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Least Squares Estimation of the Pareto Type I and II Distribution

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LEAST SQUARES ESTIMATION OF THE PARETO TYPE I AND II DISTRIBUTION

by

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ABSTRACT

Least Squares Estimation of the

Pareto Type I and II Distribution

by

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Department: Applied Statistics

The estimation of the Pareto distribution can be computationally expensive and the method is badly biased. In this work, an improved Least Squares derivation is used and the estimation will be less biased. Numerical examples and figures are provided so that one may observe the solution more clearly. Furthermore, by varying the different methods of estimation, a comparing of the estimators of the parameters is given. The improved Least Squares derivation is confidently employed for it is economic and efficient.

(37 pages)

CHAPTER I

INTRODUCTION

The Pareto distributions were developed in economics as a distribution of income over a population. Four different types of the distributions have been suggested; however, except for the type I, estimation by maximum likelihood is difficult. Maximum likelihood estimators for types II, III and IV required numerical analysis procedures which can be costly and inefficient. A least squares approach to estimation of the parameters of a Pareto type I distribution is given in Johnson and Kotz (1970). However, the method is badly biased. The primary purpose of this thesis is to derive an improved least squares approach to estimating the parameters of the type I and II distributions.

The Pareto type I and II cumulative distribution functions are defined by

$$F(x) = 1 - \left(\frac{x}{\sigma}\right)^{-\alpha} \qquad \text{(Type I) (1.1)}$$

$$F(x) = 1 - \left(1 + \frac{x - \mu}{\sigma}\right)^{-\alpha} \qquad \text{(Type II)(1.2)}$$

Objectives

1. Derive improved least squares estimators for the type I and II Pareto distribution.
2. Compare the accuracy of estimation by least squares with maximum likelihood estimators for the type I distribution.

Research Method

The least squares approach to estimation requires that the data be expressed as a function of the parameters of the distribution in the linear form

$$y = \theta_1 + \theta_2 x + \xi \tag{1.3}$$

where θ_1 and θ_2 represent the parameters of the distribution and ξ is an error random variable with zero mean (i.e., $E(y/x) = \theta_1 + \theta_2 x$).

Ordinary Least Squares methodology (Kendall & Stuart, 1961) may then be used to derive unbiased estimates of θ_1 and θ_2. It is important that the x in (1.3) be measured without error. It is shown in Chapter 2 that the parameters of the Pareto type I and II distribution can be expressed in the form (1.3) where the sample order statistics are the dependent variables and the independent variables are given by (1.4). These are derived in Chapter 3.

$$E_i = \frac{n!}{(n - i)!} \sum_{j=0}^{i-1} \frac{(-1)^j (n + j - i + 1)^{-2}}{j!(i - j - 1)!} \tag{1.4}$$

The second objective is to be accomplished using Monte Carlo simulation to compare the bias and mean squared error of the least squares estimators with maximum likelihood estimators for the Pareto type I distribution.

CHAPTER II

LEAST SQUARES ESTIMATION IN SIMPLE LINEAR REGRESSION

The least squares method of estimating is simple and very general with computer programs readily available. Estimation of the Pareto type I and II distribution will be separately derived by least squares method in this chapter.

In many practical problems, a dependent variable y can be expressed as a linear function of an independent variable x,

$$Y_i = aX_i + b + \varepsilon_i \tag{2.1}$$

Assumptions:

(1) $E(\varepsilon_i) = 0$.

(2) X_i measured without error.

L.S. estimation derives a and b which minimize

$$\Psi = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (Y_i - (aX_i + b))^2 \tag{2.2}$$

solution

$$\hat{a} = \frac{\sum_{i=1}^{n} X_i Y_i - n\bar{X}\bar{Y}}{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}$$

$$\hat{b} = \bar{Y} - \hat{a}X$$

Under the above assumptions these estimators are unbiased.

Derivation of L.S.E. for the Pareto Type I Distribution

A least squares method for estimating parameters of the Pareto type I distribution is given by Johnson and Kotz (1970).

Let X_i, $i=1, 2, \ldots, n$ be the order statistics of a random sample with CDF

$$F(x_i) = 1 - \left(\frac{x_i}{\sigma}\right)^{-\alpha}$$

$$1-F(X_i) = \left(\frac{X_i}{\sigma}\right)^{-\alpha}$$

$$\ln(1-F(X_i)) = \alpha\ln\sigma - \alpha\ln X_i \qquad (2.3)$$

The parameters α and σ are directly estimated by least squares from sample estimates of $F(x_i)$, using, as dependent variable, the logrithm of 1 minus the cumulative distribution of the sample.

$$\Psi = \sum_{i=1}^{n} \ln(1-F(X_i)) - \alpha\ln\sigma - \alpha\ln X_i)^2$$

$$\frac{\partial\Psi}{\partial r} = -2\sum_{i=1}^{n} \ln(1-F(X_i)) - \alpha r - \alpha\ln X_i)\alpha = 0 \text{ where } r = \ln\sigma$$

$$\frac{\partial\Psi}{\partial\alpha} = -2\sum_{i=1}^{n} \ln(1-F(X_i)) - \alpha r - \alpha\ln X_i)(r + \ln X_i) = 0$$

and get the solution

$$\hat{\alpha} = \frac{-n\sum_{i=1}^{n}\ln X_i \ln(1-F(X_i)) + \sum_{i=1}^{n}\ln X_i(\sum_{i=1}^{n}\ln(1-F(X_i)))}{n\sum_{i=1}^{n}(\ln X_i)^2 - (\sum_{i=1}^{n}\ln X_i)^2} \qquad (2.4)$$

$$\hat{r} = \frac{\sum_{i=1}^{n} \ln (1-F(X_i)) - \hat{\alpha} \sum_{i=1}^{n} \ln X_i}{n\hat{\alpha}} \qquad (2.5)$$

when computating $\hat{\alpha}$ and $\hat{r}$ from (2.4) and (2.5), the observed order statistics X_i are used and $E(F(x_i) = i/(n+1)$ is substituted for $F(x_i)$.

It is informative to investigate (2.3) with $E(F(x_i)) = \frac{i}{n+1}$ in place of $F(X_i)$. Equation (2.3) becomes

$$\ln \left(\frac{n-i+1}{n+1}\right) = \alpha \ln\sigma - \sigma \ln X_i + \varepsilon_i \qquad (2.6)$$

Note that since $E(\ln(1-F(x_i))) \neq \ln((n-i+1)/(n+1))$ it cannot be assumed that $E(\varepsilon_i) = 0$. In addition, for the given formulation (2.6), the term $\ln(\frac{n-i+1}{n+1})$ (associated with the dependent variable in regression) is measured without error. For these reasons the estimates (2.4) and (2.5) are found to be rather poor in applications.

Some improvement in estimation can be obtained by altering (2.6) so that $\ln(\frac{n-i+1}{n+1})$ appears as an independent variable. Thus

$$\ln X_i = -\frac{1}{\alpha}\left(\frac{n-i+1}{n+1}\right) + \ln\sigma + \varepsilon_i \qquad (2.7)$$

However, $E(\varepsilon_i) \neq 0$.

In the next section an improved L.S. estimator is derived which satisfies the requirements of the L.S. method.

Improved Least Squares Estimation

Ordinary Least Squares methodology as given by Kendall and Stuart is used in this section to derive unbiased estimates of the parameters. We take the expected value of both sides of the logrithm (2.3)

$$E_i = E(-\ln(1-F(X_i))) = \alpha E(\ln X_i) - \alpha \ln\sigma$$

Equation (2.3) can then be written

$$\ln X_i = E_i/\alpha + \ln\sigma + \varepsilon_i \qquad (2.8)$$

$$E(\ln X_i) = E_i/\alpha + \ln\sigma$$

it follows that $E(\varepsilon_i) = 0$ in (2.8).

Equation (2.8) is in the proper form for least squares estimation, i.e., $E(\varepsilon_i) = 0$ and the independent variable E_i is measured without error.

For convenience (2.8) is reparameterized so that

$$\ln X_i = \alpha' E_i + r + \varepsilon_i$$

where $\alpha' = \frac{1}{\alpha}$ and $r = \ln\sigma$.

The function to be minimized is given by

$$\Psi = \sum_{i=1}^{n} (\ln X_i - \alpha' E_i - r)^2$$

$$\frac{\partial\Psi}{\partial r} = -2\sum_{i=1}^{n} (\ln X_i - \alpha' E_i - r) = 0 \qquad (2.9)$$

From (2.9), we get

$$\sum_{i=1}^{n} \ln X_i = \hat{\alpha}' \sum_{i=1}^{n} E_i + n\hat{r}$$

$$\hat{r} = -\hat{\alpha}\overline{E} + \sum_{i=1}^{n} \ln X_i/n \qquad (2.10)$$

$$\frac{\partial\Psi}{\partial\alpha} = -2\sum_{i=1}^{n} (\ln X_i - \hat{\alpha}' E_i - r)E_i = 0 \qquad (2.11)$$

From (2.10) and (2.11), we have

$$\sum_{i=1}^{n} \ln X_i E_i - \hat{\alpha}' \sum_{i=1}^{n} E_i^2 + (\hat{\alpha}'\overline{E} - \sum_{i=1}^{n} \ln X_i/n) \sum_{i=1}^{n} E_i = 0$$

$$\sum_{i=1}^{n} \ln X_i E_i - \hat{\alpha}' \sum_{i=1}^{n} E_i^2 + (\sum_{i=1}^{n} E_i)^2/n - \sum_{i=1}^{n} \ln X_i \overline{E} = 0$$

$$\hat{\alpha}' = \frac{\sum_{i=1}^{n} \ln X_i E_i - \sum_{i=1}^{n} X_i \overline{E}}{\sum_{i=1}^{n} E_i^2 - (\sum_{i=1}^{n} E_i)^2/n}$$

<u>Derivation of Least Square Estimation for the Type II Pareto Distribution</u>

The derivation here follows closely the pattern in the previous section.

$$F(X_i) = 1 - (1 + \frac{X_i - \mu}{\sigma})^{-\alpha}$$

$$1 - F(X_i) = 1 + (\frac{X_i - \mu}{\sigma})^{-\alpha}$$

$$\ln(1-F(X_i)) = -\alpha \ln(\frac{X_i - \mu + \sigma}{\sigma})$$

Let X_i, $i=1, 2, \ldots, n$ be the order statistics of a random sample and $E_i = E(-\ln(1 - F(X_i)))$ then

$$E_i = \alpha(\ln(X_i - \beta) - r)$$

where $\beta = \mu - \sigma$, $r = \ln\sigma$

$$\ln(X_i - \beta) = \frac{E_i}{\alpha} + r + \varepsilon_i \qquad (2.12)$$

$$\Psi = \sum_{i=1}^{n} (\ln(X_i - \beta) - \alpha' E_i - r)^2$$

where $\alpha' = 1/\alpha$

$$\frac{\partial \Psi}{\partial \beta} = -2 \sum_{i=1}^{n} (\ln(X_i - \beta) - \alpha' E_i - r)\frac{1}{X_i - \beta} \quad (2.13)$$

$$\frac{\partial \Psi}{\partial \alpha} = -2 \sum_{i=1}^{n} (\ln(X_i - \beta) - \alpha' E_i - r)E_i \quad (2.14)$$

$$\frac{\partial \Psi}{\partial r} = -2 \sum_{i=1}^{n} (\ln(X_i - \beta) - \alpha' E_i - r) \quad (2.15)$$

From (2.15), we have

$$\sum_{i=1}^{n} \ln(X_i - \hat{\beta}) = \hat{\alpha}' \sum_{i=1}^{n} E_i + n\hat{r}$$

$$\hat{r} = -\hat{\alpha}'\overline{E} + \sum_{i=1}^{n} \ln(X_i - \hat{\beta})/n \quad (2.16)$$

From (2.14) and (2.16), we have

$$\sum_{i=1}^{n} \ln(X_i - \hat{\beta}) - \hat{\alpha}' \sum_{i=1}^{n} E_i^2 + (\hat{\alpha}'\overline{E} - \sum_{i=1}^{n} \ln(X_i - \beta)/n) \sum_{i=1}^{n} E_i = 0$$

$$\sum_{i=1}^{n} \ln(X_i - \hat{\beta})E_i - \hat{\alpha}' \sum_{i=1}^{n} E_i^2 + (\sum_{i=1}^{n} E_i)^2/n - \sum_{i=1}^{n} \ln(X_i - \hat{\beta})\overline{E} = 0$$

$$\hat{\alpha}' = \frac{\sum_{i=1}^{n} \ln(X_i - \hat{\beta})E_i - \sum_{i=1}^{n}(\ln(X_i - \hat{\beta})\overline{E}}{\sum_{i=1}^{n} E_i^2 - (\sum_{i=1}^{n} E_i)^2/n} \quad (2.17)$$

From (2.13), we have

$$DB = \sum_{i=1}^{n} \frac{\ln(X_i - \hat{\beta})}{X_i - \hat{\beta}} - \hat{\alpha}' \sum_{i=1}^{n} \frac{E_i}{X_i - \hat{\beta}} - \hat{r} \sum_{i=1}^{n} \frac{1}{X_i - \hat{\beta}} = 0 \qquad (2.18)$$

The estimator of B must be chosen so that Ψ is minimized for DB approaches zero. Numerical methods must be used to obtain a solution for B. An example will be given in Chapter 4.

CHAPTER III

DERIVATION OF E_i

In order to get the required form for least squares estimation the expected value E_i must be used. We will show that $-\ln(1-F(X))$ has a standard exponential distribution for any $F(X)$.

Let $F(X)$ be the cumulative distribution function for the random variable X, then

$$F(X) = \int_{-\infty}^{X} f(w)dw$$

and

$$Pr(F(X) \leq g) = P(X \leq F^{-1}(g)) = F(F^{-1}(g)) = g.$$

Let $Y = -\ln(1-F(X))$, then

$$\begin{aligned} P(Y \leq y) &= P(-\ln(1-F(X)) \leq y) \\ &= P((1-F(X)) \geq e^{-y}) \\ &= P(F(X) \leq 1 - e^{-y}) \\ &= 1 - e^{-y} \end{aligned}$$

Therefore $Y = -\ln(1 - F(X))$ has exponential distribution with $\theta = 1$.

In least squares estimation of the Pareto type I and II distribution

$$Y = ax + b + \varepsilon$$

where the expected value of the order statistics of an eponential random variable and measured without error. These expected values are known to be (Kendall & Stuart, 1961)

$$E_{ik} = \frac{n!k!}{(n - i)!} \sum_{j=0}^{i-1} \frac{(-1)^j (n + j - i + 1)^{-k-1}}{j!(i - j - 1)!}$$

where E_{ik} is the expected value of the i^{th} order statistics with power k which is equal to 1 for the standard exponential distribution. For the case k=1, E_i can also be simplified to $\sum_{j=1}^{i} (n - j + 1)^{-1}$ (A. E. Sarhan & B. G. Greenberg, 1962).

After E_i applied in the estimation, we get the linear form for least squares estimation (2.8) and (2.12). The estimator of the parameters of Pareto distribution will be unbiased after E_i is used in estimation.

CHAPTER IV

ESTIMATION OF THE PARAMETERS OF THE PARETO TYPE I AND II DISTRIBUTION

In this chapter examples of the uses of the estimation functions for the type I and type II distributions are given. Monte Carlo data is used to provide sample values.

The type I and II distributions are slightly different. Note in (1.1) and (1.2) that if $\mu = \sigma$ they are the same.

Example 1. The data in Table I is a Monte Carlo sample of 20 order statistics drawn from a Pareto type I population with shape parameter $\alpha = 2$ and scale parameter $\sigma = 1$.

Table 1

Data List for Example 1

1.0256	1.0450	1.0452	1.0504	1.1336	1.1563	1.1853	1.1854	1.2895	1.2910
0.0493	0.0843	0.0845	0.0937	0.2219	0.2520	0.2882	0.2884	0.3986	0.4000
1.3471	1.4239	1.4519	1.8322	1.8966	1.9749	2.1968	2.2936	3.6240	3.6863
0.4489	0.5068	0.5256	0.7021	0.7219	0.7436	0.7928	0.8099	0.9237	0.9264

Since $-\ln(1-F(X))$ has a standard exponential distribution for any $F(X)$, the E_i can be computed from (1.4).

Table 2

Expected Values of 20 Order Statistics

of a Standard Exponential Random Variable

0.0500	0.1026	0.1582	0.2170	0.2795	0.3462	0.4176	0.4945	0.5779	0.6688
0.7688	0.8799	1.0049	1.1478	1.3144	1.5144	1.7644	2.0977	2.5978	3.5977

From Chapter 2, the estimation of type I formula, we have

$\alpha = 2.4166130$ $\sigma = 1.0083267$

the sum squared error (Ψ) is SSE = 0.037707.

We will be comparing the SSE of Maximum Likelihood in the next chapter.

Estimation of Pareto Distribution Type II

Example 2. Suppose we have the same data in x and F(x) as Table I. The data is drawn from a Pareto type II population with shape parameter $\alpha = 2$, location parameter $\mu = 1$, scale parameter $\sigma = 1$. The same E_i as in Table II is used. From (2.16), (2.17), and (2.18) the estimated parameters α and r are

$$\hat{r} = -\alpha\overline{E} + \sum_{i=1}^{n} \ln(X_1 - \hat{\beta})/n$$

$$\hat{\alpha}' = \frac{\sum_{i=1}^{n} \ln(X_i - \hat{\beta})E_i - \sum_{i=1}^{n} \ln(X_i - \hat{\beta})\overline{E}}{\sum_{i=1}^{n} E_i^2 - (\sum_{i=1}^{n} E_i)^2/n}$$

and $\hat{\beta}$ is the solution of

$$DB = \sum_{i=1}^{n} \frac{\ln(X_i - \hat{\beta})}{X_i - \hat{\beta}} - \hat{\alpha}' \sum_{i=1}^{n} \frac{E_i}{X_i - \hat{\beta}} - \hat{r} \sum_{i=1}^{n} \frac{1}{X_i - \hat{\beta}} = 0$$

The parameter B is most difficult to estimate since an analytic solution is not available. A numerical method must be used.

Several different values of B can be used to get the respective values of SSE and plot B against SSE. From Figure (4.1) we know that the value of B which minimizes SSE is between zero and -0.5. A numerical search is then used to find the approximate minimum. The resulting solution is

$\hat{\alpha}$ = 2.8527103

$\hat{\sigma}$ = 1.3169471

$\hat{\mu}$ = -0.326500

SSE = 0.0340562

B	1.000	0.500	0.000	-0.500	-1.000
SSE	4.072	0.095	0.038	0.035	0.039

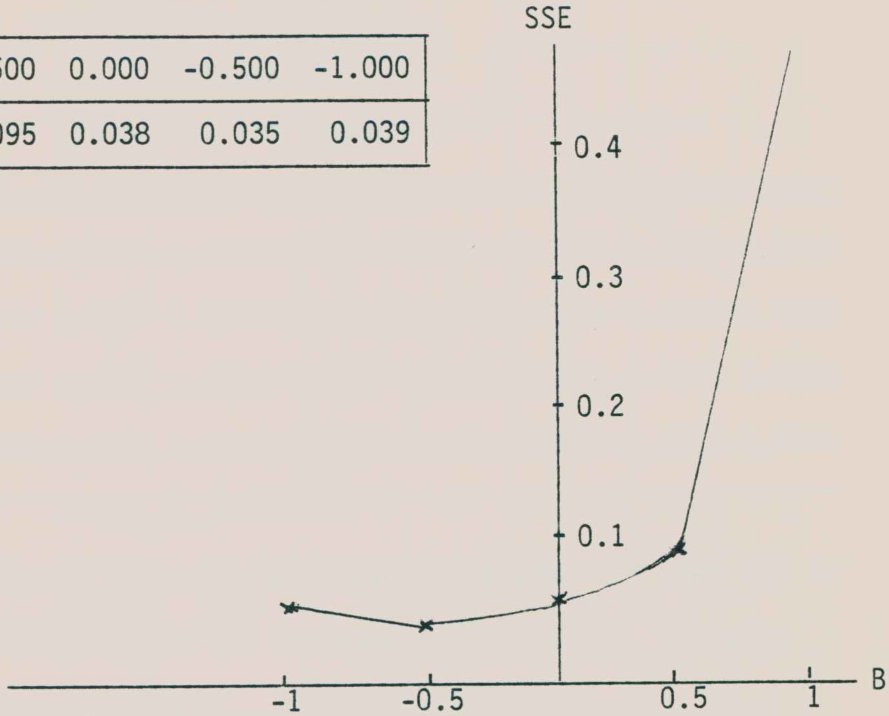

Figure 1. The regression of estimator of location and SSE for example 2.

Example 3. The data in Table III is a Monte Carlo sample of 20 order statistics x drawn from a Pareto type II population with shape parameter $\alpha = 2$, location parameter $\mu = 0$, and scale parameter $\sigma = 1$.

Table 3

Data List for Example 2

0.2561	0.4500	0.4521	0.5045	0.1336	0.1563	0.1853	0.1842	0.2895	0.2910
0.0493	0.0883	0.0846	0.0937	0.2219	0.2520	0.2882	0.2884	0.3986	0.4000
0.3471	0.4239	0.4519	0.8322	0.8966	0.9749	1.1968	1.2936	2.6204	2.6863
0.4489	0.5068	0.5256	0.7021	0.7219	0.7436	0.7928	0.8099	0.9237	0.9264

Solution: The range of B should be between -1.0 and -1.5 when SSE is minimized. The plot of B and SSE is shown in Figure 4.2. Using a numerical search method we have

$\hat{\alpha} = 2.8527103$

$\hat{\mu} = -0.00955$

$\hat{\sigma} = 1.3169471$

SSE = 0.0340562

B	0.000	-0.500	-1.000	-1.200	-1.500
SSE	4.072	0.095	0.038	0.0345	0.0346

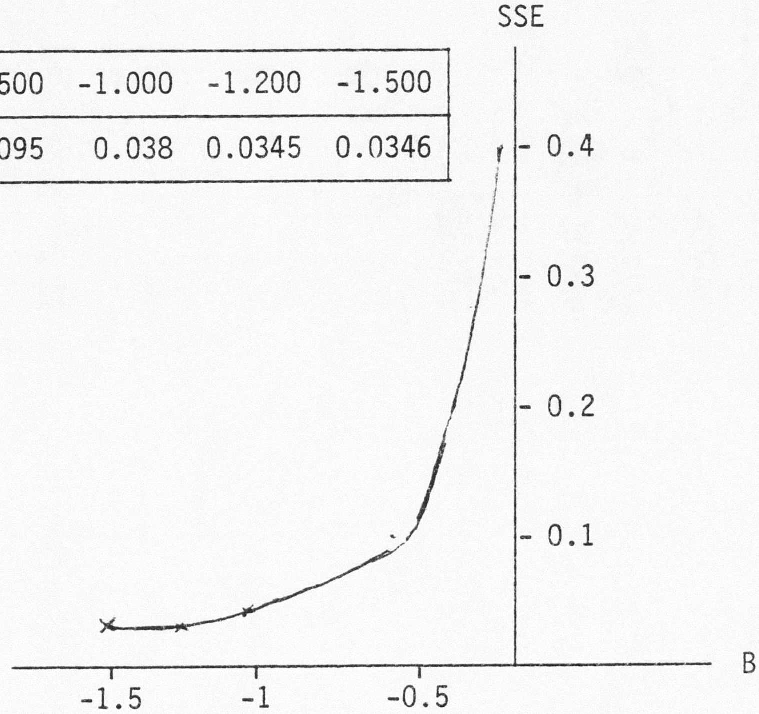

Figure 2. The regression of estimator of location and SSE for example 3.

CHAPTER V

ACCURACY OF ESTIMATION BY LEAST SQUARES AND THE MAXIMUM LIKELIHOOD

The objective of this chapter is to compare two methods of estimating the Pareto distribution.

The Pareto type I distribution is given by $F(x) = 1 - (\frac{\sigma}{x})^{\alpha}$ where $\sigma > 0$, $\alpha > 0$ and $x \geq \sigma$. Its parameters σ and α (where σ is the lower bound of the random variable x) can be estimated by a variety of methods. The maximum likelihood estimator is considered to be best and thus is a good method to compare with the method of least squares.

The Method of Maximum Likelihood

The likelihood function for a random sample $(x_1, \ldots, x_n)$ from a Pareto type I is

$$L = \prod_{i=1}^{n} f(x_i) = \frac{\alpha^n \sigma^{\alpha n}}{(\prod_{i=1}^{n} x_i)^{\alpha + 1}}$$

and taking logrithms,

$$\ln L = n\ln\alpha + \alpha\ln\sigma - (\alpha + 1) \sum_{i=1}^{n} \ln x_i$$

Hence

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + n\ln\sigma - \sum_{i=1}^{n} \ln x_i = 0$$

yielding for the estimate

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln x_i/\hat{\sigma}} \tag{5.1}$$

A maximum likelihood estimate cannot be obtained for σ by differentiating L with respect to σ since L is unbounded with respect to (5.2). But since σ is the lower bound of the random variable x, we may maximize L subject to the constraint.

$$\hat{\sigma} \leq \min x_i \tag{5.2}$$

Clearly L is maximized with respect to (5.2) when

$$\hat{\sigma} = \min x_i$$

which is, therefore, the maximum likelihood estimate for σ .

For the data of example 1 in Chapter 4, we have the estimator of shape parameter $\hat{\alpha}$ = 2.5201225, the estimator of scale parameter $\hat{\sigma}$ = 1.025611 by the method of maximum likelihood. The sum of squared error is 0.041520.

Comparison and Conclusion

To compare the accuracy of estimation by least squares with maximum likelihood estimators for the Pareto type I distribution, we use Monte Carlo method to generate the 20 sample values as in example 1. This is repeated 1,000 times. The mean and mean squared error of estimators of σ and α are computed and shown in Table IV.

Table 4

Comparison of the Estimators of the Parameter of

Pareto Type I Distribution by L.S. with M.L.

Method	$\bar{\hat{\alpha}}$	MSE $\hat{\alpha}$	$\bar{\hat{\sigma}}$	MSE $\hat{\sigma}$
Maximum Likelihood	2.24503	0.34860	1.02524	0.00137
Least Squares	2.20005	0.50112	1.00290	0.00876

Comparing the two mean squared errors of the estimator of the parameters of Pareto type I distribution, it can be seen that the maximum likelihood method performs better than least squares method. However, the averages indicate that the L.S. estimator has less bias.

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APPENDICES

Appendix A

Computer Program in Least Squares Method For Type I Pareto Distribution

```
C*      ***************************************************************
C*
C*      LEAST SQUARES METHOD FOR TYPE I PARETO DISTRIBUTION
C*      INPUT N    =NUMBER OF SAMPLE VALUES
C*            SCALE=SCALE PARAMETER OF TYPE I PARETO DISTRIBUTION
C*            ALPHA=SHAPE PARAMETER OF TYPE I PARETO DISTRIBUTION
C*      OUTPUT E(I)=EXPECTED VALUES OF ORDER STATISTICS OF A STANDARD
C*                  EXPONENTIAL RANDOM VARIABLE
C*             AHAT=ESTIMATOR OF SHAPE PARAMETER
C*             EXP(GHAT)=ESTIMATOR OF SCALE PARAMETER
C*
C*      ***************************************************************
        REAL SCALE,X(100),E(100),U(100)
        READ(5,/)N,SCALE,ALPHA,LAR
C*      GENERATE DATA BY MONTE CARLO METHOD
        TEMP=0
        DO 1 I=1,N
        V=RANDOM(LAR)
        U(I)=1.-(1.-TEMP)*V**(1./(N-I+1))
        X(I)=(EXP(-ALOG(1.-U(I))/ALPHA))*SCALE
        TEMP=U(I)
1       WRITE(6,/)X(I),U(I)
C*      COMPUTE THE EXPECTED VALUE
        E(1)=0
        DO 30 I=1,N
        DO 20 K=1,I
        TEM=FAC(N-K+2)
20      E(I)=E(I)+1/TEM

        WRITE(6,/)E(I)
30      CONTINUE
        SSE=0
        SE=0
        DO 4 I=1,N
        SE=SE+E(I)
4       SSE=SSE+E(I)*E(I)
        SSE=SSE-SE*SE/N
        SE=SE/N
        WRITE(6,10)
10      FORMAT(1X,'ALPHA',12X,'SIGMA')
        SLN=0.
        SLNE=0.
        DO 5 J=1,N
        XB=X(J)
        AXB=ALOG(XB)
        SLN=SLN+AXB
        SLNE=SLNE+AXB*E(J)
        AHAT=SSE/(SLNE-SLN*SE)
5       GHAT=-SE/AHAT+SLN/N
        WRITE(6,11)AHAT,EXP(GHAT)
```

```
11      FORMAT(2(X,E15.8))
        SUM=0
        DO 77 I=1,N
        YY=1.0-(EXP(GHAT)/X(I))**AHAT
        DY=YY-U(I)
        SUM=SUM+DY**2
77      CONTINUE
        WRITE(6,55)SUM
55      FORMAT(24HOTHE SUM SQUARE ERROR IS,E13.6)
        END
        FUNCTION FAC(M)
        FAC=1
        IF(M-1)8,8,6
6       DO 9 J=2,M
9       FAC=FAC*J
8       RETURN
        END
```

```
C*     ***************************************************************
C*
C*     LEAST SQUARE METHOD FOR TYPE I PARETO DISTRIBUTION REAPTED
C*     INPUT N    =NUMBER OF SAMPLE VALUES
C*           SCALE=SCALE PARAMETER OF TYPE I PARETO DISTRIBUTION
C*           ALPHA=SHAPE PARAMETER OF TYPE I PARETO DISTRIBUTION
C*           MC   =REPEATED TIMES
C*     OUTPUT AV  =AVERAGE OF ESTIMATOR OF SHAPE PARAMETER
C*            SV  =AVERAGE OF ESTIMATOR OF SCALE PARAMETER
C*            AD  =MSE OF ESTIMATOR OF SHAPE PARAMETER
C*            SD  =MSE OF ESTIMATOR OF SCALE PARAMETER
C*
C*     ***************************************************************
       REAL SCALE,X(100),X(100),E(100),U(100)
C*     N=20 SCALE=1 ALPHA=2 LAR=9731357 MC=1000
       READ(5,/)N,SCALE,ALPHA,LAR,MC
C*     COMPUTE THE EXPECTED VALUE
       E(1)=0
       DO 30 I=1,N
       DO 20 K=1,I
       TEM=FAC(N-K+2)
20     E(I)=E(I)+1/TEM

30     CONTINUE
       SSE=0
       SE=0
       DO 4 I=1,N
       SE=SE+E(I)
4      SSE=SSE+E(I)*E(I)
       SSE=SSE-SE*SE/N
       SE=SE/N
C*     GENERATE DATA BY MONTE CARLO METHOD
       SB=0
       AB=0
       AHATS=0
       SS=0
       DO 40 IX=1,MC
       TEMP=0
       DO 1 I=1,N
       V=RANDOM(LAR)
       U(I)=1.-(1.-TEMP)*V**(1./(N-I+1))
       X(I)=(EXP(-ALOG(1.-U(I))/ALPHA))*SCALE
       TEMP=U(I)
1      CONTINUE
C*     COMPUTE THE ESTIMATOR OF PARAMETERS
C*     COMPUTE THE AVERAGE OF THE ESTIMATOR OF PARAMETERS
C*     COMPUTE THE MSE OF THE ESTIMATOR OF PARAMETERS
       SLN=0.
       SLNE=0.
       DO 5 J=1,N
       XB=X(J)
```

```
      AXB=ALOG(XB)
      SLN=SLN+AXB
      SLNE=SLNE+AXB*E(J)
      AHAT=SSE/(SLNE-SLN*SE)
5     GHAT=-SE/AHAT+SLN/N
      AHATS=AHATS+AHAT
      AA=AHAT-ALPHA
      AB=AB+AA*AA
      SS=SS+EXP(GHAT)
      SA=EXP(GHAT)-SCALE
      SB=SB+SA*SA
40    CONTINUE
      AV=AHATS/MC
      SV=SS/MC
      AD=AB/MC
      SD=SB/MC
      WRITE(6,11)AV,SV
11    FORMAT(1X,"AHAT-AV=",E15.8,5X,"SCALEHAT-AV=",E15.8)
      WRITE(6,12)AD,SD
12    FORMAT(1X,"AHAT MSE=",E15.8,3X,"SCALEHAT MSE=",E15.8)
      END
      FUNCTION FAC(M)
      FAC=1
      IF(M-1)8,8,6
6     DO 9 J=2,M
9     FAC=FAC*J
8     RETURN
      END
```

Appendix B

Computer Program in Least Squares Method for Type II Pareto Distribution

Example II

```
C*     ************************************************************
C*
C*     LEAST SQUARES METHOD FOR TYPE II PARETO DISTRIBUTION-EXAMPLE 2
C*     INPUT N=NUMBER OF SAMPLE VALUES
C*           LOC=LOCATION PARAMETER OF TYPE II PARETO DISTRIBUTION
C*           SCALE=SCALE PARAMETER OF TYPE II PARETO DISTRIBUTION
C*           ALHPA=SHAPE PARAMETER OF TYPE II PARETO DISTRIBUTION
C*     OUPUT X(I) =THE GENERATED SAMPLE VALUES
C*           U(I) =THE FUNCTION VALUES OF THE PARETO TYPE II CDF
C*           E(I) =EXPECTED VALUES OF ORDER STATISTICS OF A STANDARD
C*                 EXPONENTIAL RANDOM VARIABLE
C*           SMALL=SUM SQUARED ERROR OF THE REGRESSION
C*           AHAT =ESTIMATOR OF SHAPE PARAMETER
C*           EXP(GHAT)=ESTIMATOR OF SCALE PARAMETER
C*           B=ESTIMATOR OF BETA(BETA=LOCATION-SCALE)
C*
C*     ************************************************************
       REAL SA(1000)
       REAL LOC,SCALE,X(100),E(100),U(100)
C*     GENERATE DATA BY MONTE CARLO METHOD
       READ(5,/)N,LOC,SCALE,ALPHA,LAR
       WRITE(6,15)N,LOC,SCALE,ALPHA,LAR
15     FORMAT(1X,'N=',I3,2X,'LOC='I3,2X,'SCALE=',I3,2X,'ALPHA=',I3
      *,2X,'LAR=',I8)
       WRITE(6,7)
7      FORMAT(1X,'X(I)',T19,'U(I)')
       TEMP=0
       DO 1 I=1,N
       V=RANDOM(LAR)
       U(I)=1.-(1.-TEMP)*V**(1./(N-I+1.))
       X(I)=(EXP(-ALOG(1.-U(I))/ALPHA)-1.)*SCALE+LOC
       TEMP=U(I)
1      WRITE(6,/)X(I),U(I)
C*     COMPUTE THE EXPECTED VALUE
       WRITE(6,10)
10     FORMAT(24H0THE EXPECTED VALUES ARE)
       E(1)=0
       DO 30 I=1,N
       DO 20 K=1,I
       TEM=FAC(N-K+2)
20     E(I)=E(I)+1/TEM
       WRITE(6,/)E(I)
30     CONTINUE
C*
       SSE=0
       SE=0
       DO 4 I=1,N
```

```
      SE=SE+E(I)
4     SSE=SSE+E(I)*E(I)
      SSE=SSE-SE*SE/N
      SEB=SE/N
      WRITE(6,40)
40    FORMAT(1X,'SSE',T19,'ALPHA',T35,'SIGMA',T51,'BETA')
C*
      B=-0.5
      DO 17 K=1,1000
      B=B+0.0005
      SLN=0
      SLNE=0
      SAOXB=0.
      SEOXB=0.
      SXB=0.
      DO 5 J=1,N
      XB=X(J)-B
      AXB=ALOG(XB)
      SLN=SLN+AXB
      SLNE=SLNE+AXB*E(J)
      SAOXB=SAOXB+AXB/XB
      SEOXB=SEOXB+E(J)/XB
5     SXB=SXB+1./XB
      AHAT=SSE/(SLNE-SLN*SEB)
      GHAT=-SEB/AHAT+SLN/N
      DB=SAOXB-SEOXB/AHAT-GHAT*SXB
      SA(K)=0
      DO 33 I=1,N
      Y=1-(EXP(GHAT)/(X(I)-B))**AHAT
      S=U(I)-Y
      SA(K)=SA(K)+S*S
33    CONTINUE
      IF(K.EQ.1)GO TO 60
      D=SMALL-SA(K)
      IF(D.LE.0.0)GO TO 65
60    SMALL=SA(K)
17    CONTINUE
65    WRITE(6,11)SMALL,AHAT,EXP(GHAT),B
11    FORMAT(4(X,E15.8))
      END
      FUNCTION FAC(M)
      FAC=1
      IF(M-1)8,8,6
6     DO 9 J=2,M
9     FAC=FAC*J
8     RETURN
      END
```

```
C*     ***************************************************************
C*
C*     LEAST SQUARES METHOD FOR TYPE II PARETO DISTRIBUTION-EXAMPLE 3
C*     INPUT N=NUMBER OF SAMPLE VALUES
C*           LOC=LOCATION PARAMETER OF TYPE II PARETO DISTRIBUTION
C*           SCALE=SCALE PARAMETER OF TYPE II PARETO DISTRIBUTION
C*           ALHPA=SHAPE PARAMETER OF TYPE II PARETO DISTRIBUTION
C*     OUPUT X(I) =THE GENERATED SAMPLE VALUES
C*           U(I) =THE FUNCTION VALUES OF THE PARETO TYPE II CDF
C*           E(I) =EXPECTED VALUES OF ORDER STATISTICS OF A STANDARD
C*                 EXPONENTIAL RANDOM VARIABLE
C*           SMALL=SUM SQUARED ERROR OF THE REGRESSION
C*           AHAT =ESTIMATOR OF SHAPE PARAMETER
C*           EXP(GHAT)=ESTIMATOR OF SCALE PARAMETER
C*           B=ESTIMATOR OF BETA(BETA=LOCATION-SCALE)
C*
C*     ***************************************************************
       REAL SA(1000)
       REAL LOC,SCALE,X(100),E(100),U(100)
C*     GENERATE DATA BY MONTE CARLO METHOD
       READ(5,/)N,LOC,SCALE,ALPHA,LAR
       WRITE(6,15)N,LOC,SCALE,ALPHA,LAR
15     FORMAT(1X,'N=',I3,2X,'LOC='I3,2X,'SCALE=',I3,2X,'ALPHA=',I3
      *,2X,'LAR=',I8)
       WRITE(6,7)
7      FORMAT(1X,'X(I)',T19,'U(I)')
       TEMP=0
       DO 1 I=1,N
       V=RANDOM(LAR)
       U(I)=1.-(1.-TEMP)*V**(1./(N-I+1.))
       X(I)=(EXP(-ALOG(1.-U(I))/ALPHA)-1.)*SCALE+LOC
       TEMP=U(I)
1      WRITE(6,/)X(I),U(I)
C*     COMPUTE THE EXPECTED VALUE
       WRITE(6,10)
10     FORMAT(24HOTHE EXPECTED VALUES ARE)
       E(1)=0
       DO 30 I=1,N
       DO 20 K=1,I
       TEM=FAC(N-K+2)
20     E(I)=E(I)+1/TEM
       WRITE(6,/)E(I)
30     CONTINUE
C*
       SSE=0
       SE=0
       DO 4 I=1,N
```

Appendix C

Computer Program in

Least Squares Method For

Type II Pareto Distribution

Example III

```
      SE=SE+E(I)
4     SSE=SSE+E(I)*E(I)
      SSE=SSE-SE*SE/N
      SEB=SE/N
      WRITE(6,40)
40    FORMAT(1X,'SSE',T19,'ALPHA',T35,'SIGMA',T51,'BETA')
C*
      B=-1.5
      DO 17 K=1,1000
      B=B+0.0005
      SLN=0
      SLNE=0
      SAOXB=0.
      SEOXB=0.
      SXB=0.
      DO 5 J=1,N
      XB=X(J)-B
      AXB=ALOG(XB)
      SLN=SLN+AXB
      SLNE=SLNE+AXB*E(J)
      SAOXB=SAOXB+AXB/XB
      SEOXB=SEOXB+E(J)/XB
5     SXB=SXB+1./XB
      AHAT=SSE/(SLNE-SLN*SEB)
      GHAT=-SEB/AHAT+SLN/N
      DB=SAOXB-SEOXB/AHAT-GHAT*SXB
      SA(K)=0
      DO 33 I=1,N
      Y=1-(EXP(GHAT)/(X(I)-B))**AHAT
      S=U(I)-Y
      SA(K)=SA(K)+S*S
33    CONTINUE
      IF(K.EQ.1)GO TO 60
      D=SMALL-SA(K)
      IF(D.LE.0.0)GO TO 65
60    SMALL=SA(K)
17    CONTINUE
65    WRITE(6,11)SMALL,AHAT,EXP(GHAT),B
11    FORMAT(4(X,E15.8))
      END
      FUNCTION FAC(M)
      FAC=1
      IF(M-1)8,8,6
6     DO 9 J=2,M
9     FAC=FAC*J
8     RETURN
      END
```

Appendix D

Computer Program in Maximum Likelihood Method For Type I Pareto Distribution

```
C*      ***************************************************************
C*
C*      MAXIMUM LIKELIHOOD METHOD FOR TYPE I PARETO DISTRIBUTION
C*      INPUT N=NUMBER OF SAMPLE VALUES
C*            SCALE=SCALE PARAMETER OF TYPE II PARETO DISTRIBUTION
C*            ALHPA=SHAPE PARAMETER OF TYPE II PARETO DISTRIBUTION
C*      OUPUT X(I) =THE GENERATED SAMPLE VALUES
C*            U(I) =THE FUNCTION VALUES OF THE PARETO TYPE II CDF
C*                  EXPONENTIAL RANDOM VARIABLE
C*            SMALL=ESTIMATOR OF SCALE PARAMETER
C*            AHAT =ESTIMATOR OF SHAPE PARAMETER
C*            SSE  =SUM SQUARED ERROR OF THE REGRESSION
C*
C*      ***************************************************************
        REAL SCALE,X(100),E(100),U(100)
        READ(5,/)N,SCALE,ALPHA,LAR
        TEMP=0
        DO 1 I=1,N
        V=RANDOM(LAR)
        U(I)=1.-(1.-TEMP)*V**(1./(N-I+1))
        X(I)=(EXP(-ALOG(1.-U(I))/ALPHA))*SCALE
        TEMP=U(I)
1       WRITE(6,/)X(I),U(I)
        SMALL=X(1)
        BIG=SMALL
        DO 10 I=2,N
        IF(X(I)-BIG)20,20,30
30      BIG=X(I)
        GO TO 10
20      IF(X(I)-SMALL)40,10,10
40      SMALL=X(I)
10      CONTINUE
        WRITE(6,/)SMALL
        TMP=0
        DO 4 I=1,N
        TMP=ALOG(X(I)/SMALL)+TMP
4       CONTINUE
        ALPHAH=N/TMP
        WRITE(6,/)ALPHAH
        SSE=0
        DO 77 I=1,N
        YY=1-(SMALL/X(I))**ALPHAH
        DY=YY-U(I)
        SSE=SSE+DY**2
77      CONTINUE
        WRITE(6,55)SSE
55      FORMAT(24H0THE SUM SQUARE ERROR IS,E13.6)
        END
```

```
C*    ****************************************************************
C*
C*    M.L. METHOD FOR TYPE I PARETO DISTRIBUTION REAPTED
C*    INPUT N     =NUMBER OF SAMPLE VALUES
C*          SCALE =SCALE PARAMETER OF TYPE I PARETO DISTRIBUTION
C*          ALPHA =SHAPE PARAMETER OF TYPE I PARETO DISTRIBUTION
C*          MC    =REPEATED TIMES
C*    OUTPUT AV   =AVERAGE OF ESTIMATOR OF SHAPE PARAMETER
C*           SV   =AVERAGE OF ESTIMATOR OF SCALE PARAMETER
C*           AD   =MSE OF ESTIMATOR OF SHAPE PARAMETER
C*           SD   =MSE OF ESTIMATOR OF SCALE PARAMETER
C*
C*    ****************************************************************
      REAL SCALE,X(100),E(100),U(100)
      READ(5,/)N,SCALE,ALPHA,LAR,MC
C*    GENERATE THE DATA BY MONTE CARLO METHOD
      SB=0
      AB=0
      AHATS=0
      SS=0
      DO 50 IX=1,MC
      TEMP=0
      DO 1 I=1,N
      V=RANDOM(LAR)
      U(I)=1.-(1.-TEMP)*V**(1./(N-I+1))
      X(I)=(EXP(-ALOG(1.-U(I))/ALPHA))*SCALE
      TEMP=U(I)
1     CONTINUE
C*    COMPUTE ESTIMATOR OF PARAMETERS
      SMALL=X(1)
      BIG=SMALL
      DO 10 I=2,N
      IF(X(I)-BIG)20,20,30
30    BIG=X(I)
      GO TO 10
20    IF(X(I)-SMALL)40,10,10
40    SMALL=X(I)
10    CONTINUE
      TMP=0
      DO 4 I=1,N
      TMP=ALOG(X(I)/SMALL)+TMP
4     CONTINUE
      ALPHAH=N/TMP
C*    COMPUTE AVERAGE OF ESTIMATOR OF PARAMETER
C*    COMPUTE MSE OF ESTIMATOR OF PARAMETER
      AHATS=AHATS+ALPHAH
      AA=ALPHAH-ALPHA
      AB=AB+AA*AA
      SS=SS+SMALL
      SA=SMALL-SCALE
```

```
      SB=SB+SA*SA
50    CONTINUE
      AV=AHATS/MC
      SV=SS/MC
      AD=AB/MC
      SD=SB/MC
      WRITE(6,11)AV,SV
11    FORMAT(1X,"AHAT-AV=",E15.8,5X,"SCALEHAT-AV=",E15.8)
      WRITE(6,12)AD,SD
12    FORMAT(1X,"AHAT MSE=",E15.8,3X,"SCALEHAT MSE=",E15.8)
      END
```