On the Geodesic Centers of Polygonal Domains

Haitao Wang*

Department of Computer Science, Utah State University, Logan, UT, USA
haitao.wang@usu.edu

Abstract

In this paper, we study the problem of computing Euclidean geodesic centers of a polygonal domain $\mathcal{P}$ of n vertices. We give a necessary condition for a point being a geodesic center. We show that there is at most one geodesic center among all points of $\mathcal{P}$ that have topologically-equivalent shortest path maps. This implies that the total number of geodesic centers is bounded by the size of the shortest path map equivalence decomposition of $\mathcal{P}$, which is known to be $O(n^{10})$. One key observation is a π-*range property* on shortest path lengths when points are moving. With these observations, we propose an algorithm that computes all geodesic centers in $O(n^{11} \log n)$ time. Previously, an algorithm of $O(n^{12+\epsilon})$ time was known for this problem, for any $\epsilon > 0$.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems, I.1.2 Algorithms, I.3.5 Computational Geometry and Object Modeling

Keywords and phrases geodesic centers, shortest paths, polygonal domains

Digital Object Identifier 10.4230/LIPIcs.ESA.2016.77

1 Introduction

Let $\mathcal{P}$ be a polygonal domain with a total of h holes and n vertices, i.e., $\mathcal{P}$ is a multiply-connected region whose boundary is a union of n line segments, forming $h+1$ closed polygonal cycles. A simple polygon is a special case of a polygonal domain with $h = 0$. For any two points s and t, a *shortest path* or *geodesic path* from s to t is a path in $\mathcal{P}$ whose Euclidean length is minimum among all paths from s to t in $\mathcal{P}$; we let $d(s,t)$ denote the Euclidean length of any shortest path from s to t and we also say that $d(s,t)$ is the *shortest path distance or geodesic distance* from s to t.

A point s is a *geodesic center* of $\mathcal{P}$ if s minimizes the value $\max_{t \in \mathcal{P}} d(s,t)$, i.e., the maximum geodesic distance from s to all points of $\mathcal{P}$.

In this paper, we study the problem of computing the geodesic centers of $\mathcal{P}$. The problem in simple polygons has been well studied. It is known that for any point in a simple polygon, its farthest point must be a vertex of the polygon [20]. It has been shown that the geodesic center in a simple polygon is unique and has at least two farthest points [18]. Due to these helpful observations, efficient algorithms for finding geodesic centers in simple polygons have been developed. Asano and Toussaint [2] gave an $O(n^4 \log n)$ time algorithm for the problem, and later Pollack, Sharir, and Rote [18] improved the algorithm to $O(n \log n)$ time. Recently, Ahn et al. [1] solved the problem in linear time.

Finding a geodesic center in a polygonal domain $\mathcal{P}$ is much more difficult. This is partially due to that a farthest point of a point in $\mathcal{P}$ may be in the interior of $\mathcal{P}$ [3]. Also, it is easy to construct an example where the geodesic center of $\mathcal{P}$ is not unique (e.g., see Fig. 1). Bae, Korman, and Okamoto [4] gave the first-known algorithm that can compute a geodesic center

* This research was supported in part by NSF under Grant CCF-1317143.

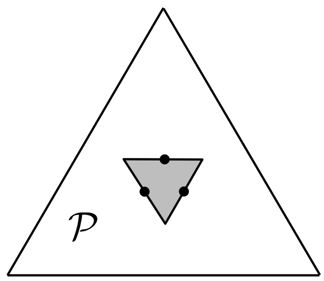

Figure 1 The boundary of $\mathcal{P}$ consists of an outer and inner equilateral triangles with their geometric centers co-located. Each of the three thick points is a geodesic center of $\mathcal{P}$.

in $O(n^{12+\epsilon})$ time for any $\epsilon > 0$. They first showed that for any point its farthest points must be vertices of its shortest path map in $\mathcal{P}$. Then, they considered the shortest path map equivalence decomposition (or SPM-equivalence decomposition) [7], denoted by $\mathcal{D}_{spm}$; for each cell of $\mathcal{D}_{spm}$, they computed the upper envelope of $O(n)$ graphs in 3D space, which takes $O(n^{2+\epsilon})$ time [10], to search a geodesic center in the cell. Since the size of $\mathcal{D}_{spm}$ is $O(n^{10})$ [7], their algorithm runs in $O(n^{12+\epsilon})$ time.

A closely related concept is the *geodesic diameter*, which is the maximum geodesic distance over all pairs of points in $\mathcal{P}$, i.e., $\max_{s,t,\in\mathcal{P}} d(s,t)$. In simple polygons, due to the property that there always exists a pair of vertices of $\mathcal{P}$ whose geodesic distance is equal to the geodesic diameter, efficient algorithms have been given for computing the geodesic diameters. Chazelle [6] gave the first algorithm that runs in $O(n^2)$ time. Later, Suri [20] presented an $O(n \log n)$-time algorithm. The problem was eventually solved in $O(n)$ time by Hershberger and Suri [11]. Computing the geodesic diameter in a polygonal domain $\mathcal{P}$ is much more difficult, partially because the diameter can be realized by two points in the interior of $\mathcal{P}$, in which case there are at least five distinct shortest paths between the two points [3]. As for the geodesic center, this makes it difficult to discretize the search space. By an exhaustive-search method, Bae, Korman, and Okamoto [3] gave the first algorithm for computing the diameter of $\mathcal{P}$, which runs in $O(n^{7.73})$ or $O(n^7(\log n + h))$ time.

Refer to [5, 8, 13, 15, 16, 17, 19] for other variations of geodesic diameter and center problems (e.g., the L_1 metric and the link distance case).

1.1 Our Contributions

We conduct a "comprehensive" study on geodesic centers of $\mathcal{P}$. We discover many interesting observations, and some of them may be even surprising. For example, we show that even if a geodesic center is in the interior of $\mathcal{P}$, it may have only one farthest point, which is somewhat counter-intuitive. We give a necessary condition for a point being a geodesic center. We show that there is at most one geodesic center among all points of $\mathcal{P}$ that have topologically-equivalent shortest path maps in $\mathcal{P}$. This immediately implies that the interior of each cell or each edge of the SPM-equivalence decomposition $\mathcal{D}_{spm}$ can contain at most one geodesic center, and thus, the total number of geodesic centers of $\mathcal{P}$ is bounded by the combinatorial size of $\mathcal{D}_{spm}$, which is known to be $O(n^{10})$ [7]. Previously, the only known upper bound on the total number of geodesic centers of $\mathcal{P}$ is $O(n^{12})$, which is suggested by the algorithm in [4].

These observations are all more or less due to an interesting observation, which we call the *π-range property* and is one key contribution of this paper. Here we only demonstrate an application of the π-range property. Let s and t be two points in the interior of $\mathcal{P}$ such

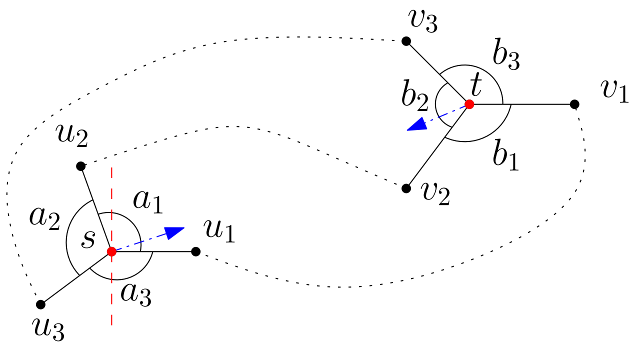

Figure 2 Illustrating the π-range property. Suppose there are three shortest s-t paths through vertices u_i and v_i with $i = 1, 2, 3$, respectively. If s and t move along the blue arrows simultaneously (possibly with different speeds), then all three shortest paths strictly decreases (it is difficult to tell whether this is true from the figure, so these two blue directions here are only for illustration purpose). The special case happens when the six angles a_i and b_i satisfy $a_i = b_i$ for $i = 1, 2, 3$.

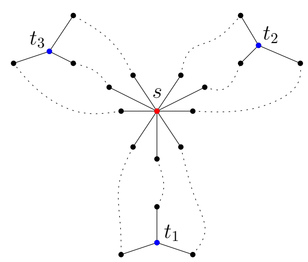

Figure 3 Illustrating a geodesic center s with three farthest points t_1, t_2, t_3 such that all these four points are in the interior of $\mathcal{P}$. There are three shortest paths from s to each of t_1, t_2, t_3.

that t is a farthest point of s in $\mathcal{P}$. Refer to Fig. 2 for an example. Suppose there are three shortest paths from s to t as shown in Fig. 2. The π-range property says that unless a special case happens, there exists an open range of exactly size π (e.g., delimited by the right open half-plane bounded by the vertical line through s in Fig. 2) such that if s moves along any direction in the range for an infinitesimal distance, we can always find a direction to move t such that the lengths of *all three* shortest paths strictly decrease. Further, if the special case does not happen, we can explicitly determine the above range of size π. In fact, it is the special case that makes it possible for a geodesic center having only one farthest point.

With these observations, we propose an exhaustive-search algorithm to compute a set S of *candidate points* such that all geodesic centers must be in S. For example, refer to Fig. 3, where a geodesic center s has three farthest points t_1, t_2, t_3 and all these four points are in the interior of $\mathcal{P}$. The nine shortest paths from s to t_1, t_2, t_3 provide a system of eight equations, which give eight (independent) constraints that can determine the four points s, t_1, t_2, t_3 if we consider the coordinates of these points as eight variables. This suggests our exhaustive-search to compute candidate points for such a geodesic center s. However, if a geodesic center s has only one farthest point (e.g., Fig. 2), then we have only three shortest paths (in the non-degenerate case), which give only two constraints. In order to determine s and t, which have four variables, we need two more constraints. It turns out the π-range property (i.e., the special case) provides exactly two more constraints (on the angles as shown in Fig. 2). In this way, we can still compute candidate points for such s. Also, if s has two farthest points, we will need one more constraint, which is also provided by the π-range property (the non-special case).

The number of candidate points in S is $O(n^{11})$. To find all geodesic centers from S, a straightforward solution is to compute the shortest path map for every point of S, which takes $O(n^{12} \log n)$ time in total. Again, with the help of the π-range property, we propose a *pruning algorithm* to eliminate most points from S in $O(n^{11} \log n)$ time such that none of the eliminated points is a geodesic center and the number of the remaining points of S is only $O(n^{10})$. Consequently, we can find all geodesic centers in additional $O(n^{11} \log n)$ time.

Although we improve the previous $O(n^{12+\epsilon})$ time algorithm in [4] by a factor of roughly $n^{1+\epsilon}$, the running time is still huge. We feel that our observations (e.g., the π-range property) may be more interesting than the algorithm itself. We suspect that they may also find other applications. The paper is lengthy, which is mainly due to a considerable number of cases depending on whether a geodesic center and its farthest points are in the interior, on an edge, or at vertices of $\mathcal{P}$, although the essential idea is quite similar for all these cases.

The rest of the paper is organized as follows. In Section 2, we introduce notation and review some concepts. In Section 3, we give our observations. We present the π-range property in Section 4. Computing the candidate points is discussed in Section 5. Finally, we find all geodesic centers from the candidate points in Section 6. Due to the space limit, many details and proofs are omitted but can be found in the full paper.

2 Preliminaries

Consider any point $s \in \mathcal{P}$. Let $d_{\max}(s)$ be the maximum geodesic distance from s to all points of $\mathcal{P}$, i.e., $d_{\max}(s) = \max_{t\in\mathcal{P}} d(s,t)$. A point $t \in \mathcal{P}$ is a *farthest point* of s if $d(s,t) = d_{\max}(s)$. Let $F(s)$ denote the set of all farthest points of s. For any two points p and q in $\mathcal{P}$, we say that p is *visible* to q if the line segment $\overline{pq}$ is in $\mathcal{P}$ and the interior of $\overline{pq}$ does not contain any vertex of $\mathcal{P}$. We use $|pq|$ to denote the (Euclidean) length of any line segment $\overline{pq}$. Note that two points s and t in $\mathcal{P}$ may have more than one shortest path between them, and if not specified, we use $\pi(s,t)$ to denote any such shortest path.

For simplicity of discussion, we make a general position assumption that any two vertices of $\mathcal{P}$ have only one shortest path and no three vertices of $\mathcal{P}$ are on the same line.

Denote by $\mathcal{I}$ the set of all interior points of $\mathcal{P}$, $\mathcal{V}$ the set of all vertices of $\mathcal{P}$, and E the set of all relatively interior points on the edges of $\mathcal{P}$ (i.e., E is the boundary of $\mathcal{P}$ minus $\mathcal{V}$).

Shortest path maps. A *shortest path map* of a point $s \in \mathcal{P}$ [7], denoted by $\mathit{SPM}(s)$, is a decomposition of $\mathcal{P}$ into regions (or cells) such that in each cell σ, the combinatorial structures of shortest paths from s to all points t in σ are the same, and more specifically, the sequence of obstacle vertices along $\pi(s,t)$ is fixed for all t in σ. Further, the *root* of σ, denoted by $r(\sigma)$, is the last vertex of $\mathcal{V} \cup \{s\}$ in the path $\pi(s,t)$ for any point $t \in \sigma$ (hence $\pi(s,t) = \pi(s,r(\sigma)) \cup \overline{r(\sigma)t}$; note that $r(\sigma)$ is s if s is visible to t). As in [7], we classify each edge of σ into three types: a portion of an edge of $\mathcal{P}$, *an extension segment*, which is a line segment extended from $r(\sigma)$ along the opposite direction from $r(\sigma)$ to the vertex of $\pi(s,t)$ preceding $r(\sigma)$, and *a bisector curve/edge* that is a hyperbolic arc. For each point t in a bisector edge of $\mathit{SPM}(s)$, t is on the common boundary of two cells and there are two shortest paths from s to t through the roots of the two cells, respectively (and neither path contains both roots). The *vertices* of $\mathit{SPM}(s)$ include $\mathcal{V} \cup \{s\}$ and all intersections of edges of $\mathit{SPM}(s)$. If a vertex t of $\mathit{SPM}(s)$ is an intersection of two or more bisector edges, then there are more than two shortest paths from s to t. The map $\mathit{SPM}(s)$ has $O(n)$ vertices, edges, and cells, and can be computed in $O(n \log n)$ time [12]. It was shown [4] that any farthest point of s in $\mathcal{P}$ must be a vertex of $\mathit{SPM}(s)$. For differentiation, we will refer to the vertices of $\mathcal{V}$ as *polygon vertices* and refer to the edges of E as *polygon edges*.

The SPM-equivalence decomposition $\mathcal{D}_{spm}$ [7] is a subdivision of $\mathcal{P}$ into regions such that for all points s in the interior of the same region or edge of $\mathcal{D}_{spm}$, the shortest path maps of s are topologically equivalent. Chiang and Mitchell [7] showed that the combinatorial size of $\mathcal{D}_{spm}$ is bounded by $O(n^{10})$ and $\mathcal{D}_{spm}$ can be computed in $O(n^{11})$ time.

Directions and ranges. We will have intensive discussions on moving points along certain directions. For any direction r, we represent r by the angle $\alpha(r) \in [0, 2\pi)$ counterclockwise from the positive direction of the x-axis. For convenience, whenever we are talking about an angle α, unless otherwise specified, depending on the context we may refer to any angle $\alpha + 2\pi \cdot k$ for $k \in \mathbb{Z}$. For any two angles α_1 and α_2 with $\alpha_1 \leq \alpha_2 < \alpha_1 + 2\pi$, the interval $[\alpha_1, \alpha_2]$ represents a *direction range* that includes all directions whose angles are in $[\alpha_1, \alpha_2]$, and $\alpha_2 - \alpha_1$ is called the *size* of the range. Note that the range can also be open (e.g., (α_1, α_2)) and the size of any direction range is no more than 2π.

Consider a half-plane h whose bounding line is through a point s in the plane. We say h *delimits* a range of size π of directions for s that consists of all directions along which s will move towards inside h. If h is an open half-plane, then the range is open as well.

A direction r for $s \in \mathcal{P}$ is called a *free direction* of s if we move s along r for an infinitesimal distance then s is still in $\mathcal{P}$. We use $R_f(s)$ to denote the range of all free directions of s. Clearly, if $s \in \mathcal{I}$, $R_f(s)$ contains all directions; if $s \in E$, $R_f(s)$ is a (closed) range of size π; if $s \in V$, $R_f(s)$ is delimited by the two incident polygon edges of s.

3 Observations

Consider any point $s \in \mathcal{P}$ and let t be any farthest point of s. Recall that t is a vertex of $\mathit{SPM}(s)$ [4]. Suppose we move s infinitesimally along a free direction r to a new point s'. Since $|ss'|$ is infinitesimal, we can assume that s and s' are in the same cell σ of $\mathcal{D}_{spm}$. Further, if s is in the interior of σ, then s' is also in the interior of σ. Regardless of whether s is in the interior of σ or not, there is a vertex $t' \in \mathit{SPM}(s')$ *corresponding to* the vertex t of $\mathit{SPM}(s)$ in the following sense [7]: If the line segment $\overline{s't'}$ is a shortest path from s' to t', then $\overline{st}$ is a shortest path from s to t; otherwise, if $s', u_1, u_2, \ldots, u_k, t'$ is the sequence of the vertices of $\mathcal{V} \cup \{s', t'\}$ in a shortest path from s' to t', then $s, u_1, u_2, \ldots, u_k, t$ is also the sequence of the vertices of $\mathcal{V} \cup \{s, t\}$ in a shortest path from s to t.

In the case that s is on the boundary of σ while s' is in the interior of σ, there might be more than one such vertex $t' \in \mathcal{D}_{spm}$ corresponding to t (refer to [7] for the details) and we use $M_t(s')$ to denote the set of all such vertices t'. We should point out that although a vertex in $\mathit{SPM}(s)$ may correspond to more than one vertex in $\mathit{SPM}(s')$, any vertex in $\mathit{SPM}(s')$ can correspond to one and only one vertex in $\mathit{SPM}(s)$ (because s' is always in the interior of σ).

We introduce the following definition which is crucial to the paper.

▶ **Definition 1.** A free direction r is an *admissible direction* of s with respect to t if as we move s infinitesimally along r to a new point s', $d(s', t') < d(s, t)$ holds for each $t' \in M_t(s')$.

For any $t \in F(s)$, let $R(s, t)$ denote the set of all admissible directions of s with respect to t; let $R(s) = \bigcap_{t \in F(s)} R(s, t)$. The following Lemma 3, which gives a necessary condition for a point being a geodesic center of $\mathcal{P}$, explains why we consider admissible directions. Before presenting Lemma 3, we introduce some notation and Observation 2.

Consider any two points s and t in $\mathcal{P}$. Suppose the vertices of $\mathcal{V} \cup \{s, t\}$ along a shortest s-t path $\pi(s, t)$ are $s = u_0, u_1, \ldots, u_k = t$. By our definition of "visibility", s is visible to t if and only if $k = 1$. If s is not visible to t, then $k \neq 1$ and we call u_1 an *s-pivot* and u_{k-1} a *t-pivot* of $\pi(s, t)$. It is possible that there are multiple shortest paths between s and t, and thus there might be multiple s-pivots and t-pivots for (s, t). We use $U_s(t)$ and $U_t(s)$ to denote the sets of all s-pivots and t-pivots for (s, t), respectively. By our above definition, for any $u \in U_s(t)$, the line segment $\overline{su}$ does not contain any polygon vertex in its interior.

We have the following observation. Similar results have been given in [3].

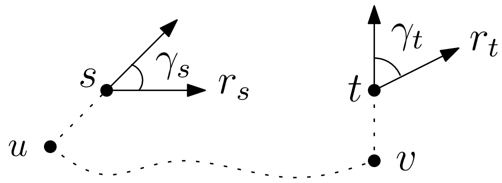

Figure 4 The definitions of the angles γ_s and γ_t.

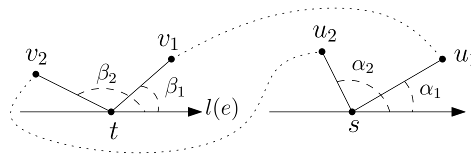

Figure 5 The definitions of the angles.

▶ **Observation 2.** Suppose t is a farthest point of a point s.

1. If t is in $\mathcal{I}$, then $|U_t(s)| \geq 3$ and t must be in the interior of the convex hull of the vertices of $U_t(s)$.
2. If t is in E, say, $t \in e$ for a polygon edge e of E, then $|U_t(s)| \geq 2$ and $U_t(s)$ has at least one vertex in the open half-plane bounded by the supporting line of e and containing the interior of $\mathcal{P}$ in the small neighborhood of e. Further, $U_t(s)$ has at least one vertex in each of the two open half-planes bounded by the line through t and perpendicular to e.

▶ **Lemma 3.** *If s is a geodesic center of $\mathcal{P}$, then $R(s) = \emptyset$.*

As explained in Section 1, we will compute candidate points for geodesic centers. As a necessary condition, Lemma 3 will be helpful for computing those candidate points.

Consider any point $s \in \mathcal{P}$. Let t be a farthest point of s. To determine the admissible direction range $R(s,t)$, we will give a sufficient condition for a direction being in $R(s,t)$. We first assume that s is not visible to t, and as will be seen later, the other case is trivial.

Let u and v respectively be the s-pivot and the t-pivot of (s,t) in a shortest s-t path $\pi(s,t)$. Clearly, $d(s,t) = |su| + d(u,v) + |vt|$. We define $d_{u,v}(s,t) = |su| + d(u,v) + |vt|$ as a function of $s \in \mathbb{R}^2$ and $t \in \mathbb{R}^2$. Suppose we move s along a free direction r_s with the unit speed and move t along a free direction r_t with a speed $\tau \geq 0$. Let γ_s denote the smaller angle between the following two rays originated from s (e.g., see Fig. 4): one with direction r_s and one with direction from u to s. Similarly, let γ_t denote the smaller angle between the following two rays originated from t: one with direction r_t and one with direction from v to t. In fact, as discussed in [3], if we consider $d(s,t)$ as a four-variate function, the triple (r_s, r_t, τ) corresponds to a vector ρ in $\mathbb{R}^4$, and the directional derivative of $d_{u,v}(s,t)$ at $(s,t) \in \mathbb{R}^4$ along ρ, denoted by $d'_{u,v}(s,t)$, and the second directional derivative of $d_{u,v}(s,t)$ at (s,t) along ρ, denoted by $d''_{u,v}(s,t)$, are

$$d'_{u,v}(s,t) = \cos\gamma_s + \tau\cos\gamma_t, \quad d''_{u,v}(s,t) = \frac{\sin^2\gamma_s}{|su|} + \tau\cdot\frac{\sin^2\gamma_t}{|tv|}. \tag{1}$$

Since $\tau \geq 0$, $d''_{u,v}(s,t) \geq 0$. Further, if $\tau \neq 0$, then $d''_{u,v}(s,t) = 0$ if and only if $\sin^2\gamma_s = \sin^2\gamma_t = 0$, i.e., each of γ_s and γ_t is either 0 or π. Below, in order to make the discussions more intuitive, we choose to use the parameters r_s, r_t, and τ, instead of the vectors of $\mathbb{R}^4$.

For each vertex $u \in U_s(t)$, there must be a vertex $v \in U_t(s)$ such that the concatenation of $\overline{su}$, $\pi(u,v)$, and $\overline{vt}$ is a shortest path from s to t, and we call such a vertex v a *coupled t-pivot* of u (if u has more than one such vertex, then all of them are coupled t-pivots of u). Similarly, for each vertex $v \in U_t(s)$, we also define its coupled s-pivots in $U_s(t)$.

The following lemma provides a sufficient condition for a direction being in $R(s,t)$.

▶ **Lemma 4.** *Suppose t is a farthest point of s and s is not visible to t.*

1. *For $t \in \mathcal{I}$, a free direction r_s is in $R(s,t)$ if there is a free direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ, each vertex $v \in U_t(s)$ has a coupled s-pivot u with either $d'_{u,v}(s,t) < 0$, or $d'_{u,v}(s,t) = 0$ and $d''_{u,v}(s,t) = 0$.*
2. *For $t \in E$, a free direction r_s is in $R(s,t)$ if there is a free direction r_t for t that is parallel to the polygon edge of E containing t with a speed $\tau \geq 0$ such that when we move s along r_s with the unit speed and move t along r_t with speed τ, each vertex $v \in U_t(s)$ has a coupled s-pivot u with either $d'_{u,v}(s,t) < 0$, or $d'_{u,v}(s,t) = 0$ and $d''_{u,v}(s,t) = 0$.*
3. *For $t \in \mathcal{V}$, a free direction r_s is in $R(s,t)$ if we move s along r_s with the unit speed, each vertex $v \in U_t(s)$ has a coupled s-pivot u with either $d'_{u,v}(s,t) < 0$, or $d'_{u,v}(s,t) = 0$ and $d''_{u,v}(s,t) = 0$.*

Lemma 4 is on the case where s is not visible to t. If s is visible to t, the result is trivial.

▶ **Observation 5.** Suppose t is a farthest point of s and s is visible to t. Then t must be a polygon vertex of $\mathcal{V}$. Further, a free direction r_s of s is in $R(s,t)$ if and only if r_s is towards the interior of $h_s(t)$, where $h_s(t)$ is the open half-plane containing t and bounded by the line through s and perpendicular to $\overline{st}$.

By Observation 5, if s is visible to t, then the range $R(s,t)$ is the intersection of the free direction range $R_f(s)$ and an open range of size π delimited by the open half-plane $h_s(t)$.

The next lemma is proved by using Lemmas 3 and 4 as well as Observation 5.

▶ **Lemma 6.** *Among all points of $\mathcal{P}$ that have topologically equivalent shortest path maps in $\mathcal{P}$, there is at most one geodesic center. This implies that each cell or edge of $\mathcal{D}_{spm}$ contains at most one geodesic center in its interior, which further implies that the number of geodesic centers of $\mathcal{P}$ is $O(|\mathcal{D}_{spm}|)$, where $|\mathcal{D}_{spm}|$ is the combinatorial complexity of $\mathcal{D}_{spm}$.*

The following corollary implies that if t is a farthest point of s, then slightly moving s along a free direction that is not in $R(s,t)$ can never obtain a geodesic center.

▶ **Corollary 7.** *Suppose t is a farthest point of s. If we move s infinitesimally along a free direction that is not in $R(s,t)$, then $d_{\max}(s)$ will become strictly larger.*

To compute the candidate points, we need to determine $R(s,t)$. It turns out that it is sufficient to determine $R(s,t)$ when t is at a non-degenerate position of s in the following sense: Suppose t is a farthest point of s; we say that t is *non-degenerate* with respect to s if there are exactly three, two, and one shortest s-t paths for t in $\mathcal{I}$, E, and $\mathcal{V}$, respectively (by Observation 2, this implies that $|U_t(s)|$ is 3, 2, and 1, respectively for the three cases).

Lemma 4 gives a sufficient condition for a direction in $R(s,t)$. The following lemma gives both a sufficient and a necessary condition for a direction in $R(s,t)$ when t is non-degenerate, and the lemma will be used to explicitly compute the range $R(s,t)$ in Section 4. Note that Observation 5 already gives a way to determine $R(s,t)$ when s is visible to t.

▶ **Lemma 8.** *Suppose t is a non-degenerate farthest point of s and s is not visible to t. Then, a free direction r_s is in $R(s,t)$ if and only if*
1. *for $t \in \mathcal{I}$, there is a free direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ, each vertex $v \in U_t(s)$ has a coupled s-pivot u with $d'_{u,v}(s,t) < 0$.*
2. *for $t \in E$, there is a free direction r_t for t that is parallel to the polygon edge containing t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ, each vertex $v \in U_t(s)$ has a coupled s-pivot u with $d'_{u,v}(s,t) < 0$.*
3. *for $t \in \mathcal{V}$, when we move s along r_s with unit speed, each vertex $v \in U_t(s)$ has a coupled s-pivot u with $d'_{u,v}(s,t) < 0$.*

4 Determining the Range $R(s,t)$ and the π-Range Property

In this section, we determine the admissible direction range $R(s,t)$ for any point s and any of its non-degenerate farthest point t. In particular, we will present the π-range property.

Depending on whether t is in $\mathcal{V}$, E, and $\mathcal{I}$, there are three cases. Recall that $R_f(s)$ is the range of free directions of s. In each case, we will show that $R(s,t)$ is the intersection of $R_f(s)$ and an open range $R_\pi(s,t)$ of size π. We call $R_\pi(s,t)$ the *π-range*. As will be seen later, $R_\pi(s,t)$ can be explicitly determined based on the positions of s, t, and the vertices of $U_s(t)$ and $U_t(s)$. In fact, for each case, we will give a more general result on shortest path distance functions. These general results will be useful for computing the candidate points.

4.1 The Case $t \in \mathcal{V}$

We first discuss the case $t \in \mathcal{V}$. The result is relatively straightforward in this case. If s is visible to t, the π-range $R_\pi(s,t)$ is defined to be the open range of directions delimited by the open half-plane $h_s(t)$ as defined in Observation 5; by Observation 5, $R(s,t) = R_f(s) \cap R_\pi(s,t)$.

Below, we assume s is not visible to t. We first present a more general result on a shortest path distance function. Let s and t be any two points in $\mathcal{P}$ such that t is in $\mathcal{V}$ and s is not visible to t. Let $\pi(s,t)$ be any shortest s-t path in $\mathcal{P}$. Let u and v be the s-pivot and t-pivot in $\pi(s,t)$, respectively. Thus, $d_{u,v}(s,t) = |su| + d(u,v) + |vt|$. Now we consider $d_{u,v}(s,t)$ as a function of s and t in the entire plane $\mathbb{R}^2$ (not only in $\mathcal{P}$; namely, when we move s and t, they are allowed to move outside $\mathcal{P}$, but the function $d_{u,v}(s,t)$ is always defined as $|su| + d(u,v) + |vt|$, where $d(u,v)$ is a fixed value).

The *π-range* $R_\pi(s,t)$ is defined with respect to t and the path $\pi(s,t)$ as follows: a direction r_s for s is in $R_\pi(s,t)$ if $d'_{u,v}(s,t) < 0$ when we move s along r_s with unit speed.

▶ **Lemma 9.** *$R_\pi(s,t)$ is exactly the open range of size π delimited by $h_s(u)$, where $h_s(u)$ is the open half-plane containing u and bounded by the line through s and perpendicular to $\overline{su}$.*

Now we are back to our original problem to determine $R(s,t)$ for a non-degenerate farthest point t of s with $t \in \mathcal{V}$. Since t is non-degenerate and t is in $\mathcal{V}$, there is only one shortest path $\pi(s,t)$ from s to t. We define $R_\pi(s,t)$ as above. Based on Observation 5 and Lemmas 8(3), we have Lemma 10, and thus $R(s,t)$ can be determined by Observation 5 and Lemma 9.

▶ **Lemma 10.** *$R(s,t) = R_f(s) \cap R_\pi(s,t)$.*

4.2 The Case $t \in E$

The analysis for this case is substantially more complicated than the previous case, although the next case for $t \in \mathcal{I}$ is even more challenging. As in the previous case, we first present a more general result that is on two shortest path distance functions.

Let s and t be any two points in $\mathcal{P}$ such that t is in E and there are two shortest s-t paths $\pi_1(s,t)$ and $\pi_2(s,t)$ (this implies that s is not visible to t). Let e be the polygon edge containing t and let $l(e)$ denote the line containing e. For each $i = 1,2$, let $\pi_i(s,t) = \pi_{u_i,v_i}(s,t)$, i.e., u_i and v_i are the s-pivot and t-pivot of $\pi_i(s,t)$, respectively. We further require the set $\{v_1, v_2\}$ to satisfy the same condition as $U_t(s)$ in Observation 2(2), i.e., $\{v_1, v_2\}$ has at least one vertex in the open half-plane bounded by $l(e)$ and containing the interior of $\mathcal{P}$ in the small neighborhood of e, and it has at least one vertex in each of the two open half-planes bounded by the line through t and perpendicular to e. We say that the two shortest paths $\pi_1(s,t)$ and $\pi_2(s,t)$ are *canonical* with respect to s and t if $\{v_1, v_2\}$ satisfies the above condition. In the following, we assume $\pi_1(s,t)$ and $\pi_2(s,t)$ are canonical. Note

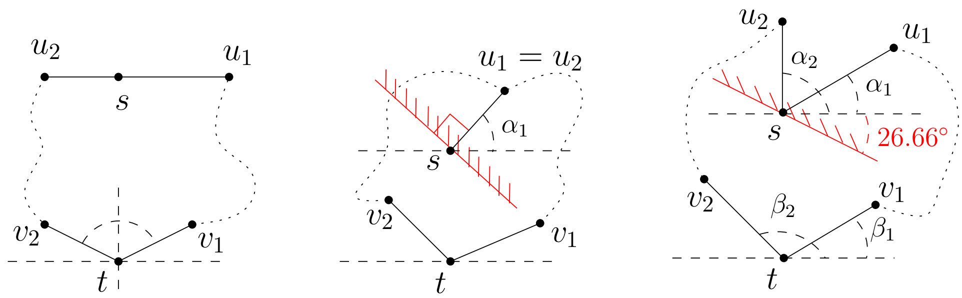

Figure 6 Illustrating some examples for Lemma 11. Left: the special case (the vertical line through t bisects $\angle v_1tv_2$); in this case, $R_\pi(s,t) = \emptyset$. Middle: the case where $\alpha_1 = \alpha_2$, and thus $\alpha = 0$, $\sin(\alpha) = 0$, and $u_1 = u_2$; in this case, $R_\pi(s,t) = (\alpha_1 - \pi/2, \alpha_1 + \pi/2)$, which is delimited by the open half-plane (marked with red color in the figure) bounded by the line through s and perpendicular to $\overline{su_1}$. Right: the most general case where $\alpha_1 = 30°$, $\alpha_2 = 90°$, $\beta_1 = 30°$, $\beta_2 = 135°$; by calculation, $\lambda \approx 1.3165$, $\arctan(\frac{\gamma}{\sin(\alpha)}) \approx 56.66°$, and thus, $R_\pi(s,t) \approx (\alpha_1 - 56.66°, \alpha_1 + 56.66°) = (-26.66°, 153.34°)$; the open half-plane that delimits $R_\pi(s,t)$ is marked with red color in the figure.

that the condition implies that $v_1 \neq v_2$. However, $u_1 = u_2$ is possible. For each $i = 1, 2$, we consider $d_{u_i,v_i}(s,t) = |su_i| + d(u_i, v_i) + |v_it|$ as a function of $s \in \mathbb{R}^2$ and $t \in e$.

In this case, the *π-range* $R_\pi(s,t)$ of s is defined with respect to t and the two paths $\pi_1(s,t)$ and $\pi_2(s,t)$ as follows: a direction r_s for s is in $R_\pi(s,t)$ if there exists a direction r_t parallel to e for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed $\tau \geq 0$, $d'_{u_i,v_i}(s,t) < 0$ holds for $i = 1, 2$.

In Section 4.1, we showed that the π-range for the case $s \in \mathcal{V}$ is an open range of size π. Here we will show a similar result in Lemma 11 unless a special case happens. Although the result in Section 4.1 is quite straightforward, the result here for two functions $d_{u_i,v_i}(s,t)$ with $i = 1, 2$ is somewhat surprising. We first introduce some notation.

For any two points p and q in the plane, define $\overrightarrow{pq}$ as the direction from p to q.

Recall that the angle of any direction r is defined to be the angle in $[0, 2\pi)$ counterclockwise from the positive direction of the x-axis. Let α_1 denote the angle of the direction $\overrightarrow{su_1}$, and let α_2 denote the angle of the direction $\overrightarrow{su_2}$ (e.g., see Fig. 5). Note that by our way of defining pivot vertices, $\alpha_1 = \alpha_2$ if and only if $u_1 = u_2$.

Note that v_1 and v_2 are in a closed half-plane bounded by the line $l(e)$. We assign a direction to $l(e)$ such that each of v_1 and v_2 are to the left or on $l(e)$. Define β_i as the smallest angle to rotate $l(e)$ counterclockwise such that the direction of $l(e)$ becomes the same as $\overrightarrow{tv_i}$, for each $i = 1, 2$ (e.g., see Fig. 5). Hence, both β_1 and β_2 are in $[0, \pi]$. Without of loss of generality, we assume $\beta_1 \leq \beta_2$ (otherwise the analysis is symmetric). Since $\{v_1, v_2\}$ contains at least one vertex in each of the open half-planes bounded by the line through t and perpendicular to e, we have $\beta_1 \in [0, \pi/2)$ and $\beta_2 \in (\pi/2, \pi]$. Further, since at least one of v_1 and v_2 is not on $l(e)$, it is not possible that both $\beta = 0$ and $\beta = \pi$ hold.

Let $\alpha = \alpha_2 - \alpha_1$. We refer to the case where $\beta_1 + \beta_2 = \pi$ and $\alpha = \pm\pi$ (i.e., α is π or $-\pi$) as the *special case*. In the special case, s is on $\overline{u_1u_2}$ and the vertical line through t and perpendicular to $l(e)$ bisects the angle $\angle v_1tv_2$.

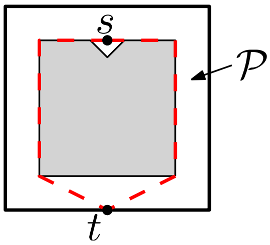

Figure 7 Illustrating an example in which a geodesic center s is in $\mathcal{I}$ and has only one farthest point t. The polygonal domain $\mathcal{P}$ is between two (very close) concentric squares plus an additional (very small) triangle so that s is in $\mathcal{I}$. The point s is at the middle of the top edge of the inner square, and t is at the middle of the bottom edge of the outer square. One can verify that s is a geodesic center and t is the only farthest point of s. The two shortest paths from s to t are shown with red dashed segments. Note that the middle point of every edge of the inner square is a geodesic center.

▶ **Lemma 11.** *The π-range $R_\pi(s,t)$ is determined as follows (e.g., see Fig. 6).*

$$R_\pi(s,t) = \begin{cases} (\alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}), \alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}) + \pi) & \text{if } \sin(\alpha) > 0, \\ (\alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)}) - \pi, \alpha_1 - \arctan(\frac{\lambda}{\sin(\alpha)})) & \text{if } \sin(\alpha) < 0, \\ (\alpha_1 - \pi/2, \alpha_1 + \pi/2) & \text{if } \sin(\alpha) = 0 \text{ and } \lambda > 0, \\ (\alpha_1 - 3\pi/2, \alpha_1 - \pi/2) & \text{if } \sin(\alpha) = 0 \text{ and } \lambda < 0, \\ \emptyset & \text{if } \sin(\alpha) = 0 \text{ and } \lambda = 0, \end{cases}$$

where $\lambda = \cos\alpha - \frac{\cos\beta_2}{\cos\beta_1}$. Further, $\alpha = \pm\pi$ and $\beta_1 + \beta_2 = \pi$ (i.e., the special case) if and only if $\sin(\alpha) = 0$ and $\lambda = 0$.

By Lemma 11, if the special case happens, $R_\pi(s,t) = \emptyset$; otherwise, it is an open range of size π. Since $\alpha = 0$ if and only if $u_1 = u_2$, the case $u_1 = u_2$ is also covered by the lemma.

Now consider our original problem of determining the range $R(s,t)$ for a non-degenerate farthest point $t \in E$ of s. By Observation 5, s is not visible to t. Further, s and t have exactly two shortest paths $\pi_1(s,t)$ and $\pi_2(s,t)$. Clearly, by Observation 2(2), the two paths are canonical. Therefore, the π-range $R_\pi(s,t)$ of s with respect to t and the two shortest paths $\pi_1(s,t)$ and $\pi_2(s,t)$ can be determined by Lemma 11. Lemma 8(2) leads to Lemma 12.

▶ **Lemma 12.** *$R(s,t) = R_\pi(s,t) \cap R_f(s)$.*

Suppose t is the only farthest point of s and t is non-degenerate. By Lemma 11, if the special case happens, $R_\pi(s,t) = \emptyset$ and $R(s,t) = \emptyset$. By Corollary 7, if we move s along any free direction infinitesimally, $d_{\max}(s)$ will be strictly increasing. Therefore, it is possible that the point s, which is in $\mathcal{I}$ and has only one farthest point, is a geodesic center. It is not difficult to construct such an example by following the left figure of Fig. 6; e.g., see Fig. 7. Hence, we have the following corollary.

▶ **Corollary 13.** *It is possible that a geodesic center is in $\mathcal{I}$ and has only one farthest point.*

4.3 The Case $t \in \mathcal{I}$

The analysis for this case is substantially more difficult than the case $t \in E$. As before, we first present a more general result that is on three shortest path distance functions.

Let s and t be two points in $\mathcal{P}$ such that t is in $\mathcal{I}$ and there are three shortest s-t paths $\pi_1(s,t)$, $\pi_2(s,t)$, and $\pi_3(s,t)$ (this implies that s is not visible to t). For each $i = 1,2,3$, let $\pi_i(s,t) = \pi_{u_i,v_i}(s,t)$, i.e., u_i and v_i are the s-pivot and t-pivot of $\pi_i(s,t)$, respectively. We say that the three paths are *canonical* with respect to s and t if they have two properties:

1. t is in the interior of the triangle $\triangle v_1v_2v_3$.
2. Suppose we reorder the indices such that v_1, v_2, and v_3 are *clockwise* around t, then u_1, u_2, and u_3 are *counterclockwise* around s (e.g., see Fig. 2).

The above first property implies that v_1, v_2, and v_3 are distinct, but this may not be true for u_1, u_2, u_3. In the following, we assume that the three shortest paths $\pi_i(s,t)$ with $1 \leq i \leq 3$ are canonical, and we reorder the indices as in the above second property. For each $i = 1,2,3$, we consider $d_{u_i,v_i}(s,t) = |su_i| + d(u_i,v_i) + |v_it|$ as a function of $s \in \mathbb{R}^2$ and $t \in \mathbb{R}^2$. In this case, the *π-range* $R_\pi(s,t)$ of s is defined with respect to t and the three paths $\pi_i(s,t)$ for $i = 1,2,3$ as follows: a direction r_s for s is in $R_\pi(s,t)$ if there exists a direction r_t for t with a speed $\tau \geq 0$ such that when we move s along r_s with unit speed and move t along r_t with speed τ, $d'_{u_i,v_i} < 0$ holds for $i = 1,2,3$.

As Lemma 11 in the previous cases, we will have a similar lemma (Lemma 14), which says that unless a special case happens $R_\pi(s,t)$ is an open range of size exactly π. The proof is much more challenging. Before presenting Lemma 14, we introduce some notation.

For each $i = 1,2,3$, let β_i denote the angle of the direction $\overrightarrow{tv_i}$ (i.e., the angle of $\overrightarrow{tv_i}$ counterclockwise from the positive x-axis). Further, we define three angles b_i for $i = 1,2,3$ as follows (e.g., see Fig. 2). Define b_1 as the smallest angle we need to rotate the direction $\overrightarrow{tv_1}$ *clockwise* to $\overrightarrow{tv_2}$; define b_2 as the smallest angle we need to rotate the direction $\overrightarrow{tv_2}$ clockwise to $\overrightarrow{tv_3}$; define b_3 as the smallest angle we need to rotate the direction $\overrightarrow{tv_3}$ clockwise to $\overrightarrow{tv_1}$.

For any two angles α' and α'', we use $\alpha' \equiv \alpha''$ to denote $\alpha' = \alpha'' \mod 2\pi$.

It is easy to see that $b_1 \equiv \beta_1 - \beta_2$, $b_2 \equiv \beta_2 - \beta_3$, and $b_3 \equiv \beta_3 - \beta_1$. Note that since t is in the interior of $\triangle v_1v_2v_3$, it holds that $b_i \in (0,\pi)$ for $i = 1,2,3$. Note that $b_1 + b_2 + b_3 = 2\pi$.

For each $i = 1,2,3$, let α_i denote the angle of the direction $\overrightarrow{su_i}$. According to our definition of pivot vertices, $u_i = u_j$ if and only if $\alpha_i = \alpha_j$ for any two $i,j \in \{1,2,3\}$. We define three angles a_i for $i = 1,2,3$ as follows (e.g., see Fig. 2). Define a_1 as the smallest angle we need to rotate the direction $\overrightarrow{su_1}$ *counterclockwise* to $\overrightarrow{su_2}$; define a_2 as the smallest angle we need to rotate the direction $\overrightarrow{su_2}$ clockwise to $\overrightarrow{su_3}$; define a_3 as the smallest angle we need to rotate the direction $\overrightarrow{su_3}$ clockwise to $\overrightarrow{su_1}$. Hence, $a_1 \equiv \alpha_2 - \alpha_1$, $a_2 \equiv \alpha_3 - \alpha_2$, and $a_3 \equiv \alpha_1 - \alpha_3$.

We refer to the case where $a_i = b_i$ for each $i = 1,2,3$ as the *special case.*

▶ **Lemma 14.** *The π-range $R_\pi(s,t)$ is determined as follows (e.g., see Fig. 8).*

$$R_\pi(s,t) = \begin{cases} (\alpha_1 - \arctan(\frac{\delta_1-\delta_2}{\delta}), \alpha_1 - \arctan(\frac{\delta_1-\delta_2}{\delta}) + \pi) & \text{if } \delta > 0, \\ (\alpha_1 - \arctan(\frac{\delta_1-\delta_2}{\delta}) - \pi, \alpha_1 - \arctan(\frac{\delta_1-\delta_2}{\delta})) & \text{if } \delta < 0, \\ (\alpha_1 - \pi/2, \alpha_1 + \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 > \delta_2, \\ (\alpha_1 - 3\pi/2, \alpha_1 - \pi/2) & \text{if } \delta = 0 \text{ and } \delta_1 < \delta_2, \\ \emptyset & \text{if } \delta = 0 \text{ and } \delta_1 = \delta_2, \end{cases}$$

where $\delta = \frac{\sin(\alpha_3-\alpha_1)}{\sin(\beta_3-\beta_1)} - \frac{\sin(\alpha_2-\alpha_1)}{\sin(\beta_2-\beta_1)}$, $\delta_1 = \frac{\cos(\beta_2-\beta_1)-\cos(\alpha_2-\alpha_1)}{\sin(\beta_2-\beta_1)}$, *and* $\delta_2 = \frac{\cos(\beta_3-\beta_1)-\cos(\alpha_3-\alpha_1)}{\sin(\beta_3-\beta_1)}$. *Further, $a_i = b_i$ for each $i = 1,2,3$ (i.e., the special case) if and only if $\delta = 0$ and $\delta_1 = \delta_2$.*

According to Lemma 14, if the special case happens, then $R_\pi(s,t)$ is empty; otherwise, it is an open range of size exactly π.

Now we are back to our original problem to determine the range $R(s,t)$ for a non-degenerate farthest point $t \in \mathcal{I}$ of s. Since there are exactly three shortest s-t paths $\pi_i(s,t)$

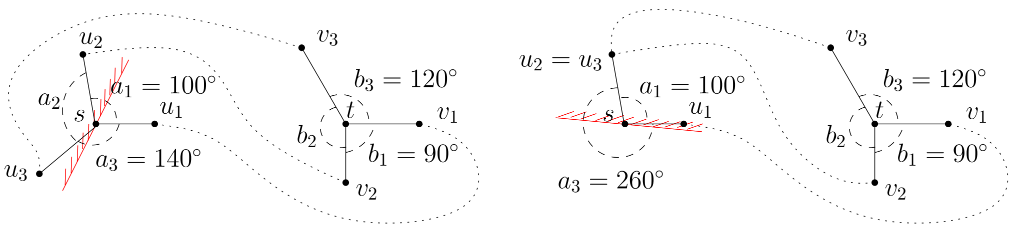

Figure 8 Illustrating two examples for Lemma 14. Left: The sizes of the angles of a_i and b_i for $1 \leq i \leq 3$ are already shown in the figure with $\alpha_1 = 0$. By calculation, $\delta \approx 0.2426$, $\delta_1 \approx -0.1736$, $\delta_2 \approx 0.3072$, $\arctan(\frac{\delta_1-\delta_2}{\delta}) \approx -63.23°$, and thus $R_\pi(s,t) \approx (\alpha_1 + 63.23°, \alpha_1 + 63.23° + 180°) = (63.23°, 243.23°)$. Right: a case where $\alpha_2 = \alpha_3$ with $\alpha_1 = 0$. Thus, $a_2 = 0$ and $u_2 = u_3$. The sizes of other angles are already shown in the figure. By calculation, $\delta \approx 2.1220$, $\delta_1 \approx -0.1736$, $\delta_2 \approx -0.3768$, $\arctan(\frac{\delta_1-\delta_2}{\delta}) \approx 5.47°$, and thus $R_\pi(s,t) \approx (\alpha_1 - 5.47°, \alpha_1 - 5.47° + 180°) = (-5.47°, 174.53°)$. The open half-planes that delimit $R_\pi(s,t)$ in both examples are marked with red color.

for $i = 1, 2, 3$, the three paths must be canonical. To see this, by Observation 2, t is in the interior of $\triangle v_1 v_2 v_3$. Further, it is easy to see that no two of the three paths cross each other since otherwise there would be more than three shortest s-t paths, this implies that the second property of the canonical paths holds. Let $R_\pi(s,t)$ be the π-range of s with respect to t and the above three shortest paths. By Lemma 8(1), we have the following lemma.

▶ **Lemma 15.** $R(s,t) = R_\pi(s,t) \cap R_f(s)$.

5 Computing the Candidate Points

In this section, with the help of the observations in Sections 3 and 4, we compute a set S of candidate points such that all geodesic centers must be in S.

Let s be any geodesic center. Recall that $F(s)$ is the set of all farthest points of s. Depending on whether s is in $\mathcal{V}$, E, or $\mathcal{I}$, the size $|F(s)|$, whether some points of $F(s)$ are in $\mathcal{V}$, E, or $\mathcal{I}$, whether s has a degenerate farthest point, there are a significant (but still constant) number of cases. For each case, we use an exhaustive-search approach to compute a set of candidate points such that s must be in the set. In particular, there are four cases, called *dominating cases*, for which the number of candidate points is $O(n^{11})$. But the total number of the candidate points for all other cases is only $O(n^{10})$. Therefore, the set S has a total of $O(n^{11})$ candidate points. We will show that S can be computed in $O(n^{11} \log n)$ time.

To find the geodesic centers in S, a straightforward algorithm works as follows. For each point $\hat{s} \in S$, we can compute $d_{\max}(\hat{s})$ in $O(n \log n)$ time by first computing the shortest path map $SPM(\hat{s})$ of $\hat{s}$ in $O(n \log n)$ time [12] and then obtaining the maximum geodesic distance from $\hat{s}$ to all vertices of $SPM(\hat{s})$. Since all geodesic centers are in S, the points of S with the smallest $d_{\max}(\hat{s})$ are geodesic centers of $\mathcal{P}$.

Since $|S| = O(n^{11})$, the above algorithm runs in $O(n^{12} \log n)$ time. Let S_d denote the set of the candidate points for the four dominating cases. Clearly, the bottleneck is on finding the geodesic centers from S_d. To improve the algorithm, when we compute the candidate points of S_d, we will maintain the corresponding *path information*. By using these path information and based on new observations, we will present in Section 6 an $O(n^{11} \log n)$ time "pruning algorithm" that can eliminate most of the points from S_d such that none of the eliminated points is a geodesic center and the number of remaining points in S_d is only $O(n^{10})$. Consequently, we can find all geodesic centers in additional $O(n^{11} \log n)$ time.

The dominating cases. In order to discuss our pruning algorithm in Section 6, we explain the four dominating cases as follows. Suppose s is a geodesic center in $\mathcal{I}$ that has the following three properties. (1) s does not have a degenerate farthest point. (2) For any farthest point t of s, the π-range $R_\pi(s,t) \neq \emptyset$ (i.e., the special cases in Lemmas 11 and 14 do not happen). (3) s has at least three farthest points. By Lemma 3, these properties further imply that s must have three farthest points t_1, t_2, t_3 such that $R_\pi(s,t_1) \cap R_\pi(s,t_2) \cap R_\pi(s,t_3) = \emptyset$ since the size of $R_\pi(s,t_i)$ is π for each t_i. Depending on whether each t_i for $i = 1,2,3$ is in $\mathcal{I}$, E, $\mathcal{V}$, there are several cases. We use (x,y,z) to refer to the case where x, y, and z points of t_1, t_2, t_3 are in $\mathcal{I}, E$, and $\mathcal{V}$, respectively, with $x+y+z=3$. For example, $(3,0,0)$ refers to the case where t_1,t_2,t_3 are all in $\mathcal{I}$. The four dominating cases are: $(3,0,0)$, $(2,1,0)$, $(1,2,0)$, and $(0,3,0)$. For any geodesic center s, if s does not belong to the dominating cases, then our algorithm guarantees that s is in $S \setminus S_d$. Below, due to the space limit, we only sketch how to compute candidate points for three "representative" cases, and other details are omitted.

Case 1. Consider the dominating case $(3,0,0)$ discussed above. We compute the candidate points for the geodesic center s as follows. For each $i = 1,2,3$, since t_i is in $\mathcal{I}$, there are three shortest paths from s to t_i: $\pi_{u_{ij},v_{ij}}(s,t)$ with $j = 1,2,3$ (i.e., u_{ij} and v_{ij} are the s-pivot and t-pivot, respectively). Hence, we have the following

$$
\begin{aligned}
&|t_1v_{11}| + d(v_{11},u_{11}) + |u_{11}s| = |t_1v_{12}| + d(v_{12},u_{12}) + |u_{12}s| = |t_1v_{13}| + d(v_{13},u_{13}) + |u_{13}s| \\
=&|t_2v_{21}| + d(v_{21},u_{21}) + |u_{21}s| = |t_2v_{22}| + d(v_{22},u_{22}) + |u_{22}s| = |t_2v_{23}| + d(v_{23},u_{23}) + |u_{23}s| \\
=&|t_3v_{31}| + d(v_{31},u_{31}) + |u_{31}s| = |t_3v_{32}| + d(v_{32},u_{32}) + |u_{32}s| = |t_3v_{33}| + d(v_{33},u_{33}) + |u_{33}s|.
\end{aligned}
$$

By considering the coordinates of s, t_1, t_2, and t_3 as eight variables, the above equations on the lengths of the nine shortest paths provide eight (independent) constraints, which are sufficient to compute all four points. Correspondingly, we compute the candidate points for s by the exhaustive-search algorithm below (similar methods were also used before, e.g., [3, 7]).

We enumerate all possible combinations of nine polygon vertices as v_{i1}, v_{i2}, v_{i3}, with $i = 1,2,3$. For each combination, we compute the overlay of the shortest path maps of the nine vertices. The overlay is of size $O(n^2)$ and can be computed in $O(n^2 \log n)$ time [3, 7]. For each cell C of the overlay, we obtain nine roots of the shortest path maps and consider them as u_{i1}, u_{i2}, u_{i3} for $i = 1,2,3$. We form the above system of eight equations and solve it to obtain a constant number of quadruples $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ and each such $\hat{s}$ is a candidate point. In this way, for each combination of nine polygon vertices, we can obtain $O(n^2)$ candidate points in $O(n^2 \log n)$ time. Since there are $O(n^9)$ combinations, we can compute $O(n^{11})$ candidate points in $O(n^{11} \log n)$ time and s must be one of these candidate points.

If this were not a dominating case, we would have done for this case. Since this is a dominating case, we need to maintain certain path information. To this end, we perform a "validation procedure" on each such quadruple $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ computed above, as follows.

In the above procedure for computing $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$, we also obtain a path length, denoted by $d(\hat{s})$, which is equal to the value in the above equations, e.g., $d(\hat{s}) = |\hat{s}u_{11}| + d(u_{11}, v_{11}) + |v_{11}\hat{t}_1|$. First, we check whether $\hat{s}$ is in C, which can be done in $O(\log n)$ time by using a point location data structure [9, 14] with $O(n^2)$ time and space preprocessing on the overlay. If yes, for each t_i with $i = 1,2,3$, we check whether $d(\hat{s})$ is equal to $d(\hat{s}, \hat{t}_i)$, which can be computed in $O(\log n)$ time by using the two-point shortest path query data structure [7] with $O(n^{11})$ time and space preprocessing on $\mathcal{P}$. If yes, we check whether v_{i1}, v_{i2}, v_{i3} satisfy the condition in Observation 2(1), i.e., whether $\hat{t}_i$ is in the interior of the triangle $\triangle v_{i1}v_{i2}v_{i3}$ for each $i = 1,2,3$. If yes, for each $\hat{t}_i$ with $i = 1,2,3$, we check whether the order of the vertices of v_{i1}, v_{i2}, v_{i3} around $\hat{t}_i$

are consistent with the order of the vertices of u_{i1}, u_{i2}, u_{i3} (we say that the two orders are *consistent* if after reordering the indices, v_{i1}, v_{i2}, v_{i3} are clockwise around $\hat{t}_i$ while u_{i1}, u_{i2}, u_{i3} are counterclockwise around s; note that this consistency is needed for determining the π-range in Lemma 14). If yes, for each $\hat{t}_i$ with $i = 1, 2, 3$, we compute the π-range $R_\pi(s, \hat{t}_i)$ determined by Lemma 14, and then check whether $R_\pi(\hat{s}, \hat{t}_1) \cap R_\pi(\hat{s}, \hat{t}_2) \cap R_\pi(\hat{s}, \hat{t}_3)$ is empty. If yes, we say that the quadruple $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ *passes* the validation procedure and we call $\hat{s}$ a *valid* candidate point and add $\hat{s}$ to the set S_d. In addition, we maintain the following *path information*: $d(\hat{s})$, $\hat{t}_i$, v_{ij}, and u_{ij}, with $1 \leq i \leq 3$ and $1 \leq j \leq 3$.

In the worst case, we can have $O(n^{11})$ valid candidate points for S_d. Note that the geodesic center s and the quadruple (s, t_1, t_2, t_3) discussed above must pass the validation procedure, and thus the quadruple (s, t_1, t_2, t_3) will be computed by our exhaustive-search algorithm and s will be computed as a valid candidate point in S_d.

Based on our validation procedure, the following observation summarizes the properties of the valid candidate points, which will be useful for our pruning algorithm in Section 6.

► **Observation 16.** Suppose $(\hat{s}, \hat{t}_1, \hat{t}_2, \hat{t}_3)$ a quadruple that passes the validation procedure, with u_{ij} and v_{ij}, $i = 1, 2, 3$ and $j = 1, 2, 3$ defined as above. Then the following hold.

1. For each $i = 1, 2, 3$, $\overline{\hat{s}u_{ij}} \cup \pi(u_{ij}, v_{ij}) \cup \overline{v_{ij}\hat{t}_i}$ is a shortest path from $\hat{s}$ to $\hat{t}_i$ for each $j = 1, 2, 3$.
2. For each $i = 1, 2, 3$, v_{i1}, v_{i2}, v_{i3} satisfy the condition of Observation 2(1), i.e., $\hat{t}_i$ is in the interior of the triangle $\triangle v_{i1}v_{i2}v_{i3}$.
3. $d(\hat{s}) = d(\hat{s}, \hat{t}_i)$ for each $i = 1, 2, 3$.
4. $R_\pi(\hat{s}, \hat{t}_1) \cap R_\pi(\hat{s}, \hat{t}_2) \cap R_\pi(\hat{s}, \hat{t}_1) = \emptyset$.

Case 2. Consider the case where s is a geodesic center that has only one farthest point t such that t is non-degenerate with respect to s, with both s and t in $\mathcal{I}$. We compute the candidate points for s as follows. We will need the π-range property in this case.

Since t is non-degenerate, there are exactly three shortest s-t paths: $\pi_{u_i,v_i}(s, t)$ with $i = 1, 2, 3$. We have $|su_1| + d(u_1, v_1) + |v_1t| = |su_2| + d(u_2, v_2) + |v_2t| = |su_3| + d(u_3, v_3) + |v_3t|$. If we consider the coordinates of s and t as four variables, the equations give two constraints, and to determine s and t, we need two more constraints, which are provided by the π-range property (this kind of situation does not appear in the previous work [3, 7] and thus they do not need the π-range property). Indeed, since t is the only farthest point of s, we have $R(s) = R(s, t)$. By Lemma 3, $R(s) = \emptyset$. Hence, $R(s, t) = \emptyset$. Since $s \in \mathcal{I}$, $R_\pi(s, t) = R(s, t) = \emptyset$. Since $t \in \mathcal{I}$, $R_\pi(s, t)$ is determined by Lemma 14. We define the angles a_i, b_i, for $i = 1, 2, 3$, in the same way as those for Lemma 14 in Section 4.3. By Lemma 14, $R_\pi(s, t) = \emptyset$ if and only if $a_i = b_i$ for $i = 1, 2, 3$. The identities of the three pairs of angles provide another two (independent) constraints. Using the above four constraints, we can determine s and t. Correspondingly, the candidate points for s can be computed in an exhaustive manner, as follows.

We enumerate all possible combinations of three polygon vertices as v_1, v_2, v_3. We compute the shortest path maps of v_1, v_2, and v_3 in $O(n \log n)$ time. Next we compute the overlay of the three shortest path maps. Then, for each cell of the overlay, we obtain the three roots of the cell in the three shortest path maps and consider them as u_1, u_2, u_2. Finally, we use the above four constraints to determine a constant number of pairs $(\hat{s}, t)$ (we assume this can be done in constant time since the angles a_i and b_i can be parameterized by the coordinates of $\hat{s}$ and t), and each such $\hat{s}$ is considered as a candidate point. In this way, for each combination of v_1, v_2, v_3, we can compute $O(n^2)$ candidate points in $O(n^2 \log n)$ time. Since there are $O(n^3)$ combinations, we can compute $O(n^5)$ candidate points in $O(n^5 \log n)$ time.

Case 3. Consider the case where s is a geodesic center that has only two farthest point t_1 and t_2 such that both t_1 and t_2 are non-degenerate, with s, t_1, t_2 all in $\mathcal{I}$.

For each t_i, $i = 1, 2$, there are exactly three shortest paths from s. The equations on the lengths of the six paths give five constraints for s, t_1, t_2 if we consider their coordinates as six variables. To determine s, t_1, t_2, we need one more constraint, which is provided by the π-range property as follows. Since s, t_1, t_2 are all in $\mathcal{I}$, for each $i = 1, 2$, $R(s, t_i) = R_\pi(s, t_i)$, and $R_\pi(s, t_i)$ is determined by Lemma 14. We assume neither $R_\pi(s, t_1)$ nor $R_\pi(s, t_2)$ is empty since otherwise the candidate points could be computed by the algorithm for the above Case 2. Hence, each $R_\pi(s, t_i)$, $i = 1, 2$, is an open range of size π. By Lemma 3, $R(s) = R(s, t_1) \cap R(s, t_2) = R_\pi(s, t_1) \cap R_\pi(s, t_2) = \emptyset$. Since each $R_\pi(s, t_i)$, $i = 1, 2$, is an open range of size π (i.e., it is delimited by an open half-plane whose bounding line contains s), to have $R_\pi(s, t_1) \cap R_\pi(s, t_2) = \emptyset$, the two bounding lines of the two half-planes delimiting the two π-ranges must be overlapped, and this provides the sixth constraint to determine s, t_1, t_2. Correspondingly, the candidate points for s can be computed in an exhaustive way.

6 Computing the Geodesic Centers

In this section, we find all geodesic centers from the candidate point set S. Recall that S_d is the set of candidate points for the four dominating cases. Let S' denote the set of candidate points for all other cases, and thus $S = S_d \cup S'$. As discussed in Section 5, $|S'| = O(n^{10})$ and we can find all geodesic centers in S' in $O(n^{11} \log n)$ time by computing shortest path maps. Below, we focus on finding all geodesic centers in S_d.

We first remove all points from S_d that are also in S', which can be done in $O(n^{11} \log n)$ time (e.g., by first sorting these points by their coordinates). Then, according to our definitions of the four dominating cases in Section 5, for any point $s \in S_d$, if s is a geodesic center, s does not have any degenerate farthest point since otherwise s was also in S' and thus would have already been removed from S_d. Recall that each point s of S_d is a valid candidate point and we have maintained its path information (in particular, the value $d(s)$).

We first perform the following *duplication-cleanup* procedure: for each point $s \in S_d$, if there are many copies of s, we only keep the one with the largest value $d(s)$ (if more than one copy has the largest value, we keep an arbitrary one). This procedure can be done in $O(n^{11} \log n)$ time (e.g., by first sorting all points of S_d by their coordinates). According to our algorithm for computing the candidate points of S_d, we have the following observation.

▶ **Lemma 17.** *After the duplication-cleanup procedure, for any point $s \in S_d$, if s is a geodesic center, then $d_{\max}(s) = d(s)$.*

Recall that all points of S_d are in $\mathcal{I}$. In the following, we give a *pruning algorithm* that can eliminate most of the points from S_d such that none of these eliminated points is a geodesic center and the number of remaining points of S_d is $O(n^{10})$. Our pruning algorithm relies on the property that each candidate point s of S_d is valid. Specifically, if s is computed for the dominating case $(3, 0, 0)$, then s is associated with the following path information $d(s)$, t_i, v_{ij}, and u_{ij} for $1 \le i \le 3$ and $1 \le j \le 3$, such that Observation 16 holds (i.e., s is $\hat{s}$ and each t_i is $\hat{t}_i$). For other three dominating cases (e.g., $(2, 1, 0)$, $(1, 2, 0)$, and $(0, 3, 0)$), there are similar properties. By using these properties, we have the following key lemmas.

▶ **Lemma 18.** *Let s be any point in S_d. If s is in the interior of a cell or an edge of $\mathcal{D}_{spm}$, then for any other point s' in the interior of the same cell or edge of $\mathcal{D}_{spm}$, it holds that $d_{\max}(s') > d(s)$.*

▶ **Lemma 19.** *For any two points s_1 and s_2 of S_d that are in the interior of the same cell or the same edge of $\mathcal{D}_{spm}$, if $d(s_1) < d(s_2)$, then s_1 cannot be a geodesic center, and if $d(s_1) = d(s_2)$, then neither s_1 nor s_2 is a geodesic center.*

Based on Lemma 19, our pruning algorithm for S_d works as follows. For each point s of S_d, we determine the cell, edge, or vertex of $\mathcal{D}_{spm}$ that contains s in its interior, which can be done in $O(\log n)$ time by using a point location data structure [9, 14] with $O(n^{10})$ time and space preprocessing on $\mathcal{D}_{spm}$. For each edge or cell, let S'_d be the set of points of S_d that are contained in its interior. We find the point s of S'_d with the largest value $d(s)$. If there are more than one such point in S'_d, we remove all points of S'_d from S_d; otherwise, remove all points of S'_d except s from S_d. By Lemma 19, none of the points of S_d that are removed above is a geodesic center. After the above pruning algorithm, S_d contains at most one point in the interior of each cell, edge, or vertex of $\mathcal{D}_{spm}$. Hence, $|S_d| = O(|\mathcal{D}_{spm}|)$. Since $|\mathcal{D}_{spm}| = O(n^{10})$ [7], we obtain $|S_d| = O(n^{10})$. Consequently, we can find all geodesic centers in S_d in $O(n^{11} \log n)$ time by computing shortest path maps.

We thus conclude that all geodesic centers of $\mathcal{P}$ can be computed in $O(n^{11} \log n)$ time.

▶ **Theorem 20.** *All geodesic centers of $\mathcal{P}$ can be computed in $O(n^{11} \log n)$ time.*

Acknowledgment. We wish to thank Yan Sun for the discussions on proving the π-range property.

References

1. H.-K. Ahn, L. Barba, P. Bose, J.-L. De Carufel, M. Korman, and E. Oh. A linear-time algorithm for the geodesic center of a simple polygon. In *Proc. of the 31st Annual Symposium on Computational Geometry (SoCG)*, pages 209–223, 2015.
2. T. Asano and G. Toussaint. Computing the geodesic center of a simple polygon. Technical Report SOCS-85.32, McGill University, Montreal, Canada, 1985.
3. S.W. Bae, M. Korman, and Y. Okamoto. The geodesic diameter of polygonal domains. *Discrete and Computational Geometry*, 50:306–329, 2013.
4. S.W. Bae, M. Korman, and Y. Okamoto. Computing the geodesic centers of a polygonal domain. In *Proc. of the 26th Canadian Conference on Computational Geometry (CCCG)*, 2014.
5. S.W. Bae, M. Korman, Y. Okamoto, and H. Wang. Computing the L_1 geodesic diameter and center of a simple polygon in linear time. *Computational Geometry: Theory and Applications*, 48:495–505, 2015.
6. B. Chazelle. A theorem on polygon cutting with applications. In *Proc. of the 23rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 339–349, 1982.
7. Y.-J. Chiang and J.S.B. Mitchell. Two-point Euclidean shortest path queries in the plane. In *Proc. of the Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 215–224, 1999.
8. H.N. Djidjev, A. Lingas, and J.-R. Sack. An $O(n \log n)$ algorithm for computing the link center of a simple polygon. *Discrete and Computational Geometry*, 8:131–152, 1992.
9. H. Edelsbrunner, L. Guibas, and J. Stolfi. Optimal point location in a monotone subdivision. *SIAM Journal on Computing*, 15(2):317–340, 1986.
10. D. Halperin and M. Sharir. New bounds for lower envelopes in three dimensions, with applications to visibility in terrains. *Discrete and Computational Geometry*, 12:313–326, 1994.
11. J. Hershberger and S. Suri. Matrix searching with the shortest-path metric. *SIAM Journal on Computing*, 26(6):1612–1634, 1997.

12 J. Hershberger and S. Suri. An optimal algorithm for Euclidean shortest paths in the plane. *SIAM Journal on Computing*, 28(6):2215–2256, 1999.

13 Y. Ke. An efficient algorithm for link-distance problems. In *Proc. of the 5th Annual Symposium on Computational Geometry (SoCG)*, pages 69–78, 1989.

14 D. Kirkpatrick. Optimal search in planar subdivisions. *SIAM Journal on Computing*, 12(1):28–35, 1983.

15 J.S.B. Mitchell. Geometric shortest paths and network optimization, in *Handbook of Computational Geometry,* J.-R Sack and J. Urrutia (eds.), pages 633–702. Elsevier, Amsterdam, the Netherlands, 2000.

16 B.J. Nilsson and S. Schuierer. Computing the rectilinear link diameter of a polygon. In *Proc. of the International Workshop on Computational Geometry - Methods, Algorithms and Applications*, pages 203–215, 1991.

17 B.J. Nilsson and S. Schuierer. An optimal algorithm for the rectilinear link center of a rectilinear polygon. *Computational Geometry: Theory and Applications*, 6:169–194, 1996.

18 R. Pollack, M. Sharir, and G. Rote. Computing the geodesic center of a simple polygon. *Discrete and Computational Geometry*, 4(1):611–626, 1989.

19 S. Schuierer. An optimal data structure for shortest rectilinear path queries in a simple rectilinear polygon. *International Journal of Compututational Geometry and Applications*, 6:205–226, 1996.

20 S. Suri. Computing geodesic furthest neighbors in simple polygons. *Journal of Computer and System Sciences*, 39:220–235, 1989.