On The Stability Analysis of Perturbed Continuous T-S Fuzzy Models

Mohammad Shekaramiz[1] and Farid Sheikholeslam[2]

[1] Dept. of Electrical and Computer Engineering, Utah State University, Utah, U.S.A.

[2] Dept. of Electrical and Computer Engineering, Isfahan University of Technology, Isfahan, Iran

***Abstract*—This paper deals with the stability problem of continuous-time Takagi-Sugeno (T-S) fuzzy models. Based on the Tanaka and Sugeno theorem, a new systematic method is introduced to investigate the asymptotic stability of T-S models in case of having second-order and symmetric state matrices. This stability criterion has the merit that selection of the common positive-definite matrix P is independent of the sub-diagonal entries of the state matrices. It means for a set of fuzzy models having the same main diagonal state matrices, it suffices to apply the method once. Furthermore, the method can be applied to T-S models having certain uncertainties. We obtain bounds for the uncertainties under which the asymptotic stability of the system is guaranteed. The obtained bounds are shown to be tight. Finally, the maximum permissible uncertainty bounds are investigated. Several examples are given to illustrate the effectiveness of the proposed method.**

***Index Terms*—Takagi-Sugeno (T-S) fuzzy model; asymptotic stability; Lyapunov function; linear matrix inequalities; eigenvalues estimation; symmetric matrices**

I. INTRODUCTION

Stability is one of the most important issues of any control system. Since fuzzy systems have been proved to be applicable to many kinds of industrial applications, analyzing the stability of such systems is of great importance. However, fuzzy systems are essentially nonlinear systems and stability analysis of these systems has been difficult. In order to tackle this problem, Takagi and Sugeno [1] proposed the T-S fuzzy model. By the T-S fuzzy model, a complex dynamic model can be composed of a set of local linear subsystems via the fuzzy inference.

Based on the Lyapunov direct method, Tanaka and Sugeno [2] showed that the stability of a Takagi-Sugeno (T-S) fuzzy model could be guaranteed by finding a common symmetric positive-definite matrix P for all the subsystems. But in many cases, especially by increasing number of fuzzy rules, finding such a common matrix for satisfying the set of Lyapunov's inequalities seems to be a daunting task. During the past decades, many researchers were attracted to reduce the conservatism of the Tanaka and Sugeno's method in finding the common positive definite matrix P [3, 5]. In this area, Kawamoto et al. [3] proposed a simple approach to find the region of existence of such a matrix for discrete-time T-S models. There was not a systematic method for obtaining the aforementioned matrix P until Narendra et al. [4] proposed one to investigate the stability of switching systems in case of having pairwise commutative state matrices. A couple of years later, Joongseon Joh et al. [5] used the pairwise commutative characteristic and introduced a systematic approach for finding a common positive-definite matrix P for discrete T-S models. They also obtained criteria under which even if the state matrices are not pairwise commutative, a common matrix P can still be found. Thanks to their conducive work, it is possible to extend it to the continuous T-S models. But, it would not be easy to find the matrix P for fuzzy systems not having pairwise commutative characteristic.

There were also introduced other approaches, on account of having difficulty in finding such a matrix to investigate the stability of T-S models [6-10]. In this area, Tanaka et al. [6] studied the stability problem by considering the time derivative property of the membership functions and the fuzzy Lyapunov function was defined by fuzzily blending quadratic Lyapunov functions. Wang and Sun [7] divided the state space into several sub-regions and obtained the local common matrix P_j for each sub-region j. In order to avoid finding the common matrix P, Louh [8] employed a robust criterion and defined the average of the state matrices as the nominal matrix of the whole subsystems. Then, the difference between each state matrix and the nominal matrix was considered as the perturbation of its relevant subsystem. Finally, based on a robust criterion, they proposed a sufficient condition that sought a matrix P satisfying just one inequality instead of the famous Lyapunov inequalities. But this criterion is also conservative and it fails determining the stability of many stable systems. Based on the matrix norm, Pang and Guu [9] offered a necessary and sufficient condition for deciding on the stability of discrete T-S models. Wang and Sun [10] investigated the stability by checking the maximum distance of two successive states in the discrete model. Although many researchers have studied the stability analysis of fuzzy systems, there is still a substantial need for having a general reliable method to verify stability or instability of fuzzy systems.

In this paper, we propose a new systematic approach to analyze the stability of continuous T-S fuzzy models in case of having second-order and symmetric state matrices. In this case, a new sufficient stability criterion in terms of Lyapunov function candidate is introduced to investigate the global asymptotic stability of T-S models. This criterion is derived from the existing methods for estimating the spectrum of Hermitian matrices. Applying the criterion leads to bounds for the sub-diagonal entries of the state matrices. It will be shown that if the sub-diagonal entries of the state matrices are within their relevant obtained bounds, then the system is asymptotically stable. This implies that for a set of fuzzy models having the same main diagonal state matrices in their subsystems, it suffices to apply the method once. Furthermore, the method is also extended to systems having uncertainties in their sub-diagonal entries of the state matrices and a sufficient condition for such systems is introduced. Finally, the maximum permissible uncertainty bounds which can guarantee the stability of the system are investigated. Several examples are given to illustrate the effectiveness of the proposed method. The work appears here has been partially published in [11,12].

This paper is organized as follows: The preliminaries are given in section II. In section III, firstly, the previous work [3] on the stability analysis is extended to continuous T-S fuzzy systems. Moreover, a sufficient criterion for the asymptotic stability of LTI systems having symmetric state matrices is introduced. Then, we propose a new systematic approach to determine the asymptotic stability of continuous T-S fuzzy systems having second-order and symmetric state matrices in their subsystems. The stability problem is also solved for T-S uncertain systems and the maximum allowable uncertainty bounds to guarantee the stability is obtained. Numerical examples are presented to demonstrate the effectiveness of the proposed methods. Finally, the concluding remarks are given in section IV.

II. PRELIMINARIES

An unforced continuous-time T-S fuzzy model can be presented as

$$Rule\ i\ : if\ x_1(t)\ is\ M_1^i\ and \ldots x_n(t)\ is\ M_n^i\ then\ \dot{X}(t) = A_i X(t),\ i = 1,2,\ldots,n. \tag{1}$$

where, $x(t)$ is the state vector, n is the number of IF-THEN rules, $Rule\,i$ is the i-th fuzzy inference rule and M_j^i is the fuzzy set. The global T-S fuzzy system is inferred as

$$\dot{x}(t) = \sum_{i=1}^{n} \omega_i(x(t)) A_i x(t) \Big/ \sum_{i=1}^{n} \omega_i(x(t)) \tag{2}$$

where, $\omega_i(x(t)) = \prod_{j=1}^{n} M_j^i(x_j(t))$ is the firing strength of the i-th rule and $M_j^i(x_j(t))$ is the membership grade of $x_j(t)$ to the fuzzy set M_j^i. By defining a normalized weight for each rule as

$$h_i(x(t)) = \frac{\omega_i(x(t))}{\sum_{i=1}^{n} \omega_i(x(t))}$$

the equation (2) can be rewritten as

$$\dot{x}(t) = \sum_{i=1}^{n} h_i(x(t)) A_i x(t) \tag{3}$$

Tanaka and Sugeno have presented the following theorem for verifying stability of the system (3).

Theorem [2]: The continuous fuzzy system described by equation (3) is globally asymptotically stable if there exists a common positive-definite symmetric matrix P such that

$$A_i^T P + P A_i \langle 0,\ \text{i=1, 2, …., n.} \tag{4}$$

The above inequalities (4) can be rewritten as

$$A_i^T P + P A_i = Q_i,\ Q_i = Q_i^T \langle 0,\ \text{i=1, 2, …., n.} \tag{5}$$

It can be seen from (4) that matrix P must satisfy n inequalities. Hence, increasing the number of rules makes it difficult to find such a matrix.

III. MAIN RESULTS

The main objective of this section is to propose a new systematic approach for analyzing the stability of continuous Takagi-Sugeno (T-S) fuzzy models in case of having symmetric and second-order state matrices. Before describing our new stability criterion, it is required to extend a previous approach for analyzing the stability of discrete-time second–order fuzzy models to the corresponding continuous-time systems. The method seeks the region of existence for common symmetric positive-definite matrix P under which the set of Lyapunov inequalities are satisfied. Then, the stability problem of linear continuous-time systems of having symmetric state matrices is considered. The stability investigation will then be accomplished using the existing methods in estimating the spectra of Hermitian matrices. Finally, the stability analysis of continuous T-S models in case of having *symmetric and second-order* state matrices is carried out. This stability criterion is derived from the aforementioned tasks.

A. EXPLORING P-REGION

As has been stated before, the process of finding a common positive-definite matrix P fulfilling the set of inequalities, as defined in (4), especially by increasing number of fuzzy rules is not an easy task. The purpose of this section is to find the region of existence in which the common positive-definite matrix P for continuous-time T-S fuzzy systems with second-order state matrices does exist. This work is the extension of Kawamoto et al. method [3] in finding the P region for discrete T-S models and we are going to apply it to the relevant continuous systems.

Consider the symmetric positive-definite matrix P as

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{6}$$

This matrix can be written as follows

$$P = |p_{12}| \begin{bmatrix} p_1 & \pm 1 \\ \pm 1 & p_2 \end{bmatrix}, \ (p_{12} \neq 0), \quad p_1 = \frac{p_{11}}{|p_{12}|}, \ p_2 = \frac{p_{22}}{|p_{12}|}$$

Please note that the existence of matrix P in case of $p_{12} = 0$ can be investigated by substituting definition (6) into the set of Lyapunov equations (5). In the rest of this paper and without loss of generality, the common matrix P is considered in the form below.

$$P = \begin{bmatrix} p_1 & \pm 1 \\ \pm 1 & p_2 \end{bmatrix} \tag{7}$$

It is easy to show that if P is a positive-definite matrix for the fuzzy system (1), then "kP" for any positive scalar "k" is also a positive-definite matrix for this system. Hence, considering matrix P as defined in (7) would not degrade the generality of the problem. For simplicity, define matrices P_P and P_N as follows

$$P_P = \begin{bmatrix} p_1 & 1 \\ 1 & p_2 \end{bmatrix}, P_N = \begin{bmatrix} p_1 & -1 \\ -1 & p_2 \end{bmatrix} \tag{8}$$

Assume that state matrices in the fuzzy system (1) are the second-order matrices and denoted by

$$A_i = \begin{bmatrix} a_1^i & a_2^i \\ a_3^i & a_4^i \end{bmatrix} \tag{9}$$

Remark 1: Substitution of matrices A_i and P_P , instead of P, into the Lyapunov equations (5) and considering the positive-definite property of matrix P_P and satisfaction of the Lyapunov inequalities provides the following lemma. In order to find the common positive-definite matrix P, required in the Tanaka-Sugeno theorem [2], it is sufficient to draw the region of existence of P_P for each subsystem, denoted by P_P^i, and then investigate whether there exists a common intersection for all the P_P^i. The same approach can be applied to obtain the common matrix P_N .

Lemma 1: Consider a continuous-time T-S fuzzy system with second-order state matrices as follows.

$$Rule\ i: if\ x_1(t)\ is\ M_1^i\ and \ldots x_n(t)\ is\ M_n^i\ then\ \dot{X}(t) = A_i X(t),\ i = 1,2,\ldots,n.$$

Where the state matrices A_i are assumed to be asymptotically stable and denoted by (9).

Then, the system is asymptotically stable, if there exists a common region for either matrix P_P or P_N in the $p_1 - p_2$ plane. These regions are obtained from the following set of inequalities.

$\forall i = 1,2,..., n.$

$$P_p^i, P_N^i : \begin{cases} p_1 \rangle 0 & (10\text{-}1) \\ p_2 \rangle \dfrac{1}{p_1} & (10\text{-}2) \\ a_1^{(i)} p_1 \pm a_3^{(i)} \langle 0 & (10\text{-}3) \\ (a_3^{(i)} p_2 + a_2^{(i)} p_1)^2 \pm 2(a_4^{(i)} - a_1^{(i)})(a_2^{(i)} p_1 - a_3^{(i)} p_2) & \\ -4(a_1^{(i)} a_4^{(i)}) p_1 p_2 + [(a_1^{(i)} + a_4^{(i)})^2 - 4a_2^{(i)} a_3^{(i)}] \langle 0 & (10\text{-}4) \end{cases}$$

Where in the sign $\pm$ in the (10-3) and (10-4), the signs + and - refer to P_P^i and P_N^i, respectively. Finally

$$P_p : \bigcap_{i=1}^{n} P_P^i \quad , P_N : \bigcap_{i=1}^{n} P_N^i \qquad (10\text{-}5)$$

Proof: The proof can be easily obtained from remark 1, the positive definite property of matrices P_P^i and P_N^i, and the negative definite property of matrices Q_i in the set of Lyapunov inequalities (5) and is omitted.

Remark 2: It is not instructive to sketch both the diagrams of P_P and P_N in the $p_1 - p_2$ plane. In other words, finding any intersection in one of these regions assures the stability of the system.

Remark 3: Choosing any arbitrary point (p_1, p_2) lie in the intersection region of the $p_1 - p_2$ plane which is obtained from common P_P region or P_N region (if exist) leads to construct a matrix P_P or P_N, as defined in Eq. (8), respectively. According to lemma 1, the obtained matrix is a common positive-definite matrix for the fuzzy system.

Example 1: Consider the T-S model (1) with nine rules which is borrowed from [8]. The state matrices are:

$$A_1 = \begin{bmatrix} -6.4 & 2 \\ 2 & -5 \end{bmatrix}, A_2 = \begin{bmatrix} -5.2 & 2 \\ 2 & -4.3 \end{bmatrix}, A_3 = \begin{bmatrix} -7 & 2 \\ 1.8 & -4 \end{bmatrix}, A_4 = \begin{bmatrix} -6 & 2 \\ 2 & -5 \end{bmatrix}, A_5 = \begin{bmatrix} -7 & 1.2 \\ 3 & -5.2 \end{bmatrix}, A_6 = \begin{bmatrix} -6.2 & 2 \\ 2 & -4.5 \end{bmatrix}, A_7 = \begin{bmatrix} -6.3 & 2.3 \\ 2 & -5 \end{bmatrix}$$

$$A_8 = \begin{bmatrix} -6.4 & 2.1 \\ 3 & -5.2 \end{bmatrix}, A_9 = \begin{bmatrix} -5.8 & 2.2 \\ 2 & -5 \end{bmatrix}$$

Again, applying lemma 1 yields to the following results:

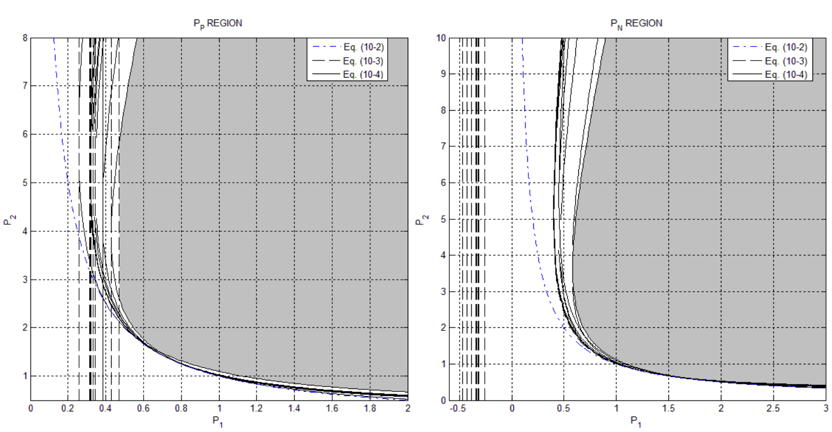

Fig 1. (a) Fig 1. (b)

Fig. (1): Fig. 1(a) and Fig. 1(b) show the region of existence for the matrices P_P and P_N, respectively.

According to lemma1, the existence of the common positive definite matrix P (shaded region) in either of P_P or P_N region assures the asymptotic stability of the system. We choose the same point ($P_1 = 1, P_2 = 3$) from the shaded regions of Fig. 1(a) and Fig. 1(b) and reach to the matrices P_P and P_N below

$$P_P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, P_N = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

Table 1: Validation of matrices P_P and P_N .

Matrix A_i	$\lambda_{1,2}[A_i^T P_P + P_P A_i]$	$\lambda_{1,2}[A_i^T P_N + P_N A_i]$
A_1	-26.648 , -8.152	-46.6207 , -4.1793
A_2	-21.945 , -6.255	-41.2191 , -2.9809
A_3	-21.199 , -9.199	-41.9207 , -3.6793
A_4	-26.487 , -7.513	-46.0238 , -3.9762
A_5	-28.991 , -7.809	-50.2094 , -3.3906
A_6	-23.483 , -7.9167	-43.7744 , -3.6256
A_7	-25.919 , -8.080	-47.1676 , -4.0324
A_8	-27.012 , -6.788	-51.2698 , -2.9302
A_9	-25.968 , -7.232	-46.1981 , -3.8019

Example 2: Consider the following T-S fuzzy system with four rules.

Rule i : *if* $x_1(t)$ *is* M_1^i *and* $x_2(t)$ *is* M_2^i *then* $\dot{X}(t) = A_i X(t)$, $i = 1,2,3,4.$

$$A_1 = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 \\ 3 & -6 \end{bmatrix}, A_3 = \begin{bmatrix} -4 & 1.5 \\ 1 & -2.5 \end{bmatrix}, A_4 = \begin{bmatrix} 2 & 4.5 \\ -4 & -5 \end{bmatrix}$$

By drawing the set of inequalities (10) corresponding to P_P in lemma 1, the region of existence for the common matrix P_P , defined in (8), is obtained and represented by the shaded region in the figure below.

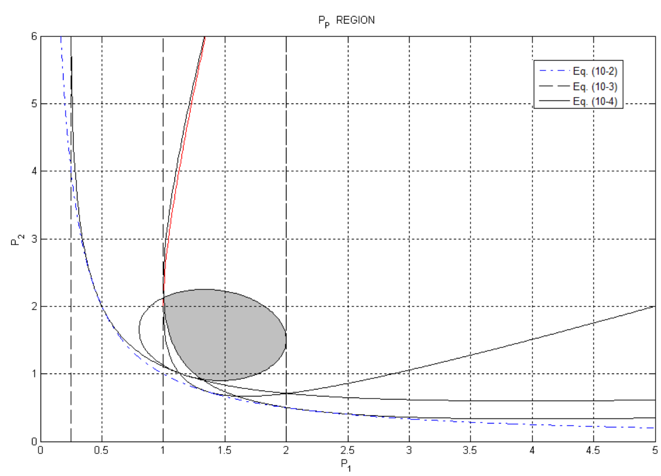

Fig. (2): Shows the region of existence for the matrix P_P .

It can be seen from the above diagram that the intersection exists for matrix P_P . Hence, according to lemma 1, choosing any arbitrary point (p_1, p_2) included in the shaded region of the $p_1 - p_2$ plane leads to a common positive-definite matrix P_P for the system. This shaded region assures that there exists a common positive-definite matrix for the system and consequently, the system is asymptotically stable. Now, we choose the point ($P_1 = 1.5$, $P_2 = 2$) from the shaded region of Fig. 2 and reach to the matrix P_P as below.

$$P_P = \begin{bmatrix} 1.5 & 1 \\ 1 & 2 \end{bmatrix}$$

The following table demonstrates that the obtained matrix P_P is in fact a common positive-definite matrix fulfilling the Lyapunov inequalities (4) and it guarantees the asymptotic stability of the T-S system.

Table 2: Validation of the selected matrix P_P .

Matrix A_i	A_1	A_2	A_3	A_4
$\lambda_{1,2}\,[A_i^T P_P + P_P A_i]$	-8.1400 , -0.8599	-20 , -3	-11.2042 , -5.7958	-12.6897 , -0.3103

Remark 4: Louh [8] suggested the following theorem for the stability analysis of T-S fuzzy models.

Theorem [8]: The equilibrium point of (3) is asymptotically stable if there exists a positive definite matrix P such that

$$\max_i |h_i| \langle \delta_{\max}^{-1} \left(\sum_{i=1}^{n} |H_i| \right) \tag{11}$$

where for i=1,2,…,n

$$A_0 = \frac{1}{n} \sum_{i=1}^{n} A_i$$
$$\Delta A_i = A_i - A_0$$
$$A_0^T P + P A_0 = -I$$
$$H_i = \Delta A_i^T P + P \Delta A_i \tag{12}$$

The table below shows the merit of the proposed method on the stability analysis in comparison to the one introduced in [8].

Table 3: Comparison of the proposed method with the one obtained in [8].

Method	***Proposed by***	***Stability Criterion***	***Example 1***		***Example 2***	
			Criterion	***Stability***	***Criterion***	***Stability***
Robust Criterion	Louh [8]	$\max_i \lvert h_i \rvert \langle \delta_{\max}^{-1} \left(\sum_{i=1}^{n} \lvert H_i \rvert \right)$ Eq.(11), Eq. (12)	$\delta_{\max}^{-1} \left(\sum_{i=1}^{9} \lvert H_i \rvert \right) = 1.006 \rangle 1$	stable	$\delta_{\max}^{-1} \left(\sum_{i=1}^{4} \lvert H_i \rvert \right) = 0.1526 \langle 1$	No results
Region of Existence	In This Paper	Finding any intersection Eq. (10)	Intersection exists	stable	Intersection exists	stable

B. ESTIMATING EIGENVALUES OF SYMMETRIC MATRICES

In this sub-section a sufficient criterion is proposed to investigate the asymptotic stability of continuous-time linear systems having symmetric and second-order state matrices. This criterion will be derived from the work of Chi-Kwang Li and Ren- Cang Li in [13] and the previous works in the estimation of eigenvalues for Hermitian matrices. In this approach, the estimation of eigenvalues is calculated from the corresponding diagonal part of its Hermitian matrix. Now, before representing the stability criterion, let us have some definitions.

Consider the following Hermitian matrix defined in [13]

$$A=\begin{matrix} & p & q \\ p & H_1 & E \\ q & E^* & H_2 \end{matrix} \tag{13}$$

where, E^* is the complex conjugate transpose of matrix E. Suppose that $\tilde{A}$ is the diagonal matrix of A and is defined as follows

$$\tilde{A}=\begin{matrix} & p & q \\ p & H_1 & 0 \\ q & 0 & H_2 \end{matrix} \tag{14}$$

Assume that $\lambda(X)$ is the spectrum of the square matrix X, and $\|X\|$ is the spectral norm of the matrix Y, i.e., the largest singular value of X. There are three common types of bounds for the eigenvalues $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{p+q}$ and $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \ldots \ge \tilde{\lambda}_{p+q}$ corresponding to the matrices A and $\tilde{A}$, respectively [13].

1. $\left|\lambda_i - \tilde{\lambda}_i\right| \le \|E\|, \; i = 1,2,\ldots,p+q.$ (15)

2. If the spectra of H_2, H_1 are disjoint, then

$$\left|\lambda_i - \tilde{\lambda}_i\right| \le \frac{\|E\|^2}{\eta}, \quad \eta \overset{def}{=} \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} \left|\mu_1 - \mu_2\right| \tag{16}$$

where $\lambda(H_i)$ is the spectra of the matrix H_i and η is the spectral gap between the spectra of H_1 and H_2.

Note that having a small value of η in Eq. (16) leads to a very large and conservative bound. Furthermore, in case of $\eta = 0$, Eq. (16) does not provide any bound!

3. **Theorem [13]:** Consider matrices A and $\tilde{A}$ defined in Eq. (13) and (14). Assume that the eigenvalues of matrices A and $\tilde{A}$ be $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{p+q}$ and $\tilde{\lambda}_1 \ge \tilde{\lambda}_2 \ge \ldots \ge \tilde{\lambda}_{p+q}$, respectively. Define η as follows

$$\eta_i \overset{def}{=} \begin{cases} \min_{\mu_2 \in \lambda(H_2)} \left|\tilde{\lambda}_i - \mu_2\right| \;, & if \quad \tilde{\lambda}_i \in \lambda(H_1) \\ \min_{\mu_1 \in \lambda(H_1)} \left|\tilde{\lambda}_i - \mu_1\right| \;, & if \quad \tilde{\lambda}_i \in \lambda(H_2) \end{cases} \tag{17}$$

$$\eta \overset{def}{=} \min_{1 \le i \le p+q} \eta_i = \min_{\mu_1 \in \lambda(H_1), \mu_2 \in \lambda(H_2)} \left|\mu_1 - \mu_2\right| \tag{18}$$

Then, for $i = 1,2,\ldots, p+q$, we have

$$\left|\lambda_i - \tilde{\lambda}_i\right| \le \frac{2\|E\|^2}{\eta_i + \sqrt{\eta_i^2 + 4\|E\|^2}} \le \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}} \tag{19}$$

Now, let us use the above theorem and propose a new sufficient criterion for continuous–time linear systems having symmetric state matrices.

Lemma 2: Assume the following continuous-time linear system with a symmetric (p+q)×(p+q) state matrix.

$\dot{X}(t) = A X(t)$

The system is asymptotically stable if the following inequality holds.

$$SCH = \max_i \{\tilde{\lambda}_i + \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}\} = \tilde{\lambda}_1 + \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}} \langle 0 \text{, for i=1,2, …, p+q.} \tag{20}$$

where, matrices A and $\tilde{A}$ have the form of definitions Eq. (13) and Eq. (14), respectively, E^* is the transpose of matrix E and $\|E\|$ is the spectral norm of matrix E. η_i and η are obtained from Eq. (17) and Eq. (18).

(SCH stands for Stability Criterion of Hermitian matrices)

Proof: By applying the method proposed in theorem [13], bounds locating the eigenvalues of matrix A were obtained from calculating the eigenvalues of its relevant diagonal matrix $\tilde{A}$. Let us revisit the bound

$$\left|\lambda_i - \tilde{\lambda}_i\right| \le \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}$$

Note that the entries of the state matrix of a continuous-time linear system are real values. Moreover, in systems having symmetric state matrices, all the eigenvalues lie on the real axis of the complex plane. Hence for inequality (19), we have

$$\lambda_i \le \tilde{\lambda}_i + \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}, \; i = 1,2,\ldots,p+q.$$

Therefore, the system is asymptotically stable if the following inequalities hold.

$$\tilde{\lambda}_i + \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}} \langle 0, \; i = 1,2,\ldots,p+q.$$

Furthermore, by having $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{p+q}$, if the largest eigenvalues of matrix A i.e., $\lambda_1(A)$ have negative upper bound, then the spectrum of matrix A are negative as well. Hence, it suffices to just investigate the following inequality for the stability.

$$SCS = \max_i \{\tilde{\lambda}_i + \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}}\} = \tilde{\lambda}_1 + \frac{2\|E\|^2}{\eta + \sqrt{\eta^2 + 4\|E\|^2}} \langle 0, \; i = 1,2,\ldots,p+q. \qquad \square$$

The merit of Eq. (17-19) will be unveiled in the next section for investigating the stability analysis of continuous-time T-S fuzzy systems.

C. NEW STABILITY ANALYSIS OF CONTINUOUS-TIME T-S MODELS

In the previous sections, the problem of exploring the P-region for T-S fuzzy models with second-order state matrices has been investigated. Moreover, a method for estimating the eigenvalues of Hermitian matrices was addressed. Then, a new sufficient criterion for analyzing the stability of continuous-time linear systems with symmetric state matrices has been proposed. In the rest of this paper, those aforementioned approaches will be applied to the continuous-time T-S fuzzy models having *second-order* and *symmetric* state matrices. In this case, a new sufficient condition in order to guarantee the stability of such systems will be introduced.

Remark 5: Consider the continuous T-S fuzzy system (1). Assume that all the state matrices A_i are *2×2, diagonal* and *stable* as well. Then, the stability conditions of Eq. (10) will be simplified as follows

$$P: \begin{cases} p_1 \rangle 0 \\ p_2 \rangle \dfrac{1}{p_1} \\ p_2 \rangle \dfrac{(a_1^i + a_4^i)^2}{4a_1^i a_4^i} \dfrac{1}{p_1}, i = 1,2,\ldots,n \end{cases} \qquad (21)$$

Now, define α_i as

$$\alpha_i = \frac{(a_1{}^i + a_4{}^i)^2}{4a_1{}^i a_4{}^i}, \quad i = 1,2,\ldots,n. \qquad (22)$$

Substituting Eq. (22) into Eq. (21) yields

$$\begin{cases} p_1 \rangle 0 \\ p_2 \rangle \dfrac{\max(\alpha_i, 1)}{p_1} \end{cases}, \quad i = 1,2,\ldots,n. \tag{23}$$

Let us have a definition.

$$\forall \varepsilon \rangle 0: \ \alpha \overset{def}{=} \max(1+\varepsilon, \max_i \ \alpha_i), \quad i = 1,2,\ldots,n. \tag{24}$$

Then for such a system, it is easy to show that the point (α,α) lies in the region of existence for matrix P in the $p_1 - p_2$ plane. In the rest of this paper, the construction of matrices P_P and P_N , defined in (8), for the point (α,α) are denoted by P_α^+ and P_α^- , respectively.

$$P_\alpha^+ = k\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}, P_\alpha^- = k\begin{bmatrix} \alpha & -1 \\ -1 & \alpha \end{bmatrix}, k \rangle 0 \tag{25}$$

Example 3: Suppose there is a continuous T-S fuzzy model, as defined in Eq. (1), with the following state matrices.

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}, \ A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \ A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

Then by applying the results of Remark 5, the region of existence for the common positive-definite matrix P, as defined in (7), can be illustrated in the $p_1 - p_2$ plane below.

Table 4: The values of α_i Eq. (22)

Rule i	***Rule 1***	***Rule 2***	***Rule 3***
α_i	$\alpha_1 = 1.5626$	$\alpha_2 = 1.0417$	$\alpha_3 = 1.3333$

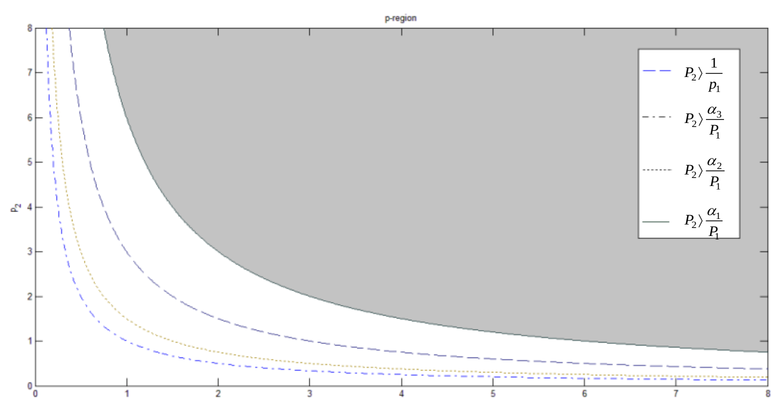

Fig. (3): The shaded region shows the region of existence for the matrix P.

Now, consider a continuous-time T-S fuzzy model with asymptotically stable, second-order and symmetric state matrices. Assume that the diagonal parts of the state matrices are asymptotically stable as well. In our next stability criterion, the diagonal part of the i-th state matrix will be considered as the nominal state matrix $A_{i,0}$ of its corresponding subsystem. It means,

$$\textit{Rule } i: \textit{if } x_1(t) \textit{ is } M_1^i \textit{ and } x_2(t) \textit{ is } M_2^i \textit{ then } \dot{X}(t) = A_i X(t), \ i = 1,2,\ldots,n.$$

where

$$A_i = A_{i,0} + \Delta A_i \ , i = 1,2,\ldots,n.$$

In the above notation, ΔA_i is the difference between the state matrix A_i and its corresponding nominal matrix $A_{i,0}$.

Remark 6: In the following theorem, the positive definite matrices P_α^+ and P_α^- will be obtained from the main diagonal of the state matrices i.e., ΔA_i . Then, the stability problem for the whole system with the state matrices A_i will be solved by applying P_α^+ and P_α^- to the Lyapunov equations (5). In this case, we will obtain bounds for a_2^i s (sub-diagonal entries) under which the stability of the system is still guaranteed.

Theorem 1: Consider a continuous-time T-S fuzzy model as follows.

$$Rule\ i\ :if\ x_1(t)\ is\ M_1^i\ and\ x_2(t)\ is\ M_2^i\ then\ \dot{X}(t) = A_i X(t)\ ,\ i = 1,2,\ldots,n.$$

where the state matrices are symmetric and defined as follows

$$A_i = \begin{bmatrix} a_1^i & a_2^i \\ a_2^i & a_4^i \end{bmatrix}, \quad i = 1,2,\ldots,n.$$

Assume that all the state matrices A_i and their corresponding diagonal matrices are *asymptotically stable*.

Then the matrix P_α^+ , defined in (25), is a common positive-definite matrix for the system if the following conditions for all the sub-diagonal entries of the state matrices i.e. a_2^i s, hold.

$\forall\ i = 1,2,\ldots,n.$

$$\begin{cases} \forall\, a_1^i \neq a_4^i\ :\ a_2^i \in E_P^i,\ E_P^i = \left(\max\{-\delta_i, \dfrac{(-\beta_i - \Delta_i)}{2\alpha}\} \quad \min\{\delta_i, \dfrac{(-\beta_i + \Delta_i)}{2\alpha}\} \right) \\ \forall\, a_1^i = a_4^i\ :\ a_2^i \in E_P'^i,\ E_P'^i = \left(-\left|a_1^i\right| \quad \left|a_1^i\right| \right) \end{cases} \tag{26}$$

and the matrix P_α^-, defined in (25), is a common positive-definite matrix for the system if all the sub-diagonal entries, a_2^i s, be in their relevant below bounds.

$\forall\ i = 1,2,\ldots,n.$

$$\begin{cases} \forall\, a_1^i \neq a_4^i\ :\ a_2^i \in E_N^i,\ E_N^i = \left(\max\{-\delta_i, \dfrac{(\beta_i - \Delta_i)}{2\alpha}\} \quad \min\{\delta_i, \dfrac{(\beta_i + \Delta_i)}{2\alpha}\} \right) \\ \forall\, a_1^i = a_4^i\ :\ a_2^i \in E_N'^i,\ E_N'^i = \left(-\left|a_1^i\right| \quad \left|a_1^i\right| \right) \end{cases} \tag{27}$$

where

$$\alpha = \max\,(1+\varepsilon, \max_i \alpha_i), \alpha_i = \frac{(a_1^i + a_4^i)^2}{4a_1^i a_4^i} \tag{28}$$

$$M_i = \max(a_1^i, a_4^i) \tag{29}$$

$$\delta_i = \sqrt{a_1^i a_4^i} \tag{30}$$

$$\beta_i = \left|a_1^i - a_4^i\right| - \left|a_1^i + a_4^i\right| \quad (31)$$

$$\gamma_i = M_i \left|a_1^i - a_4^i\right| \alpha^2 + 0.25(a_1^i + a_4^i)^2 \quad (32)$$

$$\Delta_i = \sqrt{\beta_i^2 - 4\gamma_i} \quad (33)$$

Consequently, having P_α^+ or P_α^- as a common positive-definite for the system guarantees the global asymptotic stability of the equilibrium of the fuzzy system.

Proof: see appendix

Example 4: Investigate the stability of the T-S fuzzy model described below.

$Rule\ i\ : if\ x_1(t)\ is\ M_1^i\ and\ x_2(t)\ is\ M_2^i\ then\ \dot{X}(t) = A_i X(t)\ ,\ i = 1,2,3,4.$

where

$$A_1 = \begin{bmatrix} -1.5 & 1.2 \\ 1.2 & -4 \end{bmatrix},\ A_2 = \begin{bmatrix} -2 & 1.255 \\ 1.255 & -1 \end{bmatrix},\ A_3 = \begin{bmatrix} -2 & 1.5 \\ 1.5 & -2 \end{bmatrix},\ A_4 = \begin{bmatrix} -3 & -2.5 \\ -2.5 & -3 \end{bmatrix}$$

$$A_1 = A_{1,0} + \Delta A_1\ ,\ A_{1,0} = \begin{bmatrix} -1.5 & 0 \\ 0 & -4.0 \end{bmatrix},\ \Delta A_1 = \begin{bmatrix} 0 & 1.2 \\ 1.2 & 0 \end{bmatrix}$$

$$A_2 = A_{2,0} + \Delta A_2\ ,\ A_{2,0} = \begin{bmatrix} -2.0 & 0 \\ 0 & -1.0 \end{bmatrix},\ \Delta A_2 = \begin{bmatrix} 0 & 1.255 \\ 1.255 & 0 \end{bmatrix}$$

$$A_3 = A_{3,0} + \Delta A_3\ ,\ A_{3,0} = \begin{bmatrix} -2.0 & 0 \\ 0 & -2.0 \end{bmatrix},\ \Delta A_3 = \begin{bmatrix} 0 & 1.5 \\ 1.5 & 0 \end{bmatrix}$$

$$A_4 = A_{4,0} + \Delta A_4\ ,\ A_{4,0} = \begin{bmatrix} -3.0 & 0 \\ 0 & -3.0 \end{bmatrix},\ \Delta A_4 = \begin{bmatrix} 0 & -2.5 \\ -2.5 & 0 \end{bmatrix}$$

Then, the positive definite matrix P_α^+, as defined in Eq. (25), can be easily obtained from the matrices $A_{i,0}$. Consider the positive definite matrix P_α^+ as defined in (25). According to theorem 1, we can investigate the acceptable bounds for the sub-diagonal entries of the state matrices under which the stability of the system is guaranteed. Table (5) shows those obtained permissible bounds.

Table 5: Calculation of the acceptable $E_P^i, E_P'^i$ for matrix P_α^+.

Matrix A_i	α_i ***(28)***	δ_i ***(30)***	β_i ***(31)***	γ_i ***(32)***	Δ_i ***(33)***	$E_P^i, E_P'^i$ ***(26)***	a_2^i ***Sub-diagonal entry of matrix*** A_i
A_1	$\alpha_1 = 1.2604$	$\sqrt{6}$	-3	1.6051	1.6061	E_P^1 =(0.5529 1.8271)	a_2^1 = 1.2
A_2	$\alpha_2 = 1.1250$	$\sqrt{2}$	-2	0.6613	1.1640	E_P^2 =(0.3317 1.25501)	a_2^2 = 1.255
A_3	$\alpha_3 = 1.0000$	2	-4	4.0000	0	$E_P'^3$ =(-2 2)	a_2^3 = 1.5
A_4	$\alpha_4 = 1.0000$	3	-6	9.0000	0	$E_P'^4$ =(-3 3)	a_2^4 =-2.5

Since each a_2^i is within its relevant obtained bound of the table above, the following matrix P_α^+ is certainly a common positive-definite matrix for the system satisfying the Lyapunov inequalities (5). This concludes the asymptotically stability of the system.

$$P_\alpha^+ = k\begin{bmatrix} 1.2604 & 1 \\ 1 & 1.2604 \end{bmatrix}$$

The stability of the system is validated in the following table.

Table 6: Eigenvalues of $A_i^T P_\alpha^+ + P_\alpha^+ A_i$ with k=1.

Matrix A_i	$A_i^T P_\alpha^+ + P_\alpha^+ A_i = Q_i$	$\lambda_{1,2}(Q_i)$
A_1	$Q_1 = \begin{bmatrix} -1.3813 & -2.4750 \\ -2.4750 & -7.6833 \end{bmatrix}$	-8.5391, -0.5255
A_2	$Q_2 = \begin{bmatrix} -2.5316 & 0.1636 \\ 0.1636 & -0.0108 \end{bmatrix}$	-2.5422, -0.0002
A_3	$Q_3 = \begin{bmatrix} -2.0417 & -0.2188 \\ -0.2188 & -2.0417 \end{bmatrix}$	-2.2604, -1.8229
A_4	$Q_4 = \begin{bmatrix} -12.5625 & -12.3021 \\ -12.3021 & -12.5625 \end{bmatrix}$	-24.8646, -0.2604

Example 5: Assume that the state matrices of a continuous-time T-S model are represented as

$$A_1 = \begin{bmatrix} -1 & -1.5616 \\ -1.5616 & -4 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} -4 & -2.5 \\ -2.5 & -5 \end{bmatrix}, A_4 = \begin{bmatrix} -3 & -2 \\ -2 & -5 \end{bmatrix}, A_5 = \begin{bmatrix} -7 & -3.1086 \\ -3.1086 & -2 \end{bmatrix}$$

In this example we want to show the merit of theorem 1 in investigating the stability problem via the existence of matrix P_α^- defined in (25). The obtained acceptable bounds for a_2^i s are shown in the table below.

Table 7: Calculation of the acceptable E_N^i for matrix P_α^- .

Matrix A_i	α_i ***(28)***	δ_i ***(30)***	β_i ***(31)***	γ_i ***(32)***	Δ_i ***(33)***	E_N^i ***(26)***	a_2^i ***Sub-diagonal entry of matrix*** A_i
A_1	$\alpha_1 = 1.5625$	2	-2	-1.0742	2.8804	E_N^1 =(-1.5617 , 0.2817)	a_2^1 =-1.5616
A_2	$\alpha_2 = 1.0417$	$\sqrt{6}$	-4	1.3672	3.2452	E_N^2 =(-2.3185 , -0.2415)	a_2^2 =-1.0
A_3	$\alpha_3 = 1.0125$	$2\sqrt{5}$	-8	10.4844	4.6971	E_N^3 =(-4.0631 , -1.0569)	a_2^3 =-2.5
A_4	$\alpha_4 = 1.0667$	$\sqrt{15}$	-6	1.3516	5.5311	E_N^4 =(-3.6900 , -0.1500)	a_2^4 =-2.0
A_5	$\alpha_5 = 1.4464$	$\sqrt{14}$	-4	-4.1641	5.7146	E_N^5 =(-3.10862 , 0.5487)	a_2^5 =-3.1086

Since each a_2^i is within its relevant obtained bound of the above table, the following matrix P_α^- is certainly a common positive-definite matrix for the system satisfying the Lyapunov inequalities (5).

$$P_\alpha^- = k\begin{bmatrix} 1.5625 & -1 \\ -1 & 1.5625 \end{bmatrix}, k\rangle 0$$

Table 8: Validation of the matrix P_α^- as a common positive-definite matrix for the system for k=1.

Matrix A_i	A_1	A_2	A_3	A_4	A_5
$\lambda_{1,2}(Q_i)$	-9.3783 , -0.0003	-8.253 , -3.372	-7.100 , -11.025	-12.082 , -4.918	-15.6904, -0.0002

Therefore, the fuzzy system is asymptotically stable.

Remark 7: It can be seen from Ex. 4 that the obtained upper-bounds for the sub-diagonal entries of the state matrices for the fuzzy system are tight i.e., $\lambda_1(Q_2)$ in table 6 is very close to zero. Having $A_2 = \begin{bmatrix} -2 & 1.2552 \\ 1.2552 & -1 \end{bmatrix}$ for the fuzzy system of Ex. 4 causes $\lambda_{1,2}(Q_2)$ = -2.5418, 0.0002.

It can also be seen from the Ex. 5 that the obtained lower-bounds for the sub-diagonal entries of the state matrices are tight as well.

Remark 8: It can be seen from theorem1 that in order to check the stability of a set of fuzzy models having the same diagonal entries in their state matrices, it suffices to apply the method once. Despite achieving the same bounds for the sub-diagonal entries, the stability of each system may be different depending on whether or not all the sub-diagonals lie in their relevant obtained bounds. For better understanding, look at the following example.

Example 6: Consider a set of fuzzy models with the state matrices below.

Fuzzy system 1:

$$A_1=\begin{bmatrix}-1.5 & 1.8\\ 1.8 & -4\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix},\ A_3=\begin{bmatrix}-2 & -1.5\\ -1.5 & -2\end{bmatrix},\ A_4=\begin{bmatrix}-3 & -2\\ -2 & -3\end{bmatrix}$$

Fuzzy system 2:

$$A_1=\begin{bmatrix}-1.5 & 0.7\\ 0.7 & -4\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 1.25\\ 1.25 & -1\end{bmatrix},\ A_3=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix},\ A_4=\begin{bmatrix}-3 & 2.9\\ 2.9 & -3\end{bmatrix}$$

Fuzzy system 3:

$$A_1=\begin{bmatrix}-1.5 & 1\\ 1 & -4\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 0.5\\ 0.5 & -1\end{bmatrix},\ A_3=\begin{bmatrix}-2 & 1.5\\ 1.5 & -2\end{bmatrix},\ A_4=\begin{bmatrix}-3 & -1\\ -1 & -3\end{bmatrix}$$

Fuzzy system 4:

$$A_1=\begin{bmatrix}-1.5 & 1.8\\ 1.8 & -4\end{bmatrix},\ A_2=\begin{bmatrix}-2 & 0.9\\ 0.9 & -1\end{bmatrix},\ A_3=\begin{bmatrix}-2 & 2.1\\ 2.1 & -2\end{bmatrix},\ A_4=\begin{bmatrix}-3 & 2\\ 2 & -3\end{bmatrix}$$

Since the main-diagonal of these matrices are the same as those in Ex.4, we will arrive at the same bounds of table 5. The table below shows the stability conclusions.

Table 9: Stability investigation.

Fuzzy system	$E_P^i, E_P'^i$	P_α^+	***Stability criterion***	***Stability conclusion***
Fuzzy system 1	E_P^1 =(0.5529 1.8271) E_P^2 =(0.3317 1.2550) $E_P'^3$ =(-2 2) $E_P'^4$ =(-3 3)	$P_\alpha^+=\begin{bmatrix}1.2604 & 1\\ 1 & 1.2604\end{bmatrix}$	$a_2^1=1.8\in E_P^1$ $a_2^2=1.0\in E_P^2$ $a_2^3=-1.5\in E_P'^3$ $a_2^4=-2.0\in E_P'^4$	Asymptotically stable
Fuzzy system 2	The same as the one obtained for the fuzzy system 1	The same as the one obtained for the fuzzy system 1	$a_2^1=0.7\in E_P^1$ $a_2^2=1.25\in E_P^2$ $a_2^3=1.0\in E_P'^3$ $a_2^4=2.9\in E_P'^4$	Asymptotically stable
Fuzzy system 3	The same as the one obtained for the fuzzy system 1	The same as the one obtained for the fuzzy system 1	$a_2^1=1.0\in E_P^1$ $a_2^2=0.5\in E_P^2$ $a_2^3=1.5\in E_P'^3$ $a_2^4=-1.0\in E_P'^4$	Asymptotically stable
Fuzzy system 4	The same as the one obtained for the fuzzy system 1	The same as the one obtained for the fuzzy system 1	$a_2^1=1.8\in E_P^1$ $a_2^2=-1\in E_P^2$ $a_2^3=2.1\notin E_P'^3$ $a_2^4=2.0\in E_P'^4$	No results

Remark 9: In theorem 1, we considered the diagonal part of the state matrices as nominal matrices and the stability problem was solved by applying the defined P_α^+ and P_α^- to satisfy the Lyapunov equations. In this case, we found bounds for a_2^i s (sub-diagonal arrays) under which the stability is guaranteed. It means that the approach is independent of the value of a_2^i s and we can consider the sub-diagonal entries or part of them as uncertainties.

Corollary 1: Consider the T-S fuzzy model with n symmetric uncertain rules below.

$$Rule\,i\ :if\ x_1(t)\ is\ M_1^i\ and \ldots x_n(t)\ is\ M_n^i\ then\ \dot{X}(t) = (A_i + \Delta A_i(t))\,X(t),\ i = 1,2,\ldots,n. \tag{34}$$

where $A_i \in R^{2\times 2}$ is the nominal state matrix and $\Delta A_i(t)$ is its corresponding time-varying uncertainty, and are denoted by

$$A_i = \begin{bmatrix} a_1^i & a_2^i \\ a_2^i & a_4^i \end{bmatrix}, \Delta A_i(t) = \begin{bmatrix} 0 & d_i(t) \\ d_i(t) & 0 \end{bmatrix}, \left|d_i(t)\right| \le \rho_i,\ i = 1,2,\ldots,n. \tag{35}$$

Then, the uncertain system is asymptotically stable and P_α^+ , as defined in Eq. (25), is a common positive-definite matrix for the system if the following conditions hold.

$$\forall\ i = 1,2,\ldots,n.$$
$$\begin{cases} \forall\, a_1^i \ne a_4^i :\ a_2^i \pm \rho_i \in E_P^i \\ \forall\, a_1^i = a_4^i :\ a_2^i \pm \rho_i \in E_P'^i \end{cases} \tag{36}$$

Where E_P^i and $E_P'^i$ are obtained from theorem 1.

And P_α^- , as defined in Eq. (25), is a common positive-definite matrix for the system if the below conditions hold.

$$\forall\ i = 1,2,\ldots,n.$$
$$\begin{cases} \forall\, a_1^i \ne a_4^i :\ a_2^i \pm \rho_i \in E_N^i \\ \forall\, a_1^i = a_4^i :\ a_2^i \pm \rho_i \in E_N'^i \end{cases} \tag{37}$$

Where E_N^i and $E_N'^i$ are obtained from theorem 1.

Proof: The proof can be easily obtained from theorem 1 and is omitted.

According to corollary 1, the obtained common positive-definite matrix P can tolerate some perturbations on the sub-diagonal of the state matrices.

Example 7: Consider the following T-S fuzzy system with n uncertain plant rules.

$$Rule\,i\ :if\ x_1(t)\ is M_1^i\ and \ldots x_n(t)\ is M_n^i\ then\ \dot{X}(t) = (A_i + \Delta A_i(t))\,X(t)\ ,\ i = 1,2,\ldots,n.$$

$$A_1 = \begin{bmatrix} -1 & -0.8 \\ -0.8 & -4 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}, A_3 = \begin{bmatrix} -4 & -2.5 \\ -2.5 & -5 \end{bmatrix}, A_4 = \begin{bmatrix} -3 & -2 \\ -2 & -5 \end{bmatrix}, A_5 = \begin{bmatrix} -7 & -1.5 \\ -1.5 & -2 \end{bmatrix}$$

$$\Delta A_1(t) = \begin{bmatrix} 0 & d(t) \\ d(t) & 0 \end{bmatrix}, \Delta A_2(t) = \begin{bmatrix} 0 & d(t) \\ d(t) & 0 \end{bmatrix}, \Delta A_3(t) = \begin{bmatrix} 0 & 1.4d(t) \\ 1.4d(t) & 0 \end{bmatrix}, \Delta A_4(t) = \begin{bmatrix} 0 & 3d(t) \\ 3d(t) & 0 \end{bmatrix}, \Delta A_5(t) = \begin{bmatrix} 0 & 3d(t) \\ 3d(t) & 0 \end{bmatrix}$$

where ΔA_i s are time-varying uncertainties and $\left|d(t)\right| \le r, r = 0.5.$

We first ignore the uncertainties and apply the method proposed in theorem 1. Consequently the stability problem is similar to the example 5 and we reach to the results of table 7 and the same P_α^- as follows

$$P_{\alpha}^{-} = k\begin{bmatrix} 1.5625 & -1 \\ -1 & 1.5625 \end{bmatrix}, k \rangle 0$$

Now applying corollary 1 yields

Table 10: Investigation of stability via the matrix P_{α}^{-} with k=1.

Rule i	$a_2^i \pm \rho_i$	E_N^i ***(27)***	***Validity of*** $a_2^i \pm \rho_i \in E_N^i$
Rule 1	-1.0617 , -0.2183	E_N^1 =(-1.5617 , 0.2817)	✓
Rule 2	-1.8185 , -0.7415	E_N^2 =(-2.3185 , -0.2415)	✓
Rule 3	-3.3631 , -1.7569	E_N^3 =(-4.0631 , -1.0569)	✓
Rule 4	-2.1900 , -1.6500	E_N^4 =(-3.6900 , -0.1500)	✓
Rule 5	-1.6087 , -0.9513	E_N^5 =(-3.1087 , 0.5487)	✓

Since all $a_2^i \pm \rho_i$ are within their obtained bounds in table 10, the system is asymptotically stable.

Corollary 2: consider the uncertain T-S fuzzy model having symmetric state matrices as

$$Rule\ i: if\ x_1(t)\ is M_1^i\ and \ldots x_n(t)\ is\ M_n^i\ then\ \dot{X}(t) = (A_i + \Delta A_i(t))\, X(t)\ ,\ i = 1,2,\ldots,n.$$

where $A_i \in R^{2\times 2}$ is the nominal state matrix and the corresponding $\Delta A_i(t)$ is time-varying uncertainty of the i-th rule having the following form

$$A_i = \begin{bmatrix} a_1^i & a_2^i \\ a_2^i & a_4^i \end{bmatrix}, \Delta A_i(t) = \begin{bmatrix} 0 & d_i(t) \\ d_i(t) & 0 \end{bmatrix}, \left| d_i(t) \right| \le \rho_{i,\max}, i = 1,2,\ldots,n. \quad (38)$$

If applying theorem 1 to the system having nominal state matrices A_i leads to the stability of the system, then the maximum uncertainty bounds under which the stability of the system is still guaranteed can be obtained as below:

$$\rho_{i,\max} = \min(\Delta_{Max}^i, \Delta_{Min}^i) \quad (39)$$

Where,

Case 1: the stability of the nominal system is guaranteed by P_{α}^{+} :

$\forall\ i = 1,2, \ldots, n.$

$$\begin{cases} \forall\, a_1^i \ne a_4^i : \Delta_{Max}^i = \left| E_P^i(upper-bound) - a_2^i \right|, \Delta_{Min}^i = \left| E_P^i(lower-bound) - a_2^i \right| \\ \forall\, a_1^i = a_4^i : \Delta_{Max}^i = \left| E_P'^i(upper-bound) - a_2^i \right|, \Delta_{Min}^i = \left| E_P'^i(lower-bound) - a_2^i \right| \end{cases} \quad (40)$$

Where E_P^i and $E_P'^i$ are obtained from theorem 1.

Case 2: the stability of the nominal system is guaranteed by P_{α}^{-} :

$\forall\ i = 1,2, \ldots, n.$

$$\begin{cases} \forall\, a_1^i \ne a_4^i : \Delta_{Max}^i = \left| E_N^i(upper-bound) - a_2^i \right|, \Delta_{Min}^i = \left| E_N^i(lower-bound) - a_2^i \right| \\ \forall\, a_1^i = a_4^i : \Delta_{Max}^i = \left| E_N'^i(upper-bound) - a_2^i \right|, \Delta_{Min}^i = \left| E_N'^i(lower-bound) - a_2^i \right| \end{cases} \quad (41)$$

Proof: The proof can be easily obtained from theorem 1 and corollary 1.

Example 8: The objective is to calculate the maximum uncertainty bounds of example 7.

Table 11: Calculating the maximum permissible uncertainty bounds.

Rule i	*Rule1*	*Rule 2*	*Rule 3*	*Rule 4*	*Rule 5*
Δ^i_{Min}	0.762	1.318	1.563	1.69	1.609
Δ^i_{Max}	1.0817	0.758	1.443	1.85	2.049
$\rho_{i,\max}$	0.762	0.758	1.443	1.69	1.609

IV. CONCLUSION

First, a sufficient stability condition is proposed to investigate the asymptotic stability of T-S fuzzy models with second-order state matrices. Then, the problem of estimating the spectrum of symmetric matrices was considered and a sufficient criterion for stability analysis of continuous-time linear systems having symmetric matrices was introduced. Moreover, the stability problem of T-S fuzzy models with symmetric and second-order state matrices was investigated. In our stability criterion we considered the diagonal-part of state matrices as nominal state matrices of the system. Then, the common matrix P was investigated for the system having those diagonal state matrices. The purpose was then to find the conditions under which matrices Q_i in the Lyapunov equations be negative-definite. Finally, we obtained conditions for the sub-diagonal entries of state matrices under which the system is still stable. This leads to have the ability of checking the stability of set of fuzzy systems of having the same diagonal entries in their relevant subsystems. Then, we extended the method to systems with uncertainty in their sub-diagonal entries and a sufficient condition for checking the stability was represented. Finally, the maximum uncertainty bound under which the stability of the system is guaranteed, was investigated.

APPENDIX

Proof of theorem 1:

It was shown in Remark 5 that matrices P_α^+ and P_α^- are the common positive-definite matrices for the system when all the state matrices of fuzzy model are diagonal and asymptotically stable as well. The obtained matrices P_α^+ and P_α^- satisfy the Lyapunov inequalities (4) and are revisited below.

$$P_\alpha^+ = k\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}, P_\alpha^- = k\begin{bmatrix} \alpha & -1 \\ -1 & \alpha \end{bmatrix}, k\rangle 0 \quad \text{(25) revisited}$$

where

$$\alpha_i = \frac{(a_1{}^i + a_4{}^i)^2}{4a_1{}^i a_4{}^i} \ , \quad i = 1,2,\ldots,n. \quad \text{(22) revisited}$$

$$\forall \varepsilon\rangle 0: \ \alpha \overset{def}{=} \max(1+\varepsilon, \max_i \ \alpha_i), \quad i = 1,2,\ldots,n. \quad \text{(24) revisited}$$

Now, consider a continuous-time T-S fuzzy model with asymptotically stable, second-order and symmetric state matrices. Assume that the diagonal parts of the state matrices are asymptotically stable as well. Define the diagonal part of the i-th state matrix as the nominal state matrix $A_{i,0}$ of its corresponding subsystem. It means that Eq. (1) can be rewritten as

$$Rule \ \ i: if \ x_1(t) \ is \ M_1^i \ and \ x_2(t) \ is \ M_2^i \ then \ \dot{X}(t) = (A_{i,0} + \Delta A_i) X(t), \ i = 1,2,..., n.$$

where, ΔA_i is the difference between the state matrix A_i and its corresponding nominal matrix $A_{i,0}$.

In this theorem, the positive definite matrices P_α^+ and P_α^- will be obtained from the main diagonal of the state matrices i.e., ΔA_i . Then, the stability problem for the whole system with the state matrices A_i will be solved by applying P_α^+ and P_α^- to the Lyapunov equations (5). In this case, we seek bounds for a_2^i s (sub-diagonal entries) under which the stability of the system is still guaranteed. Then, the system is proved to be asymptotically stable if all the sub-diagonal entries of the system are within their relevant obtained bounds.

At first we prove the theorem for the case of investigating the stability via the common positive-definite matrix P_α^+ as defined in (25). Consider matrices A_i , P_α^+ and the Lyapunov inequalities for the fuzzy system as follows.

$A_i = \begin{bmatrix} a_1^i & a_2^i \\ a_2^i & a_4^i \end{bmatrix}$, $P_\alpha^+ = k\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}$, $k\rangle 0$, $A_i^T P + PA_i = Q_i$, $Q_i = Q_i^T \langle 0$, i=1, 2, ..., n.

Substitution of A_i and P_α^+ into the above Lyapunov equations yields

$$Q_i = \begin{bmatrix} 2k(a_1^i\alpha + a_2^i) & k[(a_1^i + a_4^i) + 2a_2^i\alpha] \\ k[(a_1^i + a_4^i) + 2a_2^i\alpha] & 2k(a_4^i\alpha + a_2^i) \end{bmatrix}, i = 1,2,....,n.$$

According to the Tanaka and Sugeno theorem [2], if A_i is a stable matrix and P_α^+ is a positive-definite matrix for the system, then matrix Q_i is certainly a negative definite matrix. Now, we want to specify the values for the a_2^i s under which the above matrices Q_i can satisfy the Lyapunov inequalities i.e. being negative definite. In other words, it's required to have negative spectrum for all the matrices Q_i . As stated before, it is possible to estimate the location of the eigenvalues of a symmetric matrix via its corresponding diagonal matrix. According to the theorem [13], the relevant diagonal matrices of Q_i are

$$\tilde{Q}_i = \begin{bmatrix} 2k(a_1^i\alpha + a_2^i) & 0 \\ 0 & 2k(a_4^i\alpha + a_2^i) \end{bmatrix}, i = 1,2,....,n.$$

Applying the definitions of theorem [13] yields

$\eta^i = 2k\left|a_1^i - a_4^i\right|\alpha$: since $\alpha\rangle 1$ and $\eta^i\rangle 0$

$\left|E^i\right| = k\left|(a_1^i + a_4^i) + 2a_2^i\alpha\right|$

where η^i and E^i respectively denote η and E of theorem [13] related to the ith rule of the fuzzy model.

Consider the spectrum of the second-order matrices Q_i and $\tilde{Q}_i$ by $\lambda_1^i \ge \lambda_2^i$ and $\tilde{\lambda}_1^i \ge \tilde{\lambda}_2^i$, respectively. According to theorem [13]:

$$\left|\lambda_j^i - \tilde{\lambda}_j^i\right| \le \frac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}}, \quad j = 1,2., i = 1,2,...,n.$$

where the index j denotes the j-th eignvalue of matrices Q_i and $\tilde{Q}_i$ and i describes the i-th rule. In other words,

$$\begin{aligned} &\left|\lambda_1^i - 2k(\max(a_1^i, a_4^i)\alpha + a_2^i)\right| \le \frac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}} \\ &\left|\lambda_2^i - 2k(\min(a_1^i, a_4^i)\alpha + a_2^i)\right| \le \frac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}} \end{aligned} \qquad \text{(A-1)}$$

Moreover, matrices Q_i are symmetric and consequently all of their eigenvalues are real. Therefore, Eq. (A-1) can be rewritten as follows

$$\begin{cases} 2k(\max(a_1^i, a_4^i)\alpha + a_2^i) - \dfrac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}} \le \lambda_1^i \le 2k(\max(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}} \\ 2k(\min(a_1^i, a_4^i)\alpha + a_2^i) - \dfrac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}} \le \lambda_2^i \le 2k(\min(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{2{E^i}^2}{\eta^i + \sqrt{{\eta^i}^2 + 4{E^i}^2}} \end{cases}$$

The symmetric matrices Q_i are negative definite when their λ_1^i and λ_2^i are negative i.e., the right-side bounds of λ_1^i and λ_2^i in the above inequalities be negative. It means:

$$\begin{cases} \lambda_1^i \le 2k(\max(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{2E^{i^2}}{\eta^i + \sqrt{\eta^{i^2} + 4E^{i^2}}} \langle 0 \\ \lambda_2^i \le 2k(\min(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{2E^{i^2}}{\eta^i + \sqrt{\eta^{i^2} + 4E^{i^2}}} \langle 0 \end{cases} \tag{A-2}$$

where η^i has a positive value. It is obvious from (A-2) that:

$$\begin{cases} \lambda_1^i \langle 2k(\max(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{2E^{i^2}}{\eta^i + \sqrt{\eta^{i^2} + 4E^{i^2}}} \langle 2k(\max(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{E^{i^2}}{\eta^i} \\ \lambda_2^i \langle 2k(\min(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{2E^{i^2}}{\eta^i + \sqrt{\eta^{i^2} + 4E^{i^2}}} \langle 2k(\min(a_1^i, a_4^i)\alpha + a_2^i) + \dfrac{E^{i^2}}{\eta^i} \end{cases}$$

define

$$M_i \overset{def}{=} \max(a_1^i, a_4^i)$$

Consequently, for satisfying $\lambda^i{}_{1,2} < 0$, it suffices to satisfy the inequalities below.

$$a_1^i \ne a_4^i : \ 2k(M_i\alpha + a_2^i) + \frac{E^{i^2}}{\eta^i} \langle 0 \tag{A-3}$$

which concludes

$$(4\alpha^2)a_2^{i^2} + 4\alpha(\left|a_1^i - a_4^i\right| + (a_1^i + a_4^i))a_2^i + 4M_i\left|a_1^i - a_4^i\right|\alpha^2 + (a_1^i + a_4^i)^2 \langle 0 \tag{A-4}$$

Finally, solving the above inequalities for a_2^i yield

$$a_1^i \ne a_4^i : \begin{cases} \dfrac{-\beta_i - \sqrt{\beta_i^2 - 4\gamma_i}}{2\alpha} \langle a_2^i \langle \dfrac{-\beta_i + \sqrt{\beta_i^2 - 4\gamma_i}}{2\alpha} \\ \beta_i = \left|a_1^i - a_4^i\right| - \left|a_1^i + a_4^i\right| \\ \gamma_i = M_i\left|a_1^i - a_4^i\right|\alpha^2 + 0.25(a_1^i + a_4^i)^2 \end{cases} \tag{A-5}$$

Furthermore, it is assumed that matrices A_i and their corresponding diagonal matrices are asymptotically stable. Therefore,

$$a_1^i, a_4^i \langle 0,\ a_1^i a_4^i - (a_2^i)^2 \langle 0 \quad : \quad -\sqrt{a_1^i a_4^i} \langle a_2^i \langle \sqrt{a_1^i a_4^i} \quad ,\ i = 1,2,..,n \tag{A-6}$$

Then, combination of equations (A-5) and (A-6) yields:

$$\begin{cases} \max\{-\sqrt{a_1^i a_4^i}, \dfrac{-\beta_i - \sqrt{\beta_i^2 - 4\gamma_i}}{2\alpha}\} \langle a_2^i \langle \min\{\sqrt{a_1^i a_4^i}, \dfrac{-\beta_i + \sqrt{\beta_i^2 - 4\gamma_i}}{2\alpha}\} \\ \beta_i = \left|a_1^i - a_4^i\right| - \left|a_1^i + a_4^i\right| \\ \gamma_i = M_i\left|a_1^i - a_4^i\right|\alpha^2 + 0.25(a_1^i + a_4^i)^2 \\ \alpha = \max\{1 + \varepsilon, \max\limits_i \alpha_i\} \\ \alpha_i = \dfrac{(a_1^i + a_4^i)^2}{4a_1^i a_4^i} \\ M_i = \max(a_1^i, a_4^i) \end{cases} \tag{A-7}$$

Hence, for any value of a_2^i (i=1,2,…,n) lie in their corresponding intervals of the above inequalities, there exists the common positive-definite matrix P_α^+ for the fuzzy system. Notice that the assumption of $a_1^i \neq a_4^i$ has been made in the above inequalities. Otherwise, we would have $\eta^i = 0$ which leads to no bound at all.

Now, consider a special case where one or all the state matrices of the subsystems have $a_1^i = a_4^i$. In this case, $\eta^i = 0$ and the method for investigating the bound for a_2^i is aborted.

$$\left|\lambda_j(Q_i) - \lambda_j(\tilde{Q}_i)\right| \leq \left.\frac{2E^{i^2}}{\eta^i}\right|_{\eta^i=0} = ?$$

Therefore, for this special case, we apply the bound defined in (15)

$$a_1^i = a_4^i: \quad \left|\lambda_j(Q_i) - \lambda_j(\tilde{Q}_i)\right| \leq \frac{2E^{i^2}}{\eta^i + \sqrt{\eta^i + 4E^{i^2}}} \overset{\eta^i=0}{=} \left|E^i\right| \tag{A-8}$$

where

$$\lambda_{1,2}(\tilde{Q}_i) = 2(a_1^i\alpha + a_2^i)$$

Then, the definitions of M_i and E^i are simplified as:

$$M_i = \max(a_1^i, a_4^i) = a_1^i$$
$$\left|E^i\right| = 2\left|a_1^i + a_2^i\alpha\right| \tag{A-9}$$

Now, consider the inequality (A-8)

$$-\left|E^i\right| + \lambda_j(\tilde{Q}_i) \leq \lambda_j(Q_i) \leq \left|E^i\right| + \lambda_j(\tilde{Q}_i)$$

Having negative values for the right-side of the above inequalities ensure that the matrices Q_i are negative definite.

$$\lambda_j(\tilde{Q}_i) + \left|E^i\right| \langle 0$$

By substituting E_i in (A-9) into the above inequality, we have

$$a_1^i = a_4^i: \ -\left|a_1^i\right| \langle a_2^i \langle \left|a_1^i\right| \tag{A-10}$$

Therefore, for any arbitrary value of a_2^i satisfies (A-10), the matrix P_α^+ is still a common positive-definite matrix for the fuzzy system. The proof for the matrix P in the form of P_α^- is almost the same as the one for P_α^+ and is omitted.

□

REFERENCES

[1] T. Tanaka and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Systems Man Cybernetics*, vol. 15, pp. 116-132, 1985.

[2] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems", *Fuzzy sets and systems*, Vol. 2, pp. 135-156, 1992.

[3] S. Kawamoto, K. Tada, A. Ishigame, and T. Taniguchi, "An approach to stability analysis of second order fuzzy systems," *Proc. of IEEE Int. Conf. on Fuzzy Syst*., pp. 1427-1434, 1992.

[4] K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices", *IEEE Trans. on Automatic Control*, Vol. 39, No. 12, pp. 2469-2471, Dec. 1994.

[5] J. Joh, Y. H. Chen, and R. Langari, "On the stability issues of linear Takagi-Sugeno fuzzy models," *IEEE Trans. Fuzzy Syst*., vol. 6, no. 3, pp. 402-410, 1998.

[6] K. Tanaka, T. Hori and H. O. Wang, "A fuzzy Lyapunov approach to fuzzy control system design," *Proc. of the American Control Conf*., pp. 4790-4795, June 2001.

[7] W. J. Wang and C. H. Sun, "Relaxed stability and stabilization conditions for a T–S fuzzy discrete system," *Fuzzy Sets and Systems*, vol. 156, pp. 208–225, May 2005.

[8] L. Luoh, "New stability analysis of T-S fuzzy systems with robust approach," *Mathematics and computers in simulation*, vol. 59, pp. 335-340, 2002.

[9] C. T. Pang, and S. M. Guu, "Sufficient conditions for the stability of linear Takagi-Sugeno free fuzzy systems*," IEEE Trans. Fuzzy Syst*., vol. 11, no. 5, pp. 695-700, Oct. 2003.

[10] W. J. Wang and C. H. Sun, "A relaxed stability issues of linear Takagi- Sugeno Fuzzy Models," *Proc. of the 2004 IEEE International Conf. on Networking, Sensing & Control*, pp. 21-23, March 2004.

[11] M. Shekaramiz and F. Sheikholeslam, "*On the Stability of Continuous-Time T-S Model*", in Proceedings of 8th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD), IEEE, Shanghai, China, pp. 226-230, 26-28 July, 2011.

[12] F. Sheikholeslam and M. Shekaramiz, "*Stability Criterion for Takagi-Sugeno Models*", in Proceedings of 8th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD), IEEE, Shanghai, China, pp. 262-267, 26-28 July, 2011.

[13] C. K. Li and R. C. Li, "A note on eigenvalues of perturbed Hermitian matrices", *Linear Algebra and its Applications*, vol. 395, pp. 183-190 2005.

[14] G. W. Stewart and J. G. Sun, Matrix perturbation theory, *Academic Press, Boston*, 1990.