CAMP: An Algorithm to Recover Sparse Signals with Unknown Clustering Pattern Using Approximate Message Passing

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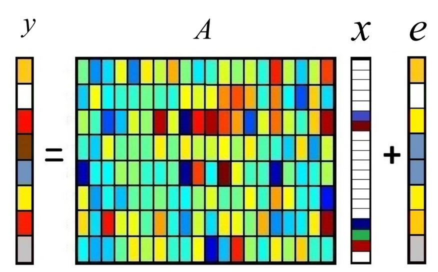
Introduction

- Single measurement vector (SMV) model:
  - $\mathbf{y} = A\mathbf{x}_s + \mathbf{e}$: linear non-adaptive noisy measurements
  - $\mathbf{x}_s \in \mathbb{R}^{N\times 1}$ is the sparse signal of interest
  - $\mathbf{y} \in \mathbb{R}^{M\times 1}$ contains measurements ($M \ll N$)
  - solving the linear inverse problem to find $\mathbf{x}_s$

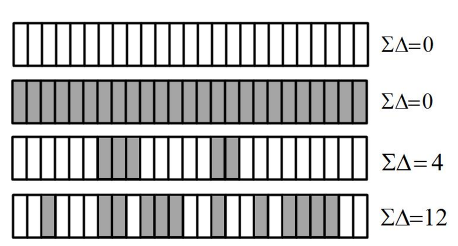


- **Assumption:**
  - *Sparse clustered pattern:* Non-zero elements of $\mathbf{x}$ may appear in clusters with an unknown structure
- **Proposed model:** $\mathbf{y} = A(\mathbf{s} \circ \mathbf{x}) + \mathbf{e}$

  $\mathbf{y} \in \mathbb{R}^{M\times 1}$, $A \in \mathbb{R}^{M\times N}$, $\mathbf{s} \in \{0,1\}^{N\times 1}$, $\mathbf{x} \in \mathbb{R}^{N\times 1}$, $\mathbf{e} \in \mathbb{R}^{M\times 1}$ ($M \ll N$)
  - $\mathbf{s}$ is the support learning vector and accounts for the non-zero locations of $\mathbf{x}$
  - In $(\mathbf{s} \circ \mathbf{x})$, $\circ$ denotes Hadamard product
- **Objective:**
  - Learning the sparsity pattern of $\mathbf{x}$
  - Recovering sparse signal $\mathbf{x}$ using the noisy SMV model
- **Proposed algorithm:**
  - *CAMP:* Algorithm to recover sparse signals with unknown clustering pattern using approximate message passing framework

Proposed Statistical Model and Defining Priors

- Measure of clumpiness [2]: $(\Sigma\Delta)_{\mathbf{s}} = \sum_{i=2}^{P} |s_i - s_{i-1}|$, where $\mathbf{s}$ is the support learning vector of the solution
  - There exist few transitions for the case where the supports of the solution have a clustered pattern
  - For example, a constant vector (all ones or all zeros) has a $\Sigma\Delta$ of 0
  - More examples:



- Change in the measure of clumpiness

$$(\Sigma\Delta)_{(\text{support of } \mathbf{x})} = \sum_{n=2}^{N} |b(x_n, T) - b(x_{n-1}, T)|,$$

  - T: a predetermined threshold
  - $b(.,.)$: returns a binary value

$$b(x_n, T) = \begin{cases} 1 & \text{if } |x_n| > T \\ 0 & \text{otherwise.} \end{cases}$$

- Prior on the solution vector $\mathbf{x}$:

$$\begin{aligned} &\forall n = 1, \ldots, N, \\ &x_n \sim \mathcal{N}(0, \alpha_n) \\ &\alpha_n \sim \mathcal{N}(e^{\left\{\frac{(\Sigma\Delta)|_{b(x_n,.)=0} - (\Sigma\Delta)|_{b(x_n,.)=1} - 1}{\theta_1}\right\}}, \theta_2), \end{aligned}$$

  - $(\Sigma\Delta)|_{b(x_n,.)=1}$: Sigma-Delta evaluation of the supports of the solution when s_n is set to be active
  - θ_1: A tuning parameter for the emphasis on the measure of clumpiness
  - θ_2: The prior variance on the variance of the variable x_n and is updated via the EM algorithm
- Joint probability distribution of the model:

$$P(\mathbf{y}, \mathbf{x}, \boldsymbol{\alpha}, \theta_1, \theta_2, \sigma^2) \propto$$
$$P(\mathbf{y}|\mathbf{x}, \sigma^2 I_N) \prod_{n=1}^{N} \Big(P(x_n; 0, \alpha_n) P(\alpha_n; e^{\left\{\frac{(\Sigma\Delta)|_{b(x_n,.)=0} - (\Sigma\Delta)|_{b(x_n,.)=1} - 1}{\theta_1},\right\}}, \theta_2) \Big)$$

- The measurement noise is assumed to be $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \sigma^2 I_N)$

Justification of the Priors

- CAMP algorithm adds an additional layer to AMP-SBL algorithm [1] using the described priors to encourage the clustered pattern
- To encourage sparsity, we assumed $x_n \sim \mathcal{N}(0, \alpha_n)$ as a prior
- The supports of the solution are then specified by the function $b(., T)$ where T is a predetermined threshold
- Based on the threshold, we discard the small-valued components of $\mathbf{x}$ from being considered as the support of the solution
- The smaller α_n is, the higher probability it provides to x_n becoming 0 i.e. (discarding s_n)

Behavior of α_n with respect to Sigma-Delta

- The reason behind assuming such prior on α_n:

$(\Sigma\Delta)\|_{b(x_n,.)=1}$	$(\Sigma\Delta)\|_{b(x_n,.)=0}$	α_n
cte	*cte*	↓
↑	*cte*	↓
cte	↓	↓
cte	↑	↑
↓	*cte*	↑

- For example consider the case where forcing either $s_n = 0$ or $s_n = 1$ does not make any change in the evaluation of Sigma-Delta
- In this case, though it promotes the clumpiness in the solution, it discourages the solution to be sparse
- Therefore, α_n needs to be decreased

CAMP Algorithm

CAMP:
Solver of linear inverse SMV problem for the clustered sparse signals:

- *Definitions*

$F_n(k_n, \alpha_n, c) = k_n \frac{\alpha_n}{c+\alpha_n}$
$G_n(\alpha_n, c) = \frac{c.\alpha_n}{c+\alpha_n}$
$F'_n(\alpha_n, c) = \frac{\alpha_n}{c+\alpha_n}$

- *Message updates using AMP*

For $n = 1, 2, \ldots, N$
  $k_n = \sum_{m=1}^{m} a^{\star}_{mn} z_m + \mu_n$
  $\mu_n = F_n(k_n, \alpha_n, c)$
  $\nu_n = G_n(\alpha_n, c)$
End
$c = \sigma^2 + \frac{1}{M}\sum_{n=1}^{N} \nu_n$
$z_m = y_m - \sum_{n=1}^{N} a_{mn}\mu_n + \frac{z_m}{M}\sum_{n=1}^{N} F'_n(\alpha_n, c), \forall m = 1, \ldots, M$

- *Parameter updates using EM algorithm*

% Updating $\boldsymbol{\alpha}$:
$\forall n = 1, 2, \ldots, N$, solve for α_n in
$\alpha_n^3 - e^{\left\{\frac{(\Sigma\Delta)|_{b(x_n,.)=0} - (\Sigma\Delta)|_{b(x_n,.)=1} - 1}{\theta_1}\right\}} \alpha_n^2 + \frac{\theta_2}{2}\alpha_n - \frac{\theta_2(\mu_n^2 + \nu_n)}{2} = 0$
which is the minimizer of

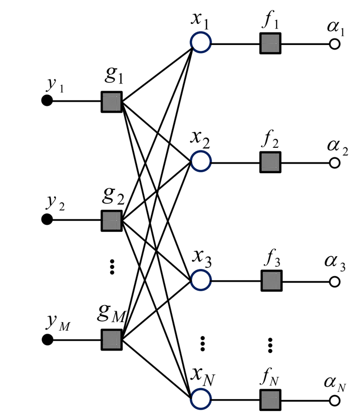
$$f(\alpha_n) = \ln(\alpha_n) + \frac{\mu_n^2 + \nu_n}{\alpha_n} + \frac{1}{\theta_2}(\alpha_n - e^{\left\{\frac{(\Sigma\Delta)|_{b(x_n,.)=0} - (\Sigma\Delta)|_{b(x_n,.)=1} - 1}{\theta_1}\right\}})^2$$

% Updating the noise variance σ^2:
$\sigma^{2[k+1]} = \frac{\|\mathbf{y} - A\boldsymbol{\mu}\|_2^2 + \sum_{n=1}^{N} \|\mathbf{a}_n\|_2^2 \nu_n}{M}$
% Updating the variance of $\boldsymbol{\alpha}$:
$\theta_2^{[k+1]} = \frac{1}{N}\sum_{n=1}^{N} \left(\alpha_n - e^{\left\{\frac{(\Sigma\Delta)|_{b(x_n,.)=0} - (\Sigma\Delta)|_{b(x_n,.)=1} - 1}{\theta_1}\right\}}\right)^2$

Factor Graph

- Under such modeling, all the distributions of the joint, conditional, and posterior densities become Gaussian

$$\begin{aligned} g_m &:= P(y_m|\mathbf{x}, \boldsymbol{\alpha}), \quad m = 1, 2, \ldots, M \\ f_n &:= P(x_n; 0, \alpha_n), \quad n = 1, 2, \ldots, N \end{aligned}$$



Message Passing

- Message from a function node to a variable node

$$M_{g_m \to x_n} \propto \mathcal{N}(a_{mn}x_n; z_{mn}, c_{mn}),$$

where

$$z_{mn} = y_m - \sum_{q \neq n} a_{mq}\mu_q, \text{ and } c_{mn} = \sigma^2 + \sum_{q \neq n} |a_{mq}|^2 \Sigma_q$$

- Message from a variable node to a function node

$$M_{x_n \to g_m} \propto \mathcal{N}\big(x_n; \sum_{l \neq m} a_{nl}z_{ln} \frac{\alpha_n}{c_n + \alpha_n \sum_{l \neq m} a_{ln}^2}, \frac{c_n\alpha_n}{c_n + \alpha_n \sum_{l \neq m} a_{ln}^2}\big)$$

- c_{nl} (under the large-system-limit) is approximated by

$$c_{nl} \simeq c_n := \frac{1}{M}\sum_{m=1}^{M} c_{mn}.$$

- Assumption: Matrix A is normalized with respect to its columns

$$\sum_{l \neq m} a_{ln}^2 \simeq \sum_{m=1}^{M} a_{mn}^2 = 1.$$

  Therefore,

$$M_{x_n \to g_m} \propto \mathcal{N}\big(x_n; \sum_{l \neq m} a_{nl}z_{ln} \frac{\alpha_n}{c_n + \alpha_n}, \frac{c_n\alpha_n}{c_n + \alpha_n}\big).$$

- Estimating the posterior on x_n

$$P(x_n|\mathbf{y}) \propto P(x_n; \alpha_n) \prod_{m=1}^{M} P(y_m|x_n)$$
$$\propto M_{f_n \to x_n} \prod_{m=1}^{M} M_{g_m \to x_n}) \propto \mathcal{N}(x_n; \mu_n, \nu_n)$$

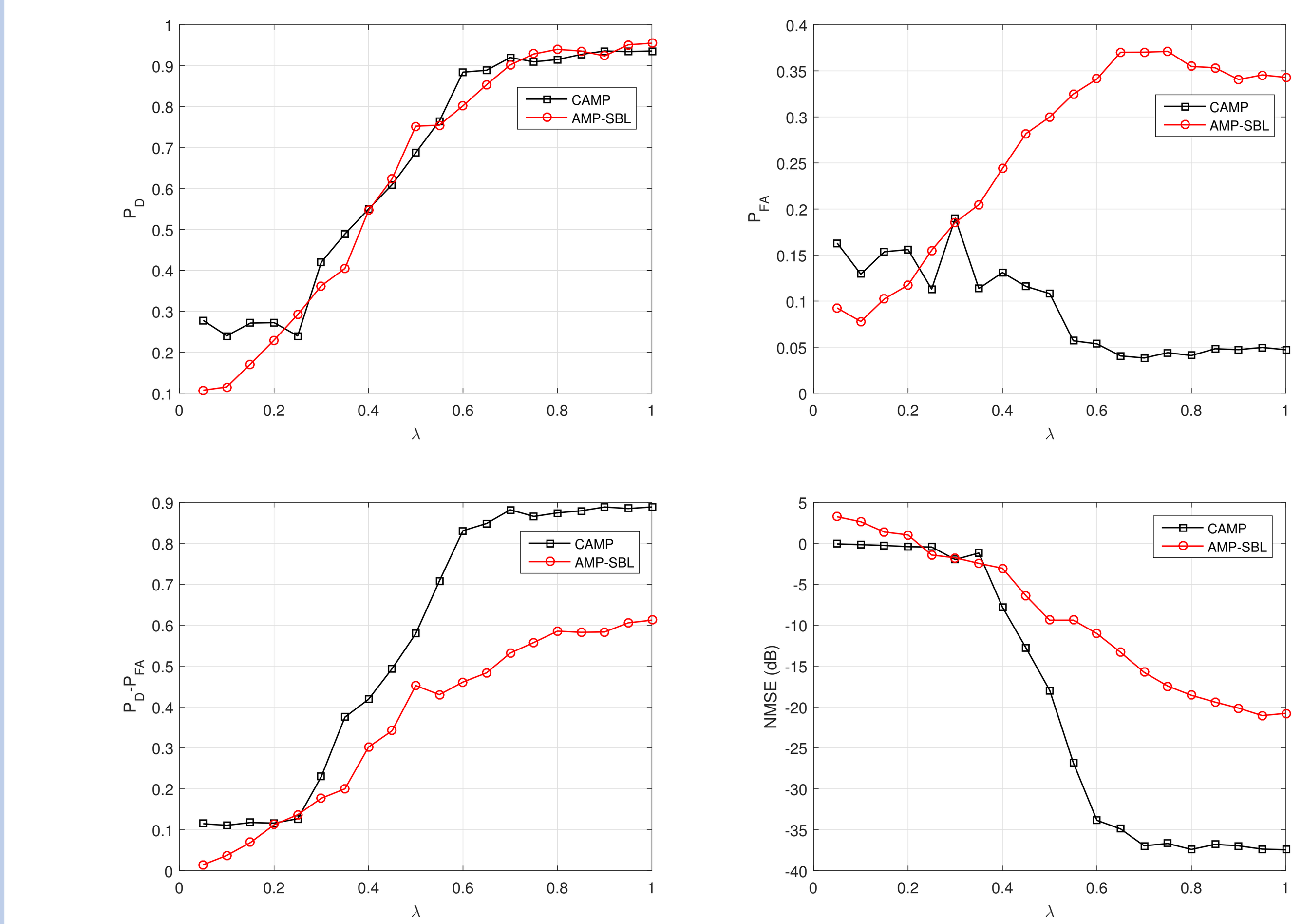
where

$$\mu_n = \sum_{m=1}^{M} a_{mn}z_{mn}(\frac{\alpha_n}{c_n + \alpha_n}), \text{ and } \nu_n = \frac{c_n\alpha_n}{c_n + \alpha_n}$$

Simulations Settings

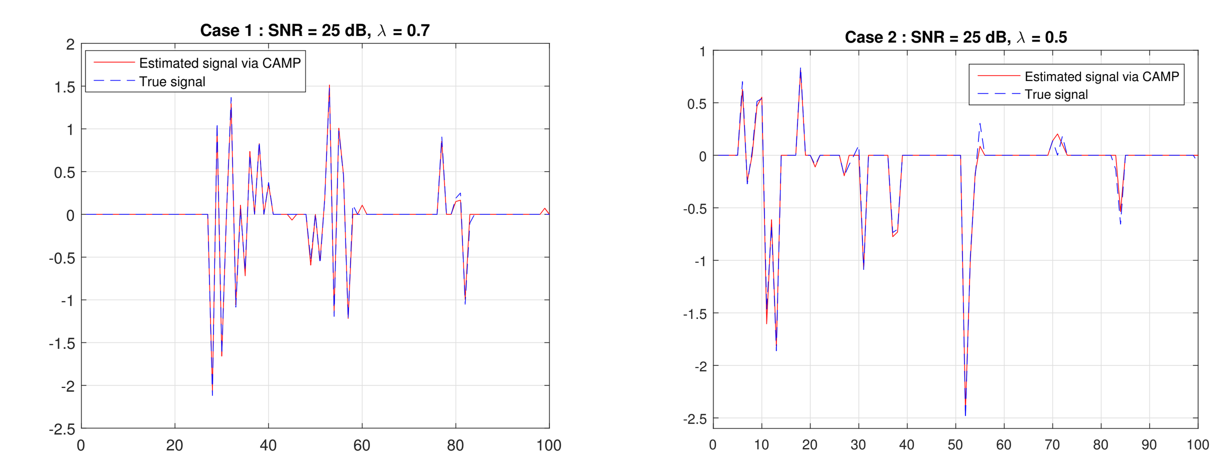
- The supports of the solution are binary and drawn from a Bernoulli distribution in such a way to have clustered sparsity structure
- The entries of $\mathbf{x}_{np}$ is drawn i.i.d. from Gaussian distribution with zero mean and variance $\sigma_x^2 = 1$
- The true solution is constructed from $\mathbf{x} = \mathbf{s} \circ \mathbf{x}_{np}$
- The sensing matrix $A \in \mathbb{R}^{M\times N}$, with $a_{mn} \sim \mathcal{N}(0, \frac{1}{\sqrt{M}})$, where M varies and $N = 100$
- The noise components are drawn i.i.d. from $\mathcal{N}(0, \sigma_n^2)$ with $SNR = 25dB$
- The measurement vector $\mathbf{y} \in \mathbb{R}^{M\times 1}$ is then computed from $\mathbf{y} = A\mathbf{x} + \mathbf{e}$
- The cardinality of $\mathbf{x}$ is set to $K_{sp} = 25$ for all the simulations
- In all the simulations, the total number of iterations is set to 1000
- To study the performance, we generate 100 random cases using the above settings and then averaging over all the obtained results
- In the figures, λ is the sampling rate and is defined as $\lambda = M/N$

Simulation Results (Performance Comparison)



Reconstruction Examples for Synthetic Data

- Parameters initialization for CAMP algorithm:
  - $\theta_1 = 10$, $c^{[0]} = 10$, $\theta_2^{[0]} = 0.5$, $T = 0.001$, and $ITER = 1000$
- *Case 1: SNR* $= 25dB$, $N = 100$, $\lambda = 0.7$, and $k_{sp} = 25$
- *Case 2: SNR* $= 25dB$, $N = 100$, $\lambda = 0.5$, and $k_{sp} = 25$



Conclusions

- A new algorithm for the recovery of sparse signals with unknown clustering pattern for the SMV problem was proposed
- **Performance evaluation:**
  - CAMP provides an overall lower false alarm rate (P_{FA}) compared to AMP-SBL
  - Based on the performance measure of ($P_D - P_{FA}$) we showed that CAMP performs better than AMP-SBL
  - The overall normalized mean-squared error (NMSE) between the true and the estimated solution for the CAMP is lower than the one for AMP-SBL

Main References

1. M. Al-Shoukairi and B. Rao, *Sparse Bayesian learning using approximate message passing*, 48th Asilomar Conf. on Sig., Syst. and Compt., pp. 1957–1961, Nov. 2014.
2. M. Shekaramiz, T. Moon, and J. Gunther, *On the block-sparse solution of single measurement vectors*, 49th Asilomar Conf. on Sig., Syst. and Compt., pp. 508–512, Nov. 2015.
3. D. L. Donoho, A. Maleki, and A. Montanari, *Message passing algorithms for compressed sensing: I. motivations and construction*, Proc. Inform. Theory Workshop, (Cairo, Egypt), pp. 1–5, Jan. 2010.