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Nonparametric Test of Fit

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NONPARAMETRIC TEST OF FIT

by

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CHAPTER I

INTRODUCTION

Most statistical methods require assumptions about the populations from which samples are taken. Usually these methods measure the parameters, such as variance, standard deviations, means, etc., of the respective populations. One example is the assumption that a given population can be approximated closely with a normal curve. Since these assumptions are not always valid, statisticians have developed several alternate techniques known as nonparametric tests. The models of such tests do not specify conditions about population parameters.

Certain assumptions, such as (1) observations are independent and (2) the variable being studied has underlying continuity, are associated with most nonparametric tests. However, these assumptions are weaker and less in number than those commonly associated with parametric tests.

Justification of the review

The chief advantages of nonparametric tests are:

1. Most probability statements are exact and accurate regardless of the shape of the population distribution (in large samples, excellent approximations are already available). Some nonparametric tests may assume that the shape of two or more populations are identical.

Others may assume that the population shapes are symmetrical. They also may assume that the underlying distribution is continuous, an assumption which they share with parametric tests.

2. Unless the population distribution is known exactly, nonparametric statistical tests are the best way to treat sample sizes as small as n = 5 or n = 6.

3. Observations from several different populations can be adequately treated by nonparametric tests, whereas unrealistic assumptions often treat such samples by parametric methods.

4. Data which are inherently in ranks or data with numerical scores having the strength of ranks can be treated. In other words, if a variable such as anxiety is considered, the researcher may only be able to state that Subject A is more anxious than Subject B without being able to say exactly how much more anxious. Even if data can be categorized only as plus or minus, better or worse, more or less, etc., they can be treated by nonparametric methods. To treat like material by parametric methods requires precarious and even unrealistic assumptions about the underlying distributions.

5. Data measured on a nominal scale are easily treated by nonparametric methods whereas parametric techniques may not be justified for such data.

6. Nonparametric statistical tests are generally much easier to learn and apply.

Nonparametric statistical tests are not without their disadvantages, however. Some of these disadvantages are:

1. Nonparametric statistical tests are wasteful of data if the measurements are sufficiently strong and if all the assumptions

of the parametric statistical model are met. The degree of wastefulness is expressed by the power-efficiency of the nonparametric test. In other words, if such a test had a power-efficiency of 90 percent, the appropriate parametric test (if all test conditions are met) would be just as effective with a 10 percent smaller sample.

2. Interactions in the analysis of variance model cannot be tested by any known nonparametric method unless special assumptions about additivity are made. It should be noted, however, that parametric tests also are forced to make the assumption of additivity. The problem of higher ordered interactions has not yet been considered in nonparametric literature.

3. Nonparametric statistical tests and their accompanying tables of significant values have the disadvantage of being widely scattered about in various publications. Because many of these publications are highly specialized, they are in many cases unavailable to behavioral scientists.

Statement of the problem

One can find most nonparametric tests of fit such as Neyman-Bartman, Smirnov or Cramer, Chi-square, and Kolmogorov, in current statistics books. The purpose of this thesis is to assemble in one paper many of the more useful nonparametric tests of fit and compare those which are similar. The characteristics on which comparison are made are (1) ease of application and (2) power. Such comparisons are not readily obtainable in most of the statistical texts now available.

Method of procedure

All but two source books used for gathering information and material are: *Nonparametric Statistics for the Behavioral Sciences* by S. Siegel; *Statistical Theory* by B. Lindgren; *The Advanced Theory of Statistics*, Vol. II, by M. Kendall and A. Stuart; and *Handbook of Nonparametric Statistics*, Vol. I, by J. Walsh. The two exceptions are papers contained in *The Annals of Mathematical Statistics* (June and December, 1962) by J. Rosenblatt. These can be found in the library of the Applied Statistics Computer Science Department.

General description of nonparametric tests of fit

The Kolmogorov-Smirnov tests are one type of the nonparametric tests of fit that several researchers have investigated. Rosenblatt (1962) has eliminated the paradox of almost sure rejection of the null hypothesis when too much data are observed and has extended the test to composite hypothesis. Previous to his work, the Kolmogorov-Smirnov tests were suitable only for testing the simple hypothesis $F = F_0$ against all alternatives. Some of his works are included in this paper.

Let x_1, x_2,..., and x_n be independent observations on a random variable with the distribution function F(x) unknown. Suppose that we wish to test the hypothesis

$$H_0:F(x) = F_0(x) \tag{1.1}$$

where $F_0(x)$ is some particular distribution function (d.f.), which may be continuous or discrete. The problem of testing (1.1) is called a goodness-of-fit problem. Any test of (1.1) is called a test of fit.

Hypotheses of fit, like parametric hypotheses, divide naturally into simple and composite hypotheses. The above hypothesis, (1.1), is simple if $F_0(x)$ is completely specified; e.g., the hypothesis that the n observations have come from a normal distribution with specified mean and variance. On the other hand, we may wish to test whether the observations have come from a normal distribution where all parameters are unspecified, and as such the hypothesis would be composite; or in this case it would often be called a test of normality. Similarly, if the normal distribution has its mean but not its variance specified, the hypothesis remains composite.

This thesis covers Neyman-Barton "smooth" and Smirnov tests in the simple hypothesis case and chi-square and Kolmogorov in both simple and composite hypothesis cases.

A general failing of most tests of hypotheses such as these is that given a sufficiently large sample, rejection is sure. This is because the true distribution being considered will usually not be distributed exactly as specified under the hypothesis. Thus, any small difference can be detected by a test of sufficient size. Therefore, a much more useful test would test the hypothesis that x is distributed approximately as $F_0(x)$. Such a test has been developed and reported in *The Annals of Mathematical Statistics* (June and December, 1962) by J. Rosenblatt.

CHAPTER II

SIMPLE HYPOTHESES

Neyman-Barton "smooth" tests

Given $H_0:F(x) = F_0(x)$, we transform each observation x_i, $i = 1, \ldots, n$ as in (2.1) by the probability integral transformation

$$Y_i = \int_{-\infty}^{x_i} f_0(\mu)du = F_0(x_i). \quad i = 1,2,\ldots,n \tag{2.1}$$

and obtain n independent observations uniformly distributed on the interval (0,1) when H_0 holds. We specify the alternative to H_0 as departures from the uniformity of the Y_i, which nevertheless remain independent on (0,1). Neymay set up a system of distributions designed to allow the alternative to vary smoothly from the H_0 (uniform) distribution in terms of a few parameters. (It is this "smoothness" of the alternatives which has been transferred, by hypallage, to become a description of the tests.) In fact, Neyman specified for the frequency function of any Y_i the alternatives

$$f(Y/H_K) = c(\theta_1, \theta_2, \ldots, \theta_k) \exp \{1 + \sum_{r=1}^{k} \theta_r \Pi r(Y)\},$$

$$0 < Y < 1, \; K = 1,2,3 \tag{2.2}$$

where c is a constant which ensures that (2.2) integrates to 1 and the

$\Pi r(Y)$ are Legendre polynomials transformed linearly so that they are orthonormal on the interval (0,1). If we write $z = y - 1/2$, the polynomials are in the fourth order:

$$\Pi_0(z) = 1$$

$$\Pi_1(z) = 3^{1/2} \cdot 2z$$

$$\Pi_2(z) = 5^{1/2} \cdot (6z^2 - 1/2) \tag{2.3}$$

$$\Pi_3(z) = 7^{1/2} \cdot (20z^3 - 3z)$$

$$\Pi_4(z) = 3 \cdot (70z^4 - 15z^2 + 3/8)$$

$$\Pi_{u+1}(z) = 2z[(2u + 3)(2u + 1)]^{1/2} \Pi_u(z)/(u + 1) - u[(2u + 3)/(2u - 1)]^{1/2}$$

$$\Pi_{u-1}(z)/(u + 1)$$

The problem now is to find a test statistic for H_o against H_k. If (2.2) is rewritten as

$$f(Y/H_k) = c(\theta) \exp \{ \sum_{r=0}^{k} \theta_r \Pi_r(Y) \}, \quad 0 < Y < 1, \; k = 0,1,2,\ldots, \tag{2.4}$$

defining $\theta_0 \equiv 1$, this includes H_o also. We may wish to test the simple

$$H_o:\theta_1 = \theta_2 = \ldots\ldots = \theta_k = 0 \qquad (2.5)$$

or equivalently

$$H_o: \sum_{r=1}^{k} \theta_r^2 = 0 \qquad (2.6)$$

against its composite negation. It will be seen that (2.4) is an alternative of the exponential family, linear in the θ_r and Π_r. The likelihood function for n independent observations is

$$L(Y/\theta) = \{c(\theta)\}^n \exp\{\sum_{r=0}^{k} \theta_r \sum_{l=1}^{n} \Pi_r(Y_i)\} \qquad (2.7)$$

Formula (2.7) clearly factorizes into k parts, and each statistic $T_r = \sum_{i=1}^{n} {}_r(Y_i)$ is sufficient for θ_r; and we therefore may confine ourselves to functions of the T_r in our search for a test statistic; or we can write $\Psi k^2 = (1/n) \Sigma_{u=1}^{k} [\Sigma_{i=1}^{n} \Pi_u F_o(x) - 1/2]^2$; and reject the null hypothesis if $\Psi_k^2 \geq \chi_\alpha^2(k)$. Table 1 in Appendix B contains values of $\chi_\alpha^2(k)$.

The main advantage of the Neyman-Barton "smooth" tests is that a system of alternative hypothesis may be specified which may be an interesting test. Unfortunately, one frequently has no very precise alternative in mind when testing fit; and if that is the case, there is no need to use a smooth test according to Kendall and Stuart (1961).

An example of smooth test from Table 1 follows: Consider testing the hypothesis that a distribution is normal with mean 32 and standard deviation 1.8, using the ten observations 31.0, 31.4, 33.3, 33.4, 33.5, 33.7, 34.4, 34.9, 36.2, 37.0 with k = 3 and α = .05.

$$\chi^2_{.05}(3) = 7.815$$

If $\Psi_k^2 > \chi^2_{.05}(3)$, we reject the hypothesis

$$\Psi_k^2 = \frac{1}{n} \sum_{\mu=1}^{k} [\sum_{i=1}^{m} \Pi_\mu \{F_o(x) - 1/2\}]^2.$$

$$F(x_1) = F(31) = \Phi(\frac{x - \mu}{\sigma}) \doteq .29$$

$$F(x_2) = F(31.4) \doteq .63$$

$$F(x_3) = F(33.3) \doteq .76$$

$$F(x_4) = F(33.4) \doteq .78$$

$$F(x_5) = F(33.5) \doteq .79$$

$$F(x_6) = F(33.7) \doteq .82$$

$$F(x_7) = F(34.4) \doteq .9$$

$$F(x_8) = F(34.9) \doteq .95$$

$$F(x_9) = F(36.2) \doteq .99$$

$$F(x_{10}) = F(37) \doteq .997$$

$$\Pi_1(y) = 2\sqrt{3}\, y$$

$$\Pi_2(y) = \sqrt{5}\,(6y^2 - 1/2)$$

$$\Pi_3(y) = \sqrt{7}\,(20y^3 - 3y)$$

$$\Psi_3^2 = \frac{1}{10} \sum_{\mu=1}^{3} [\sum_{i=1}^{10} \Pi_1(2.997)]^2$$

$$\Psi_3^2 = \frac{1}{10} \sum_{\mu=1}^{3} [(2\sqrt{3})(.29) + (2\sqrt{3})(.63) + 2\sqrt{3}\,(.76)\ldots +$$

$$+ (2\sqrt{3})(.997)(2.977)]^2$$

$$\Psi_3^2 = \frac{1}{10} \{(120.05)^2 + [\sqrt{5}(6(0.29^2 - 1/2)) \ldots\ldots (2.977)]^2$$

$$+ [(\sqrt{7}(20(.29)^3 - 3(.29)] + \sqrt{7}(20(.63)^3 - 3(.63))\ldots\}$$

By calculating the first Π_1, we can see the value is much larger than $\chi^2_{.05}(3)$; and if we finish calculating we have to add the value of Π_2 and Π_3 to the value of Π_1 which is $(120.05)^2$, and the value of Ψ_3^2 becomes larger. Since $\Psi_3^2 > \chi^2_{.05(3)}$, we reject the hypothesis.

Smirnov or Cramer-von Mises

To consider $x[i]$[1] for a notational statement of observations,

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[1] $x[i]_j$ = i^{th} order statistic for the j^{th} group ($i = 1,\ldots\ldots,n$; $j = 1,\ldots,m$). $x[i]_1 = x[i]$.

the following formulas of Smirnov or Cramer-von Mises can be used:

$$W_n^{(1)} = 1/12n + \sum_{i=1}^{n} \{F(x[i]) - (2i - 1)/2n\}^2 = n\int_{-\infty}^{\infty} [F_o(x) - F_n(x)]^2 \, dF_o(x).$$

$$W_n^{(2)} = -n - \sum_{i=1}^{n} (2i - 1)\{\log_e F_o(x[i]) + \log_e[1 - F_o(x[n + 1 - i])]\}/n = n\int_{-\infty}^{\infty} [F_o(x) - F_n(x)]^2 \{F_o(x)[1 - F_o(x)]\}^{-1} \, dF_o(x)$$

or

$$nW_n^{(2)} = \frac{1}{12n} + \sum_{i=1}^{m} \{\frac{2i - 1}{2n} - F(x[i])\}^2$$

In the first test, reject the null hypothesis if $W_n^{(1)} > W_1^{(\alpha,n)}$.

In the second test, reject the null hypothesis if $W_n^{(2)} > W_2^{(\alpha,n)}$ and $W_1(\alpha,n) \doteq W_1(\alpha,\infty) = W_1(\alpha)$ where $\alpha \leq 1/2$ and $n > 20/\sqrt{\alpha}$.

Table 2 in Appendix B contains values of $W_1(\alpha)$ for $\alpha = .001 (.001).01(.01).5$. $W_2(\alpha,n) \doteq W_2(\alpha,\infty)$ where $\alpha < 1/2$ and $n > 25/\sqrt{\alpha}$; $W_2(.1,\infty) = 1.933$, $W_2(.05,\infty) = 2.492$, $W_2(.01,\infty) = 3.857$.

The test based on $W_n^{(2)}$ emphasized the tails-of-distribution function.

The test based on $W_n^{(1)}$ should be sensitive to the alternatives expressed in the terms of metric

$$n \int_{-\infty}^{\infty} [F_0(x) - F(x)]^2 dF_0(x).$$

The test based on $W_n^{(2)}$ should be sensitive to alternatives based on metric

$$n \int_{-\infty}^{\infty} [F_0(x) - F(x)]^2 \{F_0(x)[1 - F_0(x)]\}^{-1} dF_0(x),$$ (Walsh, 1962, p. 361).

An example of the Smirnov or Cramer-von Mises tests would be to test the hypothesis that a distribution is normal with mean 32 and standard deviation 1.8, by using the ten observations: 31.0, 31.4, 33.3, 33.4, 33.5, 33.7, 34.4, 34.9, 36.2, 37.0. For x[i],

$$F(x[1]) = F(31) = \Phi\left(\frac{31 - 32}{1.8}\right) \doteq .29$$

and

$$W_n^{(1)} = \frac{1}{12n} + \sum_{i=1}^{n} \left\{F(x[i]) - \frac{(2i - 1)}{2n}\right\}^2$$

$$= \frac{1}{12n} + \left\{\frac{(2 - 1)}{20} - .29\right\}^2$$

$$= \frac{1}{120} + .057 = .065$$

Proceding in this fashion, one finds $nW_n^{(2)}$ to be about .88, whereas for $\alpha = .05$ the rejection limit is .461 (Table 2, Appendix B). The null hypothesis that the distribution is normal with mean 32 and standard deviation 1.8 is rejected at the 5 percent level (Lindgren, 1962, p. 334).

Chi-square (χ^2)

When the null hypothesis is simple, the chi-square goodness-of-fit test is based on the statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - nP_{oi})^2}{nP_{oi}} \qquad (2.8)$$

where n_i = observed number of cases categorized in the i^{th} category, nP_{oi} = expected number of cases in i^{th} category, and P_{oi} = probability of an observation falling in each class.

The degree of freedom is k - 1. Thus, Formula (2.8) ordinarily is a one-side test where the null hypothesis is rejected when this statistic is too large. Since P_{oi} are known values, the distribution of this statistic under the null hypothesis can be determined exactly. However, to avoid computational difficulties, approximations in this distribution are nearly always used.

For a small n, Pearson (1900) expressed the formula (2.8) as

$$\chi^2 = \frac{1}{n} \sum_{i} \frac{ni^2}{P_{oi}} - n \qquad (2.9)$$

which is easier to compute, but (2.8) has the advantage over (2.9) of being a direct function of the difference between the observed frequencies n_i and their hypothetical expectations nP_{oi}, differences which are themselves of obvious interest (Walsh, 1962, p. 447).

The whole of the chi-square test, which has been discussed so far, is valid. However, we can determine the K classes into which the observations are grouped for best of fit. For example, in some classical experiments on pea-breeding, Mendel observed the frequencies of different kinds of seeds in crosses from plants with round yellow seeds and plants with wrinkled green seeds. They are given below, together with the theoretical probabilities (Kendall and Stuart, 1961, p. 422).

Seeds	Observed frequency	Theoretical probability
	n_i	P_{oi}
Round and yellow	315	9/16
Wrinkled and yellow	101	3/16
Round and green	108	3/16
Wrinkled and green	32	1/16
	n = 556	1

The formula (2.9) gives

$$\chi^2 = \frac{1}{556} .16 \left\{\frac{(315)^2}{9} + \frac{(101)^2}{3} + \frac{(108)^2}{3} + \frac{(32)^2}{1}\right\} - 556$$

$$= \frac{16}{556} .19,337.3 - 556 = 0.47.$$

For (K - 1) = 3 degrees of freedom, the table of chi-square, Table 3, Appendix B, gives the probability of a value exceeding 0.47 as a number which lies between .90 and .95. Therefore, the fit of the observations to the theory is very good indeed. A test of any size $\alpha < .90$ would not reject the hypothesis (Kendall and Stuart, 1961, p. 422-423).

We now must seek some means of avoiding the unpleasant fact that there is a multiplicity of possible sets of classes, any of which will, in general, give a different result for the same data. Wald (1942) and Gumbel (1943) require a rule which is plausible and practical. Given K, choose the classes so that the hypothetical probabilities P_{0i} are all equal to 1/K. This procedure is perfectly definite and unique. This procedure requires that the data be available ungrouped for exactness.

For example, 50 random variables are obtained from the distribution:

$$df = \exp(-x)dx \quad 0 < x < \infty.$$

Arranged in order of variate-value, the observations are: 0.01, 0.01, 0.04, 0.17, 0.18, 0.22, 0.22, 0.25, 0.25, 0.29, 0.42, 0.46, 0.47, 0.47, 0.56, 0.59, 0.67, 0.68, 0.70, 0.72, 0.76, 0.78, 0.83, 0.85, 0.87, 0.93, 1.00, 1.01, 1.01, 1.02, 1.03, 1.05, 1.32, 1.34, 1.37, 1.47, 1.50, 1.52, 1.59, 1.71, 1.90, 2.10, 2.35, 2.46, 2.46, 2.50, 3.73, 4.07, 6.03.

Suppose that we wish to form four classes for a chi-square test. A natural grouping with equal-width interval would be (Kendall and Stuart, 1961, p. 432).

Variate value	Observed frequency	Hypothetical frequency
0 - 0.50	14	19.7
0.51 - 1.00	13	11.9
1.01 - 1.50	10	7.2
1.51 and over	13	11.2
	50	50

The hypothetical frequencies are obtained from the Biometrika Table (Pearson and Hartley, 1956, Vol. 1) distribution function of a chi-square variable with 2 degrees of freedom which is just twice a variable with the distribution alone. We find $\chi^2 = 3.1$ with 3 degrees of freedom, a value which would not reject the hypothetical parent distribution for any test of a size less than $\alpha = .37$. The agreement of observation and hypothesis is, therefore, very satisfactory.

Let us now consider how the same data would be treated by the method of the equal probabilities. We first determine the value of the hypothetical variable by dividing it into four equal probability classes. These are, of course, the quantiles. The Biometrika Tables give the values 0.288, 0.693, 1.386. We now group the classes as follows (Kendall and Stuart, 1961, p. 432):

Variate value	Observed frequency	Hypothetical frequency
0 - 0.28	9	12.5
0.29 - 0.69	9	12.5
0.70 - 1.38	17	12.5
1.39 and over	15	12.5
	50	50.0

Chi-square is now easier to calculate, since (2.9) reduces to

$$\chi^2 = \frac{k}{n} \sum_{i=1}^{k} n_i^2 - n$$

And since all hypothetical probabilities $P_{oi} = \frac{1}{k}$, we find here that $\chi^2 = 3.9$ would not lead to rejection unless the test size exceeded 0.27. The result is still very satisfactory, but the equal probabilities test seems more critical of the hypothesis than was the other test. There is a little extra arithmetical work involved in the equal probabilities method of carrying out the chi-square test. Instead of a regular class width, with hypothetical frequencies to be looked up in a table (or, if necessary, to be calculated), we have irregular class widths determined from the tables so that the hypothetical frequencies are equal. The equal probabilities method of forming classes for the chi-square test will not necessarily increase the power of the test (Kendall and Stuart, 1961, p. 431-433).

Let us outline some of the advantages and disadvantages of chi-square test for goodness-of-fit. First, this test has the general advantages and disadvantages of tests based on categorical type data. Namely, it has the advantages of being relatively easy to apply and being applicable to investigating probability distributions over specified restricted sets of points. Second, by grouping the data into classes, we do not need to know the values of the individual observations so long as we have k classes for which the hypothetical P_{0i} can be computed. On the other hand, the chi-square test is not consistent against general alternatives and seems to be somewhat insensitive unless the sample size is large.

The chi-square test of fit also has some additional disadvantages. First the signs of the derivations of n_i from nP_{0i}, and the order in which these signs occur, are not taken into consideration. Second, the relative locations of the disjoint sets from which the categories are determined are not considered. Third, there are some difficulties of a computational nature. Fourth, the need to group the data into classes clearly involves the sacrifice of a certain amount of information (Walsh, 1962, p. 448).

Kolmogorov

The Kolmogorov test is the most important of the general tests of fit alternative to chi-square. It is based on deviations of the sample df $S_n(x)$ from the completely specified continuous hypothetical df $F_o(x)$ (simple hypothesis). In a sample df, the distribution function is defined by

$$S_n(x) = \begin{cases} 0 & x < x_{(1)} \\ \frac{r}{n} & x_{(r)} < x < x_{(r+1)} \\ 1 & x_{(n)} < x \end{cases} \tag{2.10}$$

The $x_{(r)}$ are the order-statistics, i.e., the observations arranged to that $x_{(1)} < x_{(2)} < \ldots\ldots < x_{(n)}$. $S_n(x)$ is simple, the proportion of the observation not exceeding x.

The Kolmogorov test is defined by

$$D_n = \sup_x [S_n(x) - F_o(x)] \tag{2.11}$$

maximum absolute difference between $S_n(x)$ and $F_o(x)$.

The appearance of the modules in the definition (2.11) might lead us to expect difficulties in the investigation of the distribution of D_n, but remarkably enough the asymptotic distribution was obtained by Kolmogorov (1933) when he first proposed the statistic. The derivation which follows is due to Feller (1948) and is found in Appendix A.

For a given n, a single table is required for the distribution function of D_n and can be used for any $F_0(x)$. This table can be computed through the use of recursion formulas and has been computed for various sample sizes (Table 4, Appendix B, and Lindgren, 1962, p. 329).

For an example, consider testing the hypothesis that a distribution is normal with mean 32 and variance 3.24, by using the ten observations, 31.0, 31.4, 33.3, 33.4, 33.5, 33.7, 34.4, 34.9, 36.2, 37.0. The sample distribution function and the population distribution function being tested is sketched in Figure 1. The maximum deviation is about .56. According to Table 4, the 95th percentile of the distribution of D_n is .409. Since .567 > .409, the distribution being tested is rejected at the 5 percent level (Lindgren, 1962, p. 329-330).

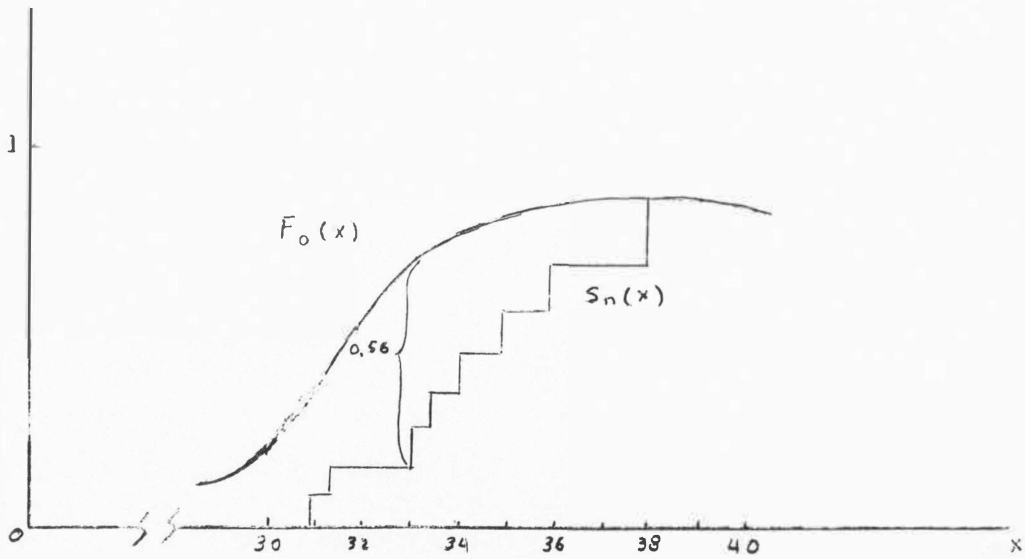

Figure 1. The absolute difference between $S_n(x)$ and $F_0(x)$.

Comparison of the chi-square and Kolmogorov tests

The chi-square test is definitely less powerful than the Kolmogorov test. For very small samples the chi-square test is not applicable at all, but the Kolmogorov test is.

The chi-square test is suitable for data which are in nominal or stronger scales. In many cases the chi-square test may not make efficient use of all information in the data. If the populations of scores are continuously distributed, the Kolmogorov test should be chosen in preference to the chi-square test. If the Kolmogorov test is used with data which do not meet the assumption of continuity, it is still suitable but it operates more conservatively; i.e., the obtained value of P in such cases will be slightly higher than it should be, and thus the probability of a Type II error will be slightly increased. If H_0 is rejected with such data, confidence can be had in the decision (Siegel, 1956, p. 47).

CHAPTER III

COMPOSITE HYPOTHESES

$H_0 : F(x) = F_0(x)$

A hypothesis is composite when the observations have come from a normal distribution where the parameters are unspecified, and it sometimes is called a test of normality. In addition, if the normal distribution has its mean but not its variance specified, the hypothesis remains composite.

Restrictions of chi-square and Kolmogorov tests

Rosenblatt (1962, p. 513) has eliminated the paradox of almost sure rejection of the null hypothesis when too much data are observed and has extended the Kolmogorov test to composite hypothesis. Previous to his work, the Kolmogorov-Smirnov tests were suitable only for testing the simple hypothesis, $H_0 : F(x) = F_0(x)$, against all alternatives.

The same paradox of almost certain rejection of the null hypothesis, when numerous observations are used, is also pointed out in the chi-square test-of-fit by Cochran (1952).

Chi-square (χ^2)

Suppose that $F_0(x)$ is specified as to its form, but that some (or perhaps all) of the parameters are left unspecified. The multinomial formulation of (30.4) on page 420 of *The Advanced Theory of Statistics* (Kendall and Stuart, 1961) is that the theoretical probabilities P_{oi} are not now immediately calculable because they are functions of the S (assumed < k - 1) unspecified parameters $\theta_1, \theta_2, \ldots, \theta_s$ which we may denote collectively by θ. Thus, we must write them $P_{oi}(\theta)$. To make progress, we must estimate θ by some vector of estimator T; to be as chi-square distributed multinomial maximum likelihood (ML) estimators of the parameters must be used (Kendall and Stuart, 1961, p. 426). However, it has been shown that ordinary ML estimators are adequate in large samples (Kendall and Stuart, 1961, p. 430) and use

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - nP_{oi})^2}{nP_{oi}(t)}$$

in the form

$$\chi^2 = \sum_{i=1}^{k} \frac{\{n_i - nP_{oi}(t)\}^2}{nP_{oi}(t)} \tag{3.1}$$

This clearly changes our distribution problem, for now the $P_{oi}(t)$ are themselves random variables, and it is not obvious that the asymptotic distribution of chi-square will be of the same form as in the case of simply H_0. In fact, the term $n_i - nP_{oi}(t)$ does not necessarily have a zero expectation. We may write chi-square identically as

$$\chi^2 = \sum_{i=1}^{k} \frac{1}{nP_{oi}(t)} [\{n_i - nP_{oi}(\theta)\}^2 + n^2\{P_{oi}(t) - P_{oi}(\theta)\}^2 -$$

$$- 2n\{n_i - nP_{oi}(\theta)\}\{P_{oi}(t) - P_{oi}(\theta)\}] \quad (3.2)$$

As an example, five "coins" with identical but unknown values of P = P (heads) are tossed together 100 times to test the hypothesis that the number of heads per toss follows a binomial distribution. (Perhaps some kind of dependence is introduced in the tossing process.) The results are given as follows (Lindgren, 1962, p. 327):

Number of heads	0	1	2	3	4	5
Frequency	3	16	36	32	11	2

The maximum likelihood estimate of P is the mean number of heads per five coins divided by five, which turns out to be 0.476. Using this to calculate the cell probabilities by the binomial formula, one obtains the following expected frequencies: 4.0, 17.9, 32.6, 29.6, 13.5, 2.4.

The value of chi-square is then found to be

$$\chi^2 = \frac{(3 - 4)^2}{4} + \dots. + \frac{(2 - 2.4)^2}{2.4} = 1.53.$$

The five percent rejection limit would be the 95th percentile of the chi-square distribution with 6-1-1-4 degrees of freedom which is 9.49. Since $1.53 < 9.49$, the null hypothesis is accepted (Lindgren, 1962, p. 327).

<u>Kolmogorov</u>

Test of approximate hypothesis for location scale parameter families of distributions are:

$H_0 = [G : G(x) = F_c(\frac{x-\mu}{\sigma})$ all x, some $\mu, \sigma > 0$, F_c a given continuous d.f.]

$H_0^* = [F \,\&\, D : d_1(F,G) \leq K,\ G \,\&\, H_0]$

H_0^* = [F element of D such that $d_1(F,G) < K$, G element H_0]

D = is the set of all one-dimensional distribution functions.

d_1 = Kolmogorov distance $d_1(F,G) = \text{Max}|F(x) - G(x)|$

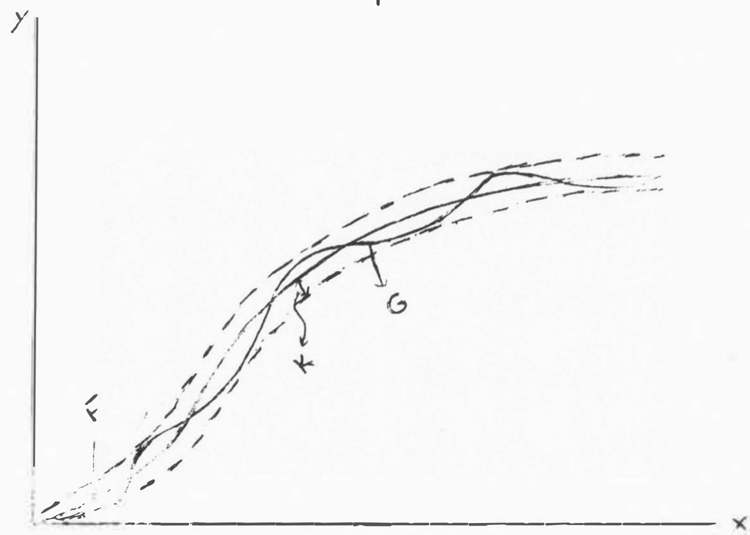

Figure 2. The determination of the number K.

The number K is determined from realistic considerations external to the mathematics. It is associated with -ln envelope shown in Figure 2. Any df contained within the dotted lines are in H_0^* for the given K. For each G & H_0, let $g_G = [H \,\&\, D : d_1(G,H) \leq K]$. If F & D and F & g_G, we have, say, $d_1(F,G) = K + \ell$, $\ell > 0$.

The test of H_0^* which is proposed by Judah Rosenblatt is

$RejH_0^*$ when $x_1(\omega),\ldots.,x_n(\omega)$ are observed $\leftrightarrow inf_{G \& H_0} d_1$ $(F_n,G)(\omega) \geq K + h_{1-\alpha,n} = q$, where F_n is an empirical function and the value of $h_{1-\alpha,n}$ is given in Table 4, Appendix B.

The test procedure has been worked out by Rosenblatt, and it is based upon the following theorem:

Theorem 1. $inf_{H \& H_0} < q$ if and only if for at least one (μ,σ), $\sigma > 0$. $(x_{[j]}(\omega) - \mu)/\sigma < b_{j,q}, (x_{[j]}(\omega) - \mu)/\sigma > a_{j,q}$ for all $j = 1,\ldots,n$, when $a_{j,q}$ and $b_{j,q}$ be any number which $F_c(a_{j,q}) = j/n - q, F_c(b_{j,q}) = (j - 1)/n - q$ for $j = 1,\ldots,n$, where $x_{[j]}(\omega)$ is the j^{th} order statistics of the sample.

The proof of the theorem is in *The Annals of Mathematical Statistics*, Volume 33, December, 1962, p. 1359. The significance of this theorem is that for each $q,x_1(\omega),\ldots,x_n(\omega)$ we can determine whether or not there is a (μ,σ), $\sigma > 0$ for which the inequalities in Theorem 1 are all satisified in a finite number of operations. This is accomplished geometrically by looking at each inequality in Theorem 1 separately and blocking out those points in the (μ,σ) plane for which each inequality cannot be satisfied. Only a straight-edge and graph paper are required. If there are any points (μ,σ), $\sigma > 0$, not blocked out for at least one of these inequalities, then we know that there is a G in H_0 for which $d_1 (F_n,G)(\omega) < q$. For $q = K + h_{1-\alpha,n}$, if there is a (μ,σ), $\sigma > 0$ for which Theorem 1 are satisfied, we accept H_0^* (Rosenblatt, 1962, p. 1358-1359).

Examples

Test the hypothesis that the following random sample comes from the populationwhich is approximately normally distributed, .41, .38, .28, .02, .39, .58, .37, .79, .21, .71, α = .95. In terms of H_o and H_o^*, the hypothesis may be written

$$H_o = [G:G(x) = F_c(\frac{x-\mu}{\sigma})]$$

$$H_o^* = [F \& D:d_1(F,G) \leq K]$$

where F_c is the standard normal distribution and let K = .001. H_o^* is the set of all d.f. such that

$$d_1(F,G) \leq .001.$$

First we rewrite the 10 samples in order: .02, .21, .28, .37, .38, .41, .58, .71, .79. Let

$$q = K = h_{1-\alpha,n} \quad j = 1,\ldots,n.$$

We get the value of $h_{1-\alpha,n} = h_{.05,10}$ = .409 from Table 4, Appendix B. Let q = .001 + .409 = .410. The $a_{j,q}$ and $b_{j,q}$ are found from the standard normal distribution in the following manner:

$$\Phi(a_{j,q}) = j/n - q$$

$$\Phi(b_{j,q}) = \frac{1 - j}{n} + q$$

The solution of $a_{j,q}$ and $b_{j,q}$ for i = 1,...,10 are shown below:

j	$\Phi(a_{j,q})$	$a_{j,q}$	$\Phi(b_{j,q})$	$b_{j,q}$
1	0	$-\infty$	.410	-.22
2	0	$-\infty$	.310	.03
3	0	$-\infty$	.210	.28
4	0	$-\infty$	.110	.56
5	.090	-.134	.010	.88
6	.190	-.87	-.090	1.34
7	.290	-.55	1	∞
8	.390	-.27	1	∞
9	.490	-.02	1	∞
10	.590	-.23	1	∞

We shall proceed by blocking out the value of μ and σ which do not satisfy the inequalities in Theorem 1.

For j = 1, the inequalities are

$$(x_1(\omega) - \mu)/\sigma < b_{1,q} \qquad (x_1(\omega) - \mu)/\sigma > a_{1,q}$$

$$(.02 - \mu)/\sigma < -.22 \qquad (.02 - \mu)/\sigma > -\infty$$

The shaded area of Figure 3 shows the values of μ and σ which do not satisfy inequalities (1) for j = 1. Similarly for j = 2, the shaded area of Figure 4 shows the vlues of μ and σ which do not satisfy inequalities (1) for j = 2.

$(.21 - \mu)/\sigma < .03 \qquad (.21 - \mu)/\sigma > -\infty.$

For j = 3, Figure 5 shows the value of μ and σ which do not satisfy inequalities (1) for j = 3.

$(.28 - \mu)/\sigma < .28 \qquad (.28 - \mu)/\sigma > -\infty.$

For j = 4, Figure 6 shows the value of μ and σ which do not satisfy inequalities (1) for j = 4.

$(.37 - \mu)/\sigma < .56 \qquad (.37 - \mu)/\sigma > -\infty.$

For j = 5, Figure 7 shows the value of μ and σ which do not satisfy inequalities (1) for j = 5.

$(.38 - \mu)/\sigma < .88 \qquad (.38 - \mu)/\sigma > -1.34.$

For j = 6, Figure 8 shows the value of μ and σ which do not satisfy inequalities (1) for j = 6.

$(.39 - \mu)/\sigma < 1.34 \qquad (.39 - \mu)/\sigma > -.87.$

For j = 7, Figure 9 shows the value of μ and σ which do not satisfy inequalities (1) for j = 7.

$(.41 - \mu)/\sigma < + \infty \qquad (.41 - \mu)/\sigma > -.55.$

For $j = 8$, Figure 10 shows the value of μ and σ which do not satisfy inequalities (1) for $j = 8$.

$$(.58 - \mu)/\sigma < +\infty \qquad (.58 - \mu)/\sigma > .27.$$

For $j = 9$, Figure 11 shows the value of μ and σ which do not satisfy inequalities (1) for $j = 9$.

$$(.71 - \mu)/\sigma < +\infty \qquad (.71 - \mu)/\sigma > -.02.$$

For $j = 10$, Figure 12 shows the value of μ and σ which do not satisfy inequalities (1) for $j = 10$.

$$(.79 - \mu)/\sigma < +\infty \qquad (.79 - \mu)/\sigma > .23.$$

All of the shaded regions given in Figures 3 to 12 are superimposed in Figure 13. Since the complete plane is not covered, we accept the hypothesis.

Comparison of Kolmogorov's statistic with chi-square

Massey (1952) established a lower bound to the power of the Kolmogorov test in large samples as follows:

Write $F_1(x)$ for the d f. under the alternative hypothesis H_1, $F_0(x)$ for the d.f. being tested in *Advanced Theory of Statistics* Kendall and Stuart, 1961, Chapter 30) and

$$\Delta = \operatorname*{Sup}_x |F_1(x) - F_0(x)| \qquad (3.3)$$

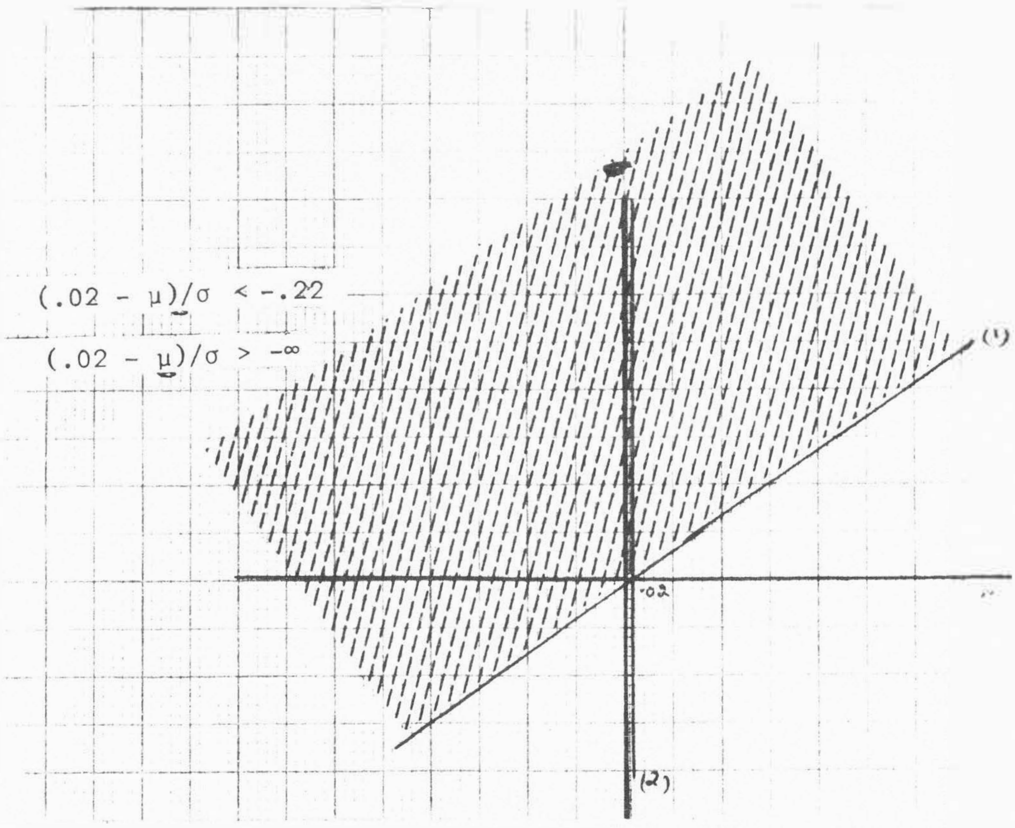

Figure 3. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 1.

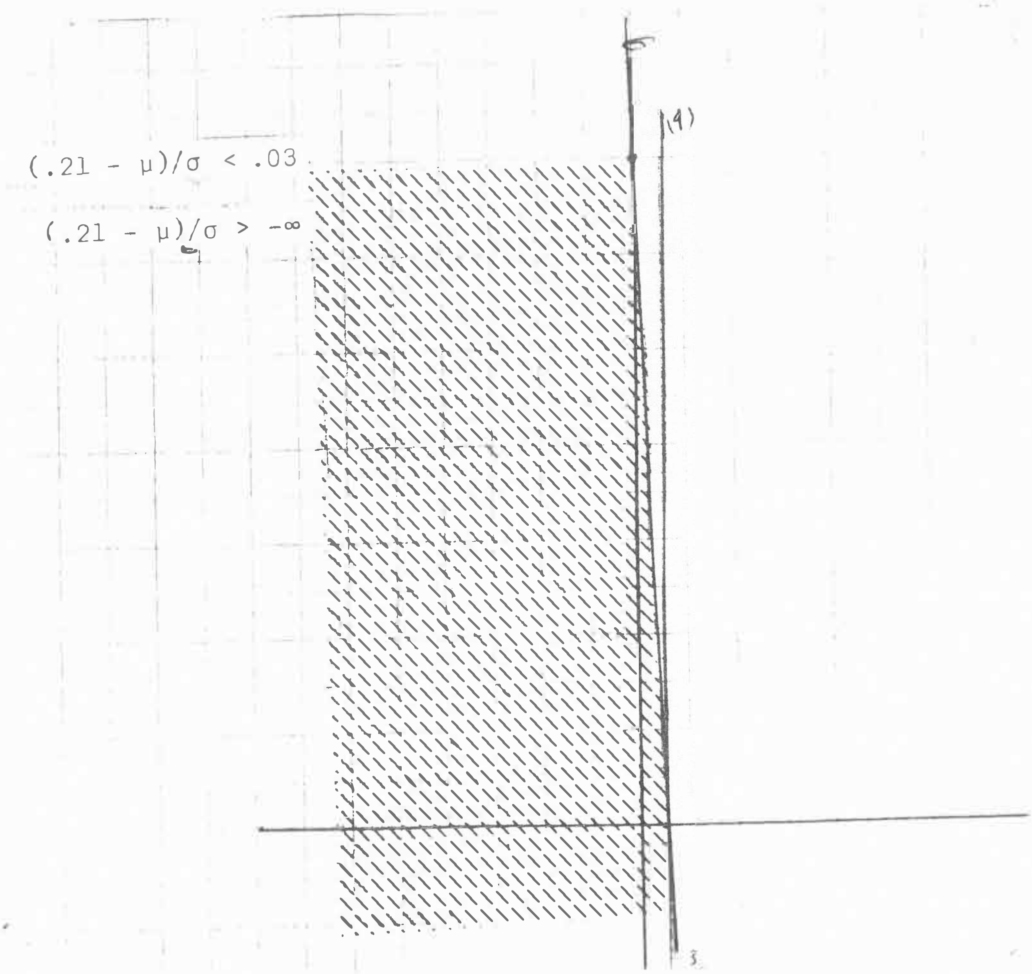

Figure 4. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 2.

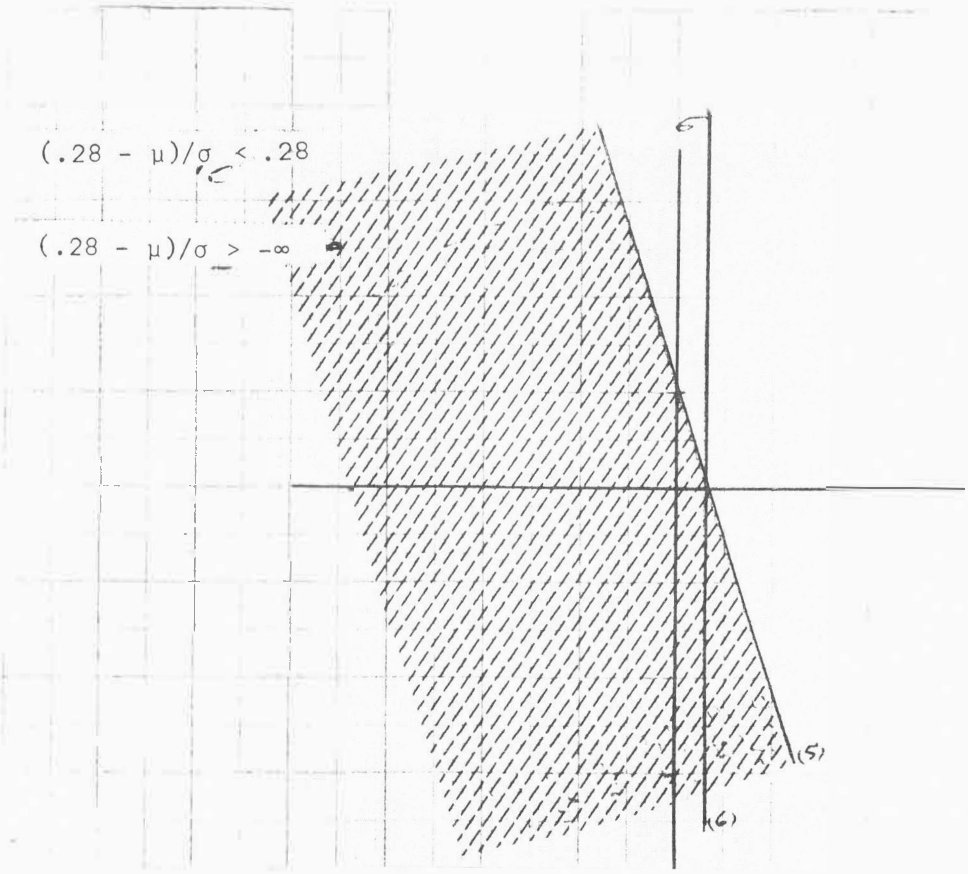

Figure 5. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 3.

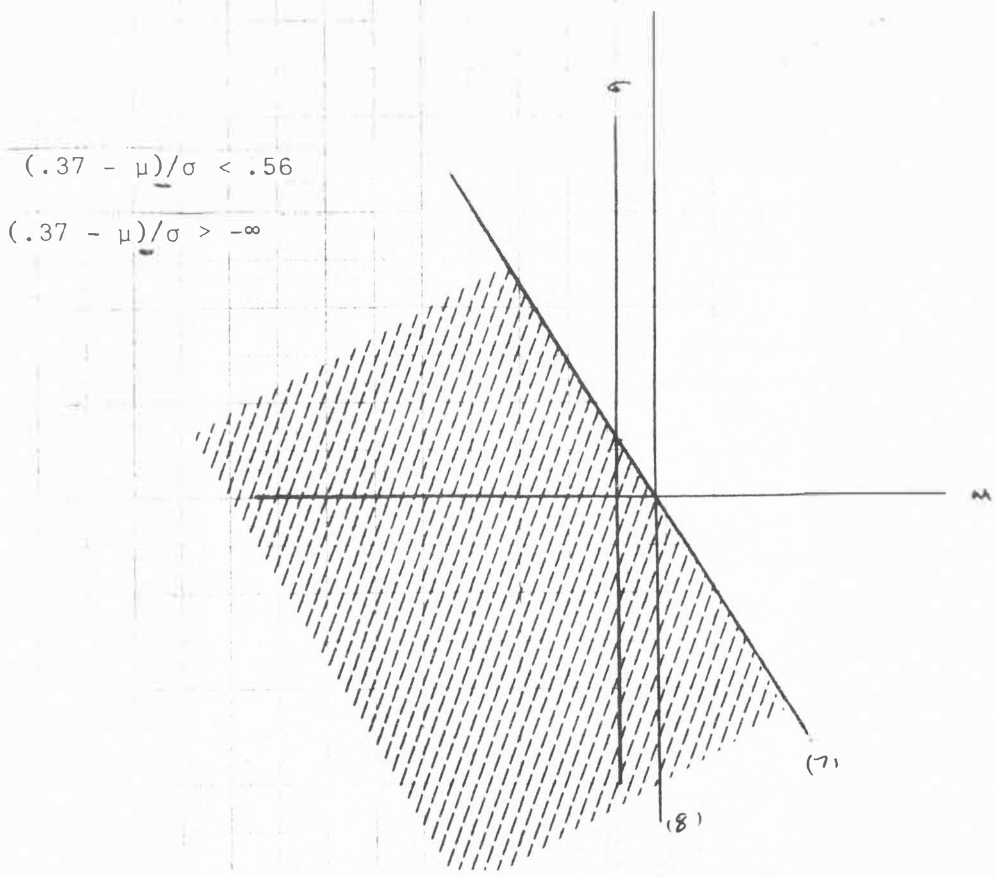

Figure 6. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 4.

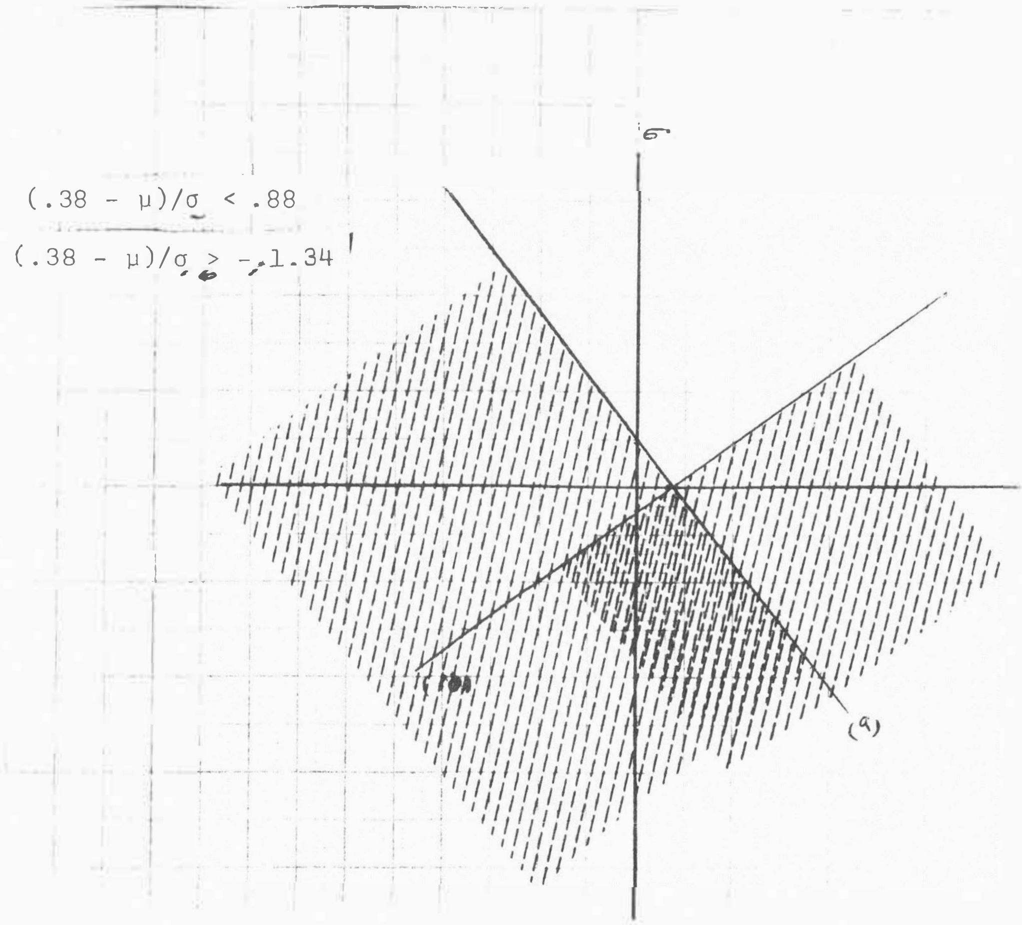

Figure 7. The values of μ and σ which do not satisfy inequalities in Theorem 1 for $j = 5$.

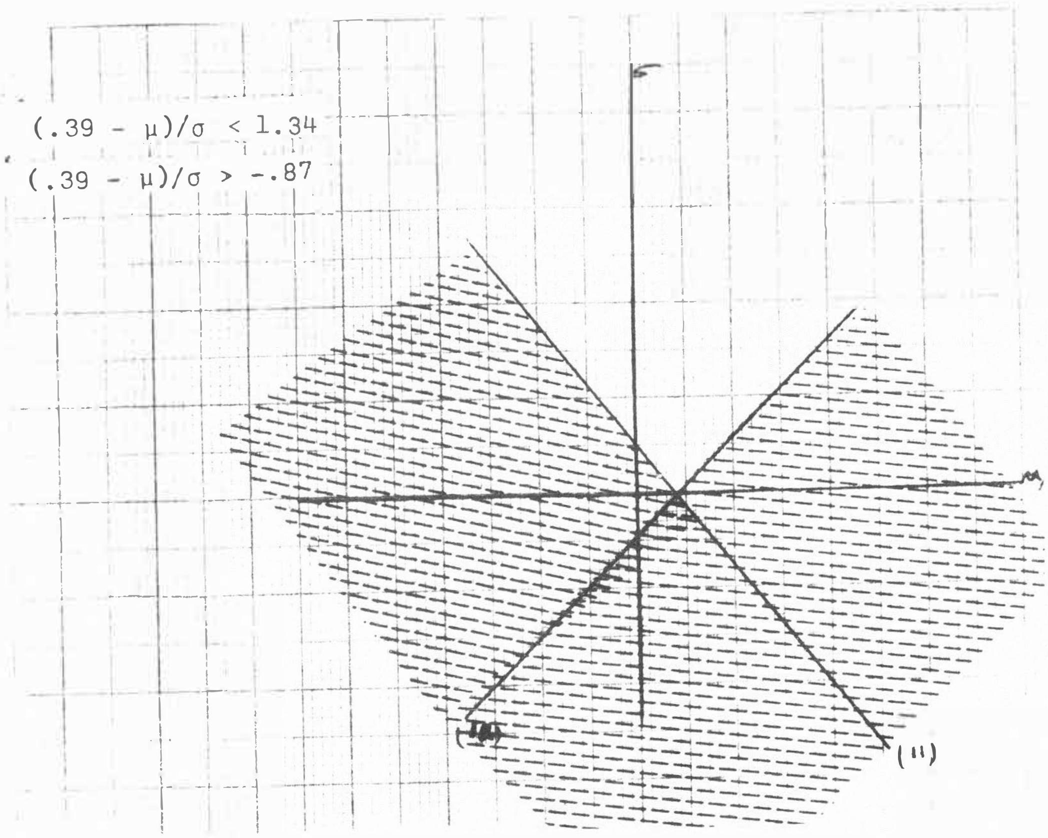

Figure 8. The values of μ and σ which do not satisfy inequalities in Theorem 1 for $j = 6$.

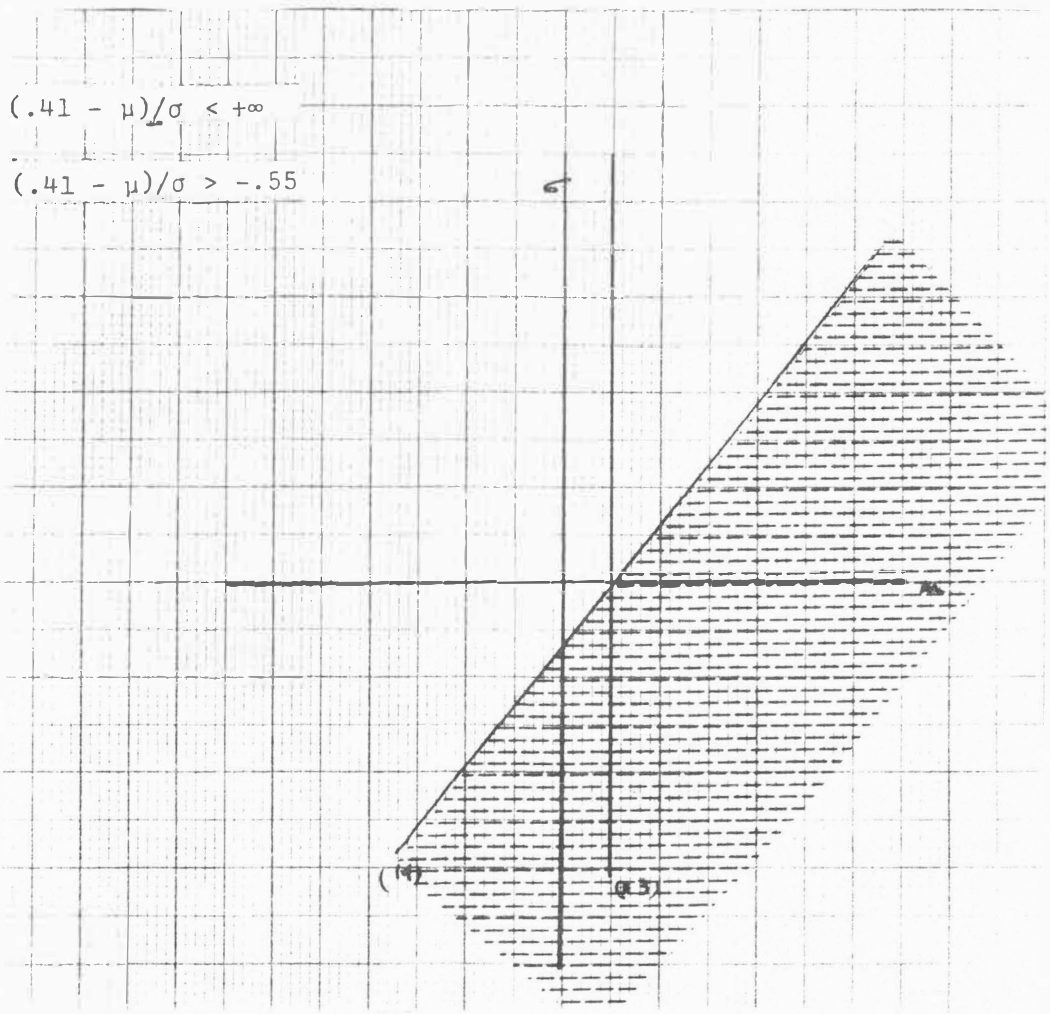

Figure 9. The values of μ and σ which do not satisfy inequalities in Theorem 1 for $j = 7$.

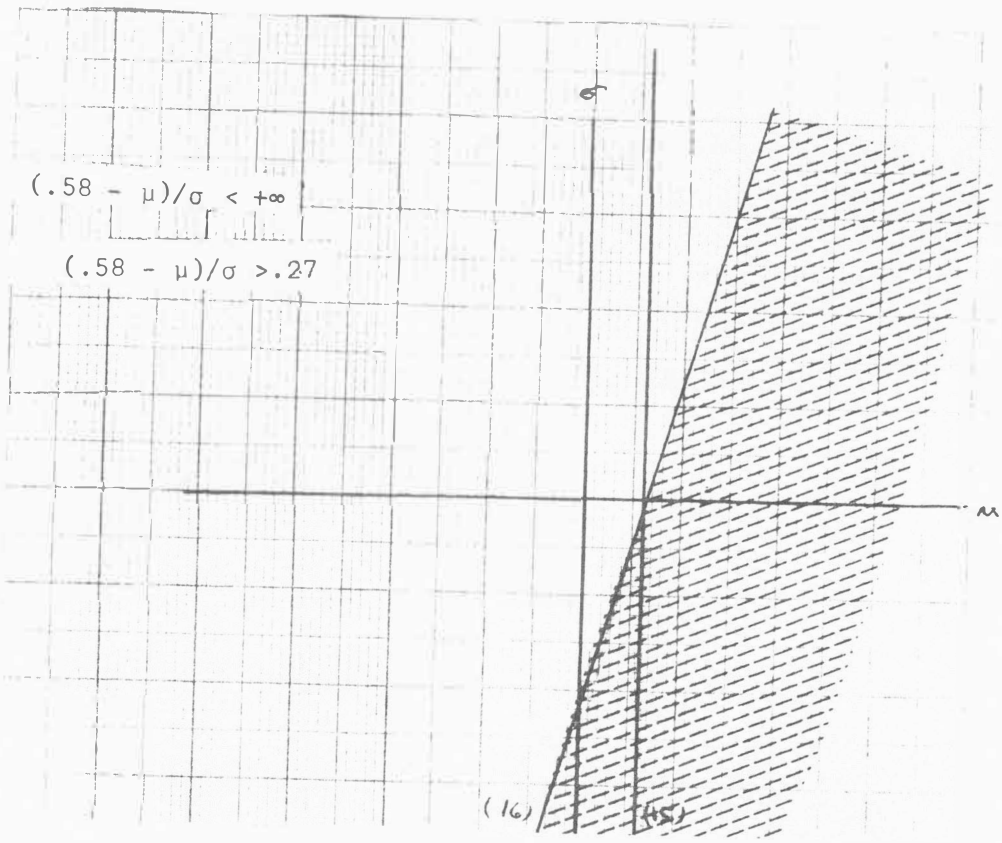

Figure 10. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 8.

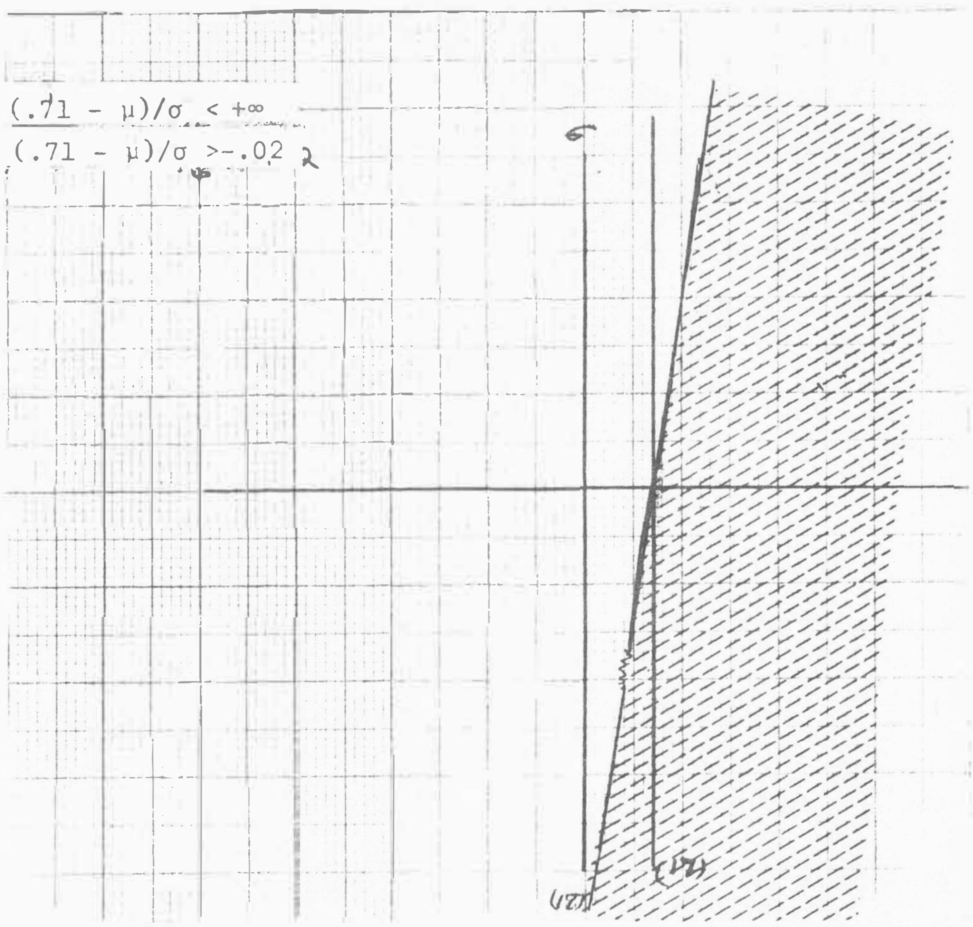

Figure 11. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 9.

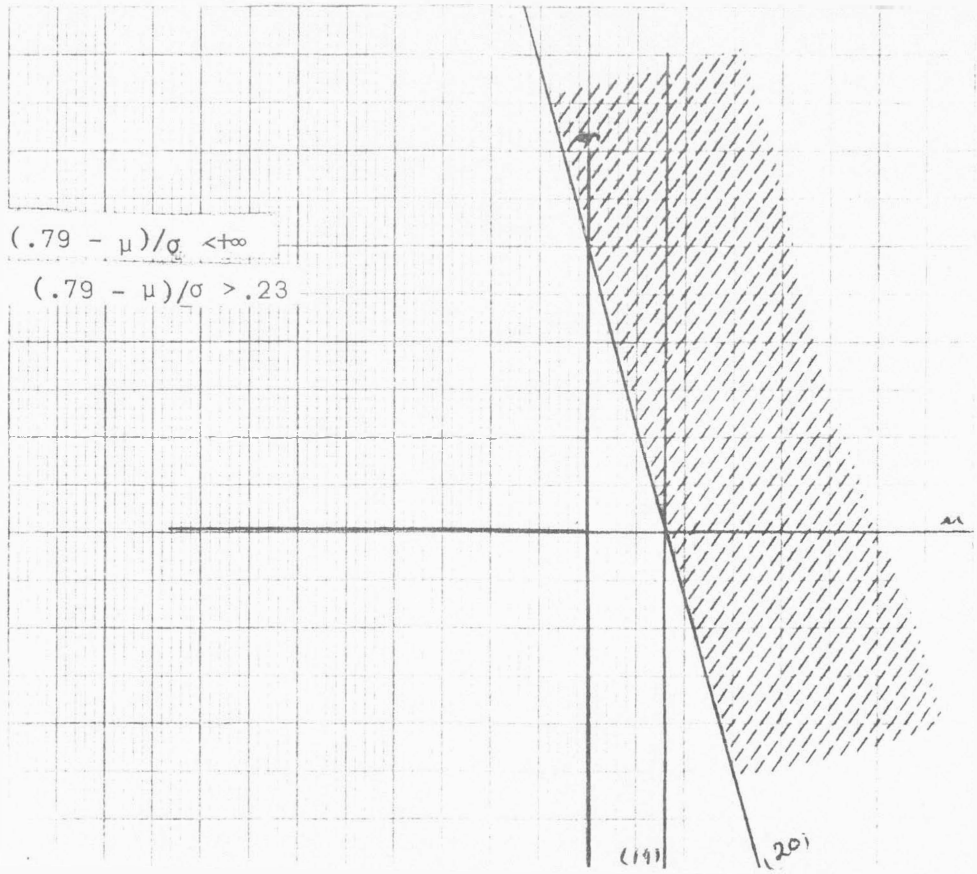

Figure 12. The values of μ and σ which do not satisfy inequalities in Theorem 1 for j = 10.

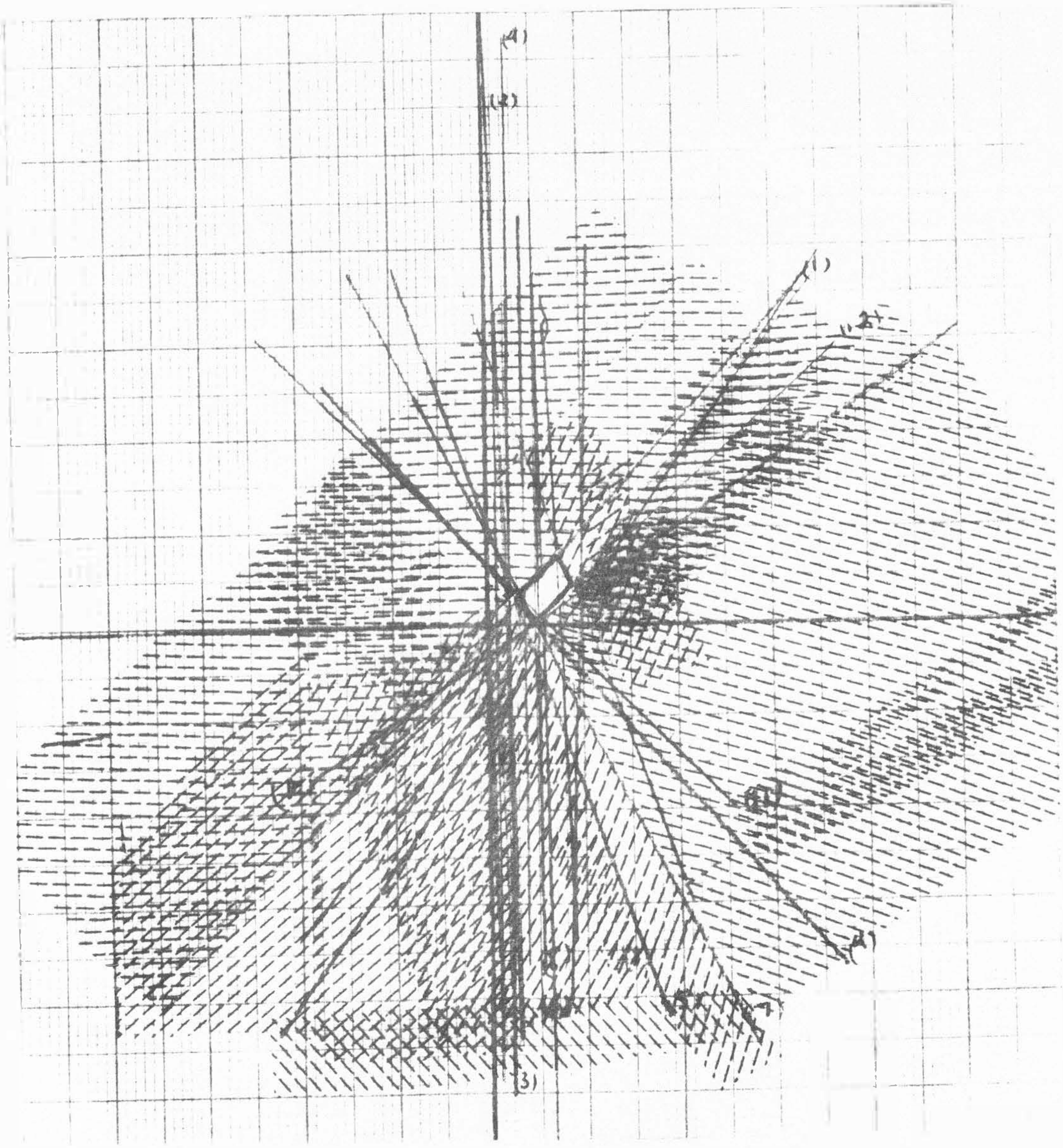

Figure 13. Figures 3 through 12 all together.

if d_α is the critical value of D_n as before, the power we require is

$$P = P\{\sup_x |S_n(x) - F_o(x)| > d_\alpha / H_1\}$$

This is the probability of an inequality arising for some x. Clearly this is no less than the probability that it occurs at any particular value of x. Let us choose a particular value, x_Δ, at which F_o and F_1 are at their farthest apart, i.e.,

$$\Delta = F_1(x_\Delta) - F_o(x_\Delta). \tag{3.4}$$

Thus we have

$$P \geq P\{|S_n(x_\Delta) - F_o(x_\Delta)| > d_\alpha / H_1\}$$

or

$$P \geq 1 - P\{F_o(x_\Delta) - d_\alpha \leq S_n(x_\Delta) \leq F_o(x_\Delta) + d_\alpha / H_1\}. \tag{3.5}$$

Now, $S_n(x_\Delta)$ is binomially distributed with probability $F_1(x_\Delta)$ of falling below x_Δ. Thus, we may approximate the right-hand side of (3.5) using the approximation to the binomial distribution, i.e., asymptotically

$$P \geq 1 - (2\Pi)^{-1/2} \int_{\frac{F_o - F_1 - d_\alpha}{\{F_1(1 - F_1)/N\}^{1/2}}}^{\frac{F_o - F_1 + d_\alpha}{\{F_1(1 - F_1)/M\}^{1/2}}} \exp(-1/2u^2)d\mu \tag{3.6}$$

F_o and F_1 are evaluated at x_Δ in (3.6) and hereafter. If F_1 is specified, (3.6) is the required lower bound for the power. Clearly, as $n \to \infty$, both limits of integration increase. If

$$d_\alpha < |F_o - F_1| = \Delta, \tag{3.7}$$

they will both tend to $+\infty$ if $F_o > F_1$ and to $-\infty$ if $F_o < F_1$. Thus the integral will tend to zero and the power to 1. As n increases, d_α declines, so (3.7) is always ultimately satisfied. Hence, the power $\to$ 1 and the test is consistent. If F_1 is not completely specified, we may still obtain a (worse) lower bound to the power from (3.6). Since $F_1(1 - F_1) < 1/4$, we have, for large enough n,

$$P \geq 1 - (2\Pi)^{-1/2} \int_{2n^{1/2}(F_o - F_1 - d_\alpha)}^{2n^{1/2}(F_o - F_1 + d_\alpha)} \exp(-1/2u^2)d\mu$$

which, using the symmetry of the normal distribution, if $F_o < F_1$, we may write as

$$P \geq 1 - (2\Pi)^{-1/2} \int_{2n^{1/2}(\Delta - d_\alpha)}^{2n^{1/2}(\Delta + d_\alpha)} \exp(-1/2u^2)d\mu \tag{3.8}$$

The bound (3.8) is in terms of the maximum deviation Δ above. Using (3.8) and calculations made by Williams (1950), Massey (1952) compared the value of Δ for which the large-sample powers of the chi-square and the D_n tests are at least 0.5. For test size $\alpha = .05$, the D_n test can detect with power 0.5 a Δ about half the magnitude of that which the chi-square test can detect with this power. Even with n = 200, the ratio of Δ's is 0.6, and it declines steadily in favor of D_n as n increases. Since this comparison is based on the poor lower bound (3.8) to the power of D_n, we must conclude that D_n is a much more sensitive test for the fit of a continuous distribution (Kendall and Stuart, 1961, p. 458).

Let us suppose that a sample of 40 observations is in hand, where values are arranged in order: 0.0475, 0.2153, 0.2287, 0.2824, 0.3743, 0.3868, 0.4421, 0.5033, 0.5945, 0.6004, 0.6255, 0.6331, 0.6478, 0.7867, 0.8878, 0.8930, 0.9335, 0.9602, 1.0448, 1.0556, 1,0894, 1.0999, 1.1765, 1.2036, 1.2344, 1.2712, 1.3515, 1.3528, 1.3774, 1.4209, 1.4304, 1.5137, 1.5288, 1.5291, 1.5677, 1.7238, 1.7919, 1.8794.

We wish to test, with $\alpha = .05$, whether the parent $F_0(x)$ is normal with mean 1 and variance 6. From Birnbaum's (1952) Table, we find for n = 40, $\alpha = .05$, that $d_\alpha = .2101$. Consider the smallest observation, $x_{(1)}$. To be acceptable, $F_0(x_{(1)})$ should lie between 0 and d , i.e., in the interval (9,0.2101). The observed value of $x_{(1)}$ is 0.0475; and from Table of the normal d.f. in *Statistical Theory* (Lindgren, 1962, p. 478), we find $F_0(x_{(1)}) = 0.0098$, within the above interval. So the hypothesis is not rejected by this observation. Further, it cannot possibly be rejcted by the next higher observations until we reach an $x_{(1)}$ for which either (a) 1/40 - 0.2101 > .0098, i.e., i > 8.796, or (b) $F_0(x_{(i)}) > .2101 +$ 1/40, i.e., $x_{(i)} > .7052$ (from the tables again). The 1/40 is added on the right of (b) because we know that $S_n(x_{(i)}) \geq 1/40$ for i > 1. Now from the data, $x_{(i)} > .7052$ for $i \geq 14$. We will not need, therefore, to examine i = 9 (from the inequality (a)). We find there the acceptance interval for $F_0(x_{(q)})$

$$(S_q(x) - d_\alpha, S_8(x) + d_\alpha) = 9/40 - .2101, 8/40 + .2101)$$

$$= (0.0149, .4101)$$

We find from the tables $F_0(x_{(q)}) = F_0(0.5945) = .1603$, which is acceptable.

To reject H_0, we now require either

$$1/40 - 0.2101 > .1603, \text{ i.e., } i > 14.82$$

or

$$F_0(x_{(i)}) > .4101 + 1/40, \text{ i.e., } x_{(i)} > .9052, \text{ i.e., } i \geq 17.$$

We therefore proceed to i = 15, and so on. One should note that only the six values, i = 1, 9, 15, 21, 27, 34, require computations in this case. The hypothesis is accepted because in every one of these six cases the value of F_0 lies in the confidence interval. It would have been rejected, and computations ended, if any one value had lain outside the interval (Kendall and Stuart, 1961, p. 460-461).

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APPENDIXES

Appendix A

Calculations

The distribution of D_n is completely distribution-free when H_o holds; because if $S_n(x)$ and $F_o(x)$ are plotted as ordinates against x as Absissa, D_n is simply the value of the largest vertical difference between them. Clearly, if we make any one-to-one transformation of x, this will not affect the vertical difference at any point and, in particular, the value of D_n will be unaffected.

Now consider the values x_{10}, $x_{20}, \ldots, x_{n-1,0}$ defined by

$$F_o(x_{ko}) = \frac{k}{N} \tag{A-1.1}$$

(If for some k, (A-1.1) holds within an interval, we take x_{ko} to be the lower end-point of the interval.) Let c be a positive integer. If, for some value x,

$$S_n(x) - F_o(x) > \frac{c}{n}, \tag{A-1.2}$$

the inequality (A-1.2) will hold for all values of x in some interval where at the upper end-point x' it becomes an equality; i.e.,

$$S_n(x') - F_o(x') = \frac{c}{n} \tag{A-1.3}$$

Since $S_n(x)$ is by definition a step-function with values which are multiples of 1/n and c is an integer, it follows from (A-1.3) that

$F_o(x')$ is a multiple of 1/n; and thus from (A-1.1), $x' = x_{ko}$ for some k, so that (A-1.3) becomes

$$S_n(x_{ko}) = F_o(x_{ko}) = c/n,$$

i.e., from (A-1.1) $S_n(x_{ko}) = \frac{k+c}{n}$. (A-1.4)

From the definition of $S_n(x)$ at (2.10) in Part 4 of Chapter 2, this means that exactly (k+c) of the observed values of x are less than x_{ko}, the hypothetical value below which k of them should fall. Conversely, if $x_{(k+c)} < x_{ko} < x_{(k+c+1)}$, (A-1.2) will follow immediately. We have, therefore, established the preliminary result that the equality

$$S_n(x) = F_o(x) > c/n$$

hold for some x if and only if for some k

$$x_{k+c} \leq x_{ko} \ x_{(k+c+1)} \quad \text{(A-1.5)}$$

We may therefore confine ourselves to consideration of the probability that (A-1.5) occurs.

We denote the event (A-1.5) by $A_k(c)$. From (2.11) in Part 4 of Chapter 2, we see that the statistic D_n will exceed c/n if and only if at least one of the 2n events

$$A_1(c),\ A_1(-c),\ A_2(c),\ A_2(-c),\ldots,A_n(c),\ A_n(-c) \quad \text{(A-1.6)}$$

occurs. We now define the 2n mutually exclusive events U_r and V_r. U_r occurs if $A_r(c)$ is the first event in the sequence (A-1.6) to occur, and V_r occurs if $A_r(-c)$ is the first. Evidently

$$P\{D_n > \frac{c}{n}\} = \sum_{r=1}^{n} [P\{U_r\} + P\{V_r\}] \qquad \text{(A-1.7)}$$

We have, from the definitions of $A_r(c)$ and U_r, V_r, the relations

$$\left.\begin{aligned} P\{A_k(c)\} &= \sum_{r=1}^{k} [P\{U_r\}P\{A_k(c)|A_r(c)\} + P\{V_r\}P\{A_k(c)|A_r(-c)\}], \\ P\{A_k(-c)\} &= \sum_{r=1}^{k} [P\{U_r\}P\{A_k(-c)|A_r(c)\} + P\{V_r\}P\{A_k(-c)|A_r(-c)\}] \end{aligned}\right]$$

(A-1.8)

From (A-1.5) and (A-1.1), we see that $P\{A_k(c)\}$ is the probability that exactly (k=c) "successes" occur in n binomial trials with probability k/n, ie.,

$$P\{A_k(c)\} = \binom{n}{k+c}\left(\frac{k}{n}\right)^{k+c}\left(1 - \frac{k}{n}\right)^{n-(k+c)} \qquad \text{(A-1.9)}$$

Similarly, for $r \leq k$,

$$\left.\begin{aligned} P\{A_k(c)|A_r(c)\} &= \binom{n-(r+c)}{k-r}\left(\frac{k-r}{n-r}\right)^{k-r}\left(1 - \frac{k-r}{n-r}\right)^{n-(k+c)} \\ P\{A_k(c)|A_r(-c)\} &= \binom{n-(r-c)}{k-r+2c}\left(\frac{k-r}{n-r}\right)^{k-r+2c}\left(1 - \frac{k-r}{n-r}\right)^{n-(k+c)} \end{aligned}\right] \qquad \text{(A-1.10)}$$

Formulas (A-1.9) and (A-1.10) hold for negative as well as positive c. Using them we see that (A-1.8) is a set of 2n linear equations for the 2n unknown $P\{U_r\}$, $P\{V_r\}$. If we solved these, and substituted into (A-1.7), we should obtain $P\{D_n > c/n\}$ for any c.

If we now unite

$$P_k(c) = e^{-k}\,\frac{k^{k+c}}{(k+c)!}\,, \tag{A-1.11}$$

we have

$$\left.\begin{aligned} P\{A_k(c)\} &= P_k(c)P_{n-k}(-c)/P_n(o). \\ P\{A_k(c)\,|\,A_r(c)\} &= P_{k-r}(0)P_{n-k}(-c)/P_{n-r}(-c) \\ P\{A_k(c)\,|\,A_r(-c)\} &= P_{k-r}(2c)P_{n-k}(-c)/P_{n-r}(c). \end{aligned}\right] \quad \text{(A-1.12)}$$

Then if we define

$$U_r = P\{U_r\}\,\frac{P_n(o)}{P_{n-r}(-c)}\,,\ V_r = P\{V_r\}\,\frac{P_n(o)}{P_{n-r}(c)} \tag{A-1.13}$$

and substitute (A-1.9--A-1.13) into (A-1,8), the latter becomes simply

$$\begin{aligned} P_k(c) &= \sum_{r=1}^{k}[U_r\,P_{k-r}(0) + V_rP_{k-r}(2c)] \\ P_k(-c) &= \sum_{r=1}^{k}[U_rP_{k-r}(-2c) + V_rP_{k-r}(0)] \end{aligned} \tag{A-1.14}$$

The system (A-1.14) is to be solved for

$$\sum_{r=1}^{n} [P\{U_r\} + P\{V_r\}] = \frac{1}{P_n(o)} \sum_{r=1}^{n} [P_{n-r}(-c)\, U_r + P_{n-r}(c)\, V_r] \tag{A-1.15}$$

We therefore define

$$p_k = \frac{1}{P_n(o)} \sum_{r=1}^{k} P_{k-r}(-c)\, U_r,\quad q_k = \frac{1}{P_n(o)} \sum_{r=1}^{k} P_{k-r}(c)\, V_r \tag{A-1.16}$$

so that, from (A-1.16)

$$\sum_{r=1}^{n} [P\{U_r\} + P\{V_r\}] = p_n + q_n \tag{A-1.17}$$

We now set up generating functions for the p_k and q_k, i.e.,

$$G_p(t) = \sum_{k=1}^{\infty} p_k\, t^k,\quad G_q(t) = \sum_{k=1}^{\infty} q_k t^k$$

If we also define generating functions for the U_k, V_k and (for convenience) $n^{-1/2}P_k(c)$, i.e.,

$$G_u(t) = \sum_{k=1}^{\infty} U_k t^k,\quad G_v(t) = \sum_{k=1}^{\infty} V_k t^k$$

and

$$G(t,c) = n^{-1/2} \sum_{k=1}^{\infty} P_k(c) t^k,$$

we have from (A-1.16), the relationships

$$\left.\begin{aligned} G_p(t) &= G_u(t)G(t,-c)n^{1/2}/P_n(o) \\ G_q(t) &= G_v(t)G(t,c)n^{1/2}/P_n(o) \end{aligned}\right] \qquad \text{(A-1.18)}$$

To consider the limiting form of (A-1.18), we put

$$c = zn^{-1/2}$$

and let $n \to \infty$ and $c \to \infty$ with it so that z remains fixed. We see from (A-1.11) that $P_k(c)$ is simply the probability of the value (k+c) for a poisson variate with parameter k, i.e., the probability of its being $c/k^{1/2}$ standard deviations above its mean. If k/n tends to some fixed value m, then as the poisson variate tends to normality.

$$P_k(c) \to (2\Pi K)^{-1/2} \exp\,(-1/2\,\frac{c^2}{k})$$

or, putting k = mn, $c = zn^{1/2}$

$$n^{1/2}P_k(zn^{1/2}) \to (2\Pi m)^{1/2} \exp\,(-1/2\,\frac{z^2}{m}). \qquad \text{(A-1.19)}$$

Now, since G(t,c) is a generating function for the $n^{1/2}P_k(c)$, we have

$$G(e^{-t/n}, zn^{1/2}) = n^{-1/2} \sum_{k=1}^{\infty} P_k(zn^{1/2})e^{-tk/n}$$

and under our limiting process this tends by (A-1.19) to

$$\lim_{n\to\infty} G(e^{-t/n}, zn^{1/2}) = (2\Pi)^{-1/2} \int_0^\infty n^{-1/2} \exp(-tm - 1/2 \frac{z^2}{m})\, dm.$$

(A-1.20)

If we differentiate the integral I on the right of (A-1.20) with respect to $1/2z^2$, we then find the simple differential equation

$$\frac{\partial I}{\partial(1/2z^2)} = - (\frac{t}{1/2z^2})\, I$$

whose solution is

$$I = (\frac{\Pi}{t})^{1/2} \exp\{-(2tz^2)^{1/2}\}.$$

Thus

$$\lim_{n\to\infty} G(e^{-t/n}, zn^{1/2}) = (2t)^{-1/2} \exp\{-2tz^2)^{1/2}\} \qquad \text{(A-1.21)}$$

(A-1.21) is an even function of z, and therefore of c.

Since from (A-1.14)

$$G(t,c) = G_u(t)G(t,o) + G_v(t)G(t,2c),$$

(A-1.22)

$$G(t,-c) = G_u(t)G(t,-2c) + G_v(t)G(t,o)$$

this evenness of (A-1.21) in c gives us

$$\lim_{n\to\infty} G_u(e^{-t/n}) = \lim_{n\to\infty} G_v(e^{-t/n})$$

$$= \frac{\lim G(e^{-t/n}, zn^{1/2})}{\lim G(e^{-t/n},0) + \lim G(e^{-t/n}, 2zn^{1/2})} \qquad \text{(A-1.23)}$$

$$= \frac{\exp\{-(2tz^2)^{1/2}\}}{1 + \exp\{-(8tz^2)^{1/2}\}}$$

by (A-1.21). Thus in (A-1.18), remembering that

$$P_n(o) \to (2\Pi n)^{1/2},$$

(A-1.21) and (A-1.23) give

$$\lim_{n\to\infty} n^{-1} G_p(e^{-t/n}) = \lim_{n\to\infty} n^{-1} G_q(e^{-t/m}) = \left(\frac{2\Pi}{2t}\right)^{1/2}$$

$$\frac{\exp\{-(8tz^2)^{1/2}\}}{1 + \exp\{-(8tz^2)^{1/2}\}} = L(t)$$

This may be expanded into geometric series as

$$L(t) = \left(\frac{2\Pi}{2t}\right)^{1/2} \sum_{r=1}^{\infty} (-1)^{r-1} \exp\{-(8tr^2z^2)^{1/2}\} \qquad \text{(A-1.24)}$$

By the same integration as at (A-1.20), L(t) is seen to be one-sided Laplace transform $\int_0^\infty e^{-mt} f(m)dm$ of the function

$$f(m) = \sum_{r=1}^{\infty} (-1)^{r-1} \exp\{-2r^2z^2/m\}. \qquad \text{(A-1.25)}$$

The formula (A-1.25) is thus the result of inverting either of the limiting generating functions of the P_k or q_k, of which the first is

$$\lim_{\to\infty} n^{-1} G_p(e^{-t/n}) = \lim n^{-1} \sum_{k=1}^{\infty} P_k e^{-tk/n} = \int_0^{\infty} (\lim P_k) e^{-tm} dm.$$

From (A-1.7) and (A-1.17), we require the value $(p_n + q_n)$. We thus put $k = n$, i.e., $m = 1$, in (A-1.25) and after multiplying by 2, obtain our final result

$$\lim_{n\to\infty} P\{D_n \; zn^{1/2}\} = 2 \sum_{r=1}^{\infty} (-1)^{r-1} \exp\{-2r^2z^2\}. \qquad \text{(A-1.26)}$$

(Kendall and Stuart, 1961, p. 458-459)

Appendix B

Tables

Tables 1 through 4 are included as an appendix because they are too cumbersome to include in the main body of the paper. They are, however, extremely valuable for obtaining values used in the various types of nonparametric tests.

Table 1. Values of $\chi^2_\alpha(k)$ for $.005 < \alpha < .995$ and $k = 1(1)(30)$ when $k > 30$ and $2/7k \leq \alpha \leq 1 - 1/2k$, $\chi^2_\alpha(k) \doteq k(1 - 2/9k + k_\alpha \sqrt{2/9k})^3$

k^a	.995	.990	.980	.975	.950	.900	.800
1	393×10^{-7}	157×10^{-6}	628×10^{-6}	982×10^{-6}	393×10^{-5}	.0158	.0642
2	.0100	.0201	.0404	.0506	.1026	.2107	.446
3	.0717	.1148	.185	.2158	.3518	.5844	1.005
4	.2070	.2971	.429	.4844	.7107	1.064	1.649
5	.4117	.5543	.752	.8312	1.145	1.610	2.343
6	.6757	.7821	1.134	1.237	1.635	2.204	3.070
7	.9893	1.269	1.564	1.690	2.167	2.833	3.822
8	1.344	1.646	2.032	2.180	2.733	3.490	4.594
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989
12	3.074	3.571	4.178	4.404	5.266	6.304	7.807
13	3.565	4.107	4.765	5.009	5.892	7.042	8.634
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467
15	4.601	5.229	5.985	6.262	7.261	8.547	10.31
16	5.142	5.812	6.614	6.908	7.962	9.312	11.15
17	5.697	6.408	7.255	7.564	8.672	10.09	12.00
18	6.265	7.015	7.906	8.231	9.390	10.86	12.86
19	6.844	7.633	8.567	8.907	10.12	11.65	13.72
20	7.434	8.260	9.237	9.591	10.85	12.44	14.58
21	8.034	8.897	9.915	10.28	11.59	13.24	15.45
22	8.643	9.542	10.60	10.98	12.34	14.04	16.31
23	9.260	10.20	11.29	11.69	13.09	14.85	17.19
24	9.886	10.86	11.99	12.40	13.85	15.66	18.06
25	10.52	11.52	12.70	13.12	14.61	16.47	18.94
26	11.16	12.20	13.41	13.84	15.38	17.29	19.82
27	11.81	12.88	14.13	14.57	16.15	18.11	20.70
28	12.46	13.56	14.85	15.31	16.93	18.94	21.59
29	13.12	14.26	15.57	16.05	17.71	19.77	22.48
30	13.79	14.95	16.31	16.79	18.49	20.60	23.36

Table 1. Continued

k[a]	.750	.700	.500	.300	.250	.200
1	.1015	.148	.4549	1.074	1.323	1.642
2	.5754	.713	1.386	2.408	2.773	3.219
3	1.213	1.424	2.366	3.665	4.108	4.642
4	1.923	2.195	3.357	4.878	5.385	5.989
5	2.675	3.000	4.351	6.064	6.626	7.289
6	3.455	3.828	5.348	7.231	7.841	8.558
7	4.255	4.671	6.346	8.383	9.037	9.803
8	5.071	5.527	7.344	9.524	10.22	11.03
9	5.899	6.393	8.343	10.66	11.39	12.24
10	6.737	7.267	9.342	11.78	12.55	13.44
11	7.584	8.148	10.34	12.90	13.70	14.63
12	8.438	9.034	11.34	14.01	14.85	15.81
13	9.299	9.926	12.34	15.12	15.98	16.99
14	10.17	10.82	13.34	16.22	17.12	18.15
15	11.04	11.72	14.34	17.32	18.25	19.31
16	11.91	12.62	15.34	18.42	19.37	20.47
17	12.79	13.53	16.34	19.51	20.48	21.62
18	13.68	14.44	17.34	20.60	21.60	22.76
19	14.56	15.35	18.34	21.69	22.72	23.90
20	15.45	16.27	19.34	22.78	23.83	25.04
21	16.34	17.18	20.34	23.86	24.93	26.17
22	17.24	18.10	21.34	24.94	26.04	27.30
23	18.15	19.02	22.34	26.02	27.14	28.43
24	19.04	19.94	23.34	27.10	28.24	29.55
25	19.94	20.87	24.34	28.17	29.34	30.68
26	20.84	21.79	25.34	29.25	30.43	31.80
27	21.75	22.72	26.34	30.32	31.53	32.91
28	22.66	23.65	27.34	31.39	32.62	34.03
29	23.57	24.58	28.34	32.46	33.71	35.14
30	24.48	25.51	29.34	33.53	34.80	36.25

Table 1. Continued

k[a]	.100	.050	.025	.020	.010	.005
1	2.760	3.841	5.024	5.412	6.635	7.879
2	4.605	5.991	7.378	7.824	9.210	10.60
3	6.251	7.815	9.348	9.837	11.34	12.84
4	7.779	9.488	11.14	11.67	13.28	14.86
5	9.236	11.07	12.83	13.39	15.09	16.75
6	10.64	12.59	14.45	15.03	16.81	18.55
7	12.02	14.07	16.01	16.62	18.48	20.28
8	13.36	15.51	17.53	18.17	20.09	21.96
9	14.68	16.92	19.02	19.68	21.67	23.59
10	15.99	18.31	20.48	21.16	23.21	25.19
11	17.28	19.68	21.92	22.62	24.72	26.76
12	18.55	21.03	23.34	24.05	26.22	28.30
13	19.81	22.36	24.74	25.47	27.69	29.82
14	21.06	23.68	26.12	26.87	29.14	31.32
15	22.31	25.00	27.49	28.26	30.58	32.80
16	23.54	26.30	28.85	29.63	32.00	34.27
17	24.77	27.59	30.19	31.00	33.41	35.72
18	25.99	28.87	31.53	32.35	34.81	37.16
19	27.20	30.14	32.85	33.69	36.19	38.58
20	28.41	31.41	34.17	35.02	37.57	40.00
21	29.62	32.67	35.48	36.34	38.93	41.40
22	30.81	33.92	36.78	37.66	40.29	42.80
23	32.01	35.17	38.08	38.97	41.64	44.18
24	33.20	36.42	39.36	40.27	42.98	45.56
25	34.38	37.65	40.65	41.57	44.31	46.93
26	35.56	38.89	41.92	42.86	45.64	48.29
27	36.74	40.11	43.19	44.14	46.96	49.64
28	37.92	41.34	44.46	45.42	48.28	50.99
29	39.09	42.56	45.72	46.69	49.59	52.34
30	40.26	43.77	46.98	47.96	50.89	53.67

Based on the table in R. A. Fisher's *Statistical methods for research workers* (12th edition), Oliver and Boyd, Ltd., and on the table of the paper: Tables of percentage points of the incomplete beta function and of the chi-square distribution, C. M. Thompson, *Biometrika* 32:188-189 (1941). Used with the kind permission of the authors, R. A. Fisher and C. M. Thompson, and of the publishers.

Table 2. Values of $W_1(\alpha)$ versus a for α = .001(.001).01(.01).5[a]

a	$W_1(\alpha)$	α	$W_1(\alpha)$	α	$W_1(\alpha)$
.001	1.168	.12	.318	.32	.1757
.002	1.039	.13	.306	.33	.1716
.003	.963	.14	.295	.34	.1677
.004	.910	.15	.284	.35	.1639
.005	.870	.16	.274	.36	.1602
.006	.836	.17	.265	.37	.1566
.007	.808	.18	.257	.38	.1532
.008	.784	.19	.249	.39	.1499
.009	.763	.20	.241	.40	.1466
.010	.743	.21	.234	.41	.1435
.020	.620	.22	.227	.42	.1405
.030	.549	.23	.221	.43	.1375
.040	.499	.24	.215	.44	.1346
.050	.461	.25	.209	.45	.1318
.060	.431	.26	.204	.46	.1291
.070	.405	.27	.1987	.47	.1265
.080	.383	.28	.1937	.48	.1239
.090	.364	.29	.1889	.49	.1213
.100	.347	.30	.1843	.50	.1189

[a]Anderson, T. W., and D. A. Darling, Asymptotic theory of certain "goodness-of-fit" criteria based on stochastic processes.

Table 3. Table of critical values of chi-square [a]

df	Probability under H_0 that $\chi^2 \leq$ chi-square						
	.99	.98	.95	.90	.80	.70	.50
1	.00016	.00063	.0039	.016	.064	.15	.46
2	.02	.04	.10	.21	.45	.71	1.39
3	.12	.18	.35	.58	1.00	1.42	2.37
4	.30	.43	.71	1.06	1.65	2.20	3.36
5	.55	.75	1.14	1.61	2.34	3.00	4.35
6	.87	1.13	1.64	2.20	3.07	3.83	5.35
7	1.24	1.56	2.17	2.83	3.82	4.67	6.35
8	1.65	2.03	2.73	3.49	4.59	5.53	7.34
9	2.09	2.53	3.32	4.17	5.38	6.39	8.34
10	2.56	3.06	3.94	4.86	6.18	7.27	9.34
11	3.05	3.61	4.58	5.58	6.99	8.15	10.34
12	3.57	4.18	5.23	6.30	7.81	9.03	11.34
13	4.11	4.76	5.89	7.04	8.63	9.93	12.34
14	4.66	5.47	6.57	7.79	9.47	10.82	13.34
15	5.23	5.98	7.26	8.55	10.31	11.72	14.34
16	5.81	6.61	7.96	9.31	11.15	12.62	15.34
17	6.41	7.26	8.67	10.08	12.00	13.53	16.34
18	7.02	7.91	9.39	10.86	12.86	14.44	17.34
19	7.63	8.57	10.12	11.65	13.72	15.35	18.34
20	8.26	9.24	10.85	12.44	14.58	16.27	19.34
21	8.90	9.92	11.59	13.24	15.44	17.18	20.34
22	9.54	10.60	12.34	14.04	16.31	18.10	21.24
23	10.20	11.29	13.09	14.85	17.19	19.02	22.34
24	10.86	11.99	13.85	15.66	18.06	19.94	23.34
25	11.52	12.70	14.61	16.47	18.94	20.87	24.34
26	12.20	13.41	15.38	17.29	19.82	21.79	25.34
27	12.88	14.12	16.15	18.11	20.70	22.72	26.34
28	13.56	14.85	16.93	18.94	21.59	23.65	27.34
29	14.26	15.57	17.71	19.77	22.48	24.58	28.34
30	14.95	16.31	18.49	20.60	23.36	25.51	29.34

Table 3. Continued

df	Probability under H_0 that $\chi^2 \leq$ chi-square						
	.30	.20	.10	.05	.02	.01	.001
1	1.07	1.64	2.71	3.84	5.41	6.64	10.83
2	2.41	3.22	4.60	5.99	7.83	9.21	13.82
3	3.66	4.64	6.25	7.82	9.84	11.34	16.27
4	4.88	5.99	7.78	9.49	11.67	13.28	18.46
5	6.06	7.29	9.24	11.07	13.39	15.09	20.52
6	7.23	8.56	10.64	12.59	15.03	16.81	22.46
7	8.38	9.80	12.02	14.07	16.62	18.48	24.32
8	9.52	11.03	13.36	15.51	18.17	20.09	26.12
9	10.66	12.24	14.68	16.92	19.68	21.67	27.88
10	11.78	13.44	15.99	18.31	21.16	23.21	29.59
11	12.90	14.63	17.28	19.68	22.62	24.72	31.26
12	14.01	15.81	18.55	21.03	24.05	26.22	32.91
13	15.12	16.98	19.81	22.36	25.47	27.69	34.53
14	16.22	18.15	21.06	23.68	26.87	29.14	36.12
15	17.32	19.31	22.31	25.00	28.26	30.58	37.70
16	18.42	20.46	23.54	26.30	29.63	32.00	39.29
17	19.51	21.62	24.77	27.59	31.00	33.41	40.75
18	20.60	22.76	25.99	28.87	32.35	34.80	42.31
19	21.69	23.90	27.20	30.14	33.69	36.19	43.82
20	22.78	25.04	28.41	31.41	35.02	37.57	45.32
21	23.86	26.17	29.62	322.67	36.34	38.93	46.80
22	24.94	27.30	30.81	33.92	37.66	40.29	48.27
23	26.02	28.43	32.01	35.17	38.97	41.64	49.73
24	27.10	29.55	33.20	36.42	40.27	42.98	51.18
25	28.17	30.68	34.38	37.65	41.57	44.31	52.62
26	29.25	31.80	35.56	38.88	42.86	45.64	54.05
27	30.32	32.91	36.74	40.11	44.14	46.96	55.48
28	31.39	34.03	37.92	41.34	45.42	48.28	56.89
29	32.46	35.14	39.09	42.56	46.69	49.59	58.30
30	33.53	36.25	40.26	43.77	47.96	50.89	59.70

[a]Table 3 is abridged from Table 4 of Fisher and Yates (1950) by permission of the authors and publishers.

Table 4. Acceptance limits for the Kolmogorov-Smirnov test of goodness of fit

Sample size (n)	Significance level				
	.20	.15	.10	.05	.01
1	.900	.925	.950	.975	.995
2	.684	.726	.776	.842	.929
3	.565	.597	.642	.708	.829
4	.494	.525	.564	.624	.734
5	.446	.474	.510	.563	.669
6	.410	.436	.470	.521	.618
7	.381	.405	.438	.486	.577
8	.358	.381	.411	.457	.543
9	.339	.360	.388	.432	.514
10	.322	.342	.368	.409	.486
11	.307	.326	.352	.391	.468
12	.295	.313	.338	.375	.450
13	.284	.302	.325	.361	.433
14	.274	.292	.314	.349	.418
15	.266	.283	.304	.338	.404
16	.258	.274	.295	.328	.391
17	.250	.266	.286	.318	.380
18	.244	.259	.278	.309	.370
19	.237	.252	.272	.301	.361
20	.231	.246	.264	.294	.352
25	.21	.22	.24	.264	.32
30	.19	.20	.22	.242	.29
35	.18	.19	.21	.23	.27
40				.21	.25
50				.19	.23
60				.17	.21
70				.16	.19
80				.15	.18
90				.14	
100				.14	
Asymptotic formula:	$\frac{1.07}{\sqrt{n}}$	$\frac{1.14}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$

Reject the hypothetical distribution $F(x)$ if $d_1(F,G) = \mathrm{SUP}_x |F(x) - G(x)|$ exceeds the tabulated value.

(For a = .01 and .05, asymptotic formulas give values which are too high by 1.5 percent for n = 80.)

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