Remarks on Legendrian Self-Linking

Chris Beasley,
Department of Mathematics, Northeastern University, Boston MA 02115, USA

Brendan McLellan,
Department of Mathematics & Statistics, Utah State University, Logan UT 84322, USA

and

Ruoran Zhang
Department of Mathematics, Northeastern University, Boston MA 02115, USA

The Thurston-Bennequin invariant provides one notion of self-linking for any homologically-trivial Legendrian curve in a contact three-manifold. Here we discuss related analytic notions of self-linking for Legendrian knots in $\mathbb{R}^3$. Our definition is based upon a reformulation of the elementary Gauss linking integral and is motivated by ideas from supersymmetric gauge theory. We recover the Thurston-Bennequin invariant as a special case.

arXiv:1503.06120v2 [math.SG] 20 Aug 2018

I. INTRODUCTION

The linking number $\mathsf{lk}(C_1, C_2)$ of disjoint, oriented curves $C_1, C_2 \subset \mathbb{R}^3$ is among the most basic invariants in knot theory, and as such it admits many different descriptions. We begin with three.

Let (x, y, z) be Euclidean coordinates on $\mathbb{R}^3$ and consider the global angular form [6]

$$\psi = \frac{1}{4\pi} \frac{x\,dy{\wedge}dz \,-\, y\,dx{\wedge}dz \,+\, z\,dx{\wedge}dy}{[x^2 \,+\, y^2 \,+\, z^2]^{3/2}} \in \Omega^2\big(\mathbb{R}^3 - \{0\}\big). \tag{1}$$

With the given normalization, $\psi = \varrho^*\omega$ is the pullback of an $SO(3)$-invariant, unit-volume form ω on $S^2 \subset \mathbb{R}^3$ under the retraction

$$\begin{aligned} \varrho : \mathbb{R}^3 - \{0\} &\longrightarrow S^2\,, \\ \varrho(x, y, z) &= \frac{(x, y, z)}{[x^2 \,+\, y^2 \,+\, z^2]^{1/2}}\,. \end{aligned} \tag{2}$$

Without loss, we take the embedded curves C_i for $i = 1, 2$ to be parametrized by smooth maps

$$X_i : S^1 \longrightarrow \mathbb{R}^3\,, \qquad i = 1, 2\,, \tag{3}$$

in terms of which we write the difference

$$\begin{aligned} \Gamma : \mathbb{R}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{R}^3\,, \\ \Gamma(X_1, X_2) &= X_2 - X_1\,. \end{aligned} \tag{4}$$

The Gauss formula for the linking number of C_1 and C_2 is then given by an integral over the torus $T^2 = S^1 \times S^1$,

$$\mathsf{lk}(C_1, C_2) = \int_{T^2} (X_1 \times X_2)^* \Gamma^* \psi\,. \tag{5}$$

Essential here, since C_1 and C_2 are disjoint space curves, the singularity of ψ at the origin is avoided, and the Gauss integrand is everywhere smooth and bounded on T^2.

Alternatively, because ω represents a generator for the cohomology $H^2(S^2; \mathbb{Z})$, the Gauss linking integral computes the topological degree

$$\mathsf{lk}(C_1, C_2) = \deg \varphi_{12} \in \mathbb{Z}\,, \tag{6}$$

where φ_{12} is the composition

$$\varphi_{12} : T^2 \longrightarrow S^2\,, \qquad \varphi_{12} = \varrho \circ \Gamma \circ (X_1 \times X_2)\,. \tag{7}$$

In its second description as the degree of φ_{12}, the linking number is clearly an integer and invariant under smooth isotopies of the curves $C_{1,2}$. The overall sign of the linking number depends upon the orientations for T^2 and S^2. Throughout, if (θ_1, θ_2) are angular coordinates on T^2, we orient the torus by $d\theta_1{\wedge}d\theta_2$. We similarly give $S^2 \subset \mathbb{R}^3$ the orientation induced from the standard orientation $dx{\wedge}dy{\wedge}dz$, for which the cohomology generator $\omega > 0$ is positive.

Famously the linking number admits a third, diagrammatic description, convenient for computations. Again without loss, we consider a generic plane projection for C_1 and C_2 in which only simple, double-point crossings are present. The index set $\mathbf{I}$ of all crossings in the planar diagram for the link divides into subsets $\mathbf{I} = \mathbf{I}_1^1 \cup \mathbf{I}_2^1 \cup \mathbf{I}_1^2 \cup \mathbf{I}_2^2$, depending upon whether the upper and lower strands at each crossing belong respectively to C_1 or C_2. To each crossing $a \in \mathbf{I}$ we attach a local writhe $\mathsf{w}_a = \pm 1$ according to the handedness, as in Figure 1. In terms of these data, the linking number is computed by any of the following sums,

$$\mathsf{lk}(C_1, C_2) = \sum_{a \in \mathbf{I}_1^2} \mathsf{w}_a = \sum_{b \in \mathbf{I}_2^1} \mathsf{w}_b = \frac{1}{2} \sum_{c \in \mathbf{I}_1^2 \cup \mathbf{I}_2^1} \mathsf{w}_c\,. \tag{8}$$

The signed sum of crossings by C_2 over C_1 can be interpreted as a signed count of preimages $\varphi_{12}^{-1}(p)$ for $p \in S^2$ at the North Pole, so the first equality in (8) follows directly from the topological description of $\mathsf{lk}(C_1, C_2)$ as the degree of φ_{12}. One can check that our orientation conventions for T^2 and S^2 are consistent with the assignment of signs for w
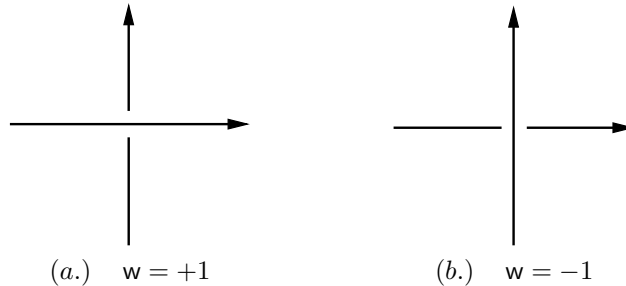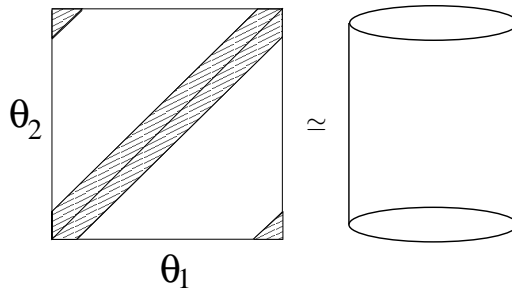
FIG. 1. Writhe at right- and left-handed crossings.

FIG. 2. Neighborhood $\Delta(\varepsilon)$ of the diagonal in T^2.

in Figure 1. The second equality in (8) follows similarly with $p \in S^2$ at the South Pole, and the third equality is the symmetric combination of the preceding two.

A.   Perspectives on self-linking

The present article is concerned not with linking but with self-linking, for which one would like to make sense of $\mathsf{lk}(C_1, C_2)$ as a knot invariant in the degenerate case $C_1 = C_2$. This problem does not have a unique solution, and several notions of self-linking already exist. Our purpose is to propose another, motivated by gauge theory [3, 19] and with an eye towards higher-order self-linking invariants, defined in the geometric style of [5] as integrals over configuration spaces of points associated to the knot.

The most naive attempt to define a self-linking number for an embedded, oriented curve $C \subset \mathbb{R}^3$ is based upon the Gauss integral in (5). Again parametrizing C as the image of a smooth map

$$X : S^1 \longrightarrow \mathbb{R}^3 \,, \tag{9}$$

we follow our nose to set

$$\mathsf{slk}_0(C) := \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X \times X)^* \Gamma^* \psi \,. \tag{10}$$

Unlike (5), the self-linking integrand is now singular along the diagonal $\Delta \subset T^2$, due to the divergence of ψ at the origin in $\mathbb{R}^3$. To deal with the singularity, we excise a tubular neighborhood $\Delta(\varepsilon)$ of width $0 < \varepsilon \ll 1$ about the diagonal, depicted as the shaded region in Figure 2, and we integrate only over the resulting cylinder $T^2 - \Delta(\varepsilon)$. We finally take the limit $\varepsilon \to 0$ to remove any dependence on the auxiliary parameter.

There are two essential statements to make about the naive self-linking integral in (10), both of which go back to the classic works by Călugăreanu [7] and Pohl [15]. First, the singularity along the diagonal Δ is integrable, so the limit defining $\mathsf{slk}_0(C) \in \mathbb{R}$ does exist.

Second, $\mathsf{slk}_0(C)$ is *not* a deformation-invariant of C, but rather varies to first-order under a change δX in the

embedding,

$$\delta\,\mathsf{slk}_0(C) \;=\; -\frac{1}{2\pi}\oint_{S^1} ds \; \epsilon_{\mu\nu\rho}\,\dot{X}^\mu\,\delta X^\nu\,\ddot{X}^\rho\,, \qquad\qquad ||\dot{X}(s)||^2 \;=\; 1\,. \tag{11}$$

To simplify the expression on the right of (11), we have taken $X^\mu(s)$ to be a regular, unit-speed parametrization, $\dot{X}^\mu \equiv dX^\mu/ds$, for which ds is the arc-length measure on C. Also, $\epsilon_{\mu\nu\rho}$ for $\mu,\nu,\rho = 1,2,3$ is the fully anti-symmetric tensor, normalized so that $\epsilon_{123} = +1$. As a corollary of (11), the value of the naive self-linking integral depends non-trivially on the geometry of $C \subset \mathbb{R}^3$, and it cannot generically be an integer, $\mathsf{slk}_0(C) \notin \mathbb{Z}$.

The naive attempt to define a self-linking invariant fails for two reasons, one topological and one analytic.

From the topological perspective, the primary difficulty is that the domain of integration $T^2 - \Delta(\varepsilon)$ has a boundary, illustrated in Figure 2. When the embedding map X varies in (10), the integrand changes by a cohomologically-trivial two-form, whose integral over the cylinder $T^2 - \Delta(\varepsilon)$ receives a non-trivial boundary contribution for $\varepsilon > 0$.

The analytic aspect of the self-linking anomaly is not as obvious, but no less important. To wit, in the detailed computation leading to (11), the boundary contribution to $\delta\,\mathsf{slk}_0(C)$ remains non-vanishing even in the limit $\varepsilon \to 0$. See the Ph.D. thesis of Bar-Natan [1] for a very nice presentation of the anomaly calculation. This phenomenon depends very much on the analytic behavior of the angular form ψ near the origin, and need not have occurred had the divergence of ψ been less rapid. This observation will be central to our work and is exemplified by the Fundamental Lemma in Section II.

Because of the anomaly, all notions of self-linking involve a modification to the naive Gauss integral in (10), as well as a refinement of the equivalence relation by smooth isotopy for the curve $C \subset \mathbb{R}^3$.

For instance, in the original approach of [7, 15], one adds to $\mathsf{slk}_0(C)$ a counterterm $T(C)$ whose variation with respect to X is exactly opposite the variation in (11),

$$\delta T(C) \;=\; -\delta\,\mathsf{slk}_0(C)\,. \tag{12}$$

The required counterterm is nothing more than the total torsion of C measured with respect to arc-length,

$$T(C) \;=\; \frac{1}{2\pi}\oint_{S^1} ds\; \tau\,, \tag{13}$$

where the local Frenet-Serret [8] torsion τ is given in terms of X by

$$\tau \;=\; \frac{\epsilon_{\mu\nu\rho}\,\dot{X}^\mu\,\ddot{X}^\nu\,\dddot{X}^\rho}{||\dot{X}\times\ddot{X}||^2}\,, \qquad\qquad \ddot{X}(s) \neq 0\,. \tag{14}$$

For sake of brevity, we do not review the proof of (12) here. A direct calculation of $\delta T(C)$ can be found in Appendix B of [3].

From the relation in (12), the sum

$$\mathsf{slk}_\tau(C) \;=\; \mathsf{slk}_0(C) \,+\, T(C) \tag{15}$$

does not change under any small deformation of C, so it provides a reasonable notion of self-linking. However, $\mathsf{slk}_\tau(C)$ is also not invariant under arbitrary smooth isotopies of C. Due to the denominator in (14), the torsion τ is only defined at points of C where $\ddot{X}(s) \neq 0$, or equivalently, where the Frenet-Serret curvature is non-zero. To make sense of $T(C)$, we assume that C has everywhere non-vanishing curvature. The latter condition is open and so holds for the generic space curve, but of course it does not hold for all space curves. As a result, $\mathsf{slk}_\tau(C)$ is only invariant under those 'non-degenerate' isotopies which preserve the condition of non-vanishing curvature along C.

Like the linking number, the self-linking number in (15) has a simple diagrammatic description. To derive it, consider the smooth isotopy

$$F_\Lambda : \mathbb{R}^3 \longrightarrow \mathbb{R}^3\,, \qquad\qquad F_\Lambda(x,y,z) = \left(x,y,\frac{z}{\Lambda}\right)\,, \tag{16}$$

where $\Lambda > 0$ is a positive real parameter. Applied to any curve $C \subset \mathbb{R}^3$, this isotopy flattens C to the xy-plane as Λ grows to infinity. Rotating C if necessary, we assume that both C and its projection to the xy-plane have everywhere non-vanishing curvature. See Figure 3 for a diagram of the trefoil knot which satisfies this condition. Then F_Λ is non-degenerate for all values of Λ, and $\mathsf{slk}_\tau(C)$ can be evaluated in the limit $\Lambda \to \infty$. For any plane curve, $\tau = 0$ identically, so $\mathsf{slk}_\tau(C)$ reduces to $\mathsf{slk}_0(C)$ in this limit. Otherwise, when Λ is large and C is nearly planar, the naive

FIG. 3. Trefoil knot with $\mathsf{w}(C) = -3$.

$(a.)$ $\quad$ $\mathsf{w}(C) = 0$ $\qquad\qquad$ $(b.)$ $\quad$ $\mathsf{w}(C) = 1$ $\qquad\qquad$ $(c.)$ $\quad$ $\mathsf{w}(C) = 2$
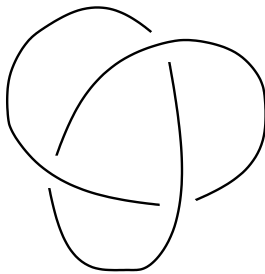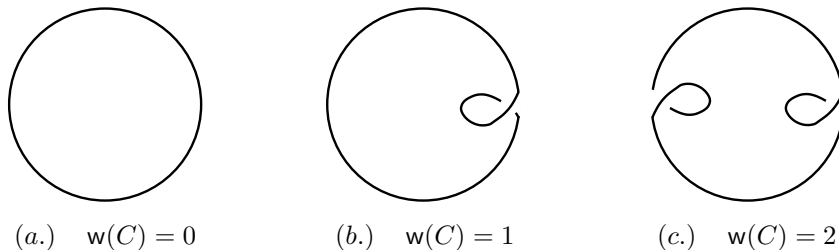
FIG. 4. Writhe for a selection of unknots.

self-linking integral in (10) reduces to a sum over the local writhe at each self-crossing of C. We omit a proof of the latter claim, which is hopefully plausible on the basis of the similar description (8) for $\mathsf{lk}(C_1, C_2)$. In Section III A, we will establish a closely related result as part of our Main Theorem.

Altogether, $\mathsf{slk}_\tau(C)$ is given by the writhe $\mathsf{w}(C)$ of the planar diagram for C,

$$\mathsf{slk}_\tau(C) \,=\, \mathsf{w}(C) \,:=\, \sum_{a \in \mathbf{I}} \mathsf{w}_a \;. \tag{17}$$

In particular, $\mathsf{slk}_\tau(C) \in \mathbb{Z}$ is an integer, which is not manifest from the analytic description. Also, because the assignment of the writhe $\mathsf{w} = \pm 1$ in Figure 1 is invariant under a reversal of orientation for both strands, $\mathsf{slk}_\tau(C)$ does not depend upon the orientation of C. Finally, the various versions of the unknot in Figure 4 illustrate that $\mathsf{slk}_\tau(C)$ cannot be a full isotopy invariant of C.

The Frenet-Serret self-linking number $\mathsf{slk}_\tau(C)$ is really a special case of the more general, and perhaps more familiar, notion of framed self-linking. By definition, a framing of $C \subset \mathbb{R}^3$ is a trivialization of the normal bundle N_C, up to homotopy. Such a trivialization can be specified by a nowhere-vanishing normal vector field $n \in \Gamma(C, N_C)$ along the knot. In turn, the normal vector field determines a new curve C_n obtained by displacing C a small amount in the direction of n, as shown in Figure 5.
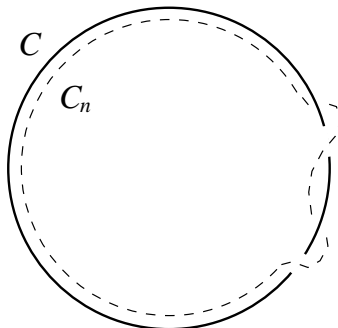
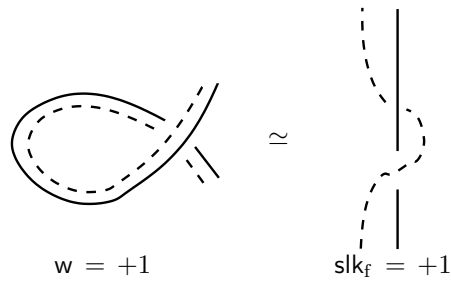FIG. 5. A framed unknot, with $\mathsf{slk}_\mathrm{f}(C, n) = 2$.

FIG. 6. Relation between writhe and Frenet-Serret framing.

Given the pair (C, n), the framed self-linking number is defined as the ordinary linking number of the two disjoint curves C and C_n,

$$\mathsf{slk}_\mathsf{f}(C, n) := \mathsf{lk}(C, C_n). \tag{18}$$

On the upside, $\mathsf{slk}_\mathsf{f}(C, n)$ is manifestly an integral isotopy invariant of the pair (C, n). On the downside, $\mathsf{slk}_\mathsf{f}(C, n)$ carries no information about the knot C itself, since the invariant takes all possible values as the winding number of n about C shifts.

Had one a canonical choice of framing, the framed self-linking number $\mathsf{slk}_\mathsf{f}(C, n)$ could be converted into an honest invariant of C. No such choice exists for all smooth curves simultaneously, but a variety of choices can be made if one restricts to special classes of curves.

We have already encountered an example in the discussion of the Frenet-Serret self-linking $\mathsf{slk}_\tau(C)$, for which we require $C \subset \mathbb{R}^3$ to have everywhere non-vanishing curvature. Not coincidentally, such curves also carry a canonical Frenet-Serret framing, with unit normal $n = \ddot{X}/||\ddot{X}||$. Indeed, a natural guess is that $\mathsf{slk}_\tau(C) = \mathsf{slk}_\mathsf{f}(C, n)$ for the Frenet-Serret normal.

This guess is correct, as can be seen most easily by considering the behavior of the Frenet-Serret frame in the planar limit $\Lambda \to \infty$ from (16). In this limit, the Frenet-Serret framing reduces to the blackboard framing in which the unit normal vector n lies everywhere in the plane of the knot diagram. After displacing C by n, one finds a planar ribbon graph, shown in the neighborhood of a positive crossing on the left in Figure 6. For such a graph, each self-crossing of C corresponds to a crossing of C by C_n with identical chirality, so automatically $\mathsf{w}(C) = \mathsf{lk}(C, C_n)$. The relation between the writhe and the Frenet-Serret framing can also be understood in three-dimensional terms, as indicated to the right in Figure 6.

B. Legendrian knots

This article serves as the companion to a longer work [3] in which we develop a new, effectively supersymmetric formulation of Chern-Simons perturbation theory. Both the Frenet-Serret and the framed self-linking invariants were rediscovered physically [14, 16, 18] in the setting of bosonic Chern-Simons theory, and our purpose is to present what one finds with the addition of $\mathcal{N} = 2$ supersymmetry, after all the baggage from gauge theory is removed. Details about the gauge theory are discussed in [3, 19].

Supersymmetry has two consequences. We discuss the first now, and we defer a discussion of the second to the next subsection.

As a first consequence of supersymmetry, $C \subset \mathbb{R}^3$ must be Legendrian with respect to the standard contact structure on $\mathbb{R}^3$, and the supersymmetric self-linking number will be a Legendrian isotopy invariant. For very enjoyable introductions to contact topology and the study of Legendrian knots, see [9, 10, 12, 13]. At the moment, we recall only the minimum necessary to state and prove our Main Theorem.

Throughout, we represent the standard contact structure on $\mathbb{R}^3$ with the contact form

$$\kappa = dz + x\,dy - y\,dx, \tag{19}$$

for which the top-form $\kappa \wedge d\kappa > 0$ is positive with respect to the standard orientation on $\mathbb{R}^3$. This choice of contact form respects many symmetries, which are important here and in [3, 19].

Manifestly, κ is preserved under translations generated by the Reeb field $\mathsf{R} = \partial/\partial z$ as well as rotations in the xy-
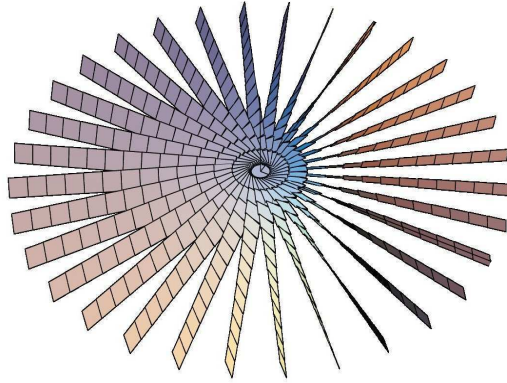
FIG. 7. The standard radially-symmetric contact structure on $\mathbb{R}^3$. (Courtesy of the `Mathematica` routine 'CSPlotter' by Matias Dahl.)

plane. Though κ is not preserved by translations in the xy-plane, κ is preserved by the left-action of the Heisenberg Lie group $\mathbb{H} \simeq \mathbb{R}^3$ on itself, where the Heisenberg multiplication $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is given by

$$\mu\big(X_1, X_2\big) = \big(x_1 + x_2,\, y_1 + y_2,\, z_1 + z_2 - x_1 y_2 + x_2 y_1\big), \qquad X_{1,2} \in \mathbb{H}. \tag{20}$$

The origin remains the identity in $\mathbb{H}$, and the Heisenberg inverse is $X^{-1} = (-x, -y, -z)$. Finally, κ is homogeneous of degree two under the parabolic scaling

$$(x, y, z) \longmapsto \big(\lambda x,\, \lambda y,\, \lambda^2 z\big), \qquad \lambda \in \mathbb{R}_+, \tag{21}$$

which commutes with Heisenberg multiplication. As a result, the parabolic scaling fixes each contact plane $H \subset \mathbb{R}^3$ in the kernel of κ.

A picture of the family of contact planes $H = \ker \kappa$ appears in Figure 7. The contact planes are approximately horizontal near $x = y = 0$, but they twist vertically as one moves outward from the origin in the xy-plane.

We also require a few facts about Legendrian knots. By definition, $C \subset \mathbb{R}^3$ is Legendrian when the tangent line $T_p C$ at any point $p \in C$ lies in the contact plane H_p at that point. Equivalently, the pullback of κ to C vanishes,

$$\kappa\big|_C = 0 \qquad \Longleftrightarrow \qquad C \text{ is Legendrian}, \tag{22}$$

or in terms of a parametrization $X : S^1 \to \mathbb{R}^3$,

$$\frac{dz}{d\theta} = y\,\frac{dx}{d\theta} - x\,\frac{dy}{d\theta}, \qquad X(\theta) \equiv \big(x(\theta),\, y(\theta),\, z(\theta)\big). \tag{23}$$

Any smooth knot admits a Legendrian representative, so the theory of Legendrian knots is extremely rich. Moreover, equivalence by Legendrian isotopy, ie. continuous isotopy through a family of Legendrian knots, strictly refines the usual topological equivalence. A given topological knot has infinitely-many inequivalent Legendrian representatives.

Unlike topological knots, Legendrian knots have canonical plane projections. For this reason, Legendrian knots behave in many ways like plane curves. Our interest lies in the so-called Lagrangian projection to the xy-plane,

$$\Pi : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \Pi(x, y, z) = (x, y), \tag{24}$$

for which the image $\Pi(C)$ of a Legendrian knot $C \subset \mathbb{R}^3$ is a smoothly immersed curve. The smoothness of $\Pi(C)$ is already a non-trivial feature of the Legendrian condition (23), since this condition implies that $\dot{z} = 0$ at any point on C where $\dot{x} = \dot{y} = 0$. Hence if X is a regular parametrization of C, then $\Pi \circ X$ is a regular parametrization of $\Pi(C)$. Trivially, $\Pi(C)$ is a Lagrangian submanifold of $\mathbb{R}^2$ with respect to the symplectic form $d\kappa = 2\,dx \wedge dy$, whence the name.
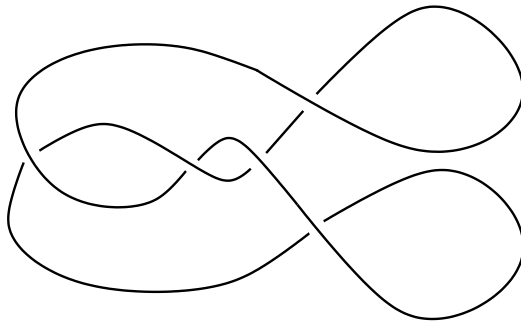
FIG. 8. Legendrian trefoil knot, with $\mathsf{tb}(C) = 1$ and $\mathsf{rot}(C) = 0$.

In Figure 8 we display the Lagrangian projection of a Legendrian trefoil knot. To guide the eye, we indicate over- and under-crossings in the figure. Unlike for topological knot diagrams, the crossing information for a Legendrian knot is redundant, since the spatial configuration of $C \subset \mathbb{R}^3$ can be completely recovered from the plane curve $\Pi(C)$ by integrating the contact relation in (23),

$$z(\theta) = z_0 + \int_0^\theta d\theta' \left[y(\theta') \, \dot{x}(\theta') - x(\theta') \, \dot{y}(\theta') \right]. \tag{25}$$

Here z_0 is the height of C at the basepoint corresponding to $\theta = 0$. Due the symmetry of κ under translations in z, this constant is both arbitrary and irrelevant.

Not every immersed curve can be the plane projection of a Legendrian knot. For instance, if we take the parameter θ in (25) to have periodicity 2π, then

$$0 = z(2\pi) - z(0) = \int_0^{2\pi} d\theta' \left[y(\theta') \, \dot{x}(\theta') - x(\theta') \, \dot{y}(\theta') \right]. \tag{26}$$

Equivalently by Stokes' Theorem, the oriented 2-chain D bounded by $\Pi(C)$ must have zero symplectic area,

$$\int_D dx \wedge dy = 0, \tag{27}$$

where each component of D is oriented consistently with $\partial D = C$. Also, if $\theta_1 \neq \theta_2$ are distinct parameter values for which $x(\theta_1) = x(\theta_2)$ and $y(\theta_1) = y(\theta_2)$, corresponding to the location of a crossing in $\Pi(C)$, then

$$0 \neq z(\theta_2) - z(\theta_1) = \int_{\theta_1}^{\theta_2} d\theta' \left[y(\theta') \, \dot{x}(\theta') - x(\theta') \, \dot{y}(\theta') \right]. \tag{28}$$

The necessary conditions in (26) and (28) are sufficient for the immersed plane curve to lift to an embedded Legendrian knot. These conditions depend upon the signed areas of the regions enclosed by $\Pi(C)$, so the Lagrangian projection cannot be manipulated in a wholly topological fashion á la Reidemeister.

Legendrian knots always carry a canonical framing by the Reeb vector field $\mathsf{R} = \partial/\partial z$. Concretely from (23), a Legendrian curve $C \subset \mathbb{R}^3$ cannot have a vertical tangent, where $\dot{x} = \dot{y} = 0$ but $\dot{z} \neq 0$. With the choice $n = \mathsf{R}$, the framed self-linking number $\mathsf{slk}_\mathrm{f}(C, n)$ can then be converted into a Legendrian invariant

$$\mathsf{tb}(C) := \mathsf{slk}_\mathrm{f}(C, \mathsf{R}) \in \mathbb{Z}, \tag{29}$$

a kind of self-linking number for C.

The Thurston-Bennequin invariant $\mathsf{tb}(C)$ can be easily computed from the Lagrangian projection of C. After a rigid rotation, the vertical framing by the Reeb field R becomes equivalent to the planar, blackboard framing of $\Pi(C)$. But again by Figure 6, the self-linking number in the blackboard framing is exactly the writhe of the knot diagram. Thus,

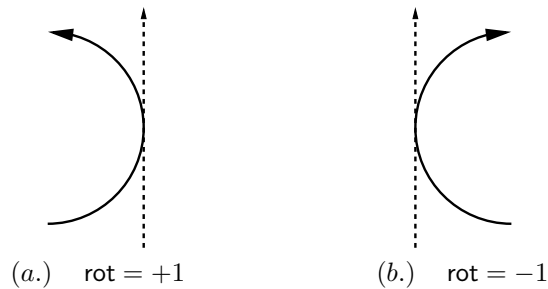$$\mathsf{tb}(C) = \mathsf{w}(\Pi(C)). \tag{30}$$

$(a.)$ $\mathsf{rot} = +1$ $(b.)$ $\mathsf{rot} = -1$

FIG. 9. Rotation number at upwards tangencies.

Since the writhe is fixed under orientation-reversal, so too is

$$\mathsf{tb}(-C) = \mathsf{tb}(C) \,. \tag{31}$$

The Thurston-Bennequin invariant is one of a pair of classical Legendrian invariants. To state the Main Theorem, we also need the other.

Because C is determined by its Lagrangian projection $\Pi(C)$, any isotopy invariant of immersed plane curves yields a Legendrian invariant of C. According to the Whitney-Graustein Theorem [17], the unique such invariant of an immersion $\gamma : S^1 \to \mathbb{R}^2$ is the rotation number

$$\mathsf{rot}(\gamma) = \deg \dot{\gamma} \,, \qquad \dot{\gamma} : S^1 \to \mathbb{R}^2 - \{0\} \,, \tag{32}$$

defined as the topological degree of the derivative $\dot{\gamma}$. Equivalently, $\mathsf{rot}(\gamma)$ is the total signed curvature (see eg. Exercise 12 in §1.5 of [8]) of the immersed plane curve,

$$\mathsf{rot}(\gamma) = \frac{1}{2\pi} \oint_{S^1} d\theta \, \frac{\dot{\gamma} \times \ddot{\gamma}}{||\dot{\gamma}||^2} \,, \tag{33}$$

where we use the shorthand '$\times$' for the scalar cross-product,

$$\dot{\gamma} \times \ddot{\gamma}(\theta) \equiv \dot{x}(\theta) \, \ddot{y}(\theta) - \dot{y}(\theta) \, \ddot{x}(\theta) \,, \qquad \gamma(\theta) = (x(\theta), y(\theta)) \in \mathbb{R}^2 \,. \tag{34}$$

As will be essential later, the formula for $\mathsf{rot}(\gamma)$ in (33) presents the rotation number as a local invariant, in the sense of being the integral of a locally-defined geometric quantity along the curve. With our conventions, $\mathsf{rot}(\gamma) = 1$ when γ is a circle traversed in the counterclockwise direction.

For the Legendrian knot $C \subset \mathbb{R}^3$, we set

$$\mathsf{rot}(C) := \mathsf{rot}(\Pi(C)) \in \mathbb{Z} \,. \tag{35}$$

See Definition 3.5.12 in [13] for an intrinsically three-dimensional characterization of $\mathsf{rot}(C)$. In terms of the diagram for $\Pi(C)$, the rotation number can be computed as a signed count of upwards vertical tangencies, as in Figure 9. For the Legendrian trefoil in Figure 8, two upwards vertical tangencies occur, but they do so with opposite signs, so $\mathsf{rot}(C) = 0$.

Note that the rotation number depends upon the orientation of the curve, and under a reversal of orientation, the rotation number changes sign. So in contrast to the behavior (31) of the Thurston-Bennequin invariant,

$$\mathsf{rot}(-C) = -\mathsf{rot}(C) \,. \tag{36}$$

We have not specified an orientation for the trefoil knot in Figure 8, but because $\mathsf{rot}(C) = 0$, the orientation does not matter.

C.   Main theorem

As its second consequence, supersymmetry modifies the angular form $\psi \in \Omega^2\big(\mathbb{R}^3 - \{0\}\big)$ which enters the elementary Gauss linking integral (5). Because the Legendrian condition on $C \subset \mathbb{R}^3$ does not respect the Euclidean action by $SO(3)$, we forgo the seemingly-natural requirement that ψ itself be $SO(3)$-invariant. In recompense, the supersymmetric version of ψ will enjoy superior analytic behavior near the origin in $\mathbb{R}^3$.

As heuristic motivation for the following, one should imagine that we alter the angular form ψ in (1) by concentrating the support for the generator of $H^2(S^2; \mathbb{Z})$ at the North Pole on the sphere. This trick is well-known to aficionados of Chern-Simons perturbation theory, but here we must take care to preserve the underlying symmetries of the contact structure on $\mathbb{R}^3$.

More precisely, we introduce the Gaussian two-form $\omega_\Lambda \in \Omega^2\big(\mathbb{R}^2\big)$ on the xy-plane,

$$\omega_\Lambda \;=\; \frac{\Lambda}{2\pi}\, e^{-\Lambda(x^2+y^2)/2}\, dx \wedge dy\,, \qquad \Lambda > 0\,. \tag{37}$$

Here Λ is a positive real parameter which sets the width of the Gaussian, and in the limit $\Lambda \to \infty$, the Gaussian becomes a delta-function concentrated at the origin in $\mathbb{R}^2$. Clearly ω_Λ is invariant under rotations, and ω_Λ is normalized so that for all Λ,

$$\int_{\mathbb{R}^2} \omega_\Lambda \;=\; 1\,. \tag{38}$$

The various factors of two in (37) are standard and could be absorbed into Λ if desired.

Though the support of ω_Λ is not compact, ω_Λ decays very rapidly at infinity. At least morally, ω_Λ should be regarded as a generator for the compactly-supported cohomology $H^2_c\big(\mathbb{R}^2; \mathbb{Z}\big) \simeq \mathbb{Z}$ of the plane, on the same footing as the unit-area form on the sphere. Because the normalization condition in (38) does not depend on Λ, neither does the cohomology class of ω_Λ. Explicitly, a small computation shows

$$\frac{\partial \omega_\Lambda}{\partial \Lambda} \;=\; \frac{1}{2\pi}\left[1 \;-\; \frac{\Lambda(x^2+y^2)}{2}\right] e^{-\Lambda\,(x^2+y^2)/2}\, dx \wedge dy \;=\; d\alpha_\Lambda\,, \tag{39}$$

where

$$\alpha_\Lambda \;=\; \frac{1}{4\pi}\, e^{-\Lambda(x^2+y^2)/2}\,(x\,dy - y\,dx) \;\in\; \Omega^1\big(\mathbb{R}^2\big). \tag{40}$$

The transgression form α_Λ will reappear in the proof of our Fundamental Lemma. In the meantime, note that α_Λ is also $SO(2)$-invariant, as required by the relation to ω_Λ in (39).

We next introduce the planar analogue for the retraction onto S^2 in (2). To preserve the parabolic scaling in (21), we consider the map $\varrho_+ : \mathbb{R}^3_+ \to \mathbb{R}^2$ defined on the upper half-space $\mathbb{R}^3_+$ by

$$\varrho_+(x,y,z) \;=\; \left(\frac{x}{\sqrt{z}}, \frac{y}{\sqrt{z}}\right), \qquad z > 0\,. \tag{41}$$

Trivially, the image of ϱ_+ is preserved under the scaling for which x and y have weight one and z has weight two.

Using the planar retraction in (41), we pull the Gaussian form $\omega_\Lambda \in \Omega^2(\mathbb{R}^2)$ back to a new two-form

$$\begin{aligned}
\chi_\Lambda \;&=\; \varrho_+^*\omega_\Lambda\,, \\
&=\; \frac{\Lambda}{2\pi z} e^{-\Lambda(x^2+y^2)/2z}\left[dx \wedge dy \;+\; \frac{1}{2}\,(x\,dy - y\,dx)\wedge\frac{dz}{z}\right], \qquad z > 0\,.
\end{aligned} \tag{42}$$

By construction, χ_Λ is invariant under the parabolic scaling and the action of $SO(2)$, but not $SO(3)$.

Of course, χ_Λ strongly resembles the heat kernel for the Laplacian in two dimensions. As with the heat kernel, so long as $x^2 + y^2 \neq 0$, the expression in (42) vanishes smoothly as $z \to 0$ from above. To define χ_Λ on the entire punctured space $\mathbb{R}^3 - \{0\}$, we simply extend by zero,

$$\chi_\Lambda \;=\; 0\,, \qquad z \leq 0\,. \tag{43}$$

With this choice, $\chi_\Lambda \in \Omega^2\big(\mathbb{R}^3 - \{0\}\big)$ is automatically closed away from $\{0\}$ and generates the cohomology $H^2\big(\mathbb{R}^3 - \{0\}; \mathbb{Z}\big)$.

For instance, over the unit sphere $S^2 \subset \mathbb{R}^3$,

$$\int_{S^2} \chi_\Lambda = \int_{S^2 \cap \mathbb{R}^3_+} \chi_\Lambda = \int_{\mathbb{R}^2} \omega_\Lambda = 1 \,. \tag{44}$$

We have yet to incorporate the Heisenberg symmetry of the contact structure. In the elementary Gauss linking integral, the abelian structure of $\mathbb{R}^3$ as a vector space enters implicitly through the definition of the difference map Γ in (4). To preserve instead the non-abelian symmetry by left-translation in $\mathbb{H} \simeq \mathbb{R}^3$, we consider a Heisenberg difference map $\widehat{\Gamma} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$, given by

$$\begin{aligned} \widehat{\Gamma}(X_1, X_2) &= \mu(X_1^{-1}, X_2) = \mu(-X_1, X_2), && X_{1,2} \in \mathbb{H}, \\ &= (x_2 - x_1, \, y_2 - y_1, \, z_2 - z_1 + x_1 y_2 - x_2 y_1) \,. \end{aligned} \tag{45}$$

Since μ is the Heisenberg multiplication, $\widehat{\Gamma}(X_1, X_2) = X_1^{-1} \cdot X_2$ in the usual shorthand. This combination of X_1 and X_2 is invariant under simultaneous left-multiplication,

$$\widehat{\Gamma}(g \cdot X_1, g \cdot X_2) = \widehat{\Gamma}(X_1, X_2), && g \in \mathbb{H}, \tag{46}$$

and we have selected the relative signs of X_1 and X_2 in (45) to agree with the convention for the abelian difference in (4).

Proposition I.1 (Heisenberg Linking) *Suppose $C_i \subset \mathbb{R}^3$ for $i = 1, 2$ are disjoint oriented curves, not necessarily Legendrian, with respective parametrizations $X_i : S^1 \to \mathbb{R}^3$. Then*

$$\mathsf{lk}(C_1, C_2) = \int_{T^2} (X_1 \times X_2)^* \widehat{\Gamma}^* \chi_\Lambda \,, && \Lambda > 0 \,. \tag{47}$$

This proposition follows from the fact that the heat form χ_Λ is equivalent in cohomology to the global angular form ψ,

$$[\chi_\Lambda] = [\psi] \in H^2\big(\mathbb{R}^3 - \{0\}; \mathbb{Z}\big) \,. \tag{48}$$

Also, the Heisenberg difference $\widehat{\Gamma}$ in (45) is homotopic to the abelian difference Γ. To see this, set

$$\mu_\hbar(X_1, X_2) = \big(x_1 + x_2, \, y_1 + y_2, \, z_1 + z_2 - \hbar\,(x_1 y_2 - x_2 y_1)\big) \,, && \hbar \in [0, 1] \,. \tag{49}$$

Then

$$\widehat{\Gamma}_\hbar(X_1, X_2) = \mu_\hbar\big(X_1^{-1}, X_2\big) = \big(x_2 - x_1, \, y_2 - y_1, \, z_2 - z_1 + \hbar\,(x_1 y_2 - x_2 y_1)\big) \tag{50}$$

smoothly interpolates from the abelian to the Heisenberg difference as the Planck constant $\hbar$ ranges over the interval from $\hbar = 0$ to $\hbar = 1$. □

Though the angular form ψ and the heat form χ_Λ agree in cohomology, they behave very differently near the origin in $\mathbb{R}^3$. This analytic distinction matters crucially for approaches to self-linking.

Let $C \subset \mathbb{R}^3$ be an oriented Legendrian curve, with regular parametrization $X : S^1 \to \mathbb{R}^3$. By analogy to the naive Gauss self-linking integral in (10), we consider a new Heisenberg self-linking integral

$$\mathsf{slk}_\kappa(C) := \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \,, && \hbar \in [0, 1] \,, && \Lambda > 0 \,. \tag{51}$$

The remainder of the article is devoted to the proof of the following Main Theorem.

Theorem I.2 (Legendrian Self-Linking) *The limit defining $\mathsf{slk}_\kappa(C)$ exists. The value of $\mathsf{slk}_\kappa(C)$ is independent of Λ and depends only upon the Legendrian isotopy class of C. In terms of the Thurston-Bennequin invariant $\mathsf{tb}(C)$ and the rotation number $\mathsf{rot}(C)$,*

$$\mathsf{slk}_\kappa(C) = \begin{cases} \mathsf{tb}(C) - \mathsf{rot}(C), & \hbar \neq 1, \\ \mathsf{tb}(C), & \hbar = 1. \end{cases} \tag{52}$$

Informally, the Main Theorem states that the framing anomaly for knots in bosonic Chern-Simons theory is absent in supersymmetric Chern-Simons theory. The corresponding statement for Seifert-fibered three-manifolds was observed previously in [2]. The Main Theorem is also consistent with results of Fuchs and Tabachnikov [11] identifying $\mathsf{tb}(C)$ and $\mathsf{rot}(C)$ as the only order ≤ 1 finite-type invariants of Legendrian knots.

The strategy of proof for Theorem I.2 is straightforward. We first demonstrate that $\mathsf{slk}_\kappa(C)$ is independent of the parameter Λ which sets the width of the Gaussian in χ_Λ. For any $\varepsilon > 0$, the derivative of $\mathsf{slk}_\kappa(C)$ with respect to Λ is given in terms of a boundary integral of the transgression form α_Λ in (40). The content of our Fundamental Lemma in Section II is to demonstrate that the potentially anomalous boundary contribution from α_Λ vanishes in the limit $\varepsilon \to 0$, due to the rapid decay of the heat kernel away from the diagonal.

In Section III we directly evaluate the self-linking integral in the limit $\Lambda \to \infty$. In this regime, the self-linking integrand is non-negligible only in the neighborhood of points on T^2 which map under the product $X \times X$ to pairs of points $p, q \in C$ that become coincident under the Lagrangian projection to the xy-plane, ie. $\Pi(p) = \Pi(q)$. Such points on T^2 correspond either to the preimage of crossings in $\Pi(C)$, or more trivially, to points on the diagonal $\Delta \subset T^2$. The local contribution from the crossings leads to the appearance of the Thurston-Bennequin invariant $\mathsf{tb}(C)$, while the local contribution from the diagonal Δ leads to the appearance of the rotation number $\mathsf{rot}(C)$ for $\hbar \neq 1$. At the special value $\hbar = 1$, the Heisenberg symmetry of the integrand is restored, and the anomalous contribution from Δ vanishes.

The Legendrian condition is used crucially throughout the analysis. Invariance under Legendrian isotopy follows *a posteriori* from the formula in (52).[1]

Let us emphasize one striking feature of Theorem I.2, which is perhaps best appreciated after one works through the localization computation in Section III A. Namely, since the coefficients of $\mathsf{tb}(C)$ and $\mathsf{rot}(C)$ in (52) are integers, so is the value of $\mathsf{slk}_\kappa(C)$! The coefficient of $\mathsf{tb}(C)$ is directly related to our normalization condition (44) on the heat form χ_Λ, required to recover the usual linking number in Proposition I.1. Thus the coefficient of $\mathsf{tb}(C)$ is fixed by fiat to unity. In contrast, the coefficient of $\mathsf{rot}(C)$ is determined by a delicate calculation near the diagonal $\Delta \subset T^2$, so its integrality for all $\hbar$ is a non-trivial feature of the Legendrian self-linking integral.

From the physical perspective, integrality of $\mathsf{slk}_\kappa(C)$ is necessary for gauge invariance as well as consistency with standard lore about non-renormalization and the infrared behavior of supersymmetric Yang-Mills-Chern-Simons theory. See §2.3 of [3] for a discussion of this statement. However, on the principle that no integer appears by chance, a simple topological explanation for the integrality of $\mathsf{slk}_\kappa(C) \in \mathbb{Z}$ would be nice to have.

For instance, the difference of classical Legendrian invariants $\mathsf{tb}(C) - \mathsf{rot}(C)$ in (52) occurs naturally in contact topology as the transverse self-linking invariant $\mathsf{slk}(C_+)$ of the canonical positive transverse push-off C_+ of C (Proposition 3.5.36 in [13]). The transverse self-linking invariant $\mathsf{slk}(C_+)$ can be interpreted in terms of a relative Euler class on a Seifert surface for C_+, so its appearance in Theorem I.2 is surely no accident.

D. Notation and conventions

For the convenience of the reader, we summarize the notation and conventions used in the rest of the paper.

- $\mathbb{R}^3$ has Euclidean coordinates (x, y, z) and is oriented by $dx \wedge dy \wedge dz$.

- The map $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection onto the xy-plane.

- $\kappa = dz + x\, dy - y\, dx$ is the standard radially-symmetric contact form, positive with respect to the orientation on $\mathbb{R}^3$.

- $C \subset \mathbb{R}^3$ is an oriented Legendrian knot, with regular parametrization $X : S^1 \to \mathbb{R}^3$.

- $\theta \sim \theta + 2\pi$ is an angular coordinate on S^1, compatible under X with the given orientation on C. We abbreviate $dX/d\theta \equiv \dot{X}(\theta)$, and so on.

- The torus $T^2 = S^1 \times S^1$ has angular coordinates (θ_1, θ_2) and is oriented by $d\theta_1 \wedge d\theta_2$. The diagonal $\Delta \subset T^2$ is the subset where $\theta_1 = \theta_2$.

- For $\varepsilon > 0$, a tubular neighborhood $\Delta(\varepsilon)$ of the diagonal $\Delta \subset T^2$ is parametrized by $\theta_1 = \phi$ and $\theta_2 = \phi + \eta$ for $|\eta| < \varepsilon$. The cylinder $T^2 - \Delta(\varepsilon)$ has boundary circles $S^1_\pm$ on which $\eta = \pm\varepsilon$, respectively.

---

[1] A direct computation of the Legendrian variation $\delta\,\mathsf{slk}_\kappa(C)$, achieved for $\delta\,\mathsf{slk}_0(C)$ in (11), involves some formidable differential algebra and did not appear practical to these authors.

- $\gamma : S^1 \to \mathbb{R}^2$ is an immersed plane curve which is the Lagrangian projection of C, ie. $\gamma = \Pi \circ X$.

- We use the abbreviation '$\times$' for the scalar cross-product on the plane, as in

$$\gamma \times \dot{\gamma}(\theta) \equiv x(\theta)\,\dot{y}(\theta) - y(\theta)\,\dot{x}(\theta)\,, \qquad \gamma(\theta) = (x(\theta), y(\theta)) \in \mathbb{R}^2\,.$$

- The contact form κ is left-invariant with respect to the Heisenberg multiplication

$$\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{H}\,, \qquad \mathbb{H} \simeq \mathbb{R}^3\,, \qquad [\hbar = 1]$$
$$\mu\big(X_1, X_2\big) = (x_1 + x_2,\, y_1 + y_2,\, z_1 + z_2 - x_1 y_2 + x_2 y_1)\,.$$

  More generally, for other values of $\hbar$ set

$$\mu_\hbar(X_1, X_2) = \big(x_1 + x_2,\, y_1 + y_2,\, z_1 + z_2 - \hbar\,(x_1 y_2 - x_2 y_1)\big)\,.$$

- The left-invariant Heisenberg difference $\widehat{\Gamma}_\hbar : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ for $\hbar \in [0,1]$ is given by

$$\widehat{\Gamma}_\hbar(X_1, X_2) = \mu_\hbar(X_1^{-1}, X_2)\,,$$
$$= (x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1 + \hbar\,(x_1 y_2 - x_2 y_1))\,.$$

- The Gaussian fundamental class $\omega_\Lambda \in \Omega^2\big(\mathbb{R}^2\big)$ of the xy-plane is given by

$$\omega_\Lambda = \frac{\Lambda}{2\pi}\,\mathrm{e}^{-\Lambda(x^2 + y^2)/2}\,dx \wedge dy\,, \qquad \Lambda > 0\,.$$

- The transgression form $\alpha_\Lambda \in \Omega^1\big(\mathbb{R}^2\big)$ satisfies $\partial\omega/\partial\Lambda = d\alpha_\Lambda$, where

$$\alpha_\Lambda = \frac{1}{4\pi}\,\mathrm{e}^{-\Lambda(x^2 + y^2)/2}\,(x\,dy - y\,dx)\,.$$

- The planar retraction $\varrho_+ : \mathbb{R}^3_+ \to \mathbb{R}^2$ is defined on the upper half-space $\mathbb{R}^3_+$ by

$$\varrho_+(x, y, z) = \left(\frac{x}{\sqrt{z}}, \frac{y}{\sqrt{z}}\right)\,, \qquad z > 0\,.$$

- The heat form $\chi_\Lambda \in \Omega^2\big(\mathbb{R}^3 - \{0\}\big)$ is the pullback

$$\chi_\Lambda = \begin{cases} \varrho_+^*\omega_\Lambda\,, & z > 0\,, \\ 0\,, & z \leq 0\,. \end{cases}$$

  Explicitly,

$$\varrho_+^*\omega_\Lambda = \frac{\Lambda}{2\pi z}\mathrm{e}^{-\Lambda(x^2+y^2)/2z}\left[dx \wedge dy + \frac{1}{2}\,(x\,dy - y\,dx) \wedge \frac{dz}{z}\right]\,, \qquad z > 0\,.$$

II. FUNDAMENTAL LEMMA

We first demonstrate that the value of the Legendrian self-linking integral $\mathsf{slk}_\kappa(C)$ does not depend upon the parameter $\Lambda > 0$ which sets the width of the Gaussian in the heat form χ_Λ.

Lemma II.1 (Fundamental Lemma) *The limit which defines the Legendrian self-linking integral* $\mathsf{slk}_\kappa(C)$ *exists,*

$$\mathsf{slk}_\kappa(C) = \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X \times X)^*\widehat{\Gamma}_\hbar^*\,\chi_\Lambda\,, \qquad \hbar \in [0,1]\,, \qquad \Lambda > 0\,, \tag{53}$$

and the value of $\mathsf{slk}_\kappa(C) \in \mathbb{R}$ *is independent of the parameter* Λ.

Precisely at the special value $\hbar = 1$, the self-linking integrand is Heisenberg-invariant. For this reason, the analysis to prove Lemma II.1 will differ slightly depending on whether $\hbar \neq 1$ or $\hbar = 1$.

Throughout, we make two extra topological assumptions about the Legendrian knot C. Both assumptions hold generically and help to simplify the proofs.

1. The Lagrangian projection $\Pi(C)$ is an immersed plane curve γ with only double-point singularities, as in Figure 8.

2. The height function $z(\theta)$ on $C \subset \mathbb{R}^3$ is Morse, with isolated non-degenerate critical points. That is, $\dot{z}$ vanishes only at isolated points $p \in C$, at which $\ddot{z} \neq 0$. Because C is Legendrian, the Morse condition on $z(\theta)$ is equivalent by (23) to the condition that the function $\gamma \times \dot{\gamma}$ vanish only at isolated points $\theta \in S^1$, at which $\gamma \times \ddot{\gamma} \neq 0$.

The first assumption is standard. Otherwise, the Morse condition is used only in the case $\hbar \neq 1$ and could possibly be relaxed, at the cost of further effort.

Before embarking on the proof of our Fundamental Lemma, let us mention an easy corollary which is handy for the gauge theory analysis in §5 of [3]. To state the corollary, we require additional notation. Let $t > 0$ be a positive scaling parameter. As in the remarks following Proposition I.1, we consider a version of the Heisenberg difference $\widehat{\Gamma}_{t\hbar}$ with rescaled Planck constant $t\hbar$ for fixed $\hbar \in [0,1]$,

$$\widehat{\Gamma}_{t\hbar}(X_1, X_2) = (x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1 + t\hbar\,(x_1 y_2 - x_2 y_1))\,. \tag{54}$$

For each value of t, we associate the contact form

$$\kappa_t = t^{-1/2}\,dz + t^{1/2}\,(x\,dy - y\,dx)\,. \tag{55}$$

The relative power of t between the two terms in (55) ensures that the contact form is left-invariant under the Heisenberg multiplication with $\hbar = t$ in (49). The overall power of t ensures that the contact condition $\kappa_t \wedge d\kappa_t = 2\,dx \wedge dy \wedge dz \neq 0$ is satisfied trivially for all values $t > 0$.

Finally, suppose that $C \subset \mathbb{R}^3$ is a Legendrian knot with respect to the standard contact form $\kappa_{t=1}$. Just as we consider the family of isotopic contact forms in (55), we would like to consider a family of isotopic knots $C_t \subset \mathbb{R}^3$, each of which is Legendrian with respect to κ_t for the given value of t. Such a family of knots is determined if we simply require the Lagrangian projection of C_t to coincide with that of C,

$$\Pi(C_t) = \Pi(C)\,, \qquad t > 0\,. \tag{56}$$

Either by integrating the contact condition as in (25) or just by scaling, the embedding map $X_t\colon S^1 \to \mathbb{R}^3$ for C_t is then related to the original embedding X for C via

$$X_t(\theta) \equiv (x_t(\theta),\, y_t(\theta),\, z_t(\theta)) = (x(\theta),\, y(\theta),\, t\,z(\theta))\,. \tag{57}$$

In particular, the abelian limit $t \to 0$ of the Heisenberg structure corresponds to a limit in which C_t flattens to a curve in the xy-plane, and the Lagrangian projection $\Pi(C)$ is realized geometrically.

Given the family of curves C_t, we consider the *three*-parameter self-linking integral

$$\mathsf{slk}_\kappa(C_t) = \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X_t \times X_t)^* \widehat{\Gamma}_{t\hbar}^* \chi_\Lambda\,, \qquad \hbar \in [0,1]\,, \qquad t, \Lambda > 0\,. \tag{58}$$

Precisely for $\hbar = 1$, the self-linking integrand is invariant under the Heisenberg symmetry with multiplication map μ_t.

Remark II.2 (Scaling Identity) *For all $t, \Lambda > 0$,*

$$(X_t \times X_t)^* \widehat{\Gamma}_{t\hbar}^* \chi_\Lambda = (X \times X)^* \widehat{\Gamma}_{\hbar}^* \chi_{\Lambda/t}\,. \tag{59}$$

By the Scaling Identity, the behavior of the self-linking integrand in limit $\Lambda \to \infty$ with fixed t is the same as the behavior in the limit $t \to 0$ with fixed Λ. Because C_t flattens to a plane curve in the latter limit, Λ plays a similar role to the parameter of the same name in (16).

Proof of Scaling Identity

The t-dependence of the pullback is $(X_t \times X_t)^* \widehat{\Gamma}_{t\hbar}^* \chi_\Lambda$ is actually very simple. In terms of the finite differences

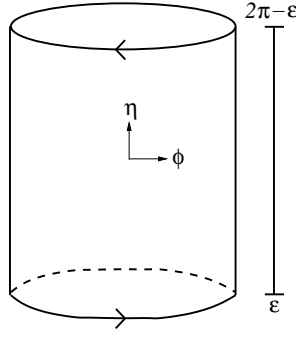$$\Delta x = x_2 - x_1\,, \qquad \Delta y = y_2 - y_1\,, \qquad \widehat{\Delta} z = z_2 - z_1 + \hbar\,(x_1 y_2 - x_2 y_1)\,, \tag{60}$$

FIG. 10. Orientations on the cylinder $T^2 - \Delta(\varepsilon)$.

the formula for the heat form χ_Λ in (42) implies

$$(X_t \times X_t)^* \widehat{\Gamma}_{t\hbar}^* \chi_\Lambda =$$
$$\frac{\Lambda}{2\pi t \widehat{\Delta} z} e^{-\Lambda(\Delta x^2 + \Delta y^2)/2t\widehat{\Delta} z} \left[d\Delta x \wedge d\Delta y + \frac{1}{2} (\Delta x \, d\Delta y - \Delta y \, d\Delta x) \wedge \frac{d\widehat{\Delta} z}{\widehat{\Delta} z} \right], \tag{61}$$

provided $\widehat{\Delta} z > 0$. Otherwise, the pullback of χ_Λ vanishes. Evidently, t just multiplies $\widehat{\Delta} z$ in (61), and all dependence on t can be absorbed by rescaling the Gaussian parameter Λ. $\square$

Corollary II.3 *The value of $\mathsf{slk}_\kappa(C_t)$ in (58) is independent of both t and Λ.*

Proof The corollary follows immediately from Lemma II.1 and the Scaling Identity in (59). $\square$

A. Analysis near the diagonal

The non-trivial aspect of our work concerns the local analysis of the self-linking integrand in the vicinity of the diagonal $\Delta \subset T^2$. In practice, this analysis amounts to the Taylor expansion of expressions such as occur in (61). Rather than scatter such expansions willy-nilly throughout the paper, we collect here the basic ingredients to be used again and again.

Let (θ_1, θ_2) be angular coordinates on T^2. To parametrize the tubular neighborhood $\Delta(\varepsilon) \subset T^2$ of the diagonal, we set

$$\theta_1 = \phi, \qquad \theta_2 = \phi + \eta. \tag{62}$$

Here ϕ is an angular coordinate along the diagonal, and $\Delta(\varepsilon)$ is the subset where $|\eta| < \varepsilon$. Our local expansions near the diagonal will then be Taylor expansions in η, appropriate for the regime $\varepsilon \ll 1$. We must be careful about orientations. In terms of the coordinates (ϕ, η), the orientation form on T^2 is given by $d\theta_1 \wedge d\theta_2 = d\phi \wedge d\eta$. Topologically, the configuration space $T^2 - \Delta(\varepsilon)$ is a cylinder with oriented boundary circles

$$S_\pm^1: \quad \eta = \pm\varepsilon \mod 2\pi. \tag{63}$$

As shown in Figure 10, the boundary orientation of S_+^1 is positive with respect to the direction of increasing ϕ and the orientation of S_-^1 is negative, so

$$\partial(T^2 - \Delta(\varepsilon)) = S_+^1 - S_-^1. \tag{64}$$

We now expand the various terms appearing in the self-linking integrand $(X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda$, given by the expression in (61) for $t = 1$. We start with

$$\Delta x = x(\theta_2) - x(\theta_1) = x(\phi + \eta) - x(\phi) = \eta \, \dot{x}(\phi) + \mathcal{O}(\varepsilon^2), \tag{65}$$

and similarly for Δy. Hence

$$\Delta x^2 + \Delta y^2 = \eta^2 \left(\dot{x}^2 + \dot{y}^2\right) + \mathcal{O}(\varepsilon^3) = \eta^2 \, ||\dot{\gamma}||^2 + \mathcal{O}(\varepsilon^3), \tag{66}$$

where $\gamma(\phi) = (x(\phi), y(\phi))$ is the Lagrangian immersion $\gamma = \Pi \circ X$ as before. For the related two-form, we can write

$$
\begin{aligned}
d\Delta x \wedge d\Delta y &= \frac{1}{2} d\Delta\gamma \times d\Delta\gamma, && \Delta\gamma \equiv (\Delta x, \Delta y), \\
&= \frac{1}{2} d(\eta \, \dot{\gamma}) \times d(\eta \, \dot{\gamma}) + \mathcal{O}(\varepsilon^3), \\
&= \frac{1}{2} \left(\dot{\gamma} \, d\eta + \eta \, \ddot{\gamma} \, d\phi\right) \times \left(\dot{\gamma} \, d\eta + \eta \, \ddot{\gamma} \, d\phi\right) + \mathcal{O}(\varepsilon^3).
\end{aligned}
\tag{67}
$$

After collecting terms in the product,

$$d\Delta x \wedge d\Delta y = -\eta \, \left(\dot{\gamma} \times \ddot{\gamma}\right) d\phi \wedge d\eta + \mathcal{O}(\varepsilon^2) \, d\phi \wedge d\eta. \tag{68}$$

Recall the definition

$$\widehat{\Delta} z = z_2 - z_1 + \hbar \left(x_1 y_2 - x_2 y_1\right). \tag{69}$$

The Legendrian condition on C is absolutely critical for our results, because it implies that the local behavior of $\widehat{\Delta} z$ near the diagonal is controlled by the geometry of the Lagrangian immersion γ. Moreover, the quadratic terms in $\widehat{\Delta} z$ are required by the Heisenberg symmetry when $\hbar = 1$.

The expansion of the abelian difference $\Delta z = z_2 - z_1$ is fixed by the Legendrian condition

$$\dot{z} = y \, \dot{x} - x \, \dot{y} = -\gamma \times \dot{\gamma}. \tag{70}$$

Thus

$$
\begin{aligned}
z_2 - z_1 &= z(\phi + \eta) - z(\phi) = \eta \, \dot{z}(\phi) + \frac{1}{2} \eta^2 \, \ddot{z}(\phi) + \frac{1}{6} \eta^3 \, \dddot{z}(\phi) + \mathcal{O}(\varepsilon^4), \\
&= -\eta \left(\gamma \times \dot{\gamma}\right) - \frac{1}{2} \eta^2 \left(\gamma \times \ddot{\gamma}\right) - \frac{1}{6} \eta^3 \left(\dot{\gamma} \times \ddot{\gamma} + \gamma \times \dddot{\gamma}\right) + \mathcal{O}(\varepsilon^4).
\end{aligned}
\tag{71}
$$

In passing to the second line of (71), we repeatedly differentiate the Legendrian condition on $\dot{z}$ in (70).

The attentive reader may wonder why we have expanded $\Delta z = z_2 - z_1$ all the way to cubic order in η. The question answers itself once we expand the remaining quadratic terms in the Heisenberg difference $\widehat{\Delta} z$,

$$
\begin{aligned}
x_1 y_2 - x_2 y_1 &= \gamma(\phi) \times \gamma(\phi + \eta), \\
&= \eta \left(\gamma \times \dot{\gamma}\right) + \frac{1}{2} \eta^2 \left(\gamma \times \ddot{\gamma}\right) + \frac{1}{6} \eta^3 \left(\gamma \times \dddot{\gamma}\right) + \mathcal{O}(\varepsilon^4).
\end{aligned}
\tag{72}
$$

So long as $\hbar \neq 1$, the expansion of $\widehat{\Delta} z$ begins at linear order in η,

$$\widehat{\Delta} z \overset{\hbar \neq 1}{=} -\eta \left(1 - \hbar\right) \left(\gamma \times \dot{\gamma}\right) + \mathcal{O}(\varepsilon^2), \tag{73}$$

as one naively expects. But precisely at $\hbar = 1$, cancellations occur in the sum of (71) and (72), and the leading term in the expansion of $\widehat{\Delta} z$ near the diagonal begins at cubic order,

$$\widehat{\Delta} z \overset{\hbar = 1}{=} -\frac{1}{6} \eta^3 \left(\dot{\gamma} \times \ddot{\gamma}\right) + \mathcal{O}(\varepsilon^4). \tag{74}$$

The cancellation in (74) is forced by the Heisenberg symmetry at $\hbar = 1$. Clearly, the quantity $\gamma \times \dot{\gamma}$ in (73) is not invariant under translations $\gamma \mapsto \gamma + \gamma_0$ for constant $\gamma_0 \in \mathbb{R}^2$. Upon projection to the xy-plane, such translations are generated by the Heisenberg action, so $\gamma \times \dot{\gamma}$ is forbidden to appear at $\hbar = 1$.

Combining the expansions in (66), (73), and (74), we see that the argument of the heat kernel is given in the

neighborhood $\Delta(\varepsilon)$ by

$$\frac{\Delta x^2 + \Delta y^2}{2\widehat{\Delta}z} = \begin{cases} -\dfrac{||\dot{\gamma}||^2\, \eta}{2\,(1-\hbar)\,(\gamma \times \dot{\gamma})} + \mathcal{O}(\varepsilon^2)\,, & [\hbar \neq 1] \\[4mm] -\dfrac{3\,||\dot{\gamma}||^2}{(\dot{\gamma} \times \ddot{\gamma})\,\eta} + \mathcal{O}(1)\,. & [\hbar = 1] \end{cases} \tag{75}$$

For generic $\hbar \neq 1$, the argument of the heat kernel in (75) vanishes linearly near the diagonal. However, at the symmetric point $\hbar = 1$, the argument instead diverges as $\eta \to 0$. In Section III, this difference will ultimately lead to the discontinuity in the value of $\mathsf{slk}_\kappa(C)$ at $\hbar = 1$.

Remark II.4 (Local Positivity) *The pullback of the heat form χ_Λ vanishes identically unless $\widehat{\Delta}z > 0$. On the neighborhood $\Delta(\varepsilon)$, positivity of $\widehat{\Delta}z$ becomes equivalent via (73) and (74) to the local sign condition*

$$\widehat{\Delta}z\big|_{\Delta(\varepsilon)} > 0 \quad \Longleftrightarrow \quad \begin{cases} \eta\,(1-\hbar)\,(\gamma \times \dot{\gamma}) < 0\,, & [\hbar \neq 1]\,, \\[2mm] \eta\,(\dot{\gamma} \times \ddot{\gamma}) < 0\,. & [\hbar = 1]\,. \end{cases} \tag{76}$$

Again, the nature of the positivity condition depends upon whether or not $\hbar = 1$. For generic values of $\hbar$, the sign of η is determined by the sign of $\gamma \times \dot{\gamma}$ and hence the sign of the derivative $\dot{z}$. For $\hbar = 1$, the sign of η is instead fixed by the sign of $\dot{\gamma} \times \ddot{\gamma}$, proportional to the plane curvature of $\Pi(C)$.

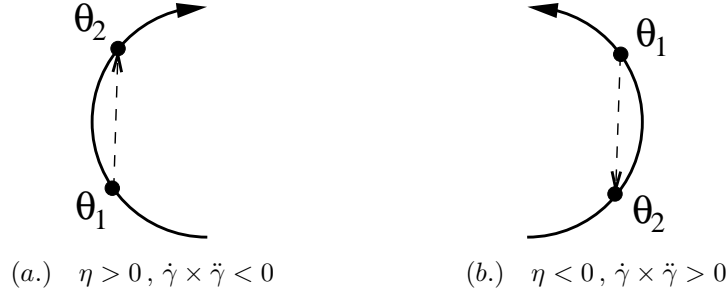(a.) $\eta > 0\,,\ \dot{\gamma} \times \ddot{\gamma} < 0$ (b.) $\eta < 0\,,\ \dot{\gamma} \times \ddot{\gamma} > 0$

FIG. 11. Local positivity condition $\widehat{\Delta}z > 0$ for $\hbar = 1$.

Since the local positivity condition depends upon the sign of η, the self-linking integrand always vanishes in the neighborhood of one or the other of the boundary circles $S^1_\pm$ in Figure 10, and the geometry of C at any given point determines on which boundary circle the integrand vanishes. See Figure 11 for a geometric illustration of the local positivity condition $\widehat{\Delta}z > 0$ in the symmetric case $\hbar = 1$. For clarity, we exaggerate the small separation between the points $\gamma(\theta_1)$ and $\gamma(\theta_2)$ in the figure. In both cases, the positivity condition is sensitive to the orientation of C, as a reversal of orientation flips the signs of $\gamma \times \dot{\gamma}$ and $\dot{\gamma} \times \ddot{\gamma}$ in (76).

Let us complete the expansion for small η of the self-linking integrand. The second bracketed term in (61) involves the angular one-form

$$\begin{aligned} \Delta x\, d\Delta y - \Delta y\, d\Delta x &= \Delta\gamma \times d\Delta\gamma\,, \\ &= (\eta\,\dot{\gamma}) \times d\,(\eta\,\dot{\gamma}) + \mathcal{O}(\varepsilon^3)\,, \\ &= \eta^2\,(\dot{\gamma} \times \ddot{\gamma})\,d\phi + \mathcal{O}(\varepsilon^3)\,. \end{aligned} \tag{77}$$

From the expansions of $\widehat{\Delta}z$ in (73) and (74),

$$(\Delta x\, d\Delta y - \Delta y\, d\Delta x) \wedge d\widehat{\Delta}z = \begin{cases} -(1-\hbar)\,(\gamma \times \dot{\gamma})\,(\dot{\gamma} \times \ddot{\gamma})\,\eta^2\, d\phi \wedge d\eta\,, & [\hbar \neq 1] \\[3mm] -\dfrac{1}{2}\,(\dot{\gamma} \times \ddot{\gamma})^2\,\eta^4\, d\phi \wedge d\eta\,, & [\hbar = 1] \end{cases} \tag{78}$$

to leading order, so

$$(\Delta x \, d\Delta y - \Delta y \, d\Delta x) \wedge \frac{d\widehat{\Delta}z}{\widehat{\Delta}z} = \begin{cases} \eta \, (\dot{\gamma} \times \ddot{\gamma}) \, d\phi \wedge d\eta \, + \, \mathcal{O}(\varepsilon^2) \, d\phi \wedge d\eta \,, & [\hbar \neq 1] \\ 3 \, \eta \, (\dot{\gamma} \times \ddot{\gamma}) \, d\phi \wedge d\eta \, + \, \mathcal{O}(\varepsilon^2) \, d\phi \wedge d\eta \,. & [\hbar = 1] \end{cases} \tag{79}$$

After a bit of algebra, one finds that the pullback (61) of χ_Λ behaves near the diagonal $\Delta \subset T^2$ for $\hbar \neq 1$ as

$$(X \times X)^* \widehat{\Gamma}_\hbar^* \, \chi_\Lambda \Big|_{\Delta(\varepsilon)} \overset{\hbar \neq 1}{=}$$
$$- \operatorname{sgn}(\eta) \, \frac{\Lambda \, (\dot{\gamma} \times \ddot{\gamma})}{4\pi \, |1 - \hbar| \, |\gamma \times \dot{\gamma}|} \, \exp\left[- \frac{\Lambda \, ||\dot{\gamma}||^2 \, |\eta|}{2 \, |1 - \hbar| \, |\gamma \times \dot{\gamma}|} \right] d\phi \wedge d\eta \, + \, \cdots \,, \tag{80}$$

assuming the sign condition $\eta \, (1 - \hbar) \, (\gamma \times \dot{\gamma}) < 0$ in (76) holds. Otherwise, the pullback is equal to zero. The sign condition ensures that the exponential in (80) is always decaying, and it leads to a non-analytic dependence on the sign of η.[2] The ellipses in (80) indicate subleading terms which vanish as $\eta \to 0$.

By contrast, at the symmetric value $\hbar = 1$,

$$(X \times X)^* \widehat{\Gamma}_\hbar^* \, \chi_\Lambda \Big|_{\Delta(\varepsilon)} \overset{\hbar = 1}{=} - \frac{3 \, \Lambda}{2\pi\eta^2} \, \exp\left[- \frac{3 \, \Lambda \, ||\dot{\gamma}||^2}{|\dot{\gamma} \times \ddot{\gamma}|} \, \frac{1}{|\eta|} \right] d\phi \wedge d\eta \, + \, \cdots \,,$$
$$= \frac{3 \, \Lambda}{2\pi} \, \exp\left[- \frac{3 \, \Lambda \, ||\dot{\gamma}||^2}{|\dot{\gamma} \times \ddot{\gamma}|} \, |\nu| \right] d\phi \wedge d\nu \, + \, \cdots \,, \tag{81}$$

under the sign condition $\eta \, (\dot{\gamma} \times \ddot{\gamma}) < 0$ in (76). In passing to the second line, we make the substitution $\nu = 1/\eta$ for clarity. In this case, the pullback of the heat form χ_Λ vanishes exponentially as $|\nu| \to \infty$, or equivalently $|\eta| \to 0$.

B.   Proof of the fundamental lemma

The proof of our Fundamental Lemma II.1 is now an exercise in calculus.

As a brief formality, we first establish that the singularity in the pullback of the heat form χ_Λ is integrable, so that the defining limit $\varepsilon \to 0$ in (53) does exist. By assumption, $||\dot{\gamma}||^2 > 0$ is everywhere non-vanishing, and the functions $|\gamma \times \dot{\gamma}|$ and $|\dot{\gamma} \times \ddot{\gamma}|$ are bounded from above on C. Integrability in the region of small $|\eta| < \varepsilon$ follows immediately from the local expressions in (80) and (81), both of which remain finite as $\eta \to 0$.

Otherwise, we must check that the value of $\mathsf{slk}_\kappa(C)$ does not depend upon the parameter $\Lambda > 0$. This argument will be equally straightforward but the result is significant; the analogue for the naive Gauss self-linking integral $\mathsf{slk}_0(C)$ is simply false. Our strategy will be to show that the derivative of $\mathsf{slk}_\kappa(C)$ with respect to Λ vanishes for all values of $\Lambda > 0$. The details differ somewhat depending upon whether $\hbar \neq 1$ or $\hbar = 1$, but the main idea is the same in both cases.

We compute

$$\frac{d \, \mathsf{slk}_\kappa(C)}{d\Lambda} = \lim_{\varepsilon \to 0} \int_{T^2 - \Delta(\varepsilon)} (X \times X)^* \widehat{\Gamma}_\hbar^* \left(\frac{\partial \chi_\Lambda}{\partial \Lambda} \right) \,,$$
$$= \lim_{\varepsilon \to 0} \int_{[T^2 - \Delta(\varepsilon)]_+} (X \times X)^* \widehat{\Gamma}_\hbar^* \, \varrho_+^* \left(\frac{\partial \omega_\Lambda}{\partial \Lambda} \right) \,. \tag{82}$$

Here $[T^2 - \Delta(\varepsilon)]_+$ indicates the closed subset of the cylinder where $\widehat{\Delta}z \geq 0$ is positive,

$$[T^2 - \Delta(\varepsilon)]_+ = \left\{ (\theta_1, \theta_2) \, \big| \, \widehat{\Delta}z(\theta_1, \theta_2) \geq 0 \right\} \,, \tag{83}$$

on which the pullback of the heat form χ_Λ is non-vanishing. Of course, the positive subset in (83) depends upon the Legendrian embedding X. We omit this dependence from the notation as X is fixed throughout.

---

[2] As usual, $\operatorname{sgn}(\eta) = +1$ for $\eta > 0$, and $\operatorname{sgn}(\eta) = -1$ for $\eta < 0$.

By the calculation in (39),

$$\frac{\partial \omega_\Lambda}{\partial \Lambda} = d\alpha_\Lambda \,, \tag{84}$$

for the transgression one-form

$$\alpha_\Lambda = \frac{1}{4\pi}\, \mathrm{e}^{-\Lambda(x^2+y^2)/2}\,(x\,dy - y\,dx)\ \in\ \Omega^1\big(\mathbb{R}^2\big)\,. \tag{85}$$

We apply the commutativity of the de Rham operator with pullback, followed by Stokes' Theorem, to reduce the bulk integral in the second line of (82) to a boundary integral,

$$\int_{[T^2-\Delta(\varepsilon)]_+} (X \times X)^* \widehat{\Gamma}_\hbar^* \,\varrho_+^*\left(\frac{\partial \omega_\Lambda}{\partial \Lambda}\right) = \int_{\partial[T^2-\Delta(\varepsilon)]_+} (X \times X)^* \widehat{\Gamma}_\hbar^* \,\varrho_+^* \alpha_\Lambda\,. \tag{86}$$

Explicitly, the boundary integrand in (86) is given, where non-zero, by

$$(X \times X)^* \widehat{\Gamma}_\hbar^* \,\varrho_+^* \alpha_\Lambda = \frac{1}{4\pi \widehat{\Delta} z}\, \mathrm{e}^{-\Lambda\left(\Delta x^2 + \Delta y^2\right)/2\widehat{\Delta} z}\big(\Delta x\, d\Delta y - \Delta y\, d\Delta x\big)\,. \tag{87}$$

This expression vanishes smoothly whenever $\widehat{\Delta} z \to 0$ from above with $\Delta x^2 + \Delta y^2 \neq 0$.

The boundary of the positive subset $\partial[T^2 - \Delta(\varepsilon)]_+$ in (86) includes those curves where $\widehat{\Delta} z = 0$ as well as the intersection of $[T^2 - \Delta(\varepsilon)]_+$ with the boundary circles $S^1_\pm$ themselves. Recall that points on $S^1_\pm$ satisfy $\eta = \pm\varepsilon$, respectively. By the preceding, only the boundary integral over the intersection $S^1_\pm \cap [T^2 - \Delta(\varepsilon)]_+$ is relevant, because the boundary integrand in (87) vanishes on the locus where $\widehat{\Delta} z = 0$.

Altogether, in terms of the boundary integral on the right in (86),

$$\frac{d\,\mathsf{slk}_\kappa(C)}{d\Lambda} = \lim_{\varepsilon\to 0}\left[\int_{S^1_+ \cap [T^2-\Delta(\varepsilon)]_+} (X \times X)^* \widehat{\Gamma}_\hbar^* \,\varrho_+^* \alpha_\Lambda - \int_{S^1_- \cap [T^2-\Delta(\varepsilon)]_+} (X \times X)^* \widehat{\Gamma}_\hbar^* \,\varrho_+^* \alpha_\Lambda\right]. \tag{88}$$

The minus sign for the boundary integral over S^1_- accounts for the relative orientation in Figure 10.

Despite the minus sign, no possibility exists for a trivial cancellation between the two boundary integrals in (88) for any fixed $\varepsilon > 0$. According to the local positivity condition in (76) for respectively $\hbar \neq 1$ or $\hbar = 1$,

$$\begin{cases} (1-\hbar)(\gamma \times \dot{\gamma}) \text{ or } \dot{\gamma} \times \ddot{\gamma} \leq 0 & \text{on}\quad S^1_+ \cap [T^2 - \Delta(\varepsilon)]_+\,,\\[4pt] (1-\hbar)(\gamma \times \dot{\gamma}) \text{ or } \dot{\gamma} \times \ddot{\gamma} \geq 0 & \text{on}\quad S^1_- \cap [T^2 - \Delta(\varepsilon)]_+\,. \end{cases} \tag{89}$$

The domains of integration over the two boundary circles $S^1_\pm$ in (88) are therefore disjoint away from the degeneracy locus where $(1 - \hbar)(\gamma \times \dot{\gamma})$ or $\dot{\gamma} \times \ddot{\gamma} = 0$, so no cancellation can occur. Generically, the degeneracy locus consists of a finite set of isolated inflection points on the curve.

Let us examine the behavior of the boundary integrand (87) via the expansion near the diagonal from Section II A.

Symmetric case $\hbar = 1$

We initially consider the Heisenberg-symmetric case $\hbar = 1$. Similar to the bulk integrand in (81), the boundary integrand behaves to leading-order at $\eta = \pm\varepsilon$ as

$$(X \times X)^* \widehat{\Gamma}_\hbar^* \,\varrho_+^* \alpha_\Lambda\Big|_{S^1_\pm} \overset{\hbar=1}{=} \mp\frac{3}{2\pi\varepsilon}\, \exp\left[-\frac{3\,\Lambda\,||\dot{\gamma}||^2}{\varepsilon\,|\dot{\gamma} \times \ddot{\gamma}|}\right] d\phi\ +\ \cdots\,, \tag{90}$$

where the omitted terms vanish more rapidly as $\varepsilon \to 0$. By a conspiracy of signs, the difference on the right of (88) can be rewritten as the single integral

$$\frac{d\,\mathsf{slk}_\kappa(C)}{d\Lambda} \overset{\hbar=1}{=} \lim_{\varepsilon\to 0}\left[-\frac{3}{2\pi\varepsilon}\oint_{S^1} d\phi\, \exp\left(-\frac{3\,\Lambda\,||\dot{\gamma}||^2}{\varepsilon\,|\dot{\gamma} \times \ddot{\gamma}|}\right)\right] = 0\,. \tag{91}$$

To deduce the vanishing of the limit $\varepsilon \to 0$, we note that the ratio $||\dot{\gamma}||^2/|\dot{\gamma} \times \ddot{\gamma}| \geq m$ is everywhere bounded from

below on S^1 by a positive constant $m > 0$, so the integrand in (91) is dominated by the exponentially-small constant

$$\exp\left(-\frac{3\,\Lambda\,||\dot{\gamma}||^2}{\varepsilon\,|\dot{\gamma} \times \ddot{\gamma}|}\right) \leq \exp\left(-\frac{3\,\Lambda\,m}{\varepsilon}\right). \tag{92}$$

Since $\Lambda > 0$ has been arbitrary throughout, $\mathsf{slk}_\kappa(C)$ is independent of Λ for $\hbar = 1$. $\qquad\square$

Generic case $\hbar \neq 1$

The analysis for generic $\hbar \neq 1$ is slightly more delicate. Here

$$(X \times X)^*\widehat{\Gamma}_\hbar^*\,\varrho_+^*\alpha_\Lambda\Big|_{S_\pm^1} \overset{\hbar \neq 1}{=} \frac{\varepsilon\,(\dot{\gamma} \times \ddot{\gamma})}{4\pi\,|1 - \hbar|\,|\gamma \times \dot{\gamma}|}\,\exp\left[-\frac{\Lambda\,||\dot{\gamma}||^2\,\varepsilon}{2\,|1 - \hbar|\,|\gamma \times \dot{\gamma}|}\right]d\phi + \cdots, \tag{93}$$

so the derivative becomes

$$\frac{d\,\mathsf{slk}_\kappa(C)}{d\Lambda} \overset{\hbar \neq 1}{=} \lim_{\varepsilon \to 0}\left[\mathrm{I}_+(\varepsilon) - \mathrm{I}_-(\varepsilon)\right], \tag{94}$$

with

$$\mathrm{I}_\pm(\varepsilon) = \frac{\varepsilon}{4\pi|1 - \hbar|}\int_{S_\pm^1 \cap [T^2 - \Delta(\varepsilon)]_+}d\phi\,\frac{\dot{\gamma} \times \ddot{\gamma}}{|\gamma \times \dot{\gamma}|}\,\exp\left(-\frac{\Lambda\,||\dot{\gamma}||^2\,\varepsilon}{2\,|1 - \hbar|\,|\gamma \times \dot{\gamma}|}\right). \tag{95}$$

The functions $\mathrm{I}_\pm(\varepsilon)$ differ only in the domain of integration over S^1, and our goal will be to show individually

$$\lim_{\varepsilon \to 0}\mathrm{I}_\pm(\varepsilon) = 0. \tag{96}$$

Were the function $\gamma \times \dot{\gamma}$ to be everywhere non-zero on S^1, the conclusion in (96) would be immediate, as we would know the integral in (95) to be bounded in magnitude even for $\varepsilon = 0$. The explicit prefactor of ε then ensures the vanishing of $\mathrm{I}_\pm(\varepsilon)$ in the limit $\varepsilon \to 0$. However, $\dot{z} = -\gamma \times \dot{\gamma}$ always vanishes for at least two points (the highest and the lowest) on the knot $C \subset \mathbb{R}^3$, and we must worry about what happens to the integral in (95) near a zero of $\gamma \times \dot{\gamma}$, when ε is very small.

Let us make an elementary simplification. Since $|\dot{\gamma} \times \ddot{\gamma}|$ is bounded from above and $||\dot{\gamma}||^2 > 0$ is bounded from below on S^1,

$$|\mathrm{I}_\pm(\varepsilon)| \leq \mathrm{J}_\pm(\varepsilon) = \int_{S_\pm^1 \cap [T^2 - \Delta(\varepsilon)]_+}d\phi\,\frac{A\,\varepsilon}{|\gamma \times \dot{\gamma}|}\,\exp\left(-\frac{B\,\varepsilon}{|\gamma \times \dot{\gamma}|}\right), \qquad A, B > 0, \tag{97}$$

for some positive constants A and B, into which we also absorb the dependence on Λ and $\hbar$ and the various other numerical factors in (95). To deduce the limit (96) for $\mathrm{I}_\pm(\varepsilon)$, we show that $\mathrm{J}_\pm(\varepsilon)$ vanishes in the same limit.

By assumption, the height function $z(\phi)$ is Morse, with isolated non-degenerate critical points. Equivalently, the function $(\gamma \times \dot{\gamma})(\phi)$ vanishes non-degenerately at an isolated set of points on S^1. By the criteria in (89), these points are precisely the endpoints of the intervals which compose each integration domain $S_\pm^1 \cap [T^2 - \Delta(\varepsilon)]_+$. Locally near such an endpoint $\phi = \phi_0$,

$$(\gamma \times \dot{\gamma})(\phi) = c_0\,(\phi - \phi_0) + \mathcal{O}(|\phi - \phi_0|^2), \qquad c_0 \neq 0. \tag{98}$$

When we examine $\mathrm{J}_\pm(\varepsilon)$ in the limit $\varepsilon \to 0$, only the contribution to the integral from a (one-sided) neighborhood of ϕ_0 can be non-zero, so we simplify further by replacing $\mathrm{J}_\pm(\varepsilon)$ by the model

$$\mathrm{K}(\varepsilon) = \int_{\phi_0}^{\phi_1}d\phi\,\frac{\varepsilon}{f(\phi)}\,\exp\left(-\frac{\varepsilon}{f(\phi)}\right). \tag{99}$$

Here ϕ_1 is an arbitrary upper cutoff, and $f(\phi)$ is now any continuous function defined on the interval $[\phi_0, \phi_1]$ such that

$$f(\phi) > 0 \text{ for } \phi > \phi_0, \qquad f(\phi_0) = 0, \quad \text{and} \quad \lim_{\phi \to \phi_0}\left[\frac{(\phi - \phi_0)}{f(\phi)}\right] > 0 \text{ exists}. \tag{100}$$

For all $\varepsilon > 0$, the integral defining $\mathrm{K}(\varepsilon)$ exists, since the integrand vanishes at the endpoint $\phi = \phi_0$. For convenience,
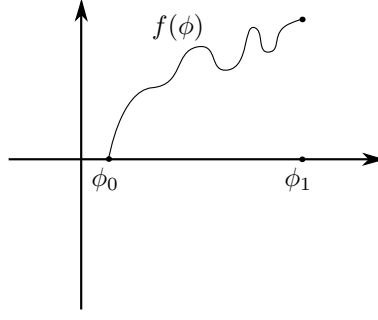
FIG. 12. A function $f(\phi)$ satisfying the assumptions in Lemma II.5.

we take $\phi_0 = 0$ and $\phi_1 = 1$ by a suitable choice of parameter. The proof of the Fundamental Lemma II.1 for generic $\hbar \neq 1$ reduces to the following claim.

Lemma II.5 *Let* $\mathrm{K}(\varepsilon)$ *and* $f(\phi)$ *be defined as in* (99) *and* (100). *Then*

$$\lim_{\varepsilon \to 0} \mathrm{K}(\varepsilon) = \lim_{\varepsilon \to 0} \left[\int_0^1 d\phi \, \frac{\varepsilon}{f(\phi)} \, \exp\left(-\frac{\varepsilon}{f(\phi)} \right) \right] = 0 \,. \tag{101}$$

Proof

We consider a succession of three cases.

(i) We start with the basic example $f(\phi) = \phi$, so that

$$\mathrm{K}(\varepsilon) = \varepsilon \int_0^1 \frac{d\phi}{\phi} \, \exp\left(-\frac{\varepsilon}{\phi} \right) \,. \tag{102}$$

After the substitution $x = \varepsilon/\phi$,

$$\mathrm{K}(\varepsilon) = \varepsilon \int_\varepsilon^\infty \frac{dx}{x} \, \mathrm{e}^{-x} \leq \varepsilon \int_\varepsilon^1 \frac{dx}{x} + \varepsilon \int_1^\infty dx \, \mathrm{e}^{-x} = \varepsilon \, |\ln \varepsilon| + \varepsilon \, \mathrm{e}^{-1} \,, \tag{103}$$

from which the limit follows.

(ii) Next, let $g(\phi)$ and $h(\phi)$ be continuous functions on the interval $[0, 1]$ obeying bounds

$$0 < m \leq g(\phi) \,, \qquad |h(\phi)| \leq M \,, \tag{104}$$

for some constants m and M. Set

$$\mathrm{K}(\varepsilon) = \varepsilon \int_0^1 \frac{d\phi}{\phi} \, h(\phi) \, \exp\left[-\varepsilon \, \frac{g(\phi)}{\phi} \right] \,. \tag{105}$$

Then

$$\mathrm{K}(\varepsilon) \leq M \, \varepsilon \int_0^1 \frac{d\phi}{\phi} \, \exp\left(-\varepsilon \, \frac{m}{\phi} \right) = M \, \varepsilon \int_0^{1/m} \frac{d\phi}{\phi} \, \exp\left(-\frac{\varepsilon}{\phi} \right) \,. \tag{106}$$

The function $\mathrm{K}(\varepsilon)$ vanishes as $\varepsilon \to 0$ by (i).

(iii) In the general case of interest,

$$\mathrm{K}(\varepsilon) = \int_0^1 d\phi \, \frac{\varepsilon}{f(\phi)} \, \exp\left[-\frac{\varepsilon}{f(\phi)} \right] = \varepsilon \int_0^1 \frac{d\phi}{\phi} \left(\frac{\phi}{f(\phi)} \right) \exp\left[-\frac{\varepsilon}{\phi} \left(\frac{\phi}{f(\phi)} \right) \right] \,. \tag{107}$$

Because $f(\phi) > 0$ for $\phi > 0$ by assumption, the function $g(\phi) = h(\phi) = \phi/f(\phi)$ is continuous and positive for all $\phi > 0$. Since the limit $\lim_{\phi \to 0} [\phi/f(\phi)] > 0$ is also assumed to exist and be non-zero, $g(\phi) > 0$ is continuous and non-vanishing throughout the unit interval. Hence $0 < m \leq g(\phi) \leq M$ for some constants m and M, and the general case follows from (ii). $\square$

III. PLANAR LIMIT

According to the Fundamental Lemma II.1, the value of the self-linking integral $\mathsf{slk}_\kappa(C)$ does not depend upon the positive parameter $\Lambda > 0$ which sets the width of the Gaussian in the heat form χ_Λ. To evaluate $\mathsf{slk}_\kappa(C)$, and in the process to show that $\mathsf{slk}_\kappa(C)$ is invariant under Legendrian isotopy, we now analyze the self-linking integral (51) in the limit $\Lambda \to \infty$. The Legendrian knot $C \subset \mathbb{R}^3$ and its regular parametrization $X : S^1 \to \mathbb{R}^3$ remain fixed throughout.

The limit $\Lambda \to \infty$ has several interpretations.

In terms of the heat kernel, this limit is the short-time limit, in which the Gaussian generator ω_Λ for the compactly-supported cohomology $H^2_c(\mathbb{R}^2; \mathbb{Z})$ concentrates to a form with delta-function support at the origin. More geometrically, by the Scaling Identity in (59), the limit $\Lambda \to \infty$ is equivalent to the limit $t \to 0$ in which the contact planes represented by κ_t in (55) and the Legendrian knot C_t in (57) flatten to the xy-plane. Simultaneously, the Planck constant $\hbar$ in the Heisenberg multiplication scales to zero, and the abelian structure of $\mathbb{R}^3$ is restored. For this reason, we refer to the limit $\Lambda \to \infty$ as the planar limit.

In the planar limit, the Legendrian self-linking integral simplifies immensely, as can be understood from the formula for the integrand

$$
\begin{aligned}
(X \times X)^* \widehat{\Gamma}^*_\hbar \chi_\Lambda \;=\;& \frac{\Lambda}{2\pi\widehat{\Delta}z}\, \mathrm{e}^{-\Lambda(\Delta x^2 + \Delta y^2)/2\widehat{\Delta}z} \;\times \\
& \times \left[d\Delta x \wedge d\Delta y \,+\, \frac{1}{2}\left(\Delta x\, d\Delta y - \Delta y\, d\Delta x\right) \wedge \frac{d\widehat{\Delta}z}{\widehat{\Delta}z} \right], \qquad \widehat{\Delta}z > 0\,.
\end{aligned}
\tag{108}
$$

Intuitively, the behavior of the integrand is controlled by the exponential factor in the first line of (108). When Λ is sufficiently large, the integrand is negligible away from the locus where

$$
\left[\Delta x^2 + \Delta y^2\right]\Big|_{(\theta_1, \theta_2)} \;\ll\; \frac{1}{\Lambda}\,, \qquad (\theta_1, \theta_2) \in T^2 - \Delta(\varepsilon)\,.
\tag{109}
$$

Because Δx and Δy are given by the differences

$$
\Delta x \;=\; x(\theta_2) - x(\theta_1)\,, \qquad \Delta y \;=\; y(\theta_2) - y(\theta_1)\,,
\tag{110}
$$

the asymptotic condition in (109) means that the pair θ_1, θ_2 map under the embedding $X : S^1 \to \mathbb{R}^3$ to points $p, q \in C$ which are nearly coincident under the Lagrangian projection to the xy-plane. Thus the point (θ_1, θ_2) either lies near the preimage of a crossing (aka double-point) on the Lagrangian projection $\Pi(C)$, or (θ_1, θ_2) lies near the diagonal Δ itself, in the boundary region that we previously analyzed in Section II A.

With this observation, our proof of the Main Theorem proceeds in three steps.

1. Estimate the contribution to $\mathsf{slk}_\kappa(C)$ from each crossing of $\Pi(C)$ when Λ is large.

2. Estimate the contribution to $\mathsf{slk}_\kappa(C)$ from the diagonal $\Delta \subset T^2$ when Λ is large.

3. Bound the contributions from elsewhere on the integration domain, as well as the errors in the preceding estimates, by a quantity δ which can be made arbitrarily small as $\Lambda \to \infty$.

Conceptually, the local estimates in the first two steps are most important, because these estimates explain why the Thurston-Bennequin invariant $\mathsf{tb}(C)$ and the rotation number $\mathsf{rot}(C)$ appear in the formula (52) for $\mathsf{slk}_\kappa(C)$. We therefore begin in Section III A with simple, informal computations for the first two steps in the proof.

The technical heart of the proof resides in the third step, when we carefully bound the errors in the preceding local computations. This step is required for a rigorous analysis, but the ideas are standard and offer no surprises. For this reason, Sections III B and III C could be omitted on the initial reading of the paper. In Section III B we introduce various geometric quantities to be used in the error analysis, and in Section III C we make the necessary bounds.

A. Local computations

We compute the contribution to $\mathsf{slk}_\kappa(C)$ from a right-handed crossing in the Lagrangian projection. We depict such a crossing on the left in Figure 13. We shall proceed softly, reserving precise inequalities for Section III C.

To first-order over the double-point, the curve C is approximated by a pair of straight lines. For our local computation, we take the lines to be parametrized by maps $X^\pm : \mathbb{R} \to \mathbb{R}^3$, where X^+ passes over X^- by convention. As in

$$(a.) \quad \mathsf{slk}_\kappa = +1 \qquad\qquad (b.) \quad \mathsf{slk}_\kappa = -1$$
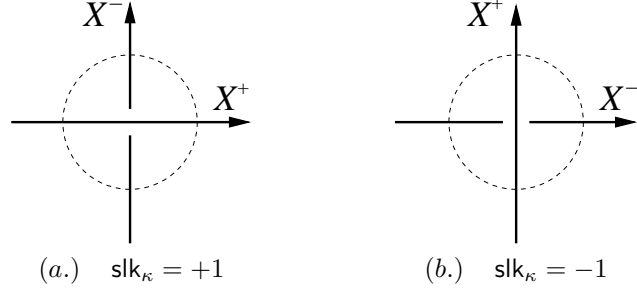
FIG. 13. Neighborhoods of right- and left-handed crossings of $\Pi(C)$.

the figure, we take $X^\pm$ to describe lines which are perpendicular and lie in parallel planes,

$$X^-(\theta_1) = (0, \theta_1, 0), \qquad X^+(\theta_2) = (\theta_2, 0, \Delta z), \qquad \theta_{1,2} \in \mathbb{R}. \tag{111}$$

Here $\Delta z > 0$ is a positive constant, the height of the overpass. With this choice, both $X^\pm$ satisfy the Legendrian condition (23) and so describe a Legendrian crossing. Because we have yet to establish isotopy-invariance of any kind, our assumptions about even the first-order geometry of C require justification. A significant portion of the analysis in Section III C will be devoted exactly to this issue.

The local contribution to $\mathsf{slk}_\kappa(C)$ from the right-handed crossing at $\{0\} \in \mathbb{R}^2$ is now given by

$$\mathsf{slk}_\kappa(C)\big|_{\{0\}} = \int_{\mathbb{R}^2} \left(X^- \times X^+\right)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda, \qquad \Lambda \gg 1, \tag{112}$$

where we integrate over all $(\theta_1, \theta_2) \in \mathbb{R}^2$, with the standard orientation $d\theta_1 \wedge d\theta_2$. Informally, the error which we make when we extend the range of integration from a small region on T^2 to the entire plane $\mathbb{R}^2$ vanishes exponentially as $\Lambda \to \infty$, due to the rapid decay of the heat form χ_Λ away from the origin. By extending over all of $\mathbb{R}^2$, we will be able to evaluate the integral (112) in closed form.

For the perpendicular lines $X^\pm$ in (111), the differences Δx, Δy, and $\widehat{\Delta} z$ in (108) become

$$\Delta x = x^+ - x^- = \theta_2, \qquad \Delta y = y^+ - y^- = -\theta_1, \tag{113}$$

and

$$\widehat{\Delta} z = z^+ - z^- + \hbar\left(x^- y^+ - x^+ y^-\right) = \Delta z - \hbar\,\theta_1\theta_2. \tag{114}$$

After a small calculation, one finds for the self-linking integrand in (108)

$$\left(X^- \times X^+\right)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda = \frac{\Lambda}{2\pi\Delta z\left(1 - \hbar\,\theta_1\theta_2/\Delta z\right)^2} \exp\left[-\frac{\Lambda\left(\theta_1^2 + \theta_2^2\right)}{2\left(\Delta z - \hbar\,\theta_1\theta_2\right)}\right] d\theta_1 \wedge d\theta_2, \tag{115}$$

assuming the positivity condition $\widehat{\Delta} z > 0 \iff \Delta z > \hbar\,\theta_1\theta_2$ (else the integrand vanishes).

Thus,

$$\mathsf{slk}_\kappa(C)\big|_{\{0\}} = \int_{\Delta z > \hbar\,\theta_1\theta_2} d\theta_1 d\theta_2 \, \frac{\Lambda}{2\pi\Delta z\left(1 - \hbar\,\theta_1\theta_2/\Delta z\right)^2} \exp\left[-\frac{\Lambda\left(\theta_1^2 + \theta_2^2\right)}{2\left(\Delta z - \hbar\,\theta_1\theta_2\right)}\right]. \tag{116}$$

Since $\Lambda \gg 1$ is large, let us rescale the integration variables to eliminate the overall factor of Λ from the argument of the exponential,

$$\mathsf{slk}_\kappa(C)\big|_{\{0\}} =$$

$$\int_{\Lambda\Delta z > \hbar\,\theta_1\theta_2} d\theta_1 d\theta_2 \, \frac{1}{2\pi\Delta z\left(1 - \hbar\,\theta_1\theta_2/\Lambda\Delta z\right)^2} \exp\left[-\frac{\theta_1^2 + \theta_2^2}{2\,\Delta z\left(1 - \hbar\,\theta_1\theta_2/\Lambda\Delta z\right)}\right]. \tag{117}$$

After we expand the integrand of (117) asymptotically in $1/\Lambda$, the local contribution to $\mathsf{slk}_\kappa(C)$ from the right-handed crossing can be evaluated as a Gaussian integral over $\mathbb{R}^2$,

$$
\begin{aligned}
\mathsf{slk}_\kappa(C)\big|_{\{0\}} &= \int_{\mathbb{R}^2} d\theta_1 d\theta_2 \, \frac{1}{2\pi\Delta z} \, \exp\left[-\frac{\theta_1^2 + \theta_2^2}{2\Delta z}\right] + \mathcal{O}(1/\Lambda), \\
&= 1 + \mathcal{O}(1/\Lambda).
\end{aligned}
\tag{118}
$$

Note that all dependence on the homotopy parameter $\hbar$ disappears as soon as we perform the asymptotic expansion in Λ.

In principle, the contribution from the right-handed crossing in Figure 13 also includes the portion of the integration domain $T^2 - \Delta(\varepsilon)$ where the roles of θ_1 and θ_2 are swapped in (111), with $X^+ \equiv X^+(\theta_1)$ and $X^- \equiv X^-(\theta_2)$. In this case, $\widehat{\Delta} z = -\Delta z + \hbar\,\theta_1\theta_2 < 0$ is negative near the origin, and the self-linking integrand vanishes identically by the definition of the heat form χ_Λ.

Finally, to evaluate the local contribution from the left-handed crossing in Figure 13, we simply swap the roles of X^+ and X^-. Apparently from (112), this swap is equivalent to an orientation-reversal on $\mathbb{R}^2$, so the sign of $\mathsf{slk}_\kappa(C)\big|_{\{0\}}$ is reversed.

Comparing to our conventions for the writhe in Figure 1, we conclude that $\mathsf{slk}_\kappa(C)\big|_{\{0\}}$ is the local writhe of the given crossing in $\Pi(C)$. In total, the local contribution to $\mathsf{slk}_\kappa(C)$ from the crossings is precisely the Thurston-Bennequin invariant of C,

$$
\sum_{a\in\mathbf{I}} \mathsf{slk}_\kappa(C)\big|_a = \mathsf{w}\big(\Pi(C)\big) = \mathsf{tb}(C),
\tag{119}
$$

where $\mathbf{I}$ indexes the set of all crossings in the Lagrangian projection. The localization computation is also consistent with Proposition I.1 regarding Heisenberg linking, together with the diagrammatic formula for $\mathsf{lk}(C_1, C_2)$ in (8).

More interesting is the local contribution to $\mathsf{slk}_\kappa(C)$ from the diagonal $\Delta \subset T^2$. This contribution does depend (weakly) on the value of $\hbar$, a small remnant of the topological anomaly. Integrating over a neighborhood of the diagonal really means integrating over the two boundary regions on the cylinder $T^2 - \Delta(\varepsilon)$, as we have already considered in our proof of the Fundamental Lemma in Section II. So we do not need to perform any new computations to evaluate the contribution from the diagonal.

We begin with the generic case $\hbar \neq 1$, for which the local expression for the self-linking integrand appears in (80). Directly for $\Lambda \gg 1$,

$$
\begin{aligned}
\mathsf{slk}_\kappa(C)\big|_\Delta \;&\overset{\hbar\neq 1}{=} \\
&-\int_{S^1_+ \cap [(1-\hbar)(\gamma\times\dot\gamma)<0]} d\phi \left[\int_0^\infty d\eta \, \frac{\Lambda\,(\dot\gamma \times \ddot\gamma)}{4\pi\,|1-\hbar|\,|\gamma \times \dot\gamma|} \, \exp\left(-\frac{\Lambda\,||\dot\gamma||^2\,\eta}{2\,|1-\hbar|\,|\gamma \times \dot\gamma|}\right)\right] + \\
&-\int_{S^1_- \cap [(1-\hbar)(\gamma\times\dot\gamma)>0]} d\phi \left[\int_0^\infty d\eta \, \frac{\Lambda\,(\dot\gamma \times \ddot\gamma)}{4\pi\,|1-\hbar|\,|\gamma \times \dot\gamma|} \, \exp\left(-\frac{\Lambda\,||\dot\gamma||^2\,\eta}{2\,|1-\hbar|\,|\gamma \times \dot\gamma|}\right)\right].
\end{aligned}
\tag{120}
$$

The integrals in the two lines of (120) describe the respective local contributions to $\mathsf{slk}_\kappa(C)$ from collar neighborhoods of the boundary circles $S^1_\pm$ on the cylinder in Figure 10. Both integrals appear with identical signs, after one takes into account the relative boundary orientations on $S^1_\pm$ and the explicit dependence of the integrand on $\mathrm{sgn}(\eta)$ in (80). We make a trivial change of variables so that the integral in the neighborhood of S^1_- also runs over positive, as opposed to negative, values of η. By the positivity condition in (76), the integral over S^1_+ runs over the subset where $(1-\hbar)(\gamma \times \dot\gamma) < 0$, and the integral over S^1_- runs over the complement. Finally, we extend the integration range over the normal coordinate η to infinity at the cost of an exponentially small error for large Λ, and we set $\varepsilon = 0$ at the lower limit of integration for η.

After integrating over η in (120),

$$
\mathsf{slk}_\kappa(C)\big|_\Delta \overset{\hbar\neq 1}{=} -\frac{1}{2\pi}\int_{S^1_+ \cap [(1-\hbar)(\gamma\times\dot\gamma)<0]} d\phi \, \frac{\dot\gamma \times \ddot\gamma}{||\dot\gamma||^2} - \frac{1}{2\pi}\int_{S^1_- \cap [(1-\hbar)(\gamma\times\dot\gamma)>0]} d\phi \, \frac{\dot\gamma \times \ddot\gamma}{||\dot\gamma||^2},
\tag{121}
$$

or put more succinctly,

$$
\mathsf{slk}_\kappa(C)\big|_\Delta \overset{\hbar\neq 1}{=} -\frac{1}{2\pi}\oint_{S^1} d\phi \, \frac{\dot\gamma \times \ddot\gamma}{||\dot\gamma||^2}.
\tag{122}
$$

So long as $\hbar \neq 1$, all dependence on $\hbar$ disappears. From the geometric expression for the rotation number in (33), we deduce

$$\mathsf{slk}_\kappa(C)\big|_\Delta \overset{\hbar \neq 1}{=} -\mathsf{rot}(C) . \tag{123}$$

On general grounds, the appearance of the rotation number in this calculation is not so surprising, as one was guaranteed to find the integral of some local geometric quantity on C. However, the integrality of the result (123) comes as a minor miracle, which is far from obvious from the definition of the Legendrian self-linking integral in (51). Remember, the value of the naive Gauss self-linking integral $\mathsf{slk}_0(C)$ is not even a deformation-invariant!

We return to our formula in (81) to evaluate the local self-linking contribution from the diagonal when $\hbar = 1$,

$$(X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda\Big|_{\Delta(\varepsilon)} \overset{\hbar = 1}{=} -\frac{3\Lambda}{2\pi\eta^2} \exp\left[-\frac{3\Lambda \, ||\dot\gamma||^2}{|\dot\gamma \times \ddot\gamma|} \frac{1}{|\eta|} \right] d\phi \wedge d\eta \, + \, \cdots . \tag{124}$$

Unlike the expressions in (120), which are non-zero for $\eta = 0$, the self-linking integrand in (124) vanishes exponentially as $\eta \to 0$ for any $\Lambda > 0$. By inspection we conclude

$$\mathsf{slk}_\kappa(C)\big|_\Delta \overset{\hbar = 1}{=} 0 . \tag{125}$$

If the Heisenberg symmetry is preserved, the diagonal does not contribute to the Legendrian self-linking integral.

At least informally, modulo precise control of the error terms, we obtain from these local computations the statement in the Main Theorem,

$$\begin{aligned}
\mathsf{slk}_\kappa(C) \; &= \; \lim_{\Lambda \to \infty} \mathsf{slk}_\kappa(C) \; = \; \sum_{a \in \mathbf{I}} \mathsf{slk}_\kappa(C)\big|_a \, + \, \mathsf{slk}_\kappa(C)\big|_\Delta \, , \\[6pt]
&= \; \begin{cases} \mathsf{tb}(C) - \mathsf{rot}(C) \, , & \hbar \neq 1 \, , \\ \mathsf{tb}(C) \, , & \hbar = 1 \, . \end{cases}
\end{aligned} \tag{126}$$

B. Some preliminaries

The informal localization computation in Section III A is useful for developing geometric intuition about the behavior of the Legendrian self-linking integral. To prove the Main Theorem, we retrace the same route philosophically, but we exercise greater care in analyzing the dependence of the error terms on Λ, at least when Λ is large.

Before we establish precise inequalities in Section III C, we need to introduce a bevy of constants related to the geometry of the knot $C \subset \mathbb{R}^3$ and its Lagrangian projection $\Pi(C)$. These constants are important, as the required bounds fall out naturally from them.

Local Neighborhoods of Crossings

The notion of a collar neighborhood for the boundary of $T^2 - \Delta(\varepsilon)$ is unambiguous, but we also need a proper notion for the neighborhood of each crossing in $\Pi(C)$. The trick will be to choose these neighborhoods to be small enough so that the geometry of C is controlled over each neighborhood. Informally, we treated this issue by linearizing C in Figure 13, but now we work nonlinearly.

As before, $\gamma : S^1 \to \mathbb{R}^2$ is the regular immersed plane curve which is the Lagrangian projection of the embedding $X : S^1 \to \mathbb{R}^3$,

$$\gamma \; = \; \Pi \circ X \, , \qquad \gamma(\theta) \; \equiv \; (x(\theta), \, y(\theta)) \, . \tag{127}$$

By assumption, γ has only a finite number n of simple double-point singularities, located at positions

$$\gamma_1, \, \gamma_2, \, \ldots, \, \gamma_n \; \in \; \mathbb{R}^2 \, . \tag{128}$$

See Figure 8 for our canonical trefoil example, with $n = 5$. Each crossing γ_a for $a = 1, \ldots, n$ lies under a pair of corresponding points (p_a, q_a) on the knot C,

$$\gamma_a \; = \; \Pi(p_a) \; = \; \Pi(q_a) \, , \qquad p_a, q_a \in C \, . \tag{129}$$
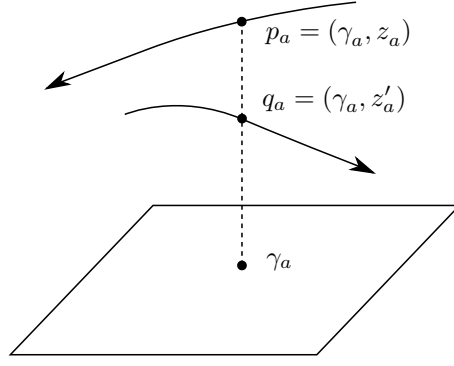
FIG. 14. Two points p_a and q_a on C with coincident Lagrangian projections.

Let z_a and z_a' be the respective heights of p_a and q_a, so that these points have coordinates in $\mathbb{R}^3$ given by

$$p_a = (\gamma_a,\, z_a), \qquad q_a = (\gamma_a,\, z_a'). \tag{130}$$

As in the informal computation, an important geometric quantity is the absolute difference Δz_a in the heights of p_a and q_a over the crossing,

$$\Delta z_a = |z_a - z_a'| > 0. \tag{131}$$

See Figure 14 for a local (nonlinear) picture of C near the points p_a and q_a.

Let $D(\gamma_a; \mathsf{h}) \equiv D_a(\mathsf{h}) \subset \mathbb{R}^2$ be the open disc of radius $\mathsf{h} > 0$ centered at the location γ_a of a given crossing in the plane. The union $\cup D_a(\mathsf{h})$ of these discs, each with the same radius h, provides an open neighborhood for all crossings in $\Pi(C)$. We now choose $\mathsf{h} > 0$ to be sufficiently small so that the following statements are true at each crossing. By continuity of X and compactness of the closure $\overline{D_a(\mathsf{h})}$, such a choice is always possible.

For ease of notation, we suppress the crossing index 'a' below.

1. The disc $D(\mathsf{h})$ intersects the immersed plane curve γ in two arcs, as shown in Figure 15. We denote these arcs by γ^+ and γ^-, where '$\pm$' indicate the respective upper and lower strands at the crossing. Over the disc, the embedding X restricts to a pair of maps $X^-(\theta_1) = (\gamma^-(\theta_1), z^-(\theta_1))$ and $X^+(\theta_2) = (\gamma^+(\theta_2), z^+(\theta_2))$, with $z^+ > z^-$. Here $X^\pm$ are nonlinear analogues of the expressions in (111).
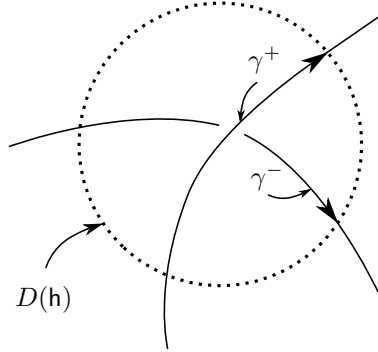

FIG. 15. Intersection of $D(\mathsf{h})$ and γ.

2. With the same arcs $\gamma^\pm$ in mind, let $\Gamma_{2d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be the two-dimensional difference map

$$\Gamma_{2d}(u, v) = v - u. \tag{132}$$

Consider the composition

$$\varphi = \Gamma_{2d} \circ (\gamma^- \times \gamma^+) = \gamma^+ - \gamma^-, \tag{133}$$

which maps the region on T^2 where $\gamma^- \times \gamma^+$ is locally defined to another region on the uv-plane. Then $\varphi \equiv (u(\theta_1, \theta_2), v(\theta_1, \theta_2))$ is a diffeomorphism to a curvy quadrilateral region $\mathcal{Q}$ around the origin in the uv-plane, as in Figure 16.

Eventually, we will use φ to make a change-of-variables to simplify the self-linking integrand in the neighborhood of the crossing.
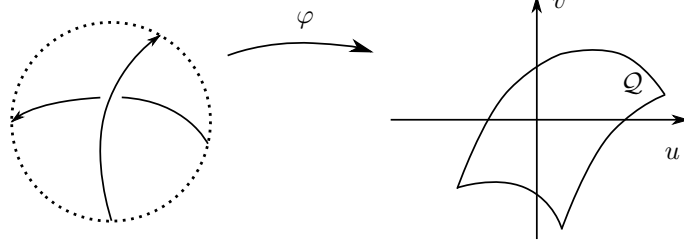

FIG. 16. Diffeomorphism $\varphi = \gamma^+ - \gamma^-$ to $\mathcal{Q}$.

3. Let J_φ be the Jacobian for the change-of-variables induced by φ from (θ_1, θ_2) to (u, v),

$$J_\varphi = \left| \frac{d\gamma^+}{d\theta_2} \times \frac{d\gamma^-}{d\theta_1} \right| . \tag{134}$$

Since φ is a diffeomorphism, $J_\varphi \neq 0$ is non-vanishing throughout the domain of φ, as illustrated in Figure 17. We go slightly further and assume that J_φ is uniformly bounded from below by a positive constant

$$0 < \mathsf{J}_0 < J_\varphi . \tag{135}$$
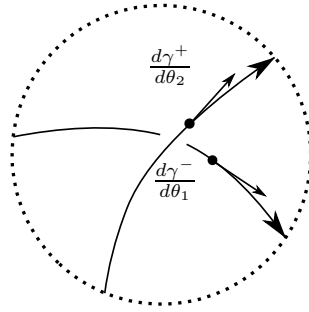
FIG. 17. The Jacobian $J_\varphi \neq 0$ of φ.

4. For the crossing labelled by 'a', consider all pairs of points on $C \subset \mathbb{R}^3$ which lie in the image of $X_a^+ \times X_a^-$ over the disc $D_a(\mathsf{h})$. Then the difference in heights $|z_a^+ - z_a^-|$ for all such pairs lies in the range

$$\left(1 - \frac{\mathsf{c}}{2}\right) \Delta z_a \; < \; |z_a^+ - z_a^-| \; < \; \left(1 + \frac{\mathsf{c}}{2}\right) \Delta z_a . \tag{136}$$

Here c is a positive constant, independent of a, bounded by

$$0 \; < \; \mathsf{c} \; < \; \frac{1}{2} . \tag{137}$$

This assumption in (136) implies that the difference $|z_a^+ - z_a^-|$ for all pairs of points on C projecting to $D_a(\mathsf{h})$

obeys[3]

$$1 - \mathsf{c} < \frac{\Delta z_a}{|z_a^+ - z_a^-|} < 1 + \mathsf{c}, \tag{138}$$

for c in the given range. Informally, the constant c controls the variation in the vertical separation between the two strands of C over the disc $D_a(\mathsf{h})$, relative to the separation Δz_a over the crossing itself. The constant '1/2' in the bound (137) is a convenient choice related to other inequalities later.

Bounds on the Integrand

Our assumption about the radius h of the discs $D_a(\mathsf{h})$ gives us adequate control on the local geometry of C above each crossing γ_a. We now introduce further constants related to the magnitude of the self-linking integrand itself.

First, let $\mathsf{Z} > 0$ be the height of the knot $C \subset \mathbb{R}^3$. More formally, Z is the maximum vertical separation between any pair of points on C,

$$\mathsf{Z} = \max_{(\theta_1, \theta_2) \in T^2} \Big[|z(\theta_2) - z(\theta_1)| \Big]. \tag{139}$$

We now fix a small positive constant $\delta > 0$ which will control the errors. In reference to the dependence of the heat form χ_Λ on $r^2 = x^2 + y^2$ and z in (42), we note the following lemma.

Lemma III.1 *Given $\delta > 0$ and sufficiently large Λ, there exists a constant $\mathsf{r}_\Lambda > 0$, depending on Λ, so that for all $r > \mathsf{r}_\Lambda$,*

$$\sup_{0 < z \leq \mathsf{Z}} \left[\frac{\Lambda}{2\pi z} \exp\left(-\frac{\Lambda r^2}{2z} \right) \right] < \delta, \tag{140}$$

and

$$\sup_{0 < z \leq \mathsf{Z}} \left[\frac{\Lambda}{4\pi z^2} \exp\left(-\frac{\Lambda r^2}{2z} \right) \right] < \delta. \tag{141}$$

Proof

The proof of the lemma is elementary, but we wish to gain precise knowledge about how r_Λ must depend upon Z and Λ for the bounds to hold. The bounds in (140) and (141) can be treated similarly; we start with the bound in (140).

We differentiate the function in (140) with respect to z,

$$\frac{\partial}{\partial z} \left[\frac{\Lambda}{2\pi z} \exp\left(-\frac{\Lambda r^2}{2z} \right) \right] = \frac{\Lambda}{2\pi z^2} \left(\frac{\Lambda r^2}{2z} - 1 \right) \exp\left(-\frac{\Lambda r^2}{2z} \right). \tag{142}$$

The derivative in (142) is positive so long as

$$\frac{\Lambda r^2}{2z} > 1, \qquad 0 < z \leq \mathsf{Z}, \tag{143}$$

which in turn is equivalent to

$$r^2 > \frac{2\mathsf{Z}}{\Lambda}. \tag{144}$$

We will actually require a stronger bound on r in regard to its dependence on Λ. We set

$$\mathsf{r}_\Lambda^2 = \frac{2\mathsf{Z}}{\sqrt{\Lambda}}. \tag{145}$$

---

[3] The bound in (138) is not sharp given (136) but will suffice for us.

So long as $\Lambda > 1$, the condition $r > r_\Lambda$ implies the bound in (144), so that

$$\frac{\partial}{\partial z}\left[\frac{\Lambda}{2\pi z}\exp\left(-\frac{\Lambda r^2}{2z}\right)\right] > 0. \tag{146}$$

The function in (140) therefore increases with z for all $r > r_\Lambda$, and the supremum is achieved at the value $z = Z$,

$$\sup_{0 < z \leq Z}\left[\frac{\Lambda}{2\pi z}\exp\left(-\frac{\Lambda r^2}{2z}\right)\right] = \frac{\Lambda}{2\pi Z}\exp\left(-\frac{\Lambda r^2}{2Z}\right), \qquad r > r_\Lambda. \tag{147}$$

For $r > r_\Lambda$, the argument of the exponential in (147) satisfies

$$\frac{\Lambda r^2}{2Z} > \frac{\Lambda r_\Lambda^2}{2Z} = \sqrt{\Lambda}. \tag{148}$$

Had we imposed the weak inequality in (144), the quantity $\Lambda r^2/2Z$ would have been bounded from below only by a constant, independent of Λ, and we would have no chance to achieve the bound by δ in (140). Instead, with the strong inequality $r > r_\Lambda$ in (145),

$$\frac{\Lambda}{2\pi Z}\exp\left(-\frac{\Lambda r^2}{2Z}\right) < \frac{\Lambda}{2\pi Z}\exp\left(-\sqrt{\Lambda}\right). \tag{149}$$

For any positive $\delta > 0$, we now take Λ sufficiently large so that

$$\frac{\Lambda}{2\pi Z}\exp\left(-\sqrt{\Lambda}\right) < \delta, \tag{150}$$

implying via (147) and (149) the desired inequality in (140).

The analysis of the function in (141) is identical, up to a factor of 2, due to appearance of the same Gaussian factor. In this case, we take $r_\Lambda^2 = 4Z/\sqrt{\Lambda}$. To treat both cases of Lemma III.1 simultaneously, we set

$$r_\Lambda^2 = \max\left\{\frac{2Z}{\sqrt{\Lambda}}, \frac{4Z}{\sqrt{\Lambda}}\right\} = \frac{4Z}{\sqrt{\Lambda}}. \tag{151}$$

With this choice for r_Λ, the lemma follows. $\qquad\square$

We require one additional quantity to make our bounds on the error. This quantity will be a function of Λ which arises in reference to the magnitude of the Gaussian integrand (118) at a crossing. Briefly, for given δ and crossing index 'a', we let $R_a(\Lambda) > 0$ be the solution to

$$\frac{\Lambda}{2\pi\Delta z_a}\exp\left[-\frac{\Lambda R_a(\Lambda)^2}{2\Delta z_a}\right] = \delta. \tag{152}$$

Explicitly,

$$R_a(\Lambda)^2 = \frac{2\Delta z_a}{\Lambda}\ln\left(\frac{\Lambda}{2\pi\Delta z_a\delta}\right). \tag{153}$$

As a function of Λ, $R_a(\Lambda)$ is decreasing for sufficiently large Λ and vanishes in the limit $\Lambda \to \infty$.

For the convenience of the reader, we summarize all these constants in Table I.

C.   Bounds on the error

We now establish the necessary bounds to prove the Main Theorem.

The most fundamental bound emerges trivially from the definition of the constant r_Λ in Lemma III.1. Let $\mathcal{U}$ and $\mathcal{V}$ be the subsets of the cylinder $T^2 - \Delta(\varepsilon)$ defined by

$$\mathcal{U} = \left\{(\theta_1, \theta_2) \in T^2 - \Delta(\varepsilon)\,\Big|\,[\Delta x^2 + \Delta y^2]|_{(\theta_1,\theta_2)} \leq r_\Lambda^2\right\}, \tag{154}$$

Λ	Width of Gaussian in χ_Λ.
M, m	Any positive constant which only depends upon $C \subset \mathbb{R}^3$. The values of M and m may differ at various places in the text.
δ	A fixed, small positive constant. An error less than $M\delta$ is negligible.
Δz_a	Vertical displacement of C over each crossing $\gamma_a \in \mathbb{R}^2$.
h	Radius of disc $D_a(\mathsf{h}) \subset \mathbb{R}^2$ centered at each crossing.
J_0	Lower bound $0 < \mathsf{J}_0 < J_\varphi$ for the Jacobian of φ.
c	Positive constant $< 1/2$ for which $1 - \mathsf{c} < \Delta z_a / |z_a^+ - z_a^-| < 1 + \mathsf{c}$.
Z	Total height of $C \subset \mathbb{R}^3$.
r_Λ	A positive number given by $\mathsf{r}_\Lambda^2 = 4\mathsf{Z}/\sqrt{\Lambda}$. For sufficiently large Λ and $r > \mathsf{r}_\Lambda$, the inequalities in Lemma III.1 are true.
$\mathsf{R}_a(\Lambda)$	Positive solution to $(\Lambda/2\pi\Delta z_a)\exp\left[-\Lambda \mathsf{R}_a(\Lambda)^2/2\Delta z_a\right] = \delta$.

TABLE I. List of Constants.

and

$$\mathcal{V} = \left\{ (\theta_1, \theta_2) \in T^2 - \Delta(\varepsilon) \,\Big|\, [\Delta x^2 + \Delta y^2]\big|_{(\theta_1, \theta_2)} > \mathsf{r}_\Lambda^2 \right\}. \tag{155}$$

That is, $\mathcal{U}$ consists of those pairs of points (θ_1, θ_2) whose images in the xy-plane under the map $\Pi \circ (X \times X)$ lie within the critical distance r_Λ, and $\mathcal{V}$ consists of those pairs separated in the xy-plane by a distance greater than r_Λ.

Since $\mathcal{U}$ and $\mathcal{V}$ are complementary subsets of the cylinder, the self-linking integral can be written as the sum

$$\mathsf{slk}_\kappa(C) = \int_{\mathcal{U}} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda + \int_{\mathcal{V}} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda. \tag{156}$$

By the very definition of r_Λ in Lemma III.1, the magnitude of the self-linking integrand (108) on $\mathcal{V}$ is everywhere bounded by δ. Automatically,

$$\left| \mathsf{slk}_\kappa(C) - \int_{\mathcal{U}} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \right| = \left| \int_{\mathcal{V}} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \right| < M\delta. \tag{157}$$

Here M is a constant, depending on the geometry of the knot $C \subset \mathbb{R}^3$ but independent of Λ. For small δ, the value of $\mathsf{slk}_\kappa(C)$ is well-approximated by the integral over the subset $\mathcal{U}$.

Our next task is to characterize the points in the domain of integration that lie in $\mathcal{U}$. If we formally set $\mathsf{r}_\Lambda = 0$ in (154), then $\mathcal{U}$ consists of those pairs (θ_1, θ_2) which become coincident after projection to the xy-plane. Such pairs either lie along the diagonal $\Delta \subset T^2$, or they lie in the preimage of a crossing in the Lagrangian projection of C. When $\mathsf{r}_\Lambda > 0$ is positive but sufficiently small, these closed sets fatten, and $\mathcal{U}$ is contained within the disjoint union of a tubular neighborhood $N_\Delta(\mathsf{w})$ of the diagonal and a collection of open balls $B_b(\mathsf{h}) \subset T^2$ associated to the crossings,

$$\mathcal{U} \subset N_\Delta(\mathsf{w}) \cup \bigcup_{b=1}^{2n} B_b(\mathsf{h}). \tag{158}$$

See Figure 18 for a schematic picture of the open set containing $\mathcal{U}$ for sufficiently small r_Λ. As in the picture, each crossing in $\Pi(C)$ has two preimages, which are exchanged when the roles of θ_1 and θ_2 swap. Thus if $\Pi(C)$ has n crossings, the index 'b' on the balls $B_b(\mathsf{h})$ runs to $2n$.

In writing $B_b(\mathsf{h})$ for the open ball in T^2, we abuse notation somewhat. By assumption, the radius of $B_b(\mathsf{h})$ is fixed so that this ball lies in the preimage of the corresponding disc $D_a(\mathsf{h}) \subset \mathbb{R}^2$ under the map $\gamma_a^- \times \gamma_a^+$ in Figure 15,

$$\left(\gamma_a^- \times \gamma_a^+\right)(B_b(\mathsf{h})) \subset D_a(\mathsf{h}), \qquad a \equiv b \mod n. \tag{159}$$

Therefore the radius of $B_b(\mathsf{h})$ is not necessarily equal to h, but it is determined by h independently of Λ. Once h is
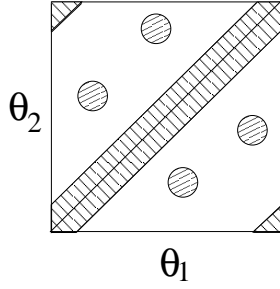
FIG. 18. Neighborhood of the subset $\mathcal{U}$ for sufficiently small $r_\Lambda > 0$.

fixed in terms of the geometry of C, we can always take Λ sufficiently large and $r_\Lambda \sim \Lambda^{-1/4}$ sufficiently small so that $B_b(h)$ contains the relevant portion of $\mathcal{U}$.

Similarly, the tubular neighborhood $N_\Delta(w)$ of the diagonal has width $w > 0$, meaning that points in $N_\Delta(w)$ satisfy $|\theta_2 - \theta_1| < w$ for the parameters in Figure 18. A crucial step will be to fix the value of w, which must be large enough so that $N_\Delta(w)$ contains the piece of $\mathcal{U}$ near the diagonal, but also small enough so that the local analysis from Section II A is applicable everywhere in $N_\Delta(w)$.

According to the next lemma, both conditions on $N_\Delta(w)$ can be simultaneously satisfied once we set

$$w = \frac{r_\Lambda \sqrt{2}}{m}, \tag{160}$$

with a positive constant $m > 0$ defined geometrically by

$$m^2 = \min_{\theta \in S^1} \left[||\dot{\gamma}(\theta)||^2 \right]. \tag{161}$$

Since $\gamma = \Pi \circ X$ is an immersion, the minimum speed in (161) is bounded away from zero, which is essential. The constant $\sqrt{2}$ is inessential and could be absorbed into the definition of m.

Lemma III.2 *For sufficiently large Λ and $w = r_\Lambda \sqrt{2}/m$, the tubular neighborhood $N_\Delta(w)$ contains the diagonal component of $\mathcal{U}$.*

Proof

The lemma says that, away from crossings, all points on the projection $\Pi(C)$ which lie within the distance r_Λ of a given point $\gamma(\theta)$ are contained within the image of the interval $\left[\theta - r_\Lambda \sqrt{2}/m, \theta + r_\Lambda \sqrt{2}/m\right]$ under γ, for all values of θ. See Figure 19 for an illustration of this claim. The minimum speed along γ naturally sets the scale of the required interval, which decreases with increasing m.
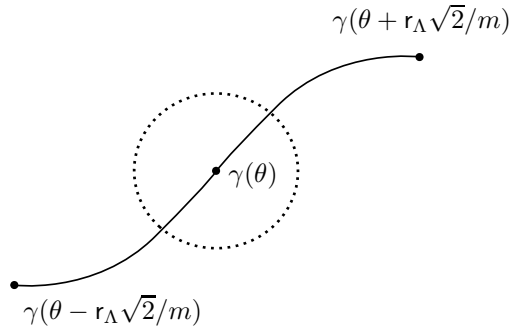

FIG. 19. Points within the distance r_Λ of $\gamma(\theta)$.

For any fixed $\theta_0 \in S^1$, define the function

$$F(\theta) = \frac{d}{d\theta} ||\gamma(\theta) - \gamma(\theta_0)||^2 = 2\left(\gamma(\theta) - \gamma(\theta_0)\right) \cdot \dot{\gamma}(\theta). \tag{162}$$

Then $F(\theta_0) = 0$, and the derivative of F is

$$\frac{dF}{d\theta}(\theta) = 2\,||\dot{\gamma}||^2 + 2\left(\gamma(\theta) - \gamma(\theta_0)\right) \cdot \ddot{\gamma}(\theta). \tag{163}$$

Since γ is an immersion, $||\dot{\gamma}|| \geq m$ is bounded below, and $||\ddot{\gamma}|| < \infty$ is bounded above. So if $||\gamma(\theta) - \gamma(\theta_0)|| \leq \mathsf{r}_\Lambda$ for sufficiently large Λ and hence sufficiently small r_Λ, then

$$\frac{dF}{d\theta}(\theta) > m^2. \tag{164}$$

Integrating the inequality in (164), we obtain

$$F(\theta) > m^2\,|\theta - \theta_0|. \tag{165}$$

Integrating once more from the definition (162) of $F(\theta)$,

$$\left|\left|\gamma(\theta) - \gamma(\theta_0)\right|\right|^2 > \frac{1}{2}m^2\,|\theta - \theta_0|^2, \tag{166}$$

or

$$\left|\left|\gamma(\theta) - \gamma(\theta_0)\right|\right| > \frac{m}{\sqrt{2}}\,|\theta - \theta_0|. \tag{167}$$

Thus if $||\gamma(\theta) - \gamma(\theta_0)|| = \mathsf{r}_\Lambda$, then $|\theta - \theta_0| < \mathsf{r}_\Lambda\sqrt{2}/m$, which is the required bound. $\qquad\square$

Because $\mathcal{U}$ is contained in the union of $N_\Delta(\mathsf{w})$ and $\cup B_b(\mathsf{h})$, we have a relation between the corresponding self-linking integrals,

$$\left| \int_{\mathcal{U}} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda - \int_{N_\Delta(\mathsf{w})} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda - \sum_{b=1}^{2n} \int_{B_b(\mathsf{h})} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \right| < M\,\delta. \tag{168}$$

This inequality again follows from Lemma III.1, because points contained in either $N_\Delta(\mathsf{w})$ or $B_b(\mathsf{h})$ but not in $\mathcal{U}$ lie in $\mathcal{V}$, where the magnitude of the self-linking integrand is bounded by δ. So for sufficiently large Λ, we just need to evaluate the self-linking integral over the balls $B_b(\mathsf{h})$ and the tubular neighborhood $N_\Delta(\mathsf{w})$ in Figure 18.

Error analysis at a crossing

We first evaluate the self-linking integral over the ball $B_b(\mathsf{h})$. By the positivity condition on the heat form χ_Λ, the self-linking integrand vanishes in exactly half the balls. Reshuffling indices as necessary, we consider only those $B_a(\mathsf{h})$ for $a = 1, \ldots, n$ on which $\widehat{\Delta}z > 0$ and the integrand is non-zero.

With malice aforethought, we have arranged that the image of $B_a(\mathsf{h})$ under the product map $\gamma_a^- \times \gamma_a^+$ lies in the disc $D_a(\mathsf{h}) \subset \mathbb{R}^2$, where we have control over the geometry of C. In particular, the map φ_a in Figure 16 restricts to a diffeomorphism from $B_a(\mathsf{h})$ to a curvy quadrilateral region $\mathcal{Q}$ about the origin in the uv-plane,

$$\varphi_a = \gamma_a^+(\theta_2) - \gamma_a^-(\theta_1) \equiv (u(\theta_1, \theta_2), v(\theta_1, \theta_2)). \tag{169}$$

Our analysis will be performed using the uv-coordinates. In these coordinates, the self-linking integrand (108) simplifies,

$$\begin{aligned}
&\frac{\Lambda}{2\pi\widehat{\Delta}z}\,\mathrm{e}^{-\Lambda(\Delta x^2 + \Delta y^2)/2\widehat{\Delta}z}\left[d\Delta x \wedge d\Delta y + \frac{1}{2}\left(\Delta x\,d\Delta y - \Delta y\,d\Delta x\right) \wedge \frac{d\widehat{\Delta}z}{\widehat{\Delta}z}\right] \\
&= \frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)}\,\mathrm{e}^{-\Lambda(u^2 + v^2)/2\widehat{\Delta}z(u,v)}\left[du \wedge dv + \frac{1}{2}\left(u\,dv - v\,du\right) \wedge \frac{d\widehat{\Delta}z(u,v)}{\widehat{\Delta}z(u,v)}\right].
\end{aligned} \tag{170}$$

Here Δx and Δy are identified with the Cartesian coordinates u and v, and $\widehat{\Delta}z$ is considered to be a function of (u,v). All unknown functional dependence of the self-linking integrand in (170) is absorbed into $\widehat{\Delta}z(u,v)$.

The integrand in (170) is a sum of two terms,

$$\Psi \ = \ \frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)}\,\mathrm{e}^{-\Lambda(u^2+v^2)/2\widehat{\Delta}z(u,v)}\,du{\wedge}dv\,, \tag{171}$$

and

$$\Xi \ = \ \frac{\Lambda}{4\pi\widehat{\Delta}z(u,v)}\,\mathrm{e}^{-\Lambda(u^2+v^2)/2\widehat{\Delta}z(u,v)}\,(u\,dv - v\,du){\wedge}\frac{d\widehat{\Delta}z(u,v)}{\widehat{\Delta}z(u,v)}\,. \tag{172}$$

Hence after making the change-of-variables in $\mathcal{Q}$,

$$\int_{B_a(\mathsf{h})}(X\times X)^*\widehat{\Gamma}_\hbar^*\chi_\Lambda \ = \ \int_{\mathcal{Q}}\Psi \ + \ \int_{\mathcal{Q}}\Xi\,. \tag{173}$$

The analysis of the two terms on the right in (173) is different. Morally, Ξ is higher-order in u and v so will be irrelevant when Λ is large and $u,v \ll 1/\Lambda$. By contrast, Ψ is always relevant. We analyze the integrals of Ψ and Ξ over $\mathcal{Q}$ in turn.
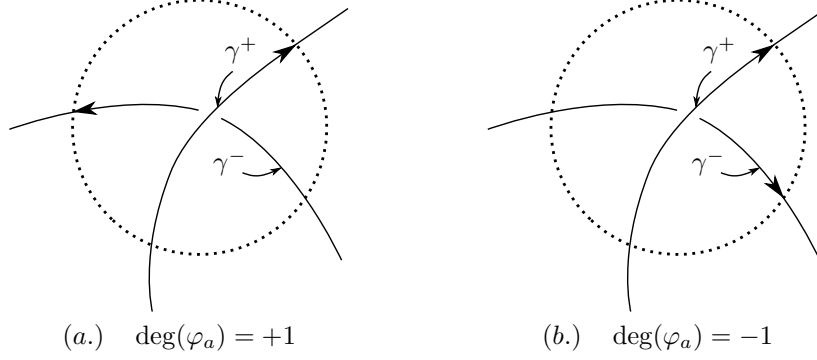

(a.) $\deg(\varphi_a) = +1$ (b.) $\deg(\varphi_a) = -1$

FIG. 20. Sign of the Jacobian for φ_a.

Integral of Ψ

Explicitly, the integral of Ψ is given by a kind of nonlinear Gaussian,

$$\int_{\mathcal{Q}}\Psi \ = \ \deg(\varphi_a)\int_{\mathcal{Q}}\frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)}\,\exp\!\left[-\frac{\Lambda\left(u^2+v^2\right)}{2\,\widehat{\Delta}z(u,v)}\right]du\,dv\,, \qquad \widehat{\Delta}z(u,v) > 0\,. \tag{174}$$

Here $\deg(\varphi_a) = \pm1$ depending upon whether the diffeomorphism $\varphi_a : B_a(\mathsf{h}) \to \mathcal{Q}$ preserves or reverses orientation. Equivalently, from the expression in (169), the sign is determined by the Jacobian in the expansion

$$du{\wedge}dv \ = \ -\left(\frac{d\gamma^-}{d\theta_1}\times\frac{d\gamma^+}{d\theta_2}\right)d\theta_1{\wedge}d\theta_2\,. \tag{175}$$

By inspection of Figure 20, $\deg(\varphi_a)$ is exactly the local writhe at the given crossing,

$$\deg(\varphi_a) \ = \ \mathsf{w}_a\,. \tag{176}$$

To suppress pernicious signs for the remainder, we assume $\mathsf{w}_a = +1$.

For large Λ, we evaluate the integral of Ψ over $\mathcal{Q}$ in two steps.

1. We replace the unknown function $\widehat{\Delta}z(u,v)$ by the constant displacement Δz_a at the crossing, with error

$$\left|\int_{\mathcal{Q}}\frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)}\,\exp\!\left[-\frac{\Lambda\left(u^2+v^2\right)}{2\,\widehat{\Delta}z(u,v)}\right]du\,dv \ - \ \int_{\mathcal{Q}}\frac{\Lambda}{2\pi\Delta z_a}\,\exp\!\left[-\frac{\Lambda\left(u^2+v^2\right)}{2\,\Delta z_a}\right]du\,dv\right| \ < \ M\delta\,. \tag{177}$$

2. We extend the subsequent range of Gaussian integration from $\mathcal{Q}$ to $\mathbb{R}^2$ so that the Gaussian integral can be performed analytically, with error

$$\left| \int_{\mathcal{Q}} \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\,\Delta z_a} \right] du\,dv - \int_{\mathbb{R}^2} \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\,\Delta z_a} \right] du\,dv \right| < \delta\,. \tag{178}$$

Of these steps, only the first is non-trivial. For the second, because the Gaussian integral over $\mathbb{R}^2$ is normalized to unity independent of Λ, we can always choose Λ sufficiently large so that

$$\int_{\mathbb{R}^2 - \mathcal{Q}} \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\,\Delta z_a} \right] du\,dv < \delta\,. \tag{179}$$

Informally, we performed both these steps in arriving at (118).

For the first step, we use heavily the positive function $\mathsf{R}_a(\Lambda)$ which satisfies

$$\frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\,\mathsf{R}_a(\Lambda)^2}{2\Delta z_a} \right] = \delta\,, \tag{180}$$

and vanishes monotonically as $\Lambda \to \infty$. The function $\mathsf{R}_a(\Lambda)$ sets the minimum distance from the origin for which the Gaussian integrand in (177) becomes negligible.

To be on the safe side in our bounds, we will have to work with a slightly larger distance $\mathsf{R}_a(\Lambda/2) > \mathsf{R}_a(\Lambda)$. Let $B_0 \equiv B_0(\mathsf{R}_a(\Lambda/2))$ be the ball of radius $\mathsf{R}_a(\Lambda/2)$ which is centered at the origin in $\mathcal{Q}$,

$$B_0: \quad u^2 + v^2 < \mathsf{R}_a^2(\Lambda/2)\,. \tag{181}$$

We shall prove that when (u, v) lies in B_0, the difference between the nonlinear and the usual Gaussian in (177) is small,

$$\left| \int_{B_0} \frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\widehat{\Delta}z(u,v)} \right] du\,dv - \int_{B_0} \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a} \right] du\,dv \right| < \delta\,. \tag{182}$$

Otherwise, when (u, v) lies outside B_0 in $\mathcal{Q}$, we show that both integrals are separately small, with

$$\int_{\mathcal{Q}-B_0} \frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\widehat{\Delta}z(u,v)} \right] du\,dv < M\,\delta\,, \tag{183}$$

and

$$\int_{\mathcal{Q}-B_0} \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a} \right] du\,dv < M\,\delta\,. \tag{184}$$

Thus the difference must also be small in $\mathcal{Q} - B_0$. This trick is the engine of asymptotic analysis. See Ch. 6 in [4] for further background on this idea.

We begin by establishing some easy bounds when (u, v) lies inside the ball $B_0 \subset \mathcal{Q}$. From the definition of $\mathsf{R}_a(\Lambda/2)$,

$$u^2 + v^2 < \mathsf{R}_a^2(\Lambda/2) = \frac{4\Delta z_a}{\Lambda} \ln\frac{\Lambda}{4\pi\Delta z_a\delta}\,, \tag{185}$$

so the argument of the Gaussian is bounded by

$$\frac{\Lambda(u^2 + v^2)}{2\Delta z_a} < 2\ln\frac{\Lambda}{4\pi\Delta z_a\delta}\,. \tag{186}$$

On the other hand, consider the difference $\widehat{\Delta}z(u,v) - \Delta z_a$. As a function of (u, v), the difference vanishes at $u = v = 0$ and is differentiable there, so

$$\left| \widehat{\Delta}z(u,v) - \Delta z_a \right| < M\sqrt{u^2 + v^2} < M\,\mathsf{R}_a(\Lambda/2)\,, \tag{187}$$

for some constant M depending on C. Immediately, since $\mathsf{R}_a^2(\Lambda/2)$ scales like $\ln \Lambda/\Lambda$, the relative fluctuations in height about Δz_a satisfy

$$\frac{\left|\widehat{\Delta}z(u,v) - \Delta z_a\right|}{\Delta z_a} \; < \; \frac{M \, \mathsf{R}_a(\Lambda/2)}{\Delta z_a} \; \sim \; \left(\frac{\ln \Lambda}{\Lambda}\right)^{1/2} . \tag{188}$$

Directly from (186) and (188),

$$\frac{\Lambda(u^2 + v^2)}{2 \, \Delta z_a} \cdot \frac{\left|\widehat{\Delta}z(u,v) - \Delta z_a\right|}{\Delta z_a} \; < \; M \, \frac{(\ln \Lambda)^{3/2}}{\Lambda^{1/2}} . \tag{189}$$

The constant M in (189) is not necessarily the same as the constant M in (188)!

Given the relative similarity between the integrands in (182), we consider their ratio

$$q \; = \; \frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\widehat{\Delta}z(u,v)}\right] \bigg/ \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a}\right] . \tag{190}$$

With some algebra, this ratio can be recast as

$$\begin{aligned}
q \; &= \; \frac{\Delta z_a}{\widehat{\Delta}z(u,v)} \exp\left[\frac{\Lambda\left(u^2 + v^2\right)}{2} \cdot \frac{\widehat{\Delta}z(u,v) - \Delta z_a}{\widehat{\Delta}z(u,v)\Delta z_a}\right] , \\
&= \; \left[1 + \frac{\widehat{\Delta}z(u,v) - \Delta z_a}{\Delta z_a}\right]^{-1} \cdot \exp\left[\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a} \cdot \frac{\widehat{\Delta}z(u,v) - \Delta z_a}{\Delta z_a} \cdot \frac{\Delta z_a}{\widehat{\Delta}z(u,v)}\right] .
\end{aligned} \tag{191}$$

By the estimates in (188) and (189), the prefactor in q approaches unity and the argument of the exponential vanishes as $\Lambda \to \infty$. Hence we can choose Λ sufficiently large so that

$$|q - 1| \; < \; \delta . \tag{192}$$

With this control over the fractional error, the difference between the nonlinear and the usual Gaussian in (182) is bounded by

$$\begin{aligned}
\int_{B_0} &\left|\frac{\Lambda}{2\pi\widehat{\Delta}z(u,v)} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\widehat{\Delta}z(u,v)}\right] - \frac{\Lambda}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a}\right]\right| du \, dv \\
&\leq \; \int_{B_0} \frac{\Lambda \, |q - 1|}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a}\right] du \, dv \\
&< \; \int_{B_0} \frac{\Lambda \, \delta}{2\pi\Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a}\right] du \, dv \; < \; \delta .
\end{aligned} \tag{193}$$

We are left to examine what happens when (u,v) lies outside the ball $B_0 \subset \mathcal{Q}$ of radius $\mathsf{R}_a(\Lambda/2)$, meaning

$$u^2 + v^2 \geq \mathsf{R}_a^2(\Lambda/2) . \tag{194}$$

To start, the bound on the Gaussian in (184) is trivial because the integrand is bounded by δ for all points (u,v) outside the ball of radius $\mathsf{R}_a(\Lambda)$, and $\mathsf{R}_a(\Lambda/2) > \mathsf{R}_a(\Lambda)$. So the real task is to establish the bound for the nonlinear Gaussian in (183).

Consider the following function on $\mathcal{Q}$,

$$\widehat{\Lambda}(u,v) \; = \; \Lambda \cdot \frac{\Delta z_a}{\widehat{\Delta}z(u,v)} . \tag{195}$$

Conceptually, we interpret $\widehat{\Lambda}$ as a fluctuating, position-dependent version of the parameter Λ, so that the width of the nonlinear Gaussian varies from point-to-point on $\mathcal{Q}$. By the estimate in (138), the relative fluctuation factor is

bounded from below everywhere on $\mathcal{Q}$ by

$$\frac{1}{2} < 1 - \mathsf{c} < \frac{\Delta z_a}{\widehat{\Delta} z(u, v)} \,. \tag{196}$$

Consequently, by the definition of $\widehat{\Lambda}$,

$$\frac{\Lambda}{2} < \widehat{\Lambda}(u, v) \,. \tag{197}$$

Associated to the local parameter $\widehat{\Lambda}(u, v)$ we have a local scale $\mathsf{R}_a(\widehat{\Lambda}(u, v))$, also a function of u and v. Since R_a is monotonically decreasing for large Λ, the lower bound on $\widehat{\Lambda}$ in (197) means that

$$\mathsf{R}_a^2(\widehat{\Lambda}(u, v)) < \mathsf{R}_a^2(\Lambda/2) < u^2 + v^2 \,. \tag{198}$$

Thus, again by the definition of $\mathsf{R}_a(\Lambda)$ in (180),

$$\frac{\widehat{\Lambda}}{2\pi \Delta z_a} \exp\left[-\frac{\widehat{\Lambda}\left(u^2 + v^2\right)}{2\,\Delta z_a}\right] < \delta \,, \tag{199}$$

or by substitution from (195),

$$\frac{\Lambda}{2\pi \widehat{\Delta} z(u, v)} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\,\widehat{\Delta} z(u, v)}\right] < \delta \,. \tag{200}$$

The bound on the nonlinear Gaussian in (200) is exactly what we need to control the integral over $\mathcal{Q} - B_0$, so that

$$\int_{\mathcal{Q} - B_0} \frac{\Lambda}{2\pi \widehat{\Delta} z(u, v)} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2 \widehat{\Delta} z(u, v)}\right] du\, dv < M\,\delta \,. \tag{201}$$

Combining with the trivial bound in (178), we deduce that the desired integral of Ψ over $\mathcal{Q}$ can be well-approximated for large Λ by the naive Gaussian integral,

$$\left| \int_{\mathcal{Q}} \Psi - \deg(\varphi_a) \int_{\mathbb{R}^2} \frac{\Lambda}{2\pi \Delta z_a} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\Delta z_a}\right] du\, dv \right| < M\,\delta \,. \tag{202}$$

Integral of Ξ

We are not finished with our error analysis at the crossing, because we still must consider the integral of Ξ over $\mathcal{Q}$ in (173). We will show that the contribution of Ξ is negligible for large Λ,

$$\left| \int_{\mathcal{Q}} \Xi \right| < M\,\delta \,. \tag{203}$$

Explicitly, from the formula in (172), the integral of Ξ is given in the (u, v)-coordinates by

$$\int_{\mathcal{Q}} \Xi = -\int_{\mathcal{Q}} \frac{\Lambda}{4\pi \widehat{\Delta} z(u, v)^2} \, e^{-\Lambda(u^2 + v^2)/2\widehat{\Delta} z(u, v)} \left(u\frac{\partial \widehat{\Delta} z}{\partial u} + v\frac{\partial \widehat{\Delta} z}{\partial v}\right) du\, dv \,. \tag{204}$$

Again, we consider the cases that (u, v) lies inside the ball B_0 and outside the ball B_0 separately. When (u, v) lies outside the ball $B_0 \subset \mathcal{Q}$, then by the definition of $\mathsf{R}_a(\Lambda)$ in (180),

$$\left| \frac{\Lambda}{4\pi \widehat{\Delta} z(u, v)^2} \, e^{-\Lambda(u^2 + v^2)/2\widehat{\Delta} z(u, v)} \left(u\frac{\partial \widehat{\Delta} z}{\partial u} + v\frac{\partial \widehat{\Delta} z}{\partial v}\right) \right| < M\,\delta \,. \tag{205}$$

Here we note that the extra factors of $1/\widehat{\Delta} z$ and $(u\,\partial/\partial u + v\,\partial/\partial v)\,\widehat{\Delta} z$ in (205) are smooth functions bounded independently of Λ on $\mathcal{Q}$. These functions do not alter the bound by δ but are absorbed into the constant M.

Otherwise, for points inside B_0, we have a bound

$$\int_{B_0} \frac{\Lambda}{4\pi \widehat{\Delta} z(u,v)^2} \exp\left[-\frac{\Lambda\left(u^2 + v^2\right)}{2\widehat{\Delta} z(u,v)}\right] du\, dv < M\,, \tag{206}$$

which follows by the same arguments used to produce the estimate in (193). Also, since $(u\,\partial/\partial u + v\,\partial/\partial v)\,\widehat{\Delta} z$ is bounded in B_0 and $|u|, |v| \leq \mathsf{R}_a(\Lambda/2) \to 0$ as $\Lambda \to \infty$, we can always choose Λ so that

$$\left| u\frac{\partial\widehat{\Delta} z}{\partial u} + v\frac{\partial\widehat{\Delta} z}{\partial v} \right| < \delta\,, \tag{207}$$

for all (u,v) in B_0. Combining the bounds in (205), (206), and (207) for outside and inside B_0, we obtain the conclusion in (203).

In summary, these bounds establish the informal localization formula in (119) for any crossing of $\Pi(C)$. $\qquad\square$

Error analysis near the diagonal

Our final goal is to evaluate the self-linking integral over the tubular neighborhood $N_\Delta(\mathsf{w})$ of the diagonal $\Delta \subset T^2$, where $\mathsf{w} \sim \Lambda^{-1/4}$ is the width set in Lemma III.2. For the informal localization computation in (120), we used the leading term in the Taylor expansion of the self-linking integrand near the diagonal to approximate the integral. Depending upon the value of the parameter $\hbar$, we abbreviate this leading term by

$$\Phi_\hbar \overset{\hbar \neq 1}{=} -\operatorname{sgn}(\eta)\, \frac{\Lambda\left(\dot\gamma \times \ddot\gamma\right)}{4\pi\left|1 - \hbar\right|\left|\gamma \times \dot\gamma\right|}\, \exp\left[-\frac{\Lambda\,||\dot\gamma||^2\,|\eta|}{2\left|1 - \hbar\right|\left|\gamma \times \dot\gamma\right|}\right] d\phi \wedge d\eta\,, \tag{208}$$

or

$$\Phi_\hbar \overset{\hbar = 1}{=} -\frac{3\,\Lambda}{2\pi\eta^2} \exp\left[-\frac{3\,\Lambda\,||\dot\gamma||^2}{\left|\dot\gamma \times \ddot\gamma\right|}\frac{1}{|\eta|}\right] d\phi \wedge d\eta\,. \tag{209}$$

In both cases we assume that the local positivity condition $\widehat{\Delta} z > 0$ is satisfied, as in (76). Otherwise, $\Phi_\hbar \equiv 0$.

To justify our localization computation, we must demonstrate for sufficiently large Λ the bound

$$\left| \int_{N_\Delta(\mathsf{w})} (X \times X)^*\widehat{\Gamma}_\hbar^* \chi_\Lambda - \int_{N_\Delta(\mathsf{w})} \Phi_\hbar \right| < M\,\delta\,. \tag{210}$$

The behavior of $\Phi_\hbar$ for small η depends very much on whether $\hbar$ is equal to one or not, so we treat the cases in (208) and (209) separately. Because the generic case $\hbar \neq 1$ is the more involved, and the more interesting, we begin with it.

Generic case $\hbar \neq 1$

In principle, the error in the leading approximation to $(X \times X)^*\widehat{\Gamma}_\hbar^* \chi_\Lambda$ for small η is controlled by the magnitude of the next-order term in the Taylor expansion. We will need this correction term for our analysis. Briefly, by the same computations leading to (75) in Section II A, the argument of the heat kernel admits the second-order expansion

$$\frac{\Delta x^2 + \Delta y^2}{2\widehat{\Delta} z} \overset{\hbar \neq 1}{=} -\frac{||\dot\gamma||^2\,\eta}{2\left(1 - \hbar\right)\left(\gamma \times \dot\gamma\right)}\left[1 + \left(\frac{\dot\gamma \cdot \ddot\gamma}{||\dot\gamma||^2} - \frac{1}{2}\frac{\gamma \times \ddot\gamma}{\gamma \times \dot\gamma}\right)\eta + \mathcal{O}\!\left(\eta^2\right)\right], \tag{211}$$

where $\dot\gamma \cdot \ddot\gamma \equiv \dot x\,\ddot x + \dot y\,\ddot y$. Similarly for the pullback of the heat form χ_Λ itself,

$$
\begin{aligned}
(X \times X)^*\widehat{\Gamma}_\hbar^* \chi_\Lambda \overset{\hbar \neq 1}{=}\ & \\
-\frac{\Lambda\,\operatorname{sgn}(\eta)\,d\phi \wedge d\eta}{4\pi\left|1 - \hbar\right|\left|\gamma \times \dot\gamma\right|} & \left[\left(\dot\gamma \times \ddot\gamma\right) + \left(\frac{1}{2}\left(\dot\gamma \times \dddot\gamma\right) - \frac{\left(\gamma \times \ddot\gamma\right)\left(\dot\gamma \times \ddot\gamma\right)}{\left(\gamma \times \dot\gamma\right)}\right)\eta + \mathcal{O}\!\left(\eta^2\right)\right] \times \\
\times \exp&\left[-\frac{\Lambda\,||\dot\gamma||^2\,|\eta|}{2\left|1 - \hbar\right|\left|\gamma \times \dot\gamma\right|}\left(1 + \left(\frac{\dot\gamma \cdot \ddot\gamma}{||\dot\gamma||^2} - \frac{1}{2}\frac{\gamma \times \ddot\gamma}{\gamma \times \dot\gamma}\right)\eta + \mathcal{O}\!\left(\eta^2\right)\right)\right].
\end{aligned}
\tag{212}
$$

We omit the computation leading to (212), since the details of this formula will not be so important. The expansion merely confirms that both the prefactor and the argument of the exponential for $\Phi_\hbar$ in (208) receive further corrections at the next order in η, as determined by the geometry of the projection $\Pi(C)$.

Validity of the leading approximation $\Phi_\hbar$ requires that the correction terms in (212) be small. At least informally,
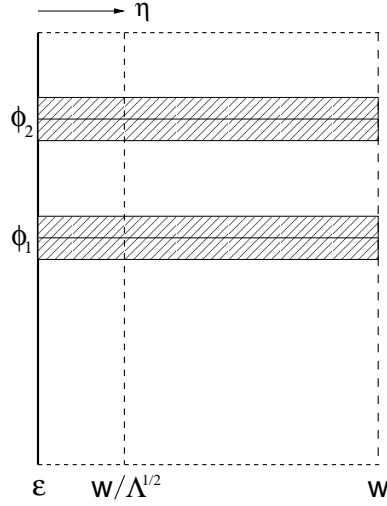
FIG. 21. Domain $N_\Delta(\mathsf{w}; \ell)$ on which $|\gamma \times \dot\gamma| > m > 0$.

for the argument of the heat kernel in (211) we require

$$\left| \left(\frac{\dot\gamma \cdot \ddot\gamma}{||\dot\gamma||^2} - \frac{1}{2} \frac{\gamma \times \ddot\gamma}{\gamma \times \dot\gamma} \right) \eta \right| \ll 1 . \tag{213}$$

By assumption, $|\eta| < \mathsf{w} \sim \Lambda^{-1/4}$ is always small on $N_\Delta(\mathsf{w})$, and $||\dot\gamma||^2 > 0$ is bounded from below. The condition in (213) is therefore only violated at points where $\gamma \times \dot\gamma = 0$. At these points, the Taylor expansion in (212) breaks down.

Despite the failure of the Taylor expansion at points where $\gamma \times \dot\gamma = 0$, these points cause no difficulty. Recall that points where $\gamma \times \dot\gamma = 0$ correspond to critical points of the height function $z(\theta)$ on C. By the Morse assumption which follows Lemma II.1, the function $(\gamma \times \dot\gamma)(\phi)$ vanishes at only a finite number of isolated critical points $\{\phi_1, \ldots, \phi_{2k}\}$ on S^1. According to the analysis at the end of Section II B, these points are precisely the endpoints of the disjoint collection of intervals $S^1_\pm \cap [T^2 - \Delta(\varepsilon)]$ in (89). At the endpoints, $\Phi_\hbar$ vanishes, and the exact integrand $(X \times X)^* \widehat\Gamma^*_\hbar \chi_\Lambda$ in (108) is exponentially small (since $\widehat{\Delta z}$ is small). Consequently, the troublesome points for the Taylor expansion can just be removed from the domain of integration, with negligible error.

Technically, about each critical point ϕ_c,

$$(\gamma \times \dot\gamma)(\phi_c) = 0 , \qquad c = 1, \ldots, 2k , \tag{214}$$

we consider a small neighborhood $(\phi_c - \ell, \phi_c + \ell)$ with fixed width $\ell > 0$. Let $\mathrm{I}_c(\ell) \subset N_\Delta(\mathsf{w})$ be the corresponding closed strip

$$\mathrm{I}_c(\ell) = \left\{ (\phi, \eta) \,\middle|\, |\phi - \phi_c| \le \ell, \, |\eta| \le \mathsf{w} \right\} , \tag{215}$$

and set

$$N_\Delta(\mathsf{w}; \ell) = N_\Delta(\mathsf{w}) - \bigcup_{c=1}^{2k} \mathrm{I}_c(\ell) . \tag{216}$$

See Figure 21 for a sketch of $N_\Delta(\mathsf{w}; \ell)$ near one boundary of the cylinder $T^2 - \Delta(\varepsilon)$. The shaded regions indicate the strips of width ℓ which have been excised about a pair of zeroes ϕ_1 and ϕ_2 of the function $\gamma \times \dot\gamma$.

We choose the width $\ell > 0$ of each strip to be small enough so that

$$\left| \sum_{c=1}^{2k} \int_{\mathrm{I}_c(\ell)} (X \times X)^* \widehat\Gamma^*_\hbar \chi_\Lambda \right| < \delta , \qquad \left| \sum_{c=1}^{2k} \int_{\mathrm{I}_c(\ell)} \Phi_\hbar \right| < \delta . \tag{217}$$

Both integrands in (217) are bounded at the points ϕ_c, so the integrals over $\mathrm{I}_c(\ell)$ can be made as small as desired by the choice of ℓ. Moreover, both integrands are decreasing functions of Λ for sufficiently large Λ, so the width ℓ can be chosen independently of Λ, our crucial requirement.

By definition, the function $(\gamma \times \dot{\gamma})(\phi)$ is now bounded away from zero everywhere on the new domain $N_\Delta(\mathsf{w}; \ell)$,

$$|\gamma \times \dot{\gamma}| \, > \, m \, > \, 0 \quad \text{on} \quad N_\Delta(\mathsf{w}; \ell) \,. \tag{218}$$

Here m is a constant which depends upon the curve C and the parameter ℓ, but not on Λ. Because the respective contributions (217) from the excised strips are small by assumption, we are free to replace $N_\Delta(\mathsf{w})$ by $N_\Delta(\mathsf{w}; \ell)$ in the inequality (210) to be proven. On the other hand, due to the lower bound in (218), we will also have uniform control of error terms such as (213) in the Taylor approximation on $N_\Delta(\mathsf{w}; \ell)$.

The remainder of the discussion proceeds in rough correspondence to the asymptotic analysis near a crossing. By analogy to the ball B_0 in (181), we introduce a smaller tubular neighborhood $N_0 \subset N_\Delta(\mathsf{w}; \ell)$ defined by

$$N_0: \quad |\eta| \, < \, \frac{\mathsf{w}}{\sqrt{\Lambda}} \, \sim \, \Lambda^{-3/4} \,. \tag{219}$$

We indicate the coaxial configuration schematically in Figure 21. For points inside the small tube N_0, we will show that the difference between the integrals of $(X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda$ and $\Phi_\hbar$ is small,

$$\left| \int_{N_0} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \, - \, \int_{N_0} \Phi_\hbar \right| \, < \, M \, \delta \,. \tag{220}$$

For points outside N_0 but inside $N_\Delta(\mathsf{w}; \ell)$, we will show that both integrals are separately small, with

$$\left| \int_{N_\Delta(\mathsf{w}; \ell) - N_0} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \right| \, < \, M \, \delta \,, \tag{221}$$

and

$$\left| \int_{N_\Delta(\mathsf{w}; \ell) - N_0} \Phi_\hbar \right| \, < \, M \, \delta \,. \tag{222}$$

The extra factor of $1/\sqrt{\Lambda}$ in the definition of N_0 is simply what is needed to ensure the inequality in (220).

We first consider the points inside N_0. From the second-order expansion in (212), the ratio of the self-linking integrand to its approximation $\Phi_\hbar$ satisfies

$$\begin{aligned}
q \, = \, \frac{(X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda}{\Phi_\hbar} \, &= \, \left[1 + \left(\frac{1}{2} \frac{\dot{\gamma} \times \dddot{\gamma}}{\dot{\gamma} \times \ddot{\gamma}} - \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta + \mathcal{O}(\eta^2) \right] \times \\
&\quad \times \exp\left[-\frac{\Lambda \, ||\dot{\gamma}||^2 \, |\eta|}{2 \, |1 - \hbar| \, |\gamma \times \dot{\gamma}|} \left(\frac{\dot{\gamma} \cdot \ddot{\gamma}}{||\dot{\gamma}||^2} - \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right) \eta + \mathcal{O}(\eta^3) \right] \,.
\end{aligned} \tag{223}$$

Again, to deal with the term $1/(\dot{\gamma} \times \ddot{\gamma})$ in the prefactor of (223), we assume that any zeroes of $\dot{\gamma} \times \ddot{\gamma}$ are isolated, and we remove small neighborhoods as necessary about those zeroes so that the functions which multiply η in both the prefactor and the argument of the exponential in (223) are bounded, independently of Λ.

For any point in the small tube N_0, the argument of the exponential in (223) is bounded in magnitude by

$$\frac{\Lambda \, ||\dot{\gamma}||^2 \, \eta^2}{2 \, |1 - \hbar| \, |\gamma \times \dot{\gamma}|} \cdot \left| \frac{\dot{\gamma} \cdot \ddot{\gamma}}{||\dot{\gamma}||^2} - \frac{1}{2} \frac{\gamma \times \ddot{\gamma}}{\gamma \times \dot{\gamma}} \right| \, < \, \Lambda \, M \, \eta^2 \, < \, M \, \mathsf{w}^2 \,, \tag{224}$$

where we apply the conditions $|\gamma \times \dot{\gamma}| > m > 0$ as well as $|\eta| < \mathsf{w}/\sqrt{\Lambda}$ in N_0. Because $\mathsf{w} \sim \Lambda^{-1/4}$, this inequality means that the argument of the exponential vanishes, and the prefactor approaches unity, in the limit $\Lambda \to \infty$. Thus for sufficiently large Λ, the fractional error is small,

$$|q - 1| \, < \, \delta \,. \tag{225}$$

By the same idea in (193),

$$\left| \int_{N_0} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda - \int_{N_0} \Phi_\hbar \right| \le |q-1| \cdot \left| \int_{N_0} \Phi_\hbar \right| < M\,\delta\,, \qquad (226)$$

since we already know the integral of $\Phi_\hbar$ to be bounded and independent of Λ by the local computation in Section III A.

The inequalities for points outside N_0 are even easier.

From the explicit expression for $\Phi_\hbar$ in (208),

$$\left| \Phi_\hbar \right| < A\,\Lambda \exp\left[- B\,\Lambda\,|\eta| \right]\,, \qquad A, B > 0\,, \qquad (227)$$

for some positive constants A and B. So in the allowed range $\Lambda^{-1/2}\,\mathsf{w} \le |\eta| \le \mathsf{w}$ on the complement of N_0,

$$\begin{aligned} \left| \Phi_\hbar \right| &< A\,\Lambda \exp\left[-B\,\Lambda^{1/2}\,\mathsf{w} \right]\,, \qquad \mathsf{w} = m\,\Lambda^{-1/4}\,, \\ &= A\,\Lambda \exp\left[-m\,B\,\Lambda^{1/4} \right]\,. \end{aligned} \qquad (228)$$

By taking Λ sufficiently large, we can make the magnitude of $\Phi_\hbar$ as small as desired on the complement of N_0 inside $N_\Delta(\mathsf{w}; \ell)$, from which the bound in (222) follows.

To establish a similar bound for the pullback of χ_Λ in (212), observe that the argument of the exponential obeys

$$\frac{||\dot\gamma||^2\,|\eta|}{2\,|1-\hbar|\,|\gamma \times \dot\gamma|} \left[1 + \left(\frac{\dot\gamma \cdot \ddot\gamma}{||\dot\gamma||^2} - \frac{1}{2}\frac{\gamma \times \ddot\gamma}{\gamma \times \dot\gamma} \right) \eta \right] > B\,|\eta|\,, \qquad B > 0\,, \qquad (229)$$

provided that Λ is sufficiently large and η sufficiently small. Here $B > 0$ is a suitable positive constant. Then according to the expansion in (212),

$$\left| (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \right| < A\,\Lambda \exp\left[- B\,\Lambda\,|\eta| \right]\,, \qquad A, B > 0\,, \qquad (230)$$

exactly as for the preceding bound on $\Phi_\hbar$ in (227). The claim in (221) now follows by an identical argument.

In total, the three inequalities in (220), (221), and (222) finish the proof of the localization formula for $\mathsf{slk}_\kappa(C)|_\Delta$ in the generic case $\hbar \ne 1$. $\quad\square$

Symmetric case $\hbar = 1$

For the Heisenberg-symmetric value $\hbar = 1$, the localization formula from Section III A states $\mathsf{slk}_\kappa(C)|_\Delta = 0$. Consistent with this result, we establish the basic bound in (210) by showing individually

$$\left| \int_{N_\Delta(\mathsf{w})} (X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \right| < \delta\,, \qquad (231)$$

and

$$\left| \int_{N_\Delta(\mathsf{w})} \Phi_\hbar \right| < \delta\,. \qquad (232)$$

Our workhorse is the next-order expansion of the self-linking integrand, which behaves differently for $\hbar = 1$. For the argument of the heat kernel, the calculations in Section II A yield

$$\frac{\Delta x^2 + \Delta y^2}{2\,\widehat{\Delta}z} \overset{\hbar=1}{\equiv} -\frac{3}{\eta} \cdot \left[\frac{||\dot\gamma||^2 + (\dot\gamma \cdot \ddot\gamma)\,\eta + \mathcal{O}(\eta^2)}{(\dot\gamma \times \ddot\gamma) + \frac{1}{2}(\dot\gamma \times \dddot\gamma)\,\eta + \mathcal{O}(\eta^2)} \right]\,. \qquad (233)$$

Similarly,

$$(X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda \overset{\hbar=1}{=} -\frac{3\Lambda \, d\phi \wedge d\eta}{2\pi\eta^2} \left[1 + \mathcal{O}(\eta^2)\right] \times$$
$$\times \exp\left[-\frac{3\,\Lambda}{|\eta|} \cdot \left| \frac{||\dot\gamma||^2 + (\dot\gamma \cdot \ddot\gamma)\,\eta + \mathcal{O}(\eta^2)}{(\dot\gamma \times \ddot\gamma) + \frac{1}{2}(\dot\gamma \times \dddot\gamma)\,\eta + \mathcal{O}(\eta^2)} \right| \right]. \tag{234}$$

For sake of brevity, we omit the calculation leading to (234). The details of this formula are not important.[4]

For the approximation $\Phi_\hbar$, recall the formula

$$\Phi_\hbar \overset{\hbar=1}{=} -\frac{3\Lambda}{2\pi\eta^2} \exp\left[-\frac{3\,\Lambda\,||\dot\gamma||^2}{|\dot\gamma \times \ddot\gamma|}\frac{1}{|\eta|}\right] d\phi \wedge d\eta. \tag{235}$$

Then $\Phi_\hbar$ vanishes smoothly for $\eta = 0$ and otherwise satisfies

$$\left|\Phi_\hbar\right| < \frac{M\Lambda}{\eta^2} \exp\left[-\frac{A\,\Lambda}{|\eta|}\right], \qquad A, M > 0, \tag{236}$$

for some positive constants A and M. For Λ sufficiently large, $\Phi_\hbar$ can be made as small as desired everywhere on $N_\Delta(\mathsf{w})$, and the inequality in (232) holds.

To treat the pullback of χ_Λ in the same fashion, note that the denominator in (233) obeys

$$\left|(\dot\gamma \times \ddot\gamma)\,\eta + \frac{1}{2}(\dot\gamma \times \dddot\gamma)\,\eta^2\right| < A\,|\eta| + B\,|\eta|^2 < 2\,A\,|\eta|, \tag{237}$$

provided $|\eta|$ is sufficiently small (with $B|\eta| < A$), as holds when Λ is sufficiently large. Also in this regime, the numerator in (233) is bounded from below by

$$\left|\,||\dot\gamma||^2 + (\dot\gamma \cdot \ddot\gamma)\,\eta\,\right| > m > 0. \tag{238}$$

Hence on the tubular neighborhood $N_\Delta(\mathsf{w})$,

$$\left|(X \times X)^* \widehat{\Gamma}_\hbar^* \chi_\Lambda\right| < \frac{M\Lambda}{\eta^2} \exp\left[-\frac{3\,m\,\Lambda}{2\,A\,|\eta|}\right]. \tag{239}$$

This inequality has the same shape as that for $\Phi_\hbar$ in (236), from which we reach the conclusion in (231).

The proof of Theorem I.2 is complete. $\square$

---

[1] D. Bar-Natan, *Perturbative Aspects of Chern-Simons Topological Quantum Field Theory*, Ph.D. thesis, Princeton University, 1991.

[2] C. Beasley and E. Witten, "Non-Abelian Localization for Chern-Simons Theory," J. Diff. Geom. **70** (2005) 183–323, hep-th/0503126.

[3] C. Beasley, B. McLellan, and R. Zhang, "Self-Linking for Legendrian Knots," to appear.

---

[4] Curiously, the order-η correction to the prefactor in (234) vanishes when $\hbar = 1$.

[4] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers, I: Asymptotic Methods and Perturbation Theory*, Springer-Verlag, New York, 1999.

[5] R. Bott and C. Taubes, "On the Self-Linking of Knots," J. Math. Phys. **35** (1994) 5247–5287.

[6] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York, 1982.

[7] G. Călugăreanu, "L'intégrale de Gauss et l'analyse des noeuds tridimensionnels," Rev. Math. Pures Appl. **4** (1959) 5–20.

[8] M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.

[9] J. B. Etnyre, "Introductory Lectures on Contact Geometry," in *Topology and Geometry of Manifolds (Athens, GA 2001)*, Proc. Sympos. Pure Math. **71**, Amer. Math. Soc., Providence, RI, 2003, `arXiv:math/0111118 [math.SG]`.

[10] J. B. Etnyre, "Legendrian and Transversal Knots," in *Handbook of Knot Theory*, pp.105–185, Elsevier B.V., Amsterdam, 2005, `arXiv:0306256 [math.SG]`.

[11] D. Fuchs and S. Tabachnikov, "Invariants of Legendrian and Transverse Knots in the Standard Contact Space," Topology **36** (1997) 1025–1053.

[12] H. Geiges, "Contact Geometry," in *Handbook of Differential Geometry, Vol.2*, pp.315–382, Ed. by F. Dillen and L. Verstraelen, North-Holland, Amsterdam, 2006, `arXiv:math/0307242 [math.SG]`.

[13] H. Geiges, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge University Press, 2008.

[14] E. Guadagnini, M. Martellini, and M. Mintchev, "Wilson Lines in Chern-Simons Theory and Link Invariants," Nucl. Phys. B **330** (1990) 575–607.

[15] W. Pohl, "The Self-Linking Number of a Closed Space Curve," J. Math. Mech. **17** (1967) 975–985.

[16] A. M. Polyakov, "Fermi-Bose Transmutations Induced by Gauge Fields," Mod. Phys. Lett. A **3** (1988) 325–331.

[17] H. Whitney, "On Regular Closed Curves on the Plane," Compositio Math. **4** (1937) 276–284.

[18] E. Witten, "Quantum Field Theory and the Jones Polynomial," Commun. Math. Phys. **121** (1989) 351–399.

[19] R. Zhang, *Gauge Theory and Self-Linking of Legendrian Knots*, Ph.D. Thesis, Northeastern University, May 2016, `hdl.handle.net/2047/D20213322`.