On the Stability Analysis of Linear Continuous-Time Distributed Systems

Mohammad Shekaramiz

Dept. of Electrical and Computer Engineering, Utah State University, Logan, Utah, USA

Course Project on Distributed Control Systems, Instructor: Dr. Sara Dadras

***Abstract*- This paper discusses the stability problem of linear continuous-time distributed systems. When dealing with large-scale systems, usually there is not thorough knowledge of the interconnection models between different parts of the entire system. In this case, a useful stability analysis method should be able to deal with high dimensional systems accompanied with bounded uncertainties for its interconnections. In this paper, in order to formulate the stability criterion for large-scale systems, stability analysis of LTI systems is first considered. Based on the existing methods for estimating the spectra of square matrices, sufficient criteria are proposed to guarantee the asymptotic stability of such systems. One of the advantages of these stability conditions is in analyzing linear systems having uncertainties. In this case, a new sufficient criterion is introduced. Back to the main purpose of the paper, it will be proved that the method can also be used for the stability investigation of large-scale systems accompanied with bounded time-variant uncertainties. Then the maximum permissible bounds for the interconnections while holding the stability will be obtained. Since in analyzing large-scale systems there is hardly thorough knowledge about the interactions between sub-systems, finding such bounds is of great importance. Unlike most of the previous work, this method is not restricted to structured uncertainties belonging to convex sets. The merit of the suggested stability analysis is illustrated via several examples.**

***Index Terms*- Large-scale systems, Stability analysis, LTI systems, Uncertainty, Gerschgorin theorem, Eigenvalue estimation.**

I. INTRODUCTION

Stability is one of the most important issues of any control system. In this case, there exist huge amount of literature on the stability of different types of control systems [1-28]. During the last few decades, systems have become larger and more complex in such a way that it makes it difficult to analyze the entire system. Therefore, researches and control designers have put much more attention to the control algorithms and stability methods to deal with such large-scale systems which might be accompanied with some sort of uncertainties and time delays as the practical sense. For the purpose of stability analysis of these systems there exist two main approaches. Most of the proposed methods are based on the previous results in small-scale systems and researchers seek to extend and apply those to large-scale systems. The other approach is to take advantage of recent developments in the analysis of networks and using graph-theory and so forth.

As is mentioned before, stability analysis of large-scale systems has been paid great attention during the last few decades. Basically, analysis of these large-scale systems is difficult and if they include either some sort of uncertainties or delays, then the problem becomes more daunting [1-7]. In this area, Suh and Bien [1] proposed a stability sufficient condition for large-scale systems having time-invariant delay. But checking the stability using their approach requires finding positive scalars satisfying set of inequalities which is generally difficult. Hmamed [2] dealt with systems of the same structure and provided a sufficient delay-independent stability criterion. This approach is based on matrix measurement and checking the obtained inequalities for all the complex numbers lie inside the unit-circle of the complex plain which is not an easy task. Huang et al. [4] extended Hmamed's result [2] and based on Lyapunov function and matrix norms, they provided a theorem for systems having time-invariant delays. Since their stability condition is related to finding matrix measurements for each sub-system, it is conservative.

Nian and Li [5] included uncertainties as well as delays in the structure of their large-scale model. They derived a sufficient stability condition from the spectra of matrices P and Q of Lyapunov method in the corresponding sub-systems. But meeting requirements of their obtained inequalities need to find some free variables which make it conservative and difficult. Xu [6] represented a lemma for stability of such systems having time-variant delays again based on matrix norm and the eigenvalues of positive-definite matrices P and Q of the Lyapunov function. In 2012, Lee and Chen discussed the stability of large-scale systems when accompanied with delay and non-linear interactions. Their proposed stability theorem requires satisfying set of inequalities that have two free scalars for each sub-system. But as aforementioned before finding these free variables make it difficult to fulfill the inequalities.

This paper aims to deal with stability analysis of large-scale systems from another point of view. When analyzing large-scale systems we usually do not have enough knowledge about the interactions. Hence in this paper the author mainly seeks to find the bounds for the interactions in which the stability of the entire system is still guaranteed. The proposed method for finding such bounds is based on methods for estimating the eigenvalues. For this purpose, we first try to apply the results to small-scale linear systems and then the extension to large-scale systems will be done.

In case of small-scale systems, stability analysis of LTI systems has been investigated via various methods. Famous methods such as Routh-Hurwitz stability criterion, Lyapunov functions, and calculating the eigenvalues of the state matrices are some of the standard approaches for checking the stability of such systems. But when having systems with high dimensional state matrices accompanied by uncertainties or time-variant parameters, using such methods seem to be abortive or time-consuming.

The stability problem of uncertain linear systems has received considerable attention and numerous criteria have been proposed for this issue during the last three decades [8-19]. It is worth noting that most of these stability conditions are based on Lyapunov method [8-17]. A single quadratic Lyapunov function is one of the traditional methods for the stability investigation of such systems. However, using it for the aim of robust stability investigation seems to be conservative. In order to reduce this conservatism, other methods such as parameter-dependent Lyapunov functions and piecewise Lyapunov functions were introduced. Based on using parameter-dependent Lyapunov functions, many criteria have been proposed to check the robust stability of systems having time-varying uncertain parameters which are less conservative than quadratic stability [11-15,17]. Let's review some of the previous works in this area.

Based on Lyapunov function and by approximating the uncertain region with a convex hyper-polyhedron, Gu et al [8] provided a necessary and sufficient condition to guarantee the quadratic stability of uncertain linear systems. Interestingly, it has also been shown that their method can be used for checking the stability of other types of systems like uncertain Takagi-Sugeno fuzzy models [20]. Fang and Loparo [10] used a Lyapunov equation for the nominal system and proposed an approach to check the robust stability of linear systems having structured uncertainty. Though improving the obtained bounds for the structured uncertainty comparing to the previous works, it requires satisfying one of the three defined inequalities which seems to be time-consuming. Romas and Peres [11] introduced a sufficient condition for the robust stability of continuous-time uncertain linear systems with convex bounded uncertainties. Based on linear matrix inequalities, they constructed a parameter dependent Lyapunov function to guarantee the stability of any matrix inside the defined uncertainty domain. Montanger et al [12] proposed a sufficient condition for the robust stability of linear systems with time-varying uncertainty. They guaranteed the stability of means of a parameter-dependent Lyapunov function and imposing bounds on the time derivatives of the uncertain parameters. Their method was based on two assumptions: the uncertain parameters belong to a polytope and the time derivatives of them are defined in certain bounds.

In terms of linear matrix inequalities and based on parameter-dependent Lyapunov function, Zhai et al [14] investigated the robust stability of such systems having real parametric uncertainty. However, the method requires checking the existence of some symmetric matrices and a skew matrix which seems not to be an easy task. Yang and Dong [15] derived stability criteria for the existence of a parameter-dependent Lyapunov function in order to guarantee the robust stability of those systems having polytopic uncertainty. Amao et al [16] considered the robust stability problem for linear uncertain systems subjected to parametric time-varying uncertainties. In this case, they made use of polyhedral Lyapunov functions and obtained less conservative results for such systems compared to the classical quadratic stability method.

It can be seen that many of those aforementioned studies on this issue have been based on the parameter dependent Lyapunov functions. However, the introduced stability conditions seem to be rather conservative and some impose very restrictive assumptions. Moreover, most of the obtained criteria for the stability problem are based on the fact that the uncertainties are structured and are confined to convex sets. These restrictions have led the researchers to think of introducing approaches which are not dependent on Lyapunov functions [18,19]. Gong and Thompson [18] considered the stability problem of systems having unstructured parameter perturbation. Their stability condition was derived from the polar decomposition of the nominal system matrix. Ren et al [19] studied linear systems having constant parameter uncertainty. By applying a Guardian map, they derived sufficient criteria to guarantee the robust stability of such systems. Though proposing a different and novel approach, their method is firstly limited to a certain type of uncertainties. Secondly, checking the stability via this method requires satisfying number of inequalities which will be increased when having higher-dimensional state matrices.

As have been stated before, most of the previous studies for the stability problem of linear systems having uncertainties are based on Lyapunov function with the assumption of having structured uncertainties belonging to convex sets. In this paper, the authors seek to propose a different method that has not the restrictions and conservativeness of the famous Lyapunov function. In this case, we will investigate the stability via estimating the eigenvalues of the perturbed system directly. The idea is inspired from [21-25,27,28], where the authors studied the estimation methods for the eigenvalues of any arbitrary matrix.

The estimation and location of eigenvalues have been always one of the important topics in matrix theory. In this area, the famous Gerschgorin disk theorem estimates all the eigenvalues of any arbitrary complex matrix in the union of defined disks [22-24]. C. K. Li and R. C. Li in [25] improved the previous results in estimating the eigenvalues of Hermitian matrices. They estimated the eigenvalues of such matrices from the eigenvalues of their corresponding diagonal matrices. In other words, the sub-diagonal entries were considered to be perturbations on the diagonal matrix which was defined as the nominal matrix. Aside from just estimating the spectra, this method can also be used for the stability investigation of linear systems. In this case, Shekaramiz and Sheikholeslam took advantage of such a method to analyze the stability of Takagi-Sugeno fuzzy models [26]. Zou and Jiang [27] showed that the spectra of an arbitrary complex matrix can be found in one closed disk. Though providing a better estimation in comparison to the previous works, it is not an easy task to compute. Xingdong et al [28] dealt with the eigenvalue estimation in order to propose a solution to the perturbed matrix Lyapunov equation. It is a novel work applicable to control theory and linear system stability. However, the method is again based on Lyapunov method and the obtained inequality seems not to be easily fulfilled.

In this paper, firstly, the common method for the stability analysis of large-scale systems is reviewed. Then, the famous Gerschgorin circle theorem and some other recent estimation methods for estimating the eigenvalues will be discussed. Gerschgorin circle theorem is a method that specifies regions in which the spectra of any complex square matrix do exist. By using the aforementioned theorem, some sufficient criteria will then be proposed in order to check the asymptotic stability of LTI systems. Moreover, the stability problem of linear systems having uncertainties will be considered. In this case, a sufficient criterion is proposed to guarantee the asymptotic stability of such systems. After building up all the required preliminaries, the main purpose of the paper will be stated. In this case, the obtained results from the small-scale systems will be extended and applied to large scale systems. Finally, we will find the maximum permissible bounds for the interactions in which the entire large-scale system is asymptotically stable.

This paper is organized as follows: Common method for stability analysis of large-scale systems and some estimation methods for the eigenvalues of arbitrary matrices will be presented in section II. In section III, sufficient criteria will be introduced to seek the asymptotic stability of LTI systems. Section IV will provide sufficient stability conditions for continuous-time linear systems having uncertainties. Furthermore, numerical examples will be presented to demonstrate the effectiveness of the proposed method. Then in section V, stability analysis of autonomous large-scale systems will be represented and the maximum allowable bounds for the interactions to hold the system stable will be derived. Finally, the concluding remarks are given in section VI.

II. PRELIMINARIES

In this section, first a brief model description of the interaction-oriented model of large-scale systems is given. Then, the common method of analyzing the stability of such systems is discussed. Finally, some existing methods for estimating the eigenvalues of a matrix are represented.

II.I INTERACTION-ORIENTED MODEL

Consider the unforced large-scale system below.

$$\dot{X}_o = A_o X_o \tag{1}$$

It is straight forward to show that the system in Eq. (1) can be represented by the following interaction-oriented model

$$\begin{cases} \dot{X}_i = A_{ii}X_i + E_iS_i \\ Z_i = C_{zi}X_i \end{cases}, \ i=1,2,\ldots,M. \tag{2}$$

where the interaction models can be defined as

$$S_i = \sum_{j=1}^{M} L_{ij}Z_j \tag{3}$$

In the above equation, L_{ij} are the interconnection matrices and the interconnection gain is defined by

$$l_{ij} = \|L_{ij}\| \tag{4}$$

such that

$$\|S_i\| \leq \sum_{j=1}^{M} l_{ij}\|Z_j\| \tag{5}$$

Now, in order to check the stability of the whole system, one must consider both the properties of each sub-system and the characterization of their interconnections. Let's review the common method for dealing with the stability of such systems.

The very first step is to verify the stability of each isolated sub-system i.e., neglecting the interactions.

$$\dot{X}_i = A_{ii}X_i \ , \ i=1,2,\ldots,M. \tag{6}$$

The stability investigation of the isolated sub-systems can be done using the Lyapunov method as follows. Systems in Eq. (6) are asymptotically stable if for all $i = 1,2,\ldots,M$ and for any $Q_i = Q_i^T > 0$ there exist $P_i = P_i^T > 0$ satisfying the set of Lyapunov equations below.

$$P_iA_{ii} + A_{ii}^TP_i = -Q_i \tag{7}$$

The second step is to identify the aggregate models describing the sub-systems' dynamics and their mutual interactions which can be found by the following non-quadratic Lyapunov function.

$$V_i(X_i) = \sqrt{X_i^TP_iX_i} \ , \ i=1,2,\ldots,M. \tag{8}$$

The followings are the key parameters and models.

$\forall \ i=1,2,\ldots,M.$

$$\begin{cases} \dot{X}_i = A_{ii}X_i + E_iS_i \\ Z_i = C_{zi}X_i \end{cases} \qquad \text{Revisited (2)}$$

$$b_{i1} = \|E_i\| \tag{9}$$

$$b_{i2} = \|C_{zi}\| \tag{10}$$

$$C_{i1} = \sqrt{\lambda_{min}(P_i)} \tag{11}$$

$$C_{i2} = \sqrt{\lambda_{max}(P_i)} \tag{12}$$

$$C_{i3} = \frac{\lambda_{min}(Q_i)}{2\sqrt{\lambda_{max}(P_i)}} \tag{13}$$

$$C_{i4} = \frac{\lambda_{max}(P_i)}{\sqrt{\lambda_{min}(P_i)}} \tag{14}$$

$$\|S_i\| \leq \sum_{j\neq i} \bar{l}_{ij}\|Z_i\| \tag{15}$$

where

$$\|L_{ij}\| \leq \bar{l}_{ij} \tag{16}$$

The third step is to verify the stability of the aggregate overall system using the aggregate parameters which can be done as follows.

$$V(X) = \begin{bmatrix} V_1(X_1) \\ \vdots \\ V_M(X_M) \end{bmatrix} \tag{17}$$

The overall aggregate model is

$$\dot{V} \leq MV \tag{18}$$

where the matrix *M* is defined as follows.

$$\mu_{ij} = \begin{cases} -\dfrac{C_{i3}}{C_{i2}} & if\ j = i \\ C_{i4} b_{i1} \bar{l}_{ij} \dfrac{b_{j2}}{C_{j1}} & if\ j \neq i \end{cases} \tag{19}$$

Finally, the stability of the whole unstructured model is guaranteed if the aggregate overall model is stable. In other words, the whole system is asymptotically stable if all the spectra of the obtained matrix *M* have negative real-part values. Moreover, using the Gerschgorin theorem (which will be defined in the next sub-section), it also can be proved that the whole system is asymptotically stable if the below set of inequalities hold.

$$\frac{C_{i3}}{C_{i2}} > C_{i4} b_{i1} \bar{l}_{ij} \frac{b_{j2}}{C_{j1}} \tag{20}$$

II.II ESTIMATION OF THE SPECTRA

Estimating the location of eigenvalues is one of the hot topics in matrix analysis. In this section, some of the estimating methods for the spectra of any arbitrary matrix are represented.

In 1909, Schur presented the following well-known estimation for the eigenvalues of an arbitrary square matrix A [29].

$$\sum_{i=1}^{n}|\lambda_i|^2 \leq \|A\|_F^2, , \quad \|A\|_F = (tr(A^*A))^{\frac{1}{2}} \tag{21}$$

Kress et al. [30] improved the upper bound of the above disk as follows

$$\sum_{i=1}^{n}|\lambda_i|^2 \leq \left\{\|A\|_F^4 - \frac{1}{2}\|[A,A^*]\|_F^2\right\}^{\frac{1}{2}} \tag{22}$$

where,

$[A,B] = AB - BA$

But since the aforementioned estimations include a disk centered at the origin of the complex plane, they cannot be used for the purpose of stability analysis.

Gerschgorin circle theorem is another estimation method that seeks the location of eigenvalues for any arbitrary square matrix. It specifies circular regions in which the entire eigenvalues of a square matrix can be found. The theorem is described below.

Gerschgorin Circle Theorem 1 [21, 22]: Consider an arbitrary complex square matrix $A_{n\times n}$ as follows.

$$A_{n\times n} = \lfloor a_{ij} \rfloor, i, j = 1,2,..., n. \tag{23}$$

Then, any eigenvalue λ of the matrix A is located in at least one of the closed disks of the complex plane centered at a_{ii} and having the defined radius R_i^R below. These disks are called Gerschgorin disks.

$$R_i^R = \sum_{\substack{j=1 \\ j\neq i}}^{n}|a_{ij}| \tag{24}$$

In other words,

$$\forall\, \lambda \in \delta(A)\ ,\exists i \quad \text{such that} \quad |\lambda - a_{ii}| \leq \mathrm{R_i^R} \tag{25}$$

where $\delta(A)$ is the spectrum of matrix A such that

$\delta(A) = \{\lambda_1, \lambda_2,..., \lambda_n\}$.

Corollary 1 [23]: Since the above result also holds for the transpose of matrix A, we can formulate a version of the Gerschgorin circle theorem based on columns sums instead of rows sums and reach to the following results.

$$\forall\, \lambda \in \delta(A)\ ,\exists\, j \quad \text{such that} \quad |\lambda - a_{jj}| \leq \mathrm{R_j^C} \tag{26}$$

where,

$$R_j^C = \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \tag{27}$$

The result is obtained from the fact that the eigenvalues of any arbitrary matrix are equal to the eigenvalues of its transpose.

Gu [31] proved that all the eigenvalues of any complex matrix $A_{n\times n}$ are located in the following disk.

$$\left\{ z \in C : \left| z - \frac{trace(A)}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(\|A\|_F^2 - \frac{|trace(A)|^2}{n} \right)} \right\} \tag{28}$$

O. Rojo and R. Soro [32] provided almost the same bound below for such estimation.

$$\left\{ \left| \lambda_i - \frac{trace(M)}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(trace(M)^2 - \frac{|trace(M)|^2}{n} \right)} \right\} \tag{29}$$

Zou and Jiang [27] proved that all the eigenvalues of arbitrarily complex matrix are located in one closed disk. Let’s review their main results.

Let $M_n(C)$ be the set of all complex matrices of order n. Let $M = (m_{ij}) \in M_n(C)$ and $M^* = (\overline{m}_{ij}) \in M_n(C)$. Denote by $\lambda(M)$ the class of all eigenvalues of M. Denote

$$\|M\|_F = \left(tr(M^*M)\right)^{\frac{1}{2}}$$

Theorem 2 [27]: Let $M \in M_n(C)$ be $n \times n$ an complex matrix partitioned as

$$M = \begin{bmatrix} A_{k\times k} & B_{k\times(n-k)} \\ C_{(n-k)\times k} & D_{(n-k)\times(n-k)} \end{bmatrix} \tag{30}$$

where $A_{k\times k}$ is a $k \times k$ principal submatrix of $M, 1 \leq k \leq n-1$. Then, all the eigenvalues of M are located in the following disk.

$$\left\{ z \in C : \left| z - \frac{trace(M)}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(\|M\|_F^2 - \frac{|trace(M)|^2}{n} - \max_{1\leq k\leq n-1} \left(\left\|B_{k\times(n-k)}\right\|_F - \left\|C_{(n-k)\times k}\right\|_F \right)^2 \right)} \right\} \tag{31}$$

In other words, all the eigenvalues of the matrix M are located in one closed disk with the following center and radius.

$$C(M) = \frac{trace\ (M)}{n} \tag{32}$$

$$R(M) = \sqrt{\frac{n-1}{n}\left(\|M\|_F^2 - \frac{|trace\ (M)|^2}{n} - \max_{1\leq k\leq n-1} \left(\left\|B_{k\times(n-k)}\right\|_F - \left\|C_{(n-k)\times k}\right\|_F \right)^2 \right)} \tag{33}$$

Now, as has been stated before, the stability problem is dependent on the spectra of matrix M defined in Eq. (19). In other words, we can redefine the stability problem of the whole system as the stability analysis of the system below.

$$\dot{X}(t) = MX(t) \tag{34}$$

Let's go back to our main purpose of this paper. Though the verification of the stability of system Eq. (34) is less demanding than the verification of the stability of the whole large-scale system Eq.(1) (since $M \ll n$: the number of states in the whole system) depending on the number of the defined sub-systems, the stability verification of system Eq. (34) can be a daunting task for large number of sub-systems. Moreover, please note that the interaction models are not thoroughly known. Therefore, the stability method should also accounts for the changes on the interactions i.e., $\bar{l}_{ij}$. This problem is now nothing more than stability analysis of semi-large scale linear systems having uncertainties. In the following section, we aim to first provide some sufficient criteria for the stability analysis of small-scale linear systems, apply them to analyze the stability of system Eq. (34), and then conclude the stability of the whole unstructured system.

III.STABILITY OF CONTINUOUS-TIME SMAILL-SCALE LTI SYSTEMS

In this section, based on Gerschgorin circle theorem, sufficient criteria will be proposed to investigate the stability of LTI systems. Please note that we are dealing with continuous-time linear systems in which all the entries of their relevant state matrices are assumed to have real values. Hence according to Gerschgorin circle theorem, all of the Gerschgorin disks for such systems will be centered on the real-axis of the complex plane. Now, consider the following definitions.

$$\overline{H} = \max_i \{a_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\} \tag{35}$$

$$\underline{H} = \min_i \{a_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\} \tag{36}$$

In the above definitions, $\overline{H}$ and $\underline{H}$ denote the maximum and minimum values that the real-part of eigenvalues of square matrix A may have possess, respectively. For better understanding, ponder the following example.

Example 1: Consider matrix A_1 as follows.

$$A_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

In the figure below the related Gerschgorin disks and the values of $\overline{H}$ and $\underline{H}$ have been illustrated.

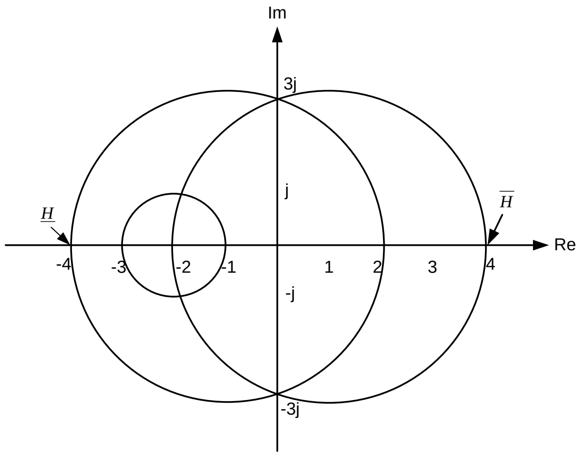

Figure 1: Gerschgorin disks for matrix A_1 and of $\overline{H}$ and $\underline{H}$ obtained from Eq. (35) and Eq. (36), respectively.

Since all the entries of matrix A_1 are real values, the centers of all Gerschgorin disks are placed on the real-axis of the complex plane. This also can be seen in Fig. (1).

In the theorem below, a sufficient criterion will be proposed in order to investigate the asymptotic stability of linear time-invariant systems.

Lemma 1: Consider the unforced LTI system $\dot{X}(t) = A_{n\times n} X(t)$. Where,

$A_{n\times n} = [a_{ij}],\ i, j = 1,2,...,n.$

Then, the system is asymptotically stable if it has a negative value for $\overline{H}$ in the following definition.

$$\overline{H} = \max_i \{a_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\} \tag{37}$$

Proof: Assume that one of the Gerschgorin disks of matrix A has been drawn as follows.

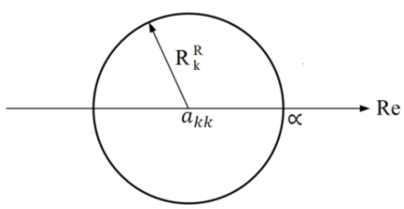

Figure 2: One of the Gerschgorin disks of matrix A in the complex plane.

Please note that all of the entries of matrix A have been assumed to have real values i.e., $a_{kk} \in \mathrm{R}$. Then, the real-part of any arbitrary point being inside or on the boundary of the above closed disk would have the value equal or less than α. Now, consider Eq. (24) and Eq. (25) defined in the Gerschgorin circle theorem.

$$\forall\ \lambda \in \delta(A)\ ,\exists i \ \text{ such that } \ \left|\lambda - a_{ii}\right| \leq \sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right|$$

By having real values for the center of Gerschgorin disks, we can then rewrite the above inequality as follows.

$$a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right| \leq \mathrm{Re}(\lambda) \leq a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right|$$

Now define $\overline{H}_i$ as

$$\overline{H}_i = a_{ii} + \sum_{\substack{j=1\\ i\neq j}}^{n} \left|a_{ij}\right| \tag{38}$$

The term $\overline{H}_i$ denotes the maximum permissible value for λ_k related to the k^{th} Gerschgorin disk. Then define

$$\overline{H} = \max_i (\overline{H}_i) \tag{39}$$

Now, having a negative value for $\overline{H}$ concludes that all of the eigenvalues of matrix A have negative real-part values. In this case, the system is asymptotically stable.

□

Example 2: Consider an LTI system having the below state matrix.

$$A_2 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & -2 & -5 \end{bmatrix}$$

By applying definition Eq. (37) to this matrix, we obtain $\overline{H}(A_2) = -1 \langle 0$. Therefore, according to lemma 1, the system is asymptotically stable. The Gerschgorin disks are shown below.

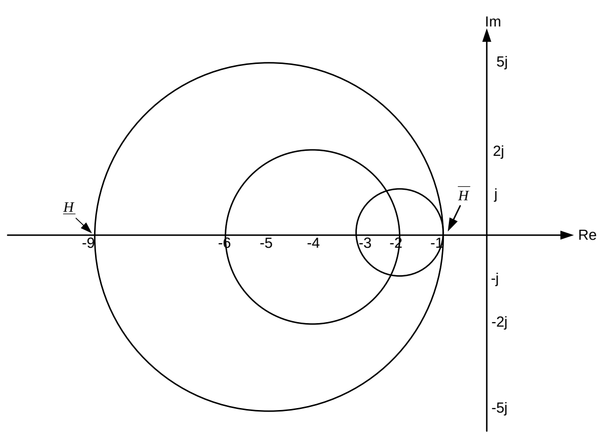

Figure 3: The Gerschgorin disks for matrix A_2 obtained from the Gerschgorin circle Theorem.

Finally, in order to verify the stability conclusion, the eigenvalues of matrix A_2 have been calculated below.

$$\delta(A_2) = -4.787 \pm 2.105\,j, -1.426$$

So far, a sufficient criterion has been proposed in order to guarantee the asymptotic stability of LTI systems. But, there exist many systems that have non-negative values for $\overline{H}$ while the systems are asymptotically stable. Let's have a look at the following example.

Example 3: Consider an LTI system with the below state matrix.

$$A_3 = \begin{bmatrix} -3 & 1 & -6 & 2.5 \\ 1.2 & -9 & 3.6 & -1 \\ 1 & 5 & -12 & 1.8 \\ -0.6 & 2 & 2 & -6 \end{bmatrix}$$

In this example, applying lemma 1 concludes no results about the stability of the system; $\overline{H}(A_3) = 6.5 \rangle 0$ while the system is asymptotic stability.

Hence, our proposed lemma 1 still seems to be conservative. Now, let's introduce another stability criterion for LTI systems.

Corollary 2: Consider the LTI system $\dot{X}(t) = AX(t)$. The system is asymptotically stable if the below definition has negative value.

$$\overline{\overline{H}} = \max_{j} \{a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|\} \tag{40}$$

Proof: The proof can be easily obtained from Gerschgorin circle theorem 1, corollary 1, and lemma 1 and is omitted.

$\square$

Example 4: Consider the system described in Ex. (3). By applying definition Eq. (40) to the system, we reach to $\overline{\overline{H}}(A_3) = -0.2 \langle 0$. Therefore according to corollary 2, the system is asymptotically stable.

Please note that the proposed criterion in corollary 2 also seems conservative. This is because of the fact that there exist many stable systems in which the stability criterion of corollary 2 is not fulfilled. Now, by taking advantage of what was introduced in lemma 1 and corollary 2, we can define the less conservative theorem below.

Theorem 3: Consider an unforced LTI system $\dot{X}(t) = A_{n\times n}\, X(t)$. Where,

$$A_{n\times n} = [a_{ij}],\ i, j = 1,2,\ldots, n.$$

Then, the system is asymptotically stable if the following inequality is satisfied.

$$\overset{\Delta}{H} = \min(\overline{H}, \overline{\overline{H}}) \langle 0 \tag{41}$$

Where,

$$\overline{H} = \max_i \{a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|\}$$ Revisited (37)

Revisited (40)

$$\overline{\overline{H}} = \max_j \{a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|\}$$

Proof: The proof is straightforward and is omitted.

□

Example 5: Consider a continuous-time LTI system having the same state matrix as Ex. (3). By applying the definitions of theorem 3, we reach to the following results.

$$\overline{H}(A_3) = 6.5,\ \overline{\overline{H}}(A_3) = -0.2 \quad \Rightarrow \overset{\Delta}{H}(A_3) = -0.2 \langle 0$$

Therefore, the system is asymptotically stable.

Now, reconsider the state matrix A_3 represented in Ex. 5. It seems that stability investigation for systems having rather higher dimension state matrices by methods such as Routh-Hurwitz criterion, Lyapunov methods, or directly calculating the eigenvalues are much more time-consuming when compared with our proposed stability criteria.

IV. STABILITY ANALYSIS OF SYSTEMS HAVING UNCERTAINTIES

So far, some sufficient criteria have been introduced in order to guarantee the asymptotic stability of LTI systems. In this section, the stability problem of systems having uncertainties will be considered. In this case, based on the Gerschgorin circle theorem and the obtained stability criteria of the previous section, a new stability condition for such systems will be proposed.

Theorem 4: Consider the following unforced continuous-time linear system.

$$\dot{X}(t) = (A_0 + \Delta A(t))\, X(t) \tag{42}$$

Where,

$$A_0 = [a_{ij}],\ \Delta A(t) = [\Delta a_{ij}(t)],\ |\Delta a_{ij}(t)| \leq M_{ij}, \quad i, j = 1,2,\ldots,n. \tag{43}$$

Matrices A_O and $\Delta A(t)$ denote the nominal state matrix and the uncertainty, respectively. Suppose that the system with its nominal state matrix A_O be asymptotically stable. Then, the system in Eq. (42) is stable if the following criterion is satisfied.

$$\overset{\Delta}{H}_{A_0+\Delta A(t)} = \min(\overline{H}_{A_0+\Delta A(t)}, \overline{\overline{H}}_{A_0+\Delta A(t)}) \langle 0 \tag{44}$$

Where,

$$R^{R}_{i,A_0+\Delta A(t)} = \sum_{\substack{j=1\\ j\neq i}}^{n} \left(\left| a_{ij} \right| + M_{ij} \right), i = 1,2,,\ldots,n. \tag{45}$$

$$R^{C}_{j,A_0+\Delta A(t)} = \sum_{\substack{i=1\\ i\neq j}}^{n} \left(\left| a_{ij} \right| + M_{ij} \right), j = 1,2,,\ldots,n. \tag{46}$$

$$C_i = a_{ii} + M_{ii} \ , i = 1,2,,...,n. \tag{47}$$

$$\overline{H}_{A_0+\Delta A(t)} = \max_{i} (C_i + R^{R}_{i,A_0+\Delta A(t)}) \ , i = 1,2,,\ldots,n. \tag{48}$$

$$\overline{\overline{H}}_{A_0+\Delta A(t)} = \max_{j} (C_j + R^{C}_{j,A_0+\Delta A(t)}) \ , j = 1,2,,\ldots,n. \tag{49}$$

The above notations represent the following definitions.

$R^{R}_{i,A_0+\Delta A(t)}$: Maximum permissible value for the radius of i[th] Gerschgorin disk corresponding to i[th] row of $A_0 + \Delta A(t)$.

$R^{C}_{j,A_0+\Delta A(t)}$: Maximum permissible value for the radius of jth Gerschgorin disk corresponding to j[th] column of $A_0 + \Delta A(t)$.

C_i : Maximum permissible value for the center of i[th] Gerschgorin disk corresponding to i[th] row of $A_0 + \Delta A(t)$.

$\overline{H}_{A_0+\Delta A(t)}$: Maximum permissible value for the real-part of spectra of $A_0 + \Delta A(t)$ based on rows sums.

$\overline{\overline{H}}_{A_0+\Delta A(t)}$: Maximum permissible value for the real-part of spectra of $A_0 + \Delta A(t)$ based on columns sums.

Proof: Having maximum values for the radii and centers of Gerschgorin disks, one can obtain the maximum permissible value for the $\overset{\Delta}{H}_{A_0+\Delta A(t)}$ defined in Eq. (44). This value represents the maximum permissible value for the real-part of spectra of matrix $A_0 + \Delta A(t)$. According to theorem 3, having negative value for $\overset{\Delta}{H}_{A_0+\Delta A(t)}$ guarantees the asymptotic stability of the system. The approach for the proof is almost the same as those described in lemma 1 and theorem 3 and is omitted.

□

Example 6: Consider the uncertain linear system defined in Eq. (42) and Eq. (43). The state matrix of the system is as follows.

$$A_0 = \begin{bmatrix} -4 & 2 & -5 & 2 \\ 0.8 & -9 & 3 & -1 \\ 1 & 3 & -12 & 1 \\ -0.5 & 1 & 2 & -6 \end{bmatrix} \text{ and } \Delta A(t) = \begin{bmatrix} a(t) & b(t) & c(t) & e(t) \\ b(t) & c(t) & d(t) & b(t) \\ a(t) & e(t) & b(t) & c(t) \\ d(t) & c(t) & b(t) & a(t) \end{bmatrix}$$

Where,

$|a(t)| \le 0.1,\ |b(t)| \le 0.3,\ |c(t)| \le 0.5,\ |d(t)| \le 0.8,\ |e(t)| \le 1.0$

Now, by applying theorem 4, the following results are obtained.

Table 1: Stability investigation of Ex.6 by applying theorem 4.

Defined terms	*Applied equation*	*Obtained values form the applied equation*	*Stability conclusion of the nominal system*	*Stability conclusion of the system having uncertainties*
$\overline{H}_{A_0}$	Eq. (37)	5	No results	-
$\overline{\overline{H}}_{A_0}$	Eq. (40)	-1.7	Asymptotically stable	-
$\overset{\Delta}{H}_{A_0}$	Eq. (41)	-1.7	Asymptotically stable	-
$\overline{H}_{A_0+\Delta A(t)}$	Eq. (48)	Not useful	-	No results
$\overline{\overline{H}}_{A_0+\Delta A(t)}$	Eq. (49)	-0.1	-	Asymptotically stable
$\overset{\Delta}{H}_{A_0+\Delta A(t)}$	Eq. (44)	-0.1	-	Asymptotically stable

Therefore by having the results obtained in Table 1, one can conclude that for any permissible values of uncertainties being in their relevant bounds, the system is asymptotically stable. In the figure below, the spectra of matrix $A_0 + \Delta A(t)$ for different values of uncertainties have been illustrated. It can also be seen in the figure that the spectra of such system are placed on the left-hand side of the imaginary axis in the complex plane.

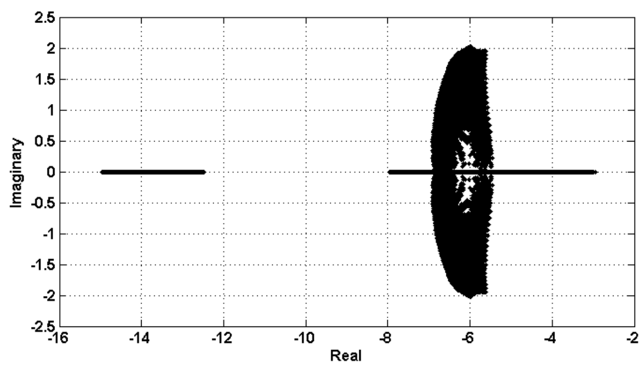

Figure 4: The spectra of matrix $A_0 + \Delta A(t)$ for all permissible values of uncertainties (precision: 0.001).

V. STABILITY ANALYSIS OF LARGE-SCALE SYSTEMS

Let's first revisit our stability problem for large-scale systems. The goal is analyzing the stability of a system as follows.

$$\dot{X}_o = A_o X_o \qquad \text{Revisited (1)}$$

where the number of state variables is in general huge and is denoted by n.

Using the aforementioned discussion in the preliminaries, the problem is then reduced to check the stability of the system below.

$$\dot{X}(t) = MX(t) \qquad \text{Revisited (34)}$$

where the matrix M is defined as follows.

$$\mu_{ij} = \begin{cases} -\dfrac{C_{i3}}{C_{i2}} & if\ j = i \\ C_{i4} b_{i1} \bar{l}_{ij} \dfrac{b_{j2}}{C_{j1}} & if\ j \neq i \end{cases} \qquad \text{Revisited (19)}$$

The above matrix M has a useful feature and that is its main diagonal entries are non-positive. In other words, the Gerschgorin disks are centered at the left-hand side of the complex-plane. Otherwise, we could not take an advantage form the aforementioned theorem. Now, by using what we obtained in the previous section, we reach to the following results.

Corollary 3: Consider the following unforced large-scale linear system.

$$\begin{cases} \dot{X}_i = A_{ii} X_i + E_i S_i \\ Z_i = C_{zi} X_i \end{cases} ,\ i=1,2,\ldots,M. \qquad \text{Revisited (2)}$$

By applying the Lyapunov-based method as was stated in the preliminaries, we reach the following matrix M defined in Eq. (19).

The entire large-scale system in Eq. (1) is asymptotically stable if the following inequality is fulfilled for the matrix M.

$$\overset{\Delta}{H} = \min(\overline{H}, \overline{\overline{H}}) \langle 0 \qquad \text{Revisited (41)}$$

Where,

$$\overline{H} = \max_i \{ \sum_{j=1}^{n} m_{ij} \} \qquad (50)$$

$$\overline{\overline{H}} = \max_j \{ \sum_{i=1}^{n} m_{ij} \} \qquad (51)$$

Proof: The proof is straightforward and is neglected.

□

Now, let’s look at the problem from another point of view. The off-diagonal entries of matrix M are related to the interconnections. But, in practical cases, we usually do not have enough knowledge about the interconnections between the sub-systems. Hence the objective is now finding the bounds for the existing interconnections, while the stability of the whole large-scale systems is guaranteed.

Corollary 4: Suppose the unforced large-scale linear system defined in Eq. (1). Assume that there exists no thorough knowledge about the interconnection matrices of Eq. (2). Applying the common Lyapunov-based method we have now the main diagonal entries of the matrix M in Eq. (19) on hand. Now, the entire system Eq. (1) is asymptotically stable if the interactions in the sub-systems belong to the following bounds.

$$\left\|L_{ij}\right\| \leq \frac{1}{n-1}\frac{C_{i3}C_{j1}}{b_{j2}C_{i4}C_{j2}} \qquad \forall\, i,j = 1,2,\dots,n. \tag{52}$$

Proof: Since we do not have enough knowledge about the interactions, the uncertainties that each main diagonal entry (spectra of the nominal matrix) can tolerate is assumed to be distributed equally between the interactions in the corresponding sub-system. This is why there is "$n-1$" in the denominator of Eq. (52). In other words, the obtained matrix M is broken in the two following parts.

$$\dot{X}_r\,(t) = MX_r\,(t) = \big(M_o + \Delta M(t)\big)X_r\,(t) \tag{53}$$

Where, matrix M_o is defined as the nominal state matrix derived from the sub-systems by neglecting the interactions, and $\Delta M(t)$ is assumed to be the effect of unknown interactions.

$$M_o = \left[\mu_{o,ij}\right], \quad \mu_{o.ij} = \begin{cases} -\dfrac{C_{i3}}{C_{i2}} & if\ j = i \\ \mathbf{0} & if\ j \neq i \end{cases} \tag{54}$$

$$\Delta M(t) = \left[\Delta\mu_{ij}\right], \quad \Delta\mu_{ij} = \begin{cases} \mathbf{0} & if\ j = i \\ C_{i4}b_{i1}l_{ij}\dfrac{b_{j2}}{C_{j1}} & if\ j \neq i \end{cases} \tag{55}$$

The proof can be easily found form the obtained previous results and is omitted.

□

Note: It is obvious that if "k" number of subsystems, have no interaction with sub-system "i", then the matrix measurement of the remained interactions in the corresponding sub-system i.e., "i", can be found as follows.

$$\left\|L_{ij}\right\| \leq \frac{1}{n-k-1}\frac{C_{i3}C_{j1}}{b_{j2}C_{i4}C_{j2}} \qquad \forall j = \{1,2,\dots,n\} - \{k\}. \tag{56}$$

Example 7: Consider the following system where the desired sub-systems are denoted by Σ_1 and Σ_2.

$$\dot{X}(t) = \begin{bmatrix} -1 & 0.1 & 0.2 & 0.1 & 0.2 \\ 0.2 & -2 & 0.5 & 0.1 & 0.1 \\ 0.1 & -1 & -3 & 0.5 & 0.4 \\ 1 & 0 & 1 & -4 & 0.2 \\ 0.2 & 0.5 & 0 & 1 & -5 \end{bmatrix} X(t)$$

Sub-systems:

$$\Sigma_1: \quad \dot{X}_1 = \begin{bmatrix} -1 & 0.1 & 0.2 \\ 0.2 & -2 & 0.5 \\ 0.1 & -1 & -3 \end{bmatrix} X_1 + \left(\begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} X_2 \right)$$

$$\Sigma_2: \quad \dot{X}_2 = \begin{bmatrix} -4 & 0.2 \\ 1 & -5 \end{bmatrix} X_2 + \left(\begin{bmatrix} 1 & 0 & 1 \\ 0.2 & 0.5 & 0 \end{bmatrix} X_1 \right)$$

In this case, as a measurement of interactions we have

$$L_{12} = 0.4455,\ L_{21} = 1.4224 \tag{57}$$

But assume that there is not enough information about the interactions on hand. Here, we want to redefine the problem as follows.

$$\Sigma_1:\quad \dot{X}_1 = \begin{bmatrix} -1 & 0.1 & 0.2 \\ 0.2 & -2 & 0.5 \\ 0.1 & -1 & -3 \end{bmatrix} X_1 + (\boldsymbol{L_{12}} X_2)$$

$$\Sigma_2:\quad \dot{X}_2 = \begin{bmatrix} -4 & 0.2 \\ 1 & -5 \end{bmatrix} X_2 + (\boldsymbol{L_{21}} X_1)$$

Applying the approach described in the preliminaries, one can reach to the following results.

$$P_1 = \begin{bmatrix} 0.5078 & 0.0230 & 0.0321 \\ 0.0230 & 0.2544 & -0.0065 \\ 0.0321 & -0.0065 & 0.1677 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.1258 & 0.0140 \\ 0.0140 & 0.1066 \end{bmatrix}$$

It worth noting that matrices P_1 and P_2 are obtained from Lyapunov equations where matrices Q_1 and Q_2 are assumed to be equal to identity.

$C_{11} = 0.4048,\ C_{12} = 0.7160,\ C_{13} = 0.6983,\ C_{14} = 1.2664,\ C_{21} = 0.3077,\ C_{22} = 0.3665,\ C_{23} = 1.3644,\ C_{24} = 0.4364,\ b_{11} = b_{12} = b_{21} = b_{22} = 1$

Having the above matrices and parameters on hand, the main diagonal entries of matrix M can be easily obtained.

$$M = \begin{bmatrix} -0.9753 & ? \\ ? & -3.7228 \end{bmatrix}$$

Now, by applying Eq. (52) in corollary 4 the bounds below for the interactions will be obtained.

$$\|L_{ij}\| \le \frac{1}{n-1} \frac{C_{i3} C_{j1}}{b_{j2} C_{i4} C_{j2}},\ \ \|L_{12}\| \le 0.4629,\ \ \|L_{21}\| \le 1.7676$$

In other words, for any arbitrary interactions having the 2-norm within the above bounds, the entire system is asymptotically stable. Since for our case (when the knowledge for interactions is complete) the interactions measurements in Eq. (57) lie in their corresponding obtained bounds, the entire system is asymptotically stable. This also shows that for any other structure of interactions satisfying their corresponding bounds, the still will still be stable.

VI. CONCLUSION

The stability problem of continuous-time LTI large-scale systems has been considered. For the purpose of the stability investigation of such systems, the stability of linear systems having uncertainties was first established. Sufficient criteria have been proposed to guarantee the asymptotic stability of such systems. The merit of this approach is mainly in linear systems having uncertainties in their state matrices or in systems having time-variant state matrices. In this case, a theorem has been proposed in order to investigate the asymptotic stability of such systems. Unlike most of the previous works in this area, our method is independent of any types of Lyapunov function. Therefore, it is not confined to having special structure for uncertainties. Finally, the obtained results were extended to check the stability of large-scale systems. In this case, the maximum permissible interactions' matrix measurements that guarantee the stability of the entire system were derived. Due to the fact that in the practical cases the interactions in the sub-systems are not usually completely specified, this is of great importance when dealing with such systems.

REFRENCES

[1] I. H. Suh and Z. Bien, "A note on the stability of large scale systems with delays," *IEEE Trans. Automat. Contr.*, 1982.

[2] A. Hmamed, "Note on the stability of large-scale system with delays," *Int. J. Syst. Sci.*, 1986.

[3] A. Hmamed, "Further results on the delay-independent asymptotic stability of linear systems," *Int. J. Syst. Sci.*, 1991.

[4] S. Huang, H. Shao, Z. Zhang, "Stability analysis of large-scale systems with delays," *Syst. and Contr., Elsevier*, 1995.

[5] X. Nian, R. Li, "Robust stability of uncertain large-scale systems with time-delay," *Int. J. of Syst. Sci.*, 2001.

[6] B. Xu, "Stability criteria for linear systems with uncertain delays," *J. Math. and Appl.*, 2003.

[7] C. Lee, C. Chen, "Further results for robust stability of homogeneous large-scale bilinear systems with time delays and uncertainties," *Compt. & Math. with Appl.* 2012.

[8] K. Gu, M. A. Zohdy, and N. Loh, "Necessary and sufficient conditions of quadratic stability of uncertain linear systems," *IEEE Trans. on Auto. Contr.,* Vol. 35, No. 5, pp. 601-604, May. 1990.

[9] A. L. Zelentsovsky, "Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems," *IEEE Trans. on Auto. Contr.,* Vol. 39, No. 1, pp. 135-138, Jan. 1994.

[10] Y. Fang and K. A. Loparo, "Stability robustness bounds and robust stability for linear systems with structured uncertainty," *Proc. of the American Contr. Conf.*, pp. 221-225, Baltimore, Maryland, June 1994.

[11] D. C. W. Ramos and P. L. D. Peres, "An LMI condition for the robust stability of uncertain continuous-time linear systems," *IEEE Trans. on Auto. Contr.*, Vol. 47, No. 4, pp. 675-678, April. 2002.

[12] V. F. Montagner and P. L. D. Peres, "A new LMI condition for the robust stability of linear time-varying systems", *Proc. of the 42nd IEEE Conf. on Decision and Contr.*, pp. 6133-6138, Hawaii, USA, Dec. 2003.

[13] D. Henrion, D. Arzelier, D. Peaucelle, and J. B. Lasserre, "On parameter-dependent Lyapunov functions for robust stability of linear systems," *43rd IEEE Conf. on Decision and Contr.*, pp. 887-892, Atlantis, Dec. 2004.

[14] D. Zhai, Y. Zhang, B. Dong, G. Liu, and L. Liang, "Stability Analysis for Uncertain Linear Systems", *Chinese Contr. and Decision Conf., IEEE,* pp. 3016-3018, China, 2008.

[15] G. H. Yang and J. Dong, "Robust stability of polytopic systems via affine parameter-dependent Lyapunov functions," *Joint 48th Conf. on Decision and Contr. Conf., IEEE,* pp. 75-80, Shanghai, Dec. 2009.

[16] F. Amato, R. Ambrosino, and M. Ariola, "Robust stability via polyhedral Lyapunov functions," *American Contr. Conf.,* USA, pp. 3736-3741, June 2009.

[17] N. Aouani, S. Salhi, G. Garcia, and M. Ksouri, "Parameter dependent Lyapunov functions for stability of linear parameter varying systems," *ICECS, IEEE,* pp. 1002-1005, 2010.

[18] C. Gong and S. Thompson, "Stability margin evaluation for uncertain linear systems," *IEEE Trans. on Auto. Contr.,* Vol. 39, No. 3, pp. 548-550, March. 1994.

[19] S. Rern, P. T. Kabamba, and D. S. Bernstein, "Guardian map approach to robust stability of linear systems with constant real parameter uncertainty," *IEEE Trans. on Auto. Contr.,* pp. 162-164, Vol. 1, No. 1, Jan. 1994.

[20] J. Joh, Y. H. Chen, and R. Langari, "On the stability issues of linear Takagi-Sugeno fuzzy models," *IEEE Trans. on Fuzzy Syst.*, Vol. 6, No. 3, pp. 402-410, August 1998.

[21] F. Szidarovszly, and A., T., Bahill, *Linear Systems Theory*, Second Edition, CRC Press, 1998, ISBN: 0849316871.
[22] R., S., Varga, *Gerschgorin and His Circles*, Springle, First Edition, 2004.
[23] Y., Saad, *Numerical methods for large eigenvalue problems*, Manchester University Press (Series in Algorithms and Architectures for Advanced Scientific Computing), Manchester, 1992.
[24] G., W. Stewart, and J., G., Sun, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.
[25] C. K. Li and R. C. Li, "A note on eigenvalues of perturbed Hermitian Matrices," *Linear Algebra and its Appl.*, *Elsevier*, Vol. 395, pp. 183-190, 2005.
[26] M. Shekaramiz and F. Sheikholeslam, "On the stability of continuous-time T-S model," *Eighth Int. Conf. on Fuzzy Syst. and Knowledge Discovery (FSKD), Proc. of IEEE*, pp. 226-230, 2011.
[27] L. Zou and Y. Jiang, "Estimation of the eigenvalues and the smallest singular value of matrices," *Linear Algebra and its Appl.*, *Elsevier*, Vol. 433, pp. 1203-1211, 2010.
[28] Y. Xingdong, D. Zhiying, Z. Jisjing, and S. Suya "On the eigenvalue estimation for solution to Lyapunov equation," *Applied Math. and Compt., Elsevier*, Vol. 217, pp. 6974-6980, 2011.
[29] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[30] R. Kress, H. L. D. Vries, R. Wegmann, "On non-normal matrices," *Linear Algebra Appl.* Vol. 8, pp. 109–120, 1974.
[31] Gu Yixi, "The distribution of eigenvalues of a matrix," *Acta Math. Appl. Sinica*, Vol. 4, pp. 501–511, 1994.
[32] O. Rojo and R. Soro, "Bounds for sums of eigenvalues and applications," *Compt. and math. with appl.*, Vol. 39, pp. 1-15, 2000.