# Canonical Coordinates on Lie Groups and the Baker Campbell Hausdorff Formula 

Nicholas Graner<br>Utah State University

Follow this and additional works at: https://digitalcommons.usu.edu/etd
Part of the Mathematics Commons

## Recommended Citation

Graner, Nicholas, "Canonical Coordinates on Lie Groups and the Baker Campbell Hausdorff Formula" (2018). All Graduate Theses and Dissertations. 7232.
https://digitalcommons.usu.edu/etd/7232

This Thesis is brought to you for free and open access by the Graduate Studies at DigitalCommons@USU. It has been accepted for inclusion in All Graduate Theses and Dissertations by an authorized administrator of DigitalCommons@USU. For more information, please

## CANONICAL COORDINATES ON LIE GROUPS AND THE BAKER CAMPBELL

## HAUSDORFF FORMULA

by

Nicholas Graner

# A thesis submitted in partial fulfillment <br> of the requirements for the degree 

of

## MASTERS OF SCIENCE

in<br>Mathematics

Approved:

Mark Fels, Ph.D.
Major Professor

Ian Anderson, Ph.D.
Committee Member

Charles Torre, Ph.D.
Committee Member

Mark R. McLellan, Ph.D.
Vice President for Research and
Dean of the School for Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

Copyright © Nicholas Graner 2018
All Rights Reserved

ABSTRACT<br>Canonical Coordinates on Lie Groups and the Baker Campbell Hausdorff Formula<br>by<br>Nicholas Graner, Master of Science<br>Utah State University, 2018

Major Professor: Mark Fels
Department: Mathematics and Statistics

Lie's third theorem states that for any finite dimensional Lie algebra $\mathfrak{g}$ over the real numbers, there is a simply connected Lie group $G$ which has $\mathfrak{g}$ as its Lie algebra. The main part of this document is concerned with the question of determining the group G, in Lie's third theorem, and its multiplication function when just $\mathfrak{g}$ is given.

The Baker-Campbell-Hausdorff formula (BCH) is implemented in the Maple Differential Geometry package for nilpotent Lie groups where the group multiplication is determined from the Lie algebra using the BCH formula. The BCH formula is also used to give a closed form formula for a local basis for the left invariant vector fields of a Lie group in canonical coordinates of the $1^{\text {st }}$ kind. It is shown how the BCH formula can be used to locally give the group multiplication in canonical coordinates of the $2^{\text {nd }}$ kind. A closed form formula for a local basis for the left invariant vector fields of a solvable Lie group is given in coordinates of the $2^{\text {nd }}$ kind. For nilpotent Lie groups these calculations are implemented in the Maple Differential Geometry package along with the calculation of local left invariant 1-forms for general Lie groups in coordinates of the $2^{\text {nd }}$ kind. Examples of the calculations are given.

## PUBLIC ABSTRACT

# Canonical Coordinates on Lie Groups and the Baker Campbell Hausdorff Formula 

Nicholas Graner

Lie Groups occur in math and physics as representations of continuous symmetries and are often described in terms of their Lie Algebra. This thesis is concerned with finding a concrete description of a Lie group given its associated Lie algebra. Several calculations toward this end are developed and then implemented in the Maple Differential Geometry package. Examples of the calculations are given.

## ACKNOWLEDGMENTS

I would like to thank my professors Ian Anderson and Charles Torre for educating me in the field of Differential Geometry. I would especially like to thank my advisor Mark Fels whose help has been invaluable in creating this thesis.

Nick Graner

## CONTENTS

ABSTRACT ..... iii
PUBLIC ABSTRACT ..... iv
ACKNOWLEDGMENTS ..... v
CHAPTER

1. Introduction ..... 1
1.1 Chapter Summary ..... 1
1.2 Literature Review and Summary of Results ..... 2
2. Lie groups and Lie algebras ..... 4
2.1 Heisenberg Example: ..... 6
3. Canonical Coordinates of the $1^{\text {st }}$ Kind ..... 8
3.1 The Exponential and Definition of the Coordinates of the $1^{\text {st }}$ Kind ..... 8
3.1.1 Heisenberg Example ..... 11
3.2 Baker Campbell Hausdorff Formula ..... 11
3.3 BCH - The Nilpotent Case ..... 16
3.3.1 Examples 3.3.1 ..... 16
3.4 Left Invariant Vector Fields in Canonical Coordinates of the $1^{\text {st }}$ Kind ..... 17
3.4.1 Example ..... 20
3.5 Left Invariant Vector Fields in Canonical Coordinates of the $1^{\text {st }}$ Kind - Nilpotent Case ..... 21
3.5.1 Examples 3.5.1 ..... 21
4. Canonical Coordinates of the $2^{\text {nd }}$ Kind ..... 23
4.1 Defining the Coordinates of the $2^{\text {nd }}$ Kind ..... 23
4.1.1 Heisenberg Example ..... 24
4.2 Group Multiplication in Coordinates of the $2^{\text {nd }}$ Kind -Nilpotent Case ..... 25
4.2.1 Heisenberg Example ..... 26
4.3 Left Invariant 1-Forms in Coordinates of the $2^{\text {nd }}$ Kind ..... 26
4.3.1 Example ..... 30
4.4 Coordinates of the $2^{\text {nd }}$ Kind - Solvable Case ..... 30
4.4.1 Left Invariant 1-Forms in Coordinates of the $2^{\text {nd }}$ Kind - Solvable Case ..... 30
4.4.2 Left Invariant Vector Fields in Coordinates of the $2^{\text {nd }}$ Kind - Solvable Case ..... 34
4.5 Left Invariant Vector Fields in Coordinates $2^{\text {nd }}$ Kind - Nilpotent Case ..... 37
4.5.1 Example ..... 37
REFERENCES ..... 38
APPENDICES ..... 40
APPENDIX A - Maple Code ..... 41
4.5.2 Maple Code for Equation (3.14) of Corollary 13 ..... 42
4.5.3 Maple Code for Equation (3.15) of Theorem 14 ..... 44
4.5.4 Maple Code for Equation (4.4) ..... 45
4.5.5 Maple Code for Equation (4.3) of Theorem 21 ..... 47
4.5.6 Maple Code for Equation (4.8) of Theorem 23 ..... 48
4.5.7 Maple Code for Equation (4.21) of Theorem 28 ..... 49
APPENDIX B - Examples ..... 53
4.5.8 Example 3.3.1.2 ..... 54
4.5.9 Example 3.5.1.2 ..... 57
4.5.10 Example 4.2.1 ..... 59
4.5.11 Example 4.3.1 ..... 61
4.5.12 Example 4.5.1 ..... 63

## CHAPTER 1

## INTRODUCTION

Lie's third theorem states that for any finite dimensional Lie algebra $\mathfrak{g}$ over the real numbers, there is a simply connected Lie group $G$ which has $\mathfrak{g}$ as its Lie algebra. The main part of this document is concerned with the question of determining the group G, in Lie's third theorem, and its multiplication function when just $\mathfrak{g}$ is given.

## 1.1

## Chapter Summary

Chapter 2 provides some background regarding Lie algebras and Lie groups and defines some basic terminology that will be used in the document.

Chapter 3 defines canonical coordinates of the $1^{\text {st }}$ kind by use of the exponential map. For any Lie group these coordinates are defined in a neighborhood of the identity. For some simply connected groups, including those with $\mathfrak{g}$ nilpotent, these coordinates are global making the group diffeomorphic to $\mathbb{R}^{n}$.

In Section 2 of Chapter 3 we introduce the Baker Campbell Hausdorff formula (BCH) which provides a local formula for the group multiplication in coordinates of the $1^{\text {st }}$ kind. For simply connected Lie groups with nilpotent $\mathfrak{g}$ this will produce the multiplication map in coordinates for the entire group. Examples are given in Section 3 and Maple code implementation is in Appendix A.

In Section 4 of Chapter 3 the BCH formula is used to determine a closed form formula for a basis for the left invariant vector fields of a Lie group in coordinates of the $1^{\text {st }}$ kind on an open set about the identity. For nilpotent $\mathfrak{g}$ these vector fields are defined on all of $\mathbb{R}^{n}$. Examples are given in Section 5 and Maple code implementation is in Appendix A.

Chapter 4 defines canonical coordinates of the $2^{\text {nd }}$ kind. For any Lie group these coordinates are defined in a neighborhood of the identity. For simply connected groups with solvable $\mathfrak{g}$ these coordinates are global making the group diffeomorphic to $\mathbb{R}^{n}$.

In Section 2 of Chapter 4 we show how the BCH formula can be used to give the group multiplication in coordinates of the $2^{\text {nd }}$ kind for simply connected Lie groups with nilpotent $\mathfrak{g}$. Examples are given and Maple code implementation is in Appendix A.

In Section 3 of Chapter 4 we modify the basis for left invariant 1 -forms given in reference ]1] to be in coordinates of the $2^{\text {nd }}$ kind. These 1-forms are defined in a neighborhood of the identity for any Lie group. Examples are given and Maple code implementation is in Appendix A.

In Section 4 of Chapter 4 we show that for Lie groups with solvable $\mathfrak{g}$ the basis of 1 -forms given in reference [5] are the same as those computed in coordinates of the $2^{\text {nd }}$ kind using the formula developed in the prior section. We then go on to show that the dual to this basis is the same as one I developed by use of the BCH formula. For simply connected Lie groups with nilpotent $\mathfrak{g}$ these vector fields are defined on all of $\mathbb{R}^{n}$. Examples are given in Section 5 and the Maple code implementation is in Appendix A.

## 1.2

## Literature Review and Summary of Results

Eugene Dynkin gave an explicit formulation of the Baker Campbell Hausdorff Formula (BCH) for expressing the multiplication in canonical coordinates of the $1^{\text {st }}$ kind. (see [4] 1.7). In this document we implement the calculation of this formula in the Maple Differential Geometry package for nilpotent Lie groups.

In this document the BCH formula is used to derive a closed form formula for a basis for the left invariant vector fields of a Lie group on a neighborhood of the identity in canonical coordinates of the $1^{\text {st }}$ kind. In developing this formula a result from reference [6] is used to write the sum of some of the coefficients from the BCH formula in terms of Bernoulli numbers. For nilpotent Lie groups these vector fields are implemented in Maple code.

In this document we use the BCH formula to derive a formula for the group multiplication for a nilpotent Lie group in coordinates of the $2^{\text {nd }}$ kind. This is implemented in Maple code.

Reference [1] gives a formula for a basis for the left invariant 1-forms in coordinates, which are similar to coordinates of the $2^{\text {nd }}$ kind, on a neighborhood of the identity for any Lie group. In this document we modify this formula so that it is given in coordinates of the $2^{\text {nd }}$ kind. This formula is implemented in Maple code.

In this document we show that for solvable Lie groups the formula for the left invariant 1forms given in [1] are, when modified to be in coordinates of the $2^{\text {nd }}$ kind, the same as the left invariant 1-forms given in [5].

In this document a formula is given for a basis for the left invariant vector fields of a solvable Lie group on a neighborhood of the identity. This formula was developed by use of the BCH formula and results from reference [6]. This formula is shown to give the same left invariant vector fields as given in [5].

## CHAPTER 2

## LIE GROUPS AND LIE ALGEBRAS

First we will introduce Lie groups and Lie algebras and state several important definitions and theorems that we will utilize.

Definition. Lie group - Let G be a group with multiplication $\mu$. The group G is a Lie group if it has the following properties:

- G is a smooth manifold
- $\mu: \mathrm{G} \times \mathrm{G} \rightarrow \mathrm{G}$ is a smooth map
- the map $\mathfrak{i}: \mathrm{G} \rightarrow \mathrm{G}$ by $\mathfrak{i}(\mathfrak{a})=\mathrm{a}^{-1}$ is a smooth map.

Definition. Lie algebra - A Lie algebra $\mathfrak{g}$ is a vector space with a multiplication $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, having the properties:

- Bilinearity: $[\mathrm{ax}+\mathrm{by}, \mathrm{z}]=\mathrm{a}[\mathrm{x}, \mathrm{z}]+\mathrm{b}[y, z],[z, \mathrm{ax}+\mathrm{by}]=\mathrm{a}[z, \mathrm{x}]+\mathrm{b}[z, y]$
- Skew Symmetry: $[x, y]=-[y, x] \Rightarrow[x, x]=0$
- Jacobi Identity: $[x,[y, z]]+[z,[x, y]]+[y,[z, x]]=0$
where $\mathrm{x}, \mathrm{y}$, and z are vectors and $\mathrm{a}, \mathrm{b} \in \mathbb{R}$.
Given $X \in \mathfrak{g}$, the Adjoint Endomorphism, adX $: \mathfrak{g} \rightarrow \mathfrak{g}$ is given by $\operatorname{adX}(Z)=[X, Z]$.

The Derived Series of a Lie algebra is a sequence of subalgebras $\mathfrak{g}_{k} \subset \mathfrak{g}$ where $\mathfrak{g}_{0}=\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{k}+1}=\left[\mathfrak{g}_{\mathrm{k}}, \mathfrak{g}_{\mathrm{k}}\right] \subset \mathfrak{g}_{\mathrm{k}}$, making each $\mathfrak{g}_{\mathrm{k}+1}$ and ideal in $\mathfrak{g}_{\mathrm{k}}$ (note that each $\mathfrak{g}_{\mathrm{k}}$ is also an ideal in $\mathfrak{g}$ (see [8] Exercise 5.1.9)) . A Lie algebra is Solvable if there is a $k$ such that the term $\mathfrak{g}_{k}=0$ in the derived series.

The Lower Central Series of a Lie algebra is a sequence of subalgebras $\mathfrak{g}_{\mathrm{k}} \subset \mathfrak{g}$ where $\mathfrak{g}_{0}=\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{k}+1}=\left[\mathfrak{g}_{\mathrm{k}}, \mathfrak{g}\right] \subset \mathfrak{g}_{\mathrm{k}}$ making each $\mathfrak{g}_{\mathrm{k}+1}$ and ideal in $\mathfrak{g}_{\mathrm{k}}$ (note that each $\mathfrak{g}_{\mathrm{k}}$ is also an ideal in $\mathfrak{g}$ based on the Jacobi identity). A Lie algebra is nilpotent if there is a k such that the term $\mathfrak{g}_{\mathrm{k}}=0$ in the lower central series. A nilpotent Lie algebra is said to be k-step nilpotent if $\mathfrak{g}_{\mathrm{k}}=0$ and $\mathfrak{g}_{\mathrm{k}-1} \neq 0$ in the lower central series.

We now describe the standard way to associate a Lie algebra with a Lie group.

Definition. Left Invariant Vector Field - Let $G$ be a Lie group. Let $L_{g}: G \rightarrow G$ be left multiplication by $g$ given by $L_{g}(h)=g h, g, h \in G$. A vector field $X$ on $G$ is a left invariant vector field if $\left(\mathrm{L}_{\mathrm{g} *} \mathrm{X}_{\mathrm{h}}\right)=X_{\mathrm{gh}}$.

Let $\mathfrak{X}$ be the vector space of vector fields on a Lie group $G$. Let $X, Y \in \mathfrak{X}$ and let $f$ be a function, $f: G \rightarrow \mathbb{R}$. Define a multiplication operation (i.e. bracket), $[X, Y](f)=X(Y(f))-Y(X(f))$. This makes $\mathfrak{X}$ an infinite dimensional Lie algebra. Let $\mathfrak{g} \subset \mathfrak{X}$ be the set of left invariant vector fields on G.

Theorem 1. The set $\mathfrak{g}$ of left invariant vector fields on $G$ with bracket, [, ] is a finite dimensional Lie algebra with $\operatorname{dim} \mathfrak{g}=\operatorname{dim} \mathrm{G}$. Additionally if $\mathrm{X}_{e} \in \mathrm{~T}_{\mathrm{e}} \mathrm{G}$ there is a unique left invariant vector field X on G with the prescribed tangent vector. (See [3] Proposition 7.1)

In particular $\mathfrak{g} \subset \mathfrak{X}$ is a finite dimensional subalgebra that we call the Lie algebra for $G$. Since $X \in \mathfrak{g}$ is uniquely determined by $X_{e}$, the Lie algebra can be thought of as either the left invariant fields or as the tangent space at the identity since these are in 1-1 correspondence and given two left invariant vector fields, $X$ and $Y$, we have $[X, Y]_{e}=\left[X_{e}, Y_{e}\right]$.

A standard theorem is the converse to Theorem 1. There are a number of forms of this but one such is the following.

Theorem 2. Lie's Third Theorem - Given any Lie algebra $\mathfrak{g}$ there is a simply connected Lie group $G$ having $\mathfrak{g}$ as its Lie algebra. (See [4] 1.14.3) In fact the simply connected group is unique up to diffeomorphism (See [10] Theorem 3.28).

## Heisenberg Example:

The Heisenberg group $H$ is the manifold $\mathbb{R}^{3}$ with the multiplication:

$$
\begin{equation*}
\left(x^{1}, x^{2}, x^{3}\right) *\left(y^{1}, y^{2}, y^{3}\right)=\left(x^{1}+y^{1}, x^{2}+y^{2}+x^{1} y^{3}, x^{3}+y^{3}\right) \tag{2.1}
\end{equation*}
$$

This multiplication gives a smooth mapping from $\mathbb{R}^{3} \times \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ making $\mathbb{R}^{3}$ into a Lie group (albeit a different Lie group then the familiar additive Abelian group). The Heisenberg Lie algebra is often defined as the vector space $\mathbb{R}^{3}$ with the Lie bracket:

$$
\begin{equation*}
\left[\left(x^{1}, x^{2}, x^{3}\right),\left(y^{1}, y^{2}, y^{3}\right)\right]=\left(0, x^{1} y^{3}-x^{3} y^{1}, 0\right) \tag{2.2}
\end{equation*}
$$

In terms of the standard basis for $\mathbb{R}^{3},\left\{e_{1}, e_{2}, e_{3}\right\}$, equation (2.2) gives non-zero products

$$
\begin{equation*}
\left[e_{1}, e_{3}\right]=e_{2},\left[e_{3}, e_{1}\right]=-e_{2} \tag{2.3}
\end{equation*}
$$

It is easy to check that [, ] satisfies the conditions for making $\mathbb{R}^{3}$ into a Lie algebra. In terms of the standard basis for $\mathbb{R}^{3}$ equation (2.2) gives:

$$
\begin{equation*}
\left[x^{i} e_{i}, y^{j} e_{j}\right]=\left(x^{1} y^{3}-x^{3} y^{1}\right) e_{2} \tag{2.4}
\end{equation*}
$$

Note that the Heisenberg Lie algebra $\mathfrak{h}$ is 2-step nilpotent since $[\mathfrak{h},[\mathfrak{h}, \mathfrak{h}]]=0$.

We now compute the Lie algebra $\mathfrak{g}$ of left invariant vector fields for $H$ and show that it is isomorphic to the one described above in equation (2.4). Let $L_{g}: H \rightarrow H$ by $L_{g}(x)=g x$, $g=\left(g^{1}, g^{2}, g^{3}\right), x=\left(x^{1}, x^{2}, x^{3}\right) \in H$. Specifically we have

$$
L_{g}\left(x^{1}, x^{2}, x^{3}\right)=\left(g^{1}+x^{1}, g^{2}+x^{2}+g^{1} x^{3}, g^{3}+x^{3}\right)
$$

A basis for the tangent space at the identity is given by $\partial_{x^{1}}, \partial_{x^{2}}, \partial_{x^{3}}$, . From these we can create a basis for left invariant vector fields by pushing them forward with $L_{g}$. We have $L_{g *}\left(\partial_{x^{1}}\right)=\partial_{x^{1}}$, $L_{g *}\left(\partial_{x^{2}}\right)=\partial_{x^{2}}$ and $L_{g *}\left(\partial_{x^{3}}\right)=\partial_{x^{3}}+g^{1} \partial_{x^{2}}$ giving a basis

$$
\left\{X_{1}=\partial_{x^{1}}, X_{2}=\partial_{x^{2}}, X_{3}=\partial_{x^{3}}+x^{1} \partial_{x^{2}}\right\}
$$

for the left invariant vector fields on H .

For $\mathrm{b}_{1}, \mathrm{~b}_{2}, \mathrm{~b}_{3}, \mathrm{c}_{1}, \mathrm{c}_{2}, \mathrm{c}_{3} \in \mathbb{R}$ and $\mathrm{X}, \mathrm{Y} \in \mathfrak{g}$ we can write

$$
\begin{array}{r}
X=c^{1} X_{1}+c^{2} X_{2}+c^{3} X_{3}=c^{1} \partial_{x^{1}}+\left(c^{2}+c^{3} x^{1}\right) \partial_{x^{2}}+c^{3} \partial_{x^{3}}  \tag{2.5}\\
Y=b^{1} X_{1}+b^{2} X_{2}+b^{3} X_{3}=b^{1} \partial_{x^{1}}+\left(b^{2}+b^{3} x^{1}\right) \partial_{x^{2}}+b^{3} \partial_{x^{3}}
\end{array}
$$

with Lie bracket

$$
\begin{equation*}
[X, Y]=\left(c^{1} b^{3}-b^{1} c^{3}\right) X_{2} \tag{2.6}
\end{equation*}
$$

Since the Lie algebra of left invariant vector fields on the Heisenberg group and the Heisenberg Lie algebra are both 3 dimensional vector spaces we can define a vector space isomorphism $\psi$ between them by $\psi\left(X_{i}\right)=e_{i}$. Comparing equations (2.4) and (2.6) we see that $[\psi(X), \psi(Y)]=\psi([X, Y])$ making $\psi$ a Lie algebra isomorphism.

## CHAPTER 3 <br> CANONICAL COORDINATES OF THE $1^{\text {ST }}$ KIND

In this section we will define the exponential map and show how it can be used to define a coordinate chart about the identity of a Lie group, referred to as canonical coordinates of the $1^{\text {st }}$ kind. We will state the Baker Campbell Hausdorff formula and show how it can be used to give the group multiplication for groups with nilpotent Lie algebras, knowing only the Lie algebra for the group. We will then express the left invariant vector fields of the group in these coordinates and show how they can be calculated algebraically in certain cases.

## 3.1

The Exponential and Definition of the Coordinates of the $1^{\text {st }}$ Kind

We first define the exponential map which is a generalization of the matrix exponential and then show how this can be used to define a coordinate chart about the identity of a Lie group.

Theorem 3. Let G be a Lie group with identity e and let $\mathfrak{g}$ be the Lie algebra for G . For $\mathrm{X} \in \mathfrak{g}$ there exists a unique curve $\gamma: \mathbb{R} \rightarrow G$ satisfying

1. $\gamma(0)=e$
2. $\gamma_{*} \frac{d}{d t}=\left.X\right|_{\gamma(t)}$
3. $\gamma(t+s)=\gamma(t) \gamma(s)$.

The curve $\gamma$ is the unique maximal integral curve of X through e. Furthermore item (3) tells us that $\gamma$ is a 1-parameter subgroup of G. (see [7] Corollary 1.5)

We use these unique integral cures to define the exponential map.

Definition. Exponential Map - Let G be a Lie group with associated Lie algebra, $\mathfrak{g}$ and identity
e. Let $X$ be a left invariant vector field on $G$. Let $\gamma$ be the integral curve of $X$ through $e$ given by Theorem 3. Define the exponential map to be $\exp : \mathfrak{g} \rightarrow G$ by $\exp (X)=\gamma(1)$.

Corollary 4. For each $X \in \mathfrak{g}$. The mapping $\phi: \mathbb{R} \rightarrow G$ by $\phi(\mathrm{t})=\exp (\mathrm{t} X)$ is the unique 1-parameter subgroup of G with $\phi(0)=e$ and $\left.\phi_{*} \frac{\mathrm{~d}}{\mathrm{dt}}\right|_{\mathrm{t}=0}=\left.\mathrm{X}\right|_{e}$.

The following theorems lay out a few fundamental properties of the exponential map.

Theorem 5. There exists an open set $\mathrm{U} \subset \mathrm{G}, \mathrm{e} \in \mathrm{U}$ and an open set $\mathrm{V} \subset \mathfrak{g}, 0 \in \mathrm{~V}$ where $\exp : \mathrm{V} \rightarrow \mathrm{U}$ is a diffeomorphism of V onto U . (see [9] 2.10.1)

Theorem 6. $\exp (0)=e$.
Proof. Let $\gamma: \mathbb{R} \rightarrow \mathrm{G}$ by $\gamma(\mathrm{t})=\mathrm{e}$. Let $\mathrm{X}=0 \in \mathfrak{g}$, then $\gamma$ is the unique integral curve of X through e. Therefore $\exp (0)=\gamma(1)=e$.

We now use the exponential map to define a coordinate chart about $e$ in $G$. Let $\left\{e_{i}\right\}_{1 \leq i \leq m}$ be a basis for $\mathfrak{g}$ and define a mapping $\lambda: \mathbb{R}^{\mathfrak{m}} \rightarrow \mathfrak{g}$ by

$$
\lambda\left(x^{1}, \ldots, x^{m}\right)=x^{i} e_{i}
$$

The function $\lambda$ defines a topology and a differential structure on $\mathfrak{g}$ by defining $\mathrm{U} \subset \mathfrak{g}$ is open iff $\lambda^{-1}(\mathrm{U})$ is open. The pair $\left(\mathbb{R}^{m}, \lambda^{-1}\right)$ gives a chart on all of $\mathfrak{g}$.

Combining the functions exp and $\lambda$ we get a function $\psi_{1}=(\exp \circ \lambda): \mathbb{R}^{m} \rightarrow G$ by

$$
\begin{equation*}
\psi_{1}\left(x^{1}, \ldots, x^{m}\right)=\exp \left(x^{i} e_{i}\right) \tag{3.1}
\end{equation*}
$$

Theorem 7. There is an open set $\mathrm{U} \subset \mathrm{G}$ and $\mathrm{W} \subset \mathbb{R}^{\mathrm{m}}, 0 \in \mathrm{~W}$ with $\psi_{1}: \mathrm{W} \rightarrow \mathrm{U}$ being a diffeomorphism making $\left(\mathrm{U}, \psi_{1}^{-1}\right)$ a coordinate chart on G with $\psi_{1}(0)=e$.

Proof. By Theorem 5 there are open sets $\mathrm{U} \subset \mathrm{G}$ and $\mathrm{V} \subset \mathfrak{g}, 0 \in \mathrm{~V}$ with $\exp (\mathrm{V})=\mathrm{U}$ a diffeomorphism. Let $W=\lambda^{-1}(V)$. If we restrict $\psi_{1}$ to $W$ we get $\psi_{1}: W \rightarrow U$ being a diffeomorphism. We also have $\psi_{1}(0)=\exp (\lambda(0))=e$.

Definition. The local coordinates $\left(\mathrm{U}, \psi_{1}^{-1}\right)$ on $G$ are called Canonical Coordinates of the $1^{\text {st }}$ Kind.

The following theorem shows that coordinate charts constructed in this way are (essentially) unique up to a change of basis.

Theorem 8. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $\left\{e_{i}\right\}_{1 \leq i \leq m}$ and $\left\{\boldsymbol{f}_{\mathfrak{i}}\right\}_{1 \leq i \leq m}$ be two basis for $\mathfrak{g}$. Let $A$ be the change of basis matrix with $A_{i}^{j} f_{j}=e_{i}$. By Theorem 7 there is a coordinate chart $\left(\mathrm{U}_{1}, \psi_{1}^{-1}\right)$ on G with $\mathrm{e} \in \mathrm{U}_{1}$ where $\psi_{1}$ is defined in equation (3.1). Similarly let $\left(\mathrm{V}_{1}, \phi_{1}^{-1}\right)$ be another coordinate chart on G with $\mathrm{e} \in \mathrm{V}_{1}$ where $\phi_{1}$ is defined in the same way as $\psi_{1}$ but with $\lambda\left(x^{1}, \ldots, x^{m}\right)=x^{i} f_{i}$, using the basis $\left\{f_{i}\right\}$, so that $\phi_{1}\left(x^{1}, \ldots, x^{m}\right)=\exp \left(x^{i} f_{i}\right)$. Let $W_{1}=\psi_{1}^{-1}\left(U_{1} \cap V_{1}\right)$ and $\mathrm{Z}_{1}=\phi_{1}^{-1}\left(\mathrm{U}_{1} \cap \mathrm{~V}_{1}\right)$. Let $\left(\mathrm{x}^{1}, \ldots, \mathrm{x}^{\mathrm{m}}\right) \in \mathrm{W}_{1}$. Then the change of coordinates function, $\rho=$ $\phi_{1}^{-1} \circ \psi_{1}: W_{1} \rightarrow Z_{1}$ is given by $\rho\left(x^{1}, \ldots, x^{m}\right)^{j}=A_{i}^{j} x^{i}$.

Proof. Let $\left(y^{1}, \ldots, y^{m}\right)=\rho\left(x^{1}, \ldots, x^{m}\right)$. We have $\psi_{1}\left(x^{1}, \ldots, x^{m}\right)=\phi_{1}\left(y^{1}, \ldots, y^{m}\right) \Rightarrow \exp \left(x^{i} e_{i}\right)=$ $\exp \left(y^{j} f_{j}\right)$. Since exp is a diffeomorphism on $U_{1} \cap V_{1}$ we have $x^{i} e_{i}=y^{j} f_{j} \Rightarrow x^{i} A_{i}^{j} f_{j}=y^{j} f_{j} \Rightarrow x^{i} \mathcal{A}_{i}^{j}=$ $y^{j}$, the desired result.

A well studied problem in Lie theory is when $\exp : \mathfrak{g} \rightarrow \mathrm{G}$ is a diffeomorphism. The answer to this is described as follows. A Lie algebra $\mathfrak{g}$ is called exponential if for some (and therefore each) Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and each root $\lambda: \mathfrak{h}_{\mathbb{C}} \rightarrow \mathbb{C}$ we have $\lambda(\mathfrak{h}) \cap 2 \pi i \mathbb{Z} \backslash\{0\}=\emptyset$ (see [11] for the following theorem).

Theorem 9. When G is simply connected the following are equivalent:

1. the exponential map is a diffeomorphism
2. the exponential map is injective
3. $\mathfrak{g}$ is exponential.

This theorem tells us that for an exponential Lie algebra, $\mathfrak{g}$, there is an associated simply connected Lie group, $G$ having $\mathfrak{g}$ as its Lie algebra and $G$ is diffeomorphic to $\mathbb{R}^{\mathfrak{m}}$, with the exponential map giving us coordinates.

When $\mathfrak{g}$ is a nilpotent Lie algebra it is itself a Cartan subalgebra with no non-zero roots $\lambda$ and is therefore exponential. This gives the following theorem.

Theorem 10. Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension $\mathfrak{n}$ and G its associated simply connected Lie group. The function $\exp : \mathfrak{g} \rightarrow \mathrm{G}$ is a diffeomorphism. Furthermore given any basis $\left\{\boldsymbol{e}_{\mathrm{i}}\right\}_{1 \leq i \leq n}$ for $\mathfrak{g}$ the function $\psi_{1}: \mathbb{R}^{n} \rightarrow G$, defined above, defines coordinates of the $1^{\text {st }}$ kind. Any other coordinates of the $1^{\text {st }}$ kind are related by a linear change of variables. (See [9] Theorem 3.6.2 for the first part and Theorem 8 for the second.)
3.1.1

## Heisenberg Example

We show how the exponential map gives coordinates of the $1^{\text {st }}$ kind for the Heisenberg group, $H$, continued from 2.1. Let $X=c^{1} \partial_{x^{1}}+\left(c^{2}+c^{3} x^{1}\right) \partial_{x^{2}}+c^{3} \partial_{x^{3}}$, which is a generic left invariant vector field on H as given in equation (2.5).

Let $\gamma(\mathrm{t}): \mathbb{R} \rightarrow \mathrm{H}$. We want $\frac{\mathrm{d}}{\mathrm{dt}} \gamma(\mathrm{t})=\left(\mathrm{c}^{1}, \mathrm{c}^{2}+\mathrm{c}^{3} \chi^{1}, \mathrm{c}^{3}\right)$ and $\gamma(0)=(0,0,0)$ so that $\gamma$ is the integral curve for the vector field $X$ passing through the identity.

Integrating gives $\gamma(t)=\left(c^{1} t, c^{2} t+\frac{1}{2} c^{3} c^{1} t^{2}, c^{3} t\right)$. Since $\gamma$ is the unique integral curve for $X$ passing through the identity we have

$$
\begin{equation*}
\exp (X)=\gamma(1)=\left(x^{1}=c^{1}, x^{2}=c^{2}+\frac{1}{2} c^{3} c^{1}, x^{3}=c^{3}\right) \tag{3.2}
\end{equation*}
$$

giving us a mapping $\psi_{1}: \mathbb{R}^{3} \rightarrow H$ by $\psi_{1}\left(c^{1}, c^{2}, c^{3}\right)=\exp (X)$ and defining a coordinate chart $\left(\mathrm{H}, \psi_{1}^{-1}\right)$ on all of H in coordinates of the $1^{\text {st }}$ kind.

## 3.2

## Baker Campbell Hausdorff Formula

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, multiplication $\mu$ and identity e. According to Theorem 5 there is an open set $\mathrm{V} \subset \mathfrak{g}$ and an open set $\mathrm{U} \subset \mathrm{G}, \mathrm{e} \in \mathrm{U}$, where $\exp : \mathrm{V} \rightarrow \mathrm{U}$ is a diffeomorphism. Let $\log : \mathrm{U} \rightarrow \mathrm{V}$ be $\exp ^{-1}$. Let $\mathrm{W}=\mu^{-1}(\mathrm{U}) \subset \mathrm{G} \times \mathrm{G}$. There is an open set $\mathrm{U}_{1} \subset \mathrm{U} \subset \mathrm{G}$ where $\mathrm{U}_{1} \times \mathrm{U}_{1} \subset \mathrm{~W}$. Let $\mathrm{V}_{1}=\log \left(\mathrm{U}_{1}\right)$. For $\mathrm{X}, \mathrm{Y} \in \mathrm{V}_{1}$ we can define:

$$
\begin{equation*}
Z=\log (\exp X \exp Y) \tag{3.3}
\end{equation*}
$$

The BCH formula provides a power series representation of the right side of equation (3.3) written out explicitly in terms of the Lie bracket on $\mathfrak{g}$, and is the following:

Theorem 11. There is an open neighborhood V of $0 \in \mathfrak{g}$ where for $\mathrm{X}, \mathrm{Y} \in \mathrm{V}$ the series:

$$
\begin{equation*}
Z=X+\sum_{\substack{k, m \geq 0 \\ r_{i}+s_{i}>0}} \frac{(-1)^{k}}{(k+1)\left(1+\sum_{i=1}^{k} s_{i}\right)} \frac{(a d X)^{r_{1}}}{r_{1}!} \frac{(a d Y)^{s_{1}}}{s_{1}!} \cdots \frac{(a d X)^{r_{k}}}{r_{k}!} \frac{(a d Y)^{s_{k}}}{s_{k}!} \frac{(a d X)^{m}}{m!} Y \tag{3.4}
\end{equation*}
$$

converges and satisfies $Z=\log (\exp X \exp Y)$, where $\mathrm{adX}, \mathrm{adY}: \mathfrak{g} \rightarrow \mathfrak{g}$ are the endomorphisms $\operatorname{adX}(\mathrm{Z})=[\mathrm{X}, \mathrm{Z}], \operatorname{adY}(\mathrm{Z})=[\mathrm{Y}, \mathrm{Z}]$. (See [8] Proposition 9.2.32)

Two alternative ways of writing the BCH formula are:

$$
Z=Y+X+\sum_{\substack{k>0 \\ r_{i}+s_{i}>0}} \frac{(-1)^{k}}{(k+1)\left(1+\sum_{i=1}^{k} r_{i}\right)} \frac{(a d X)^{r_{1}}}{r_{1}!} \frac{(a d Y)^{s_{1}}}{s_{1}!} \cdots \frac{(a d X)^{r_{k}}}{r_{k}!} \frac{(a d Y)^{s_{k}}}{s_{k}!} X
$$

(See [4] formula 1.7.3) and

$$
Z=X+\sum_{\substack{k>0 \\ r_{i}+s_{i}>0}} \frac{(-1)^{k}}{k\left(\sum_{i=1}^{k} r_{i}+s_{i}\right)} \frac{(a d X)^{r_{1}}}{r_{1}!} \frac{(a d Y)^{s_{1}}}{s_{1}!} \cdots \frac{(a d X)^{r_{k}}}{r_{k}!} \frac{(a d Y)^{s_{k}}}{s_{k}!} W
$$

where $W$ is either $X$ or $Y$. (See Encyclopedia of Mathematics - Campbell-Hausdorff formula)

In order to implement (3.4) in Maple we need to write the sum as an iterated sum. To do so, we first observe that the sum consists of polynomials in the non-commutative endomorphisms, adX and adY. We use this observation to write (3.4) as an iterated sum where one summation is over the degree of the polynomial. With this in mind we introduce some notation. Let $s_{n}=1$ and let

$$
\begin{align*}
& {\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] } \\
= & (a d X)^{r_{1}}(a d Y)^{s_{1}} \ldots(a d X)^{r_{n-1}}(a d Y)^{s_{n-1}}(a d X)^{r_{n}}(Y) \\
= & {[X,[X, \ldots[X,[Y,[Y, \ldots[Y, \ldots[X,[X, \ldots[X, Y]] \ldots],} \tag{3.5}
\end{align*}
$$

where there are $r_{1} X^{\prime}$ 's followed by $s_{1} Y^{\prime}$ 's up to $r_{n} X^{\prime}$ 's followed by $s_{n} Y^{\prime}$ s and $\left[Y^{1}\right]=Y$.

For example the non-zero terms in equation (3.4) of degree $d=3$ are then

$$
\begin{align*}
& \frac{1}{2}\left[X^{2} Y\right],-\frac{1}{2}\left[X Y^{0} X Y\right],-\frac{1}{4}\left[X^{2} Y^{0} X^{0} Y\right] \\
& \frac{1}{3}\left[X, Y^{0}, X Y^{0} X^{0} Y\right],-\frac{1}{4}\left[X^{0} Y X Y\right] \text { and } \frac{1}{6}\left[X^{0}, Y, X Y^{0} X^{0} Y\right] \tag{3.6}
\end{align*}
$$

which in Lie brackets are

$$
\begin{aligned}
& \frac{1}{2}[X,[X, Y]],-\frac{1}{2}[X,[X, Y]],-\frac{1}{4}[X,[X, Y]] \\
& \frac{1}{3}[X,[X, Y]],-\frac{1}{4}[Y,[X, Y]] \text { and } \frac{1}{6}[Y,[X, Y]] .
\end{aligned}
$$

Let $\left(t_{m}, \ldots, t_{1}\right)$ be a list of natural numbers with $t_{1}>0$. Let

$$
W\left(t_{m}, \ldots, t_{1}\right)= \begin{cases}{\left[X^{t_{m}} Y^{t_{m-1}} X^{t_{m-2}} Y^{t_{m}-3} \ldots X^{t_{2}} Y^{t_{1}}\right],} & m \text { even }  \tag{3.7}\\ {\left[Y^{t_{m}} X^{t_{m-1}} Y^{t_{m-2}} X^{t_{m-3}} \ldots X^{t_{2}} Y^{t_{1}}\right],} & m \text { odd }\end{cases}
$$

For any list of natural numbers $\left(r_{1}, s_{1}, r_{2}, s_{2}, \ldots r_{n}, s_{n}\right)$ with $s_{n}>0$ there is a unique list of positive integers $\left(t_{m}, \ldots, t_{1}\right)$ so that $W\left(t_{m}, \ldots, t_{1}\right)$ and $W\left(r_{1}, s_{1}, r_{2}, s_{2}, \ldots r_{n}, s_{n}\right)$ represent the same non-commutative polynomial in adX and adY. To see this take the ordered list of symbols $\left\langle r_{1}, s_{1}, r_{2}, s_{2}, \ldots r_{n}, s_{n}\right\rangle$ and remove any which are equal to zero. In the remaining list if an $r_{i}$ is adjacent to an $r_{j}$, let $r_{i}=r_{i}+r_{j}$ and remove $r_{j}$ from the list. Do this for the $s_{i}$ terms also and continue until we have an alternating list $<\ldots r_{i}, s_{j}, r_{k}, \ldots>$ with no terms equal to zero. Relabel these symbols $<t_{m}, \ldots, t_{1}>$ giving $W\left(t_{m}, \ldots, t_{1}\right)$, representing the same polynomial as $W\left(r_{1}, s_{1}, r_{2}, s_{2}, \ldots r_{n}, s_{n}\right)$. Let $d=\sum_{1}^{m} t_{i}$, the degree of the polynomial then $\left(t_{m}, \ldots, t_{1}\right)$ is a composition of $d$, meaning $d=\sum_{1}^{m} t_{i}$ where each $t_{i}$ is a positive integer.

For example the terms in equation (3.4) of degree $\mathbf{d}=3$, listed in (3.6) (in the same order) are then

$$
\frac{1}{2} W(2,1),-\frac{1}{2} W(2,1),-\frac{1}{4} W(2,1), \frac{1}{3} W(2,1),-\frac{1}{4} W(1,1,1), \frac{1}{6} W(1,1,1)
$$

Let $\left(a_{n_{1}}, \ldots a_{1}\right)$ and $\left(b_{n_{2}}, \ldots b_{1}\right)$ be ordered lists of natural numbers. Define an equivalence relation $\sim$ so that $\left(a_{n_{1}}, \ldots a_{1}\right) \sim\left(b_{n_{2}}, \ldots b_{1}\right)$ iff there is a sequence of positive integers $\left(t_{m}, \ldots, t_{1}\right)$ such that $W\left(a_{n_{1}}, \ldots, a_{1}\right), W\left(b_{n_{2}}, \ldots, b_{1}\right)$ and $W\left(t_{m}, \ldots, t_{1}\right)$ represent the same non-commutative polynomial in $a d X$ and $a d Y$. Let

$$
\begin{equation*}
c\left(t_{m}, \ldots, t_{1}\right)=\sum_{\substack{r_{i}+s_{i}>0 \\\left(r_{1}, s_{1}, r_{2}, s_{2}, \ldots, r_{n}, s_{n}\right) \sim\left(t_{m}, \ldots, t_{1}\right)}} \frac{(-1)^{n-1}}{n\left(\sum_{i=1}^{n} s_{i}\right) \cdot \prod_{i=1}^{n} r_{i}!s_{i}!} . \tag{3.8}
\end{equation*}
$$

For example, for the terms in equation (3.4) of degree $d=3$ we have

$$
\begin{aligned}
c(2,1) & =\frac{1}{2}-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}=\frac{1}{12} \text { and } \\
c(1,1,1) & =-\frac{1}{4}-\frac{1}{6}=-\frac{1}{12} .
\end{aligned}
$$

We can now write equation (3.4) as an iterated sum over the degree of the polynomial with the inner sum being over all compositions of $d$ where $d$ is the degree of the polynomial.

Theorem 12. With $W$ as defined in formula (3.7) and c as defined in formula (3.8), Z from Theorem 11 is given by

$$
\begin{equation*}
Z=X+\sum_{d=1}^{\infty} \sum_{\substack{\text { compositions } \\\left(t_{1}, \ldots, t_{m}\right) \\ \text { with } \\ \text { with } t_{1}=1}} c\left(t_{m}, \ldots, t_{1}\right) W\left(t_{m}, \ldots, t_{1}\right) . \tag{3.9}
\end{equation*}
$$

Equation (3.9) with all terms up to degree $d=3$ written explicitly is:

$$
\begin{aligned}
Z & =X+c(1) W(1)+c(1,1) W(1,1)+c(1,2) W(1,2)+c(1,1,1) W(1,1,1)+\ldots \\
& =X+Y+\frac{1}{2}[X, Y]+\frac{1}{12}([X,[X, Y]]-[Y,[X, Y]])+\ldots
\end{aligned}
$$

Proof. Starting with equation (3.4), using the notation from (3.5) and letting $n=k+1$ we get

$$
\begin{align*}
Z & =X+\sum_{\substack{k, m \geq 0 \\
r_{i}+s_{i}>0}} \frac{(-1)^{k}}{(k+1)\left(1+\sum_{i=1}^{k} s_{i}\right)} \frac{(a d X)^{r_{1}}}{r_{1}!} \frac{(a d Y)^{s_{1}}}{s_{1}!} \cdots \frac{(a d X)^{r_{k}}}{r_{k}!} \frac{(a d Y)^{s_{k}}}{s_{k}!} \frac{(a d X)^{m}}{m!} Y \\
& =X+\sum_{\substack{k, m \geq 0 \\
r_{i}+s_{i}>0}} \frac{(-1)^{k}}{(k+1)\left(1+\sum_{i=1}^{k} s_{i}\right)} \frac{\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{k}} Y^{s_{k}} X^{m} Y^{1}\right]}{\prod_{i=1}^{m} r_{i}!s_{i}!} \\
& =X+\sum_{\substack{n>0 \\
r_{i}+s_{i}>0}} \frac{(-1)^{n-1}}{n\left(\sum_{i=1}^{n} s_{i}\right) \cdot \prod_{i=1}^{n} r_{i}!s_{i}!}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] \tag{3.10}
\end{align*}
$$

Let $\mathrm{I}=\left(\mathrm{r}_{1}, \mathrm{~s}_{1}, \ldots, \mathrm{r}_{\mathrm{n}}, \mathrm{s}_{\mathrm{n}}\right)$ and let

$$
\begin{equation*}
B_{I}=\frac{(-1)^{n-1}}{n\left(\sum_{i=1}^{n} s_{i}\right) \cdot \prod_{i=1}^{n} r_{i}!s_{i}!} \tag{3.11}
\end{equation*}
$$

This allows equation (3.10) to be rewritten as:

$$
\begin{equation*}
Z=\log (\exp X \exp Y)=X+\sum_{\substack{n>0 \\ r_{i}+s_{i}>0}} B_{I}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] . \tag{3.12}
\end{equation*}
$$

Let $d=\sum_{i=1}^{n} r_{i}+s_{i}$, where we think of $d$ as the degree of the polynomial term $\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}}=1\right]$ in equation (3.12) in terms of the non commutative variables $X$ and $Y$. By the definition of multiple
sums, equation (3.12) can be written

$$
\begin{align*}
Z & =X+\lim _{k, l \rightarrow \infty} \sum_{n=1}^{k} \sum_{\substack{d=n \\
r_{i}+s_{i}>0}}^{l} B_{I}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] \\
& =X+\lim _{k, l \rightarrow \infty} \sum_{\substack{d=1 \\
r_{i}+s_{i}>0}}^{l} \sum_{n=1}^{m i n\{k, d\}} B_{I}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] \\
& =X+\lim _{l \rightarrow \infty} \sum_{\substack{d=1 \\
r_{i}+s_{i}>0}}^{l} \sum_{n=1}^{d} B_{I}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] \\
& =X+\sum_{d=1}^{\infty} \sum_{\substack{n=1 \\
r_{i}+s_{i}>0}}^{d} B_{I}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] \tag{3.13}
\end{align*}
$$

noting that $n>d$ is impossible based on the restriction that $r_{i}+s_{i}>0$ and the fact that terms where $k=d$ must go to the same limit as the full sequence of terms.

Using the notation (3.11) we also have

$$
c\left(t_{m}, \ldots, t_{1}\right)=\sum_{\substack{r_{i}+s_{i}>0 \text { and } \\\left(r_{1}, s_{1}, r_{2}, s_{2}, \ldots, r_{n}, s_{n}\right) \sim\left(t_{m}, \ldots, t_{1}\right)}} B_{I}
$$

allowing us to rewrite equation (3.13) as

$$
\begin{aligned}
& Z=X+\sum_{d=1}^{\infty} \sum_{\substack{n=1 \\
r_{i}+s_{i}>0}}^{d} B_{I}\left[X^{r_{1}} Y^{s_{1}} X^{r_{2}} Y^{s_{2}} \ldots X^{r_{n}} Y^{s_{n}=1}\right] \\
& =X+\sum_{d=1}^{\infty} \sum_{\begin{array}{c}
\text { compositions } \\
\left(t_{1}, \ldots, t_{m}\right) \\
\text { of d } \\
\text { with } t_{1}=1 \\
r_{i}+s_{i}>0 \text { and } \\
\left(r_{1}, s_{1}, r_{2}, s_{2}, \ldots, r_{n}, s_{n}\right) \sim\left(t_{m}, \ldots, t_{1}\right)
\end{array}} B_{I} W\left(t_{m}, \ldots, t_{1}\right) \\
& =X+\sum_{d=1}^{\infty} \sum_{\substack{\text { ompositions } \\
\left(t_{1}, \ldots, t_{m}\right) \\
\text { of } \\
\text { with } t_{1}=1}} c\left(t_{m}, \ldots, t_{1}\right) W\left(t_{m}, \ldots, t_{1}\right) .
\end{aligned}
$$

establishing the theorem.

BCH - The Nilpotent Case

One important corollary of Theorem 12 occurs when $\mathfrak{g}$ is a nilpotent Lie algebra.
Corollary 13. Let $\mathfrak{g}$ be $k$-step nilpotent, then $\mathrm{W}\left(\mathrm{t}_{\mathrm{m}}, \ldots, \mathrm{t}_{1}\right)=0$ when $\mathrm{d} \geq \mathrm{k}$ and

$$
Z=X+\sum_{d=1}^{k-1} \sum_{\substack{\text { compositions } \\\left(t_{1}, \ldots, t_{m}\right) \\ \text { off } \\ \text { with } t_{1}=1}} c\left(t_{m}, \ldots, t_{1}\right) W\left(t_{m}, \ldots, t_{1}\right)
$$

Furthermore if we write $\mathrm{Z}, \mathrm{X}$ and Y in a basis $\left\{\mathfrak{e}_{i}\right\}_{1 \leq i \leq m}$ for $\mathfrak{g}$ so that $\mathrm{Z}=z^{i} e_{i}, \mathrm{Y}=y^{i} e_{i}$ and $X=x^{i} e_{i}$ then

$$
\begin{equation*}
z^{i}=x^{i}+\left(\sum_{\substack{d=1}}^{k-1} \sum_{\substack{\text { compositions } \\ \left.\text { (t. } 1, \ldots, t_{m}\right) \\ \text { ort } \\ \text { with } t_{1}=1}} c\left(t_{m}, \ldots, t_{1}\right) W\left(t_{m}, \ldots, t_{1}\right)\right)^{i} . \tag{3.14}
\end{equation*}
$$

Let G be a simply connected Lie group associated with $\mathfrak{g}$ then equation (3.14) gives a global representation of the group multiplication in coordinates of the $1^{\text {st }}$ kind.

Formula (3.14) of Corollary 13 is implemented in Appendix A by the code 4.5.2 to produce the multiplication function for a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$.
3.3.1

## Examples 3.3.1

Example 3.3.1.1 The Heisenberg Lie algebra defined in equation (2.2) is 1 -step Nilpotent. On account of that, formula (3.4) gives us

$$
Z=X+Y+\frac{1}{2}[X, Y] .
$$

Using the standard basis $\left\{e_{1}, e_{2}, e_{3}\right\}$, the Heisenberg Lie algebra is given by the equations in (2.3). Formula (3.14) then gives:

$$
z_{1}=x^{1}+y^{1}, z_{2}=x^{2}+y^{2}+\frac{1}{2}\left(x^{1} y^{3}-x^{3} y^{1}\right), z_{3}=x^{3}+y^{3} .
$$

Example 3.3.1.2 In 4.5.8 of Appendix B a more complicated example computing formula (3.14) is shown.

Left Invariant Vector Fields in Canonical Coordinates of the $1^{\text {st }}$ Kind

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, multiplication $\mu$ and identity $e$. We will express a basis for $\mathfrak{g}$ in coordinates of the $1^{\text {st }}$ kind and show how they can be calculated by use of the BCH formula with the ultimate goal of computing the vector fields in Maple for $\mathfrak{g}$ nilpotent.

Theorem 14. Let G be a Lie group and $\left\{\mathrm{e}_{\mathrm{i}}\right\}_{1 \leq i \leq m}$ be a basis for the left invariant vector fields on G. Let $\left(\mathrm{U}, \Psi_{1}^{-1}\right)$ be a coordinate chart in coordinates of the $1^{\text {st }}$ kind as defined in Theorem 7. There is an open set $\mathrm{W} \subset \psi_{1}^{-1}(\mathrm{U}), 0 \in W$ where for $\mathrm{x}=\left(\mathrm{x}^{1}, \ldots, \mathrm{x}^{\mathrm{m}}\right) \in \mathrm{W},\left.\widehat{\mathrm{X}}_{\mathrm{k}}\right|_{x}=\left.\psi_{1_{*}}^{-1} \mathrm{e}_{\mathrm{k}}\right|_{\psi_{1}(x)}$ is given by

$$
\left[\begin{array}{c}
\left.\hat{X}_{1}\right|_{\left(x^{1}, \ldots, x^{m}\right)}  \tag{3.15}\\
\vdots \\
\left.\hat{X}_{m}\right|_{\left(x^{1}, \ldots, x^{m}\right)}
\end{array}\right]=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n}\left(\left[\operatorname{ad} x^{i} e_{i}\right]^{\top}\right)^{n}\left[\begin{array}{c}
\partial_{x^{\prime}} \\
\vdots \\
\partial_{x^{m}}
\end{array}\right]
$$

where $B_{n}$ is the $n^{\prime}$ th Bernoulli number in the standard notation $\left(B_{1}=-\frac{1}{2}\right)$ and $\left[a d x^{i} e_{i}\right]$ is the matrix representation of $\operatorname{ad} \chi^{i} e_{i}$ in the given basis.

Note that the exponential generating function for the Bernoulli numbers is

$$
\begin{equation*}
\frac{t}{e^{t}-t^{0}}=\sum_{n=0}^{\infty} \frac{t^{n}}{n!} B_{n} \tag{3.16}
\end{equation*}
$$

allowing us to write formula (3.15) as

$$
\left[\begin{array}{c}
\left.\hat{X}_{1}\right|_{\left(x^{1}, \ldots, x^{m}\right)}  \tag{3.17}\\
\vdots \\
\left.\hat{X}_{m}\right|_{\left(x^{1}, \ldots, x^{m}\right)}
\end{array}\right]=\frac{\left[a d x^{i} e_{i}\right]^{\top}}{I-e^{-\left[a d x^{i} e_{i}\right]^{\top}}}\left[\begin{array}{c}
\partial_{x^{1}} \\
\vdots \\
\partial_{x^{m}}
\end{array}\right]
$$

where $t=-\left[a d x^{i} e_{i}\right]^{\top}$ in equation (3.16). Formula (3.17) allows us to compute the vector fields in some cases where the summation in equation (3.15) is infinite. We give an example of this in 3.4.1. Function (3.16) has been well studied, for example the radius of convergence for (3.16) is $2 \pi$. Reference [4], in Lemma 1.8.2, gives a formula for right invariant vector fields which is similar to formula (3.17).

Proof. By Theorem (11) there is an opens set $V \subset \mathfrak{g}$ with exp being a diffeomorphism from $V \rightarrow$ $\mathrm{U}=\exp (\mathrm{V})$. Let $\mathrm{x}=x^{i} e_{i}, y=\left(y^{1}, \ldots, y^{m}\right)$, and $z=\left(z^{1}, \ldots, z^{\mathfrak{m}}\right)$ with $x, y \in W$. Let $X=x^{i} e_{i}$,
$\mathrm{Y}=y^{i} e_{i}$ and $Z=z^{i} e_{i}=\log (\exp X \exp Y)$. Let $\mu_{1}: U \times U \rightarrow \mathbb{R}^{m}$, by

$$
\begin{equation*}
\mu_{1}(x, y)=z=\psi_{1}^{-1}\left(\psi_{1}(x) \psi_{1}(y)\right) \tag{3.18}
\end{equation*}
$$

which is simply the group multiplication expressed in coordinates of the $1^{\text {st }}$ kind. We claim

$$
\left.\hat{X}_{k}\right|_{x}=\left.\partial_{y^{k}} \mu_{1}(x, y)^{a} \partial_{x}\right|_{y=0}
$$

To show this let $F_{x}: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ by

$$
\begin{equation*}
F_{x}(y)=\mu_{1}(x, y) \tag{3.19}
\end{equation*}
$$

giving

$$
\left.F_{x *}\right|_{y=0}\left(\partial_{y^{k}}\right)=\left.\partial_{y^{k}} \mu_{1}(x, y)^{a} \partial_{x^{a}}\right|_{y=0}
$$

Since $F_{x}(y)=\mu_{1}(x, y)=\psi_{1}^{-1}\left(\psi_{1}(x) \psi_{1}(y)\right)$ we have

$$
\begin{gathered}
F_{x *}=\psi_{1 *}^{-1} \circ L_{\psi_{1}(x) *} \circ \psi_{1 *} \\
\left.\Rightarrow F_{x *}\right|_{y=0}\left(\partial_{y^{k}}\right)=\left.\psi_{1 *}^{-1} \circ L_{\psi_{1}(x) *} e_{k}\right|_{e}=\left.\psi_{1 *}^{-1} e_{k}\right|_{\psi_{1}(x)}=\left.\widehat{X}_{k}\right|_{x} \\
\left.\Rightarrow \hat{X}_{k}\right|_{x}=\left.F_{x *}\right|_{y=0}\left(\partial_{y^{k}}\right)=\left.\partial_{y^{k}} \mu_{1}(x, y)^{a} \partial_{x^{a}}\right|_{y=0}
\end{gathered}
$$

We now use equation (3.9) to explicitly calculate $\left.\widehat{X}_{k}\right|_{x}$. Since we are going to set $y=0$ we only need the terms in the formula with exactly one Y. First we note that terms in equation (3.9) with exactly one $Y$ are of the form $c(n, 1) W(n, 1)$. Let $K$ be the sum of terms with more than one Y. Formula (3.9) becomes

$$
\begin{equation*}
Z=X+Y+\sum_{n=1}^{\infty} c(n, 1) W(n, 1)+K \tag{3.20}
\end{equation*}
$$

Let $c_{a i}^{j}$ be the structure constants for $\mathfrak{g}$ in the basis $\left\{e_{i}\right\}$ so that

$$
[\operatorname{ad} \mathbf{x}]_{k}^{a}=x^{i} c_{i k}^{a}
$$

Let $i_{a}, p_{a}$ be distinct indices for each $a$ each taking on values from $1, \ldots, m$. Let $b_{n}=c(n, 1)$. we have

$$
W(n, 1)=[X,[X, \ldots[X, Y]] \ldots]=x^{i_{1}} \cdots x^{i_{n}} y^{i_{n+1}} c_{i_{1} p_{1}}^{i} c_{i_{2} p_{2}}^{p_{1}} \cdots c_{i_{n} i_{n+1}}^{p_{n-1}} e_{i}
$$

and formula (3.20) becomes

$$
\begin{equation*}
z^{i} e_{i}=x^{i} e_{i}+y^{i} e_{i}+\sum_{n=1}^{\infty} b_{n} x^{i_{1}} \cdots x^{i_{n}} y^{i_{n+1}} c_{i_{1} p_{1}}^{i} c_{i_{2} p_{2}}^{p_{1}} \cdots c_{i_{n} i_{n+1}}^{p_{n-1}} e_{i}+K . \tag{3.21}
\end{equation*}
$$

We can express the coefficients $\boldsymbol{b}_{\mathrm{n}}$ in terms of Bernoulli numbers by the following Lemma.
Lemma 1. Let $\mathfrak{c}(\mathrm{n}, 1)$ be defined as in (3.8) then

$$
\begin{equation*}
b_{n}=c(n, 1)=\frac{(-1)^{n}}{n!} B_{n} \tag{3.22}
\end{equation*}
$$

(See [6] Theorem 3)
We can now use (3.21) to write out a formula for $\widehat{X}_{k}$. We have

$$
\mu_{1}(x, y)^{a}=z^{a}=x^{a}+y^{a}+\sum_{n=1}^{\infty} b_{n} x^{i_{1}} \cdots x^{i_{n}} y^{i_{n+1}} c_{i_{1} p_{1}}^{a} c_{i_{2} p_{2}}^{p_{1}} \cdots c_{i_{n} i_{n+1}}^{p_{n}-1}+k^{a}
$$

giving

$$
\begin{aligned}
\left.\hat{X}_{k}\right|_{x} & =\left.\frac{\partial}{\partial y^{k}}\left(x^{a}+y^{a}+\sum_{n=1}^{\infty} b_{n} x^{i_{1}} \cdots x^{i_{n}} y^{i_{n+1}} c_{i_{1} p_{1}}^{a} c_{i_{2} p_{2}}^{p_{1}} \cdots c_{i_{n} i_{n+1}}^{p_{n-1}}\right) \partial_{x^{a}}\right|_{y=0} \\
& =\left(\delta_{k}^{a}+\sum_{n=1}^{\infty} b_{n} x^{i_{1}} \cdots x^{i_{n}} c_{i_{1} p_{1}}^{a} c_{i_{2} p_{2}}^{p_{1}} \cdots c_{i_{n} i_{k}}^{p_{n-1}}\right) \partial_{x^{a}} \\
& =\left(\delta_{k}^{a}+\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!} B_{n}[\operatorname{ad} \mathbf{x}]_{p_{1}}^{a}[a d x]_{p_{2}}^{p_{1}} \cdots[a d x]_{i_{k}}^{p_{n-1}}\right) \partial_{x^{a}}
\end{aligned}
$$

which in matrix notation is (3.15) proving the theorem.

Corollary 15. Let $\mathfrak{g}$ be a Lie algebra where formula (3.9) converges for all elements in $\mathfrak{g} \times \mathfrak{g}$ and the map $\psi_{1}$ is a diffeomorphism $\mathbb{R}^{m} \rightarrow \mathrm{G}$ then the vector fields given by equation (3.15) are left invariant vector fields on $\mathbb{R}^{m}$ with the group multiplication in coordinates of the $1^{\text {st }}$ kind given by $\mu_{1}: \mathbb{R}^{m} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ by

$$
\mu_{1}(x, y)=z=\psi_{1}^{-1}\left(\psi_{1}(x) \psi_{1}(y)\right) .
$$

### 3.4.1

## Example

The following example is a case where we can compute the left invariant vector fields in coordinates of the $1^{\text {st }}$ kind by using equation (3.17) and diagonalization of the adjoint matrix.

Let $\mathfrak{g}$ be a 3 dimensional Lie algebra with basis $\left\{e_{1}, e_{2}, e_{3}\right\}$ and structure equations $\left[e_{1}, e_{2}\right]=$ $0,\left[e_{1}, e_{3}\right]=e_{1}$ and $\left[e_{2}, e_{3}\right]=e_{2}$. We have

$$
\left[\operatorname{ad} x^{i} e_{i}\right]^{\top}=\left[\begin{array}{ccc}
-x^{3} & 0 & 0 \\
0 & -x^{3} & 0 \\
x^{1} & x^{2} & 0
\end{array}\right]
$$

Putting this matrix in Jordan form gives

$$
\left[\begin{array}{ccc}
-x^{3} & 0 & 0 \\
0 & -x^{3} & 0 \\
x^{1} & x^{2} & 0
\end{array}\right]=\mathrm{QJQ}^{-1}
$$

where

$$
Q=\left[\begin{array}{ccc}
0 & \frac{x^{1}-x^{2}}{x^{1}} & -\frac{x^{2}}{x^{1}} \\
0 & 1 & 1 \\
\frac{x^{1}}{x^{3}} & -\frac{x^{1}}{x^{3}} & 0
\end{array}\right]
$$

and

$$
J=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & -x^{3} & 0 \\
0 & 0 & -x^{3}
\end{array}\right]
$$

Since J is diagonal we can now compute

$$
\frac{\left[\operatorname{ad} x^{i} e_{i}\right]^{\top}}{\mathrm{I}-\mathrm{e}^{-\left[\operatorname{ad} x^{i} e_{i}\right]^{\top}}}=\mathrm{Q}\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \frac{-x^{3}}{1-\exp \left(x^{3}\right)} & 0 \\
0 & 0 & \frac{-x^{3}}{1-\exp \left(x^{3}\right)}
\end{array}\right] \mathrm{Q}^{-1}
$$

which by equation (3.17) gives

$$
\left[\begin{array}{c}
\left.\hat{X}_{1}\right|_{\left(x^{1}, x^{2}, x^{3}\right)} \\
\left.\hat{X}_{2}\right|_{\left(x^{1}, x^{2}, x^{3}\right)} \\
\left.\hat{X}_{3}\right|_{\left(x^{1}, x^{2}, x^{3}\right)}
\end{array}\right]=\left[\begin{array}{c}
\frac{-x^{3}}{1-\exp \left(x^{3}\right)} \partial_{x^{1}} \\
\frac{-x^{3}}{1-\exp \left(x^{3}\right)} \partial_{x^{2}} \\
\frac{-x^{1}\left(1-\exp \left(x^{3}\right)+x^{3}\right)}{-x^{3}\left(1-\exp \left(x^{3}\right)\right)} \partial_{x^{1}}+\frac{-x^{2}\left(1-\exp \left(x^{3}\right)+x^{3}\right)}{-x^{3}\left(1-\exp \left(x^{3}\right)\right)} \partial_{x^{2}}+\partial_{x^{3}}
\end{array}\right] .
$$

Left Invariant Vector Fields in Canonical Coordinates of the $1^{\text {st }}$ Kind - Nilpotent Case

One important case where Corollary 15 applies is when $\mathfrak{g}$ is a nilpotent Lie algebra.

Corollary 16. Let $\mathfrak{g}$ be $k$-step nilpotent. We have $\left[\mathrm{ad} \chi^{i} \mathrm{e}_{\mathfrak{i}}\right]^{k}=0$ so that (3.15) has only finitely many terms allowing for exact calculation of the vector fields.

In the case Corollary 16 applies, formula (3.15) is implemented in Appendix A by the code 4.5.3.
3.5.1

## Examples 3.5.1

Example 3.5.1.1 As an example consider the Heisenberg Lie algebra. Using the standard basis $\left\{e_{1}, e_{2}, e_{3}\right\}$, the Heisenberg Lie algebra is given by the equations in (2.3). The transpose of the ad matrix is then given by:

$$
\left[\operatorname{ad} x^{i} e_{i}\right]^{T}=\left[\begin{array}{ccc}
0 & -x^{3} & 0 \\
0 & 0 & 0 \\
0 & x^{3} & 0
\end{array}\right]
$$

Therefore formula (3.15) gives:

$$
\begin{aligned}
{\left[\begin{array}{l}
\left.\widehat{X}_{1}\right|_{\left(x^{1}, x^{2}, x^{3}\right)} \\
\left.\widehat{X}_{2}\right|_{\left(x^{1}, x^{2}, x^{3}\right)} \\
\left.\widehat{X}_{3}\right|_{\left(x^{1}, x^{2}, x^{3}\right)}
\end{array}\right] } & =\sum_{n=0}^{1} \frac{(-1)^{n}}{n!} B_{n}\left[\begin{array}{ccc}
0 & -x^{3} & 0 \\
0 & 0 & 0 \\
0 & x^{1} & 0
\end{array}\right]^{n}\left[\begin{array}{l}
\partial_{x^{1}} \\
\partial_{x^{2}} \\
\partial_{x^{3}}
\end{array}\right] \\
& =\left(\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]-\frac{1}{2}\left[\begin{array}{ccc}
0 & -x^{3} & 0 \\
0 & 0 & 0 \\
0 & x^{1} & 0
\end{array}\right]\right)\left[\begin{array}{c}
\partial_{x^{1}} \\
\partial_{x^{2}} \\
\partial_{x^{3}}
\end{array}\right] \\
& =\left[\begin{array}{c}
\partial_{x^{1}}+\frac{1}{2} x^{3} \partial_{x^{2}} \\
\partial_{x^{2}} \\
\partial_{x^{3}}-\frac{1}{2} x^{1} \partial_{x^{2}}
\end{array}\right] .
\end{aligned}
$$

For $b_{1}, b_{2}, b_{3}, c_{1}, c_{2}, c_{3} \in \mathbb{R}$ we can write generic vector fields

$$
\begin{array}{r}
X=c^{1} \hat{X}_{1}+c^{2} \hat{X}_{2}+c^{3} \hat{X}_{3}=c^{1} \partial_{x^{1}}+\left(c^{2}+c^{3} x^{1}\right) \partial_{x^{2}}+c^{3} \partial_{x^{3}} \\
Y=b^{1} \hat{X}_{1}+b^{2} \hat{X}_{2}+b^{3} \hat{X}_{3}=b^{1} \partial_{x^{1}}+\left(b^{2}+b^{3} x^{1}\right) \partial_{x^{2}}+b^{3} \partial_{x^{3}}
\end{array}
$$

with Lie bracket

$$
[X, Y]=\left(c^{1} b^{3}-b^{1} c^{3}\right) \widehat{X}_{2}
$$

agreeing with the result we saw in equation (2.6).

Example 3.5.1.2 In 4.5.9 of Appendix B a more complicated example computing equation (3.15) is shown.

## CHAPTER 4 CANONICAL COORDINATES OF THE $2^{\text {ND }}$ KIND

In the prior section we saw how the exponential map could be used to define a chart on a neighborhood about the identity for any Lie group. In this section we will use a different method of using the exponential map to give a chart about the identity and these are called coordinates of the $2^{\text {nd }}$ kind. When the Lie group is simply connected and solvable this chart will cover the entire group.

## 4.1

Defining the Coordinates of the $2^{\text {nd }}$ Kind

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, multiplication $\mu$ and identity $e$. Let $\left\{e_{i}\right\}_{1 \leq i \leq m}$ be a basis for $\mathfrak{g}$. Let $\psi_{1}$ be defined as in equation (3.1). Let $\hat{x}^{j}=\left(0, \ldots, 0, x^{j}, 0, \ldots 0\right)$, where $x^{j}$ is the $j$ th position. We define the mapping $\psi_{2}: \mathbb{R}^{m} \rightarrow G$ by

$$
\begin{align*}
\psi_{2}\left(x^{1}, \ldots, x^{m}\right) & =\exp \left(x^{1} e_{1}\right) \exp \left(x^{2} e_{2}\right) \cdots \exp \left(x^{m} e_{m}\right) \\
& =\psi_{1}\left(\hat{x}^{1}\right) \psi_{1}\left(\hat{x}^{2}\right) \cdots \psi_{1}\left(\hat{x}^{m}\right) \tag{4.1}
\end{align*}
$$

It should be noted that the function $\psi_{2}$ utilizes the group product whereas the function $\psi_{1}$ did not. We give an example of this calculation in 4.1.1.

Theorem 17. There is an open set $\mathrm{W} \subset \mathbb{R}^{\mathrm{m}}, 0 \in \mathrm{~W}$ and an open set $\mathrm{U} \subset \mathrm{G}, \mathrm{e} \in \mathrm{U}$ such that $\psi_{2}: \mathrm{W} \rightarrow \mathrm{U}$ is a diffeomorphism, thus making $\left(\mathrm{U}, \psi_{2}^{-1}\right)$ a coordinate chart about e in G

Proof. We have $\left.\mu_{*}\right|_{e}: T_{e} G \times T_{e} G \rightarrow T_{e} G$ with $\left.\mu_{*}\right|_{e}\left(\left.e_{i}\right|_{e}, 0\right)=\mu_{*}\left(0,\left.e_{i}\right|_{e}\right)=\left.e_{i}\right|_{e}$. By induction, $\hat{\mu}: G \times \cdots \times G \rightarrow G$, by repeated group multiplication gives $\left.\hat{\mu}_{*}\right|_{e}: T_{e} G \times \cdots \times T_{e} G \rightarrow T_{e} G$ with $\left.\hat{\mu}_{*}\right|_{e}\left(0, \ldots, 0,\left.e_{i}\right|_{e}, 0, \ldots, 0\right)=\left.e_{i}\right|_{e}$.

Let $\psi_{1}$ be defined as in equation (3.1). We have

$$
\begin{aligned}
\psi_{2}\left(x^{1}, \ldots, x^{m}\right) & =\widehat{\mu}\left(\psi_{1}\left(x^{1}, 0, \ldots, 0\right), \psi_{1}\left(0, x^{2}, 0, \ldots, 0\right), \ldots, \psi_{1}\left(0, \ldots, 0, x^{m}\right)\right) \\
\left.\Rightarrow \psi_{2 *}\right|_{0}\left(\partial_{x^{i}}\right) & =\left.\hat{\mu}_{*}\right|_{e}\left(0, \ldots, 0, \psi_{1 *}\left(\partial_{x^{i}}\right), 0, \ldots, 0\right) \\
& =\left.\hat{\mu}_{*}\right|_{e}\left(0, \ldots, 0,\left.e_{i}\right|_{e}, 0, \ldots, 0\right)=\left.e_{i}\right|_{e}
\end{aligned}
$$

This makes $\left.\psi_{2 *}\right|_{0}$ invertible and therefore by the inverse function theorem there is an open set $W$, $0 \in W$ with $\psi_{2}: W \rightarrow U=\psi_{2}(W)$ a diffeomorphism. Since $\psi_{2}(0)=e$, the theorem then follows.

Definition. The coordinates $\left(\mathrm{U}, \psi_{2}^{-1}\right)$ are called Coordinates of the $2^{\text {nd }}$ Kind.

Definition. Let $\mathfrak{g}$ be a solvable Lie algebra and let $\left\{e_{1}, \ldots, e_{m}\right\}$ is a basis for $\mathfrak{g}$ where $\mathfrak{h}_{i}=\operatorname{span}\left\{e_{1}, \ldots, e_{i}\right\}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_{\mathfrak{i}}$ is an ideal in $\mathfrak{h}_{\mathfrak{i}+\boldsymbol{1}}$ then we call $\left\{e_{1}, \ldots, e_{m}\right\}$ an adapted basis.

Theorem 18. Let $\mathfrak{g}$ be a solvable Lie algebra then there exists an adapted basis for $\mathfrak{g}$. (See [7] Lemma 2.1)

Theorem 19. Let $G$ be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\left\{e_{1}, \ldots, e_{\mathrm{m}}\right\}$ be an adapted basis for $\mathfrak{g}$, then the map (4.1) is a diffeomorphism of $\mathbb{R}^{m} \rightarrow \mathrm{G}$. (See [9] Theorem 3.18.11)

The general approach used in the proof of [9] Theorem 3.18.11 is as follows. Let $\mathfrak{b}_{i}=\operatorname{span}\left\{\boldsymbol{e}_{i}\right\}$, for each $\mathfrak{i}$. In [9] Varadarajan shows that $\mathfrak{h}_{\mathfrak{m}-1}$ and $\mathfrak{b}_{\mathfrak{m}}$ are the Lie algebras for closed simply connected subgroups $H$ and $B_{m}$ of $G$ with $G=H B_{m}$ and the mapping $\left(h, b_{m}\right) \rightarrow h b_{m}, h \in H$, $b_{m} \in B_{m}$ is a diffeomorphism. By induction we get the mapping $\left(b_{1}, \ldots, b_{m}\right) \rightarrow b_{1} \cdots b_{m}, b_{i} \in B_{i}$ being a diffeomorphism. Since each $B_{i}$ is simply connected and 1-dimensional the map exp : $t \rightarrow$ $\exp \left(t \boldsymbol{e}_{\boldsymbol{i}}\right)$ is a diffeomorphism of $\mathbb{R} \rightarrow B_{i}$ giving the desired result.
4.1.1

## Heisenberg Example

As an example consider the Heisenberg group, H, defined in equation (2.1). Using the standard basis $\left\{e_{1}, e_{2}, e_{3}\right\}$, the Heisenberg Lie algebra is given by the equations in (2.3). With $\psi_{1}$ as defined
in equation (3.1) equation (4.1) then gives:

$$
\begin{aligned}
\psi_{2}\left(x^{1}, x^{2}, x^{3}\right) & =\psi_{1}\left(x^{1}, 0,0\right) \psi_{1}\left(0, x^{2}, 0\right) \psi_{1}\left(0,0, x^{3}\right) \\
& =\left(x^{1}, x^{2}+x^{1} x^{3}, x^{3}\right)
\end{aligned}
$$

Group Multiplication in Coordinates of the $2^{\text {nd }}$ Kind -Nilpotent Case

We will now show how to give the multiplication function for a simply connected Lie group with nilpotent Lie algebra in coordinates of the $2^{\text {nd }}$ kind. We first note an immediate corollary to Theorem 19.

Corollary 20. Let $G$ be a simply connected Lie group with nilpotent Lie algebra $\mathfrak{g}$ then the map $\psi_{2}$ defined in equation (4.1) is a diffeomorphism of $\mathbb{R}^{m} \rightarrow \mathrm{G}$.

Let $G$ be a simply connected Lie group with nilpotent Lie algebra $\mathfrak{g}$. Let $\left\{e_{1}, \ldots, e_{m}\right\}$ be a basis for $\mathfrak{g}$ and let $\psi_{1}$ be defined as in (3.1). By Theorem $10 \psi_{1}$ is a diffeomorphism from $\mathbb{R}^{m} \rightarrow G$. Define a function $\phi: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ by

$$
\begin{equation*}
\phi(x)=\psi_{1}^{-1} \circ \psi_{2}(x) \tag{4.2}
\end{equation*}
$$

for $x \in \mathbb{R}^{m}$. This function is the change of coordinates function between coordinates of the $2^{\text {nd }}$ kind and coordinates of the $1^{\text {st }}$ kind.

Theorem 21. Let $G$ be a simply connected Lie group with nilpotent Lie algebra $\mathfrak{g}$. Let $x, y \in \mathbb{R}^{m}$. Let $\phi$ be defined as in equation (4.2). Let $\mu_{1}$ be defined as in equation (3.18). Let $\mu_{2}: \mathbb{R}^{m} \times \mathbb{R}^{m}$ by

$$
\begin{equation*}
\mu_{2}(x, y)=\phi^{-1} \circ \mu_{1}(\phi(x), \phi(y)) \tag{4.3}
\end{equation*}
$$

then $\mu_{2}$ gives the group multiplication in coordinates of the $2^{\text {nd }}$ kind.

Proof. Since $\psi_{1}$ and $\psi_{2}$ are diffeomorphisms $\phi$ is a diffeomorphism. We have

$$
\begin{aligned}
\mu_{2}(x, y) & =\phi^{-1} \circ \mu_{1}(\phi(x), \phi(y)) \\
& =\left(\psi_{2}^{-1} \circ \psi_{1}\right) \circ \psi_{1}^{-1}\left(\psi_{1}(\phi(x)) \psi_{1}(\phi(y))\right) \\
& =\psi_{2}^{-1}\left(\psi_{2}(x) \psi_{2}(y)\right)
\end{aligned}
$$

establishing that $\mu_{2}$ gives the group multiplication in coordinates of the $2^{\text {nd }}$ kind.

We now show that for nilpotent $\mathfrak{g}$ that $\phi(x)$ and then $\mu_{2}(x, y)$ can be calculated by use of formula (3.14). Let $x=\left(x^{1}, \ldots, x^{m}\right)$. Let $\hat{x}^{j}=\left(0, \ldots, 0, x^{j}, 0, \ldots 0\right)$, where $x^{j}$ is the $j$ th position. By equations (4.2) and (4.1) we have

$$
\phi(x)=\psi_{1}^{-1} \circ \psi_{2}(x)=\psi_{1}^{-1}\left(\psi_{1}\left(\hat{\chi}^{1}\right) \psi_{1}\left(\hat{\chi}^{2}\right) \cdots \psi_{1}\left(\hat{\chi}^{m}\right)\right) .
$$

Formula (3.14) first allows us to compute

$$
\mu_{1}\left(\hat{x}^{1}, \hat{x}^{2}\right)=\psi_{1}^{-1}\left(\psi_{1}\left(\hat{x}^{1}\right) \psi_{1}\left(\hat{\chi}^{2}\right)\right) .
$$

By inductive application of formula (3.14) we can compute

$$
\begin{equation*}
\phi(x)=\psi_{1}^{-1}\left(\psi_{1}\left(\hat{x}^{1}\right) \psi_{1}\left(\hat{x}^{2}\right) \cdots \psi_{1}\left(\hat{x}^{m}\right)\right) . \tag{4.4}
\end{equation*}
$$

Formula (4.4) and the calculation of $\phi^{-1}$ is implemented in Appendix A by the code 4.5.4.

Since we can compute $\phi^{-1}, \mu_{1}$, and $\phi$ we can then by the use of formula (4.3) calculate $\mu_{2}(x, y)$.

Formula (4.3) is implemented in Appendix A by the code 4.5.5.

### 4.2.1

## Heisenberg Example

In 4.5.10 of Appendix B we show how Theorem 21 can be applied to the Heisenberg group.

## 4.3

Left Invariant 1-Forms in Coordinates of the $2^{\text {nd }}$ Kind

In equation 3.3 of [1] a basis for the left invariant 1-forms is given in coordinates similar to coordinates of the $2^{\text {nd }}$ kind. We modify this formula slightly to determine the basis in actual coordinates of the $2^{\text {nd }}$ kind.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $\left\{f_{1}, \ldots, f_{n}\right\}$ be a basis for $\mathfrak{g}$. Let $y=\left(y^{1}, \ldots, y^{n}\right)$ and define coordinates about the identity in G using

$$
\begin{equation*}
\hat{\psi}_{2}\left(y^{1}, \ldots, y^{n}\right)=\exp \left(y^{n} f_{n}\right) \exp \left(y^{n-1} f_{n-1}\right) \cdots \exp \left(y^{1} f_{1}\right) \tag{4.5}
\end{equation*}
$$

The following theorem is given in [1] p. 8.

Theorem 22. A basis for the left invariant 1-forms, dual to the $\left\{\mathrm{f}_{\mathrm{i}}\right\}$, at $\left(\mathrm{y}^{1}, \ldots, \mathrm{y}^{n}\right)$ is given by

$$
\left[\begin{array}{c}
\omega^{1}  \tag{4.6}\\
\omega^{2} \\
\omega^{3} \\
\vdots \\
\omega^{n}
\end{array}\right]=I_{n}\left[\begin{array}{c}
d y^{1} \\
0 \\
0 \\
\vdots \\
0
\end{array}\right]+M_{2}(y)\left[\begin{array}{c}
0 \\
d y^{2} \\
0 \\
\vdots \\
0
\end{array}\right]+\ldots+M_{n}(y)\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
d y^{n}
\end{array}\right]
$$

where

$$
\begin{equation*}
M_{k}(y)=\exp \left(-y^{1}\left[a d f_{1}\right]\right) \exp \left(-y^{2}\left[a d f_{2}\right]\right) \cdots \exp \left(-y^{k-1}\left[a d f_{k-1}\right]\right), k>1 \tag{4.7}
\end{equation*}
$$

We now use this theorem to obtain our desired result.

Theorem 23. Let $\beta=\left\{e_{1}, \ldots, e_{n}\right\}$ be a basis for $\mathfrak{g}$. Let $x=\left(x^{1}, \ldots, x^{n}\right)$ be coordinates of the $2^{n d}$ kind given by $\psi_{2}$ as defined in equation (4.1). A basis for the left invariant 1-forms, dual to the $\left\{\boldsymbol{e}_{i}\right\}$, at $\left(x^{1}, \ldots, x^{n}\right)$ is then given by

$$
\left[\begin{array}{c}
\omega^{1}  \tag{4.8}\\
\omega^{2} \\
\omega^{3} \\
\vdots \\
\omega^{n}
\end{array}\right]=I_{n}\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
d x^{n}
\end{array}\right]+N_{n-1}(x)\left[\begin{array}{c}
0 \\
\vdots \\
0 \\
d x^{n-1} \\
0
\end{array}\right]+\ldots+N_{1}(x)\left[\begin{array}{c}
d x^{1} \\
0 \\
0 \\
\vdots \\
0
\end{array}\right]
$$

where

$$
\begin{equation*}
N_{k}(x)=\exp \left(-x^{n}\left[\text { ad } e_{n}\right]\right) \exp \left(-x^{n-1}\left[\operatorname{ad} e_{n-1}\right]\right) \cdots \exp \left(-x^{k+1}\left[\operatorname{ad} e_{k+1}\right]\right), k<n \tag{4.9}
\end{equation*}
$$

Proof. Let Q be the $\mathrm{n} \times \mathrm{n}$ involutory anti-diagonal matrix consisting of 1 's on the anti-diagonal.

Let

$$
\left[\begin{array}{c}
y^{1}  \tag{4.10}\\
\vdots \\
y^{n}
\end{array}\right]=Q\left[\begin{array}{c}
x^{1} \\
\vdots \\
x^{n}
\end{array}\right] \text { and }\left[f_{1} \cdots f_{n}\right]=\left[e_{1} \cdots e_{n}\right] Q .
$$

In particular equation (4.10) gives $y_{n+1-k}=x_{k}$ and $f_{n+1-k}=e_{k}$. In terms of the basis $\beta^{\prime}=$ $\left\{f_{1}, \ldots, f_{n}\right\}$, using (4.5), (4.1) and (4.10), we observe that

$$
\begin{aligned}
\hat{\psi}_{2}(Q x) & =\exp \left(x^{1} f_{n}\right) \exp \left(x^{2} f_{n-1}\right) \cdots \exp \left(x^{n} f_{1}\right) \\
& =\exp \left(x^{1} e_{1}\right) \exp \left(x^{2} e_{2}\right) \cdots \exp \left(x^{n} e_{n}\right) \\
& =\psi_{2}(x)
\end{aligned}
$$

Let $\phi=\hat{\psi}_{2} \circ \psi_{2}^{-1}$ be the change of coordinates given by $y=\mathrm{Qx}$. The pullback of 1-forms by the transformation $\phi$ gives

$$
\left[\begin{array}{c}
d x^{1}  \tag{4.11}\\
\vdots \\
d x^{n}
\end{array}\right]=\phi^{*}\left(Q\left[\begin{array}{c}
d y^{1} \\
\vdots \\
d y^{n}
\end{array}\right]\right)
$$

Let

$$
\left[\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\vdots \\
\omega^{n}
\end{array}\right]=\phi^{*}\left(\mathrm{Q}\left[\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\hat{\omega}^{3} \\
\vdots \\
\hat{\omega}^{n}
\end{array}\right]\right)
$$

where $\omega$ are the left invariant 1 -forms in the coordinates $y$ given by equation (4.6). Since $Q$ is invertible and constant the $\omega^{i}$ form a basis for the left invariant 1-forms in coordinates of the
second kind. Using equations (4.6) and (4.11) we get

$$
\begin{align*}
{\left[\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\vdots \\
\omega^{n}
\end{array}\right] } & =\phi^{*}\left(\mathrm{QI}_{n}\left[\begin{array}{c}
\mathrm{d} y^{1} \\
0 \\
0 \\
\vdots \\
0
\end{array}\right]\right)+\phi^{*}\left(\mathrm{QM}_{2}(y) \mathrm{Q}^{-1} \mathrm{Q}\left[\begin{array}{c}
0 \\
d y^{2} \\
0 \\
\vdots \\
0
\end{array}\right]\right)+\ldots+\phi^{*}\left(\mathrm{QM}_{n}(y) \mathrm{Q}^{-1} \mathrm{Q}\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
d y^{n}
\end{array}\right]\right) \\
& =I_{n}\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
d x^{n}
\end{array}\right]+\phi^{*}\left(\mathrm{QM}_{2}(y) \mathrm{Q}^{-1}\right)\left[\begin{array}{c}
0 \\
\vdots \\
0 \\
d x^{n-1} \\
0
\end{array}\right]+\ldots+\phi^{*}\left(\mathrm{QM}_{n}(y) \mathrm{Q}^{-1}\right)\left[\begin{array}{c}
\mathrm{d} x^{1} \\
0 \\
0 \\
\vdots \\
0
\end{array}\right] . \tag{4.12}
\end{align*}
$$

We have from (4.7)

$$
\begin{align*}
Q M_{n+1-k}(y) Q^{-1} & =Q \exp \left(-y^{1}\left[a d f_{1}\right]\right) Q^{-1} \cdots Q \exp \left(-y^{n-k}\left[a d f_{n-k}\right]\right) Q^{-1} \\
& =\exp \left(-y^{1} Q\left[\operatorname{ad} f_{1}\right] Q^{-1}\right) \cdots \exp \left(-y^{n-k} Q\left[a d f_{n-k}\right] Q^{-1}\right) \tag{4.13}
\end{align*}
$$

Since $Q$ is the change of basis matrix from the basis $\beta^{\prime}$ to the basis $\beta$ we have

$$
\begin{equation*}
\left[\operatorname{ad} e_{n+1-k}\right]=\left[a d f_{k}\right]_{\beta}=Q\left[a d f_{k}\right]_{\beta}, Q^{-1} \tag{4.14}
\end{equation*}
$$

Using equation (4.14) in equation (4.13) gives

$$
Q M_{n+1-k}(y) Q^{-1}=\exp \left(-y^{1}\left[\operatorname{ad~}_{n}\right]\right) \cdots \exp \left(-y^{n-k}\left[\operatorname{ad} e_{k+1}\right]\right)
$$

so that

$$
\phi^{*}\left(Q M_{n+1-k}(y) Q^{-1}\right)=\exp \left(-x^{n}\left[a d e_{n}\right]\right) \cdots \exp \left(-x^{k+1}\left[\text { ad } e_{k+1}\right]\right)
$$

We observe that $N_{k}(x)=\phi^{*}\left(Q M_{n+1-k}(y) Q^{-1}\right)$, where $N_{k}(x)$ is defined as in equation (4.9). This demonstrates that equation (4.12) is the same as equation (4.8), thus establishing the theorem.

Formula (4.8) of Theorem 23 is implemented in Appendix A by the code 4.5.6.
4.3.1

## Example

In 4.5.11 of Appendix B we show an example of equation (4.8) being calculated fro so(3). Testing indicated that with an Intel I5-6400 processor the left invariant 1-forms were calculated in a few seconds for so(5), about 3 minutes for g 2 and in about 5 minutes for so(6). Testing so(7) was halted after about an hour.

## 4.4

Coordinates of the $2^{\text {nd }}$ Kind - Solvable Case

We will now demonstrate a additional formulas for left invariant 1-forms and left invariant vector fields expressed in coordinates of the $2^{\text {nd }}$ kind.

### 4.4.1

## Left Invariant 1-Forms in Coordinates of the $2^{\text {nd }}$ Kind - Solvable Case

Let $G$ be a Lie group with solvable Lie algebra $\mathfrak{g}$. The following theorem found on page 10 of [5], gives a basis for the left invariant 1-forms of G.

Theorem 24. Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be an adapted basis for a solvable Lie algebra $\mathfrak{g}$. Let $\mathfrak{t}_{s}=\operatorname{span}\left\{e_{1}, \ldots, e_{n-s}\right\}$.
Note that this numbering of sub algebras differs from that used when we defined adapted basis in Definition 4.1, having $\mathfrak{t}_{0}=\mathfrak{g}$. Let $\operatorname{ad}_{s}\left(\mathrm{e}_{\mathrm{n}-\mathrm{s}}\right)$ be the restriction of $\operatorname{ad}\left(\mathrm{e}_{\mathrm{n}-\mathrm{s}}\right)$ to the subspace $\mathrm{t}_{\mathrm{s}}$. Let $\hat{\mathrm{A}}_{\mathrm{s}}$ be the $\mathrm{n}-\mathrm{s}$ by $\mathrm{n}-\mathrm{s}$ matrix representation of $\operatorname{ad}\left(e_{n-s}\right)$ in the basis for $\mathfrak{t}_{s}$. Let $\mathrm{I}_{\mathrm{s}}$ be the $\mathrm{s} \times \mathrm{s}$ identity matrix and $0_{\mathrm{s}}$ the $\mathrm{n}-\mathrm{s}$ by s zero matrix. The following formula gives us a basis, $\tau_{1}, \ldots, \tau_{n}$ for the left invariant 1-forms on G , dual to the $\left\{\mathrm{e}_{\mathrm{i}}\right\}$,

$$
\left[\begin{array}{c}
\tau_{1}  \tag{4.15}\\
\vdots \\
\tau_{n}
\end{array}\right]=e^{-\hat{A}_{0}^{\top} x^{n}}\left[\begin{array}{cc}
e^{-\hat{A}_{1}^{\top} x^{n-1}} & 0_{1} \\
0_{1}^{\top} & 1
\end{array}\right] \cdots\left[\begin{array}{cc}
e^{-\hat{A}_{n-2}^{\top} x^{2}} & 0_{n-2} \\
0_{n-2}^{\top} & I_{n-2}
\end{array}\right]\left[\begin{array}{c}
d x^{1} \\
\vdots \\
d x_{n}
\end{array}\right]
$$

We now show that for $\mathfrak{g}$ solvable formula (4.15) matches formula (4.8), thus establishing that the vector fields from Theorem 24 are given in coordinates of the $2^{\text {nd }}$ kind.

Theorem 25. Assume that $\mathfrak{g}$ is solvable and that $\left\{e_{1}, \ldots, e_{n}\right\}$ is an adapted basis as defined in Definition 4.1 then equations (4.8) and (4.15) are identical.

Proof. Let $\mathfrak{t}_{s}, \hat{\mathcal{A}}_{s}, I_{s}$ and $\mathfrak{O}_{s}$ be defined as in Theorem 24. Since $\mathfrak{g}$ is solvable, with the choice of basis given in the theorem, the matrix [ad $e_{n-j}$ ] is of the form

$$
\left[\begin{array}{ll}
\hat{A}_{j} & * \\
0_{j}^{\top} & 0_{j}
\end{array}\right]
$$

and $\exp \left(-x^{n-j}\left[a d e_{n-j}\right]\right)$ is of the form

$$
\left[\begin{array}{cc}
e^{\hat{A}_{j}\left(-x^{n-j}\right)} & *  \tag{4.16}\\
0_{j}^{T} & I_{j}
\end{array}\right],
$$

where $*$ is an $\boldsymbol{n}-\mathfrak{j} \times \mathfrak{j}$ matrix. We are going to show that

$$
N_{k}(x)\left[\begin{array}{c}
0  \tag{4.17}\\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]=e^{A_{0}\left(-x^{n}\right)}\left[\begin{array}{cc}
e^{A_{1}\left(-x^{n-1}\right)} & 0_{1} \\
0_{1}^{\top} & 1
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{A_{n-2}\left(-x^{2}\right)} & 0_{n-2} \\
0_{n-2}^{\top} & I_{n-2}
\end{array}\right]\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right],
$$

where $\mathrm{N}_{\mathrm{k}}(\mathrm{x})$ is defined in equation (4.9), which will allow us to prove the theorem. We note that based on equation (4.16) we have

$$
\exp \left(-x^{k+1}\left[\begin{array}{ll}
\text { ad } \left.\left.e_{k+1}\right]\right)
\end{array}\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]=\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{n-(k+1)}\left(-x^{k+1}\right)} & 0_{n-(k+1)} \\
0_{n-(k+1)}^{\top} & I_{n-(k+1)}
\end{array}\right]\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right] .\right.\right.
$$

Assume that

$$
\begin{aligned}
& {\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-s}\left(-x^{n-(r-s)}\right)} & * \\
0_{r-s}^{\top} & \mathrm{I}_{r-s}
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-1}\left(-x^{n-(r-1)}\right)} & * \\
0_{r-1}^{\top} & \mathrm{I}_{r-1}
\end{array}\right]\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r}\left(-x^{n-r}\right)} & 0 \\
0_{r}^{\top} & I_{r}
\end{array}\right] } \\
= & {\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-s}\left(-x^{n-(r-s)}\right)} & 0_{r-s} \\
0_{r-s}^{\top} & \mathrm{I}_{r-s}
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-1}\left(-x^{n-(r-1)}\right)} & 0_{r-1} \\
0_{r-1}^{\top} & I_{r-1}
\end{array}\right]\left[\begin{array}{cc}
e^{\left.\hat{\mathcal{A}}_{r\left(-x^{n-r}\right.}\right)} & 0 \\
0_{r}^{\top} & I_{r}
\end{array}\right] }
\end{aligned}
$$

for $s<r$. We observe that

$$
\begin{aligned}
& {\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-(s+1)}\left(-x^{n-(r-(s+1))}\right)} & * \\
0_{r-s}^{\top} & \mathrm{I}_{\mathrm{r}-(s+1)}
\end{array}\right]\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-s}\left(-x^{n-(r-s)}\right)} & 0 \\
0_{r-s}^{\top} & \mathrm{I}_{r-s}
\end{array}\right] } \\
= & {\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-(s+1)}\left(-x^{n-(r-s)}\right)} & 0_{r-(s+1)} \\
0_{r-s}^{\top} & I_{r-s}
\end{array}\right]\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-s}\left(-x^{n-(r-s)}\right)} & 0_{r-s} \\
0_{r-s}^{\top} & I_{r-s}
\end{array}\right] }
\end{aligned}
$$

so that

$$
\begin{aligned}
& {\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-(s+1)}\left(-x^{n-(r-(s+1))}\right)} & * \\
0_{r-s}^{T} & \mathrm{I}_{r-(s+1)}
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-1}\left(-x^{n-(r-1)}\right)} & * \\
0_{r-1}^{T} & I_{r-1}
\end{array}\right]\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r}\left(-x^{n-r}\right)} & 0 \\
0_{r}^{T} & I_{r}
\end{array}\right] } \\
= & {\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-(s+1)}\left(-x^{n-(r-s)}\right)} & 0_{r-(s+1)} \\
0_{r-s}^{\top} & I_{r-s}
\end{array}\right] \cdots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r-1}\left(-x^{n-(r-1)}\right)} & 0_{r-1} \\
0_{r-1}^{T} & I_{r-1}
\end{array}\right]\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{r}\left(-x^{n-r}\right)} & 0 \\
0_{r}^{T} & I_{r}
\end{array}\right] . }
\end{aligned}
$$

We can then conclude by induction that

$$
\begin{aligned}
& e^{\hat{A}_{0}\left(-x^{n}\right)}\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{1}\left(-x^{n-1}\right)} & * \\
0_{n-1}^{T} & I_{n-1}
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{k+1}\left(-x^{n-(k+1)}\right)} & 0 \\
0_{n-1}^{T} & I_{n-1}
\end{array}\right] \\
= & e^{\hat{A}_{0}\left(-x^{n}\right)}\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{1}\left(-x^{n-1}\right)} & 0 \\
0_{n-1}^{T} & I_{n-1}
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{k+1}\left(-x^{n-(k+1)}\right)} & 0 \\
0_{n-1}^{T} & I_{n-1}
\end{array}\right]
\end{aligned}
$$

and therefore

$$
N_{k}(x)\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]=e^{\hat{\mathcal{A}}_{0}\left(-x^{n}\right)}\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{1}\left(-x^{n-1}\right)} & 0_{1} \\
0_{1}^{T} & I_{1}
\end{array}\right] \ldots\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{n-(k+1)}\left(-x^{k+1}\right)} & O_{n-(k+1)} \\
O_{n-(k+1)}^{T} & I_{n-(k+1)}
\end{array}\right]\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right] .
$$

Since $\mathfrak{g}$ is solvable, based on our choice of basis, $\hat{\mathcal{A}}_{n-k}$ has zeros in column $k$ and in row $k$. So
$e^{\hat{\mathcal{A}}_{n-k}\left(-x^{k}\right)}$ has zeros in column $k$ except for a 1 in column $k$, row $k$. This implies that

$$
\mathbf{N}_{\mathrm{k}}(x)\left[\begin{array}{c}
0 \\
\vdots \\
\mathrm{~d} x^{k} \\
\vdots \\
0
\end{array}\right]=\mathbf{N}_{\mathrm{k}}(x)\left[\begin{array}{cc}
e^{\hat{\mathcal{A}}_{n-k}\left(-x^{k}\right)} & 0_{n-k} \\
0_{n-k}^{\top} & I_{n-k}
\end{array}\right]\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right] .
$$

Since each matrix

$$
\left[\begin{array}{cc}
e^{\hat{A}_{n-j}\left(-x^{j}\right)} & 0_{n-j} \\
0_{n-j}^{T} & I_{n-j}
\end{array}\right], j<k
$$

has a $k^{\text {th }}$ column with all zeros except for a 1 in row $k$ we see that

$$
\left[\begin{array}{cc}
e^{A_{1}\left(-x^{n-k}\right)} & 0_{1} \\
0_{1}^{\top} & 1
\end{array}\right] \cdots\left[\begin{array}{cc}
e^{A_{n-2}\left(-x^{2}\right)} & 0_{n-2} \\
0_{n-2}^{T} & I_{n-2}
\end{array}\right]\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]=\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]
$$

and therefore

$$
N_{k}(x)\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]=e^{A_{0}\left(-x^{n}\right)}\left[\begin{array}{cc}
e^{A_{1}\left(-x^{n-1}\right)} & 0_{1} \\
0_{1}^{T} & 1
\end{array}\right] \cdots\left[\begin{array}{cc}
e^{A_{n-2}\left(-x^{2}\right)} & 0_{n-2} \\
0_{n-2}^{T} & I_{n-2}
\end{array}\right]\left[\begin{array}{c}
0 \\
\vdots \\
d x^{k} \\
\vdots \\
0
\end{array}\right]
$$

thus establishing equation (4.17). By expanding formula (4.15) out and utilizing equation (4.17) we see that

$$
\left[\begin{array}{c}
\tau^{1} \\
\vdots \\
\tau^{n}
\end{array}\right]=I_{n}\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
d x^{n}
\end{array}\right]+N_{n-1}(x)\left[\begin{array}{c}
0 \\
\vdots \\
0 \\
d x^{n-1} \\
0
\end{array}\right]+\ldots+N_{1}(x)\left[\begin{array}{c}
d x^{1} \\
0 \\
0 \\
\vdots \\
0
\end{array}\right]=\left[\begin{array}{c}
\omega^{1} \\
\vdots \\
\omega^{n}
\end{array}\right]
$$

establishing the theorem.

### 4.4.2

## Left Invariant Vector Fields in Coordinates of the $2^{\text {nd }}$ Kind - Solvable Case

We now demonstrate that an alternative formula, derived by the use of the BCH formula, yields a basis of left invariant vector fields dual to the basis of left invariant 1-forms shown in Theorem 24 .

The following equation found on page 10 of [5], gives a basis for the left invariant vector fields of a solvable Lie group G, which we have established in Theorem 25 to be in coordinates of $2^{\text {nd }}$ kind.

Theorem 26. Let $\mathfrak{g}$ be a solvable Lie algebra of dimension $\mathfrak{n}$. Let $\left\{X_{i}\right\}$ be vector fields on $\mathbb{R}^{\mathfrak{m}}$ given by

$$
\left[\begin{array}{c}
X_{1}  \tag{4.18}\\
\vdots \\
X_{n}
\end{array}\right]=e^{\hat{\mathcal{A}}_{0}^{\top} x^{n}}\left[\begin{array}{cc}
e^{\hat{A}_{1}^{\top} x^{n-1}} & 0_{1} \\
0_{1}^{\top} & 1
\end{array}\right] \cdots\left[\begin{array}{cc}
e^{\hat{A}_{n-2}^{\top} x^{2}} & 0_{n-2} \\
0_{n-2}^{\top} & I_{n-2}
\end{array}\right]\left[\begin{array}{c}
\partial_{x^{1}} \\
\vdots \\
\partial_{x_{n}}
\end{array}\right]
$$

with $\hat{\mathcal{A}}_{s}$ defined as in Theorem 24, then $\left\{X_{i}\right\}$ form a basis for $\mathfrak{g}$ in coordinates of the $2^{\text {nd }}$ kind.

We will now show that a formula built using BCH methods gives the same left invariant vector fields as those given in equation (4.18). Let $G$ be a Lie group of dimension $m$ with identity $e$ and solvable Lie algebra $\mathfrak{g}$. The mapping $\psi_{1}$ as defined in equation (3.1) and the mapping $\psi_{2}$ defined in equation (4.1) gives us coordinate charts about $e \in G$. We need a subset of the intersection of these charts with some specific properties. This is given by the next theorem.

Theorem 27. Let $\left(\mathrm{U}_{1}, \psi_{1}^{-1}\right)$ and $\left(\mathrm{U}_{2}, \Psi_{2}^{-1}\right)$ be charts about $\mathrm{e} \in \mathrm{G}$ as given by Theorem 11 and Theorem 17, then there is an open ball W about 0 in $\mathbb{R}^{\mathrm{m}}$ with $\mathrm{U}=\psi_{1}(\mathrm{~W}) \subset \mathrm{U}^{\prime}=\mathrm{U}_{1} \cap \mathrm{U}_{2}$ with $\mathrm{a}^{-1} \in \mathrm{U}$ for $\mathrm{a} \in \mathrm{U}$ and $\mathrm{abc} \in \mathrm{U}^{\prime}$ for $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{U}$.

Proof. Let $V=\mu^{-1}\left(\mu^{-1}\left(\mathrm{U}^{\prime}\right)\right) \subset \mathrm{G} \times \mathrm{G} \times \mathrm{G}$. We have V open with $\mathrm{abc} \in \mathrm{U}^{\prime}$ for $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{V}$. Let $W^{\prime}=\psi_{1}^{-1}(V)$. There is an open ball $W \subset W^{\prime}$ about 0 . Let $U=\psi_{1}(W)$. Let $a, b, c \in U$, then $a, b, c \in V \Rightarrow a b c \in U^{\prime}$ and there is an $x \in W$ with $a=\psi_{1}(x)$, which implies $-x \in W \Rightarrow a^{-1}=$ $\psi_{1}(-x) \in U$, which proves the theorem.

Let $\mathfrak{g}$ be a Lie algebra and let $\left\{e_{i}\right\}_{1 \leq i \leq m}$ be an adapted basis of of $\mathfrak{g}$ as defined in 4.1 with $\mathfrak{h}_{j}=\operatorname{span}\left\{\mathrm{e}_{\mathrm{i}}\right\}_{1 \leq i \leq j}$. Let

$$
\begin{equation*}
\operatorname{ad}_{\mathfrak{j}}\left(e_{\mathfrak{i}}\right)=\left.\operatorname{ad} \mathrm{e}_{\mathfrak{i}}\right|_{\mathfrak{h}_{\mathfrak{j}}}, \quad 1 \leq \mathfrak{i} \leq \mathfrak{j} \leq m . \tag{4.19}
\end{equation*}
$$

Let $0_{j}$ be a $j$ by $m-j$ matrix of zeros. Let $I_{m-j}$ be the $m-j$ identity matrix.

Let

$$
A_{x^{j}}=\left[\begin{array}{cc}
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n}\left(\left[\operatorname{ad}_{j} x^{j} e_{j}\right]^{\top}\right)^{n} & 0_{j}  \tag{4.20}\\
0_{j}^{\top} & I_{n-j}
\end{array}\right]
$$

where there is no summation over $\mathfrak{j}$ and the 0 and I matrices are omitted for $\mathfrak{j}=\mathrm{m}$.

Theorem 28. Let $G$ be a simply connected Lie group with solvable Lie algebra $\mathfrak{g}$ and let $\left\{\mathrm{e}_{\mathrm{i}}\right\}_{1 \leq \mathrm{i} \leq \mathrm{m}}$ be an adapted basis for $\mathfrak{g}$. Let $\mathcal{A}_{g^{j}}$ be defined as in (4.20). Let $\left[a d \chi^{i} e_{i}\right]$ be the matrix representation of $\operatorname{ad} x^{i} e_{i}$ in the basis $\left\{\mathrm{e}_{\mathrm{i}}\right\}$. Let $\left(\mathrm{U}_{1}, \psi_{1}^{-1}\right),\left(\mathrm{U}_{2}, \psi_{2}^{-1}\right)$, and U be defined as in Theorem 27. Then for $\mathrm{a} \in \mathrm{U}, \mathrm{x}=\left(\mathrm{x}^{1}, \ldots, \mathrm{x}^{\mathrm{m}}\right)=\psi_{2}^{-1}(\mathrm{a}),\left.\widehat{X}_{\mathrm{k}}\right|_{\mathrm{x}}=\left.\psi_{2 *}^{-1} \mathrm{e}_{\mathrm{k}}\right|_{\mathrm{a}}$ is given by

$$
\left[\begin{array}{c}
\left.\widehat{X}_{1}\right|_{x}  \tag{4.21}\\
\vdots \\
\left.\widehat{X}_{m}\right|_{x}
\end{array}\right]=\left(A_{-x^{m}}^{-1} A_{x^{m}}\right)\left(A_{-x^{m-1}}^{-1} A_{x^{m-1}}\right) \cdots\left(A_{-x^{2}}^{-1} A_{x^{2}}\right)\left[\begin{array}{c}
\partial_{x^{1}} \\
\vdots \\
\partial_{x^{m}}
\end{array}\right]
$$

Proof. We will prove this theorem by establishing that formula (4.21) gives the same vector fields as given by equation (4.18).

Formula (4.21) of Theorem 28 is implemented in Appendix A by the code 4.5.7. There is an example shown in 4.5.1.

Theorem 29. The basis for left invariant vector fields given in formulas (4.18) and (4.21) are identical.

Proof. To prove the theorem we only need to show that

$$
\left[\begin{array}{cc}
e^{\hat{A}_{n-j}^{\top} x^{j}} & 0_{j} \\
0_{j}^{\top} & I_{j}
\end{array}\right]=A_{-x^{j}}^{-1} A_{x^{j}}
$$

for $2 \leq \mathfrak{j} \leq \mathrm{n}$. In other words we will show

$$
A_{-x^{j}}\left[\begin{array}{cc}
e^{\hat{A}_{n-j}^{\top} x^{j}} & 0_{j}  \tag{4.22}\\
0_{j}^{\top} & I_{j}
\end{array}\right]=A_{x^{j}} .
$$

We first note that using the notation of formula (4.19) that $\left[\operatorname{ad}_{j}\left(e_{j}\right)\right]=\hat{A}_{n-j}$ in the notation of formula (4.15) so that

$$
A_{x^{j}}=\left[\begin{array}{cc}
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n}\left(\hat{A}_{n-j}^{\top} x^{j}\right)^{n} & 0_{j} \\
0_{j}^{\top} & I_{n-j}
\end{array}\right]
$$

Let $M=\hat{A}_{n-j}^{\top} \chi^{j}$. In order to establish (4.22) we must show that

$$
\left(\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n}(-M)^{n}\right) e^{M}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n} M^{n}
$$

Inserting the power series for exp gives

$$
\left(\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n}(-M)^{n}\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!} M^{n}\right)=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n} M^{n}
$$

Using the Cauchy product of power series formula on the left hand side we then need to show

$$
\sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^{i}}{i!} B_{i}(-1)^{i} \frac{1}{(n-i)!} M^{n}=\sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} B_{n} M^{n}
$$

which is equivalent to showing

$$
\sum_{i=0}^{n} \frac{B_{i}}{i!(n-i)!}=\frac{(-1)^{n}}{n!} B_{n}
$$

For $n=0$ we get $1=1$ and For $n=1$ we get $1-\frac{1}{2}=\frac{1}{2}$. For $n>1, B_{n}=0$ for $n$ odd so that $(-1)^{n} B_{n}=B_{n}$, so we only need to show that

$$
\sum_{i=0}^{n-1} \frac{B_{i}}{i!(n-i)!}=0
$$

but this is a known identity for Bernoulli numbers (See Wolfram Math World - Bernoulli Number) thus establishing equation (4.22) and proving the theorem.

Left Invariant Vector Fields in Coordinates $2^{\text {nd }}$ Kind - Nilpotent Case

One important case is where Theorem 28 applies is if $\mathfrak{g}$ is a nilpotent Lie algebra.

Corollary 30. Let $\mathfrak{g}$ be $k$-step nilpotent. We have $\left[\operatorname{ad} \chi^{i} e_{i}\right]^{k}=0$ so that (4.21) has only finitely many terms allowing for exact calculation of the vector fields in terms of finite sums of brackets.
4.5.1

## Example

In 4.5.12 of Appendix B we show an example of equation (4.21) of Theorem 28 being calculated.

## REFERENCES

[1] Alexy A. Magazev, Vitaly V. Mikheyev, Igor V. Shirokov, Computation of Composition Functions and Invariant Vector Fields in Terms of Structure Constants of Associated Lie Algebras, SIGMA 11 (2015).
[2] Boothby, William M., An Introduction to Differential Manifolds and Riemannian Geometry, Academic Press 2003.
[3] Daniel Bump, Lie Groups, Graduate Texts in Mathematics Vol 225.
[4] J.J. Duistermaat, J. A. C. Kolk, Lie Groups, Springer 2000.
[5] M.E. Fels, On the construction of simply connected solvable Lie groups, J. Lie Theory 27 (2017), no. 1, 193-215.
[6] Karl Goldberg, The formal power series for $\log \left(e^{x} e^{y}\right)$, Math. Comput. 48 (1987) 265-271.
[7] Sigurdur Helgason, Differential Geometry, Lie Groups and Symetric Spaces, Graduate Studies in Mathematics, Vol 34.
[8] Joachim Hilgert, Karl-Hermann Neeb Structure and Geometry of Lie Groups, Springer 2012.
[9] Varadarajan, V.S., Lie Groups, Lie Algebras and Their Representations, Springer 1984.
[10] Warner, F, Foundations of differentiable manifolds and Lie groups, Springer 2010.
[11] Michael Wustner, On the Surjectivity of the Exponential Function of Solvable Lie Groups, Math. Nachr., 192: 255267. doi:10.1002/mana.19981920115.

APPENDIX A

Maple Code
4.5.2

Maple Code for Equation (3.14) of Corollary 13

## Initialize Packages.

```
Procedures
[Input: A Lie algebra.
[Output: The result of equation 3.14 for generic points x and y in coordinates of the 1st kind.
Alg > NilpotentBCHmult:=\mathbf{proc(alginput)}
    local i, X, Y, Z, algbasis, adim, basis;
    if Query(alginput, "Nilpotent") then
            ChangeLieAlgebraTo(alginput);
            adim := DGinformation(alginput,"LieAlgebraDimension");
            basis := DGinfo(alginput, "FrameBaseVectors");
            X:= evalDG(0.basis[1]);
            Y:= evalDG(0.basis[1]);
            for i}\mathrm{ from 1 to adim do
            X:= evalDG(X+x|i* basis[i]);
            Y:= evalDG(Y+y|i* basis[i]);
            end do;
            Z:= evalDG(X+Y+BCH(X,Y,Y,1,1,0,1,1,0));
            algbasis:= [el];
            for i}\mathrm{ from 2 to adim do algbasis := [op(algbasis), e|i] end do;
            GetComponents(Z, algbasis);
        else
            print("Not a nilpotent Lie Algebra");
        end if;
    end:
```

The following function is an internal subroutine for NilpotentBCHmult. It is designed to be called recursively. It is not for external use.
FAlg $>B C H:=\boldsymbol{p r o c}(X, Y, r t$, sum, prod, leadx, leady, $n$, ind $)$
local $l x, l x 1, l x 2, l x 3, l x 4, l y, l y 1, l y 2, l y 3, l y 4, \ln , \ln 1, \ln 2$;
$\ln :=n$;
$l x 1:=0 ; l x 2:=0 ; l x 3:=0 ; l x 4:=0 ;$
ly $1:=0 ; \operatorname{ly} 2:=0 ; \operatorname{ly} 3:=0 ; \operatorname{ly} 4:=0$;
$l x:=\operatorname{LieBracket}(X, r t)$;
if $l x \neq 0$ \&mult el then
if leadx $=0$ then
$\ln 1:=(-1)^{\wedge} \operatorname{trunc}((n+$ ind $) / 2) *(\operatorname{trunc}((n+$ ind $) / 2)+1) ;$
$\ln 2:=(-1)^{\wedge} \operatorname{trunc}((n+$ ind +2$) / 2) *(\operatorname{trunc}((n+$ ind +2$) / 2)$
$+1) ;$
$\operatorname{lx} 1:=\operatorname{evalDG}(l x *(1 / \ln 1+1 / \ln 2) /($ sum $) /$ prod/leady! $) ;$
$l x 2:=B C H(X, Y, l x$, sum, prod* leady!, 1, $0, \ln +1$, ind $)$;
$l x 3:=B C H(X, Y, l x$, sum, prod* leady!, $1,0, \ln +3$, ind $) ;$

## else

$\ln 1:=(-1)^{\wedge}(\operatorname{trunc}((n+1) / 2)-1) * \operatorname{trunc}(((n+1) / 2)) ;$
$\ln 2:=(-1)^{\wedge}(\operatorname{trunc}((n+3) / 2)-1) * \operatorname{trunc}(((n+3) / 2))$;
$l x 1:=\operatorname{evalDG}(l x / \ln 1 /($ sum $) / \operatorname{prod} /($ lead $x+1)!) ;$
$l x 2:=B C H(X, Y, l x$, sum, prod, leadx $+1,0, \ln$, ind $) ;$
lx3 $:=\operatorname{evalDG}(l x / \ln 2 /($ sum $) /$ prod/leadx $!)$;
$l x 4:=B C H\left(X, Y, l x\right.$, sum, prod $^{*}$ leadx $!, 1,0, \ln +2$, ind $) ;$
end if;
end if;
$l y:=\operatorname{LieBracket}(Y, r t)$;
\#print( $6, \ln , Y, r t$ );
if $l y \neq 0$ \&mult el then
if leady $=0$ then
$\ln 1:=(-1)^{\wedge} \operatorname{trunc}((n+$ ind $) / 2) *(\operatorname{trunc}((n+$ ind $) / 2)+1) ;$
ly1 $:=$ evalDG(ly/ln1/(sum +1$) /$ prod/leadx $!) ;$
ly $2:=B C H\left(X, Y\right.$, ly, sum +1, prod ${ }^{*}$ leadx, $0,1, \ln +1$, ind $) ;$
else
$\ln 1:=(-1)^{\wedge}(\operatorname{trunc}((n+$ ind +1$) / 2)-1) * \operatorname{trunc}(((n+$ ind
+1)/2) ;
$\ln 2:=(-1)^{\wedge}(\operatorname{trunc}((n+$ ind +3$) / 2)-1) * \operatorname{trunc}(((n+$ ind
$+3) / 2)$ );
lyl $:=\operatorname{evalDG}($ ly $/ \ln 1 /(\operatorname{sum}+1) / \operatorname{prod} /(\operatorname{leady}+1)!) ;$
$l y 2:=B C H(X, Y$, ly, sum +1, prod, 0 , leady $+1, \ln$, ind $) ;$
ly $3:=$ evalDG $($ ly/ln2/(sum +1$) /$ prod/leady! $) ;$
ly4 $:=B C H\left(X, Y, l y\right.$, sum +1, prod $^{*}$ leady!, $0,1, \ln +2$, ind $) ;$
end if;
end if;
$e v a l D G(l x 1+l x 2+l x 3+l x 4+l y 1+l y 2+l y 3+l y 4) ;$
end:
4.5.3

Maple Code for Equation (3.15) of Theorem 14

```
Input: A Lie algebra.
    A manifold of the same dimension as the Lie algebra.
    OOtput: The vector fileds given by formula (3.15) in the thesis using the default basis for
    the Lie algebra input.
M > NilpotentLivflstKind:= proc(alginput,manifold)
    local i, b, dv, gen_element, A, z, adim, coords, vects, basis;
    if Query(alginput, "Nilpotent") then
        ChangeLieAlgebraTo(alginput);
        adim := DGinformation(alginput, "LieAlgebraDimension");
        basis := DGinformation(alginput, "FrameBaseVectors");
        dv := [ ];
        gen_element := evalDG(0.el);
        A := IdentityMatrix(adim);
        coords := DGinformation(M, "FrameIndependentVariables");
        vects := DGinformation(M, "FrameBaseVectors");
        for i}\boldsymbol{from}1\mathrm{ to adim do
                gen_element := evalDG(gen_element + coords[i]\cdotbasis[i]);
                dv:= [op (dv), vects[i]];
        end do;
        for i from 1 to adim do
            A:=A+\frac{(-1)\mp@subsup{)}{}{i}}{i!}\cdot\operatorname{bernoulli}(i)\cdot(Adjoint(gen_element )}\mp@subsup{}{}{+}\mp@subsup{)}{}{i}
        end do;
        A.Matrix(dv) + ;
    else
        print("Not a nilpotent Lie Algebra");
    end if;
    end:
```


## Maple Code for Equation (4.4)

Input: A Lie algebra
A point in $\mathrm{R}^{\wedge} \mathrm{n}$ (where n is the dimensiont of the Lie algebra) entered as a list
If first_or_second="second" the function changes coordinates of the 2nd kind to coordinates of the first kind
otherwise the fuction changes coordinates of the 1 st kind to coordinates of the second kind.
[Output: The result of equation (4.4) in the thesis or its inverse.
>> NilpotentCanonicalCoordinateChange $:=\mathbf{p r o c}($ alginput, coordinate, first_or_second)
local i, X, Y, W, Z, adim, basis, dgsetupvarsx, dgsetupvarsy, rt, sc, ab, ls, liedata, phi, psi,
Mult, wvars, zvars, subvars, Mult2;
if Query(alginput, "Nilpotent") then
ChangeLieAlgebraTo( alginput);
adim $:=$ DGinformation (alginput, "LieAlgebraDimension");
$l s:=\operatorname{Series}($ "Lower");
$a b:=l s[\operatorname{nops}(l s)-1]$;
for $i$ from $\operatorname{nops}(l s)-1$ by -1 to 2 do
$a b:=[o p(a b), o p($ ComplementaryBasis(ls[i],ls[i-1]))];
end do;
liedata $:=$ LieAlgebraData(ab, alginput $) ;$
DGsetup (liedata);
basis $:=$ DGinfo(alginput, "FrameBaseVectors");
$r t:=y 1 \cdot$ basis [1];
for $i$ from 2 to adim do
$X:=r t ;$
$Y:=y \| i \cdot \operatorname{basis}[i] ;$
$r t:=e v a l D G(r t+Y+B C H(X, Y, Y, 1,1,0,1,1,0)) ;$
end do;
$s c:=$ GetComponents(rt, basis);
dgsetupvarsx $:=[] ;$
dgsetupvarsy $:=[] ;$
wvars := [ ];
zvars $:=[] ;$
subvars := [];
for $i$ from 1 to adim do
dgsetupvarsx $:=[o p($ dgsetupvarsx $), x \| i] ;$
dgsetupvarsy $:=[o p(d g s e t u p v a r s y), y \| i] ;$
wvars $:=[$ op(wvars), w\|i];
zvars $:=[o p(z v a r s), z \| i] ;$
subvars $:=[o p($ subvars $), x \| i=s c[i]] ;$
end do ;
DGsetup (dgsetupvarsx, M);
DGsetup (dgsetupvarsy, $N$ );
phi $:=$ Transformation ( $N, M$, subvars);
psi := InverseTransformation(phi);
if first_or_second $=$ "second" then
$\overline{A p} p l y$ Transformation (phi, coordinate);
else
ApplyTransformation(psi, coordinate);
end if;
else
print("Not a nilpotent Lie Algebra");
end if; end:
4.5.5

Maple Code for Equation (4.3) of Theorem 21

```
[Input: A Lie algebra
Output: The result of equation 4.3 for generic points x and y in coordinates of the 2nd kind
Alg > NilpotentBCHmult2 := proc(alginput)
    local i, X, Y, W, Z, adim, basis, dgsetupvarsx, dgsetupvarsy, rt, sc, ab, ls,
        liedata, phi, psi, Mult, wvars, zvars, subvars, Mult2;
        if Query(alginput, "Nilpotent") then
            adim := DGinformation(alginput, "LieAlgebraDimension");
            wvars:= [ ];
            zvars := [ ];
            for ifrom 1 to adim do
                wvars := [op(wvars), w|i];
                zvars := [op(zvars), z|i];
            end do ;
            Mult := NilpotentBCHmult(alginput);
            W:= NilpotentCanonicalCoordinateChange(alginput, wvars, "second");
            Z:= NilpotentCanonicalCoordinateChange(alginput, zvars, "second");
            subvars:= [ ];
            for i}\boldsymbol{from}1\mathrm{ to adim do
                        subvars:= [op(subvars), x|i=W[i]];
                    subvars := [op(subvars), y|i=Z[i]];
            end do;
            Mult2 := subs(subvars, Mult);
            Mult2 := NilpotentCanonicalCoordinateChange(alginput, Mult2,
        "first");
            subvars := [ ];
            for i from 1 to adim do
                subvars := [op(subvars),w|i=x|i];
                    subvars := [op(subvars),z|i=y|i];
            end do;
            Mult2:= subs(subvars, Mult2)
        else
            print("Not a nilpotent Lie Algebra");
        end if;
        end:
```

[Input: A Lie algebra.
A manifold of the same dimension as the Lie algebra.
Output: The 1 -form fields given by equation (4.8) in the thesis using the default basis for the Lie algebra input.
M > CalcLeftInv1Forms :=, proc(alginput, manifold)
local $i, j, B, V$, adim, ls, ab, liedata, forms;
ChangeLieAlgebraTo(alginput);
adim $:=$ DGinformation ( alginput,"LieAlgebraDimension");
forms $:=$ DGinformation(manifold, "FrameBaseForms");
$B:=$ [ ];
for $i$ from 1 to adim do $B:=[o p(B), 0]$ end do;
$B:=\operatorname{convert}(B, \text { Matrix })^{+}$;
for $i$ from 1 to adim do
$V:=[$ ];
for $j$ from 1 to adim do
if $j=i$ then
$V:=[o p(V)$, forms $[j]] ;$
else
$V:=[o p(V), 0] ;$
end if;
end do;
$V:=\operatorname{convert}(V, \text { Matrix })^{+}$;
if $i<\operatorname{adim}$ then
$B:=$ evalD $G(B+$ calc_Nk(alginput, manifold, adim, $i) . V) ;$
else
$B:=\operatorname{evalDG}(B+V) ;$
end if;
end do;
end:
[The following function is an internal subroutine for calc_left_forms. It is not for external use.
[ $\mathbf{M}>$ calc_Nk $:=\mathbf{p r o c}($ alginput, manifold, adim, cnt)
local $i, N k$, basis, coords;
basis $:=$ DGinformation(alginput, "FrameBaseVectors");
coords $:=$ DGinformation (manifold, "FrameIndependentVariables");
$N k:=\operatorname{AdjointExp}((-\operatorname{coords}[$ adim $]) \cdot(\operatorname{basis}[$ adim $])) ;$
if adim $-1>$ cnt then for $i$ from $\operatorname{adim}-1$ by -1 to $1+$ cnt do
$N k:=\operatorname{Multiply}(N k, \operatorname{AdjointExp}((-\operatorname{coords}[i]) \cdot(\operatorname{basis}[i])))$ end do;
else return $N k$;
end if;
end:
4.5.7

Maple Code for Equation (4.21) of Theorem 28

```
Input: A Lie algebra.
    A manifold of the same dimension as the Lie algebra.
    Output:The vector fileds from equation (4.21) in the thesis using a basis adapted to lower
    central series for the Lie algebra input.
    M > NilpotentLivf2ndKind:= proc(alginput,manifold)
    local i,dv, A, adim, ls, ab, liedata, vects;
        if Query(alginput, "Nilpotent") then
            ChangeLieAlgebraTo(alginput);
            adim := DGinformation(alginput, "LieAlgebraDimension");
            ls:=Series("Lower");
            ab:= ls[nops(ls)-1];
            for i from nops(ls) - 1 by -1 to 2 do
                        ab:=[op(ab),op(ComplementaryBasis(ls[i],ls[i-1]))];
            end do;
            liedata := LieAlgebraData(ab, alginput);
            DGsetup(liedata);
            dv := [ ];
            A := IdentityMatrix(adim);
            vects := DGinformation(manifold, "FrameBaseVectors");
            for ifrom 1 to adim do
                dv}:=[op(dv),vects[i]]
            end do;
            for ifrom adim by -1 to 2 do
                A := A.Admatrix(alginput, manifold, adim, i,-1) -1.Admatrix(alginput,
        manifold, adim, i, 1);
            end do;
            A:= A.Matrix (dv) +}
        else
            print("Not a nilpotent Lie Algebra");
        end if;
    end:
```

```
[The following function is an internal subroutine for NilpotentLivf2ndKind. It is not for external use.
M > Admatrix:=\mathbf{proc(alginput, manifold, initdim,adim, s)}
    local i,j, b, A, A0, L2, dgsetupvars, coords, basis;
        dgsetupvars := [ ];
        A := IdentityMatrix(adim);
        A0 := IdentityMatrix(initdim, compact=false);
        for i from 1 to adim do
            dgsetupvars := [op(dgsetupvars), e|i];
        end do;
        L2 := LieAlgebraData(dgsetupvars, alginput);
        DGsetup(L2);
        basis := DGinformation(alginput, "FrameBaseVectors");
        coords := DGinformation(manifold, "FrameIndependentVariables");
        for ifrom 1 to adim do
            A:=A+\frac{(-1\mp@subsup{)}{}{i}}{i!}\cdot\mathrm{ bernoulli (i)}\cdot(Adjoint(evalDG(s\cdotcoords[adim ]
        -basis[adim]) ) }\mp@subsup{}{}{+}\mp@subsup{)}{}{i}\mathrm{ ;
        end do;
        for i from 1 to adim do
            for }j\mathrm{ from 1 to adim do
                        AO[i,j] := A[i,j];
            end do;
        end do;
        A0;
    end:
```

[The following are functions used for testing.
Alg $>$ AdaptedBasis $:=\mathbf{p r o c}(l s)$
local basis, $i$;
basis $:=l s[\operatorname{nops}(l s)-1]$;
for $i$ from $\operatorname{nops}(l s)-1$ by -1 to 2
do basis $:=[o p($ basis $)$, op(ComplementaryBasis(ls[i], ls[i-1])) $]$;
end do;
end:
$>$ Livf_from_mult $:=\mathbf{p r o c}($ mult $)$
local adim, i, j, dgsetupvars, subvars, vf, l, L;
dgsetupvars := [ ];
subvars $:=[$ ];
vf:= [ ];
adim $:=$ DGinformation(Alg,"LieAlgebraDimension") :
for $i$ from 1 to adim do
dgsetupvars $:=[o p(d g s e t u p v a r s), x \| i] ;$
subvars $:=[$ op(subvars), $y \| i=0]$;
end do :
DGsetup(dgsetupvars, M) :
for $i$ from 1 to adim do
$l:=\operatorname{subs}($ subvars, $\operatorname{diff}($ mult, $y \| i)) ;$
$L:=\operatorname{evalDG}\left(l[1] \cdot D \_x l\right)$;
for $j$ from 2 to adim do
$L:=\operatorname{evalD} G\left(L+l[j] \cdot D \_x \| j\right)$
end do;
$v f:=[o p(v f), L]$
end do;
end:
CheckAssociativity : $=\mathbf{p r o c}(\mathrm{mu})$
local $i$, subvars, a, ans1, ans2;
subvars := [ ];
$a:=[] ;$
for $i$ from 1 to $\operatorname{nops}(\mathrm{mu})$ do subvars $:=[o p($ subvars $), y\|i=w\| i, x\|i=y\| i]$ end do:
for $i$ from 1 to $\operatorname{nops}(\mathrm{mu})$ do $a:=\operatorname{simplify}([o p(a)$, subs(subvars, mu[i])]) end do:
subvars $:=[$ ];
for $i$ from 1 to nops (mu) do subvars $:=[o p($ subvars $), y \| i=a[i]]$ end do: ans $1:=\operatorname{simplify}(s u b s($ subvars, mu) );
$a:=\mathrm{mu}$;
subvars $:=[$ ];
for $i$ from 1 to nops (mu) do subvars $:=[o p($ subvars $), y\|i=w\| i]$ end do: $a:=\operatorname{subs}($ subvars, $a)$;
subvars $:=[$ ];
for $i$ from 1 to $\operatorname{nops}(\mathrm{mu})$ do subvars $:=[o p($ subvars $), x \| i=m u[i]]$ end do: ans $2:=\operatorname{subs}($ subvars, $a)$;
simplify(ans1 - ans2);
end:

```
\([\mathbf{M}>\) CheckLeftInvariant \(:=\boldsymbol{p r o c}(X, \mathrm{mu})\)
    local \(i\), subvars, left_mult, \(y X\), ans, \(X \_a t \_x y\);
    subvars \(:=\) [ ];
    for \(i\) from 1 to \(\operatorname{nops}(\mathrm{mu})\) do subvars \(:=[o p(\) subvars \(), x \| i=\mathrm{mu}[i]]\) end do:
    X_at_xy \(:=\operatorname{subs}(\) subvars, \(X\) );
    left_-̄ult \(:=\) Transformation( \(N, M\), subvars);
    subvars \(:=[\) ];
    for \(i\) from 1 to nops (mu) do subvars \(:=\left[o p(\right.\) subvars \(), D_{-} x\left\|i=D_{-} y\right\| i, x \| i=y\)
    \(\| i]\) end do:
    \(y X:=\operatorname{subs}(\) subvars, \(X)\);
    ans \(:=\) [ ];
    for \(i\) from 1 to \(\operatorname{nops}(\mathrm{mu})\) do
        ans \(:=[o p(\) ans \()\), Pushforward(left_mult, \(y X[i])]\)
    end do;
    evalDG(ans - X_at_xy);
    end:
```


## APPENDIX B

Examples
4.5.8

## Example 3.3.1.2

## Synopsis

## 1. Multiplication in Canonical Coordinates of the 1st Kind Nilpotent Case.

[We retrieve a Lie algebra from Winternitz.

$$
\left[\begin{array}{l}
>\text { LieAlg }:=\text { Retrieve }(" W i n t e r n i t z ", 1,[6,2], \text { Alg }) \\
\text { LieAlg }:=[e 1, e 2]=e 3,[e 1, e 3]=e 4,[e 1, e 4]=e 5,[e 1, e 5]=e 6,[e 1, e 6]=0,[e 2, e 3]=0,  \tag{5.1}\\
\quad[e 2, e 4]=0,[e 2, e 5]=0,[e 2, e 6]=0,[e 3, e 4]=0,[e 3, e 5]=0,[e 3, e 6]=0,[e 4, e 5 \\
\quad]=0,[e 4, e 6]=0,[e 5, e 6]=0
\end{array}\right.
$$

$\stackrel{>}{ } \rightarrow$ Gsetup (LieAlg) :
[We swith to an adapted basis for convienience in comparing vector fields in coordinates of the 1 st kind and coordinates of the second kind.
[> DGsetup(LieAlgebraData(AdaptedBasis(Series("Lower")), Alg)) :
> MultiplicationTable( )

| Alg | $e 1$ | $e 2$ | $e 3$ | $e 4$ | $e 5$ | $e 6$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $e 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $e 2$ | 0 | 0 | 0 | 0 | $e 1$ | 0 |
| $e 3$ | 0 | 0 | 0 | 0 | $e 2$ | 0 |
| $e 4$ | 0 | 0 | 0 | 0 | $e 3$ | 0 |
| $e 5$ | 0 | $-e 1$ | $-e 2$ | $-e 3$ | 0 | $e 4$ |
| $e 6$ | 0 | 0 | 0 | 0 | $-e 4$ | 0 |

EWe check to see that our Lie algebra is nilpotent.
>> Query(Alg, "Nilpotent")
true
[We use the BCH formula to calculate $\mathrm{z}=\mathrm{xy}$, where $\mathrm{z}, \mathrm{x}$, and y are group elements in coordinates of the 1st kind.
F $>z:=$ NilpotentBCHmult (Alg)

$$
\begin{align*}
z:= & {\left[x 1+y 1+\frac{1}{2} \times 2 y 5-\frac{1}{2} x 5 y 2-\frac{1}{24} \times 4 \times 5 y 5^{2}+\frac{1}{24} x 5^{2} y 4 y 5-\frac{1}{180} x 5^{2} y 5^{2} y 6\right.}  \tag{5.4}\\
& +\frac{1}{180} x 5 x 6 y 5^{3}+\frac{1}{12} x 3 y 5^{2}-\frac{1}{12} \times 5 y 3 y 5-\frac{1}{12} x 3 x 5 y 5+\frac{1}{12} x 5^{2} y 3 \\
& +\frac{1}{720} x 5^{4} y 6-\frac{1}{720} x 5^{3} x 6 y 5+\frac{1}{180} x 5^{3} y 5 y 6-\frac{1}{180} x 5^{2} x 6 y 5^{2}-\frac{1}{720} x 5 y 5^{3} y 6 \\
& +\frac{1}{720} x 6 y 5^{4}, x 2+y 2+\frac{1}{2} x 3 y 5-\frac{1}{2} x 5 y 3-\frac{1}{24} x 5^{2} y 5 y 6+\frac{1}{24} x 5 x 6 y 5^{2} \\
& +\frac{1}{12} x 4 y 5^{2}-\frac{1}{12} x 5 y 4 y 5-\frac{1}{12} x 4 x 5 y 5+\frac{1}{12} x 5^{2} y 4, x 3+y 3+\frac{1}{2} x 4 y 5
\end{align*}
$$

$$
\begin{aligned}
& -\frac{1}{2} \times 5 y 4+\frac{1}{12} \times 5 y 5 y 6-\frac{1}{12} \times 6 y 5^{2}-\frac{1}{12} x 5^{2} y 6+\frac{1}{12} \times 5 x 6 y 5, x 4+y 4 \\
& \left.+\frac{1}{2} \times 5 y 6-\frac{1}{2} \times 6 y 5, x 5+y 5, x 6+y 6\right]
\end{aligned}
$$

[We verify that this is a group multiplicaiton.
[> adim $:=$ DGinformation (Alg,"LieAlgebraDimension") :
> identvars $:=[]:$
[> invvars $:=$ [ ]:
$>$ for $i$ from 1 to adim do
identvars $:=[o p($ identvars $), y \| i=0] ;$
invvars $:=[$ op (invvars),$y\|i=-x\| i] ;$
end do:
Setting $\mathrm{y}=0$ in the multiplication gives x , verifiing that $(0,0,0,0,0,0)$ is the identity.
$>\operatorname{simplify}($ subs(identvars, z) )

$$
\begin{equation*}
[x 1, x 2, x 3, x 4, x 5, x 6] \tag{5.5}
\end{equation*}
$$

[Setting $\mathrm{y}=(-\mathrm{x} 1,-\mathrm{x} 2,-\mathrm{x} 4,-\mathrm{x} 4,-\mathrm{x} 5,-\mathrm{x} 6)$ gives 0 verifying that $(-\mathrm{x} 1,-\mathrm{x} 2,-\mathrm{x} 4,-\mathrm{x} 4,-\mathrm{x} 5,-\mathrm{x} 6)$ is the inverse.
> subs(invvars, $z$ )

$$
\begin{equation*}
[0,0,0,0,0,0] \tag{5.6}
\end{equation*}
$$

[We check the associative property by computing $\mathrm{x}(\mathrm{yw})-(\mathrm{xy}) \mathrm{w}$.
> CheckAssociativity (z)

$$
\begin{equation*}
[0,0,0,0,0,0] \tag{5.7}
\end{equation*}
$$

[We compute left invariant vector fields by taking derivatives of the multiplication function.
$>$ Livf_mult $:=$ convert $\left(\right.$ Livf_from_mult(z), Matrix) ${ }^{+}$
Livf_mult $:=\left[\left[\partial_{x 1}\right]\right.$,
$\left[-\frac{x 5}{2} \partial_{x 1}+\partial_{x 2}\right]$,
$\left[\frac{x 5^{2}}{12} \partial_{x 1}-\frac{x 5}{2} \partial_{x 2}+\partial_{x 3}\right]$,
$\left[\frac{x 5^{2}}{12} \partial_{x 2}-\frac{x 5}{2} \partial_{x 3}+\partial_{x 4}\right]$,
$\left[\left(\frac{1}{2} x 2-\frac{1}{12} x 3 x 5-\frac{1}{720} x 5^{3} x 6\right) \partial_{x 1}+\left(\frac{x 3}{2}-\frac{x 4 x 5}{12}\right) \partial_{x 2}+\left(\frac{x 4}{2}+\frac{x 5 x 6}{12}\right) \partial_{x 3}\right.$
$\left.-\frac{x 6}{2} \partial_{x 4}+\partial_{x 5}\right]$,
$\left.\left[\frac{x 5^{4}}{720} \partial_{x 1}-\frac{x 5^{2}}{12} \partial_{x 3}+\frac{x 5}{2} \partial_{x 4}+\partial_{x 6}\right]\right]$
WWe create a new Lie algebra with these vector fields.
$>L 2:=$ LieAlgebraData(convert(Livf_mult, list), Alg2)
$L 2:=[e 1, e 2]=0,[e 1, e 3]=0,[e 1, e \overline{4}]=0,[e 1, e 5]=0,[e 1, e 6]=0,[e 2, e 3]=0,[e 2, e 4$
$]=0,[e 2, e 5]=e 1,[e 2, e 6]=0,[e 3, e 4]=0,[e 3, e 5]=e 2,[e 3, e 6]=0,[e 4, e 5]=e 3$, $[e 4, e 6]=0,[e 5, e 6]=e 4$
> DGsetup(L2)

LWe see that they give the same multiplication table that we started with.
>> MultiplicationTable( )

| Alg2 | $e 1$ | $e 2$ | $e 3$ | $e 4$ | $e 5$ | $e 6$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $e 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $e 2$ | 0 | 0 | 0 | 0 | $e 1$ | 0 |
| $e 3$ | 0 | 0 | 0 | 0 | $e 2$ | 0 |
| $e 4$ | 0 | 0 | 0 | 0 | $e 3$ | 0 |
| $e 5$ | 0 | $-e 1$ | $-e 2$ | $-e 3$ | 0 | $e 4$ |
| $e 6$ | 0 | 0 | 0 | 0 | $-e 4$ | 0 |

## 2. Left Invariant Vector Fields in Coordinates of the 1st Kind by Algebraic Formula

- Nilpotent Case.
[We set up a coordinate manifold.
[> dgsetupvars:= []:
$[>$ for $i$ from 1 to adim do dgsetupvars := $[o p(d g s e t u p v a r s), x \| i]$ end do :
[> DGsetup (dgsetupvars, $M$ ) :
[We compute left invariant vector fields by formula [1] from the slides.
>Livf_algebraic := NilpotentLivflstKind $(\operatorname{Alg}, M)$
Livf_algebraic : $=\left[\left[\partial_{x 1}\right]\right.$,
$\left[-\frac{1}{2} x 5 \partial_{x 1}+\partial_{x 2}\right]$,
$\left[\frac{1}{12} x 5^{2} \partial_{x 1}-\frac{1}{2} x 5 \partial_{x 2}+\partial_{x 3}\right]$,
$\left[\frac{1}{12} x 5^{2} \partial_{x 2}-\frac{1}{2} x 5 \partial_{x 3}+\partial_{x 4}\right]$,
$\left[\left(\frac{1}{2} x 2-\frac{1}{12} x 3 x 5-\frac{1}{720} x 5^{3} x 6\right) \partial_{x 1}+\left(\frac{1}{2} x 3-\frac{1}{12} x 4 x 5\right) \partial_{x 2}+\left(\frac{1}{2} x 4\right.\right.$

$$
\left.\left.+\frac{1}{12} x 5 x 6\right) \partial_{x 3}-\frac{1}{2} x 6 \partial_{x 4}+\partial_{x 5}\right],
$$

$$
\left.\left[\frac{1}{720} x 5^{4} \partial_{x 1}-\frac{1}{12} x 5^{2} \partial_{x 3}+\frac{1}{2} x 5 \partial_{x 4}+\partial_{x 6}\right]\right]
$$

We see that these are the same vector fields given by taking derivatives of the multiplicaiton function.
[> evalDG(Livf_mult - Livf_algebraic)
$\left[\begin{array}{cc}0 \\ 0 & \partial_{x I} \\ 0 & \partial_{x I} \\ 0 & \partial_{x I} \\ 0 & \partial_{x 1} \\ 0 & \partial_{x I}\end{array}\right]$
[We verify that these vector fields are left invariant.
LWe set up second coordinate manifold.
[> dgsetupvars $:=[]:$

## $>$ for $i$ from 1 to adim do dgsetupvars $:=[o p(d g s e t u p v a r s), y \| i]$ end do :

 —> DGsetup (dgsetupvars, $N$ ):[We check left invariance by pushing forward Livf_algebraic evaluated at y to xy and see that it is the same vector as Liff_algebraic evaluated at xy.
This test is similar to the one illustrated on the online help for GroupActions [IInvariantVectorsAndForms].
> CheckLeftInvariant(convert(Livf_algebraic, list), z)

$$
\begin{equation*}
\left[0,0 \bar{\partial}_{x l}, 0 \partial_{x l}, 0 \partial_{x l}, 0 \partial_{x l}, 0 \partial_{x l}\right] \tag{7.3}
\end{equation*}
$$

## 3. Multiplication in Canonical Coordinates of the 2nd Kind Nilpotent Case.

[We caluclate coordinates of the second kind for a generic point in coordinates of the 1st kind.
—> coordinate_1st $:=[]:$
$\gg$ for $i$ from 1 to adim do coordinate_1st $:=[$ op(coordinate_1st), $x \| i]$ end do :
[> NilpotentCanonicalCoordinateChange(Alg, coordinate_1st, "first")

$$
\left[\begin{array}{l}
{\left[\frac{1}{30} x 5^{4} \times 6-\frac{1}{24} \times 5^{3} \times 4+\frac{1}{6} \times 5^{2} \times 3-\frac{1}{2} \times 5 \times 2+x 1,-\frac{1}{8} x 5^{3} \times 6+\frac{1}{6} x 5^{2} \times 4-\frac{1}{2} \times 3 \times 5\right.}  \tag{9.1}\\
\left.\quad+x 2, \frac{1}{3} x 5^{2} x 6-\frac{1}{2} \times 4 \times 5+x 3,-\frac{1}{2} \times 5 x 6+x 4, x 5, x 6\right]
\end{array}\right.
$$

[We check that converting back to coordinates of the 1st kind gives the original point we started with.
[> simplify(NilpotentCanonicalCoordinateChange(Alg, \%, "second"))

$$
\begin{equation*}
[x 1, x 2, x 3, x 4, x 5, x 6] \tag{9.2}
\end{equation*}
$$

[We calculate $\mathrm{z}=\mathrm{xy}$, where $\mathrm{z}, \mathrm{x}$, and y are group elements in coordinates of the 2 nd kind.
[> z2 := NilpotentBCHmult2 (Alg) :
LWe verify that this is a group multiplicaiton.
Setting $\mathrm{y}=0$ in the multiplication gives x , verifiing that $(0,0,0,0,0,0)$ is the identity.
[> simplify(subs(identvars, z2))

$$
\begin{equation*}
[x 1, x 2, x 3, x 4, x 5, x 6] \tag{9.3}
\end{equation*}
$$

Calculate the Inverse of a generic point by switching to coordinates of the 1 st kind and taking the negative of each coordinate
then switching bback to coordinates of the 2nd kind.
> xuars $:=[]:$
$>$ invvars $:=[]:$
$>$ for $i$ from 1 to adim do xvars := [op(xvars), $x \| i]$ end do:
[> inverse $:=$ NilpotentCanonicalCoordinateChange(Alg, xvars, "second") :
$>$ for $i$ from 1 to adim do invvars $:=[o p($ invvars $)$, -inverse $[i]]$ end do:
> inverse $2:=$ NilpotentCanonicalCoordinateChange(Alg, invvars, "first") :
[We check that this is in fact the inverse.
[> subvars := []:
$[>$ for $i$ from 1 to adim do subvars $:=[$ op(subvars), $y \| i=$ inverse $2[i]]$ end do:
> simplify(subs(subvars, z2))

$$
\begin{equation*}
[0,0,0,0,0,0] \tag{9.4}
\end{equation*}
$$

[We check the associative property by computing $x(y w)-(x y) w$.
> CheckAssociativity(z2)

$$
\begin{equation*}
[0,0,0,0,0,0] \tag{9.5}
\end{equation*}
$$

[We compute left invariant vector fields by taking derivatives of the multiplication function.
$>$ Livf_mult $2:=$ convert(Livf_from_mult(z2), Matrix) ${ }^{+}$

$$
\text { Livf_mult2 }:=\left[\begin{array}{c}
\partial_{x 1} \\
-x 5 \partial_{x 1}+\partial_{x 2} \\
\frac{x 5^{2}}{2} \partial_{x 1}-x 5 \partial_{x 2}+\partial_{x 3} \\
-\frac{x 5^{3}}{6} \partial_{x 1}+\frac{x 5^{2}}{2} \partial_{x 2}-x 5 \partial_{x 3}+\partial_{x 4} \\
\frac{x 5^{3} x 6}{6} \partial_{x 1}-\frac{x 5^{2} x 6}{2} \partial_{x 2}+x 5 x 6 \partial_{x 3}-x 6 \partial_{x 4}+\partial_{x 5} \\
\partial_{x 6}
\end{array}\right]
$$

[We see that they give the same multiplication table that we started with. > MultiplicationTable( )

| Alg3 | $e 1$ | $e 2$ | $e 3$ | $e 4$ | $e 5$ | $e 6$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $e 1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $e 2$ | 0 | 0 | 0 | 0 | $e 1$ | 0 |
| $e 3$ | 0 | 0 | 0 | 0 | $e 2$ | 0 |
| $e 4$ | 0 | 0 | 0 | 0 | $e 3$ | 0 |
| $e 5$ | 0 | $-e 1$ | $-e 2$ | $-e 3$ | 0 | $e 4$ |
| $e 6$ | 0 | 0 | 0 | 0 | $-e 4$ | 0 |

4.5.11

Example 4.3.1

## 4. Calculate Left Invariant 1-Forms.

[We retrieve a Lie algebra so(3) from Winternitz.
[>LieAlgF :=Retrieve("Winternitz", 1, [3, 9], FormAlg)
$\operatorname{LieAlg} F:=[e 1, e 2]=e 3,[e 1, e 3]=-e 2,[e 2, e 3]=e 1$
>> DGsetup (LieAlgF) :
> MultiplicationTable( )

| FormAlg | $e 1$ | $e 2$ | $e 3$ |
| ---: | ---: | ---: | ---: |
| $e 1$ | 0 | $e 3$ | $-e 2$ |
| $e 2$ | $-e 3$ | 0 | $e 1$ |
| $e 3$ | $e 2$ | $-e 1$ | 0 |

[We set up a coordinate manifold.
$\boxed{>}$ adimF $:=$ DGinformation (FormAlg,"LieAlgebraDimension") :
$>$ dgsetupvars $:=[]:$
$\gg$ for $i$ from 1 to adimF do dgsetupvars $:=[o p(d g s e t u p v a r s), x \| i]$ end do :
> DGsetup (dgsetupvars, MF) :
[We compute left invariant 1-forms by formula [2] from the slides.
$>$ omega $:=$ CalcLeftInv1Forms(FormAlg, MF)

$$
\omega:=\left[\begin{array}{c}
\cos (x 3) \cos (x 2) d x 1+\sin (x 3) d x 2  \tag{11.3}\\
-\sin (x 3) \cos (x 2) d x 1+\cos (x 3) d x 2 \\
\sin (x 2) d x 1+d x 3
\end{array}\right]
$$

[We calculate the dual vectors to these 1-forms.
[> omegavars $:=$ [ ]:
$>$ for $i$ from 1 to adimF do omegavars $:=[$ op(omegavars), omega[ $i, 1]]$ end do :
$>$ Livf_ from_lforms $:=\operatorname{simplify}\left(\right.$ convert(DualBasis (omegavars), Matrix) ${ }^{+}$)

$$
\text { Livf_from_lforms }:=\left[\begin{array}{c}
\frac{\cos (x 3)}{\cos (x 2)} \partial_{x 1}+\sin (x 3) \partial_{x 2}-\frac{\cos (x 3) \sin (x 2)}{\cos (x 2)} \partial_{x 3}  \tag{11.4}\\
-\frac{\sin (x 3)}{\cos (x 2)} \partial_{x 1}+\cos (x 3) \partial_{x 2}+\frac{\sin (x 3) \sin (x 2)}{\cos (x 2)} \partial_{x 3} \\
\partial_{x 3}
\end{array}\right]
$$

[We see that they give the same multiplication table that we started with.
> MultiplicationTable( )

| $\operatorname{Alg} 4$ | $e 1$ | $e 2$ | $e 3$ |
| ---: | ---: | ---: | ---: |
| $e 1$ | 0 | $e 3$ | $-e 2$ |
| $e 2$ | $-e 3$ | 0 | $e 1$ |
| $e 3$ | $e 2$ | $-e 1$ | 0 |

## 5. Left Invariant Vector Fields in Coordinates of the 2nd Kind by Algebraic Formula <br> - Nilpotent Case.

[We compute left invariant vector fields by formula [3] from the slides.
$[>$ Livf2_algebraic $:=$ NilpotentLivf2ndKind $($ Alg, $M)$

$$
\text { Livf2_algebraic }:=\left[\begin{array}{c}
\partial_{x 1}  \tag{13.1}\\
-x 5 \partial_{x 1}+\partial_{x 2} \\
\frac{1}{2} x 5^{2} \partial_{x 1}-x 5 \partial_{x 2}+\partial_{x 3} \\
-\frac{1}{6} x 5^{3} \partial_{x 1}+\frac{1}{2} x 5^{2} \partial_{x 2}-x 5 \partial_{x 3}+\partial_{x 4} \\
\frac{1}{6} x 5^{3} x 6 \partial_{x 1}-\frac{1}{2} x 5^{2} x 6 \partial_{x 2}+x 5 x 6 \partial_{x 3}-x 6 \partial_{x 4}+\partial_{x 5} \\
\partial_{x 6}
\end{array}\right]
$$

We see that these are the same vector fields given by taking derivatives of the multiplicaiton function in coordinates of t he 2nd kind.
$>$ evalDG (Livf_mult $2-$ Livf2_algebraic)
$\left[\begin{array}{c}0 \\ 0 \partial_{x 1} \\ 0 \partial_{x 1} \\ 0 \partial_{x 1} \\ 0 \partial_{x 1} \\ 0\end{array}\right]$
[We check left invariance by pushing forward Livf_algebraic evaluated at y to xy and see that it is the same vector as Liff_algebraic evaluated at xy.
>> CheckLeftInvariant(convert(Livf2_algebraic, list), z2)

$$
\begin{equation*}
\left[0,0 \partial_{x l}, 0 \partial_{x l}, 0 \partial_{x l}, 0 \partial_{x l}, 0 \partial_{x I}\right] \tag{13.3}
\end{equation*}
$$

