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Classifications of Plane Continua

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CLASSIFICATIONS OF PLANE CONTINUA

by

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INTRODUCTION

In the course of studying continua in the plane it has been asked if a given continuum has uncountably many disjoint duplications in the plane, and if so, what are the consequences of the existence of such a collection. The object of this paper is to study these problems and to develop some machinery useful in their resolution. In Section I, we review the definition of convergence and homeomorphic convergence of point sets in a metric space S. We then consider the space π of all continuous functions from a compact metric space P to a separable metric space Q under the "sup metric." The famous Borsuk Theorem [3], which shows that π is separable, is then proven. As a consequence of the separability of π, we show that in any uncountable collection G of compact sets in a separable metric space S, there must exist an element of G that is converged to by a sequence of elements of G. Furthermore, if the elements of G are pairwise homeomorphic, then G contains an element which is converged to homeomorphically by some sequence in G. These theorems prove to be important in the study of disjoint embeddings of continua in separable metric spaces. In Section I, as throughout this paper, definitions are motivated and illustrated by numerous examples.

A continuum M may have uncountably many homeomorphic images in a given space S. We say that two such images are equivalent if there exists a homeomorphism h of S onto itself that takes one image of M onto the other. If M lies in a space S, it may be that there are uncountably many disjoint pairwise equivalent homeomorphic images of M in S. In such a case, M is said to be thinly embedded in S. In Section II, we introduce and illustrate this concept along with the idea of a continuum being slender, thick, and wide. We also give an example of a continuum, in the plane E^2, having uncountably many embeddings (homeomorphic images) in E^2 such that no two are equivalent.

The examples and definitions in Section II point out that criteria are needed for testing a given continuum (or its embedding in a space S) to determine which of the classes, thin, slender, wide, or thick, the continuum (or its embedding) falls into. Although no such complete conditions are known, we develop in Sections III and IV some tests that can be used on a particular class of continua, namely the chainable continua. Bing has proven [1] that each chainable continuum is slender in E^2; i.e., each chainable continuum has uncountably many disjoint embeddings in the plane. His method of proof was to show that each chainable continuum has an embedding in E^2 that is chainable-with-nice-links; and then he applied a widely misunderstood result of Roberts [8]. In Section III, we introduce and illustrate the definition of chainable and chainable-with-nice-links, and in Section IV, we state and prove the result by Roberts. The result by Roberts gives a sufficient condition for a continuum to be thinly embedded in E^2.

We conclude the paper by exhibiting two homeomorphic continua M_1 and M_2 where M_2 is thinly embedded in E^2, but M_1 is not. These continua, first introduced by Bing [1], illustrate how the Roberts Theorem is often misunderstood. We also include a list of questions that have come up during the writing of this paper. These questions may or may not be difficult, although we found no immediate answer.

It is recommended that the reader who is mainly interested in embeddings of continua in the plane skip a detailed reading of Section

I. In fact, the crucial part of Section I to the paper is the understanding of the concept of homeomorphic convergence and Theorem B.

I. CONVERGENCE AND HOMEOMORPHIC CONVERGENCE

Given a sequence of point sets $\{X_n\}$, in a metric space S, we define the notion of convergence of $\{X_n\}$ to a point set L in terms of <u>limit inferior</u> (abbreviated "lim inf X_n") and <u>limit superior</u> (abbreviated "lim sup X_n") of the sequence $\{X_n\}$.

<u>Definition</u>. Let $\{X_n\}$ be a sequence of subsets of S. The set "lim inf X_n" consists of all points y in S, such that each open set containing y intersects all but a finite number of the sets X_n.

The set "lim sup X_n" consists of all points y in S such that each open set containing y intersects infinitely many of the sets X_n.

If $\{X_n\}$ is a sequence of sets such that lim inf X_n = lim sup X_n = $L \neq \emptyset$, then we say that the sequence X_n <u>converges</u> to the point set L.

Figure 1 demonstrates the case where lim inf X_n is empty while lim sup X_n is not. Consequently, Figure 1 illustrates the fact that lim sup X_n is not necessarily a subset of lim inf X_n, even though it is the case that lim inf $X_n \subset$ lim sup X_n. Thus, to show $\{X_n\}$ converges, it suffices to show lim inf X_n = L is not empty and that lim sup $X_n \subset$ lim X_n, for then $\{X_n\}$ converges to L.

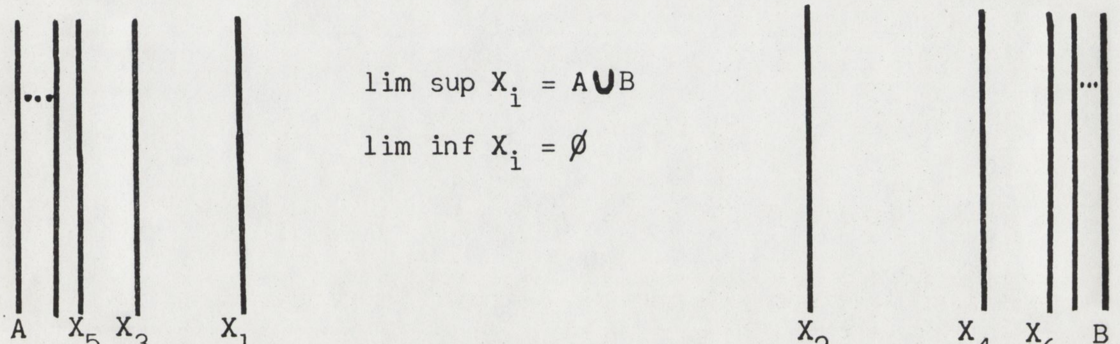
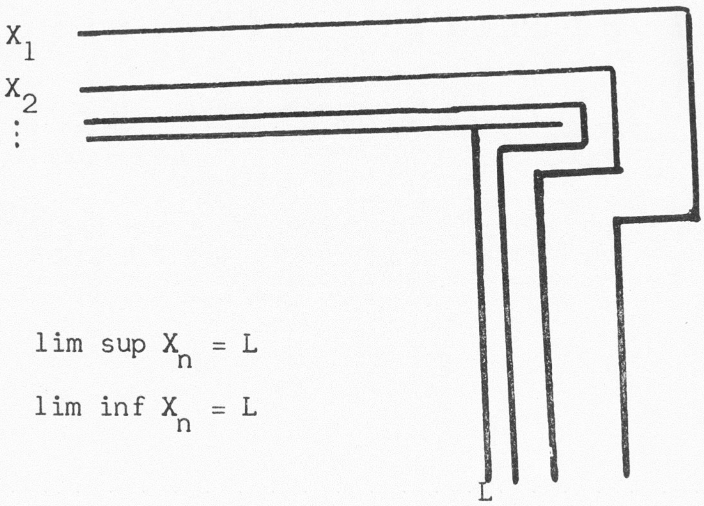


Figure 1



$\lim\sup X_n = L$

$\lim\inf X_n = L$

Figure 2

$\lim\sup X_i = \lim\inf X_i = L$

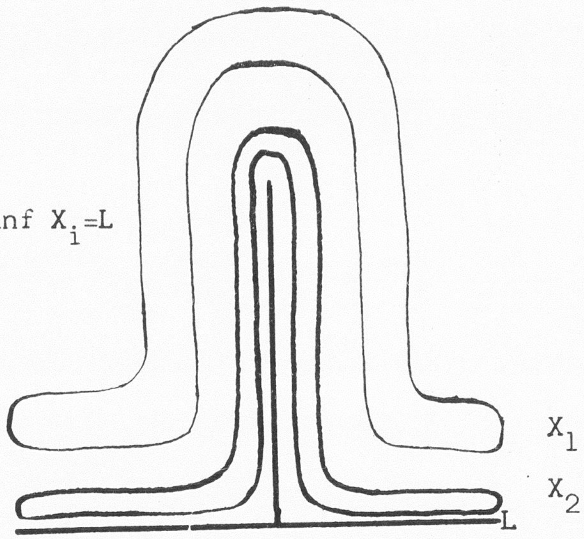


Figure 3

$\lim\sup X_i = A$

$\lim\inf X_i = A$

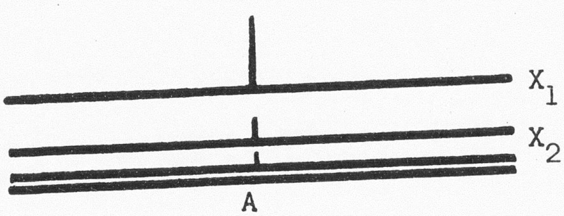


Figure 4

Figures 2 and 3 illustrate that it is possible to have a sequence of arcs as well as simple closed curves converge to a triod. Furthermore, it is possible to have a sequence of triods converge to an arc (See Figure 4).

In Figure 1, for each integer n, X_n is a connected subset of space, but the lim sup X_n fails to be connected. It follows from the next theorem that in a compact space no such example exists unless lim inf X_n is empty.

<u>Theorem 1</u>. If $\{X_n\}$ is a sequence of connected subsets of a compact metric space S such that lim inf $X_n \neq \emptyset$, then lim sup X_n is connected.

Proof. Let lim sup X_n = L, and suppose that L is not connected. Thus, there exists two mutually separated sets M and N such that $L = M \cup N$

Suppose that L is not a closed set; that is, there exists a point p, which is not in L, such that every open set about p intersects L. Let $N(p, \varepsilon)$ be such a neighborhood. Then we can find a point q in $N(p, \varepsilon) \cap L$. Since $q \in L$, $N(p, \varepsilon)$ has a non-empty intersection with an infinite collection of X_n's from $\{X_n\}$. It follows from the definition of lim sup X_n that $p \in$ lim sup X_n = L and, hence, that L is closed.

We have that M and N are disjoint closed sets in a normal space S, and hence, we can find disjoint open sets H and K such that M and N lie entirely within H and K, respectively.

Suppose that $H \cup K$ contains no infinite subsequence of $\{X_n\}$. Then there exists a subsequence $\{X_{k_n}\}$ of $\{X_n\}$ such that $X_{k_n} - (H \cup K)$ is non-empty for each integer k_n. Choose a point $a_n \in \{X_{k_n} - (H \cup K)\}$ and consider the resulting sequence of points $\{a_n\}$. Since S is compact, it follows that the sequence $\{a_n\}$ contains a subsequence which converges to some point p. The point p must be an element of lim sup X_n and, consequently, must lie in $H \cup K$. This is a contradiction to the fact that p belongs to the closed set $S - (H \cup K)$.

We now have $H \cup K$ containing an infinite subsequence $\{X_{k_n}\}$ of the connected sets $\{X_n\}$. It therefore follows that the set K contains an infinite collection of connected sets, for if not then points of N fail to be elements of lim sup X_n. It also follows that if X_{k_j} is not a subset of K then X_{k_j} is the union of the non-empty disjoint sets $(K \cap X_{k_j})$ and $(H \cap X_{k_j})$ which contradicts the fact that X_{k_j} is connected. The same facts are true of H.

Let $x \in \liminf X_n$. Accordingly, $x \in \limsup X_n$ and, hence, is an element of H. Since the set H is open, it follows from the definition of lim inf X_n that H intersects all but a finite number of elements of $\{X_n\}$, which contradicts the fact that the open set K contains an infinite collection of elements from the sequence $\{X_n\}$. Thus, the Theorem follows.

We now have examples and theorems illustrating the definition and consequences of lim inf X_n, lim sup X_n, where $\{X_n\}$ is a sequence of subsets of S. Another type of convergence which is utilized in this paper is that of homeomorphic convergence.

<u>Definition</u>. A sequence $\{M_i\}$ of sets is said to <u>converge homeomorphically</u> to a set M if for each positive number δ there exists an integer N such that if $n > N$ there is a δ - homeomorphism h_n of M_n onto M. A δ - homeomorphism $h: M_n \to M$ is a homeomorphism such that $d(x, h(x)) < \delta$ for all x in M_n.

If the sequence $\{X_n\}$ converges homeomorphically to X, then for each integer n, X_n is homeomorphic to X. Hence, in Figure 2, since each X_n

is a triod and L is an arc, the sequence $\{X_n\}$ fails to converge homeomorphically to X. Similar arguments can be used to show that the sequences in Figures 3 and 4 fail to converge homeomorphically to L and A, respectively.

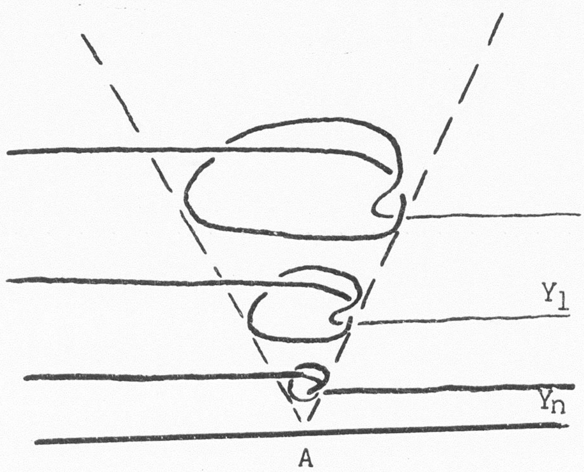


Figure 5

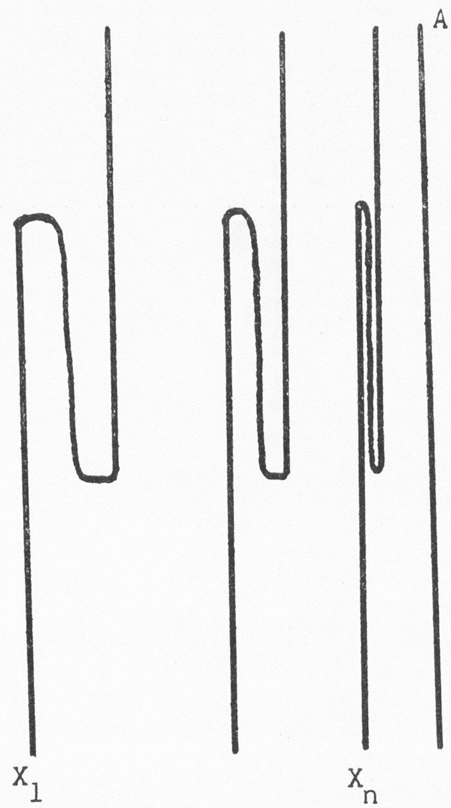


Figure 6

The sequence $\{Y_n\}$ in Figure 5 converges homeomorphically to A even though each arc Y_n is knotted. In Figure 6, we have a sequence of arcs $\{X_n\}$, each of which is homeomorphic to A, but for each integer n, X_n has a fold which maintains a fixed height as n varies. It is this fold that allows us to see that the proper δ - homeomorphisms in the definition of homeomorphic convergence fail to exist; hence, the sequence $\{X_n\}$ fails to converge homeomorphically to A even though $\lim\inf X_n = \lim\sup X_n = A$. This example will be of more importance when we discuss examples offered in Section IV.

Another important aspect of homeomorphic convergence can be found in the following example.

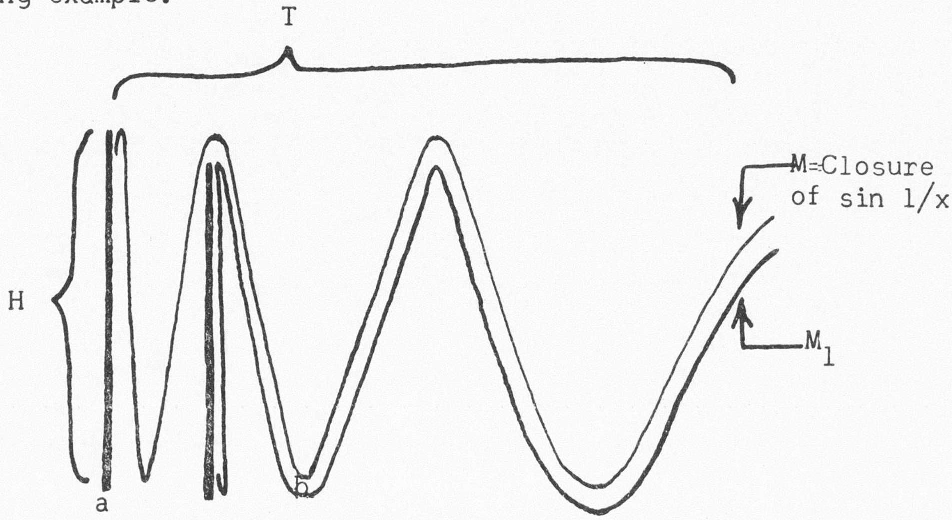


Figure 7

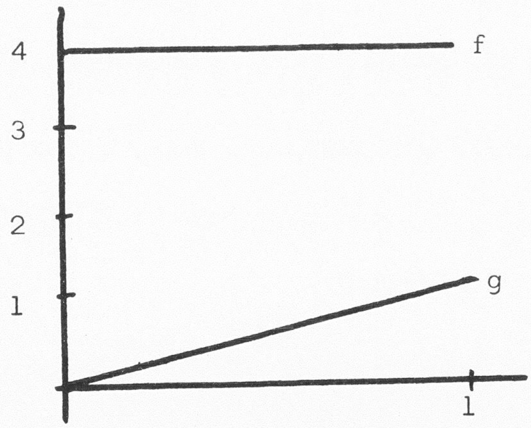
In Figure 7, we have pictured a continuum M which is made up of what we call the "head" H and the "tail" T. Let $\delta = 1/n$ be given and suppose that $d(a, b) < \delta/2$. We can now construct a continuum M_1 (see Figure 7) which is homeomorphically close to M, that is, there exists a homeomorphism h_1 such that $h_1(M) = M_1$, and $d(x, h_1(x)) < \delta$ for each $x \in M_1$. Let $\delta = 1/n$ where $n = \{1, 2, 3, \ldots\}$. Inductively construct the continuum M_n and the homeomorphism h_n, so that $d(x, h_n(x)) < \delta$ for each $x \in M_n$. By construction the sequence $\{M_n\}$ converges homeomorphically to M.

<u>Theorem 2.</u> If $\{X_n\}$ is a sequence of compact subsets of a separable metric space S which converges homeomorphically to X, then the sequence $\{X_n\}$ converges to X.

Proof. Let $\{h_n\}$ be a sequence of homeomorphisms satisfying the definition of $\{X_n\}$ converging homeomorphically to X. It follows that for a positive number ε, there exists an N such that, for $n > N$, $d(h_n(x), x) < \varepsilon$ for every $x \in X_n$. This shows that $X \subset \liminf X_n$. Let y be an element of $\limsup X_n$ and suppose that there exists an ε_0

such that $d(y, X) > 2\varepsilon_o$. Notice that corresponding to ε_o there exists an N_o such that, for $m > N_o$, $d(h_m(x), x) < \varepsilon_o$ for each $x \in X_m$, hence, $N(y, \varepsilon_o)$ intersects at most a finite number of X_n in the sequence $\{X_n\}$. Consequently, y is not an element of lim sup X_n, and the sequence $\{X_n\}$ converges to X.

Before proceeding into the Borsuk Theorem, we will consider the following space $\pi = \{f{:}f$ maps the compact set P continuously into the separable metric space Q.$\}$ We define a function $d: \pi \times \pi \to E^1$ as $d(f, g) = \sup \{\rho(f(x), g(x)) \mid x \in P\}$ where ρ is the metric for the space Q. It is easily verified that d is a metric for π. In the remainder of the paper, we use "<u>sup metric</u>" to mean the metric d just defined above.



$P = [0, 1] \qquad Q = E^1$

$f(x) = 4$

$g(x) = x$

$\rho(f, g) = (f(0), g(0)) = 4$

Figure 8

The following theorem can be found in most topology texts (See [6] for example.) We give no proof here, but note that it is a very special case of Theorems A, B, and 6 which follow.

Theorem 3. If every uncountable subset G of a metric space S has a limit point, then S is separable.

A proof of the converse of Theorem 3 follows.

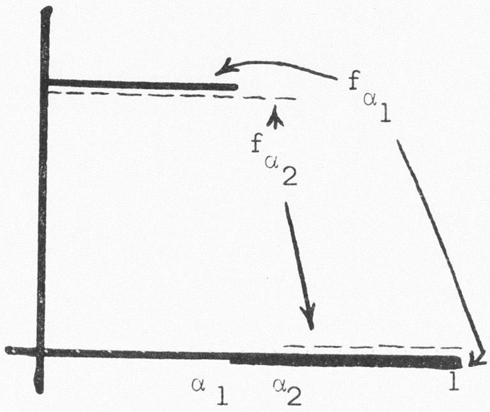
Theorem 4. If S is a separable metric space and C is an uncountable subset of S, then some point of C is a limit point of C.

Proof. Suppose that C fails to contain such a limit point. Let $p \in C$. Then there exists a positive number ε such that $N(p,\varepsilon) \cap C = \{p\}$. Consider the set $G = \{N(p, \varepsilon/2) \mid p \in C$ and ε satisfies the conditions above$\}$. We now have G as an uncountable collection of disjoint open sets in S which contradicts the fact that S is separable. Consequently, C contains some point g such that g is a limit point of C.

An observation that will accent the importance of the Borsuk Theorem follows. Consider the set $\pi^1 = \{f{:}f$ is bounded and maps the compact set P into the separable metric space Q$\}$, where d is the sup metric identified previously. We will now show that π^1 need not be separable. Let $P = [0,1]$, and $Q = E^1$. Consider the set $C = \{f_\alpha{:}f_\alpha(x) = 1$ for $0 \leq x \leq \alpha$, and $f_\alpha(x) = 0$ for $\alpha < x \leq 1$ where α is an irrational number greater than 1/2 and less than 3/4$\}$. It follows from Theorem 3 that if π^1 is separable then the set C, which is uncountable, must contain a point g which is a limit point of C. But, as Figure 9 illustrates, under the sup metric C fails to contain such a point g because C inherits the discrete topology. Hence, π^1 is not separable.

Let $\alpha_1 < \alpha_2$.

Then $d(f_{\alpha_1}, f_{\alpha_2}) = d(f_{\alpha_1}(\alpha_2), f_{\alpha_2}(\alpha_2)) = 1$



$$f_{\alpha_1} = \begin{cases} 1 \text{ if } 0 \leq x \leq \alpha_1 \\ 0 \text{ if } \alpha_1 < x \leq 1 \end{cases}$$

$$f_{\alpha_2} = \begin{cases} 1 \text{ if } 0 \leq x \leq \alpha_2 \\ 0 \text{ if } \alpha_2 < x \leq 1 \end{cases}$$

Figure 9

<u>Theorem (Borsuk).</u> If P is a compact metric space, Q is a separable metric space, and π is the collection of all continuous functions from P to Q with the sup metric, then π is a separable space.

Proof. Let the set $R = \{C_1, C_2, C_3, \ldots\}$ represent a countable dense subset of Q, and the set $B = \{B_1, B_2, B_3, \ldots\}$ be a countable basis for the compact metric space P.

Let Σ be the collection of all finite subsets $\{n_1, n_2, \ldots, n_k\}$ of the set N of positive integers such that $P \subset \bigcup_{i=1}^{k} B_{n_i}$. Notice that Σ is countable, furthermore, since P is compact, Σ is nonempty. Now with $\sigma = \{n_1, n_2, \ldots, n_k\} \in \Sigma$ define the sets $W_{n_1}, W_{n_2}, \ldots, W_{n_k}$ by letting $W_{n_1} = B_{n_1}$, $W_{n_2} = B_{n_2} - B_{n_1}$ and in general $W_{n_i} = B_{n_i} - \{\bigcup_{j=1}^{i-1} B_{n_i}\}$. Notice that such a collection of sets is pairwise disjoint and has a union which contains P.

Define $\pi_\sigma = \{f \mid f: P \to R$ where f is a constant on $W_{n_i}\}$. To show that π_σ is countable, it suffices to see that f assumes at most a countable number of values on each W_{n_i}, or that π_σ is the finite union of countable sets. Hence, for a given element σ in Σ there exists a countable collection of functions in π_σ.

Since the countable union of countable sets is countable, it follows that $\pi^1 = \bigcup_{\sigma \in \Sigma} \pi_\sigma$ is countable, since Σ is countable.

Consider the metric space ψ formed by the union of π and π^1 with the sup norm. We shall now show that ψ is separable by showing that π^1 is a countable dense subset of ψ. Let $g \in \pi$. Since g is continuous and P is compact, we have that g is uniformly continuous. Let $\varepsilon > 0$ be given. Then there exists a positive number δ such that if $p(x,y) < \delta$ then $d(g(x), g(y)) < \varepsilon/3$.

Let $B_{k_1}, B_{k_2}, \ldots, B_{k_n}$ be a finite collection of basis elements, each having diameter less than δ, which covers P. As before, define $W_{k_1} = B_{k_1}$ and $W_{k_i} = B_{k_i} - \{\bigcup_{j=1}^{i-1} B_{k_j}\}$ and recall that not only is the collection $\{W_{k_1}, \ldots, W_{k_n}\}$ pairwise disjoint, but the union contains P.

Choose points $x_{k_i} \in W_{k_i}$ for $i = 1, 2, \ldots, n$ and let $C_i \in R$ be a point in Q such that $d(g(x_{k_i}), C_i) < \varepsilon/3$. Let F be the function in π^1 defined by $f(x) = C_i$ for all $x \in W_{k_i}$.

Now for $x \in P$ we have $d(f(x), g(x)) \leq d(f(x), f(x_{k_i})) + d(f(x_{k_i}), g(x_{k_i})) + d(g(x_{k_i}), g(x)) < 0 + \varepsilon/3 + \varepsilon/3 < \varepsilon$ if $x \in W_{k_i}$. Thus, sup $\{d(f(x), g(x))$ for all $x \in P\} < \varepsilon$ and g belongs to the closure of π^1. Hence, π^1 is a countable dense subset of ψ.

The fact that π is separable falls as a consequence to the fact that ψ is a separable metric and that separability is hereditary in a metric space.

<u>Theorem A.</u> If G is an uncountable collection of compact sets in a separable metric space S, then some sequence of elements of G converges to an element of G.

Proof. Let P be a Cantor set. For each G_α in $G = \{G_\alpha \mid \alpha \epsilon A\}$ there exists a continuous function $f_\alpha: P \to G_\alpha$ of P onto G_α [6].

Let $\pi = \{f: f$ is a continuous map of P into S$\}$. By Borsuk Theorem, we know that π is separable, hence, if we define $R = \{f_\alpha: \alpha \epsilon A\}$ then R is an uncountable subset of π. It follows from Theorem 3 that R contains a point g which is a limit point of R. Since g is a limit point of R in the metric space π, it follows that there exists a sequence of points $\{f_n\}$ in R which converges to g.

Corresponding to g is a compact set X in G, where $g: P \to X$. Similarly, there is a sequence of compact sets $\{X_n\}$ in G so that f_n maps P continuously onto X_n.

We now intend to demonstrate that the sequence $\{X_n\}$ converges to X, that is; $\lim\inf X_n = \lim\sup X_n = X$.

First we show that $X \subset \lim\inf X_n$. Let $x \in X$ and let O be any open set containing x. Since the sequence $\{f_n\}$ converges to g in π, we know that the sequence of points $\{f_n(x)\}$ must then converge to q(x) in S. Thus, the point x belongs to $\lim\inf X_n$.

We now have that $X \subset \lim\inf X_n$, hence, if the $\lim\sup X_n \subset X$ the sequence $\{X_n\}$ converges to X. Let $y \epsilon \lim\sup X_n$, and suppose $y \notin X$. Since X is compact, there is a positive number ε such that $d(y, X) > 2\varepsilon$. Now there exists an integer N so that for $n \geq N$ $\sup\limits_{x \varepsilon p} \{d(f_n(x), f(x))\} < \varepsilon$. Thus, $N(y, \varepsilon)$ can intersect at most a finite number of $\{X_n\}$, and y cannot be in $\lim\sup X_n$. It follows that $\lim\sup X_n = X$, from which the theorem follows.

A consequence of Theorem A follows.

<u>Theorem 5.</u> If G is an uncountable collection of compact sets in a separable metric space S, then there exists a countable subcollection G^1 of elements of G such that for every $g \varepsilon G - G^1$ there exists a sequence $\{g_n\}$ of elements of G^1 converging to g.

Proof. Let $G = \{X_\alpha \mid \alpha \epsilon A\}$ and let $G^1 = \{X_B \mid X_B$ is not converged to by a sequence of elements in G$\}$. Suppose G^1 is an uncountable set. It therefore follows from Theorem A that there exists an $X \epsilon G^1$ such that X is converged to by a sequence of sets in G^1. Since such an X in G^1 contradicts the construction of G^1, it follows that G^1 is at most countable.

Theorem A gives us sufficient conditions for convergence of a sequence $\{X_n\}$ of compact subsets of S, where S is a separable metric space. The following theorem outlines sufficient conditions for homeomorphic convergence of the sequence $\{X_n\}$ in S.

<u>Theorem B.</u> If G is an uncountable collection of pairwise homeomorphic subsets of a separable metric space S, then some sequence of elements of G converges homeomorphically to an element of G.

Proof. Let $X_o \epsilon G$ where $G = \{X_\alpha \mid \alpha \epsilon A\}$. For each $\alpha \epsilon A$, there is a homeomorphism f_α of X_o onto X_α. Let $R = \{f_\alpha \mid \alpha \epsilon A\}$.

Define $\pi = \{f \mid f$ is a continuous map of X_o into S$\}$. Since R is an uncountable subset of π it follows from Theorem 3 that R contains a point f_{α_o} such that f_{α_o} is a limit point of R, hence, there exists a sequence $\{f_{\alpha_i}\}$ in R which converges to f_{α_o}. For convenience in notation, we let $f_{\alpha_o} = f$, $f_{\alpha_i} = f_i$, $X_{\alpha_i} = X_i$, and $X_{\alpha_o} = X$ for each integer i.

For each positive integer n define a function $g_n = f_n \circ f^{-1}: X \to X_n$. Notice that the sequence $\{g_n\}$ is a sequence of homeomorphisms between X and X_n.

We now intend to show that the sequence $\{X_n\}$ converges homeomorphically to X. Recall that the sequence $\{f_n\}$ converges to f in π, hence, given a $\delta > 0$ there exists an integer N such that for $n > N$, $\sup_{x \in X_o} \{d(f_n(x), f(x))\} < \delta$. Consequently, if $y \in X$, then there exists an $x_o \in X_o$ such that $f(x_o) = y$, hence, for $n > N$ $d(g_n(y), y) = d(f_n \circ f^{-1}(y), y) = d(f_n(x_o), f(x_o)) < \delta$.

We now have that the sequence $\{g_n\}$ is a sequence of δ-homeomorphisms, thus, $\{X_n\}$ converges homeomorphically to X.

Given a set G with certain properties in a separable metric space, Theorem B guarantees the existence of a sequence $\{X_n\}$, where for each integer n, $X_n \in G$, and a set $X \in G$ such that the sequence $\{X_n\}$ converges homeomorphically to X. The following theorem says that all but a countable number of elements in G are converged to homeomorphically by a sequence from G.

<u>Theorem 6.</u> If G is an uncountable collection of pairwise homeomorphic compact sets in a separable metric space S, then there exists a countable subcollection G^1 of elements of G such that for every $g \in G - G^1$ there exists a sequence $\{g_n\}$ of elements of G converging homeomorphically to g.

Proof. Let $G = \{X_\alpha \mid \alpha \in A\}$ and let $G^1 = \{X_B \in G \mid X_B$ is not converged to homeomorphically by a sequence of elements in G$\}$. Suppose G^1 is an uncountable set. It therefore follows from Theorem B that there exists $X \in G^1$ such that X is converged to homeomorphically by a sequence of sets in G^1. Since such an X in G^1 contradicts the construction of G^1, it follows that G^1 is at most countable.

In concluding Section I, we will state a theorem which deals with a restricted type of continuum M, a continuum M which divides the space S into two sets (complementary domains). Examples of such continua are the (n-1)-dimensional spheres $S^{(n-1)}$ in Euclidean n-dimensional space E^n ($n = 2, 3, 4, \ldots$), and the singleton sets on the real line E^1.

A consequence of Theorem 7 is that there do not exist uncountably many disjoint wild 2-spheres in E^3 [2]. Furthermore, Bryant [4] proved that there do not exist uncountably many n-cells with disjoint boundaries in E^n ($n \geq 5$) using Theorem 7.

<u>Theorem 7.</u> Let G be an uncountable collection of disjoint homeomorphic continua in E^n and suppose that each G_α in G has exactly two complementary domains V_α and U_α. Then G contains a countable subcollection G^1 such that for each G_α in $G - G^1$ there exists two sequences $\{G_{\alpha_i}\}$ and $\{G_{B_i}\}$ of elements of G each converging homeomorphically to G_α such that $G_{\alpha_i} \subset V_\alpha$ and $G_{B_i} \subset U_\alpha$ for each i.

Proof. It follows from Theorem 6 that there exists a countable set K such that for every element g of G - K there exists a sequence $\{g_n\}$ in G which converges homeomorphically to g.

Divide the set G - K into two sets U and V where $U = \{G_\alpha \in G - K \mid$ there exists a sequence of sets $\{G_{\alpha_i}\}$ in U_α which converges homeomorphically to $G_\alpha\}$ and $V = \{G_\alpha \in G - K \mid G_\alpha$ is converged to homeomorphically by some sequence $\{G_{B_i}\}$ in G, where $G_{B_i} \subset V_\alpha$ for each integer i$\}$.

Since G - K is known to be uncountable, it follows that U or V is uncountable since $G - K = U \cup V$. Therefore, suppose V is uncountable.

Define the set $G_1 = \{G_\alpha \in G - K \mid$ there exists a $G_B \in G$ such that $U_\alpha \supset U_B$ and there is a homeomorphism f from G_B onto G_α such that $d(x, f(x)) < 1$ for each $x \in G_B\}$. Let $G_\alpha \in U$. Then there exists a sequence $\{G_{\alpha_i}\}$ in U_α which converges homeomorphically to G_α. Notice that, for each integer i, $U_\alpha \supset U_{\alpha_i}$. Consequently, $G_\alpha \in G_1$ and hence, $U \subset G_1$.

Let $M_1 = \{G_\alpha \mid G_\alpha \in V$ but $G_\alpha \notin G_1\}$. Suppose now that M_1 is uncountable. It follows from Theorem 6 that there exists a $G_\alpha \in M_1$ and a sequence $\{G_{\alpha_i}\}$ in M_1 such that $\{G_{\alpha_i}\}$ converges homeomorphically to G_α. If each $G_{\alpha_i} \in U_\alpha$ then $G_\alpha \in G_1$. Furthermore, if each $G_{\alpha_i} \in V_\alpha$ then $V_{\alpha_i} \subset V_\alpha$ which implies $U_{\alpha_i} \supset U_\alpha$, and if $\delta = 1$ there exists a δ-homeomorphism f_{α_n} such that $d(f_{\alpha_n}(x), x) < \delta$ for each $x \in U_{\alpha_n}$ and $n \geq N$. Consequently, $G_{\alpha_n} \in G_1$ for each integer greater than N. In either case, we have a contradiction.

Inductively define a sequence of sets $\{G_i\}$ where, for each integer i, $G_i = \{G_\alpha \in G - K \mid$ there exists a $G_B \in G$, and a homeomorphism $f_B: G_B \to G_\alpha$ such that $U_\alpha \supset U_B$ and $d(x, f(x)) < 1/i$ for each $x \in G_B\}$. Some of the properties acquired by each G_i are that $U \subset G_i$ for each integer i, and G_i contains all but a countable number of elements from V.

Let $N = \bigcap_1^\infty G_i$. It follows from above that $U \subset N$ and that all but a countable number of elements in V also lie in N. Let $G_\alpha \in N$. Let $\varepsilon > 0$ be given. Then there exists an integer n such that $\varepsilon > 1/n$. Hence, we can find a G_B such that $U_B \subset U_\alpha$ and $d(x, f(x)) < 1/n < \varepsilon$ for each $x \in G_B$, where f maps G_B homeomorphically to G_α. Consequently, every element G_α in N has a sequence of sets converging homeomorphically to G_α in U_α.

A similar argument can be used to define a set N^1 where for each element G_α in N^1 there exists a sequence of sets converging homeomorphically to G_α in V_α. Consequently, $N \cap N^1$ describes a subcollection of G which contains all but a countable subset of G. Furthermore for each G_α in $N \cap N^1$ there exists two sequences $\{G_{\alpha_i}\}$ and $\{G_{B_i}\}$ of elements of G, each converging homeomorphically to G_α such that $G_{\alpha_i} \subset V_\alpha$ and $G_{B_i} \subset U_\alpha$ for each integer i.

II. EQUIVALENT CONTINUA

We begin Section II by defining the relationship of equivalent between two homeomorphic continua X_1 and X_2 in a space S.

Definition. Two homeomorphic continua M_1 and M_2 in a space S are said to be equivalent (equivalently embedded) in S if there exists a homeomorphism h of S onto S such that $h(M_1) = M_2$.

A result that we state without proof is that every two simple closed curves in E^2 are equivalent in E^2. [9]

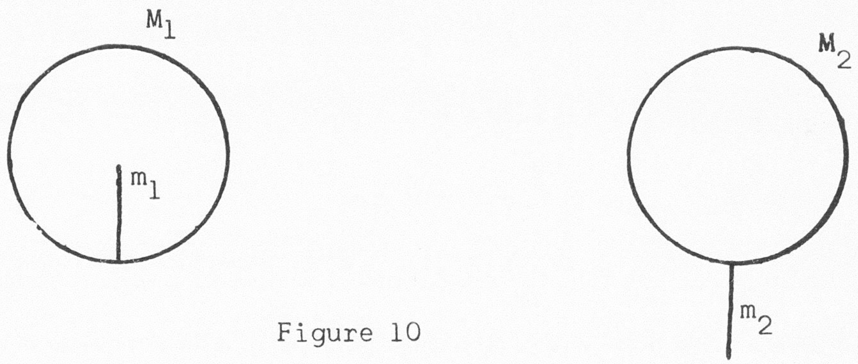


Figure 10

In Figure 10, we have two continua, M_1 and M_2, each of which contains a simple closed curve and an arc intersecting at a single point. Since M_1 and M_2 are homeomorphic, it is possible that M_1 and M_2 are equivalent in E^2, that is; there exists a homeomorphism h which maps E^2 onto E^2 and $h(M_1) = M_2$. Since h is a homeomorphism, it must take the simple closed curve of M_1 to a simple closed curve in M_2. Furthermore, h must map the interior of M_1 (Int M_1) to the interior of M_2 (Int M_2) because h maps the compact set $M_1 \cup$ Int M_1 onto a compact set. Hence, the arc m_2 must lie in the interior of M_2. Consequently, M_1 and M_2 are not equivalent in E^2.

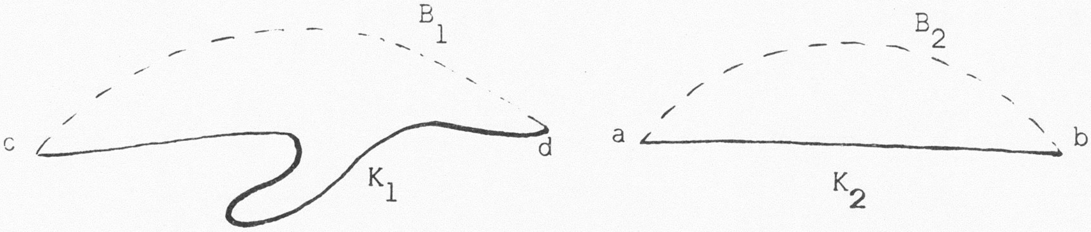
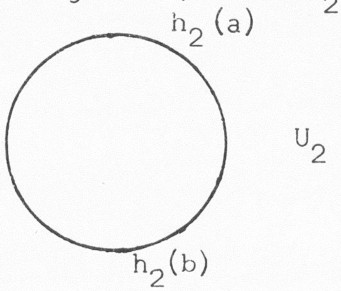


Figure 11

To show that every two arcs K_1 and K_2 in E^2 are equivalent in E^2, we construct a simple closed curve from K_2 by attaching an arc B_2 to K_2 at the points $\{a,b\}$ (See Figure 11). Now $K_2 \cup B_2$ is equivalent to a circle U_2,



that is; there exists a homeomorphism h_2 which maps E^2 onto E^2 and $h_2(K_2 \cup B_2) = U_2$. Similarly, there exists an arc B_1 and a homeomorphism h_1 which maps E^2 onto E^2 so that $h_1(K_1 \cup B_1) = U_2$. Now let h_3 be a homeomorphism of E^2 onto itself such that $h_3(h_1(K_1)) = h_2(K_2)$. The composition $h_2^{-1} \cdot h_3 \cdot h_1$ is a homeomorphism of E^2 onto E^2 that takes K_1 onto K_2.

<u>Definition.</u> A continuum M is <u>thin</u> in a space S (or thinly embedded in S) if there exists an uncountable collection of pairwise disjoint equivalent images of M in S.

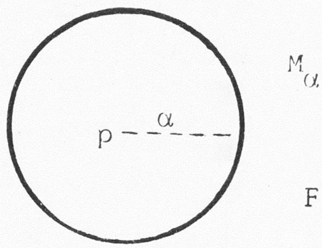


Figure 12

In Figure 12, M_α is a circle centered at the point p with radius α in E^2. Let α assume any real number in the interval $[0, 1]$ and we have an uncountable collection of disjoint equivalent simple closed curves in E^2. Thus, each simple closed curve in E^2 is thin in E^2.

Figure 13

As a further illustration, we now show that each arc in E^2 is thin in E^2. Since every two arcs in E^2 are equivalent, it suffices to exhibit an uncountable collection of disjoint arcs in E^2. For each real number α let K_α be a vertical arc of length 1 with its lower end point on the x-axis at the point $(\alpha, 0)$. (See Figure 13.) Then $G = \{K_\alpha \mid \alpha \in [0,1]\}$ is the desired collection.

<u>Definition.</u> A continuum M is <u>slender</u> in a space S if there exist uncountably many disjoint homeomorphic images of M in S.

Notice that if a continuum M is thin in S, then M is also slender in S. Therefore, the continua in Figures 12 and 13 are slender in E^2. Later we will construct continua M_1 and M_2 which are homeomorphic, but M_2 will be thin in E^2 while M_1 is slender but fails to be thin in E^2.

<u>Definition.</u> A continuum M is <u>thick</u> in S if it is not slender in S.

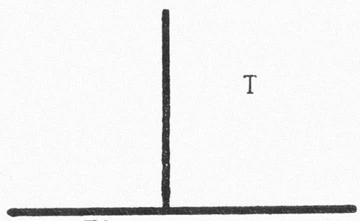


Figure 14

The triod (See Figure 14) is a continuum which is not slender in E^2 and so it must be thick in E^2. It is also the case that any continuum

M which contains a triod as a subset will be thick in E^2. (See Figure 15). These results are proven in [8]. Easier proofs of these facts are given in [7].

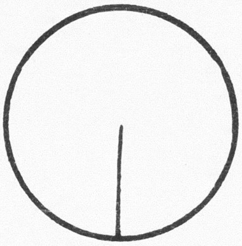


Figure 15

Definition. A continuum M is wide in S if M contains an open subset of S.

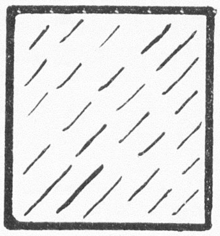


Figure 16

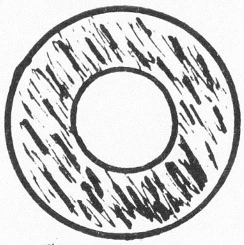


Figure 17

The continua pictured in Figures 16 and 17 are wide in E^2. Notice that a continuum M that is wide in a separable metric space S must also be thick in S, but some thick continua in E^2 are not wide in E^2. (See Figure 15).

Given a continuum M which contains an open subset of E^2, it is clear why there fails to exist an uncountable collection of disjoint homeomorphic images of M in E^2.

As was seen in Figure 10, given a continuum M it is sometimes possible to have more than one embedding of M in S. It is natural to ask if there is a continuum M in E^2 which has an infinite number of inequivalent embeddings in S.

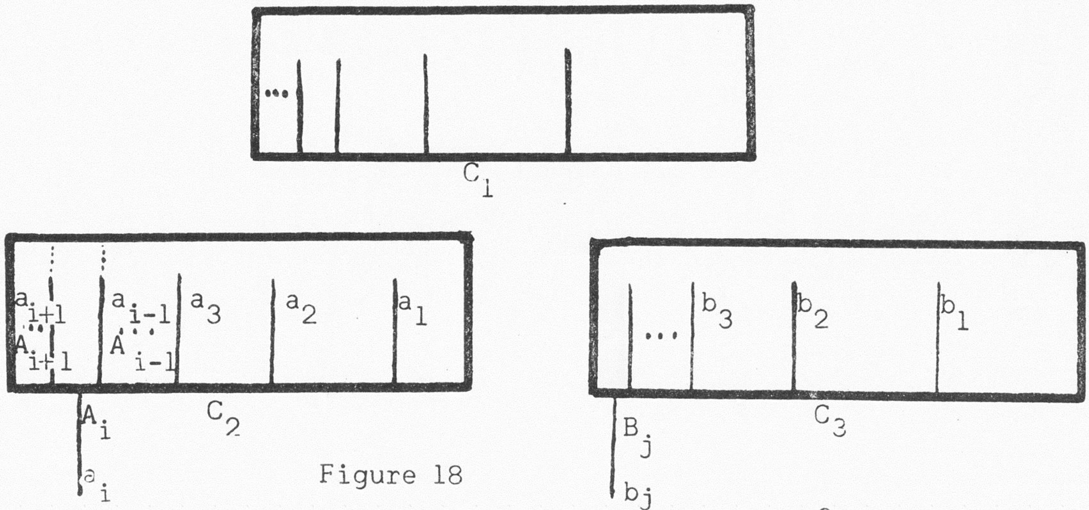


Figure 18

Three homeomorphic continua C_1, C_2, and C_3 in E^2 are pictured in Figure 18. From considerations similar to those used in connection with Figure 10, it is clear that C_i $(i = 2, 3)$ is not equivalent to C_1. We shall now show that C_2 and C_3 are not equivalent whenever $i \neq j$. Suppose there is a homeomorphism h of E^2 onto E^2 such that $h(C_2) = C_3$. It follows from previous considerations (see Figure 10) that, under h the vertical arc A_i, whose end point is a_i, must go onto the vertical arc B_j, whose end point is b_j. Now extend the vertical arcs A_{i+1} and A_{i-1}, with end points a_{i+1} and a_{i-1} respectively, upward as indicated by the dotted lines in Figure 18, and let J_{i-1} be the simple closed curve containing $A_{i+1} \cup A_{i-1}$ and otherwise lying in C_2. Since h takes J_{i-1} onto a simple closed curve, it is easily seen that $h(a_{i-1}) = b_{j-1}$. An inductive argument can now be applied to show that $h(a_1) = b_{j+k}$ for some $k > 0$ assuming $i > j$. Now there is nowhere for h^{-1} to map b_{j+k-1}. A similar contradiction exists by assuming $j < i$. Now we see that the continuum C_1 has a countably infinite number of different embeddings in E^2.

A similar example can be constructed to produce a continuum L with uncountably many inequivalent embeddings in E^2. Let L be the continuum pictured in Figure 19

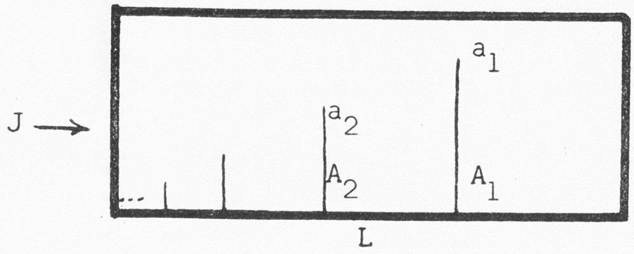


Figure 19

In L there is only one simple closed curve J while there is a countable collection of disjoint vertical arcs $\{A_1, A_2, \ldots\}$ each having an end point a_i in the interior J. Now $L = J \cup (\bigcup_{i=1}^{\infty} A_i)$. For each sequence $\alpha = \{X_n\}$ of real numbers where $X_n \in \{0, 1\}$ we now describe an embedding L_α of L in E^2. If $\alpha = 0, 0, 0, \ldots$ then $L_\alpha = L$. One further example will suffice to make the correspondence between α and L_α clear. Suppose $\alpha = 0, 1, 0, 0, 1, 0, 0, \ldots$ Then L_α is pictured in Figure 20.

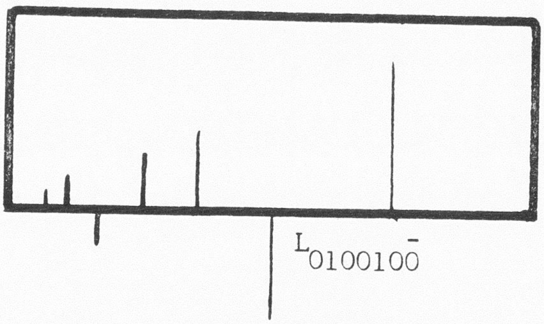


Figure 20

Since there are uncountably many sequences of zeros and ones, it is clear that we have an uncountable collection G of homeomorphic continua in E^2. Now we give an indication of why no two elements of G are equivalent.

Choose two different sequences α and β, and let n be the least integer such that the n^{th} term of α and β are different. This means that L_β and L_α have their vertical arcs A_n in opposite complementary domains of their simple closed curve J. Now we apply the inductive argument above to show that this is impossible.

We conclude Section II with a theorem that coordinates the concept of homeomorphic convergence from Section I and equivalent continua.

Theorem 8. If $\{X_n\}$ is a sequence of continua converging homeomorphically to M_1 in a metric space S and h is a homeomorphism of S onto itself, then $\{h(X_n)\}$ converges homeomorphically to $h(M_1)$.

Proof. Restrict h to M_1 and the sequence $\{X_n\}$. Now h is not only continuous but uniformly continuous. Let $\{h_n\}$ be a sequence of homeomorphisms such that $h_n : X_n \to M_1$ and h_n moves points less than a distance $1/n$. There exists a positive number δ such that if $d(x, y) < \delta$, then $d(h(x), h(y)) < \varepsilon$. Choose N so that h_N moves points of X_n less than a distance δ. Then $d(h(h_N(x)), h(x)) < \varepsilon$ since $d(h_N(x), x) < \delta$. Since this is also true for all $n > N$ we see that the sequence $\{h \circ h_n\}$ of homeomorphisms suffices for the homeomorphic convergence of $\{h(X_n)\}$ to $h(M_1)$.

III. CHAINABLE CONTINUA

In Section III we introduce the concepts of "ε-chain," "chainable," and "chainable-with-nice-links."

Definition. Given a positive number ε, a continuum M has an ε-chain if there exists a finite sequence of open sets, $L_1, L_2, \ldots, L_n$, each of which is called a link, such that $M \subset \bigcup_{i=1}^{n} L_i$, each link has a diameter less than ε, and $L_i \cap L_j$ is nonempty if and only if $|i - j| = 1$.

A continuum M is chainable if, for each positive ε, M has an ε-chain.

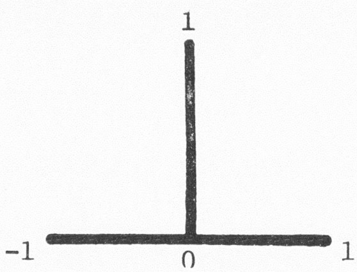


Figure 21

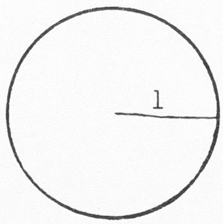


Figure 22

Both the triod in Figure 21 and the circle in Figure 22 have ε-chains for restricted values of ε, but both fail to have a 1/2-chain. Hence, neither the triod nor the simple closed curve is chainable.

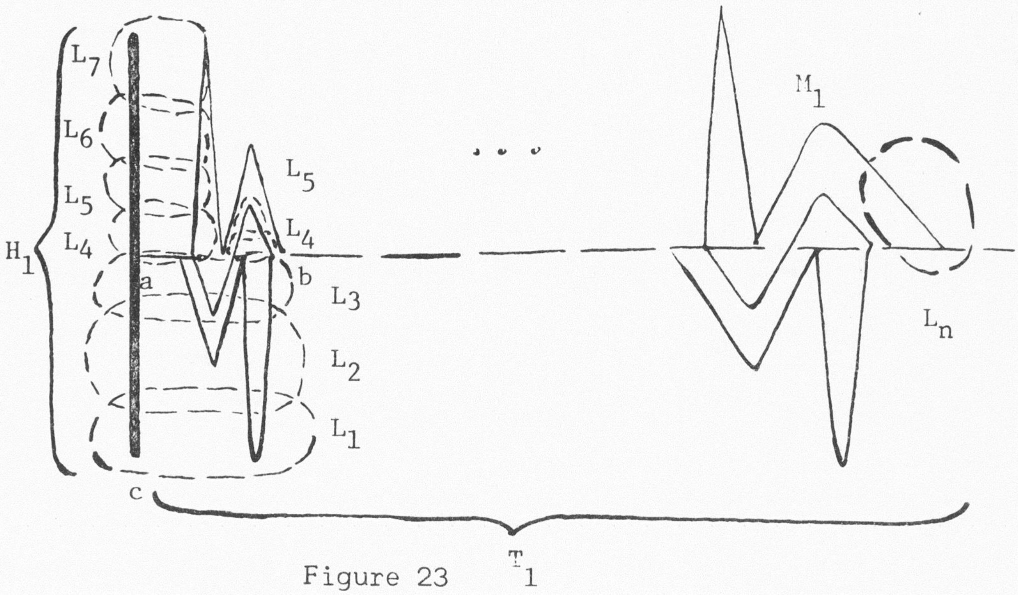


Figure 23

In Figure 23, we have pictured a continuum M_1 [1] which is the union of H_1 and T_1. H_1 is the vertical interval, and T_1 is the "wiggly" open ray which "converges" to H_1. The vertical line segment H_1 will be referred to as the head of M_1, while everything to the right of H_1, labeled T_1, is the tail of M_1. The horizontal line is not part of the continuum M_1. Only two cycles of T_1 are pictured.

At this time we wish to convince the reader that M_1 is actually chainable. Therefore, let ε be given. It follows from the construction of T_1 that there exists a point b such that $d(a, b) < \varepsilon/2$. (See Figure 23). We construct the first link L_1 of our ε-chain about the point c of H_1 and proceed along H_1 with the connected links L_2 and L_3. Notice that links L_4 and L_5, although disconnected, maintain a diameter less than ε. We complete the ε-chain about M_1 with the connected links L_6 through L_n.

Pictured in Figure 24 is one cycle of M_1 illustrating the first nine links in the ε-chain about M_1.

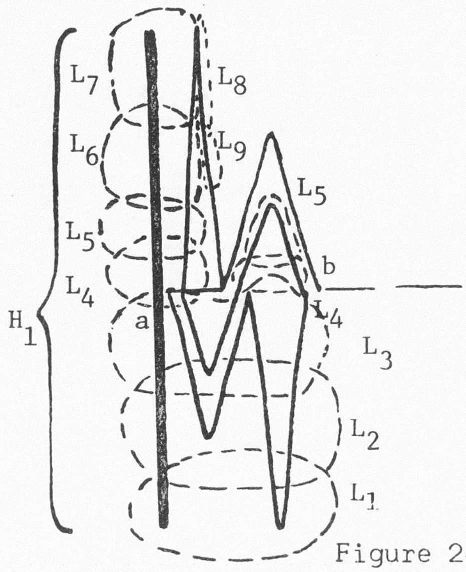


Figure 24

Since such a collection of links $L_1, L_2, \ldots, L_n$ can be constructed for any positive number ε, it follows that M_1 is a chainable continuum.

We will now define a more restrictive type of chain which will lead to the concept of "chainable-with-nice-links."

Definition. A disk K is any set which is homeomorphic to the set $K = \{(x, y) \mid x^2 + y^2 \leq 1\}$ in E^2. The boundary, Bd K, of K and the interior, Int K, of K are the sets homeomorphic with $\{(x, y) \mid x^2 + y^2 = 1\}$ and $\{(x, y) \mid x^2 + y^2 < 1\}$, respectively.

Definition. A continuum M in E^2 is chainable-with-nice-links if, for each positive number ε, there is a collection of disks $D_1, D_2, \ldots, D_n$ in E^2 such that $\{\text{Int } D_1, \text{Int } D_2, \ldots, \text{Int } D_n\}$ forms an ε-chain of M, $\text{Bd } D_i \cap \text{Bd } D_j = \emptyset$ if $|i - j| > 1$, and $\text{Bd } D_i \cap \text{Bd } D_j$ consists of two points if $|i - j| = 1$.

It follows from the definition that if a continuum M is not chainable then it also fails to be chainable-with-nice-links. Therefore, the triod in Figure 21 and the circle in Figure 22 fail to be chainable-with-nice-links.

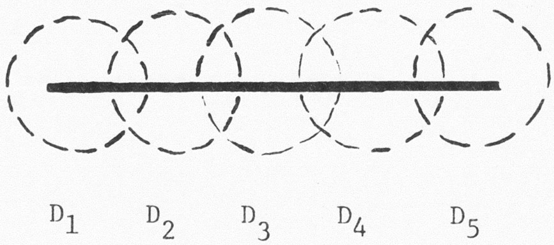


Figure 25

The arc, as shown in Figure 25, is a continuum which is chainable-with-nice-links.

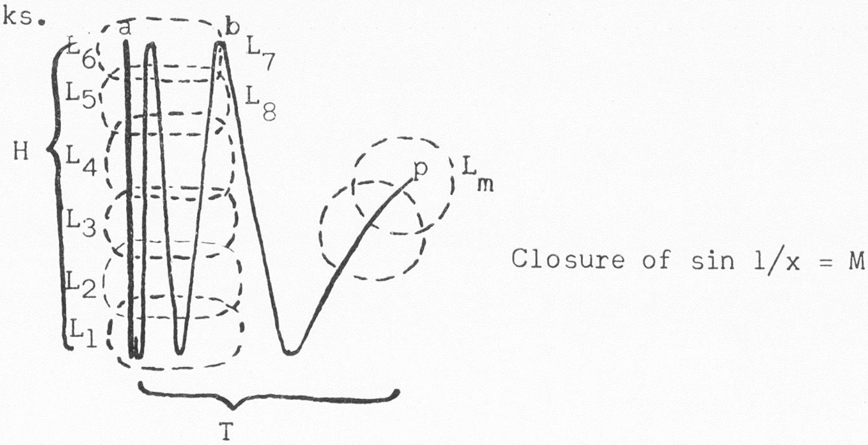


Figure 26

The continuum M of Figure 26 is divided into two parts; the head H, and the tail T. Let ε be given. To show that M is chainable-with-nice-links we locate the point b (see Figure 26) where $d(a\ b) < \varepsilon/2$. We construct our ε-chain about M by first covering H. It is essential that none of the links L_1 through L_6 intersect any portion of T to the right of b. Notice that the links $L_1, \ldots, L_m$ form an ε-chain about M. Furthermore, since such a collection $L_1, L_2, \ldots, L_m$ of disks exists for any ε, we see that M is chainable-with-nice-links.

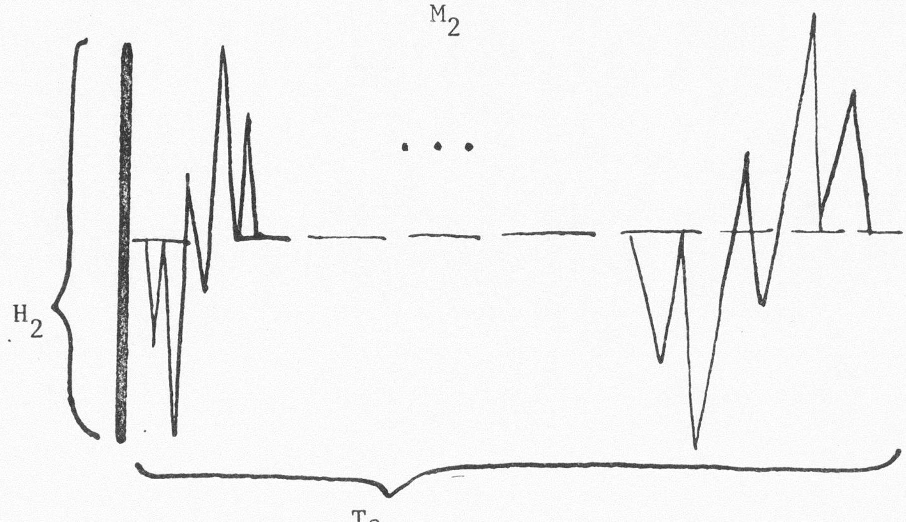


Figure 27

The continuum M_2, pictured in Figure 27, is borrowed from [1] where it is described using trigonometric functions. Using the previous chaining techniques, it is easy to see that M_2 is chainable-with-nice-links.

At this point we wish to recall the continuum M_1 (see Figure 23) and indicate why M_1 fails to be chainable-with-nice-links. Let ε be 1/4.

The reader must first convince himself that a 1/4-chain of M_1 constructed as in Figure 23 fails to satisfy the definition of chainable-with-nice-links. Furthermore, if M_1 is chainable-with-nice-links then such an ε-chain of M_1 will be constructed in a manner similar to that of M in Figure 26.

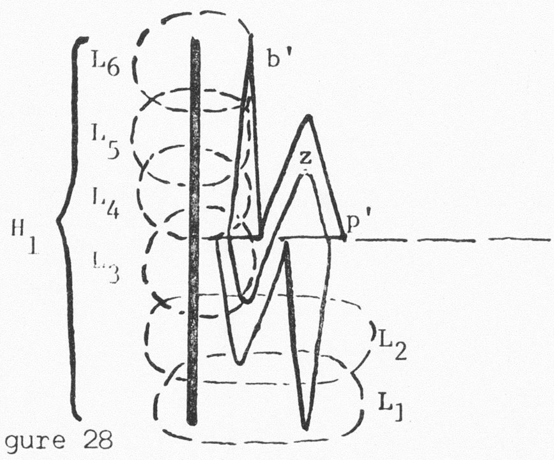
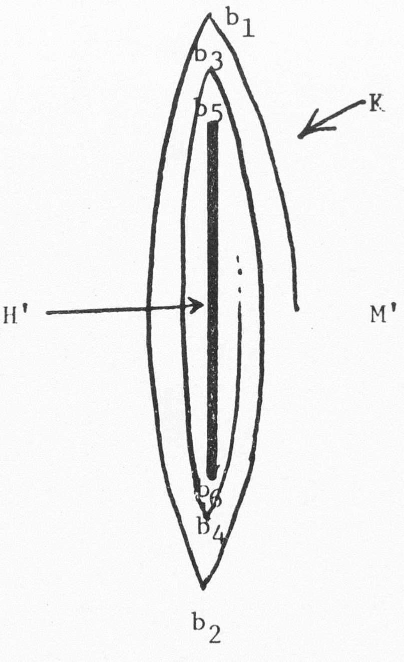


Figure 28

Notice that in the construction of the links L_1, L_2, ..., L_6 in Figure 26 it was important that these links leave the tail T of M uncovered from the point b to p. Similarly, it is essential that in covering H_1 of M_1 the arc between b' and p' must be left uncovered. Notice that with these restrictions upon our ε-chain it is impossible for any link to contain the point z in Figure 28. Based upon this initial hint the reader will convince himself that M_1 fails to be chainable-with-nice-links.

We would now like to draw the readers attention to the following fact. In Figure 23 we have described a continuum M_1 which is chainable but not chainable-with-nice-links. In Figure 27 we have described a continuum M_2 which is chainable-with-nice-links. Furthermore, it can be shown that M_2 is homeomorphic to M_1. Since M_1 and M_2 will be observed further in Section IV, it is of importance that the above results be understood.

We now describe two embeddings of a continuum M in E^2 and note that both are chainable-with-nice-links. These continua will be referred to in the next section.



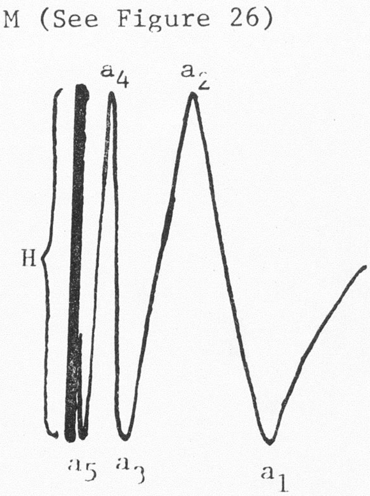


Figure 29

In Figure 29 we have described a continuum M' which is homeomorphic to the closure of sin 1/x = M (see Figure 26). The homeomorphism h, from M to M', can be constructed as follows: h maps H of M horizontally onto H' of M'. Then h maps the tail T of M onto the spiral K of M' by mapping the arc with endpoints (a_i, a_{i+1}) to the arc with endpoints (b_i, b_{i+1}) in K using a "uniform stretch." It is readily seen that this may be carried out so that h preserves the convergence of sequences and, hence, is a homeomorphism.

Notice that the continuum M' of Figure 29 is also chainable-with-nice-links as was the continuum in Figure 26.

To conclude this section, we state a consequence of a problem solved in Bing [1]. Given a chainable continuum K there exists a homeomorphism h which maps K onto a continuum L in E^2 where L is chainable-with-nice-links. Thus, every chainable continuum K has an embedding L in E^2, which is chainable-with-nice-links. In Section IV we shall show that if a continuum M is chainable-with-nice-links, then M is thin in E^2.

IV. ROBERTS' THEOREM

Section IV is divided into three parts. Part 1 deals with the development of the necessary tools to prove Roberts' Theorem. In part 2 we will prove Roberts' Theorem, after which, in Part 3 we will consider the consequences of the Roberts' Theorem on certain continua in E^2.

<u>Lemma 1.</u> If f is a continuous bijection of E^2 to E^2, then f is a homeomorphism.

Proof. The 2-sphere S is the one-point compactification of E^2. We extend f to a continuous bijection $\bar{f}$ of S onto itself by letting $\bar{f}(\infty) = \infty$ where "∞" is the ideal point of $S - E^2$. Since $\bar{f}$ is a continuous bijection defined on the compact space S, it follows that $\bar{f}$ is a homeomorphism, and f, the restriction of $\bar{f}$, is a homeomorphism also.

The following well known plane topology lemma appears in [8].

<u>Lemma 2.</u> If M is chainable-with-nice-links and C forms such an ε-chain, then there exists two homemorphisms T_1 and T_2 of the plane onto itself such that (1) both T_1 and T_2 reduce to the identity map for points not in any link of the chain C, (2) both T_1 and T_2 map the i^{th} link of C onto itself, and (3) $T_1(M)$ and $T_2(M)$ have no points in common.

We now have the necessary tools to prove the following theorem by Roberts [8].

<u>Theorem (Roberts).</u> If a continuum M in E^2 is chainable-with-nice-links, then there exists an uncountable set G of mutually disjoint

continua in E^2 all equivalent to M.

Proof. Let H_1 be a collection of open disks covering M such that H_1 satisfies the definition of chainable-with-nice-links with ε replaced by 1/2. Let T_1 and T_2 be two homeomorphisms satisfying the conclusion of Lemma 2; that is, (1) T_1 and T_2 reduce to the identity function for points outside the links of the chain H_1, (2) both T_1 and T_2 map the i^{th} link of H_1 onto itself, and (3) $T_1(M)$ and $T_2(M)$ have no points in common.

Let H_2 be a collection of open disks covering M, such that H_2 satisfies the definition of chainable-with-nice-links with ε replaced by a number less than $1/2^2$ and small enough so that, (1) $T_1(\bar{H}_2)$ and $T_2(\bar{H}_2)$ have no points in common, (2) the links of the chains $T_1(H_2)$ and $T_2(H_2)$ are of diameter less than $1/2^2$, and (3) $\bar{H}_2$ lies within H_1. To satisfy condition (1) we let U and V be two disjoint open sets containing $T_1(M)$ and $T_2(M)$ respectively, such that $\bar{U} \cap \bar{V} = \emptyset$. Notice that the existence of U and V follows from the fact that $T_1(M)$ and $T_2(M)$ are disjoint compact sets. The open set $T_1^{-1}(U) \cap T_2^{-1}(V) = O$ contains M. Now when H_2 is constructed it will be sufficient to keep the links of H_2 as subsets of O.

Notice that $\bar{H}_1$ is a compact set. Hence, T_1 and T_2 are uniformly continuous on $\bar{H}_1$. Therefore, we can satisfy condition (2) by constructing an appropriate δ-chain about M. Condition (3) is consequence of the normality of E^2.

Let T_{11}, T_{12}, and T_{21}, T_{22} denote homeomorphisms satisfying the conclusion of Lemma 2 with respect to the chains $T_1(H_2)$ and $T_2(H_2)$, respectively. Let H_3 be a collection of open disks covering M, such the H_3 satisfies the definition of chainable-with-nice-links with ε

replaced by a number less than $1/2^3$ and small enough so that no two of the sets $T_{11}[T_1(\bar{H}_3)]$, $T_{12}[T_1(\bar{H}_3)]$, $T_{21}[T_2(\bar{H}_2)]$, and $T_{22}[T_2(\bar{H}_2)]$ have any point in common. Notice that the preceding condition follows as a consequence of the argument used for $T_1(\bar{H}_2)$ and $T_2(\bar{H}_2)$.

Continue this process inductively. Let k be any sequence of the digits 1 and 2, of which there is an uncountably collection. Let k_n be the first n digits of k. We have the sequence H_1, $T_{k_1}(H_2)$, $T_{k_2}[T_{k_1}(H_3)]$, ... By construction each chain of this sequence covers the succeeding chain including its boundary; hence, if M_k is the intersection of the closure of the above sets, it follows that M_k is a continuum. We will show that, for each k, the continuum M_k is equivalent to M.

Let T_k be a mapping defined as follows: (1) For a point p not in M let n_p be the first integer such that p does not lie in any element of the chain H_{n_p} and let $T_k(p)$ be the point $T_{k_{n_p}}[\ldots T_{k_2}[T_{k_1}(p)]]$. (2) For each p in M there is an infinite sequence of open disks R_1, R_2, ..., one from each of the chains H_1, H_2, ..., such that $p = \bigcap_1^\infty R_i$. It is clear that p is the only point common to all of the open disks R_1, R_2, ... If in each of the corresponding open disks R_1, $T_{k_1}(R_2)$, $T_{k_2}[T_{k_1}(R_3)]$ ... there is constructed a corresponding sequence of points $\{y_n\}$, then this sequence is Cauchy; that is, given a positive number ε there exist an N, such that $1/2^N < \varepsilon$ and, hence, for m, n > N we have $d(y_m, y_n) < \varepsilon$. We therefore have a unique point g which is converged to by the sequence $\{y_n\}$. We define $T_k(p) = g$.

We shall now show that T_k is injective. Suppose to the contrary that two points p and g exist in E^2 such that $T_k(p) = T_k(g)$. If p and g are not in M then $T_k(p) = T_{k_{n_p}}[\ldots T_{k_2}[T_{k_1}(p)]] = T_{k_{n_g}}[\ldots T_{k_2}[T_{k_1}(g)]] =$

$T_k(g)$. If $n_g = n_p$ then $p = g$. If $n_g < n_p$ then $T_k(p) \neq T_k(g)$ since $T_{k_{n_p}}(p) \neq T_{k_{n_g}}(g)$. Similar arguments suffice for $p \in M$ and $g \notin M$. Therefore, let p and g be elements of M. Then there exists an m such that $p \in R_i$ and $g \in R_j$, $i \neq j$, where R_i and $R_j \in H_m$ and $R_i \cap R_j = \emptyset$, hence, $T_k(p) \neq T_k(g)$. Thus, T_k is an injective map.

To show that T_k is a surjective map, we let g be a point which is not in M_k. Then there exists two open sets U and V such that $M_k \subset V$, $g \in U$, and $\bar{U} \cap \bar{V} = \emptyset$. Since $M_k \subset U$ it follows that there exists an m such that the chain $T_{k_{m-1}}[\ldots T_{k_2}[T_{k_1}(H_m)]]$ whose links have diameter less than $1/2^m$, lies in U. If n_g is the least of such integers m, then for any x not in a link of H_{n_g}, $T_k(x) = T_{n_g}[\ldots T_{k_2}[T_{k_1}(x)]]$. Since g is such a point and T_k is the composition of surjections, we see that there exists a point p such that $T_k(p) = g$.

If g is an element of M_k, then g is an element of each of the sets R_1, $T_{k_1}(R_2)$, $T_{k_2}[T_{k_1}(R_3)]\ldots$, where $R_1, R_2, R_3, \ldots$, is an infinite sequence of open disks, one from each of the chains $H_1, H_2, H_3, \ldots$ Let $p \in \bigcap_1^\infty \bar{R}_i$; then by condition (2) p is the only element in $\bigcap_1^\infty \bar{R}_i$. Thus, $p = \bigcap_1^\infty R_i$, and hence, $T_k(p) = g$. Consequently, T_k is a surjection.

To show that T_k is continuous we suppose that we have a sequence $\{X_n\}$ converging to p where p fails to lie in M. As we have seen before, conditions (2) and (3) guarantee us the existence of an integer n_p, such that $p \in E^2 - \bar{H}_{n_k}$: hence, $T_k(p) = T_{k_{n_p}}[\ldots T_{k_2}[T_{k_1}(p)]]$. Since $\bar{H}_{n_p}$ is a closed set, we have that $E^2 - \bar{H}_{n_p}$ is open; hence, there exists an N such that for $m \geq N$ we have $x_m \in E^2 - \bar{H}_{n_p}$. Since $T_k(x_m) = T_{n_p}[\ldots T_{k_2}[T_{k_1}(x_m)]]$ for all $x_m \in E^2 - \bar{H}_{n_p}$, and each T_{k_i} is a homeomorphism, we see that the sequence $\{T_k(x_n)\}$ will converge to $T_k(p)$.

Consider the sequence $\{x_n\}$ converging to p where $p \in M$. Corresponding to p is an infinite sequence of open disks R_1, R_2, ... where each R_i is a link of H_i. Suppose that $T_k(p) = g$. Since R_i is an open set, there exists an N such that for $n \geq N$ we have $x_n \in R_i$; hence, $T_{k_i}[\ldots T_{k_2}[T_{k_1}(x_n)]] \in T_{k_i}[\ldots T_{k_2}[T_{k_1}(R_i)]] \subset N(g, 1/2^i)$. Furthermore, $T_{k_i}[\ldots T_{k_2}[T_{k_1}(x_n)]] \in N(g, 1/2^i)$ for each $n \geq N$. From this we can conclude that the sequence $\{T_k(x_n)\}$ converges to g. Consequently, T_k is a continuous bijection from E^2 to E^2.

It follows from Lemma 1 that T_k is a homeomorphism, thus M and M_k are equivalent in E^2.

<u>Corollary.</u> If M is a chainable continuum (not necessarily in the plane), then there is an uncountable collection G of pairwise disjoint equivalent continua in E^2 such that each is homeomorphic to M. Thus, each chainable continuum has a thin embedding in E^2.

Proof. Recall from Section III that if a continuum M is chainable in a space S, then there exists a continuum M^1 in E^2 which is homeomorphic to M such that M^1 is chainable-with-nice-links [1]. To the continuum M^1 apply Roberts' Theorem to obtain the set G.

It should be noted that one cannot conclude that if the continuum M of the above corollary is already in E^2, then G can be found so that each element of G is equivalent to M. Examples M_1 and M_2 which follow illustrate this.

Roberts' Theorem offers sufficient conditions for a continuum M to be thin in E^2. An example of a continuum which fails to satisfy the hypothesis of Roberts' Theorem yet is thin in E^2 is that of the simple closed curve (see Figure 12).

Consider the following question: Does there exist a continuum M in E^2 which fails to be thin in E^2 yet has a homeomorphic image which is thin in E^2? The following examples show that the answer to the above question is affirmitive.

In Section III (see Figure 27) we constructed a continuum M_2 in E^2 which is chainable-with-nice-links. Next we consider the continuum M_1 (see Figure 23) in E^2 which is homeomorphic to M_2 but fails to be chainable-with-nice-links. We intend to show that (1) M_1 and M_2 are not equivalent in E^2 and (2) that M_1 is not thin in E^2.

Assume that M_1 and M_2 are equivalent in E^2, that is, there exists a homeomorphism h from E^2 to E^2 which takes M_2 onto M_1. Since M_2 is thin in E^2, it follows from Theorem 7 that there exists a sequence of disjoint continua $\{L_n\}$, each equivalent to M_2, in E^2 which converges homeomorphically to some continuum M_2' equivalent to M_2. We may as well assume $M_2' = M_2$ since they are equivalent. Applying Theorem 9, we see that $\{h(L_n)\}$ converges homeomorphically to M_1. We shall now show that such convergence is impossible.

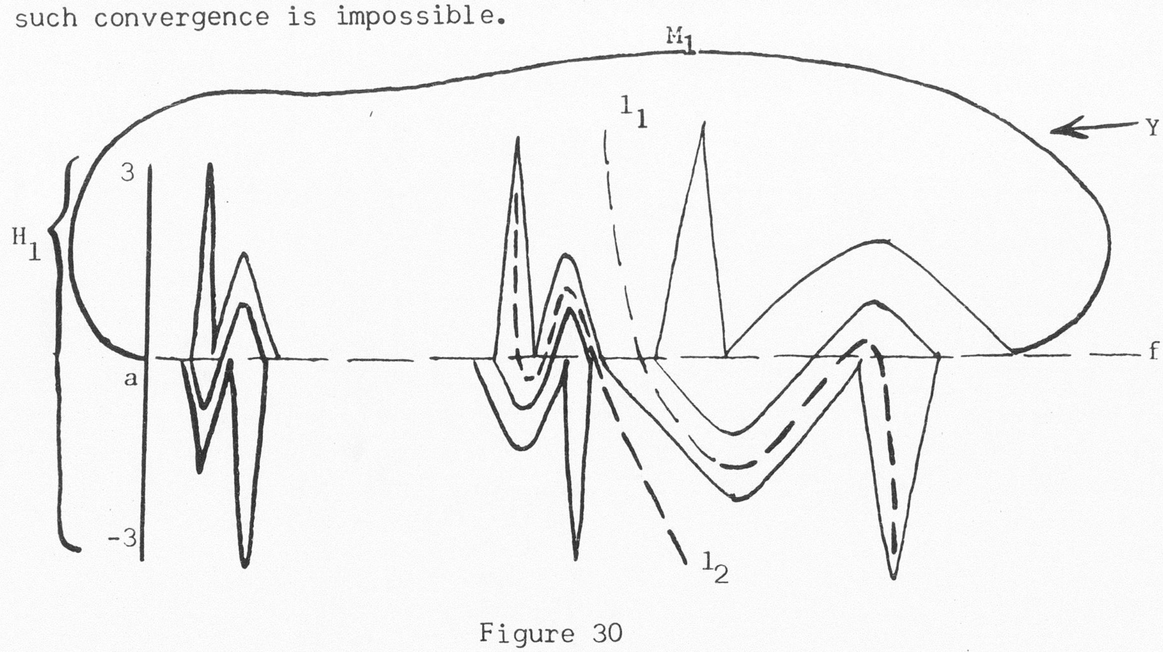


Figure 30

Notice that if the sequence $\{h(L_n)\}$ converges to M_1, it must do so from the right hand side (as M_1 is pictured in Figure 30). As was seen in the example of the sin 1/x continuum (see Figure 7, Section I), the head of $h(L_n)$, which we shall call K_n, must converge homeomorphically to the head H_1. Next construct an arc Y from the point f on M_1 to the point a on H_1 in such a way that the arc Y formed intersects no element of the sequence $\{h(L_n)\}$. Notice that $M_1 \cup Y$ separates E^2 into the two complementary domains U and V. Let U designate the complementary domain which is bounded. It can now be argued that for sufficiently large n each K_n must have a vertical length very close to 6 units. Thus, we see that if $K_n \subset U$ then K_n appears as l_1 is pictured in Figure 30. Similarly, if $K_n \subset V$ then K_n must fold as l_2 does in Figure 30. In either case each K_n has a fixed fold. As was seen in Figure 6, such a sequence of arcs cannot converge homeomorphically to the unfolded vertical arc H_1.

Consequently, M_1 and M_2 are not equivalent in E^2. Using arguments as above, we also see that no sequence $\{M_i\}$ of disjoint continua (equivalent or not) can converge homeomorphically to M_1. Thus, it follows from Theorem 7 that no uncountable collection of disjoint continua, each homeomorphic to M_1, can contain uncountably many continua equivalent to M_1.

The situation here is in contrast to properties claimed in [5] about the continuum M_1.

V. SOME RELATED QUESTIONS

We have compiled a collection of ten questions which have arisen during the preparation of this paper. These questions may be of interest to anyone who has completed a study of Sections I through IV. We further add that although some of the questions have been answered, there are questions for which we found no answer.

Question 1. If $\{M_n\}_o^\infty$ is a sequence, of chainable continua equivalent in E^2, which converges homeomorphically to M_o, then is M_o chainable-with-nice-links?

Question 2. Does there exist a continuum M which has an uncountable number of inequivalent thin embeddings in E^2?

Answer. Yes. An argument similar to that used in connection with the continuum in Figure 20 shows us that the continuum C below has an uncountable number of inequivalent embeddings in E^2, all of which are chainable-with-nice-links. It follows from Roberts' Theorem that each embedding is thin.

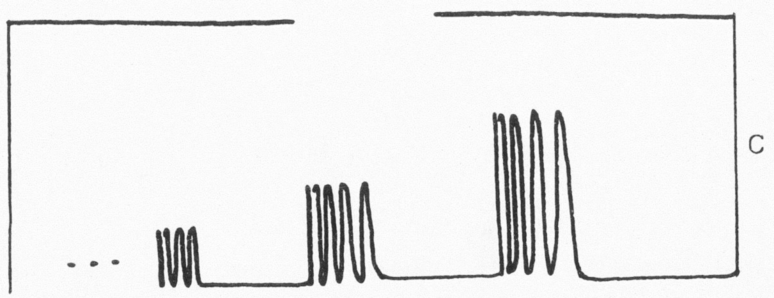


Figure 31

Question 3. If G is an uncountable collection of disjoint equivalent chainable continua in E^2, then is each element of G chainable-with-nice-links?

Notice that Question 3 is actually the converse to Roberts' Theorem in Section IV. An affirmative answer to Question 1 would supply an affirmative answer here, in view of Theorem B.

Question 4. Is there a continuum with uncountably many pairwise disjoint inequivalent embeddings in E^2? Is the continuum in Figure 31 such an example?

Question 5. Does there exist a sequence of disjoint continua $\{M_n\}_o^\infty$ converging homeomorphically to M_o, such that no two elements of $\{M_n\}$ are equivalent.

Answer. Yes. Consider the continuum C in Figure 31. As before, we can associate the sequence $\langle 0, 0, 0, \ldots \rangle$ with C (see Figure 19). Define the continuum M_n corresponding to $\langle 0, 0, \ldots, 0, 1, 0, \ldots \rangle$, where 1 is in the n^{th} position, by "flipping" the n^{th} sin 1/x curve of C to the opposite side of C and making M_n disjoint from C and the other M_i's. The reader should be able to convince himself that the sequence $\{M_n\}$ can be constructed to converge homeomorphically to C.

Question 6. If $\{M_i\}_o^\infty$ is a sequence of disjoint equivalently embedded continua in E^2 converging homeomorphically to M_o, then is M_o thin in E^2?

Cannon and Wayment [5] claimed a negative answer to this question, but we have shown that one of the counter examples they cited (namely the continuum M_1 in Figure 23) does not satisfy the conditions of the question. Furthermore, there is evidence that the other proposed counter example referred to in [5] fails for the same reason.

Question 7. If a chainable continuum M in E^2 is free from itself, then is M chainable-with-nice-links?

A continuum M is *free from itself* in E^2 if, for each positive number δ, there exists a δ-homeomorphism h of E^2 onto itself such that $h(M) \quad M = \emptyset$.

Question 8. If a continuum M in E^2 is chainable-with-nice-links, then is M free from itself?

Answer. Yes. A technique such as described in the proof of Roberts' Theorem seems to yield a proof.

Question 9. If a continuum M in E^n is free from itself in E^n, then is M thin?

Answer. Yes. It is not difficult to extend the technique in the proof of the main theorem in [8] to establish the affirmative answer.

Question 10. If $\{M_i\}_0^\infty$ is a sequence of disjoint equivalent continua in E^2 converging homeomorphically to M_0, then is M_0 free from itself?

An answer here would be of interest in light of Question 6.

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