a certain value.

It is obvious that no practical system will have an infinite loop gain. Hence, the inability of the simplified stability criteria to put restrictions on the maximum value of loop gain represents the most serious disadvantage of the hitherto known simplified stability criteria. The zero-shifting technique developed in this report eliminates this disadvantage. Even though it has been necessary to modify the first canonic form of the system differential equations in order to accomplish the zero-shifting, new simplified stability criteria have been developed which can be used to establish sufficient conditions for asymptotic stability in systems where the maximum value of the equivalent gain of the nonlinear element is known.

A new method of constructing Liapunov's functions by means of pseudo-canonic transformations has been presented. It appears that the pseudo-canonic transformation retains the advantages of the canonic transformation, and at the same time simplifies the mathematical analysis considerably.

All the methods discussed in this report may be used to prove stability (or asymptotic stability) or to design an equalizer which will make an autonomous system stable (or asymptotically stable). While asymptotic stability of systems in the presence of initial disturbances only is a very important control system quality, the total stability (i.e., stability in the presence of bounded driving functions) is most frequently the desired system quality. For systems in which the nonlinear element appears at the end of the feedback path (as shown in Fig. 4.1), a proof of global asymptotic stability is, according the Theorem 1.4, equivalent to a proof of total stability. In other cases where the nonlinear element is followed by some linear elements, it may not be possible to separate the terms describing the driving function from the remainder of the system differential equations, and, consequently, Theorem 1.4 may not be applicable. Even though it could be argued intuitively that asymptotic stability still implies total stability in such single nonlinear element systems, no theoretical proof to this effect is available at the present time.

While Liapunov's "Second Method" appears to be one of the most promising advances in the area of nonlinear control system analysis, its applicability is at the present time limited to a relatively small percentage of practical control systems. This report represents an attempt

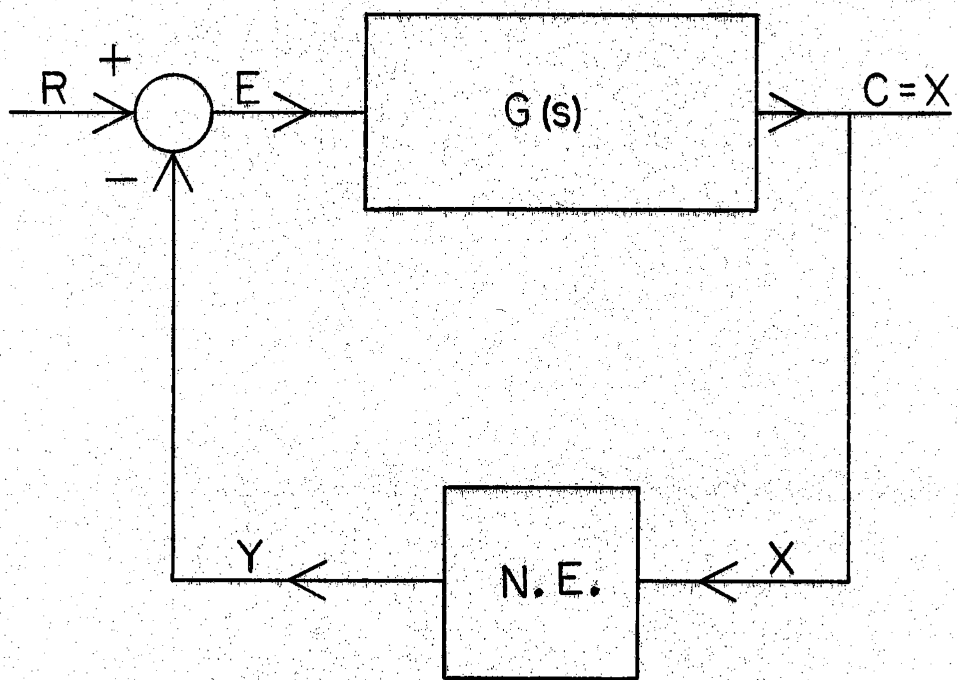

Figure 4.1

Block Diagram of a System with the Nonlinear Element in the Feedback Path.

to increase the applicability of the "Second Method". Any further results in this direction would be a most welcome addition to the limited number of nonlinear system analysis methods available to control engineers. It would be impossible to list all possible extensions of the method to all possible system configurations. Thus, only a few directions of extension of the "Second Method" for autonomous nonlinear systems will be suggested.

1. A majority of the known Liapunov's functions that are applicable to higher order systems yield sufficient and not necessary conditions for stability. It seems that at least in systems with a single nonlinear gain element suitable Liapunov's functions, together with the utilization of the root-locus concept for the linear part of the system, may also yield necessary conditions for stability.
2. While the first canonic transformation is applicable directly to systems with two or more nonlinear elements in series (as shown in Fig. 4.2) there are no known methods of finding a suitable Liapunov's function for such systems. Letov [5] proposes a canonic form of system differential equations and a Liapunov's function for systems with two actuators (in parallel). It is to be hoped that a similar procedure could be found for systems with several nonlinear elements in series.
3. While Liapunov's theorems are applicable directly to only the disturbed system responses with respect to static equilibrium states (singularities), it can easily be seen that an equation describing the boundary of the stability (or instability) region could as well serve the purpose of a Liapunov's function for systems exhibiting stable (or unstable) limit cycle oscillations [15], [22].

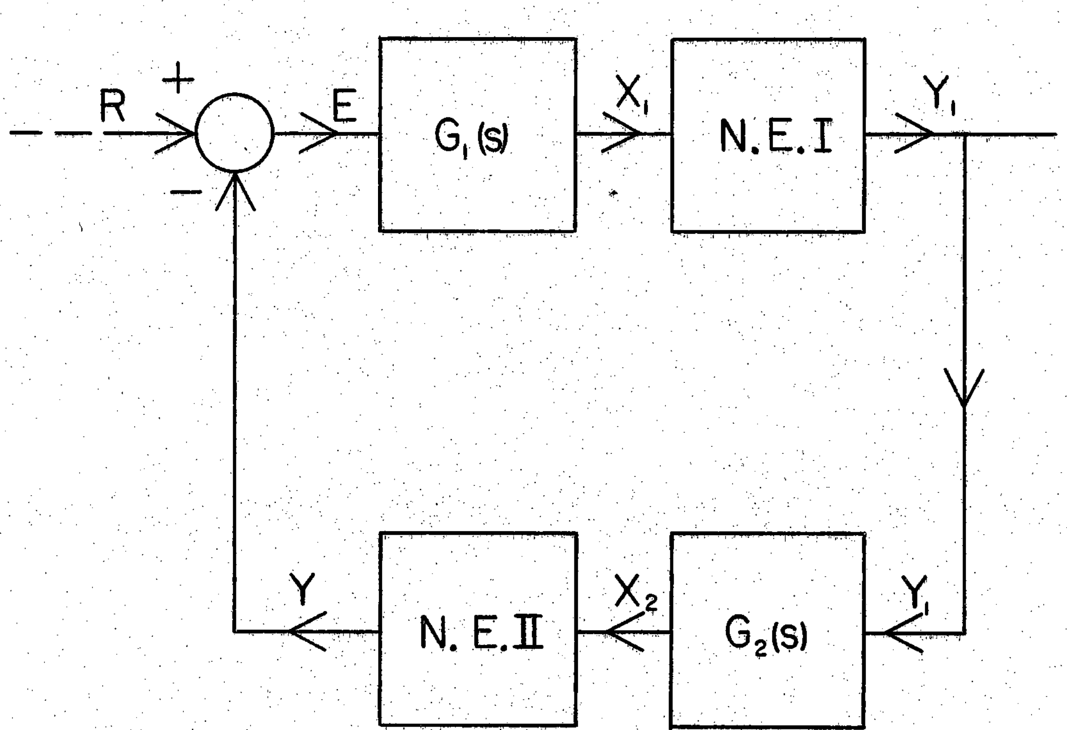

Figure 4.2

Block Diagram of a System with

Two Nonlinearities in Series.

From the engineering viewpoint, the exact determination of the limit cycle is frequently unnecessary. In most cases, it would be sufficient to estimate a region in the phase space in which a limit cycle is located. To accomplish this, the Liapunov's functions would not have to match exactly the path of the limit cycle. If one could find methods to construct such functions, then it would be possible to analyze the majority of practical control systems by the "second method". Once this analysis problem is solved, it will inevitably yield useful nonlinear synthesis procedures. The knowledge about nonlinear systems gained by such analytical methods could then be utilized to define important and meaningful specifications for nonlinear control systems.

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APPENDIX A

Solution of Stability Equations for The Second and Third Order Systems

1. The stability equation of the type

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_1 + \lambda_j} = \alpha_i \qquad i = 1, 2, \ldots n \qquad (2.20)$$

can be simplified by means of the expression

$$\sum_{i=1}^{n} a_i = \sqrt{\sum_{i=1}^{n} \beta_i} \qquad (A\text{-}1)$$

which is obtained by multiplying Eq. 2.20 by h_i and performing the summation from i=1 to n. By dividing Eq. 2.20 by h_i and then adding the equations from i=1 through n, one obtains

$$\sum_{i=1}^{n} \frac{a_i}{\lambda_i} = \sqrt{\sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i}} \; . \qquad (A\text{-}2)$$

Substitution of the above two equations into Eq. 2.20 yields for a third order system (n = 3)

$$a_1 = \frac{-\lambda_1^2 A + B + \sqrt{\left[\lambda_1^2 A - B\right]^2 - \alpha_1 \lambda_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} ; \qquad (A\text{-}3)$$

$$a_2 = \frac{-\lambda_2^2 A + B + \sqrt{\left[\lambda_2^2 A - B\right]^2 - \alpha_2 \lambda_2 (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 + \lambda_1)(\lambda_2 + \lambda_3)}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} ; \qquad (A\text{-}4)$$

$$a_3 = \frac{-\lambda_3^2 A + B + \sqrt{\left[\lambda_3^2 A - B\right]^2 - \alpha_3 \lambda_3 (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 + \lambda_1)(\lambda_3 + \lambda_2)}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} ; \qquad (A\text{-}5)$$

where

$$A = \sqrt{\beta_1 + \beta_2 + \beta_3} \tag{A-6}$$

and

$$B = \lambda_1 \ \lambda_2 \ \lambda_3 \sqrt{\frac{\alpha_1}{\lambda_1} + \frac{\alpha_2}{\lambda_2} + \frac{\alpha_3}{\lambda_3}} \ . \tag{A-7}$$

2. For the second order systems (n = 2) the solutions of Eq. 2.20 become

$$a_1 = \frac{-\lambda_1 \sqrt{\beta_1 + \beta_2} + \lambda_1 \lambda_2 \sqrt{\frac{\alpha_1}{\lambda_1} + \frac{\alpha_2}{\lambda_2}}}{\lambda_1 - \lambda_2} \ ; \tag{A-8}$$

$$a_2 = \frac{-\lambda_2 \sqrt{\beta_1 + \beta_2} + \lambda_1 \lambda_2 \sqrt{\frac{\alpha_1}{\lambda_1} + \frac{\alpha_2}{\lambda_2}}}{\lambda_2 - \lambda_1} \ . \tag{A-9}$$

3. The solution of the equation

$$2a_i \sum_{j=1}^{n} \frac{a_j}{\lambda_i + \lambda_j} = \beta_i \qquad i = 1, 2, \ldots n \tag{2.15}$$

can be obtained from the solutions of the preceding equation (Eq. 2.20) by replacing α_i by β_i and β_i by $\lambda_i \beta_i$ in the above solutions.

Solutions of other stability equations, such as Eq. 2.17, can be obtained in an analogous fashion by making appropriate substitutions in the solution of Eq. 2.20.

APPENDIX B

Global Stability of the Solution on a System of Nonlinear Differential Equations*

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A criterion for stability, under any initial conditions, of the trivial solution of a system of n nonlinear equations, the right-hand sides of which are independent of time, is described in this paper. This criterion represents some extension to nonlinear equations of the well-known theorem of A. M. Ljapunov ([1], p. 107) for linear systems; thus the sufficient conditions for global stability, developed in this paper become necessary and sufficient in the case of linear systems.

Consider the system of equations

$$\frac{dx_i}{dt} = X_i(x_i, \ldots x_n) \qquad (i = 1, \ldots n) \qquad \text{(B-1)}$$

where the right-hand sides X_i are continuous and differentiable functions in the entire space $-\infty < x_i < \infty$ $(i = 1, \ldots n)$, converging to zero at the point O (o, ... o).

Let us designate by $\frac{\partial X}{\partial x}$ the Jacobian matrix of the function X_i, i.e.,

$$\frac{\partial X}{\partial x} = \left\| \begin{matrix} \frac{\partial X_1}{\partial x_1} \cdots \frac{\partial X_1}{\partial x_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial X_n}{\partial x_1} \cdots \frac{\partial X_n}{\partial x_n} \end{matrix} \right\| \qquad \text{(B-2)}$$

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*Translated by Z. V. Rekasius from "Prikladnaja Matematika i Mekanika (P.M.M.), Vol. 18, 1954, pp. 735-737.

Theorem. In order that the trivial solution of Eq. B-1 be globally asymptotically stable, it is sufficient that there exist a positive symmetric matrix

$$A = \begin{Vmatrix} a_{11} & \cdot & \cdot & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdot & \cdot & a_{nn} \end{Vmatrix} \qquad \text{(B-3)}$$

with positive eigenvalues, such that the symmetric matrix of the products

$$\frac{1}{2}\left(\left[A\frac{\partial X}{\partial x}\right]_{ik} + \left[A\frac{\partial X}{\partial x}\right]_{ki}\right) \qquad \text{(B-4)}$$

has the eigenvalues $\lambda_i\ (x_1, \ldots x_n)$, $(i = 1, \ldots n)$, which satisfy, in the entire space $[x_i]$, the inequality

$$\lambda_i < -\delta \qquad (i = 1, \ldots n) \qquad \text{(B-5)}$$

where δ is a positive constant.

Proof. According to the Liapunov's theorem ([1], p. 82) the point 0 is asymptotically stable in the sense of Liapunov if condition (B-5) is satisfied. Let us start with the converse assumption that the region G of convergence towards point 0 does not enclose the entire space $-\infty < x < \infty$ $(i = 1, \ldots n)$.

Let us investigate the point p located on the boundary of G. The trajectory $f(p,t)$ that goes through the point p at $t = 0$ is completely contained within the boundaries of G (Erugin [2]). Consider two possibilities.

1. The trajectory $f(p,t)$ is inside the sphere

$$x_1^2 + x_2^2 + \ldots x_n^2 = R^2 \qquad \text{(B-6)}$$

for all $t > 0$, where R is a sufficiently large number. Only a finite number of singularities of the system (B-1) can be contained inside the sphere (B-6). Every singularity of the system (B-1) is, obviously, asymptotically stable in the sense of Liapunov. This fact can be established for every singularity in the same way as has been done for the point O. Consequently, every singularity of the system (B-1) possesses some region of convergence, i.e., remains isolated. Let us number the singularities which are inside and on the boundary of the sphere (B-6) as $q_1, \ldots q_k$ and surround each of these singularities by a neighborhood u_j ($j = 1, \ldots k$) contained entirely in the region of convergence of the respective singularity. The trajectory f(p,t) remains inside the sphere (B-6) and outside the neighborhood u_j ($j = 1, \ldots k$), since the boundary trajectory of G cannot belong to the region of convergence of the singularity q_j because the region of convergence is an open quantity. Hence, the trajectory f(p,t) remains, for $t > 0$, in a region governed by the inequality

$$x_1^2 + \ldots + x_n^2 > \ell \qquad \text{(B-7)}$$

where ℓ is a positive constant. Let us evaluate the time derivative along f(p,t) of the function

$$v(x_1, \ldots x_n) = \left(\sum_{i=1, j=1}^{n} a_{ij} x_i x_j \right)^{\frac{1}{2}} . \qquad \text{(B-8)}$$

The matrix (B-2) has positive eigenvalues, hence in the region under consideration the form (B-8), as a result of (B-7), does not converge to zero. Hence, we have

$$\frac{dv}{dt} = \left(\sum_{i,j=1}^{n} b_{ij}(x_1, \ldots x_n) x_i x_j \right) : \left(\sum_{i,j=1}^{n} a_{ij} x_i x_j \right)^{\frac{1}{2}} \qquad \text{(B-9)}$$

where the coefficients $b_{ij}(x_1, \ldots x_n)$ of the form are equal to the corresponding elements of the matrix (B-4). Thus, as a result of (B-5) and (B-7) in the region under consideration the following holds

$$\frac{dv}{dt} < -k^2 \qquad \text{(B-10)}$$

where k^2 is a positive constant. In the region under consideration the trajectory $f(p,t)$ is continuous in the interval $0 \leqslant t < \infty$.

Integrating (B-10) we obtain $v(t) - v(0) < -k^2 t$, which contradicts the inequality $v(x_1, \ldots x_n) \geqslant 0$ at sufficiently large values of t.

2. Let us consider the second possibility. As before, it is possible to show that the trajectory $f(p,t)$ can only be in the region in which the inequality

$$X_1^2 + \ldots + X_n^2 > 0 \qquad \text{(B-11)}$$

holds. As a result of (B-5) the form of the numerator of (B-9) remains negative definite, i.e., it satisfies the condition

$$\frac{dv}{dt} < -\frac{m^2(X_1^2 + \ldots + X_n^2)}{M(X_1^2 + \ldots + X_n^2)^{\frac{1}{2}}} < -n^2(X_1^2 + \ldots + X_n^2)^{\frac{1}{2}} \qquad \text{(B-12)}$$

where m^2 is the minimum and M^2 the maximum of the corresponding quadratic forms on the sphere

$$X_1^2 + \ldots + X_n^2 = 1.$$

Integrating (B-12) along $f(p,t)$ we get

$$v(t) - v(0) < -\int_0^t n^2(X_1^2 + \ldots + X_n^2)^{\frac{1}{2}}\,dt = -\int_0^s n^2 ds \qquad \text{(B-13)}$$

where s is the length of the curve $f(p,t)$ on the interval $(0,t)$; consequently,

$$ds = (X_1^2 + \ldots + X_n^2)^{\frac{1}{2}}\,dt \quad .$$

Under the assumed conditions $s \rightarrow \infty$ as time increases, hence it follows from (B-13) that $v(s) \rightarrow -\infty$ as $s \rightarrow \infty$, which contradicts the inequality $v(x_1, \ldots x_n) \geqslant 0$. This proves the theorem.

If an identity matrix E is selected in place of A then from the proven theorem it follows that, for global stability, it is sufficient that the symmetric Jacobian matrix of the right-hand sides of the system of equations (B-1) has negative eigenvalues satisfying inequality (B-5) in the entire space.

We will show now that in case of linear systems, the proven theorem becomes the referenced theorem of Liapunov ([1], pp. 82, 107). It is obvious that in the linear case all the quadratic forms of the variables X_i of this paper, after the substitution

$$X_i = C_{i1}x_1 + \ldots + C_{in}x_n, \quad (i = 1, \ldots n) \qquad \text{(B-14)}$$

become quadratic forms of the variables $x_1, \ldots x_n$, which satisfy Liapunov's theorems, and vice versa. In particular, the resolution of equation (B-14) with respect to x_j in case of asymptotic stability of the trivial solution of the linear differential equations follows from the fact that in this case the determinant $\delta = \|C_{ij}\|$ differs from zero.

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