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DECENTRALIZED ALGORITHMS FOR NASH EQUILIBRIUM PROBLEMS – APPLICATIONS TO MULTI-AGENT NETWORK INTERDICTION GAMES AND BEYOND

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DECENTRALIZED ALGORITHMS FOR NASH EQUILIBRIUM PROBLEMS – APPLICATIONS TO MULTI-AGENT NETWORK INTERDICTION GAMES AND BEYOND

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To my father, Prof. G. Sreekumaran.

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ABSTRACT

Sreekumaran, Harikrishnan PhD, Purdue University, December 2015. Decentralized Algorithms for Nash Equilibrium Problems – Applications to Multi-Agent Network Interdiction Games and Beyond. Major Professor: Andrew L. Liu.

Nash equilibrium problems (NEPs) have gained popularity in recent years in the engineering community due to their ready applicability to a wide variety of practical problems ranging from communication network design to power market analysis. There are strong links between the tools used to analyze NEPs and the classical techniques of nonlinear and combinatorial optimization. However, there remain significant challenges in both the theoretical and algorithmic analysis of NEPs. This dissertation studies certain special classes of NEPs, with the overall purpose of analyzing theoretical properties such as existence and uniqueness, while at the same time proposing decentralized algorithms that provably converge to solutions. The subclasses are motivated by relevant application examples.

One of the driving factors in our research is the need to design provably convergent decentralized methods to solve NEPs. While theoretical results about the convergence of such schemes for general games are unavailable, the methods are vastly popular amongst practitioners. Aside from the advantage of being relatively intuitive and easy to implement, decentralized methods also provide the means to analyze the process by which real agents interact strategically. From a computational perspective, these methods are also eminently suited towards distributed computing as well as parallel high performance architectures. Both these properties make them eminently suitable for solving large scale problems on modern computing platforms.

We start our exposition by introducing *decentralized network interdiction games*, which model the interactions among multiple interdictors with differing objectives op-

erating on a common network. These games can be seen as instances of GNEPs with non-shared constraints. We initially focus on *decentralized shortest path network interdiction games* (DSPI), and analyze the existence of equilibria for such games under both discrete and continuous interdiction strategies. We show that under continuous interdiction actions the game can be reformulated as a linear complementarity problem and solved by Lemke's algorithm. In addition, we present decentralized heuristic algorithms based on best response dynamics for games under both continuous and discrete interdiction strategies. Finally, we establish theoretical bounds on the worst-case efficiency loss of equilibria in these games, and use our decentralized algorithms to empirically study the average-case efficiency loss. We also formulate decentralized maximum flow interdiction (DMFI) as well as decentralized minimum cost flow interdiction (DMCFI) games. Unlike DSPI games, analysis is much more challenging for DMFI and DMCFI problems.

In the second half of the dissertation, we analyze computation of equilibria to Nash equilibrium problems under exogenous uncertainty. We restrict our attention to the class of potential games that has garnered great interest due to the large variety of practical problems, such as network games and Nash Cournot equilibria, that fall into the framework. We study the problem of computing the solutions to potential games where each player's objective function is assumed to depend on some exogenous uncertainty. Under the assumption that the players are risk-neutral expected-cost minimizers, we analyze the convergence of decentralized algorithms to equilibria. The primary tools we use to show convergence are that of the recently introduced multi-epiconvergence concept, as well as some recent results in decomposition schemes for player-wise convex global optimization problems. We show that under some reasonable assumptions, suitable approximation schemes combined with parallel or sequential best response type mechanisms produce *consistent* estimates of the Nash equilibria to potential games under uncertainty. We illustrate our algorithms by presenting numerical results on two practical application examples - power market equilibria under uncertainty and stochastic network traffic routing games.

1. INTRODUCTION

1.1 Background and motivation

Classically, optimization theory deals with decision making problems wherein an agent attempts to achieve certain goals while attempting to satisfy constraints on his/her decisions. Competition, i.e. interactions between multiple noncooperative agents each with their own optimization problems, is a natural extension of the typical optimization framework. The research presented in this dissertation is motivated by the need for efficient, decentralized algorithms for the computation of solutions to competition problems, especially in the context of network models.

Formally a Nash equilibrium problem (NEP) is characterized by a set of players, their objective functions and the set of feasible strategies allowed to each player. Let there be F players each controlling their own variables $x_f \in \mathbb{R}^{n_f}$. We denote the combined decision vectors of all players by $x = (x_1, x_2, \ldots, x_F)^T \in \mathbb{R}^n$, where $n = \displaystyle\sum_{f=1}^{F} n_f$. The set of players is denoted by $\mathcal{F} = \{1, \ldots, F\}$. With some slight abuse of notation, we often denote $x = (x_f, x_{-f})$ to stress the fact that x is a combination of an individual player's decision vector combined with the decisions of the *other* players.

Each player f has an objective function $\theta_f : \mathbb{R}^n \to \mathbb{R}$ that depends on both her own variables x_f and the other players' variables x_{-f}. Furthermore, the decisions of each player must fall into feasible sets $X_f \subseteq \mathbb{R}^{n_f}$. Note that if $X_f(\cdot)$ is a set-valued mapping that is in fact parametrized by x_{-f}, then we obtain a so-called generalized Nash game, also known as a generalized Nash equilibrium problem (GNEP).

Given the other players' decisions x_{-f} each player f solves the following problem.

$$\begin{array}{ll} \underset{x_f}{\text{minimize}} & \theta^f(x_f, x_{-f}) \\ \text{subject to} & x_f \in X_f. \end{array} \tag{1.1}$$

The field of non-cooperative game theory primarily deals with the resolution of NEPs, such as those defined by (1.1). At the core of this research is the solution concept for such problems proposed by Nobel laureate John Nash [78, 79]. *Nash equilibria* to the NEP defined by (1.1) are formally defined below.

Definition 1.1.1 *x_N is a Nash equilibrium (NE) to the NEP defined by* (1.1) *if*

$$\begin{array}{rcl} x_f^N & \in & X_f, \quad \textit{and} \\ \theta^f(x_f^N, x_{-f}^N) & \leq & \theta_f(y_f, x_{-f}^N), \ \forall \ y_f \in X_f. \end{array} \tag{1.2}$$

holds for each $f \in \{1, \ldots, F\}$.

The first condition in (1.2) ensures that the individual player variables are feasible to their respective problems, while the second condition ensures that the local decisions are optimal to each player's problem.

There has been a strong tie between the fields of mathematical programming and game theory, particularly the computation of Nash equilibria, since the development of linear programming techniques in the 50s. However, this link between optimization theory and equilibrium problems has become increasingly significant in recent years. This rise in interest in equilibrium problems from the operations research community is primarily motivated by the myriad applications that naturally admit equilibrium models. These include deregulated electricity markets [46, 56, 57, 116], telecommunication networks [2, 19, 98, 118], network design [3, 31, 83], traffic routing [39, 48, 92] amongst others.

Traditionally, optimization theorists have focussed on developing tractable reformulations for NEPs in the form of variational inequalities [77], optimization reformulations [82, 110, 111] etc. However, a significant portion of game theory literature, especially by economists, has focussed on decentralized algorithms for computing

Nash equilibria [44, 89]. Instead of focusing on computing equilibria via a centralized reformulation, these methods are based on simple iterative processes wherein each player iteratively updates their decisions by applying some local optimization rule. These decentralized methods for computing equilibria are becoming increasingly important in the context of the availability of high performance parallel computing platforms. However theoretical convergence results for decentralized methods are difficult to obtain, with the notable exception of certain special classes of games such as potential games [75, 88] and supermodular games [73, 107].

The main motivation for our research comes from a set of application problems, such as network interdiction, traffic routing, network design and power market equilibrium, that can be modelled using the NEP framework. In practical settings, these problems usually involve large underlying networks with thousands of nodes and edges. There could also realistically be scenarios involving a large number of heterogeneous players, each with their own objectives. Another important characteristic of such problems is that many problem parameters, such as customer demand, cost functions etc often cannot be estimated with sufficient accuracy. In this context, some form of uncertainty modeling must be incorporated into the framework, and suitable solution methodologies and approximation schemes must be designed.

Our work examines several aspects of these issues. One of the motivating problem classes for our research involves equilibrium problems arising from network security. We refer to the broad class of games as network interdiction problems, i.e. problems on which multiple players act against adversaries operating on a network. All the problems in this class involve bilevel optimization problems for each individual player. This usually results in non-differentiability of player objectives, which presents significant challenges for theoretical analysis and computation.

Another issue we address is that of approximating equilibria in games under uncertainty. Specifically, we analyze the problem of designing provably convergent approximation schemes, which can be combined with decentralized algorithms, to compute equilibria to games involving certain special uncertainty structures.

Before presenting our contributions in more detail, we first review related literature in each of these problem classes and outline the gaps and open questions.

1.2 Related literature

1.2.1 Multiagent Network Interdiction Games

Interdiction models involve adversarial situations in which an agent attempts to limit the actions of an adversary operating on a network. Such problems are usually modeled in the *Stackelberg* framework of leader-follower games and can be formulated as bi-level optimization problems. These models have been used in various military and homeland security applications such as breaking up drug traffic networks, prevention of nuclear smuggling and planning tactical air strikes. Interdiction models have also found use in non-military applications such as controlling the spread of pandemics and defending attacks on computer communication networks.

Network interdiction was first studied in a military setting for interdicting supply lines during the cold war [42, 43, 53]. Since then, the formulation has been applied to such varied real world settings as the spread of pandemics [6], controlling drug traffic networks [104, 114], protecting power infrastructure systems [94, 95], protecting communications networks [5, 74, 102], targeting air strikes [49, 72, 113], and preventing nuclear smuggling [114].

Traditionally interdiction problems have been analyzed from a centralized perspective. In other words, a single agent is assumed to analyze, compute and implement the strategies for interdiction. However in many situations it might be desirable and even necessary to have a *decentralized* perspective on the interdiction problem. For instance, such situations might arise when the computational resources utilized for solving the problem are distributed across the network. There are also problems in which a supervising body in control of a common network might assign various adversaries on the network to individual agents. In such situations, the study of the equilibrium solutions arising out of selfish, uncoordinated interdictors operating on a common network against individual adversaries becomes warranted. The idea of decentralization also has links with learning concepts from game theory such as best/better response mechanisms.

To the best of our knowledge, there has been no previous research on decentralized network interdiction games. As a result, not much is known about the inefficiency of equilibria for these games or intervention strategies to reduce such inefficiencies. There has been a considerable amount of work, however, on interdiction problems from a centralized decision-maker's perspective. As mentioned earlier, interdiction problems have been studied in the context of various military and security applications. For extensive reviews of the existing academic literature on interdiction problems, we refer the readers to Church et al. [20] and Smith and Lim [102].

One potential reason for the lack of attention paid to decentralized network interdiction games may be that such games often involve nondifferentiable objective functions, as each interdictor's optimization problem usually entails a max-min type of objective functions. Games involving nondifferentiable functions are generally challenging, in terms of both theoretical analysis of their equilibria and computing an equilibrium. While in some cases (such as in the case of shortest path interdiction), a smooth formulation (through total unimodularity and duality) is possible, such a reformulation will lead the resulting network game to the class of GNEPs, in which both the agents' objective functions as well as their feasible action spaces depend on other agents' actions. Although the conceptual framework of GNEPs can be dated to Debreu [26], rigorous theoretical and algorithmic treatments of GNEPs only began in recent years [33]. Several techniques have been proposed to solve GNEPs, including penalty-based approaches [34, 45], variational-inequality-based approaches [77], Newton's method [30], projection methods [120], and relaxation approaches [70, 108]. Most of the work on GNEPs has focused on games with shared constraints due to their tractability [32, 52]. In such games, a set of identical constraints appear in each agent's feasible action set. However, as will be seen later, in a typical decentralized network interdiction game, the constraints involving multiple agents' actions that appear in each agent's action space are not identical. As a result, such games give rise to more challenging instances of GNEPs.

Given the concept of selfish behavior amongst strategically interacting players, it is important to quantify the inefficiency of such behavior relative to perfect socially optimal centralized solutions. There have been many studies on the inefficiency of equilibria in other game-theoretic settings. Most of the efforts have been focused on routing games [10, 84, 112], in which selfish agents route traffic through a congested network, and congestion games [88], a generalization of routing games. Some examples include [8, 17, 18, 21, 24, 41, 91, 93, 105]. Several researchers have also studied the inefficiency of equilibria in network formation games, in which agents form a network subject to potentially conflicting connectivity goals [1, 3, 4, 27, 31]. The inefficiency of equilibria has been studied in other games as well, such as facility location games [109], scheduling games [69], and resource allocation games [63, 64]. Almost all of the work described above study the worst-case inefficiency of a given equilibrium concept. Although a few researchers have studied the average inefficiency of equilibria, either theoretically or empirically, and have used it as a basis to design interventions to reduce the inefficiency of equilibria [23, 106], research in this direction has not received much attention. One of the main motivations of our work is to utilize decentralized algorithms to efficiently quantify the average-case inefficiencies associated with equilibria, in realistic instances of network interdiction games.

1.2.2 Approximating equilibria under exogenous uncertainty

The main motivation for our work on games under exogenous uncertainty is the growing relevance of equilibrium models in the context of a large variety of engineering and economic problems. Much of the early work on Nash's equilibrium model focussed on the economic theory of how rational firms acted in various markets. In recent times, due to the deregulation of infrastructure markets such as electricity [55, 116, 119], gas [51] etc., the NEP model has taken on growing relevance as an analytical tool for studying the behavior of market participants. Most realistic applications involving the NEP model involve various problem parameters, such as customer demand or manufacturing costs, that can only be estimated using some probability distribution.

Typically, algorithms to solve stochastic NEPs involve using a suitable reformulation, such as variational inequality reformulations [37], complementarity reformulations, Nikaido-Isoda function based optimization reformulations [111] etc. Closely related to our work is the topic of solving stochastic variational inequalities (SVIs). The exponential convergence of SAA methods for SVI problems has been shown in [115]. There has also been a growing interest in designing Stochastic Approximation (SA) schemes for SVI problems. Such schemes, where function values and derivatives are approximated via simulation, have been shown to converge under mild conditions [51, 62]. However, most of the solution methodologies proposed for games under uncertainty that use VI or optimization reformulations fall under the centralized algorithm framework. Very little work has been done on the idea of developing distributed computational schemes that may be combined with suitable approximation methods that can compute solutions of stochastic NEPs. In this context, the closest work to ours is an investigation of extensions of SA type methods for SVIs, including some distributed methods, have been investigated in [117]. However, in contrast to our approach, even distributed methods for solving SVIs usually require some degree of coordination between the players in choosing step-lengths or other algorithmic parameters.

1.3 Contributions

1.3.1 Network interdiction games

One of the major contributions of our work is the formulation of decentralized network interdiction games in which we model multiple agents operating on a common network interdicting individual adversaries. To the best of our knowledge, this work represents the first attempt to analyze network interdiction from the perspective of strategic interactions between multiple interdictors.

Specifically, we propose to study three classes of decentralized network interdiction games.

- **Decentralized Shortest Path** interdiction (DSPI) game - In this game each player f is assumed to be protecting a node t^f from an adversary at node s^f by increasing the distance between the two nodes as much as possible. The adversaries solve a shortest path problem between the respective nodes. Each leader must then maximize the shortest path corresponding to their adversaries. The feasible interdiction strategies for each player and interdiction costs per arc for each player may also depend on the strategies of other players involved.

- **Decentralized Maximim Flow** interdiction (DMFI) game - In this game each player attempts to minimize the maximum flow of some undesirable substance that its adversary is trying to push across the network from a source node s^f to a sink node t^f. The players utilize some constrained resource in order to reduce the capacities on the arcs of the network.

- **Decentralized Minimum Cost Flow** interdiction (DMCFI) game - This game is a generalization of the previous two games. Each player attempts to interdict an adversary that wishes to organize the flows on a network such that the demands at each node are satisfied while trying to minimize the cost. The players utilize some constrained resource to increase the cost of flow for the adversaries on each arc. The objective of the interdictors is to maximize the minimum cost of flow across the network.

The major contributions of our work on interdiction games are as follows. First, we establish the existence of equilibria for DSPI games with continuous interdiction. In DSPI games with discrete interdiction, the existence of a pure strategy Nash equilibrium (PNE) is more subtle. We first demonstrate that a PNE does not necessarily exist in general discrete DSPI games. However, when all agents have the same source-target pairs (i.e., multiple agents try to achieve a common goal independently), a PNE exists in discrete DSPI games.

Second, for DSPI games under continuous interdiction, we show that each agent's optimization problem can be reformulated as a linear programming problem. As a result, the equilibrium conditions of the game can be reformulated as a linear complementarity problem with some favorable properties, allowing it to be solved by the well-known Lemke algorithm. For discrete DSPI games (and for continuous games as well), we present decentralized algorithms for finding an equilibrium, based on the well-known best-response dynamics (or Gauss-Seidel iterative) approach. While such an approach is only a heuristic method in general, convergence can be established for the special case when the agents have common source-target pairs. For more general cases, we obtain encouraging empirical results for the performance of the method on several classes of network structures.

Third, in measuring the efficiency loss of DSPI games due to the lack of coordination among noncooperative interdictors, as compared to a centralized interdiction strategy (that is, a strategy implemented by a single interdictor with respect to all the adversaries), we establish a theoretical lower bound for the worst-case price of anarchy of DSPI games under continuous interdiction. Such an efficiency loss measure, however, may be too conservative, and we therefore use the decentralized algorithms to empirically quantify the *average-case* efficiency loss over some instances of DSPI games. These results can help central authorities design mechanisms to reduce such efficiency losses for practical instances.

We also present formulations for decentralized max-flow interdiction and min-cost flow interdiction games, both under continuous and discrete interdiction. While these problems share several characteristics with DSPI games, we summarize why they present significantly more challenging instances of GNEPs.

1.3.2 Potential Games Under Exogenous Uncertainty

Although the initial focus of our research is on the properties of equilibria in deterministic network interdiction games under complete information, the second half of this dissertation deals with the topic of exogenous uncertainty in more general games. Specifically, we consider games where each agent's objective function is subject to some common underlying uncertainty factor. The exogeneity assumption essentially means that information about the uncertainty, such as the probability distribution of the random variables involved, is common knowledge to all the players.

Formally, we analyze decentralized approximation algorithms for computing solutions to Nash equilibrium problems (NEPs) under *exogenous* uncertainty. We consider games involving a set of F players, wherein each player f controls some decision variables x_f and solves the following optimization problem:

$$\begin{array}{ll} \underset{x_f}{\text{minimize}} & \phi_f(x) = \mathbb{E}\left[\theta_f(x_f, x_{-f}; \xi)\right] \\ \text{subject to} & \qquad\qquad x_f \in X_f. \end{array} \tag{1.3}$$

The convergence of decentralized algorithms to equilibria is difficult to prove for the general class of games under uncertainty. However by restricting our attention to games where the underlying deterministic version possesses an exact potential function, we are able to provide meaningful theoretical results. Since potential games arise from a wide variety of applications from network design games to Nash-Cournot market equilibrium problems, our research is well motivated. We consider risk-neutral players solving expected value optimization problems, where each agent's objective function depends on the other players' decisions as well as common random variables.

Our research is motivated by the need to analyze decentralized approximation schemes for potential games under exogenous uncertainty. Specifically, we present sample average approximation type schemes, where the expectation terms in each agent's objective function is resolved using an i.i.d sampling of the underlying random variable. We first show that under suitable conditions, solutions to appropriate SAA equilibrium problems converge asymptotically to the solutions of the true stochastic equilibrium problem. We then combine this SAA scheme with decentralized solution methods, such as Gauss Jacobi type parallel best-response or Gauss Seidel type sequential best-response. Under some fairly mild assumptions, we are able to prove that decentralized approximation schemes provide *consistent* estimates of the equilibria to the true problem, i.e. that they converge in probability to the true solution as the sampling size for the approximations increase.

The main motivating applications for our research on games under exogenous uncertainty come from electricity markets and communication systems. For the former, we analyze the strategic interactions between power generators competing in a power market. The generating firms are assumed to bid supply quantities to an independent system operator (ISO) who then dispatches power to meet the demand at each node of the network. In general, the generating firms often anticipate the effect of the ISO's problem while deciding their supply bids. However, this results in an "endogenous" model where each generation firm solves a bilevel optimization problem, resulting in an equilibrium problem with equilibrium constraints (EPEC) model for the market. Due to the intractability of EPECs, we focus instead on an "exogenous" model where the generating firms and the ISO interact in a static game. In other words, the firms and the ISO are assumed to act simultaneously, rather than sequentially. We show that this results in a potential game formulation for the market equilibrium. Our decentralized approximation methods are applied to this model, under uncertainty in the inverse demand functions. We prove that the conditions required for provable convergence of these methods are satisfied for realistic instances of the model. Further, we present some numerical results on small instances of the problem. An interesting observation from our numerical experiments is that schemes where different samples are used by different players result in convergence similar to the case with uniform sampling.

The second application we consider is that of atomic network traffic routing games, which arise in telecommunication applications. These games arise naturally in the context of network flow problems where agents wish to route flow between nodes on a common congested network. The latencies on the arcs on the network depend on the total amount of flow on the arc, and thus on the routing decisions of all the players. Quite often, the latencies are subject to uncertainty in the form of weather or other such external factors. As such, it can be shown that routing games under uncertainty in latencies, fall naturally into the framework of potential games under exogenous uncertainty. We show that these models satisfy conditions required for the convergence of decentralized approximation schemes and present numerical results on a variety of network topologies. In particular we present some preliminary results on the scaling properties of our regularized parallel best response based approximation algorithm.

The approximation scheme we present requires that the solutions computed to the SAA version of the stochastic NEP are exact. But most numerical procedures require finite termination criteria. We present a preliminary analysis of the convergence of approximate solutions to the SAA problem as well as inexact solves at each iteration.

In our theoretical results, we assume that all the players use the same samples of the underlying random vector. However it is reasonable that different players might utilize their own samples drawn from the same underlying distribution in order to compute equilibria. While we do not have theoretical convergence results for such sampling schemes, we present a simple example in which decentralized algorithms when combined with disparate sampling exhibits good empirical performance.

1.4 Summary

The remainder of this dissertation is organized as follows. Chapter 2 summarizes some background theory for NEPs that is required for the analysis presented in the remainder of the dissertation. A brief survey of decentralized algorithms is given, along with pseudocodes for parallel and sequential best response algorithms that are the focus of much of our work. Some popular centralized reformulations for NEPs are also summarized.

Chapter 3 provides detailed formulations of the three classes of DNI games and offers some theoretical results regarding the properties of such games. Focus is primarily on the shortest path interdiction game, for which we give detailed analysis on the theoretical properties such as existence and uniqueness of equilibria. Reformulation of the continuous version of the problem as an LCP is presented, and a proof of the applicability of Lemke's algorithm is given. We also present details of heuristic decentralized algorithms, that can be shown to converge to equilibria in DSPI games with common adversaries, while having good empirical performance even for the general case. Formulations for DMFI and DMFCI games are also presented and the challenges involved in their analysis are summarized.

In Chapter 4, we introduce player-wise convex potential games under exogenous uncertainty and state various properties of such games that we use in our analysis. The SAA approximation scheme for these games is outlined and convergence of regularized best-response schemes as sampling size increases is proven using the tool of multi-epiconvergence.

Chapter 5 discusses two important practical applications of the NEP under uncertainty model presented in Chapter 4. Details of the power market equilibrium problem between generation firms and independent system operators are given. The model for selfish atomic network routing is also given. Numerical results are presented both from problems in literature as well as example networks that we generate.

Chapter 6 outlines some ongoing work on extensions of the decentralized approximation scheme presented in Chapter 4. The issue of approximate solutions of subproblems, and convergence of this modified procedure to approximate equilibria is addressed. The question of whether disparate sampling schemes, where each player uses distinct samples of the common random vector is examined. Finally, possible extensions of the multi-epiconvergence concept to the GNEP domain is presented.

Chapter 6 summarizes the dissertation and provides some concluding remarks on possible future research directions.

2. BACKGROUND THEORY

In this chapter, we review and summarize some key theoretical results that we use in our study of Nash equilibrium problems. We give the formal definition of an NEP, as well as its generalization to the case involving constraint interactions. A brief review of decentralized algorithms for NEPs is presented. We also identify a key subclass of NEPs, known as potential games, that we later use in our analysis. We conclude the chapter by briefly reviewing some centralized methods to solve NEPs.

2.1 Problem definition

In this section, we give some definitions, assumptions and classifications for NEPs which we use in the later sections for various reformulations. Recall the NEP defined in section 1.1, where each player solves problem (1.1). To standardize our notation, we will henceforth refer to this problem as NEP$(\theta_f, X_f)_{f=1}^F$.

Typically the constraint sets for each player, i.e X_f, are specified using parametric constraints. The general form for such sets is given by -

$$X_f = \{x_f \in \mathbb{R}^{n_f} : g_f(x_f) \leq 0\}. \tag{2.1}$$

Note that the constraints for each player are given by the vector valued functions $g^f : \mathbb{R}^{n_f} \to \mathbb{R}^{m_f}$. We denote by $\mathbf{X}$ the cartesian product of the feasible sets for each player. Formally,

$$\mathbf{X} = \prod_{f=1}^{F} X_f. \tag{2.2}$$

Unless otherwise stated, we make the following blanket assumption throughout the remainder of this report.

Assumption 1 *The players' objective functions $\theta^f(x_f, x_{-f})$ and the constraint functions $g_f(x_f)$ are continuous, and convex with respect to the player's own variables x_f.*

2.2 Decentralized algorithms for NEPs

The primary focus of this thesis is on developing decentralized algorithms for solving NEPs. As such, we introduce several classes of such algorithms. Since the fundamental principle behind the Nash equilibrium concept is that a solution point is a fixed point for the so called "best-response" mapping, perhaps the most natural decentralized method to solve NEPs is the best-response based approach.

2.2.1 Best Response Algorithms

Best-response algorithms are methods where each player updates their decisions solely based on their payoffs and constraints, given the current state of the game as represented by the other players' decisions. Typically best-response algorithms are classified based on whether the player updates happen sequentially or in parallel.

Sequential best-response algorithms are similar to non-linear Gauss-Seidel type algorithms. The broad scheme is given in Algorithm 1.

Algorithm 1 Sequential best-response (Gauss-Seidel)

Step 0: Initialize - Set $x^0 \leftarrow (x_f^0)_{f=1}^F$, $k \leftarrow 0$.

Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**

Step 2: Main Iteration:

FOR $f = 1, \ldots, F$, let x_f^{k+1} solve

$$\begin{array}{ll} \underset{x^f}{\text{minimize}} & \theta_f(x_1^{k+1}, \ldots, x_{f-1}^{k+1}, x_f, x_{f+1}^k, \ldots, x_F^k) \\ \text{subject to} & x_f \in X_f. \end{array} \tag{2.3}$$

Step 3: Update: $x^{k+1} = (x_f^{k+1})_{f=1}^F$.

In the initialization step, it is assumed that x^0 is jointly feasible. Typically the termination condition checks for closeness between successive iterates, either in the payoff space or the decision space.

In contrast to the sequential best-response algorithm (Algorithm 1), Gauss Jacobi type methods update player decisions in parallel. In other words, at the k_{th} iteration of the algorithm, all the players update their decisions simultaneously taking as given the decisions of the other players from the previous iteration, i.e. x_{-f}^{k-1}. This Gauss-Jacobi type scheme is given below in Algorithm 2.

Algorithm 2 Parallel best-response (Gauss-Jacobi)

Step 0: Initialize - Set $x^0 \leftarrow (x_f^0)_{f=1}^F$, $k \leftarrow 0$.

Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**

Step 2: Main Iteration:

FOR $f = 1, \ldots, F$, let x_f^{k+1} solve

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \theta_f(x_f, x_{-f}^k) \\ \text{subject to} \quad & x_f \in X_f. \end{aligned} \tag{2.4}$$

Step 3: Update: $x^{k+1} = (x_f^{k+1})_{f=1}^F$.

Best-response algorithms in either of the two forms given above provide natural and intuitive methods to approach the NEP. However, in general these methods cannot be guaranteed to converge to equilibria [38]. Provable convergence of the methods is only available if we restrict our attention to special classes of games. In the following discussion, we present one such class of games, namely potential games, that plays an important role in our research.

2.2.2 Potential games

Potential games were first introduced by Rosenthal [88], in the context of congestion games wherein he showed the existence of pure strategy equilibria utilizing the potential function approach. The essential characteristic of a potential game, is the existence of a function that captures on a global level, any unilateral deviation in an individual player's cost function. In this paper, we focus on a sub-class of potential games that possess exact potential functions, in which unilateral deviations in a given player's cost function is reflected by an equal change in the potential function. In their seminal paper on potential games, Monderer and Shapley [75] formalize the concept and give various classifications. They also prove some important results about potential games characterizing the equilibria to such games using best response dynamics.

The formal definition of potential functions in the context of NEPs is given below.

Definition 2.2.1 *A continuous function $P : \mathbb{R}_n \to \mathbb{R}$ is said to be an* ordinal *potential function to the NEP$(\theta_f, X_f)_{f=1}^F$ if for each $x \in \mathbf{X}$ we have*

$$\theta_f(x_f, x_{-f}) < \theta_f(y_f, x_{-f}) \quad \textit{iff} \quad P(x_f, x_{-f}) < P(y_f, x_{-f}), \ \forall y_f \in X_f, \tag{2.5}$$

for each player $f \in \mathcal{F}$.

Definition 2.2.2 *A continuous function $P : \mathbb{R}_n \to \mathbb{R}$ is said to be a* weighted *potential function to the NEP$(\theta_f, X_f)_{f=1}^F$ if for each $x \in \mathbf{X}$ we have*

$$\theta_f(x_f, x_{-f}) - \theta_f(y_f, x_{-f}) = w_f(\ P(x_f, x_{-f}) - P(y_f, x_{-f})), \ \forall y_f \in X_f, \tag{2.6}$$

for each player $f \in \mathcal{F}$ for some vector of positive weights $w = (w_1, \ldots, w_F)$.

If the NEP admits a weighted potential function where the weights are all unity, then it is said to admit an *exact* potential function. If the NEP is such that the objective functions of the individual player problems do not involve other players' variables, i.e. $\theta_f(x) = \theta_f(x_f)$, then the sum of the objective functions provides an *exact* potential function.

Another extension of the potential function concept is given below.

Definition 2.2.3 *A continuous function $P : \mathbb{R}_n \to \mathbb{R}$ is said to be an generalized ordinal potential function for NEP$(\theta_f, X_f)_{f=1}^F$ if for each $x \in \mathbf{X}$ we have*

$$\theta_f(x_f, x_{-f}) < \theta_f(y_f, x_{-f}) \implies P(x_f, x_{-f}) < P(y_f, x_{-f}), \ \forall y_f \in X_f$$

for each player $f \in \mathcal{F}$.

Generalized ordinal potential games are, in some sense, the broadest class of potential games. Clearly ordinal potential games are a subclass of generalized ordinal potential games, because we impose the reverse implication in (2.5). Moreover, since the weights w_f are strictly positive for a weighted potential game, (2.6) implies (2.5). Therefore, weighted potential games form a subclass of ordinal potential games.

For NEPs with potential functions, there is an immediate characterization of equilibria as global minimizers of the potential function over the Cartesian product of the feasible sets X_f.

Theorem 2.2.1 *[75] Suppose NEP$(\theta_f, X_f)_{f=1}^F$ admits an ordinal potential function P . x^N is an equilibrium solution if it solves the following optimization problem.*

$$\min_{x \in \mathbf{X}} \; P(x).$$

Proof Suppose $x^N \in \operatorname*{argmin}_{x \in \mathbf{X}} P(x)$. Then $x_f^N \in X_f$ by the definition of $\Omega(x)$. Furthermore,

$$\begin{aligned} P_f(x_f^N, x_{-f}^N) - P_f(y_f, x_{-f}^N) &\leq 0 \quad \forall\, y_f \in X_f. \\ \implies \theta_f(x_f^N, x_{-f}^N) - \theta_f(y_f, x_{-f}^N) &\leq 0 \quad \forall\, y_f \in X_f. \end{aligned}$$

where the first equation follows from the minimality of x_f^N. ■

Note that the converse relationship does not hold true. Not every solution to the NEP is also a global solution to the potential-optimization problem. This is easy to see since the only requirement for a NE x^N is that no unilateral deviations are allowed. However it is possible that there exists points $y \in \mathbf{X}$ such that $P(y) < P(x^N)$. Indeed different NEs might have different P values.

Since weighted and exact potential functions are special cases of ordinal potential functions, the result is immediately extendable to NEPs with weighted/exact potential functions.

There is also an inherent link between potential games and networks. Shapley and Monderer [75] show that every finite potential game[1] is isomorphic to a *congestion* game as defined by Rosenthal [88]. It was also shown in [75] that for finite games, best response paths[2] in the action space converge to Nash equilibria in a finite number of steps.

For NEPs where each player's objective function θ_f is continuously differentiable in the joint decision variable x, potentiality may be characterized using gradients. In this case, a function P is an exact potential for NEP$(\theta_f, X_f)_F^{f=1}$ if and only if P is continuously differentiable and

$$\nabla_{x_f}\theta_f(x_f, x_{-f}) = \nabla_{x_f}P(x_f, x_{-f}), \quad \forall f \in \mathcal{F}. \tag{2.7}$$

Suppose now that θ_f are twice continuously differentiable. Then NEP$(\theta_f, X_f)_{f=1}^F$ has an exact potential function if and only if

$$\frac{\partial^2 \theta_f}{\partial x_f^{i_f} \partial x_{f'}^{j_{f'}}} = \frac{\partial^2 \theta_{f'}}{\partial x_f^{i_f} \partial x_{f'}^{j_{f'}}}, \quad \forall f, f' \in \mathcal{F}. \tag{2.8}$$

Here i_f and $j_{f'}$ are any components of x controlled by f and f' respectively.

In either of the differentiable game classes discussed above, it is easy to show that exact potential functions are identical up to a constant.

2.2.3 Non best-response based decentralized algorithms

Best-response mechanisms, such as the Gauss-Seidel and Gauss-Jacobi methods presented in Algorithms 1 and 2, constitute the most fundamental forms of decentralized algorithms for NEPs. Game theory literature is rich with references for various other types of learning-based mechanisms such as fictitious play [15], rational or Bayesian learning [66], reinforcement learning [65], no-regret learning [54], and other evolutionary dynamics. We refer the interested reader to [44, 101] for details on learning theory in games.

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[1] A game/NEP is said to be finite if each player f's action space X_f is a finite set.

[2] Best response paths consist of sequences of decision vectors $\{x^k = (x_1^k, x_2^k, \ldots, x_F^k)_T\}$ such that $x_f^{k+1} = \underset{x_f \in X_f(\bar{x}_{-f})}{\operatorname{argmax}} \theta_f(x_f, \bar{x}_{-f})$, where $\bar{x} = (x_1^{k+1}, \ldots, x_{f-1}^{k+1}, x_f^k, \ldots, x_F^k)$.

However as mentioned before, convergence analysis for these decentralized learning mechanisms is by no means easy. Usually, theoretical convergence results are restricted to special classes of games for each learning method. For instance, while it has been shown that fictitious play converges to Nash equilibria in finite zero-sum, potential and supermodular games [58], Shapley [100] provides a class of games for which the mechanism fails to converge. In fact, in several cases, learning mechanisms can only be shown to converge to equilibrium concepts weaker than Nash equilibria. The case of regret matching is an example for this type of a result, wherein convergence of the algorithm to *correlated* equilibria[3] was shown in [54]. More recently, the theory of differential inclusions has been used to show the convergence of learning methods such as no-regret learning and fictitious play to correlated equilibria [11].

2.3 Centralized algorithms for NEPs

The analysis of NEPs and computation of their solutions usually involves reformulating them as optimization problems, variational inequalities, fixed point problems or other problems for which there are known solution methodologies. We present here some such common approaches.

First, we define below varational inequalities [35] and quasi-variational inequalities [81] that we use in reformulations for NEPs.

Definition 2.3.1 *The variational inequality problem* $\boldsymbol{VI}(X, F(x))$ *consists of finding a vector* $\bar{x} \in X$ *such that* $(y - \bar{x})^T F(\bar{x}) \geq 0$ *for all* $y \in X$.

An important concept in the theoretical analysis of NEPs is that of the Nikaido-Isoda (NI) function [80], also known as the Ky-Fan function.

Definition 2.3.2 *The NI function for the NEP*$(\theta_f, X_f)_{f=1}^F$ *is given by*

$$\Psi(x, y) := \sum_{f=1}^{F} \left[\theta_f(x_f, x_{-f}) - \theta_f(y_f, x_{-f})\right]. \tag{2.9}$$

[3]Correlated equilibrium is a more general solution concept for games than Nash equilibrium. The essential idea is that players choose their actions based on their observations of some random event. Cf. [7].

The individual terms within the summation capture the gains for each player if she changes her decision from x_f to y_f while all the other players keep their decisions at x_{-f}. These gains are then summed up over all the players to obtain the NI function.

Variational Inequality Reformulations

If a NEP satisfies assumptions (1), then its solutions are linked to a variational inequality according to the theorem given below.

Theorem 2.3.1 *[33] Suppose the NEP*$(\theta_f, X_f)_F^{f=1}$ *satisfies assumptions* (1)*. Then a point* x^N *is a solution to NEP*$(\theta_f, X_f)_{f=1}^F$ *if and only if it is a solution of* $\boldsymbol{VI}(\mathbf{X}, \mathbf{F}(x))$ *where* $\mathbf{F}(x) := \nabla_{x_f}\theta_f(x)_F^{f=1}$.

Optimization reformulations

At any NE to NEP$(\theta_f, X_f)_{f=1}^F$, it is impossible for any player to improve their objectives by unilaterally deviating to another feasible solution. Since the NI function captures such improvements, it is intuitive to use it to characterize NEs.

Let

$$\hat{V}(x) := \sup_{y \in \mathbf{X}} \Psi(x, y). \tag{2.10}$$

Then we may use the function $\hat{V}$ to construct an equivalent formulation for the NEP$(\theta_f, X_f)_F^{f=1}$. Formally we have

Theorem 2.3.2 *[111]* $\hat{V}(x) \geq 0$ *for all* $x \in \mathbf{X}$*. Furthermore,* $\hat{V}(x^N) = 0$ *if and only if* x^N *is an NE to NEP*$(\theta_f, X_f)_{f=1}^F$.

Theorem (2.3.2) implies that NEs to NEP$(\theta_f, X_f)_{f=1}^F$ may be computed using the following optimization problem.

$$\begin{aligned} &\min && \hat{V}(x) \\ &\text{subject to} && x \in \mathbf{X}. \end{aligned} \tag{2.11}$$

Thus an NE x^N to a convex GNEP is also a global optimum of the optimization problem (2.11) with zero objective function [110].

Note however that this problem is still difficult since $\hat{V}(x)$ is usually nonsmooth and possibly discontinuous. In order to avoid these type of issues, a standard technique is to use a regularized version of the NI function defined below.

$$\Psi_\gamma(x,y) := \sum_{F}^{f=1} \left[\theta_f(x_f, x_{-f}) - \theta_f(y_f, x_{-f}) - \frac{\gamma}{2}\left\|x_f - y_f\right\|_2\right], \tag{2.12}$$

given a positive parameter γ. For a convex GNEP, let

$$\begin{aligned} V_\gamma(x) &:= \max_{y \in \mathbf{X}} \Psi^\gamma(x,y) \\ &= \max_{y \in \mathbf{X}} \sum_{F}^{f=1} \left[\theta_f(x_f, x_{-f}) - \theta_f(y_f, x_{-f}) - \frac{\gamma}{2}\left\|x - y\right\|_2\right]. \end{aligned} \tag{2.13}$$

This merit function V_γ can be used to characterize equilibria of player-wise convex NEPs as given in the following theorem.

Theorem 2.3.3 *[111] For a player-wise convex NEP, $V_\gamma(x) \geq 0$ for all $x \in \mathbf{X}$, and x^N is an equilibrium if and only if $x^N \in \mathbf{X}$ and $V_\gamma(x^N) = 0$. Furthermore, for every $x \in \mathbf{X}$ there exists a unique maximizer $y_\gamma(x)$ to the following problem.*

$$\max_{y \in \mathbf{X}} \Psi_\gamma(x,y).$$

If θ_f are continuously differentiable, the mapping V_γ is continuously differentiable. ▲

Using the difference of two regularized NI functions with different parameters, it is also possible to characterize the solutions to NEPs as unconstrained optimization problems. See [111] for details.

Regularized NI function refomulations for characterizing the equilibria of player-wise convex NEPs have been used to develop locally fast and globally convergent Newton-type methods [110].

Fixed-point Reformulations

The solutions to an NEP may also be characterized using a fixed-point inclusion. Recall that

$$\hat{V}(x) := \sup_{y \in \mathbf{X}} \Psi(x, y).$$

Denote by $\hat{Y}(x)$ the set that contains vectors where the supremum for $\hat{V}$ is attained. Then we have the following result.

Theorem 2.3.4 *[33] Let $\hat{Y}(x) := \{y^x \in \mathbf{X} \mid \hat{V}(x) = \Psi(x, y^x)\}$. Then a vector x^N is a solution to the NEP$(\theta_f, X_f)_{f=1}^F$ if and only if $x^N \in \hat{Y}(x^N)$.*

In other words solutions to NEP$(\theta_f, X_f)f = 1^F$ are also fixed points to the point-to-set mapping $x \mapsto \hat{Y}(x)$.

A similar formulation may be obtained using the mapping $\bar{V}_\gamma$ for all solutions and $\hat{V}^\gamma$ for normalized solutions.

KKT systems

It is also possible to characterize the solutions to NEPs using the KKT systems of the individual player problems. Suppose that x^N is a solution to the NEP$(\theta_f, X_f)_{f=1}^F$. If a suitable constraint qualification (such as the Mangasarian-Fromovitz constraint qualification) holds at a given point x^N for each player f, then there exist dual multipliers $\lambda_f^N \in \mathbb{R}_m$ such that

$$\begin{aligned} \nabla_{x_f} L_f(x_f, x_{-f}^N, \lambda_f) &= 0, \quad \text{and} \\ 0 \leq \lambda_f \perp -g_f(x_f) &\geq 0 \end{aligned}$$

are satisfied by (x_f^N, λ_f^N). Here $L_f(x, \lambda_f) = \theta_f(x) + g_f(x_f)_T \lambda_f$, the Langrangian of the optimization problem for player f.

The individual KKT systems may be concatenated to obtain the following system.

$$\begin{aligned} \mathbf{L}(x, \lambda) &= 0, \\ 0 \leq \lambda \perp -\mathbf{g}(x) &\geq 0, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \lambda &= (\lambda_1, \ldots, \lambda_F)_T \\ \mathbf{g}(x) &= (g_1(x_1)_T, \ldots g_F(x_F)_T)_T \\ \mathbf{L}(x, \lambda) &= (\nabla_{x_1} L_f(x_1, x^N_{-1}, \lambda_1), \ldots, \nabla_{x_F} L_f(x_F, x^N_{-F}, \lambda_F))_T \end{aligned}$$

We then have the following characterization of solutions to (1.1) using the system (2.14).

Assumption 2 *The functions $\theta_f(x)$ are continuously differentiable for each player $f \in \mathcal{F}$. The functions $g^i_f(x_f)$ are continuously differentiable for $i = 1, \ldots, m^f$ for each player $f \in \mathcal{F}$.*

Theorem 2.3.5 *[33] Suppose that the NEP$(\theta_f, X_f)^{f=1}_F$ satisfies Assumption* (2)*, then for every solution x^N at which all the players' feasible regions satisfy a suitable constraint qualification, there exists a vector of multipliers λ^N such that (x^N, λ^N) solves* (2.14)*.*

Suppose further that the NEP$(\theta_f, X_f)^{f=1}_F$ satisfies the assumption (1)*. In this case, if (x^N, λ^N) satisfies* (2.14)*, x^N is a solution to the NEP.*

3. MULTI-AGENT DECENTRALIZED NETWORK INTERDICTION GAMES

In this chapter, we describe one of the main motivating problem classes for this thesis, namely multi-agent network interdiction games. We formulate three specific problems within this class of problems and analyze each formulation using optimization and game theoretic techniques. Aside from studying theoretical properties such as existence and uniqueness of equilibria, we also present an empirical analysis of the efficiency of equilibria for these problems utilizing various algorithmic techniques.

3.1 Introduction

A typical network interdiction problem involves interactions between two players, an adversary and an interdictor, with conflicting interests. The *adversary* operates on a network and attempts to maximize some objective such as the flow of goods between two nodes. The *interdictor* tries to limit the adversary's objective by intentionally disrupting certain components of the network. Such interactions have historically been viewed from a *Stackelberg game* perspective in which the interdictor acts as the *leader* while the adversary acts as the *follower*. Essentially it is assumed that the interdictor acts first and the follower chooses his decisions after observing the effects of the interdictor's actions on the network. From the interdictor's perspective this captures the pessimistic viewpoint of guarding against the worst possible result given her actions.

Traditionally, interdiction problems have been analyzed from a centralized perspective. In other words, a single agent is assumed to analyze, compute and implement interdiction strategies. In many situations, however, it might be desirable and even necessary to consider an interdiction problem from a decentralized perspective. For instance, a supervising body, in control of multiple agents in a common system, may assign each agent to an adversary of interest. Each agent is then responsible for computing and implementing its own interdiction strategy against the designated adversary. Other situations may involve multiple independent agents, such as security agencies of different countries, trying to achieve a common goal on a shared network. Without any coordination between the agents, one might expect that a decentralized interdiction strategy may be inefficient compared to one determined by a central decision maker. This paper is focused on modeling and analyzing such settings and the inefficiencies that may arise.

In this chapter, we introduce *decentralized network interdiction (DNI) games*, in which multiple agents with differing objectives are interested in interdicting parts of a common network. We investigate various properties of equilibria in DNI games, including their existence and uniqueness, and propose algorithms to compute equilibria of these games. Using these algorithms, we also conduct empirical studies on the efficiency loss of equilibria in one class of DNI games, in comparison to optimal solutions obtained through centralized decision-making.

3.1.1 Application Examples

Before introducing the decentralized network interdiction model, we first present some examples of how decentralized interdiction games may be used to model problems in these application areas.

Smuggling interdiction - Network interdiction has been used in the past to study strategies to control the flow of illegal material. Various models have been proposed to study such problems in the context of nucleur material smuggling, drug networks, border control etc. Consider such a problem where adversaries attempt to maximize the flow of some illegal material on a network. If the network encompasses a large geographic area, as is reasonable for instance in drug networks, the interdiction resources may be spread across multiple jursisdictions, not merely geographically but also organizationally. The overall objective of the interdictors is to minimize the flow that the adversaries can push across this network.

We may model the problem as a decentralized maximum flow interdiction problem by decomposing the players into adversary-interdictor pairs. Each adversary attempts to maximize the flow between two nodes on the network, while the interdictor attempts to limit this maximum flow by employing resources to curtail the capacity of the arcs along the flow network. Such resources may include monitoring mechanisms such as patrolling or remote sensing equipment. Individual interdictors may be constrained by budgets on their resources as well as restrictions on the locations of arcs they may interdict. For instance, on large geographic networks, each interdictor might only have the ability to interdict nodes within a certain radius of his target node.

Infectious disease control - Consider a scenario where an infectious disease such as the avian flu is in danger of becoming a global pandemic. In this scenario, spread of the disease may be modelled using social and transportation networks along which contact results in a high probability of disease transmission. The resources to be deployed to control such a disease spread would be split among various nationalities and disparate organizational umbrellas.

Various disease control agencies take measures that would reduce the probability of disease transmission along links of the transmission network. Such measures might include shutting down arcs along transportation networks, or mitigating strategies such as masks to be worn by children attending school. The objective of each agency is to use its resources to maximize the minimum probability route along which the disease may spread from its source nodes to the target nodes. Keeping in mind that the probabilities are multiplicative, taking logarithms on the transmission probabilities will result in a model where each agency maximizes the shortest probability path from its source nodes to its target nodes.

Air strike targeting - Network interdiction models have been extensively used to capture situations in the military arena where logistical networks play a key role. Consider a situation where planners intend to use air strikes to disrupt the logistical network of the enemy. The enemy's objective is to minimize the cost of moving necessary battle supplies across a transportation network.

The nodes on the network are advance outposts on the battle field as well as supply depots and embarkation points. Each node either has excess supply or a demand that must be satisfied. The initial cost for each arc depends on what type of arc it is - road, rail or water. It is also assumed to depend on the length of the arc. The arcs are assumed to have a positive capacity.

The interdictor's decisions involve picking which arcs to interdict with the limited resources available for air strikes. The air strikes result in increase in the cost of flow across each arc as well as decrease in capacity for a fixed period of time. However the repair time for each arc also needs to be factored in.

Although this problem has been studied from a centralized planner's perspective, rapidly changing battle scenarios and the increase in computational resources available on modern military platforms necessitates the decentralized perspective which could be effective in replanning the interdiction strategy in a dynamic manner.

3.2 Decentralized Network Interdiction Games

In this work, we consider strategic interactions among multiple interdictors who operate on a common network. The interdictors may each have their own adversary or have a common adversary. If there are multiple adversaries, we assume there is no strategic interaction among the adversaries. We also assume that the interdictors are allies in the sense that they are not interested in deliberately impeding each other.

Formally, we have a set $\mathcal{F} = \{1, \ldots, F\}$ of interdictors or agents, who operate on a network $G = (V, A)$, where V is the set of nodes and A is the set of arcs. Each agent's actions or decisions correspond to interdicting each arc of the network with varying intensity: the decision variables of agent $f \in \mathcal{F}$ are denoted by $x^f \in X^f \subset \mathbb{R}^{|A|}$, where X^f is an abstract set that constrains agent f's decisions[1]. For any agent $f \in \mathcal{F}$, let x^{-f} denote the collection of all the other agents' decision variables; in other words, $x^{-f} = (x^1, \ldots, x^{f-1}, x^{f+1}, \ldots, x^F)$. The network obtained after every agent executes its decisions or interdiction strategies is called the *aftermath network*. The strategic interaction between the agents occurs due to the fact that the properties of each arc in the aftermath network are affected by the combined decisions of all the agents.

In addition to the abstract constraint set X^f, we assume that each agent $f \in \mathcal{F}$ faces a total interdiction budget of $b^f > 0$. The cost of interdicting an arc is linear in the intensity of interdiction; in particular, agent f's cost of interdicting arc (u, v) by x^f_{uv} units is $c^f_{uv} x^f_{uv}$. We assume that $c^f_{uv} > 0$ for all $(u, v) \in A$ and $f \in \mathcal{F}$. To rule out uninteresting cases, we also assume that the feasible set for each agent is also nonempty (meaning that each agent has the budget to at least interdict one arc).

The optimization problem for each agent $f \in \mathcal{F}$ is:

$$\begin{aligned} \underset{x^f}{\text{maximize}} \quad & \theta^f(x^f, x^{-f}) \\ \text{subject to} \quad & \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \le b^f, \\ & x^f \in X^f, \end{aligned} \tag{3.1}$$

where the objective function θ^f is agent f's *obstruction function*, or measure of how much agent f's adversary has been obstructed. Henceforth, we refer to the game in which each agent $f \in \mathcal{F}$ solves the above optimization problem (3.1) as a *decentralized network interdiction (DNI) game*. The obstruction function θ^f can capture various types of interdiction problems. Typically θ^f is the (implicit) optimal value function of the adversary's network optimization problem parametrized by the agents' decisions, which usually minimizes flow cost or path length subject to flow conservation, arc capacity and side constraints.

[1] In contrast to the rest of this dissertation, player indices are given as superscripts in this chapter. Subscripts are used for other indices.

Suppose that a central planner, with a comprehensive view of the network and the agents' objectives, could pool the agents' interdiction resources and determine an interdiction strategy that maximizes some global measure of how much the agents' adversaries have been obstructed. Let $\boldsymbol{\theta^c}(x^1, \ldots, x^F)$ represent the global obstruction function for a given interdiction strategy $(x^1, \ldots, x^F)$. The central planner's problem corresponding to the DNI game (3.1) is then as follows:

$$\begin{aligned} \underset{x^1,\ \ldots,\ x^F}{\text{maximize}} \quad & \boldsymbol{\theta^c}(x^1, \ldots, x^F) \\ \text{subject to} \quad & \sum_{f \in \mathcal{F}} \sum_{(u,v) \in A} c^f_{uv} x^f_{uv} \leq \sum_{f \in \mathcal{F}} b^f, \\ & x^f \in X^f \quad \forall f \in \mathcal{F}. \end{aligned} \tag{3.2}$$

We refer to (3.2) as the centralized problem, and focus primarily on when the global obstruction function is *utilitarian*; that is,

$$\boldsymbol{\theta^c}(x^1, \ldots, x^F) := \sum_{f \in \mathcal{F}} \theta^f(x^f, x^{-f}).$$

As mentioned earlier, one of the goals of this work is to quantify the inefficiency of an equilibrium of a DNI game – a decentralized solution to problem (3.1) – relative to a centrally planned optimal solution – an optimal solution to problem (3.2). A commonly used measure of such inefficiency is the *price of anarchy*.

Formally speaking, let $\mathcal{N}_I$ be the set of all equilibria corresponding to a specific instance I. (In the context of DNI games, an instance consists of the network, obstruction functions, interdiction budgets, and costs.) For the same instance I, let $(x^{1^*}, \ldots, x^{F^*})$ denote a global optimal solution to the centralized problem (3.2). Then the price of anarchy of the instance I is defined as

$$p(I) := \max_{(x^1_N, \ldots, x^F_N) \in \mathcal{N}_I} \frac{\boldsymbol{\theta^c}(x^{1^*}, \ldots x^{F^*})}{\boldsymbol{\theta^c}(x^1_N, \ldots, x^F_N)}. \tag{3.3}$$

Let $\mathcal{I}$ be the set of all instances of a game. We assume implicitly that for all $I \in \mathcal{I}$, the set $\mathcal{N}_I$ is nonempty and a global optimal solution to the centralized problem exists. By convention, p is set to 1 if the worst equilibrium as well as the global optimal solution to the centralized problem both have zero objective value. If the worst equilibrium has a zero objective value while the global optimum is nonzero, p is set to be infinity. In addition to the price of anarchy for an instance of a game, we also define the worst-case price of anarchy over all instances of the game (denoted as $w.p.o.a$) as follows:

$$w.p.o.a := \sup_{I \in \mathcal{I}} \; p(I). \tag{3.4}$$

Since we wish to study properties of a class of games such as DNI games, rather than a particular instance of a game, we are more interested in the worst-case price of anarchy. However, there are two major difficulties associated with such an efficiency measure. First, it is well-known that the worst-case price of anarchy may be a very conservative measure of efficiency loss, since the worst case may only happen with pathological instances. Second, explicit theoretical bounds on the worst-case price of anarchy may be difficult to obtain for general classes of games. Indeed most of the related research has focused on identifying classes of games where such bounds may be derived. In this work, we show how our proposed decentralized algorithms can be used to empirically study the *average-case efficiency loss* (denoted by $a.e.l$). Let $\mathcal{I}'$ denote a finite set such that $\mathcal{I}' \subset \mathcal{I}$, and let $|\mathcal{I}'|$ denote the cardinality of the the set $\mathcal{I}'$. Then

$$a.e.l(\mathcal{I}') := \frac{1}{|\mathcal{I}'|} \sum_{I \in \mathcal{I}'} \; p(I). \tag{3.5}$$

In other words, the average-case efficiency loss is the average value of $p(I)$ as defined in (3.3) over a set of sampled instances $\mathcal{I}' \subset \mathcal{I}$ of a game.

As mentioned above, the generic form of problem (3.1) can be used to describe various network interdiction settings, such as maximum flow interdiction. To start with models that are both theoretically and computationally tractable, we focus on decentralized shortest-path interdiction games, which we describe in detail next.

3.3 Decentralized Shortest Path Interdiction Games

As the name suggests, *decentralized shortest path interdiction (DSPI)* games involve players or interdictors whose adversaries are interested in the shortest path between source-target node pairs on a network. Interdictors act in advance to increase the length of the shortest path of their respective adversaries by interdicting (in particular, lengthening) arcs on the network.

To describe these games formally, we build upon the setup for the general decentralized network interdiction game described in Section 3.2. Each agent $f \in \mathcal{F}$ has a target node $t^f \in V$ which it wishes to protect from an adversary at source node $s^f \in V$ by maximizing the length of the shortest path between the two nodes. The agents achieve this goal by committing some resources (e.g. monetary spending) to increase the individual arc lengths on the network: the decision variable x^f_{uv} represents the contribution of agent $f \in \mathcal{F}$ towards lengthening arc $(u, v) \in A$. The arc length $d_{uv}(x^f, x^{-f})$ of arc $(u, v) \in A$ in the aftermath network depends on the decisions of all the agents.

We consider two types of interdiction. The first type of interdiction is *continuous*: in particular,

$$X^f := \{x^f \in \mathbb{R}^{|A|} : x^f_{uv} \geq 0 \quad \forall (u, v) \in A\}$$

and the arc lengths after an interdiction strategy $(x^1, \dots, x^F)$ has been executed are

$$d_{uv}(x^1, \dots, x^F) = d^0_{uv} + \sum_{f \in \mathcal{F}} x^f_{uv} \quad \forall (u, v) \in A, \tag{3.6}$$

where $d^0_{uv} > 0$ is the initial length of arc (u, v). The initial arc lengths d^0 are assumed to be positive. We note that if we allow negative arc lengths, as long as the graph does not possess negative length circuits (in which case shortest paths are not defined), it is possible to modify the arc lengths to be positive and preserve shortest paths (Cf. [22]).

The second type of interdiction is *discrete*: in this case,

$$X^f := \{x^f \in \mathbb{R}^{|A|} : x^f_{uv} \in \{0, 1\} \quad \forall (u, v) \in A\}$$

and the arc lengths in the aftermath network are

$$d_{uv}(x^1, \ldots, x^F) = d^0_{uv} + e_{uv} \max_{f \in \mathcal{F}} x^f_{uv} \quad \forall (u,v) \in A, \tag{3.7}$$

where $e_{uv} \in \mathbb{R}_{\geq 0}$ is the fixed extension of arc (u, v). In other words, the length of an arc is extended by a fixed amount if at least one agent decides to interdict it.

The optimization problem solved by each agent $f \in \mathcal{F}$ in a DSPI game is given by (3.1), where

$$\theta^f(x^f, x^{-f}) := \left(\begin{array}{ll} \min\limits_{z^f} & \sum\limits_{(u,v) \in A} z^f_{uv}\, d_{uv}(x^f, x^{-f}) \\ \text{s.t.} & \sum\limits_{v \in V} z^f_{uv} - \sum\limits_{v \in V} z^f_{vu} = \begin{cases} 1 & \text{if } u = s^f \\ 0 & \text{if } u \neq s^f, t^f \\ -1 & \text{if } u = t^f \end{cases} \\ & z^f_{uv} \in \{0, 1\} \quad \forall (u,v) \in A \end{array} \right) \tag{3.8}$$

where binary variable z^f_{uv} in (3.8) represents whether an arc $(u, v) \in A$ is in the shortest s^f-t^f path. In other words, agent f's optimization problem is a bilevel optimization problem, where the inner minimization problem (3.8) is its adversary's shortest path problem. Although the inner minimization problem is an integer program with binary variables, it is well known that the constraint matrix is totally unimodular (e.g. [96]), rendering the integer program equivalent to its linear programming relaxation. Therefore, once the interdictors' variables $(x^1, \ldots, x^F)$ are fixed, we can use linear programming duality to transform the inner minimization problem to a maximization problem [61] and reformulate agent f's optimization problem (3.1) as:

$$\begin{aligned} \underset{x^f,\ y^f}{\text{maximize}} \quad & y^f_{t^f} - y^f_{s^f} \\ \text{subject to} \quad & y^f_v - y^f_u \leq d_{uv}(x^f, x^{-f}) \quad \forall (u,v) \in A, \\ & \sum_{(u,v) \in A} c^f_{uv} x^f_{uv} \leq b^f, \\ & x^f \in X^f. \end{aligned} \tag{3.9}$$

It is well known (see, for example, [9, 68]) that at optimality, the term $y_u^f - y_{s^f}^f$ is equal to the length of the shortest s^f-u path in the aftermath network. This fact has several important implications for problem (3.9). For instance, it allows us to restrict the y^f variables to be integral if the underlying network data is integral, since at optimality all path lengths would also be integral. Moreover, as we show below, it also allows us to bound the y^f variables.

When interdiction is continuous, the largest possible length in the aftermath network for any arc is bounded by the largest interdiction possible on that arc. Keeping the budgetary constraints in mind, the maximum interdiction possible on any arc is bounded by

$$F \cdot \max_{f \in \mathcal{F},\ (u,v) \in A} \left\{ \frac{b^f}{c_{uv}^f} \right\}.$$

As a result, the maximum length of any arc $(u, v) \in A$ in the aftermath network is bounded by

$$d_{uv}^0 + F \cdot \max_{f \in \mathcal{F},\ (u,v) \in A} \left\{ \frac{b^f}{c_{uv}^f} \right\}.$$

Therefore, the lengths of every path in the aftermath network are bounded above by

$$M = \sum_{(u,v) \in A} d_{uv}^0 + |A|\ F \cdot \max_{f \in \mathcal{F},\ a \in A} \left\{ \frac{b^f}{c_a^f} \right\}.$$

On the other hand, when interdiction is discrete, the length of any path in the aftermath network is bounded above by

$$M = \sum_{(u,v) \in A} (d_{uv}^0 + e_{uv}).$$

Since only the differences $y_v^f - y_u^f$ across arcs (u, v) are relevant to the formulation (3.9), we may always replace y_u^f by $y_u^f - y_{s^f}^f$ for each $u \in V$ to obtain a feasible solution with equal objective value. Therefore we can then add the constraints $-M \leq y_u^f \leq M$

for all $u \in V$ to the problem (3.9) to obtain an equivalent formulation of a DSPI game, where each agent $f \in \mathcal{F}$ solves the following problem.

$$\begin{aligned}
\underset{x^f,\ y^f}{\text{maximize}} \quad & y^f_{t^f} - y^f_{s^f} \\
\text{subject to} \quad & y^f_v - y^f_u \leq d_{uv}(x^f, x^{-f}) \quad \forall (u,v) \in A, \\
& \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \leq b^f, \\
& -M \leq y^f_u \leq M \quad \forall u \in V, \\
& x^f \in X^f.
\end{aligned} \tag{3.10}$$

When analyzing the DSPI game from a centralized decision-making perspective, we assume that the global obstruction function is utilitarian, i.e., the sum of the shortest s^f-t^f path lengths over all the agents $f \in \mathcal{F}$. We also assume that the resources are pooled among all the agents, resulting in a common budgetary constraint. Thus the centralized problem for DSPI games can be given as follows:

$$\begin{aligned}
\underset{x,\ y}{\text{maximize}} \quad & \sum_{f\in\mathcal{F}} \left(y^f_{t^f} - y^f_{s^f}\right) \\
\text{subject to} \quad & y^f_v - y^f_u \leq d_{uv}(x^f, x^{-f}) \quad \forall (u,v) \in A, f \in \mathcal{F}, \\
& \sum_{f\in\mathcal{F}} \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \leq \sum_{f\in\mathcal{F}} b^f \\
& -M \leq y^f_u \leq M \quad \forall u \in V, f \in \mathcal{F}, \\
& x^f \in X^f \quad \forall f \in \mathcal{F}.
\end{aligned} \tag{3.11}$$

Since y^f is bounded for all $f \in \mathcal{F}$, a globally optimal solution of (3.11) exists regardless of whether x^f is continuous or discrete for all $f \in \mathcal{F}$. In the continuous case, Weierstrass's extreme value theorem applies since all the functions are continuous and the x^f variables are bounded due to the non-negativity and budgetary constraints. In the discrete case, there are only a finite number of values that the x^f variables can take.

3.3.1 Game Structure and Analysis

Generalized Nash Equilibrium Problems

The formulation (3.10) gives us some insight into the structure of strategic interactions among agents in a DSPI game. Note that in formulation (3.10), the objective function for each agent $f \in \mathcal{F}$ only depends on variables indexed by f (in particular, $y^f_{s^f}$ and $y^f_{t^f}$). However, the constraint set for each agent f is parametrized by other agents' variables x^{-f}.

It is straightforward to see how the DSPI game in (3.10) translates into a GNEP problem: for all $f \in \mathcal{F}$,

$$\chi^f = (x^f, y^f),$$

$$\theta^f(\chi^f, \chi^{-f}) = \left(y^f_{t^f} - y^f_{s^f}\right),$$

$$\Xi^f(\chi^{-f}) = \left\{ \chi^f = (x^f, y^f) \;\middle|\; \begin{aligned} & y^f_v - y^f_u \leq d_{uv}(x^f, x^{-f}) \quad \forall (u,v) \in A, \\ & \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \leq b^f, \\ & -M \leq y^f_u \leq M \quad \forall u \in V, \\ & x^f \in X^f \end{aligned} \right\}. \tag{3.12}$$

Note that $\chi = (\chi^1, \ldots, \chi^F) \in \mathbb{R}^n$, where $n = F\left(|V| + |A|\right)$.

As mentioned in Chapter 2, GNEPs in general are more challenging than regular Nash equilibrium problems, both theoretically and computationally. While results on the existence of (pure-strategy) generalized Nash equilibria have been established, few results exist on uniqueness of such equilibria, largely due to the fact that few GNEPs have a unique equilibrium [33]. We will rely on both existing analytic results and the special structure of DSPI games to determine the existence of equilibria and their uniqueness in these games in later subsections.

Existence of Equilibria

We first consider the existence of equilibria in the DSPI game when interdiction decisions are continuous. In this case each interdictor's optimization problem (3.10), with the arc length function $d_{uv}(\cdot)$ defined as in (3.6), is a linear program. The following result regarding general convex GNEPs can therefore be applied here.

Theorem 3.3.1 (Ichiishi [60]) *Given a simultaneous-move GNEP with each agent $f \in \mathcal{F}$ solving* (1.1)*, assume that the following conditions hold:*

(i) There exist nonempty, convex and compact sets $K^f \subseteq \mathbb{R}^{n_f}$ for each agent $f \in \mathcal{F}$ such that for every $\chi = (\chi^1, \chi^2, \ldots, \chi^F) \in \prod_{f=1}^{F} K^f$, $\Xi^f(\chi^{-f})$ is nonempty, closed and convex. In addition, $\Xi^f(\chi^{-f}) \subseteq K^f$ and $\Xi^f(\cdot)$ is both upper and lower semicontinuous as a point-to-set map.

(ii) For every agent $f \in \mathcal{F}$, the function $\theta^f(\cdot, \chi^{-f})$ is quasi-concave on $\Xi^f(\chi^{-f})$.

Then a (pure-strategy) generalized Nash equilibrium exists.

To apply Theorem 3.3.1, the following result will be useful.

Theorem 3.3.2 (Rockafellar and Wets [86]) *For every $f \in \mathcal{F}$, suppose*

$$\Xi^f(\chi^{-f}) = \{\chi^f \mid g_i^f(\chi^f, \chi^{-f}) \leq 0 \quad \text{for } i = 1, \ldots, m^f\}$$

where $g_1^f, \ldots, g_{m^f}^f$ are finite, continuous functions and $g_1^f(\chi^f, \chi^{-f}), \ldots, g_{m^f}^f(\chi^f, \chi^{-f})$ are convex in χ^f for each χ^{-f}. If for $\bar{\chi}^{-f}$ there is a point $\bar{\chi}^f$ such that $g_i^f(\bar{\chi}^f, \bar{\chi}^{-f}) < 0$ for $i = 1, \ldots, m^f$, then Ξ^f is continuous not only at $\bar{\chi}^{-f}$ but at every χ^{-f} in some neighborhood of $\bar{\chi}^{-f}$.

We now use these results to show the existence of equilibria for the DSPI game.

Proposition 3.3.1 *Under continuous interdiction, a generalized Nash equilibrium exists for the DSPI game* (3.10)*.*

Proof Recall the representation of a DSPI game as a GNEP in (3.12). We show that conditions (i) and (ii) in Theorem 3.3.1 hold. First, condition (ii) in Theorem 3.3.1 is immediately apparent from (3.10) and (3.12) since the objective functions are linear.

To show condition (i) holds, we define the set K^f for each $f \in \mathcal{F}$ as follows:

$$K^f = \left\{ (x^f, y^f) \;\middle|\; \begin{array}{r} \sum_{(u,v)\in V} c^f_{uv} x^f_{uv} \leq b^f, \\ -M \leq y^f_u \leq M \quad \forall u \in V, \\ x^f_{uv} \geq 0, \quad \forall (u,v) \in A. \end{array} \right\}. \tag{3.13}$$

Clearly K^f is nonempty for all $f \in \mathcal{F}$ as it contains the zero vector. It is also easy to see that $\Xi^f(x^{-f}, y^{-f}) \subseteq K^f$ and is a polyhedron, and therefore closed and convex for any given (x^{-f}, y^{-f}). The set $\Xi^f(x^{-f}, y^{-f})$ is also nonempty, since we can construct a feasible solution by setting $x^f_{uv} = 0$ for all $(u,v) \in A$ and y^f_u to be the length of the shortest path from s^f to u for all $u \in V$. Now all that remains to be shown for condition (i) in Theorem 3.3.1 to hold are the continuity properties of the point-to-set mapping $\Xi^f(\cdot)$.

To do so we use Theorem 3.3.2. Consider the following assignment of the x^f variables.

$$\bar{x}^f_{uv} = \frac{1}{2|A|} \min_{(u,v)\in A} \frac{b^f}{c^f_{uv}}.$$

Clearly we must then have

$$\sum_{(u,v)\in A} c^f_{uv} \bar{x}^f_{uv} \leq \frac{b^f}{2} < b^f$$

. If we now set $\bar{y}^f_u = 0$ for any $u \in V$, we obtain a tuple $(\bar{x}, \bar{y})$, that is now feasible and strictly interior to $\Xi^f(\cdot)$ for any χ^{-f}, under our original assumptions of positivity on b^f, c^f and d^0.

Combined with the linearity of the constraints, this then allows us to directly apply Theorem 3.3.2 to claim the continuity of the point-to-set mapping $\Xi^f(\cdot)$ in some neighborhood of (x^{-f}, y^{-f}), which certainly implies its continuity at (x^{-f}, y^{-f}).

Since this property holds at any (x^{-f}, y^{-f}), the mapping $\Xi^f(\cdot)$ is both lower and upper semi-continuous on its domain. ■

We may also analyze the existence of equilibria in DSPI games under continuous interdiction using a path based NEP formulation as given below.

Let $P^f = \{p^f_1, p^f_2, \ldots, p^f_{k^f}\}$ be the set of $s^f - t^f$ paths available to agent $f \in \mathcal{F}$. The length of a path $p \in P^f$ is given by

$$d_p(x^1, \ldots, x^F) = \sum_{(u,v)\in p} d_{uv}(x^1, \ldots, x^F), \tag{3.14}$$

where $d_{uv}(x^1, \ldots, x^F)$ is as defined in equation (3.6) for continuous interdiction, and as defined in (3.7) for the discrete case.

The optimization problem for each interdicting agent $f \in \mathcal{F}$ is then:

$$\begin{aligned} \underset{x^f}{\text{maximize}} \quad & \theta^f(x^f, x^{-f}) \equiv \min_{p\in P_f} d_p(x^f, x^{-f}) \\ \text{subject to} \quad & \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \le b^f, \\ & x^f \in X^f. \end{aligned} \tag{3.15}$$

Under continuous interdiction and the general assumption made ealier that X_f is nonempty, convex and compact, the feasible strategy set for agent f, given by $\{x^f \in X^f | \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \le b^f\}$ is also convex and compact. Given an x^{-f}, the objective function in (3.15) is the minimum of a set of affine functions of x^f, and therefore continuous in x^f. Thus, by Weirstrass's extreme value theorem, each agent has an optimal strategy given the strategies of the other agents. Note, however, that the objective function in (3.15) is not differentiable with respect to x_f in general.

The key is to show that the objective function in (3.15), $\theta^f(x^f, x^{-f})$, is concave in x^f, despite the fact that it is not differentiable.

Proposition 3.3.2 *Given that each agent $f \in \mathcal{F}$ solves the problem* (3.15)*, with $d_p(x^f, x^{-f})$ defined as in* (3.14) *and* (3.6)*, and assume that the abstract set X_f in* (3.15) *is nonempty, convex and compact for each $f \in \mathcal{F}$, the DSPI game under continuous interdiction has a pure strategy Nash equilibrium.*

Proof Based on the assumption, the feasible region in (3.15) is nonempty, convex and compact. With a fixed x^{-f}, the objective function of agent f is the minimum of a finite set of affine functions in x^f, and therefore, is concave with respect to x^f, by the well-known fact in convex analysis (Cf. [14]). Consequently, the DSPI game belongs to the class of "concave games," introduced in Rosen [87], and it is shown in [87] that a pure-strategy Nash equilibrium always exists for a concave game. ■

Under discrete interdiction, the existence of a PNE is not always guaranteed when different interdictors are competing against different adversaries. We illustrate the nonexistence of PNE in Example 1[2] [103] below.

Example 1 Consider the network given in Figure 3.1.

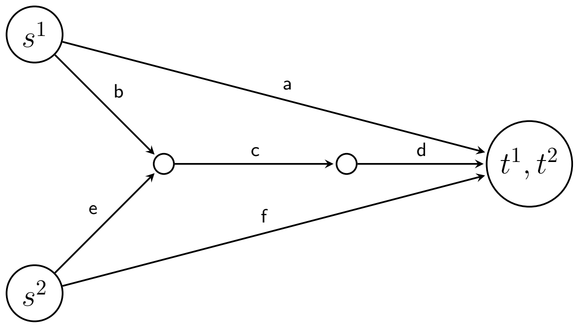

Figure 3.1. Network topology for DSPI game in Example 1.

In this game, there are two agents – agent 1 and agent 2 – who are attempting to maximize the lengths of the s^1-t^1 paths and s^2-t^2 paths respectively. Note that $t^1 = t^2$. The data for the problem, including initial arc lengths, cost of interdiction and arc extensions are given below in Table 3.1.

Suppose $b^1 = 8$ and $b^2 = 15$. As a result, player 1 can either interdict the arcs a, b and c one at a time, or the arcs a and c simultaneously. Similarly, player 2 can either interdict arc d or arc f.

[2]The example network was constructed by our co-authors Ashish Hota and Dr. Shreyas Sundaram.

Table 3.1.
Network data for Example 1

Arc tag	Initial length	Arc extension	Cost to player 1	Cost to player 2
a	7	0.5	3	20
b	0	2	6	20
c	0	1.5	5	20
d	0	6	15	15
e	0	1	20	20
f	1	6	15	15

Thus, player 1 has four feasible pure strategies and player 2 has two feasible pure strategies. The strategy tuples along with the corresponding pay-offs for each player are summarized in Table 3.2. It is easy to verify that for any joint strategy profile, there is a player who would prefer to deviate unilaterally. Therefore, this instance of the DSPI game does not possess a NE.

Table 3.2.
Pay-off combinations for Example 1

P_1/P_2 strategies	d	f
a	$6, 1$	$0, 0$
c	$7, 1$	$1.5, 1.6$
(a, c)	$7.5, 1$	$1.5, 1.5$
b	$7, 1$	$2, 0$

In the previous example, the agents have a common target node, but different source nodes. However, in the class of games in which the interdictors have a common adversary, i.e., when each agent maximizes the shortest path between a common source-target pair, we can show that DSPI games under discrete interdiction possess a PNE.

Consider the DSPI game where each agent is trying to maximize the shortest path lengths between nodes s and t. Since the objective function of each agent is the same, we can write the following centralized optimization problem to maximize the shortest $s-t$ path distance subject to the individual agents' budget constraints. Let P^{st} be the set of $s-t$ paths in the network. The centralized optimization problem is:

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & \min_{p \in P^{st}} d_p(x^1, x^2, \ldots, x^F) \\ \text{subject to} \quad & \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \le b^f \quad \forall f \in \mathcal{F}, \\ & x^f_{uv} \in \{0,1\} \quad \forall (u,v) \in A, f \in \mathcal{F}. \end{aligned} \tag{3.16}$$

The feasible solution space of the above problem is finite under individual agents' budget constraints. Therefore, the centralized problem always has a maximum. Furthermore, the optimal solution to this problem is a PNE of the DSPI game as we show in the following result.

Proposition 3.3.3 *Suppose the source and target for each agent in a DSPI problem are the same. Let x^* denote the optimal solution of the centralized problem* (3.16)*. Then x^* is a PNE to the DSPI game under discrete interdiction.*

Proof Assume the contrary, and suppose that there is an agent h for whom there exists a unilateral deviation x^f that strictly increases the path distance $s-t$. By assumption, x^h is feasible to the budgetary constraints for agent h. Therefore, $\bar{x} \equiv (x^h,\ x^{*-h})$ is feasible to (3.16) with a strictly larger objective value. Clearly this is a contradiction to the optimality of x^* to (3.16). ■

Uniqueness of equilibria

Establishing conditions under which a DSPI game has a unique equilibrium is quite difficult. However, it is easy to show that there exist simple instances of DSPI games for which multiple equilibria exist. We give several such examples below.

Example 2 Consider the following instance, based on the network in Figure 3.2. There are 2 agents: agent 1 has an adversary with source node 1 and target node 5; agent 2 has an adversary with source node 1 and target node 6. The initial arc lengths are 0, interdiction is continuous, and the interdiction costs are the same for both agents and are given in the arc labels in Figure 3.2. Both agents have a budget of 1.

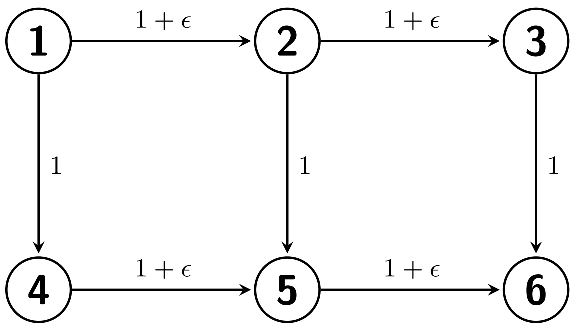

Figure 3.2. Network topology for DSPI game in Example 2.

Consider the case when $\epsilon = 2$. One generalized Nash equilibrium occurs when agent 1 interdicts the arcs $(1,4)$ and $(2,5)$ by $1/2$ each, and agent 2 interdicts arcs $(1,4)$ and $(2,5)$ by $1/6$, and arc $(3,6)$ by $2/3$. In this case both agents end up with a shortest path length of $2/3$. It is easy to see that any unilateral deviation will result in a smaller shortest path length for the deviating agent. In fact, it is straightforward to see that the source-target path lengths for each agent must be equal at an equilibrium: if the path lengths are unequal, an agent could improve its objective function by equalizing the path lengths. Therefore, in this example, any combination of decision variables that results in a shortest path length of $2/3$ for each agent will be a generalized Nash equilibrium, and there is a continuum of such decision variable combinations.

Example 3 A variant of this instance under discrete interdiction exhibits some interesting properties. Consider the same instance, except under discrete interdiction, with all of the arc extension lengths are equal to 1, and the budget for each agent equal to $1 + \epsilon$ where ϵ is an integer that is at least 1. One possible equilibrium is for agent 1 to interdict the arc $(1, 4)$ and for agent 2 to interdict the arc $(1, 2)$. If $\epsilon = 1$, then this will result in a shortest path length of 1 for each agent. Note that agent 1 does not use its entire budget. A similar equilibrium occurs when agent 1 interdicts arcs $(1, 4)$ and $(2, 5)$ and agent 2 interdicts arc $(3, 6)$. In this case agent 2 ends up with unused budget.

Example 4 A more interesting situation occurs when $\epsilon = 0$ and the budget is 1. In this case, the 2 agents can interdict at most 1 arc each. Agent 1 interdicting arc $(1, 4)$ and agent 2 interdicting arc $(1, 2)$ results in an equilibrium in which both agents have shortest path lengths of 1. The potential function value for this equilibrium is 2.

However, there exist other equilibria in which one agent is worse off than the other. For instance, suppose agent 1 interdicts the arc $(5, 6)$ and agent 2 interdicts $(3, 6)$: agent 1's shortest path length is 0, and agent 2's shortest path length is 1. Although agent 1's shortest path length is now 0, it has no incentive to deviate since there is no possible unilateral deviation that would allow it to increase its shortest path length. A similar situation occurs when agent 1 interdicts $(2, 5)$ and agent 2 interdicts $(4, 5)$. It is interesting to note that in this game, zero interdiction by both agents is also a generalized Nash equilibrium with a potential function value of 0.

3.3.2 Computing a Nash Equilibirum

In this section we focus on algorithms to compute equilibria of DSPI games. As discussed above, a DSPI game is a special case of a generalized Nash equilibrium problem. Computational methods to find an equilibrium for GNEPs include reformulations as quasi-variational inequalities [81], optimization reformulations using the Nikaido-Isoda function [29, 110, 111], direct methods using KKT systems [28] and penalty methods [34, 36], among others. We refer to the above methods as centralized algorithms, as they all attempt to find an equilibrium by tackling the game as a whole: for instance, by solving an equivalent variational inequality or complementarity problem. Such methods are usually computationally intensive.

Motivated by the observation that DSPI games admit potential functions, we also present decentralized algorithms based on best-response dynamics. Such decentralized algorithms have several advantages over centralized algorithms. First, the computational burden at each iteration is much smaller than with centralized algorithms, since only a single agent's optimization problem is solved with others agents' decisions fixed. Second, a decentralized algorithm may provide insight into how an equilibrium is achieved among agents' strategic interactions. Such insight is particularly useful when multiple equilibria exist, as is the case for many GNEPs. It is well-known (for example, [76]) that a game may possess unintuitive Nash equilibria that would never realistically be the outcome of the game. A centralized algorithm would not be able to distinguish between a meaningful and a meaningless equilibrium, and may end up computing such unintuitive equilibria. A decentralized algorithm, on the other hand, depicts how an equilibrium is achieved from a particular starting point through iterative interactions among agents, should the algorithm converge. Third, decentralized algorithms naturally lead to multithreaded implementations that can take advantage of a high performance computing environment. In addition, different threads in a multithreaded implementation may be able to find different equilibria of a game, making such an algorithm particularly suitable for computationally quantifying the average efficiency loss of decentralized strategies. Nevertheless, despite these favorable properties, best-response based algorithms suffer from a major drawback: it is difficult to theoretically prove these algorithms converge to equilibria for general classes of GNEPs.

In the following discussion, we propose solving the continuous DSPI game using a linear complementarity problem (LCP) reformulation. The reformulation is constructed using the Karush-Kuhn-Tucker (KKT) optimality conditions for each agent's optimization problem. We show that the resulting LCP has favorable properties, allowing the use of Lemke's pivoting algorithm.

Linear Complementarity Formulation

Before presenting the LCP formulation for the DSPI game, we introduce some basic notation and definitions. Formally, given a vector $q \in \mathbb{R}^d$ and a matrix $M \in \mathbb{R}^{d\times d}$, a linear complementarity problem LCP(q, M) consists of finding a decision variable vector $w \in \mathbb{R}^d$ such that

$$w \geq 0, \tag{3.17}$$

$$q + Mw \geq 0, \tag{3.18}$$

$$w^T(q + Mw) = 0. \tag{3.19}$$

The LCP(q, M) is said to be feasible if there exists a $w \in \mathbb{R}^d$ that satisfies (3.17) and (3.18). Any w satisfying (3.19) is called complementary. If w is both feasible and complementary, it is called a *solution* of the LCP. In this case, we say $w \in \mathrm{SOL}(q, M)$ to denote that w is in the solution set for the LCP. The LCP is said to be solvable if it has a solution. A thorough exposition of the theory underlying LCPs and various algorithmic techniques to solve such problems can be found in [25].

Consider now the DSPI game with continuous interdiction, introduced in Section 3.3. We restate the formulation (3.9) for the optimization problem of agent $f \in \mathcal{F}$ as follows:

$$\begin{aligned}
\underset{x^f,\ y^f}{\text{minimize}} \quad & y^f_{s^f} - y^f_{t^f} \\
\text{subject to} \quad & y^f_u - y^f_v + x^f_{uv} \geq -d^0_{uv} - \sum_{\substack{f' \in \mathcal{F} \\ f' \neq f}} x^{f'}_{uv} \quad \forall (u,v) \in A, \\
& \sum_{(u,v)\in A} -c^f_{uv} x^f_{uv} \geq -b^f, \\
& x^f_{uv} \geq 0 \quad \forall (u,v) \in A, \\
& y^f_u \geq 0 \quad \forall u \in V.
\end{aligned} \tag{3.20}$$

As observed earlier in Section 3.3, the y^f variables are essentially free variables. In order to simplify analysis, we restrict these variables to be non-negative while ignoring the bounds added in the formulation (3.10). As we shall see later, it is possible to construct a solution to (3.20) given a solution to (3.10). When the interdiction decisions of the agents $f' \neq f$ are fixed, agent f's optimization problem (3.20) is a linear program (LP). In this case, the KKT conditions are both necessary and sufficient for a given feasible solution to be optimal.

We introduce the following notation to present the KKT conditions for the LP (3.20) compactly. Let $|V| = n$ and $|A| = m$. Denote by $\mathcal{G}$ the arc-node incidence matrix of the graph G. Further let $\mathcal{I}$ denote an identity matrix, and $\mathbf{0}$ be vectors or matrices of all zeros, of appropriate dimensions, respectively. The objective coefficients for the LP (3.20), denoted by $o^f \in \mathbb{R}^{m+n}$ can be given as follows:

$$o^f = \begin{bmatrix} \mathbf{0}_m \\ \nu^f \end{bmatrix}, \quad \text{where} \quad \nu^f = \begin{cases} 1 & \text{if } u = s^f \\ 0 & \text{if } u \neq s^f, t^f \\ -1 & \text{if } u = t^f \end{cases}.$$

The right hand sides for the constraints in (3.20) are denoted using the vector $r^f(x^{-f}) \in \mathbb{R}^{m+1}$:

$$r^f(x^{-f}) = \begin{bmatrix} -d^0 \\ -b^f \end{bmatrix} - \sum_{\substack{f' \in \mathcal{F} \\ f' \neq f}} \left[\begin{array}{c|c} \mathcal{I}_m & \mathbf{0}_{m \times n} \\ \hline \mathbf{0}_m^T & \mathbf{0}_n^T \end{array}\right] \begin{bmatrix} x^{f'} \\ y^{f'} \end{bmatrix}.$$

The constraint matrix itself, denoted as $A^f \in \mathbb{R}^{(m+1)\times(m+n)}$, is

$$A^f = \left[\begin{array}{c|c} \mathcal{I}_m & \mathcal{G} \\ \hline -c^{f^T} & \mathbf{0}_n^T \end{array}\right].$$

Using this notation, the LP (3.20) can be restated as follows:

$$
\begin{aligned}
\underset{x^f, y^f}{\text{minimize}} \quad & {o^f}^T \begin{bmatrix} x^f \\ y^f \end{bmatrix} \\
\text{subject to} \quad & A^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq r^f(x^{-f}), \\
& \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq 0.
\end{aligned} \tag{3.21}
$$

Let the dual variables for the LP (3.20) be $(\lambda^f, \beta^f, \upsilon^f)$, where λ^f are the multipliers for the arc potential constraints, β^f the multiplier for the budgetary constraint and υ^f the multipliers for the non-negativity constraints. The KKT conditions for (3.21) are given by the following system.

$$
\begin{aligned}
r^f(x^{-f}) \leq A^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} \quad &\perp \quad \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \geq 0, \\
0 \leq \begin{bmatrix} x^f \\ y^f \end{bmatrix} \quad &\perp \quad \upsilon^f \geq 0, \\
o^f - {A^f}^T \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} - \upsilon^f \quad &= \quad 0.
\end{aligned} \tag{3.22}
$$

The KKT system (3.22) can be rewritten in the following form:

$$
\begin{aligned}
\upsilon^f &= o^f - {A^f}^T \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \geq 0, \quad & \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq 0, \quad & \begin{bmatrix} x^f \\ y^f \end{bmatrix}^T \upsilon^f = 0, \\
t^f &= -r^f(x^{-f}) + A^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq 0, \quad & \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \geq 0, \quad & {t^f}^T \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} = 0.
\end{aligned} \tag{3.23}
$$

In this form, it is easy to recognize that for a fixed value of x^{-f}, the KKT system is equivalent to the LCP$(q^f(x^{-f}), M^f)$ where

$$
q^f(x^{-f}) = \begin{bmatrix} o^f \\ -r^f(x^{-f}) \end{bmatrix} \quad \text{and} \quad M^f = \left[\begin{array}{c|c} \mathbf{0}_{(m+n)\times(m+n)} & -{A^f}^T \\ \hline A^f & \mathbf{0}_{(m+1)\times(m+1)} \end{array}\right]. \tag{3.24}
$$

The decision variable vector for the LCP is the vector of combined decision variables

$$w^f = \begin{bmatrix} x^f \\ y^f \\ \lambda^f \\ \beta^f \end{bmatrix}. \tag{3.25}$$

Each agent's KKT system (3.23) is parametrized by the collective decisions of other agents. As mentioned earlier, the optimization problem (3.20) is completely equivalent to the KKT system (3.23). In other words, given (x^{-f}, y^{-f}), an agent's decisions (x^f, y^f) is optimal if and only if it satisfies the system (3.23). Using this fact, it is straightforward to show that a candidate point $(\chi^1, \chi^2, \ldots, \chi^F)$, where $\chi^f = (x^f, y^f)$, is an equilibrium to the DSPI game where each agent solves (3.20) if and only if it solves the KKT systems (3.23) for each player $f \in \mathcal{F}$. As a consequence, the equilibrium problem for the DSPI game under consideration is equivalent to the complementarity problem obtained by stacking the F systems of (3.23) for $f \in \mathcal{F}$. In this case the decision variable is the combined set of primal and dual variables for each agent, denoted by $(w^1, w^2, \ldots, w^F)$.

With some algebraic manipulation, it can be shown that the complementarity system obtained by stacking the F KKT systems is itself an LCP. Consider the following system obtained from (3.23) by expanding $r^f(x^{-f})$.

$$\begin{aligned} &\upsilon^f = o^f - {A^f}^T \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \geq 0, && \begin{bmatrix} x^f \\ y^f \end{bmatrix} \geq 0, \quad \begin{bmatrix} x^f \\ y^f \end{bmatrix}^T \upsilon^f = 0, \\ &t^f = \begin{bmatrix} d^0 \\ b^f \end{bmatrix} + A^f \begin{bmatrix} x^f \\ y^f \end{bmatrix} + \sum_{\substack{f' \in \mathcal{F} \\ f' \neq f}} \left[\begin{array}{c|c} \mathcal{I}_m & \mathbf{0}_{m \times n} \\ \hline \mathbf{0}_m^T & \mathbf{0}_n^T \end{array} \right] \begin{bmatrix} x^{f'} \\ y^{f'} \end{bmatrix} \geq 0, && \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} \geq 0, \quad {t^f}^T \begin{bmatrix} \lambda^f \\ \beta^f \end{bmatrix} = 0. \end{aligned} \tag{3.26}$$

The interactions between the agent f's decision variables (x^f, y^f) and the KKT system of any other agent $f' \neq f$ can be represented using the matrix $\bar{M}^f$ given below.

$$\bar{M}^f = \left[\begin{array}{c|c|c|c} \mathbf{0}_{m\times m} & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times 1} \\ \hline \mathbf{0}_{n\times m} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times m} & \mathbf{0}_{n\times 1} \\ \hline \mathcal{I}_m & \mathbf{0}_{m\times n} & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times 1} \\ \hline \mathbf{0}_{1\times m} & \mathbf{0}_{1\times n} & \mathbf{0}_{1\times m} & 0 \end{array}\right]. \tag{3.27}$$

Using this notation, the stacked KKT systems (3.26) for agents $f = 1, \ldots, F$ can be formulated as LCP(q, M). Here, the vector q is given by

$$q = \begin{bmatrix} \bar{q}^1 \\ \bar{q}^2 \\ \vdots \\ \bar{q}^F \end{bmatrix}, \quad \text{where} \quad \bar{q}^f = \begin{bmatrix} o^f \\ d^0 \\ b^f \end{bmatrix}, \tag{3.28}$$

and the matrix M is given by

$$M = \left[\begin{array}{c|c|c|c|c} M^1 & \bar{M}^2 & \bar{M}^3 & \cdots & \bar{M}^F \\ \hline \bar{M}^1 & M^2 & \bar{M}^3 & \cdots & \bar{M}^F \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bar{M}^1 & \bar{M}^2 & \cdots & \bar{M}^{F-1} & M^F \end{array}\right]. \tag{3.29}$$

Methods for solving LCPs fall broadly into two categories: (i) pivotal methods such as Lemke's algorithm and (ii) iterative methods such as splitting schemes and interior point methods. The former class of methods are finite when applicable, while the latter class converge to solutions in the limit. In general, the applicability of these algorithms depends on the structural properties of the matrix M. In the following analysis, we show that LCP(q, M) for the DSPI game, as defined in (3.28) and (3.29), possesses two properties that allow us to use Lemke's pivotal algorithm: (i) the matrix M is a copositive matrix, and (ii) $q \in (\mathrm{SOL}(0, M))^*$. Here, given a set $K \in \mathbb{R}^d$, the set K^* is the *dual* cone of K, i.e. $K^* = \{y \in \mathbb{R}^d \ : \ y^T x \geq 0 \ \forall x \in K\}$.

We first show that M is copositive. Recall that a matrix $M \in \mathbb{R}^{d\times d}$ is said to be *copositive* if $x^T M x \geq 0$ for all $x \in \mathbb{R}^d_+$.

Lemma 3.3.3 *Let the vector q and the matrix M be as defined in* (3.28) *and* (3.29) *respectively. Then the matrix M is copositive.*

Proof Let $w \in \mathbb{R}^{2m+n+1}_+$. Using the block structure of M given in (3.29), $w^T M w$ can be decomposed as follows.

$$w^T M w = \sum_{f=1}^{F} {w^f}^T M^f w^f + \sum_{f=1}^{F} \sum_{\substack{f'=1 \\ f' \neq f}}^{F} {w^f}^T \bar{M}^{f'} w^{f'}. \tag{3.30}$$

We analyze the terms under the two summations separately. First consider ${w^f}^T M^f w^f$ for any agent f. Let the dual variables (λ^f, β^f) be collectively denoted by δ^f.

$$\begin{aligned} {w^f}^T M^f w^f &= \begin{bmatrix} {\chi^f}^T & {\delta^f}^T \end{bmatrix} \left[\begin{array}{c|c} \mathbf{0} & -{A^f}^T \\ \hline A^f & \mathbf{0} \end{array}\right] \begin{bmatrix} \chi^f \\ \delta^f \end{bmatrix} \\ &= -{\chi^f}^T {A^f}^T \delta^f + {\delta^f}^T A^f \chi^f \\ &= 0. \end{aligned} \tag{3.31}$$

Now consider any term of the form ${w^f}^T \bar{M}^{f'} w^{f'}$.

$$\begin{aligned} {w^f}^T \bar{M}^{f'} w^{f'} &= \begin{bmatrix} {x^f}^T & {y^f}^T & {\lambda^f}^T & {\beta^f}^T \end{bmatrix} \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \end{array}\right] \begin{bmatrix} x^{f'} \\ y^{f'} \\ \lambda^{f'} \\ \beta^{f'} \end{bmatrix} \\ &= \begin{bmatrix} {x^f}^T & {y^f}^T & {\lambda^f}^T & \beta^f \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ x^{f'} \\ 0 \end{bmatrix} \\ &= {\lambda^f}^T x^{f'}. \end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32) we obtain

$$w^T M w = \sum_{f=1}^{F} \sum_{\substack{f'=1 \\ f' \neq f}}^{F} {\lambda^f}^T x^{f'}. \tag{3.33}$$

Clearly ${w^f}^T \bar{M}^{f'} w^{f'} \geq 0$, if $w^f \geq 0$ and $w^{f'} \geq 0$. Thus $w \geq 0$ implies that $w^T M w \geq 0$.

■

We now show condition (ii), that $q^T w \geq 0\ \forall w \in \text{SOL}(0, M)$.

Lemma 3.3.4 *Let the vector q and the matrix M be as defined in* (3.28) *and* (3.29) *respectively. Then $q \in (SOL(0, M))^*$.*

Proof Consider any $w \in \text{SOL}(0, M)$. Clearly then, w^f must solve the system (3.26) for $f = 1, \ldots, F$, with o^f, d^0 and b^f taking zero values. In this case, considering the primal feasibility of w^f to this system, we obtain the following.

$$\left.\begin{array}{rcl} \sum_{a \in A} c_a^f x_a^f & \leq & 0 \\ y_u^f - y_v^f + \sum_{f=1}^{F} x_{u,v}^f & \geq & 0 \quad \forall (u,v) \in A \end{array}\right\} \quad \text{for } f = 1, \ldots, F. \tag{3.34}$$

Recall that $c_a^f \geq 0$ for all $a \in A$ and $f = 1, \ldots, F$ by assumption. Therefore, (3.34) implies that $x^f = 0$ for any player f. It is easy to see that in this case, we must have

$$y_u^f - y_v^f \geq 0\ \forall (u,v) \in A, \text{ for } f = 1, \ldots F. \tag{3.35}$$

The inner-product $q^T w$ can be decomposed as follows.

$$\begin{array}{rcl} q^T w & = & \sum_{f=1}^{F} {\bar{q}^f}^T w^f \\ & = & \sum_{f=1}^{f} \left({o^f}^T \begin{bmatrix} x^f \\ y^f \end{bmatrix} + {d^0}^T \lambda^f + b^f \beta^f \right) \\ & = & \sum_{f=1}^{F} (y_{s^f}^f - y_{t^f}^f) + {d^0}^T \lambda^f + b^f \beta^f. \end{array} \tag{3.36}$$

Since $w \in (\text{SOL}(0, M))$, $\lambda^f, \beta^f \geq 0$. Furthermore, $d^0, b^f \geq 0$ by assumption. Clearly, ${d^0}^T \lambda^f + b^F \beta^f \geq 0$ for $f = 1, \ldots F$.

Thus it only remains to verify that the objective function terms are non-negative. Consider any $s^f - t^f$ path $\mathcal{P}^f$. By assumption, there must be at least one such path for each player f. By summing up the inequalities (3.35) over the arcs in the path $\mathcal{P}^f$, we obtain the desired result. In other words,

$$\sum_{(u,v)\in\mathcal{P}^f} y_u^f - y_v^f = y_{s^f}^f - y_{t^f}^f \geq 0. \tag{3.37}$$

We have thus shown that $\bar{q}^{f^T} w^f \geq 0$ for $f = 1, \ldots, F$. Summing up over the players, the proof is completed. ■

Using the Lemmas 3.3.3 and 3.3.4, we can now state the following result (Theorem 4.4.13 in [25]) about Lemke's method as it applies to LCP(q, M).

Theorem 3.3.5 ([25]) *If M is copositive and $q \in (SOL(q, M))^*$, then Lemke's method will always compute a solution, if the problem is nondegenerate*[3]

In contrast to the LCP approach for solving DSPI games under continuous interdiction, we have not found a simple or efficient reformulation to solve DSPI games under discrete interdiction (in a centralized manner). In this context, we note that the two-player DSPI game with discrete interdiction may be formulated as a bimatrix game. It is well known that the mixed strategy Nash equilibria to a bimatrix game can be found by solving an LCP. However, for general DSPI games with more than two players, there is no equivalent bimatrix-like game formulation. Motivated by these difficulties, we also explore decentralized approaches to solve DSPI games, which we describe in the next section.

Gauss-Seidel Algorithm

We first present the simplest form of a best-response-based algorithm. The idea is simple: starting with a particular feasible decision variable vector for each agent $\chi_0 = (\chi_0^1, \chi_0^2, \ldots, \chi_0^F)$, solve the optimization problem of a particular agent, say, agent 1, with all of the other agents' actions fixed. Assume an optimal solution exists to

[3] A detailed discussion of degeneracy and cycling in Lemke's method can be found in Section 4.9 of [25].

this optimization problem, and denote it as χ^{1*}. The next agent, say, agent 2, solves its own optimization problem, with the other agents' actions fixed as well, but with χ_0^1 replaced by χ^{1*}. Such an approach is often referred to as a diagonalization scheme or the Gauss-Seidel iteration, and for the remainder of this paper we use the latter name to refer to this simple best-response approach.

Consider applying the Gauss-Seidel iteration to a GNEP, with each agent solving the optimization problem (1.1). We will refer to agent f's individual's problem as $\mathcal{P}(\chi^{-f})$. The Gauss-Seidel iterative procedure is presented in Algorithm 1 below.

Algorithm 3 Gauss-Seidel Algorithm for a GNEP

Initialize. Choose $\chi_0 = (\chi_0^1, \ldots, \chi_0^F)$ with $\chi_0^f \in \Xi^f(\chi_0^{-f})\ \forall f \in \mathcal{F}$. Set $k \leftarrow 0$.

Step 1:

for $f = 1, 2, \ldots, F$ **do**

Set $\chi_{k,f}^{-f} \leftarrow (\chi_{k+1}^1, \ldots, \chi_{k+1}^{f-1}, \chi_k^{f+1}, \ldots, \chi_k^F)$;

Solve $\mathcal{P}(\chi_{k,f}^{-f})$ to obtain χ_{k+1}^f.

end for

Set $\chi_{k+1} \leftarrow (\chi_{k+1}^1, \ldots, \chi_{k+1}^F)$.

Set $k \leftarrow k+1$.

Step 2:

if χ_k satisfies termination criteria, **then STOP**.

else GOTO Step 1.

The Gauss-Seidel algorithm can be directly applied to compute an equilibrium of a DSPI game with discrete interdiction. Note that updates in agent f's decisions occur at iteration k only if there is a strict increase in the agent's payoff at the iteration. For finite termination, we fix a tolerance parameter ϵ and use the following stopping criterion:

$$\|\chi_k - \chi_{k-1}\| \leq \epsilon. \tag{3.38}$$

Since the variables χ_k are integral for discrete interdiction problems, choosing $\epsilon < 1$ will ensure that the algorithm terminates only when successive outer iterates are equal.

Proposition 3.3.4 *Suppose that the Gauss-Seidel algorithm (Algorithm 4) is applied to the DSPI game with discrete interdiction, and the termination criterion* (3.38) *is used with $\epsilon < 1$. If the algorithm terminates at χ_k, then χ_k is an equilibrium to this problem.*

Proof Since the variables χ_k are integral for discrete interdiction problems, choosing $\epsilon < 1$ for the termination criterion will ensure that the algorithm terminates only when successive outer iterates are equal. Consequently, by the assumption, $\chi_{k-1} = \chi_k$ at termination. This also implies that $\chi_{k-1,f}^{-f} = \chi_k^{-f}$ for $f = 1, \ldots, F$. By construction of χ_k, we must then have

$$\chi_k^f = \operatorname*{argmin}_{\chi^f \in \Xi^f(\chi_k^{-f})} \theta^f(\chi^f, \chi_k^{-f}).$$

Clearly, χ_k must then be an equilibrium. ■

Proposition 3.3.4 establishes Algorithm 3 as a heuristic to solve the DSPI game with discrete interdiction. However, there is no guarantee that the algorithm will in fact converge, despite the fact that the DSPI game possesses a potential function. While general results on convergence of best response dynamics for potential games have been well-established in literature, the difficulty here in showing convergence lies in the fact that we are dealing with a GNEP, in which each agent's feasible region is affected by other agents' actions, and such coupling constraints do not have the same functional form for each agent. As a result, any intermediate points resulting from an agent's best responses need not be feasible in the other agents' problems.

We note however that it is possible to detect when the algorithm fails to converge. Recall that $\Xi^f(\chi^{-f}) \subseteq K^f$ for each agent $f \in \mathcal{F}$, where K^f is defined in (3.13). Moreover, the set $\prod_{f=1}^F K^f$ is finite. Any intermediate point χ_k generated by Algorithm 3 must certainly satisfy the budgetary constraints on x_k^f and the bound constraints on y_k^f for each agent f. Therefore $\chi_k \in \prod_{f=1}^F K^f$. In other words, the set of possible points χ_k generated by Algorithm 3 lies in a finite set. This means that if the algorithm fails to converge, it must generate a sequence that contains at least one cycle. The existence of such cycles in non-convergent iterate paths can then be used to detect situations in which the algorithm might fail to converge.

Proposition 3.3.4 is likely the best one can do for general DSPI games under discrete interdiction. However, for the subclass of such games with common source-target pairs, we can in fact prove that the best response dynamics always terminates in a NE in a finite number of steps.

Proposition 3.3.5 *Consider a DSPI game with discrete interdiction with common source-target pairs, and assume that the initial arc lengths d and arc extensions e are integral. Suppose that Algorithm 4 is applied to such a problem, and the termination criteria* (3.38) *is used with $\epsilon < 1$. Then the algorithm will terminate finitely at an equilibrium.*

Proof Denote the common source node as s, and the common target node as t. The set of joint feasible strategies in x under the given assumptions is a finite set. Moreover, all the agents attempt to minimize the common objective, namely the s-t path length. Note that at any iteration k at which an update occurs for any agent's decision, there must then be a strict increase in the s-t path length. Thus there can be no cycles in the sequence $\{\chi_k\}$. Furthermore, since the set of joint feasible strategies is finite, the sequence must terminate at some point χ^*. It is easy to show that χ^* must be an equilibrium (cf. Proposition 3.3.4). ■

For DSPI games with continuous interdiction, establishing the convergence result for a best-response type algorithm is more involved. Even for typical Nash equilibrium problems with no constraint interactions, the simple implementation of the Gauss-Seidel algorithm described in Algorithm 1 may not work. To obtain better convergence properties, we need a regularization scheme, as shown next.

Regularized Gauss-Seidel Algorithm

It can be shown that the basic diagonalization scheme of solving individual agent problems in sequence and updating agent decision variables at each step may not converge to an equilibrium even for GNEPs with favorable properties such as continuously differentiable and convex objective functions. However, Facchinei et al. [38] showed that under certain assumptions, we can overcome this issue by adding a regularization term to the individual agent's problem solved in a Gauss-Seidel iteration.

The regularized version of the optimization problem for agent $f \in \mathcal{F}$ is

$$\begin{aligned} \underset{\chi^f}{\text{maximize}} \quad & \theta^f(\chi^f, \chi^{-f}) - \tau \left\| \chi^f - \overline{\chi}^f \right\|^2 \\ \text{subject to} \quad & \chi^f \in \Xi^f(\chi^{-f}), \end{aligned} \tag{3.39}$$

where τ is a positive constant. Here the regularization term is evaluated in relation to a candidate point $\overline{\chi}^f$. Note that the point $\overline{\chi}^f$ and the other agents' decision variables χ^{-f} are fixed when the problem (3.39) is solved in a regularized Gauss-Seidel iteration. For ease of notation, we will refer to problem (3.39) as $\mathcal{R}(\chi^{-f}, \overline{\chi}^f)$.

The regularized Gauss-Seidel procedure is given in Algorithm 4. As the name indicates, the agents' optimization problems contain a regularization term, and are solved in sequence. The regularization term ensures that problem (3.39) has a unique optimal solution so that the algorithm is well defined. The algorithm is set to terminate when the outer iterates become sufficiently close to each other; in other words, the termination condition is given in (3.38).

This version of the algorithm was originally presented in [38] to solve potential GNEPs with shared constraints. Since DSPI games are GNEPs of non-shared constraints, we use Algorithm 4 as a heuristic algorithm to solve DSPI games under continuous interdiction.

Proposition 3.3.6 *Let* $\{\chi_k\}$ *be the sequence generated by applying Algorithm 4 to the DSPI problem under continuous interdiction, wherein each agent solves* (3.10)*. Suppose* $\{\chi_k\}$ *converges to* $\bar{\chi}$*. Then* $\bar{\chi}$ *is an equilibrium to the DSPI problem.*

Algorithm 4 Regularized Gauss-Seidel Algorithm for a GNEP

Initialize. Choose $\chi_0 = (\chi_0^1, \ldots, \chi_0^F)$ with $\chi_0^f \in \Xi^f(\chi_0^{-f})$ $\forall f \in \mathcal{F}$. Set $k \leftarrow 0$.

Step 1:

for $f = 1, 2, \ldots, F$ **do**

  Set $\chi_{k,f}^{-f} \leftarrow (\chi_{k+1}^1, \ldots, \chi_{k+1}^{f-1}, \chi_k^{f+1}, \ldots, \chi_k^F)$;

  Set $\overline{\chi}^f \leftarrow \chi_k^f$;

  Solve $\mathcal{R}(\chi_{k,f}^{-f}, \overline{\chi}^f)$ to obtain χ_{k+1}^f.

end for

Set $\chi_{k+1} \leftarrow (\chi_{k+1}^1, \ldots, \chi_{k+1}^F)$.

Set $k \leftarrow k + 1$.

Step 2:

if χ_k satisfies termination criteria, **then STOP**.

else GOTO Step 1.

Proof Since $\chi_k \to \bar{\chi}$ we must have $\chi_k^f \to \bar{\chi}^f$ and

$$\lim_{k\to\infty} \left\| \chi_{k+1}^f - \chi_k^f \right\| = 0. \tag{3.40}$$

By construction of $\chi_{k,f}$, (3.40) implies that

$$\lim_{k\to\infty} \chi_{k,f} = \bar{\chi}. \tag{3.41}$$

By Step 1 of Algorithm 4, $\chi_{k+1}^f \in \Xi^f(\chi_{k,f}^{-f})$. Since $\chi_{k+1}^f \to \bar{\chi}^f$, $\chi_k^{-f} \to \bar{\chi}^{-f}$, and $\Xi^f(\chi_{k,f}^{-f})$ is defined by linear inequalities parametrized by $\chi_{k,f}^{-f}$, it is straightforward to see by continuity arguments that $\bar{\chi}^f \in \Xi^f(\bar{\chi}^{-f})$. In other words, $\bar{\chi}$ is feasible for every agent's optimization problem (1.1).

We claim that for each agent $f \in \mathcal{F}$

$$\theta^f(\bar{\chi}^f, \bar{\chi}^{-f}) \geq \theta^f(\chi^f, \bar{\chi}^{-f}), \quad \forall\, \chi^f \in \Xi^f(\bar{\chi}^{-f}).$$

For the purposes of establishing a contradiction, let there be an agent $\bar{f}$ and a vector $\bar{\xi}^{\bar{f}} \in \Xi^{\bar{f}}(\bar{\chi}^{-\bar{f}})$ such that

$$\theta^{\bar{f}}(\bar{\chi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}) < \theta^{\bar{f}}(\bar{\xi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}).$$

We established in the proof of Proposition 3.3.1 that the set valued mapping $\Xi^{\bar{f}}(\cdot)$ satisfies inner semicontinuity relative to its domain. Using the definition of inner semicontinuity (cf. [86] Chapter 5, Section B), and because $\bar{\chi}^{-\bar{f}} \in \mathrm{dom}(\Xi^{\bar{f}}(\cdot))$, we then have

$$\liminf_{\xi^{-\bar{f}} \to \bar{\chi}^{-\bar{f}}} \Xi(\xi^{-\bar{f}}) \supset \Xi(\bar{\chi}^{-\bar{f}}), \tag{3.42}$$

where the limit in (3.42) is given by the following:

$$\liminf_{\xi^{-\bar{f}} \to \bar{\chi}^{-\bar{f}}} \Xi(\xi^{-\bar{f}}) = \left\{ u^{\bar{f}} \mid \forall \chi_k^{-\bar{f}} \to \bar{\chi}^{-\bar{f}}, \exists u_k^{\bar{f}} \to u \text{ with } u_k^{\bar{f}} \in \Xi^{\bar{f}}(\chi_k^{-\bar{f}}) \right\}. \tag{3.43}$$

Since $\bar{\xi}^{\bar{f}} \in \Xi^{\bar{f}}(\bar{\chi}^{-\bar{f}})$, equations (3.41), (3.42) and (3.43) allow us to construct a sequence $\xi_k^{\bar{f}} \in \Xi^{\bar{f}}(\chi_{k,\bar{f}}^{-\bar{f}})$ such that $\xi_k^{\bar{f}} \to \bar{\xi}^{\bar{f}}$ as $k \to \infty$.

Let $d^{\bar{f}} = (\bar{\xi}^{\bar{f}} - \bar{\chi}^{\bar{f}})$. Then by the subdifferentiality inequality for concave functions we must have

$$\theta'^{\bar{f}}(\bar{\chi}^{\bar{f}}, \bar{\chi}^{-\bar{f}}; d^{\bar{f}}) > 0. \tag{3.44}$$

Denote by Φ^f the regularized objective function for agent f's subproblem. In other words,

$$\Phi^f(\chi^f, \chi^{-f}, z) = \theta^f(\chi^f, \chi^{-f}) - \tau \left\| \chi^f - z \right\|^2.$$

We then have

$$\Phi'^f(\chi^f, \chi^{-f}, z; d^f) = \theta'^f(\chi^f, \chi^{-f}; d^f) - 2\tau(\chi^f - z)^T d^f.$$

Note that $\chi_{k+1}^{\bar{f}}$ is obtained by solving the problem $\mathcal{R}(\chi_{k,\bar{f}}^{-\bar{f}}, \chi_k^{\bar{f}})$. In other words, $\chi_{k+1}^{\bar{f}}$ maximizes $\Phi^{\bar{f}}(\xi_{k,\bar{f}}^{\bar{f}}, \chi_{k,\bar{f}}^{-\bar{f}}, \chi_k^{\bar{f}})$ over the set $\Xi^{\bar{f}}(\chi_{k,\bar{f}}^{-\bar{f}})$. Since this is a concave maximization problem, we then apply first order optimality conditions to obtain the following.

$$\begin{aligned} \Phi'^{\bar{f}}(\chi_{k+1}^{\bar{f}}, \chi_{k,\bar{f}}^{-\bar{f}}, \chi_k^{\bar{f}}; (\xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}})) &= \theta'^{\bar{f}}(\chi_{k+1}^{\bar{f}}, \chi_{k,\bar{f}}^{-\bar{f}}, \chi_k^{\bar{f}}; (\xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}})) \\ &\quad + \ 2\tau(\chi_{k+1}^{\bar{f}} - \chi_k^{-\bar{f}})(\xi_k^{\bar{f}} - \chi_{k+1}^{\bar{f}}) \qquad (3.45) \\ &\le 0. \end{aligned}$$

Passing to the limit $k \to \infty$, $k \in K$ and using (3.41) we obtain $0 \ge \theta'^{\bar{f}}(\bar{\chi}^f, \bar{\chi}^{-f}; (\bar{\xi}^f - \bar{\chi}^f))$ which contradicts (3.44).

■

The proof of this theorem, though a straightforward adaptation of Theorem 4.3 in Facchinei et al. [38] does differ in one key aspect. We assume that the entire sequence $\{\chi_k\}$ converges to $\bar{\chi}$. This is a strong assumption in the sense that it also requires that all the intermediate points $\chi_{k,f}$ to converge to $\bar{\chi}$, a fact key to proving that $\bar{\chi}$ is indeed an equilibrium. In contrast, for GNEPs with shared constraints, the feasibility of the intermediate points ensures their convergence even to cluster points of the sequence generated by the algorithm.

3.3.3 Numerical Results

We use the algorithms presented in the previous section to study several instances of DSPI games. The decentralized algorithms were implemented in MATLAB R2010a with CPLEX v12.2 as the optimization solver. The LCP formulation for the DSPI game with continuous interdiction was solved using the MATLAB interface for the complementarity solver PATH [40]. Computational experiments were carried out on a desktop workstation with a quad-core Intel Core i7 processor and 16 GHz of memory running Windows 7.

For DSPI games with discrete interdiction, we used Algorithm 3. For DSPI games with continuous interdiction, we applied a combination of Algorithm 3 and Algorithm 4. In particular, we first tried Algorithm 3, and then used Algorithm 4 if Algorithm 3 failed to converge to an equilibrium. We pursued this strategy since the number of outer iterations until Algorithm 4 converged was found to be quite sensitive to the regularization parameter τ, typically resulting in slow convergence rates for the algorithm. Since the running time for Algorithm 3, especially with the outer iterations restricted to a maximum of 1000, was quite short relative to the running time for Algorithm 4, it seemed reasonable to try using Algorithm 3 first, and use Algorithm 4 only as necessary.

Computing Equilibria

First, we applied the algorithm to Example 3 in Section 3.3.1, which is a DSPI game with continuous interdiction. In particular, the network is given in Figure 3.2 and there are 2 agents: agent 1 has an adversary with source node 1 and target node 5, and agent 2 has an adversary with source node 1 and target node 6. Both agents have an interdiction budget of 1. The initial arc lengths are 0, and the interdiction costs are equal for both agents and are given as the arc labels in Figure 3.2, with $\epsilon = 2$. We set the regularization parameter $\tau = 0.01$. We were able to obtain a solution within an accuracy of 10^{-6} in 3 outer iterations.

Furthermore, we obtained multiple Nash equilibria by varying the starting point of the algorithm. All the equilibria obtained resulted in the same shortest path lengths for each agent. Some of the equilibria obtained are given in Table 3.3. The column x_0 represents the starting interdiction vector for each agent; while x_N^1 and x_N^2 give the equilibrium interdiction vectors for agents 1 and 2, respectively. The seven components in the vectors of x_0, x_N^1 and x_N^2 represent the interdiction actions at each of the seven arcs in Figure 3.2, with the arcs being ordered as follows: first, the top horizontal arcs $(1,2)$ and $(2,3)$, then the vertical arcs $(1,4)$, $(2,5)$ and $(3,6)$, and finally the bottom horizontal arcs $(4,5)$ and $(5,6)$. The remaining two columns in Table 3.3 , p_1 and p_2, give the shortest path lengths for agents 1 and 2 respectively, at the equilibrium χ_N.

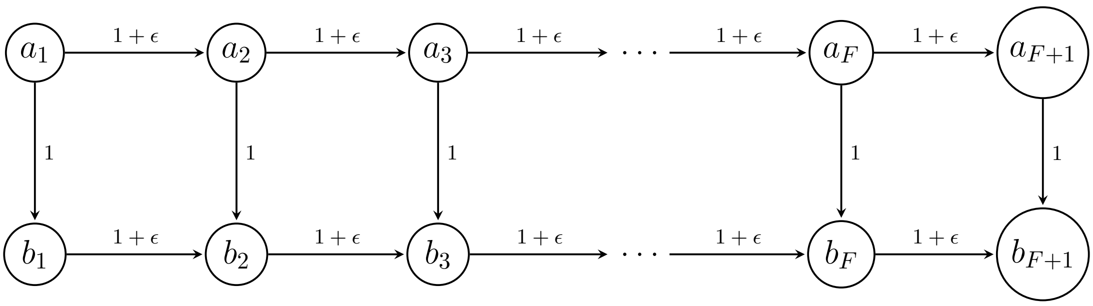

Figure 3.3. Network structure for DSPI Example 5

Table 3.3.
Multiple equilibria for the instance of the DSPI game in Example 2

x_0	x_N^1	x_N^2	p_1	p_2
$(0, 0, 0, 0, 0, 0, 0)$	$(0, 0, 0.5, 0.5, 0, 0, 0)$	$(0, 0, 0.1667, 0.1667, 0.6667, 0)$	0.6667	0.6667
$(0.2, 0.2, 0, 0, 0, 0, 0)$	$(0, 0, 0.6, 0.4, 0, 0, 0)$	$(0, 0, 0.0667, 0.2667, 0.6667, 0)$	0.6667	0.6667
$(0, 0, 0, 0, 0, 0.2, 0.2)$	$(0, 0, 0.4, 0.6, 0, 0, 0)$	$(0, 0, 0.2667, 0.0667, 0.6667, 0)$	0.6667	0.6667
$(0, 0, 0, 0, 0, 0.3, 0.3)$	$(0, 0, 0.35, 0.65, 0, 0, 0)$	$(0, 0, 0.3167, 0.0167, 0.6667, 0)$	0.6667	0.6667
$(0.3, 0.3, 0, 0, 0, 0, 0)$	$(0, 0, 0.65, 0.35, 0, 0, 0)$	$(0, 0, 0.0167, 0.3167, 0.6667, 0)$	0.6667	0.6667
$(0.25, 0.25, 0, 0, 0, 0, 0)$	$(0, 0, 0.625, 0.375, 0, 0, 0)$	$(0, 0, 0.0417, 0.2917, 0.6667, 0)$	0.6667	0.6667
$(0, 0, 0, 0, 0, 0.25, 0.25)$	$(0, 0, 0.375, 0.625, 0, 0, 0)$	$(0, 0, 0.2917, 0.0417, 0.6667, 0)$	0.6667	0.6667
$(0, 0, 0, 0, 0, 0.15, 0.15)$	$(0, 0, 0.425, 0.575, 0, 0, 0)$	$(0, 0, 0.2417, 0.0917, 0.6667, 0)$	0.6667	0.6667
$(0.15, 0.15, 0, 0, 0, 0, 0)$	$(0, 0, 0.575, 0.425, 0, 0, 0)$	$(0, 0, 0.0917, 0.2417, 0.6667, 0)$	0.6667	0.6667

Example 5 To test the algorithm on problems with larger scale, we expanded the network in Example 3 with varying graph sizes and numbers of agents. For F agents, the graph contains $2(F + 1)$ vertices with the edges as shown in Figure 3.3. The source vertex for all agents is a_1. The target vertex for a given agent f is b_{f+1}. The initial arc lengths are all assumed to be zero. The interdiction costs are the same for all the players and are given as the arc labels in Figure 3.3. All the agents are have an interdiction budget of 1. The cost parameter ϵ is chosen as 2. For discrete interdiction on these graphs, arc extensions are assumed to be by a length of 1.

The running time and iterations required to compute equilibria for these instances are summarized in Table 3.4 and Table 3.5. Table 3.4 gives the number of outer iterations and runtime for Algorithm 3 over these instances with continuous interdiction. For an empirical comparison between the decentralized and centralized approaches, the performance of Lemke's method for the LCP formulation is given in the last column of the table. The results indicate that the running time for the centralized method increases monotonically with the problem size. However, the running time for the decentralized method depends not just on the problem size but also on the number of outer iterations. In general, there is no correlation between these two parameters. Indeed the algorithm is observed to converge in relatively few iterations even for some large problem instances. This is in stark contrast to the rapid increase in running time observed for the LCP approach as problem size increases.

It must be noted that the order in which the individual agent problems are solved in Algorithm 3 plays an important role. Indeed it was found that the algorithm could fail to converge for certain agent orders, while succeeding to find equilibria quickly for the same instance with a different ordering of agents. For instance, for a network of size 25, solving the agent problems in their natural order $\{1, 2, \ldots, 25\}$ resulted in the failure of Algorithm 3 to converge even after 1000 outer iterations. However, with a randomized agent order, the algorithm converged in as few as 13 iterations. It is encouraging to note that for the same agent order that resulted in the failure of Algorithm 3, the regularized method Algorithm 4 converged to a GNE within 394 outer-iterations with a runtime of 28 wall-clock seconds.

Table 3.4.
Number of iterations and running times for DSPI Example 5 under continuous interdiction.

	Decentralized		LCP
# Agents	# Iterations	Runtime (s)	Runtime (s)
5	3	0.0205	0.0290
10	5	0.0290	0.1833
15	11	0.1103	0.7534
20	5	0.0723	2.1106
25	13	0.2609	4.8167
30	15	0.4070	10.2256
35	10	0.3605	17.7387
40	41	1.7485	30.2382
45	12	0.6601	48.6280
50	12	0.7981	75.0420

Table 3.5.
Number of iterations and running times for DSPI Example 5 under discrete interdiction.

# Agents	# Iterations	Runtime (s)
5	5	0.1776
10	3	0.1627
15	3	0.2419
20	3	0.3164
25	3	0.4005
30	3	0.5155
35	3	0.5948
40	3	0.7387
45	3	0.8794
50	3	1.0385

Computation of Efficiency Losses

Using the decentralized algorithm and its potential to find multiple equilibria by starting at different points, we empirically study the efficiency loss of decentralized interdiction strategies in DSPI games. We focus first on Example 5, with the underlying network represented in Figure 3.3. Before computing the average efficiency losses empirically using our algorithms, we first establish a theoretical bound on the worst-case price of anarchy, for the purpose of comparison.

Recall that there are F agents and the source-target pair for agent f is (a_1, b_{f+1}). Noting that all paths for all agents contain either the arc (a_1, a_2) or the arc (a_1, b_1), one feasible solution to the centralized problem is for each agent to interdict both these arcs by $1/(2+\epsilon)$ for a total cost of 1. In this case the length of both arcs become $n/(2+\epsilon)$, giving a shortest path length of $n/(2+\epsilon)$ for each agent. Note that this is not an equilibrium solution as agent 1 can deviate unilaterally to interdict arcs (a_1, b_1) and (a_2, b_2) by 1/2 to obtain a shortest path length of $(n + \epsilon/2)/(2 + \epsilon)$.

A Nash equilibrium to this problem is given by the following solution. Agent f interdicts the vertical arcs $(a_1, b_1), \ldots, (a_f, b_f)$ by $1/(f(f+1))$ and the arc (a_{f+1}, b_{f+1}) by $f/(f+1)$. Each agent then has a shortest path length of $n/(n+1)$. Note that all the s^f-t^f paths are of equal length for every agent. Therefore diverting any of the budget to any vertical arc will result in unequal path lengths and a shorter shortest path for any agent. Obviously, diverting the budget to interdict any of the horizontal arcs is cost inefficient because of their higher interdiction cost $1 + \epsilon$. Thus no agent has an incentive to deviate from this solution.

We now have a feasible solution to the centralized problem that has an objective value of $n/(2+\epsilon)$ for each agent, and a Nash equilibrium that has an objective value of $n/(n+1)$ for each agent. Therefore, the worst-case price of anarchy for the DSPI game depicted in Figure 3.3 must be at least $(n+1)/(2+\epsilon)$.

Using the regularized Gauss-Seidel algorithm we also compute lower bounds on the worst-case price of anarchy and average efficiency losses for the same network topology with varying number of agents. The instances we consider are obtained by varying ϵ uniformly in the range of $(1.5, 10)$. For purposes of comparison, the numerical results are plotted in Figure 3.4 below. Note that the average-case efficiency loss is much lower than the worst-case price of anarchy. For the particular graph structure under consideration, we observe that the average efficiency loss grows at a much lower rate than the worst-case efficiency loss. However this observation cannot be generalized to other graph structures and such patterns may only be discernible by applying the computational framework we presented.

Example 6 We further tested the decentralized algorithms for continuous interdiction on random graphs to study average efficiency losses of equilibria of DSPI games on networks with different topologies. To generate random graphs, we took the number of vertices and the density of the graph – the number of arcs in the random graph divided by the maximum possible number of arcs – as inputs. The number of agents

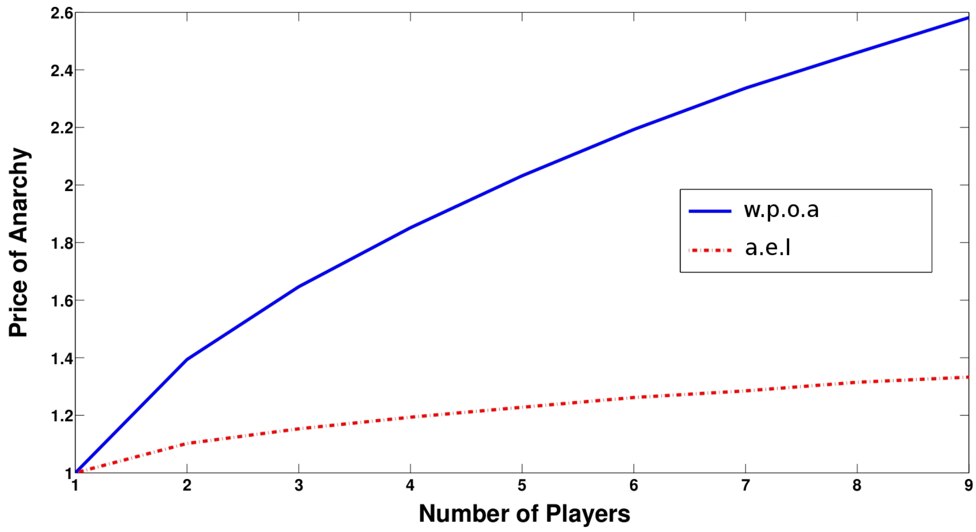

Figure 3.4. Efficiency loss with respect to the number of agents.

was chosen randomly from the interval $(0, |V|/2)$; one agent set size was chosen per vertex set size. Source-target pairs were chosen at random for each interdictor. Fixing the vertex set, we populated the arc set by successively generating source-target paths for the players until the desired density was reached. We thus ensured connectivity between the source-target pairs for each player. Costs, initial arc lengths and interdiction budgets were chosen from continuous uniform distributions. Arc interdiction costs were assigned uniformly in the range $[1, 5]$. The budget for each agent was chosen uniformly from the interval $[b^f/10, b^f/2]$, where $b^f = \sum_{a \in A} c_a^f$. The initial length of each arc was chosen uniformly from $[1, 5]$.

For each combination of vertex set size, agent set size and graph density, we generated 25 random instances by drawing values from the uniform distributions described above for the various network parameters. For each instance, we used 10 different random permutations of the agents to run the decentralized algorithms in an attempt to compute multiple equilibria. The lower bound on the price of anarchy for the game was computed as the worst case efficiency loss over these 25 instances.

The average efficiency loss over these instances was also computed. The results are summarized in Table 3.6. Our experiments indicate that the average efficiency loss and the worst-case price of anarchy tend to grow as the number of vertices and number of agents increases; on the other hand, these measures of efficiency loss sometimes do not appear to be monotonically increasing or decreasing in the density of the underlying network.

Table 3.6.
DSPI Continuous Interdiction - Random Graphs

# Vertices	# Agents	Density	Avg. Run Time (s)	# Avg Iter.	*a.e.l*	*p.o.a*
5	3	0.25	0.0037	3	1.3133	1.5561
5	3	0.5	0.0038	3	1.3265	1.9529
5	3	0.75	0.0040	3	1.5099	2.3829
10	3	0.25	0.0065	4	1.5366	2.2078
10	3	0.5	0.0176	11	1.4538	2.3114
10	3	0.75	0.0132	8	1.4273	2.1342
15	4	0.25	0.0263	11	1.7091	2.9246
15	4	0.5	0.0939	33	1.7524	2.7904
15	4	0.75	0.1267	42	1.5695	2.1425
20	5	0.25	0.1269	34	2.1907	3.2885
20	5	0.5	0.2087	43	1.8523	2.7906
20	5	0.75	0.5416	100	1.7967	2.3782
25	7	0.25	0.7167	105	2.5631	4.8788
25	7	0.5	1.9564	207	2.3022	5.5794
25	7	0.75	1.8476	158	1.9884	2.4423

3.4 Decentralized Max-Flow Interdiction Games

In this section, we analyze decentralized max-flow interdiction games (DMFI) models. These models may be used to describe situations in which adversaries wish to maximize the flow of some good between two nodes on a network. The interdictors' aim is to minimize this maximum flow by means of taking actions to reduce arc capacities on the network.

Formally, let the network on which the various agents act be described by the digraph $G = (V, A)$. Each arc $(i, j) \in A$ has a capacity u_{ij}. For agent $f \in \mathcal{F}$, the adversary wishes to maximize the flow between source node s^f and target node t^f. Agent f then wishes to minimize the maximum flow $s^f - t^f$ flow. Each agent f then solves the following optimization problem:

$$\begin{aligned}
\min_{x^f} \quad & \theta^f(x^f, x^{-f}) = \begin{pmatrix} \max\limits_{y} & y^f_{t^f s^f} \\ \text{s.t.} & \sum\limits_{v \in V} y^f_{uv} - \sum\limits_{v \in V} y^f_{vu} = 0, \ \forall u \in V \\ & y^f_a \leq u_a(x^f, x^{-f}), \ \forall a \in A \backslash \{(t^f, s^f)\} \end{pmatrix} \\
\text{s.t.} \quad & \sum_{(u,v) \in A} c^f_{uv} x^f_{uv} \leq b^f \\
& x^f_{uv} \geq 0, \quad \forall (u, v) \in A.
\end{aligned} \tag{3.46}$$

The x variables represent the actions of the interdictor, while the y variables represent the decisions of the adversaries. That is, y_{uv} is the flow on arc (u, v) on the aftermath network. Note that by a standard construction, we add an artificial arc (t, s) to the network and maximize the back flow on this arc.

As in the DSPI game, the effect of interdiction decisions (x^f, x^{-f}) on the arc capacities depend on the type of interdiction. If interdiction is assumed to be continuous and additive, then the relationship is given as follows:

$$u_{ij}(x^f, x^{-f}) = \max \left\{ (u^0_{ij} - \sum_{f=1}^{F} x^f_{ij}), 0 \right\}, \tag{3.47}$$

where x^f is restricted to be component-wise non-negative.

If on the other hand interdiction is binary, the arc capacities are given by

$$u_{ij}(x^f, x^{-f}) = u^0_{ij}(1 - \max_{f \in \{1,\ldots,F\}} x^f_{ij}), \tag{3.48}$$

and the x^f variables are assumed component-wise binary.

We leverage LP duality on the inner max flow problem to reformulate the DMFI problem for player f as follows.

$$
\begin{aligned}
\min_{x^f,\beta^f,\alpha^f} \quad & \sum_{(i,j)\in A} \beta_{ij}^f u_{ij}(x^f, x^{-f}) \\
\text{subject to} \quad & \beta_{ij}^f + \alpha_i^f - \alpha_j^f \geq 0 \quad \forall (i,j) \in A \\
& \alpha_{t^f}^f - \alpha_{s^f}^f \geq 1 \\
& \beta_{ij}^f \geq 0, \quad \forall (i,j) \in A.
\end{aligned}
\tag{3.49}
$$

Note that the strategic interaction between the players are restricted to the objective functions of each player. Thus the model presented above is a typical NEP. However, for DMFI games under continuous interdiction, the capacities u_{ij} are linear functions of the interdiction variables x. Thus the objective function in problem (3.49) contains non-convex bilinear terms. This makes the analysis of the DMFI model difficult. Note that by standard combinatorial arguments, it is possible to argue that the variables α and β can be restricted to be binary. However, in this case we end up with integer programming models for each player's optimization problem. We look at such a formulation as an alternative to (3.49).

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in A} u_{ij}\beta_{ij}^f \\
\text{subject to} \quad & \alpha_j^f - \alpha_i^f \leq \beta_{ij}^f + \frac{u_{ij} - u_{ij}(x^f, x^{-f})}{u_{ij}}, \ \forall (i,j) \in A \\
& \alpha_{t^f}^f - \alpha_{s^f}^f \geq 1 \\
& \sum_{(i,j)\in A} c_{ij}^f x_{ij}^f \leq b^f \\
& x_{ij}^f \geq 0, \ \forall (i,j) \in A \\
& 0 \leq \beta_{ij}^f \leq 1, \ \forall (i,j) \in A \\
& \alpha_i^f \in \{0,1\}, \quad \forall i \in V.
\end{aligned}
\tag{3.50}
$$

The variables $\alpha \in \{0,1\}^{|V|}$ represent an $s^f - t^f$ cut on the graph G, α_i being 1 if i is on the t^f side of the cut and 0 other wise. The parameter u_{ij}^f is the initial capacity of the arc (i,j). The variables β_{ij}^f capture the fractional capacity remaining in the arc

(i,j) on the cut defined by α. Thus the objective minimizes the capacity of the cut, which is equivalent to maximizing the $s^f - t^f$ flow.

3.4.1 Game Structure and Analysis

Generalized Nash Equilibrium Problem formulation

It is fairly straightforward to see that the model for the DMFI game where each player's problem is formulated as (3.49) is a typical NEP. On the other hand the alternative formulation given in (3.50) is a GNEP, where the variables and functions can be given as follows.

$$\chi^f = (x^f, \beta^f, \alpha^f),$$

$$\theta^f(\chi^f, \chi^{-f}) = \sum_{(i,j)\in A} u_{ij}\beta^f_{ij},$$

$$\Xi^f(\chi^{-f}) = \left\{ \chi^f = (x^f, \beta^f, \alpha^f) \;\middle|\; \begin{array}{l} \alpha^f_j - \alpha^f_i \leq \beta^f_{ij} + \dfrac{u_{ij} - u_{ij}(x^f, x^{-f})}{u_{ij}}, \ \forall (i,j) \in A \\ \alpha^f_{t^f} - \alpha^f_{s^f} \geq 1 \\ \sum\limits_{(i,j)\in A} c^f_{ij} x^f_{ij} \leq b^f \\ x^f_{ij} \geq 0, \ \forall (i,j) \in A \\ 0 \leq \beta^f_{ij} \leq 1, \ \forall (i,j) \in A \\ \alpha^f_i \in \{0,1\}, \quad \forall i \in V. \\ x^f \in X^f \end{array} \right\}. \tag{3.51}$$

Existence of Equilibria

Existence analysis for the general DMFI game is difficult for the following reasons. Under continuous interdiction, the formulation presented in (3.49) is an NEP with non-convex bilinear objective functions. Although existence results can be shown for

NEPs with quasiconvex or pseudoconvex objective functions, results for more general nonconvex games are usually not available. Thus, in the absence of a straightforward method to place the formulation within a tractable subclass such as potential or supermodular games, providing an existence guarantee for this formulation remains an open question.

On the other hand the formulation given in (3.50) is a GNEP where the functions involved are linear in all the variables under continuous interdiction (after a suitable transformation of the positive part term in the capacity functions). It seems possible in this case to apply Theorem 3.3.1 to this formulation. However, the main difficulty in this case is that while the functions involved are linear, there are integer variables in the formulation. Thus the semicontinuity of the feasible set becomes questionable. As such, we have not been able to provide the required semicontinuity proof that guarantees existence of solutions to this GNEP formulation for the DMFI game.

As with the DSPI game, if we restrict our attention to case with common adversaries, i.e. a DMFI game where each interdictor attempts to minimize the maximum flow between a common pair of nodes s and t, we can provide stronger results. In essence, since the source and sink are the same for all the players, we can omit the superscript f from the inner minimization problem in (3.46). In this case the objective function (without the superscript) $\theta(x)$ provides a natural potential function for the primal NEP formulation for the game given in (3.46). Since the feasible set for each player is compact under the continuous interdiction case, and finite under the discrete interdiction case, we can guarantee existence of a solution which is the potential minimizer. We formally present the result below.

Proposition 3.4.1 *For the DMFI game, an equilibrium always exists if the source and target nodes are the same for each player, i.e.* $s^f = s$ *and* $t^f = t$ *for* $f = 1, \ldots, F$.

3.5 Minimum Cost Flow Interdiction

In the minimum cost flow interdiction game, the adversaries attempt to minimize a linear cost flow function. The leaders attempt to maximize this min cost function for their adversaries by using their resources to increase the adversary's costs on each arc.

3.5.1 Formulation

The players operate on the same network structure given by $G = (V, A)$. We are given demands at each node d_i such that $\sum_{i \in V} d_i = 0$ and cost functions for each adversary given by the vector $\bar{c}^f$. The min cost flow interdiction problem for each interdictor may then be formulated as

$$\begin{array}{ll} \max\limits_{x^f} & \theta^f(x^f, x^{-f}) = \left(\begin{array}{lrcl} \min\limits_{z} & \sum\limits_{(i,j) \in A} z_{ij}^f \ \bar{c}_{ij}^f & & \\ \text{s.t.} & \sum\limits_{j \in V} z_{ij}^f - \sum\limits_{j \in V} z_{ji}^f & = & d_i^f \\ & z_{ij}^f & \leq & u_{ij}^f(x^f, x^{-f}), \ \forall (i,j) \in A \\ & z_{ij}^f & \geq & 0, \ \forall (i,j) \in A. \end{array} \right) \\ \text{s.t.} & \sum\limits_{(i,j) \in A} c_{ij}^f x_{ij}^f \leq b^f. \end{array} \tag{3.52}$$

Using the dual of the inner minimization problem, we may reformulate (3.52) into the following single level problem.

$$\begin{array}{ll} \max\limits_{x^f, \alpha^f, \beta^f} & \sum\limits_{i \in V} d_i^f \alpha_i^f - \sum\limits_{(i,j) \in A} u_{ij}^f(x^f, x^{-f}) \beta_{ij}^f \\ \text{s.t.} & \alpha_i^f - \alpha_j^f - \beta_{ij}^f \leq \bar{c}_{ij}^f, \ \forall (i,j) \in A \\ & \beta^f \geq 0 \\ & \sum\limits_{a \in A} c_a^f x_a^f \leq b^f. \end{array} \tag{3.53}$$

If interdiction is additive and continuous, then the arc capacities are given by

$$u_{ij}(x^f, x^{-f}) = \max\left\{u_{ij}^0 - \sum_{f=1}^{F} x_{ij}^f, 0\right\}.$$

Under binary interdiction the arc capacities are given by

$$u_{ij}(x^f, x^{-f}) = u_{ij}^0(1 - \max_{f \in \{1,\ldots,F\}} x_{ij}^f).$$

3.5.2 Game Structure and Analysis

Out of the three types of decentralized network interdiction games, the minimum cost flow version presents the most difficulties for analysis. This is not surprising considering the fact that we cannot restrict the variables α and β to be binary. The objective function also contains a biliear term in x and β. In the form given in (3.53), the problem is an NEP with a non-convex (bilinear) objective and a polyhedral feasible set.

One possible approach to analyze this problem is to push the bilinear term containing the x variables into the constraint set. In this case we end up with a GNEP with a linear objective and a non-convex feasible set, which nonetheless consists of inequalities with continuous functions. This formulation has a potential function much like the formulations for the DSPI and DMFI games since the objective functions end up independent of each other. We may then use the machinery of the objective function and use the regularized Gauss-Seidel algorithm for non-convex problems. However this formulation yields us no insight to prove existence of equilibria.

Another avenue to show existence of equilibria under continuous interdiction is to look at the formulation (3.52) directly and show that it admits a potential function.

Under discrete interdiction, if we can show that the dual variables α and β take discrete values, then existence of equilibria as well as convergence of best response chains will follow from the potentiality property applied to finite games.

4. POTENTIAL GAMES UNDER EXOGENOUS UNCERTAINTY

The models presented in the previous chapter focus primarily on decentralized decision making in the deterministic setting, i.e. when all the problem parameters and functions are known to all the players with certainty. The remainder of this dissertation looks into the problem of studying the convergence of decentralized algorithms in cases when some of the underlying parameters in each player's problem is subject to uncertainty. Specifically, we consider scenarios where the uncertainty is external, or as we call it "exogenous", by which we mean that the structure of the uncertainty is known to all the players a priori. In other words, information such as the probability distributions of random vectors is available to all the players.

The main goal of our research is to design appropriate sampling or approximation schemes, which may be used in conjunction with decentralized algorithms, to compute the solutions of NEPs under exogenous uncertainty. We utilize some recent advances in the theory of approximations for NEPs, as well as decomposition methods for certain structured non-convex optimization problems, to obtain the desired convergence results for the important class of potential games.

4.1 Introduction

Formally, we consider games involving a set of F players, wherein each player f controls some decision variables x_f and solves the following optimization problem:

$$\begin{array}{ll} \underset{x_f}{\text{minimize}} & \phi_f(x) = \mathbb{E}\left[\theta_f(x_f, x_{-f}; \xi)\right] \\ \text{subject to} & x_f \in X_f. \end{array} \tag{4.1}$$

The objective function for player f depends not only on her own decision variables x_f and the other players' variables denoted by x_{-f}, but also on a common random vector ξ that represents underlying uncertainty factors that affect all the players' decisions. Given the other players' decisions, each player f attempts to minimize the expected value of her objective function over the probability space $(\Omega, \mathcal{F}, P)$ for ξ. In this sense, the players are assumed to be risk-neutral.

Following standard notation for Nash equilibrium problems as described in Chapter 2, we denote the game defined above as $\text{NEP}(\phi_f, X_f)_{f=1}^F$. A set of player decision variables $x^* = (x_1^*, \ldots, x_F^*)^T \in \mathbf{X} = \prod_{f=1}^F X_f$ is a solution to $\text{NEP}(\phi_f, X_f)_{f=1}^F$ if no player has an incentive to deviate from x^*. In other words, we have

$$\phi_f(x_f^*, x_{-f}^*) \leq \phi_f(y_f, x_{-f}^*) \quad \forall y_f \in X_f. \tag{4.2}$$

Recall that a point $x^* \in \mathbf{X}$ that satisfies (4.2) is called a solution or a Nash equilibrium (NE) to the $\text{NEP}(\phi_f, X_f)_{f=1}^F$.

For notational clarity, we refer to the problem defined above as $\text{SNEP}(\theta_f, X_f)_{f=1}^F$. This is to stress the fact that the NEP in question, i.e. $\text{NEP}(\phi_f, X_f)_{f=1}^F$ is an equilibrium problem under exogenous uncertainty. However, note that the two problems are completely equivalent.

Our interest lies in designing provably convergent *decentralized* schemes for computing solutions to $\text{SNEP}(\theta_f, X_f)_{f=1}^F$. To be precise, we wish to design algorithmic mechanisms wherein given the state of the game as represented by all the players' decisions, each player may update their decisions solely based on their own payoffs and constraints.

Specifically, we focus on the two main types of "best-response" mechanisms described in Section 2.2 in Chapter 2, i.e. schemes in which at any stage of the algorithm, each player computes an optimal solution to her own problem (4.1), given the latest information about the decisions of the other players. The first algorithm we consider is similar to a non-linear Gauss-Seidel type iterative scheme, where the players take turns updating their solutions. The scheme is given in Algorithm 1. The second type of algorithm we consider updates player decisions in parallel, as opposed to the sequential update scheme in Algorithm 1. In other words, at the k^{th} iteration of the algorithm, all the players update their decisions simultaneously taking as given the decisions of the other players from the previous iteration, i.e. x_{-f}^{k-1}. This Gauss-Jacobi type scheme is given below in Algorithm 2.

Aside from their normative value in capturing the selfish rationality of Nash players, decentralized schemes such as Algorithms 1 and 2 also have the advantage of being intuitive and easy to implement in practical settings. This is because of the minimal requirements of information sharing amongst the players. Decentralized schemes are also attractive computationally, since they are suited for distributed or parallel computations. For instance, Algorithm 1 may be implemented in such a way that each player's problem is solved on local machines while updates are handed out via minimal communication mechanisms. On the other hand, since player updates are computed simultaneously in Algorithm 2, the Gauss-Jacobi scheme is eminently suited for implementation in a parallel computing setting.

Due to these and other advantages, decentralized algorithms have found widespread popularity amongst practitioners for solving NEPs. However, there remain significant theoretical challenges in the analysis of such schemes. Indeed, even for the simple best-response type mechanisms presented in Algorithms 1 and 2, proving convergence for general NEPs remains an elusive goal. Even in the deterministic setting, these schemes have been shown to converge to equilibria only for certain classes of games.

In this chapter, our focus is on providing such convergence results for potential games under exogenous uncertainty. To be precise, the expectation terms in the objective functions for each player are resolved by a sampling based approximation scheme. Formally, we consider the following approximation to player f's problem:

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \hat{\phi}_f(x) = \frac{1}{N}\sum_{j=1}^{N}[\theta_f(x_f, x_{-f}, \xi_j)] \\ \text{subject to} \quad & x_f \in X_f. \end{aligned} \tag{4.3}$$

Here, player f's objective function $\phi_f(x)$ is approximated by $\hat{\phi}_f(x)$ by using a sample $\{\xi_j\}_{j=1}^N$ of size N from $(\Omega, \mathcal{F}, P)$. For the purposes of our analysis, we assume that this sample is provided to all the players. The approximation of $\text{SNEP}(\theta_f, X_F)_{f=1}^F$ where each player solves (4.3) instead of (4.1) is referred to as $\text{SAANEP}(\hat{\phi}_f, X_f)_{f=1}^F$.

The main questions we seek to answer about the approximation scheme are the following -

1. If we increase the sample size N, can we establish the asymptotic convergence of solutions to $\text{SAANEP}(\hat{\phi}_f, X_f)_{f=1}^F$ to solutions of the original game $\text{SNEP}(\theta_f, X_f)_{f=1}^F$?
2. Given a sample $\{\xi_j\}_{j=1}^N$ of size N, can we solve $\text{SAANEP}(\hat{\phi}_f, X_f)_{f=1}^F$ in a decentralized fashion with provable convergence?

4.1.1 Related Literature

There has recently been a lot of interest from the optimization community on the topic of designing computational schemes to compute solutions to NEPs under uncertainty. Closely related to our work is the problem of solving stochastic variational inequalities (SVIs). The exponential convergence of SAA methods for SVI problems has been shown in [115]. There has also been a growing interest in designing Stochastic Approximation (SA) schemes for SVI problems. Such schemes, where function values and derivatives are approximated via simulation, have been shown to converge under mild conditions [51, 62]. Various generalizations and extensions of SA type methods for SVIs, including some distributed methods, have been investigated in [117]. However, in contrast to our approach, even distributed methods for solving SVIs usually require some degree of coordination between the players in choosing step-lengths or other algorithmic parameters.

Additionally, in the case of games under exogenous uncertainty, there is also the question of resolving the expectation terms that distinguish $\text{SNEP}(\theta_f, X_f)_{f=1}^F$ from a deterministic NEP. The approach we use is that of using a Sample Average Approximation (SAA), $\hat{\phi}_f(\cdot)$ in place of the expectation function $\mathbb{E}[\phi_f(\cdot)]$. While we use the SAA functions to directly approximate each player's optimization problem, SAA can also be used in conjunction with various reformulations to SNEP$(\theta_f, X_f)_{f=1}^F$, including variational inequality reformulations [37], complementarity reformulations, Nikaido-Isoda function based optimization reformulations [111] etc. Thus all the reformulation techniques outlined above lead to centralized algorithms to compute Nash equilibria under uncertainty. In contrast, we wish to focus on decentralized algorithms.

Best-response mechanisms, such as the Gauss-Seidel and Jacobi methods presented in Algorithms 1 and 2, constitute the most fundamental forms of decentralized algorithms for NEPs. As mentioned in Chapter 2, the main difficulty associated with these algorithms is the theoretical challenge associated with proving their convergence to equilibria, even in the deterministic setting.

4.1.2 Contributions

The main contribution of the research outlined in this chapter is the design and analysis of a decentralized approximation scheme for computing the solutions to SNEP$(\theta_f, X_f)_{f=1}^F$ with provable asymptotic convergence. We propose a Sample Average Approximation (SAA) method to resolve the expected value terms in each player's objective function. We utilize the theoretical tool of multi epi-convergence to show the convergence of solutions to the SAA problem SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ to those of the true problem.

Furthermore, by restricting our attention to potential games, we are able to show the convergence of both sequential and parallel best-response sequences to Nash equilibria under fairly mild assumptions. We also present an implementation scheme for both approaches within a high performance computing environment, to speed up the computation of equilibria significantly.

4.1.3 Outline

The remainder of the chapter is organized as follows. In Section 4.2, we introduce player-wise convex potential games and state various properties of such games that we use in our analysis. While we restrict our attention to this class of games for our work, we also note that many problems of practical interest fall under this class. Section 4.3 discusses the SAA approximation scheme for SNEP$(\theta_f, X_f)_{f=1}^F$ in detail and introduces the notion of multi epi-convergence, the main theoretical tool that we use to show convergence of approximate solutions. In Section 4.4, details of the sequential and parallel best-response algorithms are presented along with the necessary regularity and convexity assumptions required to prove their convergence to equilibria. The main theoretical results of our work, namely the convergence results for each algorithm, are also given in this section.

4.2 Potential games under exogenous uncertainty

In this section, we introduce the notion of potential games under exogenous uncertainty formally. We also establish conditions under which SNEP$(\theta_f, X_f)_{f=1}^F$ admits a potential function. The idea of player-wise convexity in objective functions is presented and a key result regarding the correspondence between solutions to potential games and stationary points of the relevant potential minimization problem is given.

In order to establish the potentiality property for SNEP$(\theta_f, X_f)_{f=1}^F$, we first consider the deterministic variant obtained by fixing the random vector ξ to a particular realization $\bar{\xi}$. We refer to this problem as NEP$(\bar{\theta}_f, X_f)_{f=1}^F$, where each player f solves -

$$\begin{array}{ll} \underset{x_f}{\text{minimize}} & \bar{\theta}_f(x) = \theta_f(x_f, x_{-f}; \bar{\xi}) \\ \text{subject to} & x_f \in X_f. \end{array} \tag{4.4}$$

For the remainder of this paper, we make the following assumption on the underlying deterministic NEP$(\bar{\theta}_f, X_f)_{f=1}^F$.

Assumption 3 *There exists a function $P : \mathcal{X} \times \Omega \to \mathbb{R}$ such that*

$$\theta_f(x_f, x_{-f}; \bar{\xi}) - \theta_f(y_f, x_f; \bar{\xi}) = P(x_f, x_{-f}; \bar{\xi}) - P(y_f, x_{-f}; \bar{\xi}), \tag{4.5}$$

holds for $f = 1, \ldots, F$ and any fixed $\bar{\xi} \in \Omega$.

Assumption 3 essentially states that for fixed values of the parameter ξ, the NEP$(\bar{\theta_f}, X_f)_{f=1}^F$ has an exact potential function P. With Assumption 3 in place, we may readily give the following results regarding the existence of a potential function for both SNEP$(\theta_f, X_f)_{F=1}^F$, and SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

Lemma 4.2.1 *Under Assumption 3, SNEP$(\theta_f, X_f)_{F=1}^F$ has an exact potential function -*

$$\bar{P}(x) = \mathbb{E}_\xi[P(x, \xi)].$$

Proof Integrating both sides of equation (4.5) over Ω, we obtain

$$\mathbb{E}_\xi(\theta_f(x_f, x_{-f}; \xi)) - \mathbb{E}_\xi(\theta_f(y_f, x_f; \xi)) = \mathbb{E}_\xi(P(x_f, x_{-f}; \xi)) - \mathbb{E}_\xi(P(y_f, x_{-f}; \xi)), \tag{4.6}$$

for any $x \in \mathcal{X}$ and for any player $f = 1, 2, \ldots, F$. In other words, $\bar{P}$ is an exact potential for SNEP$(\theta_f, X_f)_{F=1}^F$. ■

Lemma 4.2.2 *Under Assumption 3, SAANEP$(\hat{\phi}_f, X_f)_{F=1}^F$ has an exact potential function -*

$$\hat{P}(x) = \frac{1}{N} \sum_{j=1}^{N} [P(x, \xi_j)].$$

Proof By Assumption (3) we have

$$\theta_f(x_f, x_{-f}; \xi_j) - \theta_f(y_f, x_f; \xi_j) = P(x_f, x_{-f}; \xi_j) - P(y_f, x_{-f}; \xi_j),$$

for any $y_f \in X_f$ and for any player $f = 1, \ldots F$. Adding these equations over $j = 1, \ldots, N$ and dividing by N we obtain

$$\begin{aligned} \hat{\phi}_f(x_f, x_{-f}) - \hat{\phi}_f(y_f, x_f) &= \frac{1}{N} \sum_{j=1}^{N} \left(\theta_f(x_f, x_{-f}; \xi_j) - \theta_f(y_f, x_{-f}; \xi_j)\right) \\ &= \frac{1}{N} \sum_{j=1}^{N} \left(P(x_f, x_{-f}; \xi_j) - P(y_f, x_{-f}; \xi_j)\right). \end{aligned} \tag{4.7}$$

In other words the function $\hat{P}(x) = \frac{1}{N}\sum_{j=1}^{N} P(x,\xi_j)$ is an exact potential function to $\mathrm{NEP}(\hat{\phi}_f, X_f)_{f=1}^F$. ■

Note that we may relax Assumption 3 to hold for almost every $\xi \in \Omega$ and still have valid exact potential functions for $\mathrm{SNEP}(\theta_f, X_f)_{F=1}^F$ and $\mathrm{SAANEP}(\hat{\phi}_f, X_f)_{f=1}^F$.

Example 1: Nash-Cournot Equilibrium

In order to illustrate the principles explained above, we consider the Nash-Cournot equilibrium problem. Formally we have F participants or firms in a Cournot oligopoly. Each firm chooses its production quantity $q_f \in Q_f \subset \mathbb{R}$. Here the set Q_f is assumed to be a nonempty, convex and compact set that represents the production constraints for firm f. The inverse demand function is given by $F(Q) = a - b\sum_{f=1}^F q_f$. The parameters a and b are assumed to be positive. Each firm's production costs are given by the strongly convex quadratic cost function $c_f(q_f)$.

For this problem, the Nash-Cournot equilibrium quantities for the firms come out as a solution to the $\mathrm{NEP}(\theta_f, Q_f)_{f=1}^F$, where each firm f solves -

$$\begin{array}{ll} \underset{q_f}{\text{maximize}} & \theta_f(q) = q_f\left(a - b\sum_{f=1}^F q_f\right) - c_f(q_f) \\ \text{subject to} & \qquad q_f \in Q_f. \end{array} \tag{4.8}$$

It is well known (Cf. [75]) that an exact potential function for $\mathrm{NEP}(\theta_f, Q_f)_{f=1}^F$ is given as follows -

$$P(q) = a\sum_{f=1}^F q_f - b\sum_{f=1}^F q_f^2 - b\sum_{1\le f<f'\le F} q_f q_{f'} - \sum_{f=1}^f c_f(q_f). \tag{4.9}$$

Now suppose that the inverse demand function is subject to uncertainty. Formally, suppose the parameters a and b depend on some random vector ξ with a probability space $(\Omega, \mathcal{F}, P)$. Then the stochastic version of the Nash-Cournot game is given by SNEP$(\theta_f, Q_f)_{f=1}^F$. In our standard notation, we then have

$$\begin{aligned}
\phi_f(q) &= \mathbb{E}[\theta_f(q;\xi)] = \mathbb{E}\left[q_f\left(a(\xi) - b(\xi)\sum_{f=1}^F q_f\right) - c_f(q_f)\right] \\
\hat{\phi}_f(q) &= \frac{1}{N}\sum_{j=1}^N \left[q_f\left(a(\xi_j) - b(\xi_j)\sum_{f=1}^F q_f\right) - c_f(q_f)\right].
\end{aligned} \tag{4.10}$$

It is trivial to verify that exact potential functions for the problems SNEP$(\theta_f, Q_f)_{f=1}^F$ and SAANEP$(\hat{\phi}_f, Q_f)_{f=1}^F$ are given respectively by -

$$\begin{aligned}
\bar{P}(q) &= \mathbb{E}\left[a(\xi)\sum_{f=1}^F q_f - b(\xi)\sum_{f=1}^F q_f^2 - b(\xi)\sum_{1\le f<f'\le F} q_f q_{f'} - \sum_{f=1}^f c_f(q_f)\right], \\
\hat{P}(q) &= \frac{1}{N}\sum_{j=1}^N\left[a(\xi_j)\sum_{f=1}^F q_f - b(\xi_j)\sum_{f=1}^F q_f^2 - b(\xi_j)\sum_{1\le f<f'\le F} q_f q_{f'} - \sum_{f=1}^f c_f(q_f)\right].
\end{aligned} \tag{4.11}$$

We now introduce the notion of player-wise convexity in each player's objective function for the NEP$(\theta_f, X_f)_{f=1}^F$. The fundamental idea behind player-wise convexity is that once the other players' decisions x_{-f} are fixed, $\theta_f(\cdot)$ is convex in the player's own variables x_f. The formal definition is given below.

Definition 4.2.1 *The NEP$(\theta_f, X_f)_{f=1}^F$ is said to be player-wise convex if the sets X_f are compact, convex sets and for each $f = 1, \ldots, F$,*

$$\theta_f\left(\lambda x_f + (1-\lambda)y_f, x_{-f}\right) \le \lambda\ \theta_f(x_f, x_{-f}) + (1-\lambda)\ \theta_f(y_f, x_{-f}) \quad \forall \lambda \in (0,1), \tag{4.12}$$

holds for any $x_f, y_f \in X_f$ and $x_{-f} \in X_{-f}$.

The class of player-wise convex NEPs is an important subclass both from theoretical and practical perspectives. In general, existence of equilibria can be guaranteed only for player-wise convex NEPs. Moreover, if a player's objective function is non-convex in its own variables, then there is a possibility of multiple optimal solutions for that player's problem. This fact complicates both the theoretical analysis, as well as practical implementations of the equilibrium concept for such problems.

Example 1 continued: Consider the Nash-Cournot game in Example 1. Once the other players' decisions q_{-f} are fixed, player f's objective function is given as follows -

$$\theta_f(q_f, q_{-f}) = \left(a - b \sum_{\substack{f'=1 \\ f' \neq f}}^{F} q_{f'} \right) q_f - b q_f^2 - c_f(q_f).$$

Since $b > 0$, $\theta_f(\cdot)$ is easily verified to be strongly concave once q_{-f} is fixed. Because the objective is to maximize $\theta_f(\cdot)$, the Nash-Cournot game is player-wise convex.

We now establish an important property of player-wise convex potential games. Note that an exact potential function P for NEP$(\theta_f, X_f)_{f=1}^F$ is said to be player-wise convex if for each player $f = 1, \ldots F$, the following holds true for all $x_f, y_f \in X_f$ and each $x_{-f} \in X_{-f}$.

$$P\left(\lambda x_f + (1-\lambda) y_f, x_{-f}\right) \leq \lambda\ P(x_f, x_{-f}) + (1-\lambda)\ P_f(y_f, x_{-f}) \quad \forall \lambda \in (0,1). \quad (4.13)$$

Lemma 4.2.3 *Suppose the NEP$(\theta_f, X_f)_{f=1}^F$ is player-wise convex and has an exact potential function P, then P must also be player-wise convex.*

Proof Since NEP$(\theta_f, X_f)_{f=1}^F$ is player-wise convex, for any $x_f, y_f \in X_f$ and for any $x_{-f} \in X_{-f}$ we have

$$\theta_f\left(\lambda x_f + (1-\lambda) y_f, x_{-f}\right) \leq \lambda\ \theta_f(x_f, x_{-f}) + (1-\lambda)\ \theta_f(y_f, x_{-f}) \quad \forall \lambda \in (0,1). \quad (4.14)$$

Let $z_f = \lambda x_f + (1-\lambda) y_f$. Then the inequality (4.14) is equivalent to the following statement.

$$\lambda\left(\theta_f(z_f, x_{-f}) - \theta_f(x_f, x_{-f})\right) \leq (1-\lambda)\left(\theta_f(y_f, x_{-f}) - \theta_f(z_f, x_{-f})\right). \quad (4.15)$$

Indeed, one can simply replace the inequality (4.12) with the inequality (4.15) in Definition 4.2.1.

Furthermore, NEP$(\theta_f, X_f)_{f=1}^F$ admits an exact potential function P. We must therefore have

$$\theta_f(x_f, x_{-f}) - \theta_f(y_f, x_{-f}) = P(x_f, x_{-f}) - P(y_f, x_{-f}) \forall x_f, y_f \in X_f, \quad \forall x_{-f} \in X_{-f}. \quad (4.16)$$

Combining the equation (4.16), with the inequality (4.15), and noting that $z_f \in X_f$ due to the convexity of X_f,we obtain the following result for the potential function.

$$\begin{aligned}\lambda\left(P(z_f, x_{-f}) - P(x_f, x_{-f})\right) \;&\leq\; (1-\lambda)\left(P(y_f, x_{-f}) - P(z_f, x_{-f})\right) \\ &\forall \lambda \in (0,1),\ \forall x_f, y_f \in X_f,\ \text{and } \forall x_{-f} \in X_{-f}.\end{aligned} \tag{4.17}$$

Clearly then, the potential function P must be player-wise convex. ■

This result is not surprising since it is easy to see that the exact potentiality requirement means that once the other players' variables x_{-f} are fixed, the shape of the potential function $P(\cdot, x_{-f})$ must reflect the shape of player f's objective function $\theta_f(\cdot, x_{-f})$. To make this more precise, consider the following reasoning. Fix any $\bar{x}_f \in X_f$.

$$\begin{aligned} P(x_f, x_{-f}) &= P(\bar{x}_f, x_{-f}) + (P(x_f, x_{-f}) - P(\bar{x}_f, x_{-f})) \\ &= P(\bar{x}_f, x_{-f}) + (\theta_f(x_f, x_{-f}) - \theta_f(\bar{x}_f, x_{-f})) \\ &= (P(\bar{x}_f, x_{-f}) - \theta_f(\bar{x}_f, x_{-f})) + \theta_f(x_f, x_{-f}). \end{aligned} \tag{4.18}$$

Therefore,

$$P(x_f, x_{-f}) = C_f + \theta_f(x_f, x_{-f}), \tag{4.19}$$

where $C_f = (P(\bar{x}_f, x_{-f}) - \theta_f(\bar{x}_f, x_{-f}))$. Thus, fixing the other players' variables x_{-f}, the potential function $P(\cdot, x_{-f})$ is an *affine* transformation of $\theta_f(\cdot, x_{-f})$, an operation that preserves convexity.

Example 1 continued: For the Nash-Cournot equilibrium problem given in Example 1, once the other players' decisions q_{-f} are fixed, then the potential function can be given as

$$\begin{aligned} P(q) &= \left[q_f \left(a - b \sum_{f=1}^{F} q_f \right) - c_f(q_f) \right] \\ &\quad - \left[a \sum_{\substack{f'=1 \\ f' \neq f}}^{F} q_{f'} - b \sum_{\substack{f'=1 \\ f' \neq f}}^{F} q_{f'}^2 - b \sum_{\substack{1 \leq \bar{f} < f' \leq F \\ \bar{f}, f' \neq f}} q_{\bar{f}} q_{f'} - \sum_{\substack{f'=1 \\ f' \neq f}}^{F} c_f(q_f) \right]. \\ &= \theta_f(q_f, q_{-f}) + K(q_{-f}). \end{aligned}$$

Since $K(q_{-f})$ does not depend on q_f and since $\theta_f(q_f, q_{-f})$ has already been shown to be convex in q_f for fixed values of q_{-f}, the potential function $P(q)$ must be player-wise convex.

For player-wise convex potential games with continuous objective functions for each player, there is an important relationship between Nash equilibria and the set of stationary points of the potential minimization problem. Before explaining this result, we first state the continuity assumption.

Assumption 4 *Each $\theta_f(\cdot, x_{-f}; \xi)$ is continuously differentiable for any $x_{-f} \in X_{-f}$ and for any $\xi \in \Xi$.*

Given Assumption (4), we have the following result.

Lemma 4.2.4 *Suppose that NEP$(\theta_f, X_f)_{f=1}^F$ is a player-wise convex NEP with an exact potential function P and satisfies Assumption 4. Suppose further that $x^* \in X$ is a stationary solution to*

$$\begin{array}{ll} \underset{x}{\text{minimize}} & P(x) \\ \text{subject to} & x \in X. \end{array} \tag{4.20}$$

Then x^* is a solution to the NEP$(\theta_f, X_f)_{f=1}^F$.

Proof First note that under Assumption 4, it is easy to show the following:

$$\nabla_{x_f} \theta_f(x) = \nabla_{x_f} P(x) \quad \forall x \in \mathcal{X}. \tag{4.21}$$

Indeed this is a simple consequence of holding all but one component of x_f fixed in equation (4.16) and taking limits.

Suppose now that x^* is a stationary solution to the potential minimization problem (4.20). In this case we have

$$(y - x^*)^T \ \nabla_x P(x^*) \geq 0 \quad \forall y \in X. \tag{4.22}$$

If we decompose the gradient into components, the inequality (4.22) is equivalent to

$$\sum_{f=1}^{F} (y_f - x_f)^T \ \nabla_{x_f} P(x^*) \geq 0. \tag{4.23}$$

Fix a player f. Denote $x^* = (x^*_f, x^*_{-f})$. Consider $y = (y_f, x^*_{-f})$, where $y_f \in X_f$. Clearly $y \in \mathcal{X}$. Moreover, we have $y_{f'} - x_{f'} = 0$ for any player $f' \neq f$. The inequality (4.23) then implies the following.

$$\begin{aligned} (y_f - x_f)^T\, \nabla_{x_f} P(x^*) &\geq 0 \quad \forall y_f \in X_f \\ \Longrightarrow (y_f - x_f)^T\, \nabla_{x_f} \theta_f(x^*) &\geq 0 \quad \forall y_f \in X_f, \end{aligned} \tag{4.24}$$

where the implication follows from equation (4.21). But this in turn means that x^* is a stationary solution to player f's problem (4.18) with the other agents' decisions fixed at x^*_{-f}. By the player-wise convexity of the Nash equilibrium problem, this also means that x^*_f is *optimal* to player f's problem.

Since the above reasoning holds for any player $f = 1, \ldots, F$, x^* is simultaneously optimal to each of the players. Thus x^* must solve the NEP$(\theta_f, X_f)_{f=1}^F$. ■

Remark 4.2.5 *Note that the converse statement is not easily shown. The main difficulty is that if we start with a Nash equilibrium point x^*, in order to show its stationarity with respect to the potential minimization problem, we need to show that $(y - x^*)^T \nabla_x P(x^*) \geq 0$ for* ***any*** *$y \in X$. But we only have $(y - x)^T \nabla_x P(x^*) \geq 0$ for any $y = (y^f, x^*_{-f})$.*

4.3 Approximating equilibria and multi-epiconvergence

One of the fundamental questions about SNEP$(\theta_f, X_f)_{f=1}^F$ we seek to answer in this paper is the following - if we approximate the expectation functions in each player's objective function using a fixed sample of the random vector ξ of size N, what type of convergence properties might we expect as N is taken to ∞? The conditions under which such convergence properties may be analyzed depend heavily on precisely how we solve approximations of SNEP$(\theta_f, X_f)_{f=1}^F$.

For instance, under the potentiality assumption (Assumption 3), we may solve the problem

$$\underset{x \in \mathcal{X}}{\text{minimize}}\ \bar{P}(x) = \mathbb{E}_\xi[P(x, \xi)], \tag{4.25}$$

to obtain a solution of $\text{SNEP}(\theta_f, X_f)_{f=1}^F$. In this case we may consider approximating the optimization problem (4.25) with the following SAA problem

$$\underset{x\in\mathcal{X}}{\text{minimize}}\, \hat{P}(x) = \frac{1}{N}\sum_{j=1}^{N}[P(x,\xi_j)]. \tag{4.26}$$

For this approach to approximating $\text{SNEP}(\theta_f, X_f)_{f=1}^F$, standard results for SAA methods for optimization problem may then be applied (Cf. Chapter 5 [99]). However, a major disadvantage of this approach is that while the individual player's objective functions may be convex in her own variables, the potential functions $\bar{P}(\cdot)$ and $\hat{P}(\cdot)$ are usually not convex in the combined vector x. This is clearly illustrated in Example 1. In this case, both solving the potential minimization SAA problem (4.26) and establishing conditions for convergence of solutions of (4.26) to those of (4.25) become significantly challenging tasks.

An alternative approach to solving $\text{SNEP}(\theta_f, X_f)_{f=1}^F$ is to establish conditions under which the expectation functions $\phi_f(x_f, x_{-f})$ are convex in the player's own variables x_f. In this case, the stochastic equilibrium problem can be seen to be player convex, and we may stack the first order variational conditions on each player's optimization problem to obtain a variational inequality reformulation for $\text{SNEP}(\theta_f, X_f)_{f=1}^F$. To be precise, $\text{SNEP}(\theta_f, X_f)_{f=1}^F$ can be shown to be equivalent to the stochastic variational inequality[1] $\text{SVI}(\mathcal{X}, \mathbf{F}(x))$ where

$$\mathbf{F}(x) = \left(\nabla_{x_f}\mathbb{E}[\theta_f(x_f, x_{-f};\xi)]\right).$$

Sample average approximation methods for stochastic variational inequalities has been a subject of recent research [16,51] and exponential convergence of such methods has been reported for certain classes of SVI problems [115]. However, this approach remains a centralized method for solving $\text{SNEP}(\theta_f, X_f)_{f=1}^F$ and takes no advantage of possible potentiality or supermodularity properties.

The primary goal of this paper is to create approximations of $\text{SNEP}(\theta_f, X_f)_{f=1}^F$ and to solve these approximations in a decentralized way. Our basic strategy then

[1]The variational inequality problem $\text{VI}(\mathcal{X}, \mathbf{F}(x))$ requires finding a vector $x \in \mathcal{X}$ such that $(y - x)^T\mathbf{F}(x) \geq 0$ for all $y \in \mathcal{X}$.

is to directly approximate the objective functions $\phi_f(x)$ with the sample average functions $\hat{\phi}_f(x)$. In other words, we wish to study the behavior of solutions to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ in relation to solutions of SNEP$(\theta_f, X_f)_{f=1}^F$ as the sample size $N \to \infty$. If we then impose Assumption 3, we may then solve SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ using decentralized methods to obtain candidate solutions to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

In order to show the desired convergence properties of SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$, we rely on the theoretical property known as multi-epiconvergence [50]. This property is an extension of the well known epiconvergence concept into the multi-agent domain. The formal definition of multi-epiconvergence is given below.

Definition 4.3.1 *[50] The family of functions $\{\hat{\phi}_f\}_{f=1}^F$, where each $\hat{\phi}_f : \mathbb{R}^{n_f} \to \mathbb{R}$, is said to multi-epiconverge to the functions $\{\phi_f\}_{f=1}^F$ on the set $\mathcal{X}$ if the following conditions hold for every $f = 1, \dots, F$ and every $x \in \mathcal{X}$:*

1. *For every sequence $\{x_{-f}^k\} \in X_{-f}$ that converges to x_{-f}, a sequence $\{x_f^k\} \in X_f$ exists such that*

$$\mathop{lim\ sup}_{k \to \infty} \hat{\phi}_f(x^k) \leq \phi_f(x).$$

2. *For every sequence $\{x^k\} \in \mathcal{X}$ converging to x,*

$$\mathop{lim\ inf}_{k \to \infty} \hat{\phi}_f(x^k) \geq \phi_f(x).$$

Pang and Gurkan [50] proved that if the family $\{\hat{\phi}_f\}_{f=1}^F$ multi-epiconverges to the family $\{\phi^f\}_{f=1}^F$, then solutions to the sampled problem SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ converge to solutions of the true problem SNEP$(\theta_f, X_f)_{f=1}^F$. The formal result is given below.

Theorem 4.3.1 *[50] Suppose the sets X_f are closed for $f = 1, \dots, F$. Suppose further that the family of approximate functions $\{\hat{\phi}_f\}_{f=1}^F$ multi-epiconverges to $\{\phi_f\}_{f=1}^F$ on the set $\mathcal{X}$. If x_N is a Nash equilibrium to SAANEP$(\hat{\phi}_f, X_f)$ and $x_N \to x^*$ as $N \to \infty$, then x^* is a Nash equilibrium to the stochastic problem SNEP$(\theta_f, X_f)_{f=1}^F$.*

The required multi-epiconvergence of the family $(\hat{\phi}_f)_{f=1}^F$ to $(\phi_f)_{f=1}^F$ can be shown under some straightforward regularity conditions as stated in the assumptions below.

Assumption 5 *For any $x \in \mathcal{X}$, the functions $\theta_f(\cdot;\xi)$ is continuous at x for almost every $\xi \in \Omega$, for each $f = 1, \ldots, F$.*

Assumption 6 *For each $f = 1, \ldots, F$, the functions $\theta_f(x, \xi)$ is dominated by an integrable function, i.e. there exists a nonnegative valued measurable function $g(\xi)$ with $\mathbb{E}[g(\xi)] < \infty$ such that for each $x \in \mathcal{X}$, $|\theta_f(x, \xi)| \leq g(\xi)$ holds with probability 1.*

By Theorem 2 [50] the continuous convergence of $\hat{\phi}_f$ to ϕ_f on $\mathcal{X}$ is a sufficient condition for the required multi-epiconvergence. We use Theorem 7.48 [99], to state conditions under which the continuous convergence may be shown.

Theorem 4.3.2 *[99] Suppose that $\mathcal{X}$ is a nonempty compact set and that Assumptions 5 and 6 hold. Suppose further that the samples used to construct the functions $\hat{\phi}_f(\cdot)$ are i.i.d. Then the expected value functions $\phi_f(x)$ is finite valued and continuous on $\mathcal{X}$. Furthermore, $\hat{\phi}(x)$ converges continuously (and uniformly) to $\phi_f(x)$ on $\mathcal{X}$ with probability 1.*

Example 1 continued: For the stochastic Nash Cournot equilibrium problem, consider the case where $a(\xi)$ and $b(\xi)$ has bounded support, i.e. $\max_\xi |a(\xi)| < \infty$, and $\max_\xi |b(\xi)| < \infty$.

Under this assumption, it is easy to verify the multi-epiconvergence of the family $\{\hat{\phi}_f(q)\}_{f=1}^F$ to the family $\{\phi_f(q)\}_{f=1}^F$ on $\mathcal{Q} = \prod_{f=1}^F Q_f$. Indeed Assumption 5 is trivially satisfied since for any fixed value of ξ, $\theta_f(q)$ is a bilinear polynomial in $q = (q_1, \ldots, q_F)^T$. Since $a(\xi)$ has bounded support, we may set $g(\xi) = \bar{q}_f \ \bar{a}$ where

$$\bar{q}_f = \max_{q_f \in Q_f} |q_f| \quad \text{and} \quad \bar{a} = \max_\xi |a(\xi)|.$$

Clearly $g(\xi)$ is integrable since it is a constant independent of ξ. Furthermore, we have

$$\begin{aligned} \theta_f(q, \xi) &= q_f \left(a(\xi) - b(\xi) \sum_{f=1}^F q_f \right) - c_f(q_f) \\ &\leq q_f \ a(\xi) \leq \bar{q}_f \ \bar{a} = g(\xi), \end{aligned} \tag{4.27}$$

where the first inequality follows from the fact that $b(\xi) > 0$, $c_f(q_f) \geq 0$ and $q_f \geq 0$ for all $q_f \in Q_f$ for each firm $f = 1, \ldots, F$.

Since Q_f is a nonempty, closed, bounded, convex set for each f by assumption, we may apply Theorems 4.3.2 and 4.3.1 to show the convergence of solutions of SAANEP$(\hat{\phi}_f(q), Q_f)_{f=1}^F$ to those of SNEP$(\theta_f(q), Q_f)_{f=1}^F$.

4.4 Algorithms and convergence

Our basic strategy is to compute solutions to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ as candidate solutions to SNEP$(\theta_f, X_f)_{f=1}^F$. Under Assumption 3, SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ can easily be seen to be a game with an exact potential function $\hat{P}(\cdot)$ (see Lemma 4.2.2). In this case, we intend to utilize decentralized methods such as Gauss-Seidel or Gauss-Jacobi best-response schemes to solve SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

However it is possible to construct examples of games with exact potential games where naive decomposition algorithms such as Algorithm 1 and 2 fail to converge to equilibria [38]. Note also that in both the algorithms above, the update step involves solving each player's problem given the latest information available for the other players' decisions. However, if $\phi_f(\cdot, x_{-f})$ is not strongly convex, there may exist multiple solutions to player f's problem. In this case there is also the natural issue of which solution to choose and whether convergence can be established irrespective of such choices.

A standard approach to tackle both issues mentioned in the last paragraph is to use a regularization term on each player's objective function. In other words, instead of solving

$$\underset{x_f}{\text{minimize}} \;\; \phi_f(x_f, x_{-f}^k) \quad \text{subject to } x_f \in X_f,$$

we instead solve the following regularized problem -

$$\underset{x_f}{\text{minimize}} \;\; \phi_f(x_f, x_{-f}^k) + \tau_f \|x_f - \bar{x}_f\|_2 \quad \text{subject to } x_f \in X_f. \tag{4.28}$$

Here $\bar{x}_f$ is some reference point, usually taken as the current decision vector of the player x_f^k. With the regularization scheme, we can show convergence of Gauss-Seidel and Gauss-Jacobi schemes for SAANEP$(\hat{\phi}_f, X_F)_{f=1}^F$ under various conditions.

Before stating the algorithms formally, we first state the following Lipschitz condition on the gradients of each player's objective function, required for showing convergence of the regularized decomposition methods.

Assumption 7 *For each $f = 1, \ldots, F$, $\nabla_x \phi^f(\cdot, \xi)$ is is Lipschitz continuous for each $\xi \in \Omega$.*

We then have the following result regarding SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ based on the assumptions on NEP$\theta_f(\cdot, \xi), X_f)_{f=1}^F$.

Lemma 4.4.1 *Suppose NEP$(\theta_f(\cdot, \xi), X_f)_{f=1}^F$ is a player-wise convex NEP for each $\xi \in \Omega$ and satisfies Assumptions 3, 4 and 7, then we have the following*

(i) Each X_f is a compact, convex set.

(ii) Each $\hat{\phi}_f$ is continuously differentiable on $\mathcal{X}$.

(iii) Each $\nabla_x \hat{\phi}_f$ is Lipschitz continuous on $\mathcal{X}$.

(iv) The potential function $\hat{P}$ is continuously differentiable on $\mathcal{X}$ with Lipschitz gradients.

Proof By the player-convexity assumption, part (i) is trivially true. Parts (ii) and (iii) are a straightforward consequence of the fact that $\hat{\phi}_f(x)$ is an affine combination of the functions $\theta_f(x, \xi_j)$ for $j = 1, \ldots, N$. Thus continuity and differentiability properties as well as Lipschitz properties of the gradient are preserved.

For part (iv), we begin by noting that by Assumption 3, the potential function $P(x, \xi)$ must be continuously differentiable with Lipschitz gradients for any fixed values of ξ. We may then apply the affine combination argument once more to argue that these properties must also hold true for $\hat{P}(x)$. ∎

4.4.1 Regularized Gauss-Jacobi algorithm

We first consider the regularized version of Algorithm 2 to solve the problem SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$. Pseudocode for the algorithm is given below in Algorithm 5.

Algorithm 5 Regularized Gauss-Jacobi Algorithm for SNEP$(\theta_f, X_f)_{f=1}^F$

Step 0: Initialize - Set $N > 0$, $\tau > 0$, $x^0 \leftarrow (x_f^0)_{f=1}^F \in \mathcal{X}$, $k \leftarrow 0$.

Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**

Step 2: Main Iteration:

FOR $f = 1, \ldots, F$, let x_f^{k+1} solve

$$\begin{array}{ll} \underset{x_f}{\text{minimize}} & \hat{\phi}_f(x_f, x_{-f}^k) + \tau \left\| x_f - x_f^k \right\|^2 \\ \text{subject to} & x_f \in X_f. \end{array} \tag{4.29}$$

Step 3: Update: $x^{k+1} = (x_f^{k+1})_{f=1}^F$.

The algorithm is initialized by setting a sample size N for the approximation, as well as choosing an initial feasible point $x^0 \in \mathcal{X}$. At each major iteration step say $k+1$, each player takes as given the decisions of the players from the previous iteration, i.e x^k. Each player f then solves the problem of minimizing the approximate function $\hat{\phi}(\cdot, x_{-f}^k)$ with an added regularization term.

The convergence of iterates of Algorithm 5 to approximate solutions of the problem SNEP$(\theta_f, X_f)_{f=1}^F$ is shown in two steps. Firstly, we leverage results on parallel block update schemes for non-convex optimization problems in [97] to show that iterates of the algorithm converge to stationary points of the potential minimization problem, and thus solutions of SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$. In the second step, we establish consistency of the limit points of the algorithmic sequence to solutions of SNEP$(\theta_f, X_f)_{f=1}^F$ as the sample size $N \to \infty$ using multi-epiconvergence.

We begin by showing that solving the regularized problem (4.28), is equivalent to minimizing a similarly regularized potential function in player f's variables, while keeping the other players' decisions fixed.

Lemma 4.4.2 *Suppose Assumption 3 holds for SNEP*$(\theta_f, X_f)_{f=1}^F$. *Fix* $x_{-f} \in X_{-f}$, $x_f^k \in X_f$ *and* $\tau_f > 0$. *Then*

$$\underset{x_f}{argmin}\left(\theta_f(x_f, x_{-f}) + \tau_f \left\|x_f - x_f^k\right\|_2^2\right) = \underset{x_f}{argmin}\left(P(x_f, x_{-f}) + \tau_f \left\|x_f - x_f^k\right\|_2^2\right). \tag{4.30}$$

Proof Suppose $x^* \in \underset{x_f}{\operatorname{argmin}}\left(\theta_f(x_f, x_{-f}) + \tau_f \left\|x_f - x_f^k\right\|_2^2\right)$. This is equivalent to saying that for any $x_f \in X_f$,

$$\begin{array}{rrcl} & \left(\theta_f(x_f^*, x_{-f}) + \tau_f \left\|x_f^* - x_f^k\right\|_2^2\right) & \leq & \left(\theta_f(x_f, x_{-f}) + \tau_f \left\|x_f - x_f^k\right\|_2^2\right) \\ \Leftrightarrow & \left(\theta_f(x_f^*, x_{-f}) - \theta_f(x_f, x_{-f})\right) & \leq & \tau_f\left(\left\|x_f - x_f^k\right\|_2^2 - \left\|x_f^* - x_f^k\right\|_2^2\right) \\ \Leftrightarrow & \left(P(x_f^*, x_{-f}) - P(x_f, x_{-f})\right) & \leq & \tau_f\left(\left\|x_f - x_f^k\right\|_2^2 - \left\|x_f^* - x_f^k\right\|_2^2\right) \\ \Leftrightarrow & \left(P(x_f^*, x_{-f}) + \tau_f \left\|x_f^* - x_f^k\right\|_2^2\right) & \leq & \left(P(x_f, x_{-f}) + \tau_f \left\|x_f - x_f^k\right\|_2^2\right). \end{array} \tag{4.31}$$

This is the same as saying $x^* \in \operatorname{argmin}_{x_f}\left(P(x_f, x_{-f}) + \tau_f \left\|x_f - x_f^k\right\|_2^2\right)$. ■

The convergence result for Algorithm 5 is summarized in the theorem below.

Theorem 4.4.3 *Suppose NEP*$(\theta_f(\cdot,\xi), X_f)_{f=1}^F$ *is a player-wise convex NEP for each* $\xi \in \Omega$ *and satisfies Assumptions 3, 4 and 7. Suppose further that* τ *is chosen such that*

$$2\ c_\tau = \min_{f\in\mathcal{F}}\ \inf_{x\in\mathcal{X}}\ c_{\tau_f}(x) \geq L_{\nabla P}, \tag{4.32}$$

where c_{τ_f} *is the co-efficient of strong convexity of* $\hat{P}(x_f, x_{-f}^k) + \tau \left\|x_f - x_f^k\right\|^2$. *Then the following statements hold true:*

1. *Every limit point* $\hat{x}$ *of the sequence* x^k *generated by Algorithm 5 is an equilibrium to SAANEP*$(\hat{\phi}_f, X_f)_{f=1}^F$.
2. *Suppose further that the required multi-epiconvergence in Theorem 4.3.1 is satisfied. In this case, if* $\hat{x} \to x^*$ *as* $N \to \infty$, *then* x^* *solves SNEP*$(\theta_f, X_f)_{f=1}^F$.

Proof For part (i), we begin by noting that by Lemma 4.4.2, we may replace the objective functions in Step 2 of Algorithm 5 with $P(x_f, x_{-f}^k) + \tau \left\|x_f - x_f^k\right\|^2$. Given this fact, we show that Algorithm 5 effectively computes a stationary point of the potential minimization problem (4.26) using Theorem 3 [97]. Indeed, it is easily seen that Algorithm 5 is equivalent to the Exact Jacobi SCA Algorithm in [97], with $f_1(x) = \hat{P}(x)$, $\mathcal{I} = \{1, \ldots, F$, $C_i = I_f = \{1\}$, $\mathcal{K}_f = X_f$ with the step lengths $\gamma = 1$, and the regularization matrix $\mathbf{H}(x) = \mathbf{I}$ the identity matrix.

In order to show convergence of Algorithm 5 to a stationary point of (4.26), we merely need to verify that the required regularity conditions are satisfied. But this is easily verified as a consequence of Lemma 4.4.1.

The proof of part (i) is completed by noting that by Lemma 4.2.4, any stationary point of (4.26) is also an equilibrium to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

Part (ii) is a direct consequence of the multi-epiconvergence property and Theorem 4.3.1. ■

4.4.2 Regularized Gauss-Seidel algorithm

As with the parallel best response scheme, we may also consider the regularized version of the sequential best response method. This regularized Gauss-Seidel algorithm is given below.

Algorithm 6 Regularized Sequential best-response (Gauss-Seidel)

Step 0: Initialize - Set $x^0 \leftarrow (x_f^0)_{f=1}^F$, $k \leftarrow 0$.

Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**

Step 2: Main Iteration:

FOR $f = 1, \ldots, F$, let x_f^{k+1} solve

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \hat{\phi}_f(x_1^{k+1}, \ldots, x_{f-1}^{k+1}, x_f, x_{f+1}^k, \ldots, x_F^k) + \tau \left\| x_f - x_f^k \right\|^2 \\ \text{subject to} \quad & x_f \in X_f. \end{aligned} \tag{4.33}$$

Step 3: Update: $x^{k+1} = (x_f^{k+1})_{f=1}^F$.

While convergence of Algorithm 6 to solutions of SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ can be established under conditions similar to those for Algorithm 5, one can in fact weaken the assumptions required using the analysis in [38].

Theorem 4.4.4 *Suppose NEP*$(\theta_f(\cdot, \xi), X_f)_{f=1}^F$ *is a player-wise convex NEP for each* $\xi \in \Omega$ *and satisfies Assumptions 3 and 4.Then the following statements hold true:*

1. *Every limit point* $\hat{x}$ *of the sequence* x^k *generated by Algorithm 6 is an equilibrium to SAANEP*$(\hat{\phi}_f, X_f)_{f=1}^F$.

2. *Suppose further that the required multi-epiconvergence in Theorem 4.3.1 is satisfied. In this case, if* $\hat{x} \to x^*$ *as* $N \to \infty$*, then* x^* *solves SNEP*$(\theta_f, X_f)_{f=1}^F$.

Proof Note that under the given assumptions, SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ has player-wise convex, continuously differentiable objective functions. Since there is no constraint interactions amongst the player problems, we may directly employ Theorem 4.3 [38] to establish part (i), noting that NEPs are special cases of generalized potential games.

The proof for part (ii) follows the same lines as Theorem 4.4.3. ■

5. NUMERICAL RESULTS

In this chapter we use the algorithms developed in Section 4.4 to solve several examples of potential games under exogenous uncertainty. The main focus of this chapter is on two practical applications, namely power market equilibrium under demand uncertainty, and stochastic selfish routing games under exogenous uncertainty in edge latencies.

Before presenting detailed numerical results for the examples, we first discuss the computational set up used for our numerical experiments.

5.1 Computational scheme

One of the main advantages of Gauss-Jacobi type algorithms, such as Algorithm 2 and Algorithm 5, is the fact that the computation of player decision updates for each iteration may be done in parallel. This is possible since finding x_f^{k+1} only requires x_{-f}^k. This scheme naturally lends itself to parallel computing architectures.

On the other hand, sequential best response schemes such as Algorithm 1 and Algorithm 6 do not inherently lend themselves to parallelization. However, it must be noted that in several interesting cases such as the traffic routing game to be described below, multiple equilibria exist. Moreover such equilibria can often be reached by varying the order of player updates within the Gauss-Seidel scheme, or using different initial points for the algorithm. Such explorations of multiple update-orders or sequences with varying starting points can also be done utilizing parallel computing.

Our computational experiments are carried out on a desktop machine with a 3.4 GHz Intel Core i7-2600K quad core CPU and 8 GB of RAM. To solve the player subproblems, we use the CPLEX 12.2 quadratic programming solver in MATLAB.

Parallelization for the Gauss-Jacobi algorithms is implemented using the MATLAB parallel computing toolbox functionality.

5.2 Nash-Cournot equilibrium in competitive power markets

In this section we consider a noncooperative game model for strategic interactions between large power generating firms who bid into wholesale electricity markets and the independent system operator (ISO) that controls and coordinates the operation of the electric grid. Such models have become increasingly useful for the analysis of deregulated electricity markets in the U.S. and elsewhere.

In general, electricity markets are resolved by a bidding process wherein generating firms (GenCos) bid their supply functions while demand aggregators such as utility companies bid their demand functions. The ISO then clears the market and assigns quantities to be produced at each location on the grid and the corresponding nodal prices. An equilibrium model for this problem, where the GenCos bid piecewise linear cost functions, while consumers bid piecewise linear demand functions is studied in [59]. However, in this work, we wish to study the strategic behavior of GenCos in the presence of demand uncertainty. Therefore, we use the model presented in [71], where the pricesat each node are determined by a linear inverse demand function. We then model the underlying demand uncertainty by using random parameters in the inverse demand function.

5.2.1 Model

Formally, consider a transmission network $G = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is the set of nodes (buses) and $\mathcal{A}$ is the set of arcs (transmission lines). Suppose there are F GenCos in the market. We denote the set of GenCos as $\mathcal{F}$. We assume that each GenCo f controls generation plants at multiple locations. In this sense, the decision vector for the GenCo g_{fi} captures how much power is produced at a node $i \in \mathcal{N}$. The cost of generation at a node i for GenCo f is captured by a non-negative, convex, quadratic function $C_{fi}(g_{fi})$. The inverse demand at a node i is a function of the total electricity available at u. We assume that the inverse demand is linear, with the price given by

$$p_i = (a_i(\xi) - b_i(\xi)(y_i + g_i)), \tag{5.1}$$

where $g_i = \sum_{f=1}^{F} g_{fi}$ is the total power generation at node i while y_i is the net outflow from the node.

The model we use incorporates demand variability by using the explicit dependence of the inverse demand parameters a_i and b_i on the random variable ξ. We assume throughout that the support sets of $a(\xi)$ and $b(\xi)$ are bounded as follows.

$$\begin{aligned} -\infty < \underline{a}_i \le a_i(\xi) \le \bar{a}_i < \infty, \\ 0 < \underline{b}_i \le b_i(\xi) \le \bar{b}_i < \infty, \end{aligned} \tag{5.2}$$

for each $i \in \mathcal{N}$.

Each GenCo's expected profit maximization problem can then be given as follows.

$$\underset{g_f \in X_f}{\text{maximize}} \quad \phi_f(g_f, g_{-f}, y) = \mathbb{E}\left[\theta_f(g_f, g_{-f}, y; \xi)\right], \tag{5.3}$$

where

$$\theta_f(g_f, g_{-f}, y; \xi) = \left[\sum_{i \in \mathcal{N}} \left[a_i(\xi) - b_i(\xi)(y_i + \sum_{t=1}^{F} g_{ti})\right] g_{fi} - \sum_{i \in \mathcal{N}} C_{fi}(g_{fi})\right]. \tag{5.4}$$

The set X_f captures the production constraints for GenCo f. This set is assumed to be a nonempty, convex, compact subset of the non-negative orthant. Note that the flow variables y are set by the ISO, which solves the following social utility maximization problem.

$$\begin{aligned} \underset{y}{\text{maximize}} \quad & \phi_{\text{ISO}}(g, y) = \mathbb{E}[\theta_{\text{ISO}}(g, y; \xi)] \\ \text{subject to} \quad & \sum_{i \in \mathcal{N}} y_i = 0 \\ & \left|\sum_{i \in \mathcal{N}} \text{PTDF}_{ki} y_i\right| \le T_k, \ \forall k \in \mathcal{A}, \end{aligned} \tag{5.5}$$

where

$$\theta_{\text{ISO}}(g, y, \xi) = \sum_{i \in \mathcal{N}} \left[\int_0^{y_i + G_i} (a_i(\xi) - b_i(\xi)\tau_i) d\tau_i - \sum_{f \in \mathcal{F}} C_{fi}(g_{fi})\right] \tag{5.6}$$

The first constraint in (5.5) captures flow balance, while the second constraint represents the line capacities. The Kirchoff current and voltage laws are captured using the power transmission distribution factor (PTDF) matrix. These constraints represent the commonly used linearized DC flow model which is an approximation of the actual AC power flow equations.

The objective function for the ISO represents the area under the inverse demand function minus the total cost of generation, i.e. the system surplus. Once the integral is resolved the objective function simplifies to the following.

$$\begin{aligned}
\theta_{\text{ISO}}(g, y; \xi) &= \sum_{i \in \mathcal{N}} \left[a_i(\xi)(y_i + \sum_{f \in \mathcal{F}} g_{fi}) - \frac{b_i(\xi)}{2}(y_i^2 + \sum_{f \in \mathcal{F}} g_{fi}^2) \right] \\
&\quad - \left[b_i(\xi) \left(y_i \sum_{f \in \mathcal{F}} g_{fi} + \sum_{\substack{f \in \mathcal{F} \\ t \in \mathcal{F} \\ t \neq f}} g_{fi} g_{ti} \right) - \sum_{f \in \mathcal{F}} C_{fi}(g_{fi}) \right] \\
&= \sum_{i \in \mathcal{N}} \left[a_i(\xi) y_i - \left(\frac{b_i(\xi)}{2} \right) y_i^2 - b_i(\xi) y_i \sum_{f \in \mathcal{F}} g_{fi} \right] \\
&\quad + \sum_{i \in \mathcal{N}} \left[a_i(\xi) \sum_{f \in \mathcal{F}} g_{fi} - \frac{b_i(\xi)}{2} \sum_{f \in \mathcal{F}} g_{fi}^2 - b_i(\xi) \sum_{\substack{f \in \mathcal{F} \\ t \in \mathcal{F} \\ t \neq f}} g_{fi} g_{ti} - \sum_{f \in \mathcal{F}} C_{fi}(g_{fi}) \right]
\end{aligned} \tag{5.7}$$

We are interested in solutions to the game in which the GenCos and ISO simultaneously solve their respective optimization problems. That is, we wish to compute solutions (g, y) such that g_f is a solution to (5.3) for each Genco f given the decisions of the other firms (g_{-f}) and the ISO (y). At the same time y solves the ISO's problem (5.5) given g.

We note here that this model assumes somewhat naive behavior on the part of the generation firms, since typically the generation decisions are taken in advance of the ISO's flow decisions. In this case it is possible to consider scenarios where sophisticated GenCos attempt to anticipate the effect of their decisions on the ISO's problem. Thus one could model the market equilibrium by including the ISO's problem (5.5) explicitly within each GenCo's problem (5.3). This results in an equilibrium problem where each GenCo solves a bilevel optimization problem or an MPEC. Details of this "endogenous" model may be found in [71].

We restrict our attention to the "exogenous" equilibrium model, where the GenCos do not anticipate the ISO's decisions, for multiple reasons. The endogenous model is an equilibrium problem with equilibrium constraints (EPEC) for which even existence of solutions is an unresolved question. Even if there exist solutions to the problem, computing such equilibria is widely regarded to be a difficult task. So it is not unreasonable to assume that GenCos look instead to the "exogenous" model as a reasonable approximation of the market equilibrium.

In the following discussion, we show that the exogenous equilibrium model presented above belongs to the class of player-wise convex potential games, and that the algorithms developed in the previous sections are indeed applicable to the problem.

Firstly, consider each GenCo's objective function given in (5.4). For fixed values of $a(\xi)$ and $b(\xi)$, convexity of θ_f in g_f is guaranteed as long as $b_i(\xi) > 0$ for each $i \in \mathcal{N}$. Under the same condition, one can also show the convexity of the ISO's objective θ_{ISO} in the flow variables y. This takes care of the player-wise convexity requirements. Moreover, the objective functions for each GenCo, as well as that of the ISO, is a quadratic function of the g and y variables. As such, each of these functions is continuously differentiable and has Lipschitz gradients.

Consider the following function:

$$\begin{aligned} P(g, y; \xi) = \; & \sum_{i \in \mathcal{N}} \left[a_i(\xi)(y_i + \sum_{f \in \mathcal{F}} g_{fi}) - b_i(\xi) y_i \sum_{f \in \mathcal{F}} g_{fi} - \left(\frac{b_i(\xi)}{2} \right) y_i^2 \right] \\ & - \left[b_i(\xi) \sum_{\substack{f \in \mathcal{F} \\ t \in \mathcal{F} \\ t \neq f}} g_{fi} g_{ti} + \sum_{f \in \mathcal{F}} C_{fi}(g_{fi}) \right] \end{aligned} \tag{5.8}$$

Suppose g_{-f} and y are fixed. Then we have

$$\begin{aligned} P(g_f, g_{-f}, y; \xi) - P(h_f, g_{-f}, y; \xi) &= \sum_{i \in \mathcal{N}} \Bigg[a_i(\xi)(g_{fi} - h_{fi}) - b_i(\xi) y_i (g_{fi} - h_{fi}) \\ &\qquad - b_i(\xi) \sum_{\substack{t \in \mathcal{F} \\ t \neq f}} g_{ti}(g_{fi} - h_{fi}) - (C_{fi}(g_{fi}) - C_{fi}(h_{fi})) \Bigg] \\ &= \theta_f(g_f, g_{-f}, y; \xi) - \theta_f(h_f, g_{-f}, y; \xi). \end{aligned} \tag{5.9}$$

Now if g is fixed, then we have

$$\begin{aligned} P(g, y; \xi) - P(g, x; \xi) &= \sum_{i \in \mathcal{N}} \left[a_i(\xi)(y_i - x_i) - b_i(\xi) g_i (y_i - x_i) - \left(\frac{b_i}{2} \right) (y_i^2 - x_i^2) \right] \\ &= \theta_{\text{ISO}}(g, y; \xi) - \theta_{\text{ISO}}(g, x; \xi). \end{aligned} \tag{5.10}$$

Thus we have shown that $P(\cdot)$ is an exact potential function for the exogenous equilibrium model, for fixed realizations of ξ.

In order to establish the applicability of the decentralized approximation schemes, Algorithm 5 and Algorithm 6 to solving this problem, all that remains to be shown is that the objective functions are dominated by an integrable function. Recall that the parameter $b_i(\xi)$, the cost function C_{fi} and the generation quantities g_{fi} are all assumed non-negative. We also assume that the variables y and g are (explicitly or implicitly) bounded. Under these assumptions it is easy to show the required bounds on the objective functions.

Lemma 5.2.1 *Suppose the parameters $a(\xi)$ and $b(\xi)$ are bounded. Suppose further that the feasible generation set X_f is bounded for each GenCo $f \in \mathcal{F}$. Then the objective functions $\theta_{ISO}(g, y)$ and $\theta_f(g, y)$ satisfy Assumption 6*

Proof Under the given assumptions we must have $|g_f| < \infty$ for each $f \in \mathcal{F}$. In this case, we may restrict the flow variables y to some bounded set Y. This is easily seen by examining the ISO's objective function in 5.7 and noting that since $b_i(\xi) > 0$, setting any $y_i = \infty$ would result in $\theta_{\text{ISO}}(g, y) = -\infty$, which clearly cannot be optimal.

Thus we may safely assume the following implicit bounds for each $i \in \mathcal{N}$ and $f \in \mathcal{F}$ -

$$\begin{aligned} -\infty < \underline{g}_{fi} \leq g_{fi} \leq \bar{g}_{fi} < \infty, \\ -\infty < \underline{y}_i \leq y_i \leq \bar{y}_i < \infty. \end{aligned} \tag{5.11}$$

Observe that the objective functions in question are polynomial functions of $a(\xi)$, $b(\xi)$, g and y. Thus the desired boundedness on the absolute values of the functions may be obtained from (5.2) and (5.11) by repeated application of the product and sum rules for absolute values[1]. ■

5.2.2 Experiments

In order to illustrate the performance of Algorithms 5 and 6, we first consider the small three node example network given in Figure 5.1. There are three nodes on the network all of which have positive demand. We assume that the reactances of all the lines are equal. There are two GenCos acting on the network. Genco 1 has a generation unit at node 1, while Genco 2 has a unit at node 2. Both firms are assumed to have generation costs given by $C_i(g_i) = c_i g_i + d_i g_i^2$. We assume that the only constraint on generation is capacity of the production unit, i.e. $0 \leq g_i \leq K_i$. We assume that the inverse demand co-efficients $a_i(\xi)$ and $b_i(\xi)$ are drawn from uncorrelated truncated normal distributions. The various parameters of the problem are summarized in Tables 5.1, 5.2 and 5.3. Table 5.1 presents the nodal generation cost and inverse demand functions. The power transmission distribution factor matrix, as well as the line capacities are given in Table 5.2, while Table 5.3 summarizes the details of the inverse demand parameters, including the mean, standard deviation and end points of the truncated normal distributions from which the intercepts $a_i(\xi_i)$ are drawn.

To begin with, we take a fixed sample size of $N = 5000$. After sampling from the distributions for $a_i(\xi_i)$, we employ Algorithms 5 and 6 to compute the solutions to the deterministic SAA market equilibrium problem. From preliminary tests, we choose

[1] $|a + b| \leq |a| + |b|$ and $|ab| = |a||b|$.

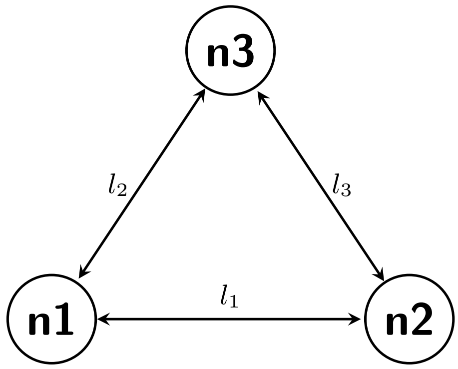

Figure 5.1. Network 1

Node	GenCo	Cost	Demand
1	1	$5g_1 + g_1^2$	$a_1(\xi_1) - b(q_1 + y_1)$
2	2	$2g_2 + 1.5g_2^2$	$a_2(\xi_2) - b(g_2 + y_2)$
3	-	-	$a_3(\xi_3) - by_3$

Table 5.1.
Parameters for Network 1.

Node	l_1	l_2	l_3
n_1	2/3	-1/3	0
n_2	1/3	1/3	0
n_3	2/3	2/3	0
T_k	1.5	1.8	1.2

Table 5.2.
PTDFs for Network 1.

to use a regularization parameter $\tau = 0$, since both algorithms converge without requiring regularization. We speculate that this result is due to the strong concavity

Node	$\mu(a_i)$	$\sigma(a_i)$	$\underline{a}_i$	$\bar{a}_i$	b_i
n_1	25	5	15	35	1.2
n_2	26.5	5	16.5	36.5	1.3
n_3	28	5	18	38	1.5

Table 5.3.
Inverse demand parameters for Network 1

of the payoff functions for all the players involved. Numerical results for this one shot sampling problem is summarized in Table 5.4.

Generation - (g_1^*, g_2^*)	(10.2364, 8.6107)
Flow - y	(-1.5, -0.6, -1.2)
Nodal price - p	(15.2364, 14.9161, 25.3)
GenCo 1 optimal payoff	115.2615
GenCo 2 optimal payoff	103.8022
Gauss-Seidel major iterations	4
Gauss-Seidel running time	0.0272 secs
Gauss-Jacobi major iterations	5
Gauss-Jacobi running time	0.0136 secs

Table 5.4.
Results for Network 1 ($N = 5000$)

Note that while the sequential Gauss-Seidel algorithm takes one less major iteration to converge, it is approximately 2 times slower than the parallel Gauss-Jacobi algorithm. Both algorithms converge to the same equilibrium point. At the equilibrium point, both lines 1 and 3 are congested.

In order to empirically examine the effects of sample size on the accuracy of the approximation algorithms, we conduct experiments with multiple sampling runs at multiple sample sizes. The results are presented in Figures 5.2 through 5.5

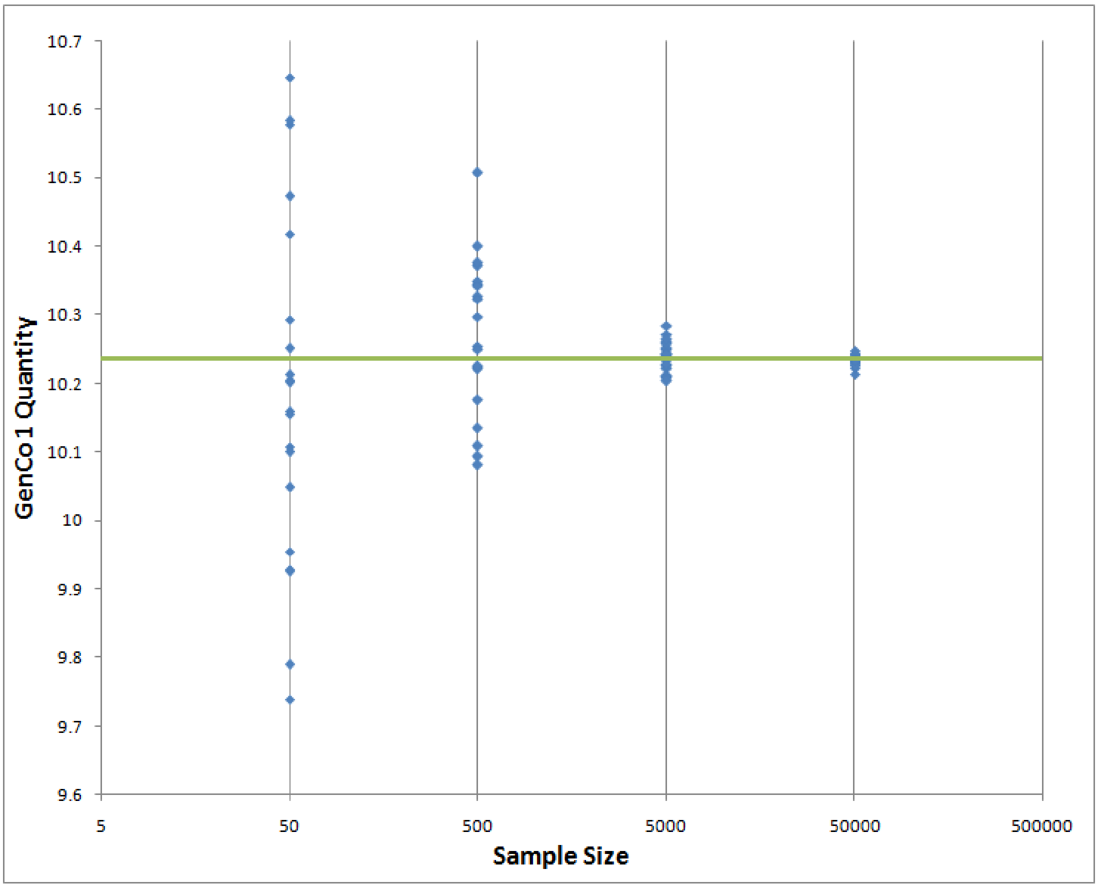

Figure 5.2. Optimal g_1 vs N

For each fixed sample size $N \in \{50, 500, 5000\}$, we conduct 10 sampling runs and solve the equilibrium problem using the approximation algorithms for each sample. Figures 5.2 and 5.3 provide scatter plots of the optimal generation quantities g_1 and g_2 respectively, while figures 5.4 and 5.5 provide similar plots for the optimal payoff values for GenCo 1 and GenCo 2. As expected, larger sample sizes result in significantly less variance in the estimated solution quantities and payoffs.

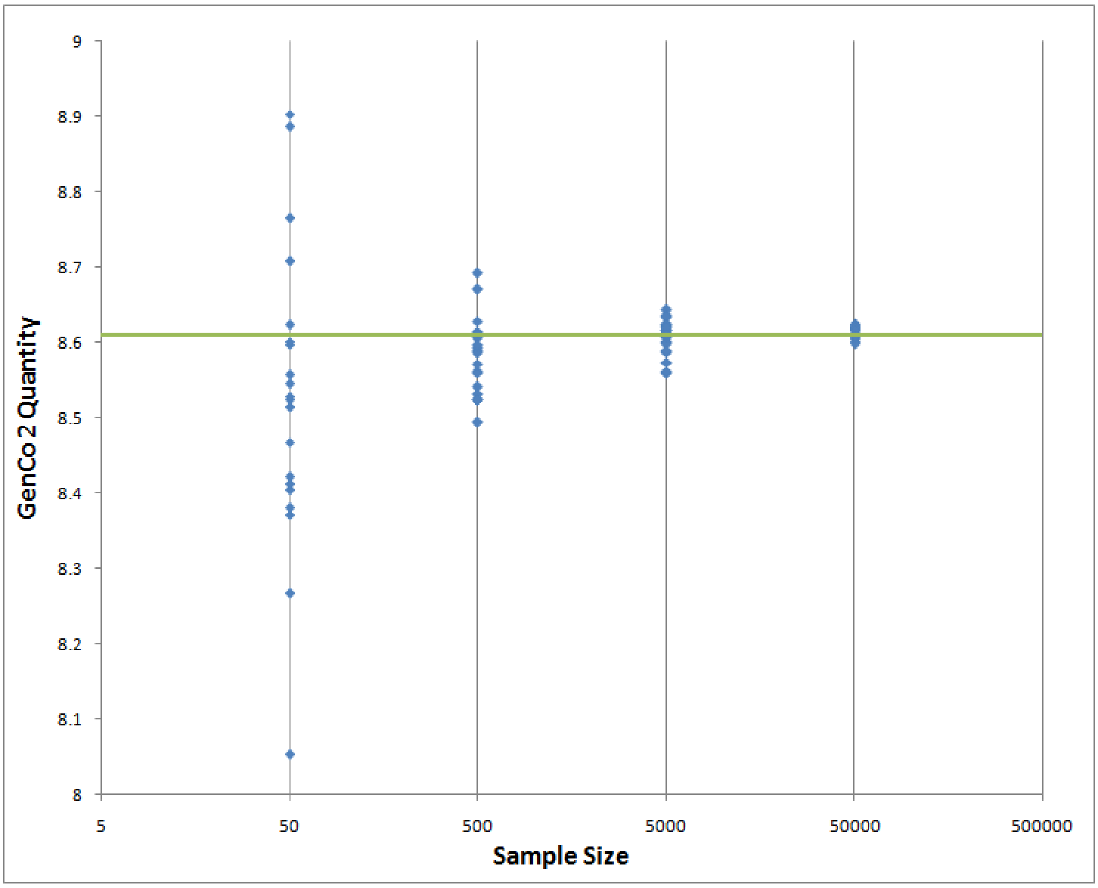

Figure 5.3. Optimal g_2 vs N

5.3 Stochastic routing games

In this section, we consider the problem of routing traffic along the arcs of a network, such as a road network or a communication network. The rich and varied literature on research into such problems, and their strategic counterparts namely selfing routing games, gives testament to their practical significance. We refer the reader to the thesis [90] and the references [12, 13, 24, 47] for a review on the analysis of selfish routing games. However, our focus is on the so-called *atomic* selfish routing games, where the costs or latencies associated with sending flow along an arc in the

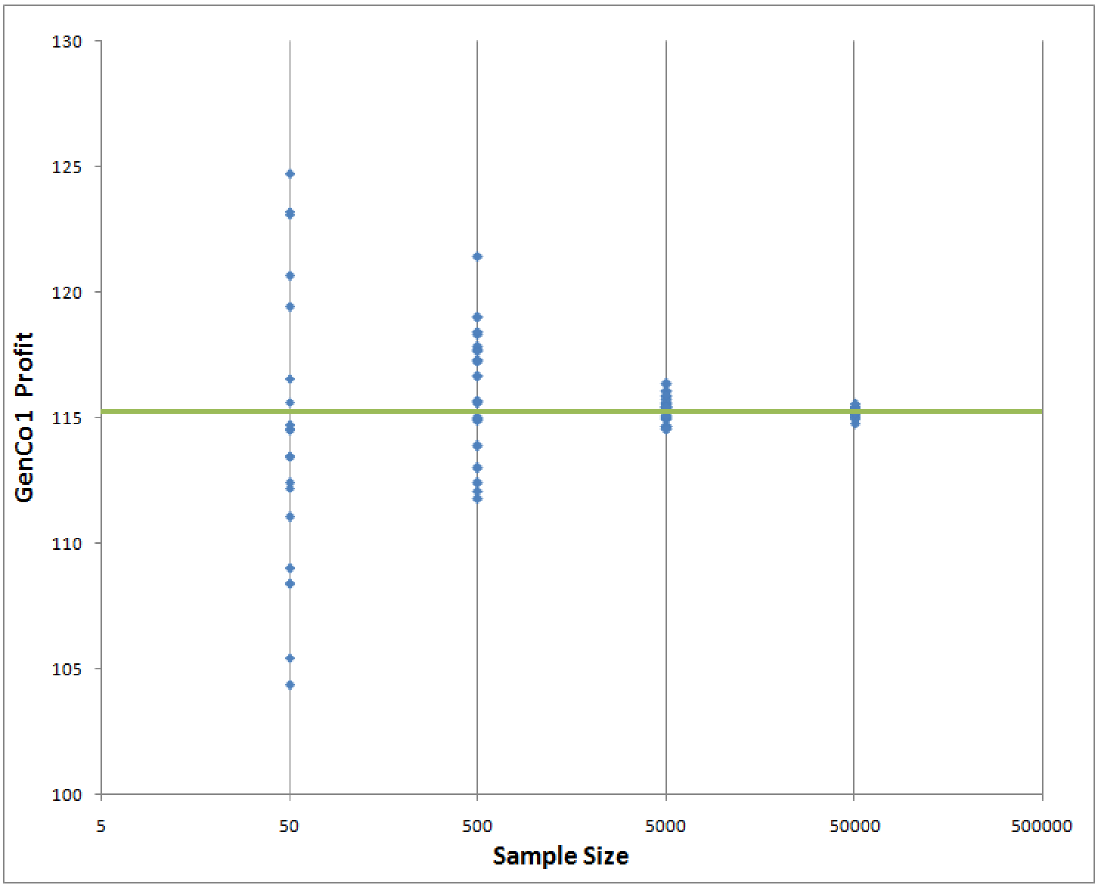

Figure 5.4. Optimal payoff for GenCo 1 vs N

network is subject to some exogenous source of uncertainty, such as weather. The setting of the games in question is described below.

5.3.1 Model

We consider a directed network $G = (V, A)$, with vertex set V and arc (edge) set A. We consider F players acting on this network. Player f wishes to direct d_f units of flow from arc s_f to t_f. Each player is assumed to control a significant percentage of the total flow on the network. In traffic routing literature, this property is referred to as *atomic* traffic routing. We also assume that player f is able to split the d_f units

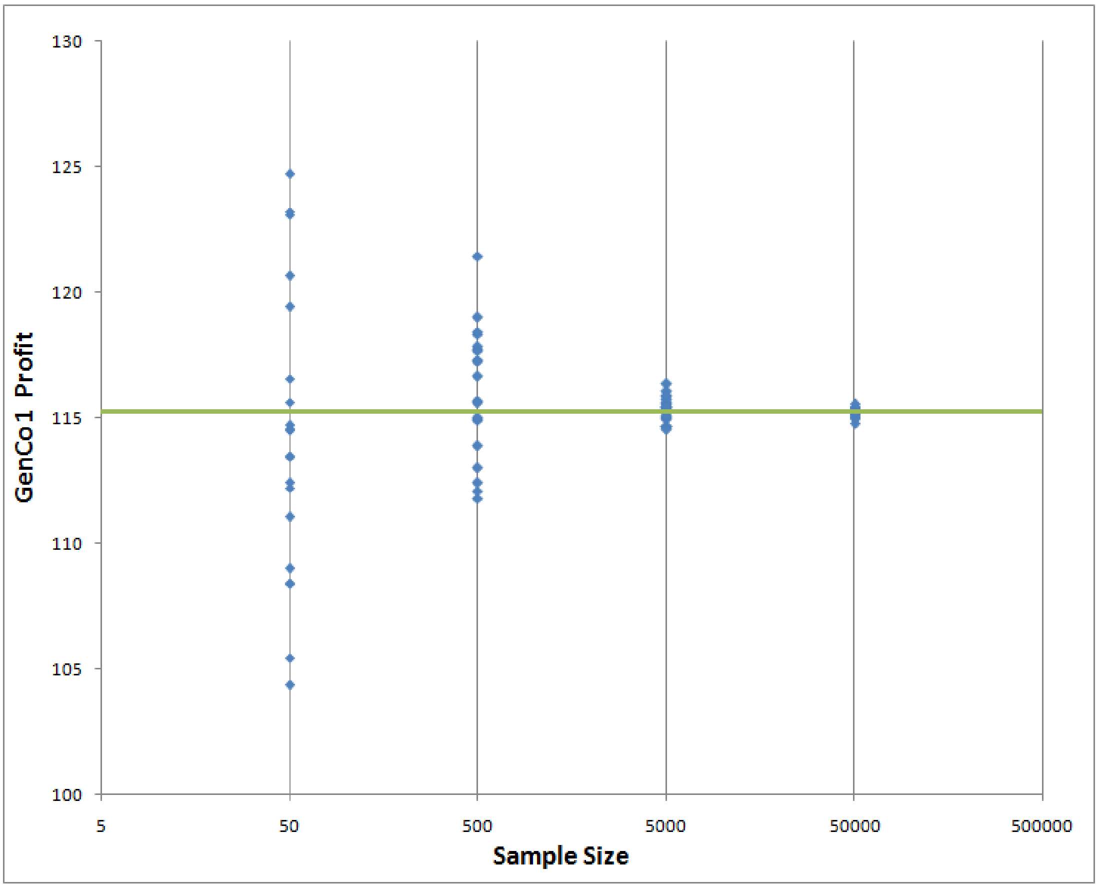

Figure 5.5. Optimal profit for GenCo 2 vs N

of flow amongst any paths between s_f and t_f. In this sense, we only consider atomic traffic routing games with splittable flow.

Each of the f players wish to minimize the cost of routing the desired flow across the network. The cost of flow is most often associated with the delay in crossing arcs of the network, also known as “latency”. We assume that the latencies are dependent on the congestion on the network, i.e. that the latency on each arc is a factor of the *total* flow on the arc.

Let the decision variable for player f be denoted by $x_f = (x_f^1, \dots, x_f^{|A|})$. Here x_f^e represents the amount of flow that f directs on arc e. A set of flow decisions $x_f \geq 0$ is feasible to player f if the following flow conservation conditions hold.

$$\sum_{(u,v)\in A} x_f^{uv} = \begin{cases} d_f & \text{if } u = s_f, \\ 0 & \text{if } u \notin \{s_f, t_f\}, \\ -d_f & \text{if } u = t_f. \end{cases}$$

Given the set of decisions of all the players, denoted by $x = (x_1, \dots x_F)$, the latency on any edge $e \in A$ is given by the function

$$c^e(x^e) = c^e\left(\sum_{f=1}^{F} x_f^e\right).$$

In general, the function $c(\cdot)$ is assumed to be nonnegative, continuous and nondecreasing. Specifically, we consider the case where the cost function is linear in x^e, i.e.

$$c^e(x^e) = a^e(\xi)x^e + b^e(\xi). \tag{5.12}$$

Note that the cost function is assumed to depend on the random vector ξ, which captures the underlying sources of uncertainty such as weather. As in the previous section, we assume that $a^e(\xi)$ and $b^e(\xi)$ are assumed to be positive, bounded functions of ξ.

The objective of each player is to minimize the average total latency of her flow on the network. We denote this objective by θ_f, i.e.

$$\begin{aligned} \theta_f(x_f, x_{-f}; \xi) &= \sum_{e\in A} c^e(x^e)x_f^e \\ &= \sum_{e\in A}\left[a^e(\xi){x_f^e}^2 + \left(b^e(\xi) + a^e(\xi)\sum_{\substack{f'=1\\ f'\neq f}}^{F} x_{f'}^e\right)x_f^e\right]. \end{aligned} \tag{5.13}$$

Since $a_e(\xi) > 0$ by assumption, it is easy to see that $\theta_f(x_f, x_{-f})$ is convex in x_f for any fixed value of x_{-f}. The optimization problem for each player f is then given by

$$\begin{aligned}
\underset{x_f}{\text{minimize}} \quad & \phi_f(x_f, x_{-f}) = \mathbb{E}\left[\theta_f(x_f, x_{-f})\right] = \mathbb{E}\left[\sum_{e\in A} c^e(x^e)x_f^e\right] \\
\text{subject to} \quad & \sum_{(u,v)\in A} x_f^{uv} = \begin{cases} d_f & \text{if } u = s_f, \\ 0 & \text{if } u \notin \{s_f, t_f\}, \\ -d_f & \text{if } u = t_f. \end{cases} \\
& x_f^{uv} \geq 0 \quad \forall (u,v) \in A.
\end{aligned} \tag{5.14}$$

For fixed values of ξ, an exact potential function for the selfish routing game $\text{NEP}(\theta_f, X_f)_{f=1}^F$ is given by

$$P(x,\xi) = \sum_{e\in A}\left[a^e(\xi)\sum_{f=1}^{f} {x_f^e}^2 + b^e(\xi)\sum_{f=1}^{f} x_f^e + a^e(\xi)\sum_{f=1}^{F}\sum_{\substack{f'=1 \\ f'\neq f}}^{F} x_f^e x_{f'}^e\right]. \tag{5.15}$$

Thus the stochastic routing game described above is a player-wise convex game with an exact potential function. Moreover, the objective function for each player f, i.e. $\theta_f(x_f, x_{-f})$ is continuously differentiable in both x_f and x_{-f}. The gradients of the objective function are linear in x for each ξ and therefore easily seen to be Lipschitz continuous.

We can employ techniques similar to those used in the previous power market example, to leverage the boundedness on the parameters a and b as well as the variables x, in order to show that the routing game satisfies Assumption 6. We have thus verified all the conditions required for the applicability of Algorithms 5 and 6 to the problem of computing equilibria for the stochastic traffic routing game.

5.3.2 Experiments

We begin our numerical experiments with a small 2 player problem defined on the network given in Figure 5.6. There are two players acting on the network. Player 1 wishes to route 10 units of flow from node $n1$ to node $n5$. Player 2 on the other

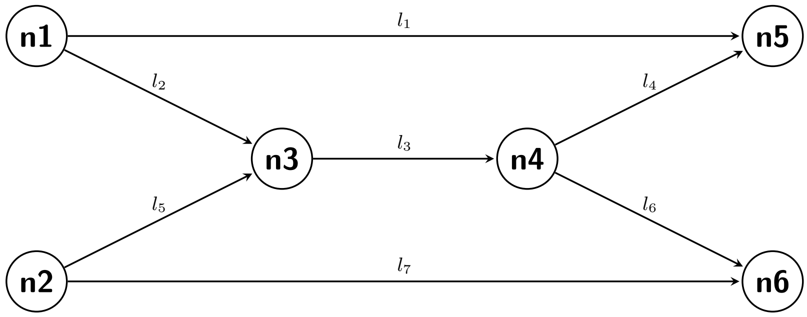

Figure 5.6. Network 2

hand wishes to route 10 units of flow from node $n2$ to node $n6$. The cost parameters associated with the edges on the network are given in Table 5.5. Note that the constant terms a_e are assumed to be deterministic, while the linear coefficient b_e is drawn from a truncated normal distribution with bounds $[\underline{b}_e, \bar{b}_e]$, mean $\mu(b_e)$ and standard deviation $\sigma(b_e)$.

Edge ID	a_e	$\underline{b}_e$	$\bar{b}_e$	$\mu(b_e)$	$\sigma(b_e)$
l_1	10	1.5	4.5	3	3
l_2	1	0.5	1.5	1	1
l_3	1	0.5	1.5	1	1
l_4	1	0.5	1.5	1	1
l_5	1	0.5	1.5	1	1
l_6	1	0.5	1.5	1	1
l_7	10	1.5	4.5	3	3

Table 5.5.
Cost data for Network 2

As with the previous section, we begin our analysis by fixing the sample size at $N = 5000$ and using both Algorithms 5 and 6 to solve the network traffic assignment equilibrium problem. The results are summarized in Table 5.6. Note that in this case, the flow numbers capture the split flows between the 2 available options to each player. For instance, the optimal flow for player 1 is to route 6.2498 units of flow directly along the arc $(n1, n5)$ and the remainder along the path ($n1$-$n3$-$n4$-$n5$). Unlike the power market problem, the speedup for the Gauss-Jacobi scheme is only 1.15 for our traffic assignment example. This is explained by the fact that most of the speedup due to parallelization is lost due to the increased number of major iterations required for the Gauss Jacobi algorithm relative to the sequential Gauss Seidel method.

Flow for P1	(6.2498, 3.7502)
Flow for P2	(6.2501, 3.7499)
P1 optimal payoff	53.9125
P2 optimal payoff	53.9109
Gauss-Seidel major iterations	9
Gauss-Seidel running time	0.0312 secs
Gauss-Jacobi major iterations	16
Gauss-Jacobi running time	0.0273 secs

Table 5.6.
Results for Network 1 ($N = 5000$)

We then run our experiments with sampling sizes drawn from $N \in \{50, 500, 5000\}$ and study the variance of the approximation results if we solve multiple runs of the problem at each sample size. The results are presented in Figures 5.7 and 5.8, which plot the optimal costs for the two players for various sampling runs.

Next we consider a larger 19 node network given in Figure 5.9. Consider a 6 player selfish atomic routing game on this network. All the players wish to route 10 units of flow from their source nodes to the target node 19. The source nodes are 1,3,5,7,9, and 11 respectively. The cost parameters a_e and b_e are drawn from truncated normal distributions with data as given in Table 5.7 For this network, the naive Gauss Seidel algorithm (Algorithm 1) in 36 iterations. However, the naive Gauss Jacobi algorithm

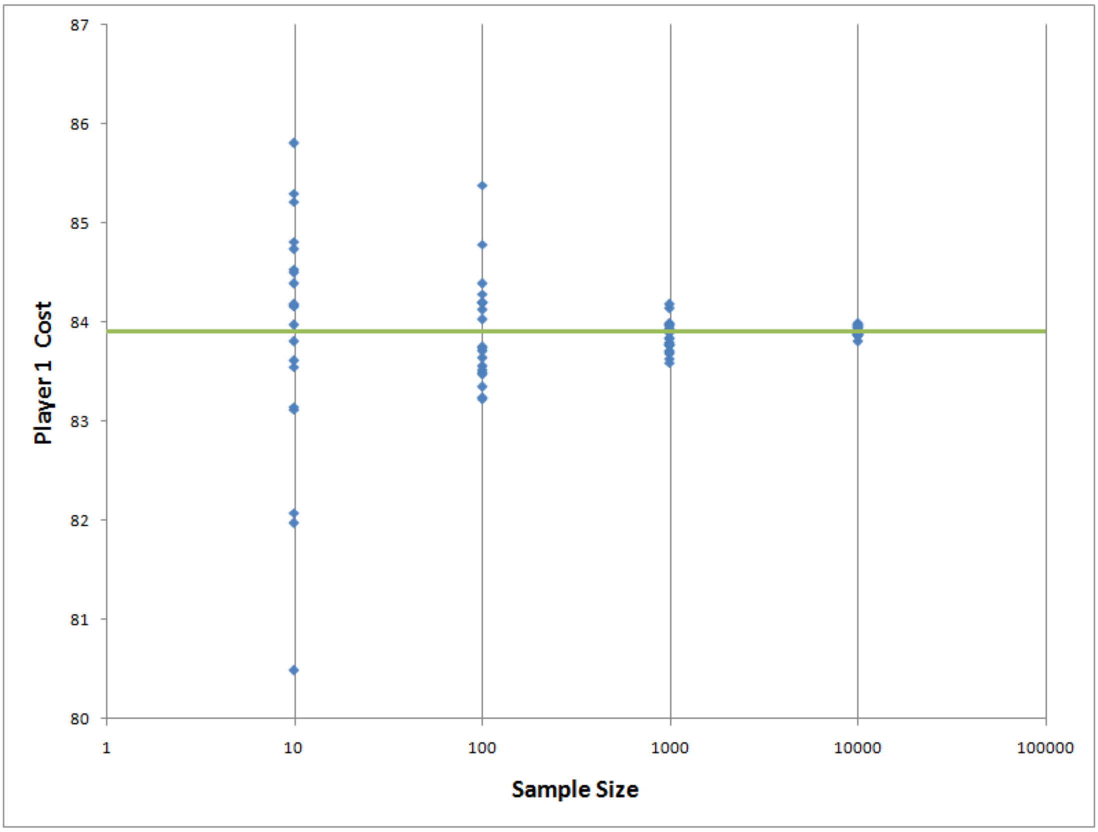

Figure 5.7. Optimal cost for P1 vs sample size N

Edge	a_e	$\underline{b}_e$	$\bar{b}_e$	$\mu(b_e)$	$\sigma(b_e)$
Outer circle edges	1	0.5	1.5	1	1
Inter circle edges	3	1.5	4.5	3	3
Inner circle edges	1	0.5	1.5	1	1
Inner circle to target edges	5	2.5	7.5	5	5

Table 5.7.
Cost data for Network 3

fails to converge even at the limit of 1 million major iterations. In this case, we use the regularized Gauss Jacobi algorithm (Algorithm 5). By choosing a maximum

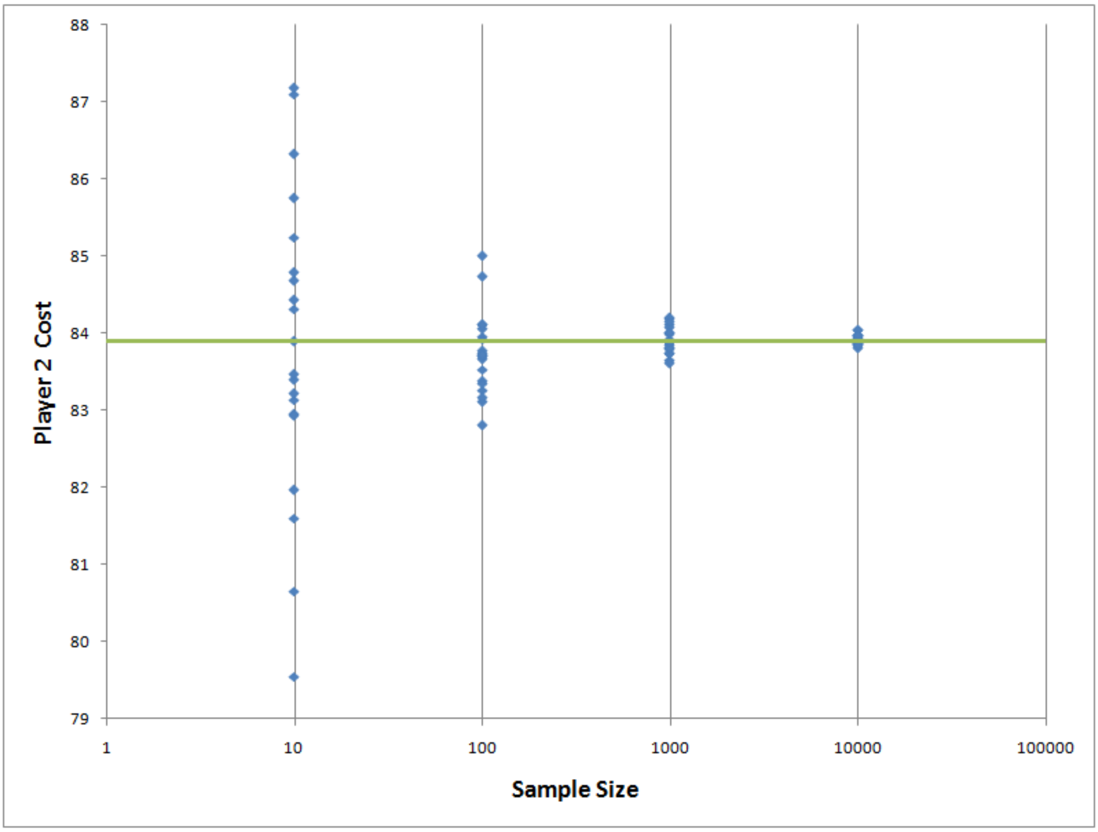

Figure 5.8. Optimal cost for P2 vs sample size N

major iteration limit of 1000, and adaptively increasing the regularization parameter τ, we find that the algorithm converges for $\tau > 0.7297$ for an error tolerance in x of 10^{-6}.

For this network, we study the variation in convergence time and the number of major iterations required for convergence for various combinations of regularization parameter τ and the error tolerance in x. The results are given in Figures 5.11 through 5.15.

Figures 5.11 and 5.10 plot the convergence time and major iterations against τ. Here the regularization parameter is varied between the range $0.75 \leq \tau \leq 1.95$. From Figure 5.10, we observe that the number of major iterations the algorithm takes to

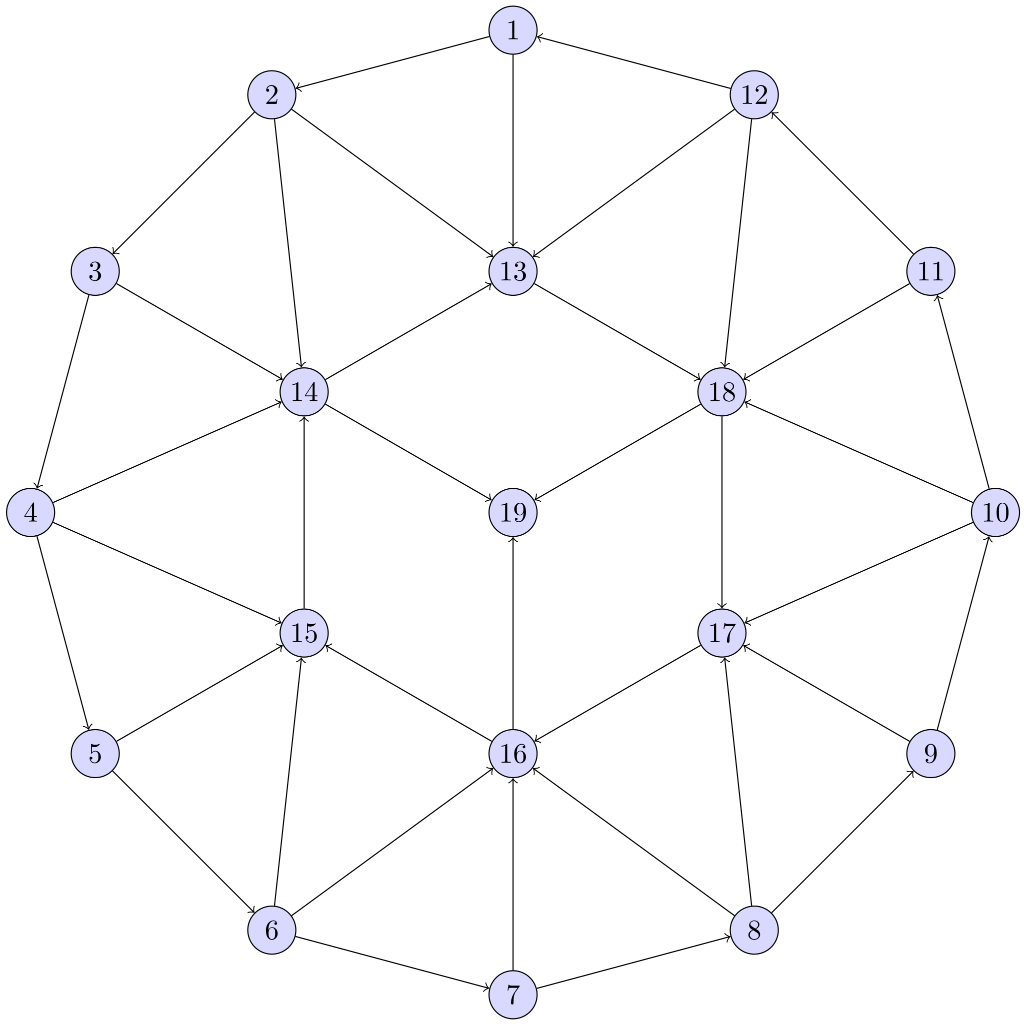

Figure 5.9. Network 3: example network for stochastic routing

converge to equilibrium reaches a minimum at $\tau = 0.9$ approximately, and then starts to rise as we increase τ.

The importance of the regularization parameter is made even clearer by Figures 5.13 and 5.12. As τ increases beyond 1, both the time to convergence and the number of major iterations to convergence increases. At $\tau = 10$, the algorithm is approximately 6.7 times slower than at $\tau = 1$. These results underline the need to compute

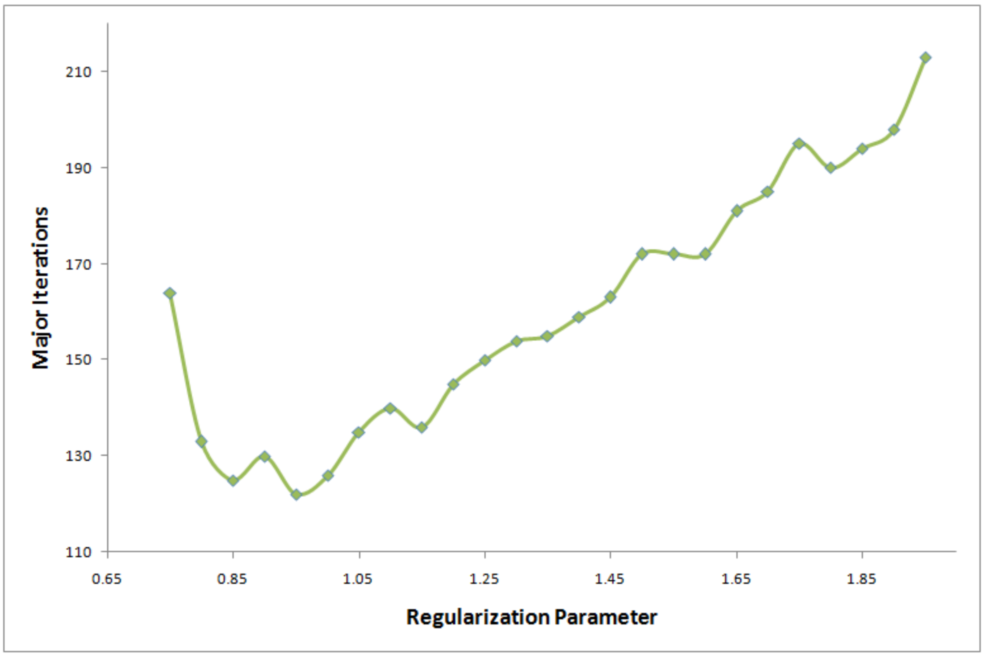

Figure 5.10. Network 3: Algorithm 5 major iterations vs τ

the appropriate τ to ensure good performance of the regularized method Algorithm 5

We also conduct an empirical study of how our parallel best response scheme scales as we employ more and more processors. For the 6 player Network 3, the results of scaling are presented in Figure 5.16. The blue curve marks the actual convergence time as we increase the number of cores used. The red curve represents a perfect linear scaling relative to the sequential convergence time. We observe that as the number of processors increases, the speedup is approximately linear (i.e. if we emply N processors, the time taken t_N is approximately t_1/N where t_1 is the serial processing time). The variations from linearity are mainly due to parallel overhead, as well as the inconsistency in solution times between player problems at each iteration which cases non-uniformity in job completion times across processors.

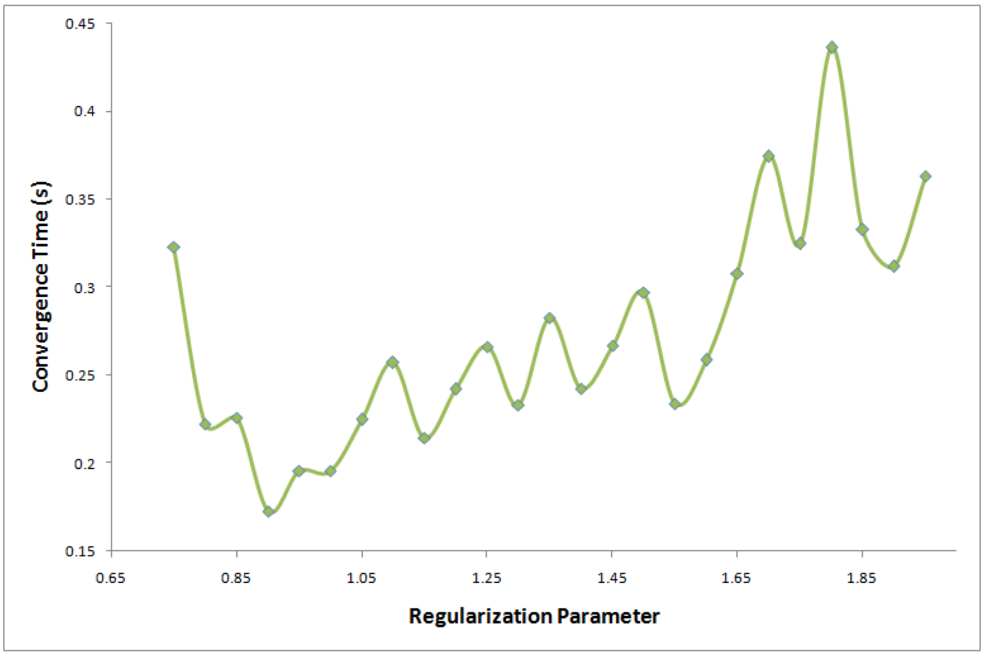

Figure 5.11. Network 3: Algorithm 5 convergence time vs τ

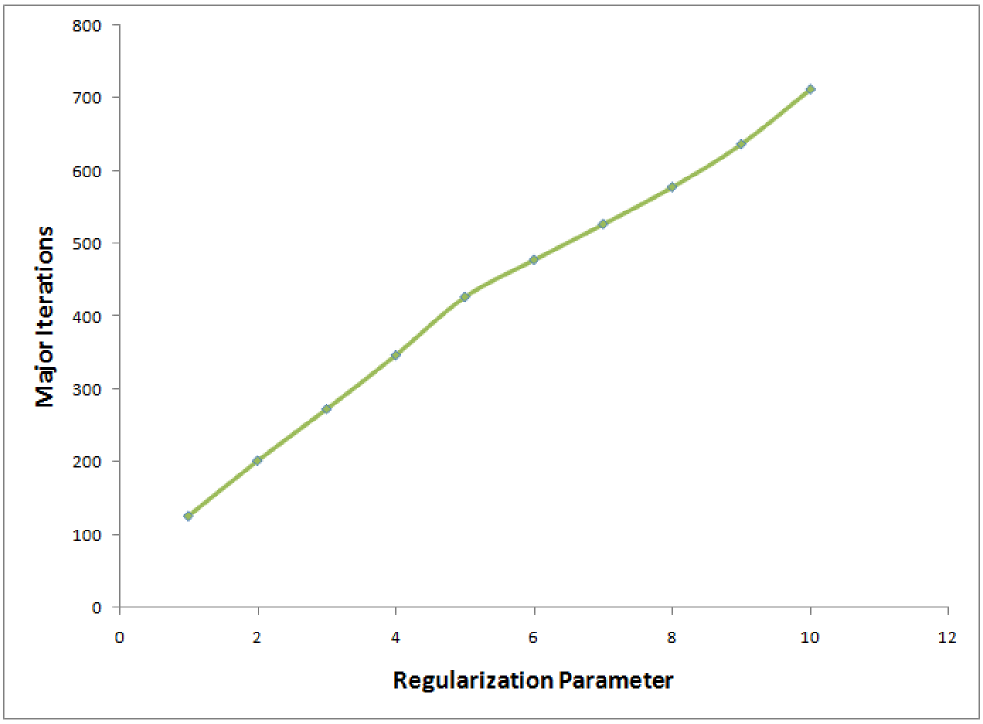

Figure 5.12. Network 3: Algorithm 5 major iterations vs τ

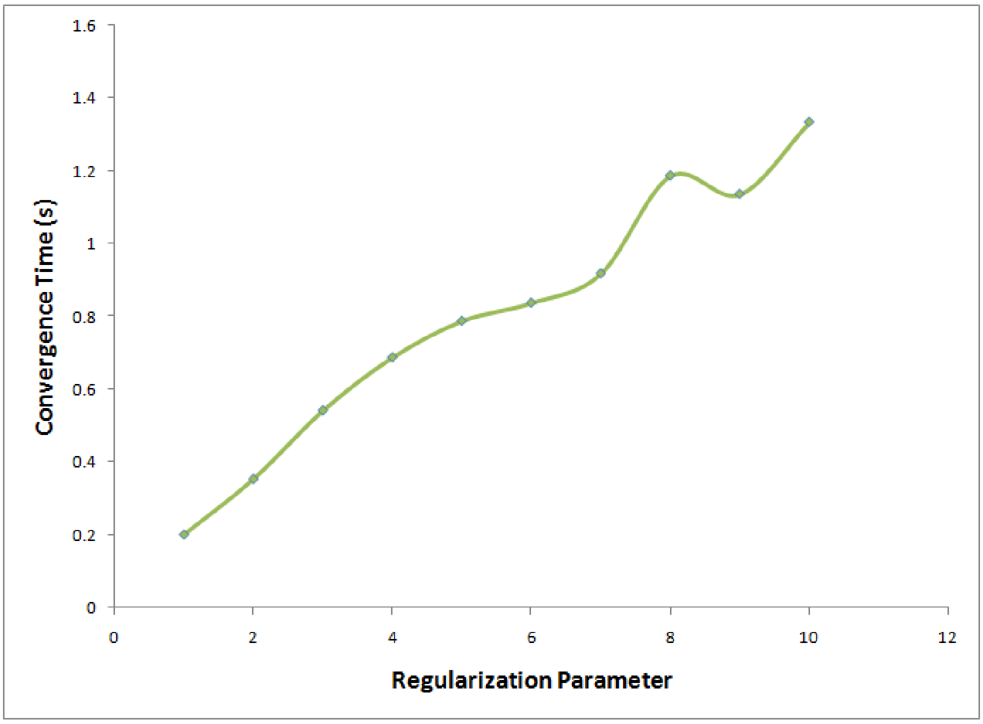

Figure 5.13. Network 3: Algorithm 5 convergence time vs τ

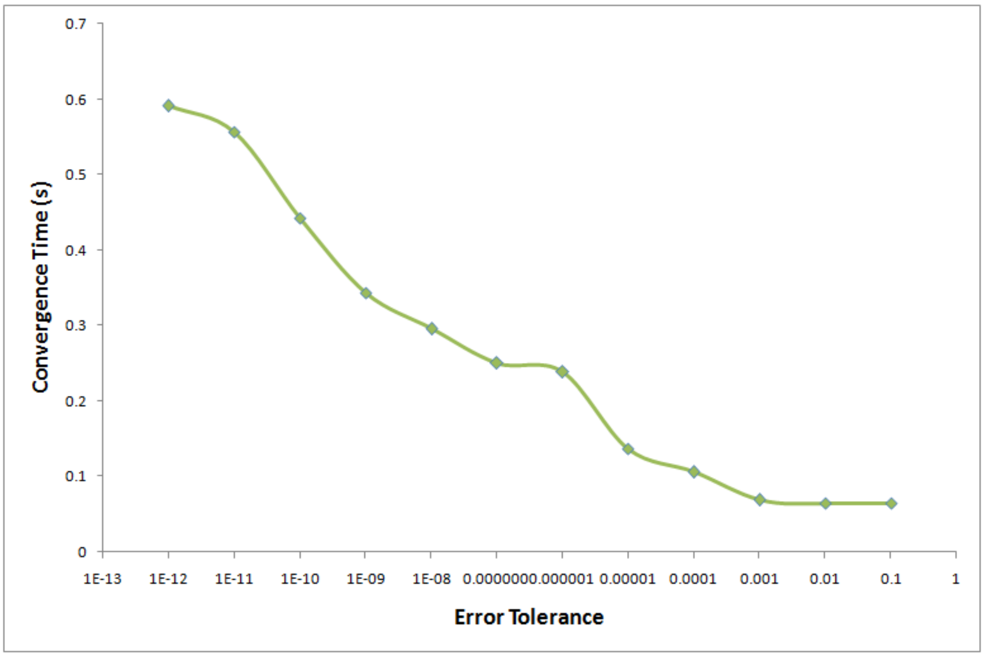

Figure 5.14. Network 3: Algorithm 5 convergence time vs error tolerance

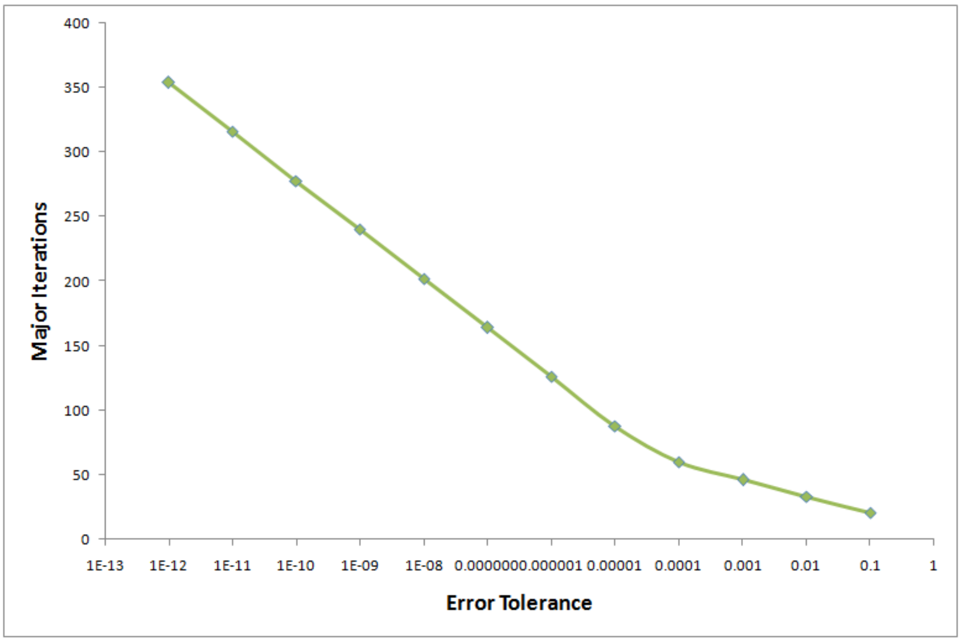

Figure 5.15. Network 3: Algorithm 5 major iterations vs error tolerance

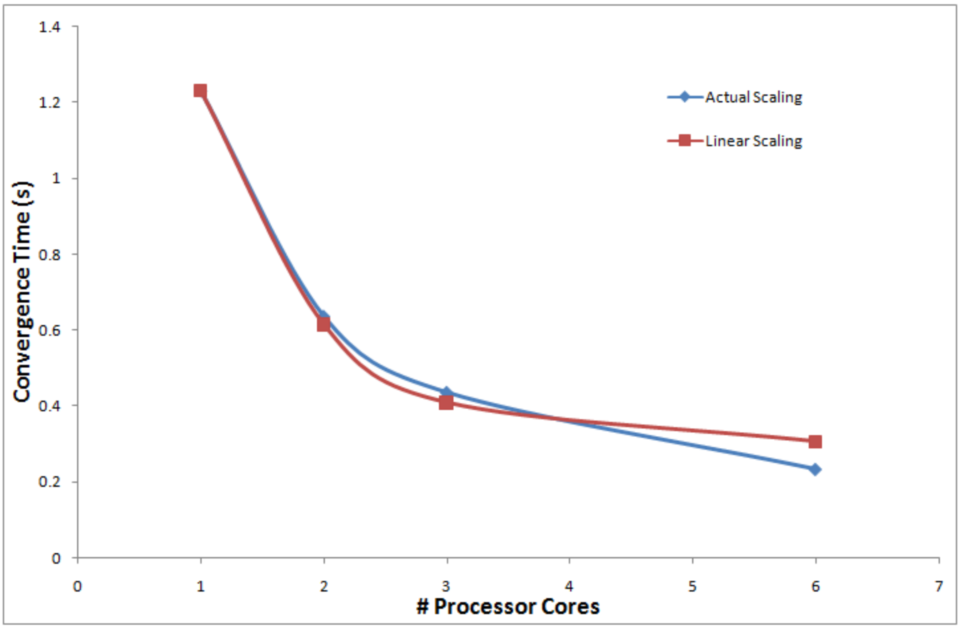

Figure 5.16. Network 3: Algorithm 5 Convergence time vs # processor cores

6. EXTENSIONS

In this chapter, we discuss several important extensions to the theory presented in Chapter 4. To begin with, we consider the question of solving SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ approximately and whether such solutions are asymptotically consistent with respect to solutions of SNEP$(\theta_f, X_f)_{f=1}^F$. We also discuss briefly the idea of modified best response schemes, wherein each player's problem is solved inexactly.

Another natural extension to the sampling-based approximation and decentralized computations presented in Chapter 4, is to consider disparate sampling amongst the players. The convergence of this kind of approximation schemes is difficult to establish. We briefly discuss some promising research directions and recent research related to this problem.

Finally, we also consider the issue of approximating equilibria in the context of generalized Nash games. The challenges associated with extending multi-epiconvergence to the GNEP domain are outlined.

6.1 Approximate and inexact best response algorithms

The theoretical results for the convergence of the regularized best response algorithms presented in Chapter 4 assume the computation of *exact* solutions to the sampled problem SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$. However, in a practical setting, such a strategy is difficult, if not outright impossible to implement. In most cases, bounds on the difference in norms between successive iterates, either in the decision space or the payoff space, are used for finite termination. In this context the question of asymptotic convergence of approximate solutions to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$, as well as solution quality of the candidate output by a best response algorithm takes on relevance.

In this section, we first consider the case where we use approximate solutions of the best response mapping for SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ as candidate solutions to SNEP$(\theta_f, X_f)_{f=1}^F$, and analyze the asymptotic convergence of these candidates. Specifically, consider using the following termination criteria for the regularized Gauss-Jacobi scheme (Algorithm 5) -

$$\sum_{f=1}^{F} \left[\hat{\phi}_f(x_f^k) - \hat{\phi}^f(x_f^{k+1}) - \frac{\tau}{2} \left\| x_f^k - x_f^{k+1} \right\|^2 \right] \leq \epsilon^N. \tag{6.1}$$

The condition given above in (6.1) may easily be verified by two function evaluations for each player f, which may be computed in parallel.

Suppose now that as we consider larger and larger sample sizes N for computing $\phi_f(\cdot)$, we also use progressively smaller tolerance bounds ϵ^N. We wish to answer the question of the asymptotic convergence of solution points generated by Algorithm 5 under the termination condition (6.1), to solutions of SNEP$(\theta_f, X_f)_{f=1}^F$. The main tool we use is the convergence of approximate solutions to optimization reformulations SNEP$(\theta_f, X_f)_{f=1}^F$ based on the sampled Nikaido-Isoda function (defined in 2.3.2) , as given in [51]. We state the result formally below.

Theorem 6.1.1 *Assume that NEP$(\theta_f(\cdot,\xi), X_f)_{f=1}^F$ is a player-convex NEP for all $\xi \in \Omega$. Let the point $x^{(k_N,N)}$ be generated by applying Algorithm 5, with a sample of size N, to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$. Suppose that the required multi-epiconvergence in Theorem 4.3.1 is satisfied. If $\epsilon^N \to 0$ and $x^{(k_N,N)} \to x^*$ as $N \to \infty$, then x^* solves SNEP$(\theta_f, X_f)_{f=1}^F$.*

Proof Consider the value function associated with the Nikaido-Isoda function for SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

$$V_\gamma(x) = \max_{y \in \mathbf{X}} \sum_{f=1}^{F} \left[\hat{\phi}_f(x_f, x_{-f}) - \hat{\phi}_f(y_f, x_{-f}) - \frac{\tau}{2} \left\| x_f - y_f \right\|^2 \right]. \tag{6.2}$$

We first show that $V_\gamma(x^{(k_N,N)}) \leq \epsilon^N$. Indeed we can rearrange (6.2) as follows.

$$V_\gamma(x) = \sum_{f=1}^{F} \left[\hat{\phi}_f(x_f, x_{-f}) - \min_{y_f \in X_f} \left(\hat{\phi}_f(y_f, x_{-f}) + \frac{\tau}{2} \left\| x_f - y_f \right\|^2 \right) \right]. \tag{6.3}$$

Therefore we have

$$\begin{aligned} V_\gamma(x^{(k_N,N)}) &= \sum_{f=1}^{F}\left[\hat{\phi}_f(x_f^{(k_N,N)}, x_{-f}^{(k_N,N)})\right] \\ &\quad -\sum_{f=1}^{F}\left[\min_{y_f\in X_f}\left(\hat{\phi}_f(y_f, x_{-f}^{(k_N,N)}) + \frac{\tau}{2}\left\|x_f^{(k_N,N)} - y_f\right\|^2\right)\right] \\ &= \sum_{f=1}^{F}\left[\hat{\phi}_f(x_f^{(k_N,N)}, x_{-f}^{(k_N,N)})\right] \\ &\quad -\sum_{f=1}^{F}\left[\left(\hat{\phi}_f(x_f^{(k_{N+1},N)}, x_{-f}^{(k_N,N)}) + \frac{\tau}{2}\left\|x_f^{(k_N,N)} - x_f^{(k_{N+1},N)}\right\|^2\right)\right] \\ &\leq \epsilon^N. \end{aligned} \tag{6.4}$$

Here the first inequality follows from the fact that $x^{(k_{N+1},N)}$ is generated by solving the regularized Gauss-Jacobi iteration with respect to $x^{(k_N,N)}$, while the second inequality follows from the termination condition (6.1).

We have thus shown that the point $x^{(k_N,N)}$ is an approximate solution to the NI function reformulation for SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$. The proof can now be completed by simply invoking Theorem 5 in [51], and noting that the required convexity and multi-epiconvergence assumptions are equivalent to the given assumptions. ■

In its current form, the termination condition (6.1) does not allow a result similar to Theorem 6.1.1 for the regularized Gauss-Seidel method (Algorithm 6). This is because an iterate component x_f^k of Algorithm 6 is a best response to the updated rival strategy tuple, rather than the older tuple x_{-f}^{-1}. Thus, establishing an error bound on the Nikaido-Isoda function for the terminal point x^{k_N} of Algorithm 6 is much harder.

The convergence result in Theorem 6.1.1 can be used to form a computational framework where we vary both sample size N and the error bound ϵ_N across multiple outer iterations. The idea is similar to the stochastic approximation framework [67, 85], in that initial outer iterations involve using small sample sizes and solving the resulting deterministic problem to a relatively low precision level. The output is then used as a starting point for the next outer iteration. The hope is that as the sample size increases and the error tolerances become tighter, providing a "good" starting point eases the computational burden of solving the larger problem to greater precision. However, at the present time, it is an open question whether this scheme can have good theoretical or practical convergence properties.

The approach outlined above considers approximation on a global level in terms of the Nikaido-Isoda function via the termination condition (6.1). In contrast, we might also think about using approximation in the individual player problems solved at each iteration within the decentralized schemes. However, in this case we require a modification to the algorithms. Specifically, rather than updating the player decisions to locally optimal solutions given the previous iterate, we use the tuple of players' best responses as a candidate descent direction. We state this version of the update rule for the Gauss-Seidel algorithm below.

Formally, suppose given $\bar{x} \in \mathbf{X}$, x_f^{BR} solves player f's regularized problem, i.e.

$$x_f^{\mathrm{BR}}(\bar{x}_{-f}, \tau) = \operatorname*{argmin}_{x_f \in X_f} \left[\theta^f(x_f, \bar{x}_{-f}) + \tau \left\| x_f - \bar{x}_f \right\|^2 \right].$$

Pseudocode for the inexact Gauss Seidel method is given in Algorithm 7.

Algorithm 7 Inexact Sequential best-response (Gauss-Seidel)

Step 0: Initialize - Set $x^0 \leftarrow (x_f^0)_{f=1}^F$, $k \leftarrow 0$.

Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**

Step 2: Main Iteration:

FOR $f = 1, \ldots, F$,

Compute z_f^k such that $\left\| z_f^k - x_f^{BR}(x_{-f}^{k,f}, \tau) \right\| \leq \epsilon_f^k$.

Set $x_f^{k+1} = x_f^k + \gamma^k z_f^k$.

Set $k \leftarrow k + 1$.

GOTO Step 1.

Recall that $x_{-f}^{k,f} = (x_1^{k+1}, \ldots, x_{f-1}^{k+1}, x_{f+1}^k, \ldots, x_F^k)$. Convergence of Algorithm 7, depends on the choice of the tolerance parameter ϵ_f^k as well as the step size parameter

γ^k. The two parameters need to be chosen carefully and in tandem in order to ensure good performance.

Theorem 6.1.2 *Suppose NEP$(\theta_f(\cdot,\xi), X_f)_{f=1}^F$ satisfies all the assumptions for Theorem 4.4.4. Suppose further that the potential function P is coercive over $\mathbf{X}$ and that the parameters γ and ϵ satisfy the following properties - (i) $\gamma^k \in (0,1]$, (ii) $\gamma^k \to 0$, (iii) $\sum_{k=1}^{\infty} \gamma^k < \infty$, (iv) $\sum_{k=1}^{\infty} (\gamma^k)^2 < \infty$ and (v) $\sum_{k=1}^{\infty} \epsilon_f^k \gamma^k < \infty$ for each $f = 1, \ldots, F$. Then the following results hold.*

1. *Every limit point $\hat{x}$ of the sequence x^k generated by Algorithm 7 is an equilibrium to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.*

2. *Suppose further that the required multi-epiconvergence in Theorem 4.3.1 is satisfied. In this case, if $\hat{x} \to x^*$ as $N \to \infty$, then x^* solves SNEP$(\theta_f, X_f)_{f=1}^F$.*

Proof For part (i), we begin by noting that by Lemma 4.4.2, $x^f_{textBR}(\bar{x}, \tau) \in \operatorname{argmin}_{x_f \in X_f}[P(x_f, \bar{x_f}) + \tau \left\| x_f - \bar{x}_f \right\|^2]$. Given this fact, it is easy to show that Algorithm 7 computes a stationary point of the potential minimization problem (4.26) using Theorem 4 [97]. Indeed, it is easily seen that Algorithm 7 is equivalent to the Inxact Gauss-Seidel SCA Algorithm in [97], with $f_1(x) = \hat{P}(x)$, $\mathcal{I} = \{1, \ldots, F\}$, $C_i = I_f = \{1\}$, $\mathcal{K}_f = X_f$ with the regularization matrix $\mathbf{H}(x) = \mathbf{I}$ the identity matrix.

In order to show convergence of Algorithm 7 to a stationary point of (4.26), we merely need to verify that the required regularity conditions are satisfied. But this is easily verified as a consequence of Lemma 4.4.1.

The proof of part (i) is completed by noting that by Lemma 4.2.4, any stationary point of (4.26) is also an equilibrium to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

Part (ii) is a direct consequence of the multi-epiconvergence property and Theorem 4.3.1. ■

Similarly, we can also think of an inexact version of Algorithm 5, as given below.

The requirements for the algorithmic parameters γ and ϵ for Algorithm 8 are the same as those for Algorithm 7. We state the convergence result for Algorithm 8 below without proof.

Algorithm 8 Inexact Parallel best-response (Gauss-Jacobi)

Step 0: Initialize - Set $x^0 \leftarrow (x_f^0)_{f=1}^F$, $k \leftarrow 0$.

Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**

Step 2: Main Iteration:

FOR $f = 1, \ldots, F$,

Compute z_f^k such that $\left\| z_f^k - x_f^{BR}(x_{-f}^k, \tau) \right\| \leq \epsilon_f^k$.

Set $x_f^{k+1} = x_f^k + \gamma^k z_f^k$.

Set $k \leftarrow k + 1$.

GOTO Step 1.

Theorem 6.1.3 *Suppose NEP$(\theta_f(\cdot, \xi), X_f)_{f=1}^F$ satisfies all the assumptions for Theorem 4.4.4. Suppose further that the potential function P is coercive over $\mathbf{X}$ and that the parameters γ and ϵ satisfy the following properties - (i) $\gamma^k \in (0,1]$, (ii) $\gamma^k \to 0$, (iii) $\sum_{k=1}^{\infty} \gamma^k < \infty$, (iv) $\sum_{k=1}^{\infty} (\gamma^k)^2 < \infty$ and (v) $\sum_{k=1}^{\infty} \epsilon_f^k \gamma^k < \infty$ for each $f = 1, \ldots, F$. Then the following results hold.*

1. *Every limit point $\hat{x}$ of the sequence x^k generated by Algorithm 8 is an equilibrium to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.*

2. *Suppose further that the required multi-epiconvergence in Theorem 4.3.1 is satisfied. In this case, if $\hat{x} \to x^*$ as $N \to \infty$, then x^* solves SNEP$(\theta_f, X_f)_{f=1}^F$.*

One of the interesting open research questions that arise out of our discussion in this section is whether the inexact best-response schemes outlined in Algorithms 8 and 7 can be combined with a termination condition like 6.1 to design a sequential sampling inexact best-response algorithm. If such an algorithm is convergent, it would provide a natural way to ease the computational burden of finding approximate equilibria to SNEP$(\theta_f, X_f)_{f=1}^F$ to some predetermined precision.

6.2 Disparate sampling schemes

All the algorithms presented in this dissertation operate under the common assumption that the sample paths $\{\xi^k\}$ used by each player at each iteration are exactly the same. This is a reasonable assumption if the application in question is an engineered system, where the Nash model is usually used for the purpose of designing distributed algorithms. In this case, the sampling might be done in common by an external or central agency and distributed to the agents.

However in many real life scenarios, it is more reasonable to consider disparate sampling schemes, where each agent is assumed to utilize different samples of the random vector ξ, all drawn from the same distribution. This is certainly the case where the agents involved are separate entities or people. Even in the case of distributed implementation for engineered systems, it is reasonable to assume that the sampling is done locally using a pseudo-random number generator with different seeds on different machines.

In light of this issue, a natural question to ask is the following - do the convergence properties of Algorithms 5 and 6 hold if the samples used by each agent f are different, i.e. agent f uses a sample $\{\xi_f^k\}$, instead of a common sample $\{\xi^k\}$. It is intuitive to think that this will indeed be the case, since under the potentiality assumption if the players draw samples of the same size (say N), then as $N \to \infty$ the error between the potential function values computed by each player will reduce to zero.

We provide an example to illustrate the reasoning presented above. For the three node Nash Cournot power market equilibrium example presented in section 5.2.2, we compare the output for Algorithm 6 computed using uniform sampling as well as disparate sampling across the players. The results are summarized in Figures 6.1 through 6.4.

The figures illustrate the fact that in general for a fixed sample size N, there is more variance in the solutions computed using disparate sampling. However as the

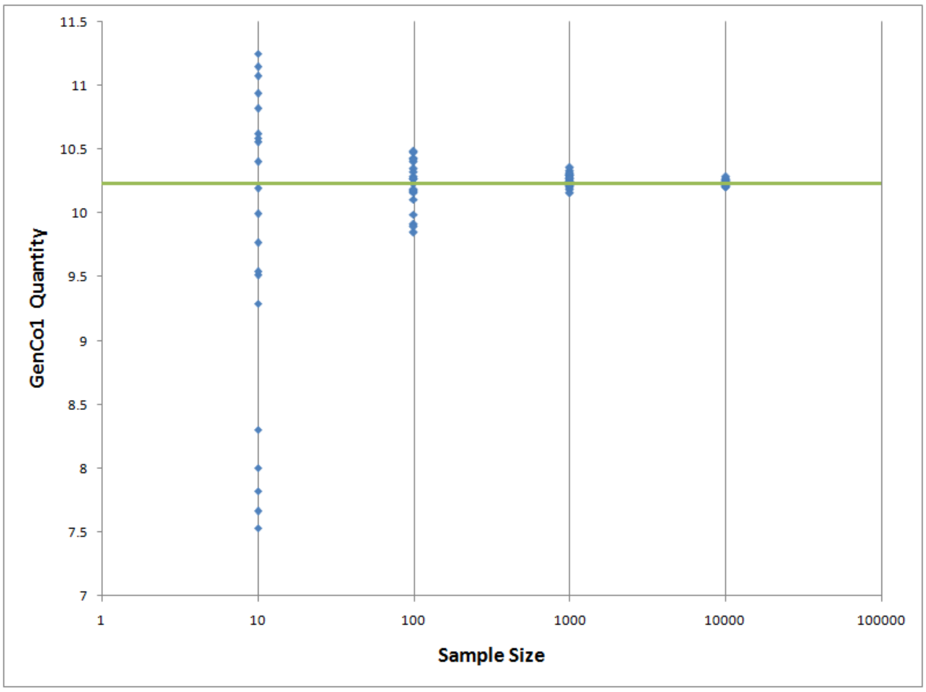

Figure 6.1. Genco 1 equilibrium quantity vs N (Disparate Sampling)

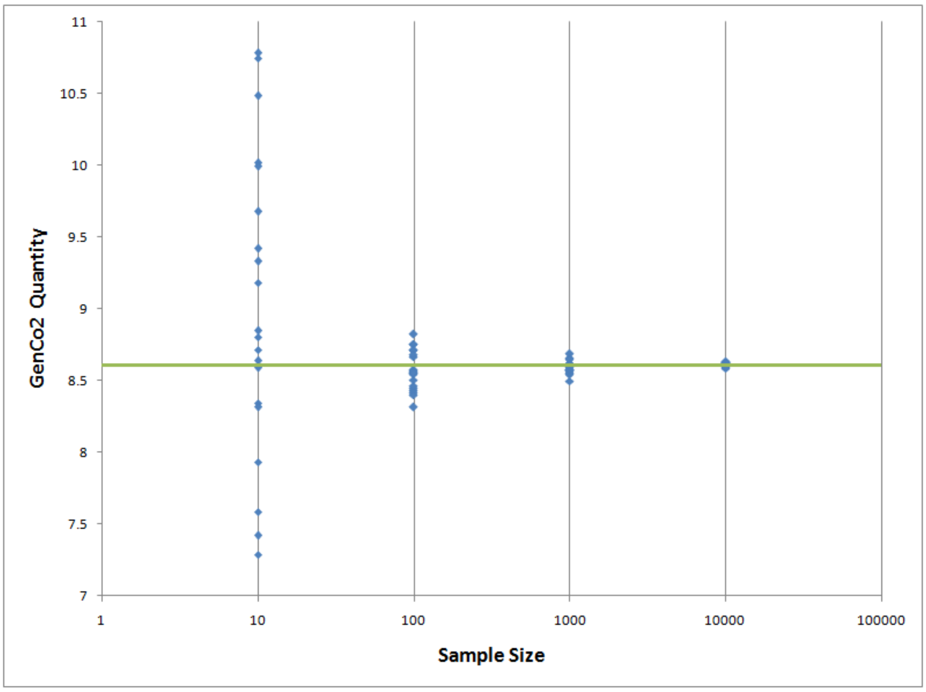

Figure 6.2. Genco 2 equilibrium quantity vs N (Disparate Sampling)

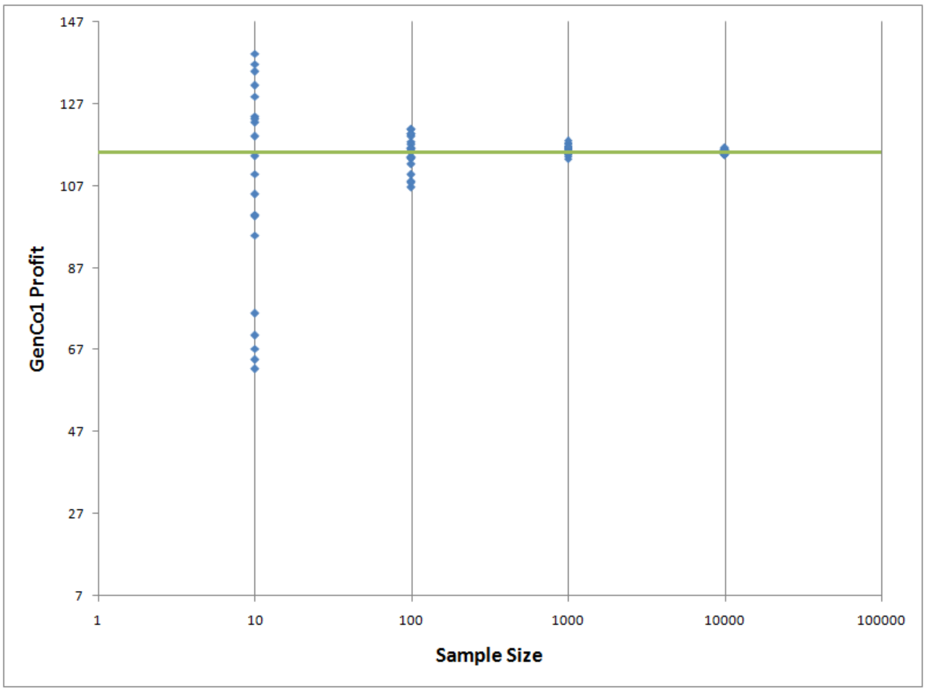

Figure 6.3. Genco 1 equilibrium profit vs N (Disparate Sampling)

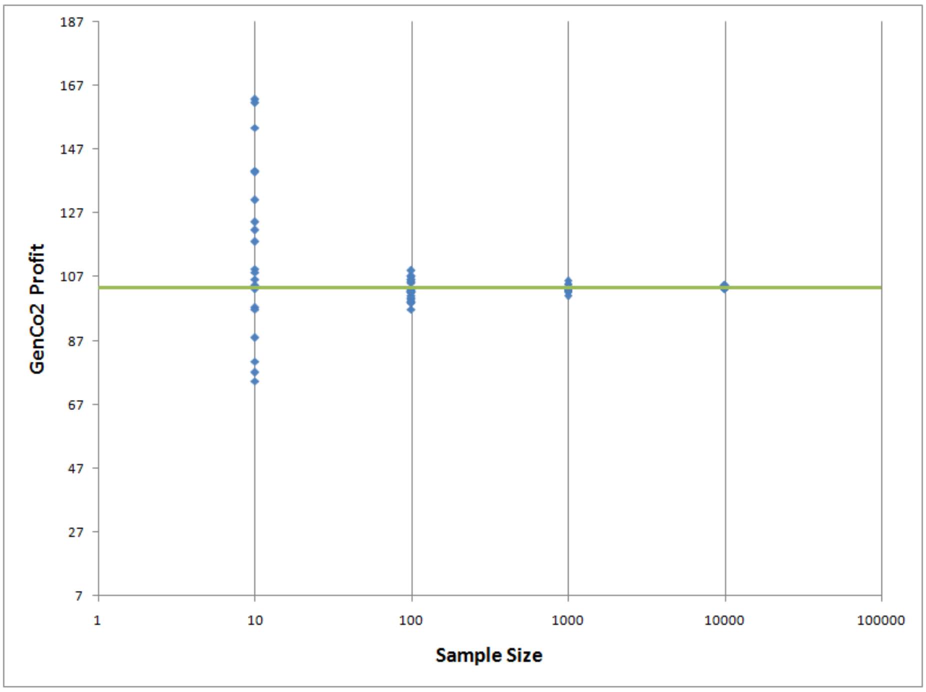

Figure 6.4. Genco 2 equilibrium profit vs N (Disparate Sampling)

sample size N increases, the solutions of the algorithm tend to converge to the true solution for both uniform as well as disparate sampling schemes.

These experiments indicate that at least for the small example that we considered of a player-wise convex potential game under uncertainty, disparate sampling schemes provide convergent approximation algorithms. We are currently exploring research on establishing these convergence properties in theory. The main challenge in this problem is that if the agents use different samples, the sampled deterministic equivalent equilibrium problem SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$ need not necessarily have a potential function. In this case, we must rely on error estimation results to analyze the difference between potential function values $\hat{P}_f$ estimated by each player given a sample of fixed size N, relative to the true potential function value P. We leave this problem for future work.

6.3 Approximation of generalized Nash equilibria

In this section, we examine whether the decentralized approximation schemes presented in Chapter 4 may be extended to the realm of generalized Nash equilibrium problems. The primary motivation behind the following discussion is the increasing number of generalized Nash models being used in application areas such as wireless communications networks and traffic network analysis.

One of the main challenges associated with proving convergence results of decentralized algorithms for GNEPs is the presence of coupled constraints. Indeed, it is very easy to construct examples where best responses result in infeasibility for player problems. However, we consider the case where the linking constraints amongst the player problems are "shared". Specifically, consider the GNEP$(\theta_f, X_f)_{F=1}^F$ where each player solves the following problem:

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \theta_f(x_f, x_{-f}) \\ \text{subject to} \quad & x_f \in X_f(x_{-f}). \end{aligned} \tag{6.5}$$

This GNEP is said to have shared constraints if there exists a nonempty, closed set $\mathcal{X} \in \mathbb{R}^n$ such that for each player $f = 1, \ldots, F$,

$$X_f(x_{-f}) = \{x_f \in \mathbb{D}_f : (x_f, x_{-f}) \in \mathcal{X}\}, \tag{6.6}$$

where $D_f \in \mathbb{R}^{n_f}$ is a nonempty closed set that captures the non-shared constraints for player f. In the form of parametric constraints, typically we have $\mathcal{X} = \{x \in \mathbb{R}^n : g(x) \leq 0\}$, where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a vector function that is usually assumed to be convex in x.

It is easy to see that if we start with a jointly feasible $x \in \mathcal{X} \cap \prod_{f=1}^{F} D_f$ and let a player f deviate unilaterally, the resulting decision tuple will still be feasible for the other players. This is precisely because of the shared nature of the linking constraints. Note that if we let multiple players best respond to a given x simultaneously, the resulting decision tuple might not necessarily be feasible to the GNEP. In essence, this means that the convergence results for regularized Gauss-Seidel algorithms go through for player-wise convex GNEPs[1] with shared constraints [38]. However, no such convergence result is possible for the regularized Gauss-Jacobi algorithm.

In view of these facts, it is natural to think of applying sampling schemes, combined with Algorithm 6 to solve these special GNEPs in the presence of exogenous uncertainty in the objective functions. The main difficulty however is that the multi-epiconvergence property defined in [50] is designed for NEPs. Extending the multi-epiconvergence concept into the domain of GNEPs is a nontrivial task. Indeed, the main tool used in proving the convergence result summarized in Theorem 4.3.1 is the following fact.

Theorem 6.3.1 *[50] The family of functions $\{\hat{\phi}_f\}_{f=1}^F$ multi-epiconverges to the family $\{\phi_f\}_{f=1}^F$ on $\mathbf{X}$ if and only if for every $f = 1, \ldots, F$ and every sequence $\{x_{-f}^k\} \subset X_{-f}$ converging to some $x_-^\infty f \in X_{-f}$, the sequence of uni-component functions $\{\psi_f^k\}$ where*

$$\psi_f^k(x_f) = \hat{\phi}_f(x_f, x_{-f}^k), \quad x_f \in X_f$$

[1] A GNEP is player-wise convex if the objective function $\theta_f(\cdot, x_{-f}$ and the feasible set $X_f(\cdot, x_{-f}$ are convex.

epiconverges to the uni-component function

$$\psi_f^\infty(x_f) = \hat{\phi}_f(x_f, x_{-f}^\infty), \quad x_f \in X_f$$

on the set X_f.

Theorem 6.3.1 allows us to use the machinery of epiconvergence argue that if we construct a sequence $\{x_f^k\}$ such that $x_f^k \in \operatorname{argmin}_{x_f \in X_f} \psi_f^k(x_f)$, and $x_f^k \to x_f^* \in X_f$, then $x_f^* \in \operatorname{argmin}_{x_f \in X_f} \psi_f^\infty(x_f)$. Since this holds for each $f = 1, \ldots, F$ simultaneously, x^* must be an equilibrium for $\text{NEP}(\hat{\phi}_f, X_f)_{f=1}^F$.

However, in the case of $\text{GNEP}(\phi_f, X_f)_{f=1}^F$, the sequence $\{x^k\}$ is constructed such that

$$x_f^k \in \underset{x_f \in X_f(x_{-f}^k)}{\operatorname{argmin}} \psi_f^k(x_f).$$

Thus epiconvergence of $\{\psi_f^k\}$ to ψ_f^∞ is insufficient to ensure convergence of solutions of $\text{GNEP}(\hat{\phi}_f, X_f)_{f=1}^F$ to solutions of the true $\text{GNEP}(\phi_f, X_f)_{f=1}^F$. We need some appropriate extension of epiconvergence to cases where the underlying feasible sets also vary, albeit perhaps in a structured fashion. We are currently pursuing research on this issue.

7. SUMMARY AND FUTURE WORK

The main focus of this dissertation is on the analysis of decentralized algorithms for computing solutions of Nash games. Our research was motivated by the myriad applications that have recently come into focus where Nash equilibrium is used as a modeling framework for strategic interactions between selfish agents. In many such scenarios, examining the convergence properties of decentralized schemes, such as best-response methods, is an important task. From a computational perspective, our work has important implications in the implementation of best-response methods for certain classes of Nash games, especially in high performance computing framework. On the other hand, our analysis also paves the way for obtaining a better understanding of how real agents reach a particular equilibrium.

A summary of our work on the two major topics of this dissertation, namely network interdiction games and potential games under exogenous uncertainty, is given below. Each section also presents ideas for future research.

7.1 Network interdiction games

In Chapter 3, we introduced decentralized network interdiction (DNI) games and gave formulations for three classes of games – decentralized shortest path interdiction (DSPI) games, decentralized maximum flow interdiction (DMFI) games and decentralized minimum cost flow interdiction (DMCFI) games. We analyzed the theoretical properties of DSPI games: in particular, we gave conditions for the existence of equilibria and examples where multiple equilibria exist. We also showed that DSPI games belong to a special class of games called potential games. This property was key in establishing several of the theoretical results we presented.

We showed that the DSPI game under continuous interdiction is equivalent to a linear complementarity problem, which can be solved by the Lemke's algorithm. This constitutes a convergent centralized method to solve such problems. We also presented decentralized heuristic algorithms to solve DSPI games under both continuous and discrete interdiction. Finally, we used these algorithms to empirically evaluate the worst case and average efficiency loss of DSPI games.

We also presented formulations for other classes of network interdiction games where the agents' obstruction functions are related to the maximum flow or minimum cost flow in the network. Establishing theoretical results and studying the applicability of the decentralized algorithms to other classes of decentralized network interdiction games are natural and interesting extensions of our work on DSPI games.

In our study of DSPI games, we made the assumption that the games have complete information; that is, the normal form of the game – the set of agents, agents' feasible action spaces, and their utility functions – is assumed to be common knowledge to all agents. In addition, we made the implicit assumption that all input data are deterministic. However, data uncertainty and lack of observability of other agents' preferences or actions are prevalent in real-world situations. For such settings, we need to extend our work to accommodate games with exogenous uncertainties, as well as cases with incomplete information.

One might also be interested in designing interventions to reduce the loss of efficiency resulting from decentralized control. This leads to the topic of mechanism design. Such a line of work also defines a very important and interesting future research direction.

7.2 Potential games under exogenous uncertainty

In Chapter 4, we presented decentralized approximation schemes for a class of games called potential games under exogenous uncertainty. The primary motivation for our work is the need for efficient algorithms to compute equilibria to games where the players' actions are subject to uncertainty whose sources are external and knowledge of whose structure is public.

We leveraged some recent results in the analysis of approximations of Nash equilibria, especially the concept of multi-epiconvergence to show that sampling based approximation schemes can be successfully used to build computational methods for games under exogenous uncertainty. Such approximation schemes, when combined with decentralized algorithms were then shown to converge to equilibria as the sample size grows large. Specifically, we gave convergence results for parallel and sequential best response algorithms, under suitable regularization schemes. The conditions we require for the convergence results are fairly mild.

In order to test the decentralized approximation algorithms presented in Chapter 4, we present two relevant application examples, stochastic traffic routing games and power market equilibrium problems. In the former we consider atomic selfish routing games with splittable flow. We perform numerical experiments for the routing games that illustrate good empirical performance for our algorithms. For power market equilibria, we consider equilibrium between electricity generation companies and an independent system operator. Under the assumption that the generation companies cannot anticipate the actions of the ISO, we show that the market equilibrium is in fact a potential game. We also present numerical results for a small test case.

Numerical experiments for both applications are carried out using CPLEX and MATLAB's parallel computing toolbox. For the instance of power market equilibrium problem that we tested, as well as small instances of the stochastic traffic routing games, the results show good empirical performance of the naive Gauss-Seidel algorithm (Algorithm 1) as well as the naive Gauss-Jacobi algorithm (Algorithm 2). However, for larger instances of the stochastic routing game, we find it necessary to employ the regularized Gauss-Jacobi method (Algorithm 5), since the naive version fails to converge. In this case, we observe that the convergence properties of the algorithm are highly dependent on the regularization parameter τ. This parameter needs to be carefully tuned in order to ensure good practical performance of the algorithm. A preliminary analysis of the parallelization efficiency for Algorithm 5 is also presented.

We note that the approach presented here marks a starting point for investigations on the convergence properties of decentralized algorithms for various games under uncertainty. From a computational perspective, there are several natural extensions and questions that arise from our work. While we have established the asymptotic consistency of estimators generated by the approximation scheme, an important question that remains to be answered is rate of convergence as well as the quality of solutions generated using finite sample sizes.

It is reasonable to assume in some scenarios, such as distributed optimization settings, that all the players are able to draw the same sample from the underlying random vector's probability distribution. However in the more general setting it is natural to ask whether the convergence properties of the algorithms would hold up if the players draw distinct samples from the distribution.

From the modelling perspective, our analysis assumes risk neutral behavior from the players. It is an open question whether the algorithms will also perform well for risk averse equilibrium models. Additional conditions might be required, for instance on the structure and properties of the risk measure, to ensure that favorable properties such as potentiality for the underlying deterministic game carries over to the risk-averse formulation. Another open question is whether the analysis may be extended to games of incomplete information, i.e. for the computation of Bayesian Nash equilibria. While we conjecture that a direct extension is possible for Bayesian potential games with finite type spaces, the case of continuous type spaces presents a challenging research direction.

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