5-1971

Torus Knots

David S. Bradley

TORUS KNOTS

by

David S. Bradley

A report submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

Plan B

UTAH STATE UNIVERSITY
Logan, Utah

1971

ACKNOWLEDGEMENTS

I wish to express my sincere thanks to those professors and colleagues who have influenced my study in the preparation of this report. I am deeply indebted to my major professor Dr. L.D. Loveland for his careful reading of the manuscript and his many valuable suggestions and criticisms. Also, I am grateful to Dr. Loveland for the encouragement and assistance he offered me while writing this report.

I should like partcilularly to acknowledge Steven Matthews and Joe Peck whose help and friendship has been an important factor in my success. Finally, I thank my wife and daughter for their patience and love.

David S. Bradley

TABLE OF CONTENTS

INTRODUCTION

We have compiled here many interesting results concerning a particular collection of knots called torus knots. Torus knots are merely simple closed curves imbedded in an unknotted torus T^2 in E^3. We show that the fundamental group of T^2, $\pi(T^2)$, is the direct product of the additive group of integers with itself. The ordered pair (p,q) in $Z \otimes Z$ determines an equivalence class of loops on the torus, and we show in Section II that the class $[(p,q)]$ contains a loop whose image is a simple closed curve if and only if p and q are relatively prime. A torus knot in the loop class $[(p,q)]$ is denoted $K_{p,q}$. It is natural to ask which of the knots $K_{p,q}$ are equivalently imbedded in E^3.

One means of answering this question is to observe the algebraic structures of the corresponding knot groups $\pi(E^3 - K_{p,q})$. If it can be shown that $\pi(E^3 - K_{p,q})$ and $\pi(E^3 - K_{r,s})$ are not isomorphic, then it follows that $E^3-K_{p,q}$ and $E^3 - K_{r,s}$ are not homeomorphic; consequently $K_{p,q}$ and $K_{r,s}$ are not equivalent knots. The definition and general properties of the fundamental group of a topological space are discussed in Section I of this report. In Section IV the fundamental group of $E^3-K_{p,q}$ is shown to have the group presentation $\{ a,b \mid a^p = b^q \}$. We will show that these groups are determined by the integers p and q, from which it follows that there are infinitely many non-equivalent torus knots.

Illustrations are used extensively to aid the reader, and an entire section is devoted to the development of an algorithm for picturing torus knots. This algorithm, Section III, provides us with an intuitive feeling for the significance of p and q in determining $K_{p,q}$. Finally, in Section V, a second knot type invariant, called the genus of the knot, is developed. The genus of a knot is a nonnegative integer assigned to the knot in a particular way. We will show that there exist torus knots of arbitrary genus and construct an infinite collection of knots, all having genus 1, none of which is a torus knot.

The material in this report comes from many sources. In many cases the proofs and illustrations were created by the author. We do not know of any similar compilation of facts relating to a specific class of knots and we hope that this report might be of use to other students of knot theory.

I. THE FUNDAMENTAL GROUP

For a topological space X and a point x_1 in X we will define an associated group. We define a path in X to be a continuous map $f: [0,1] \to X$. If f and g are two paths in X such that $f(1) = g(0)$, then the product of f and g is defined to be $(f\cdot g)(t) = \begin{cases} f(2t), & \text{for } t \in [0,\frac{1}{2}] \\ g(2t-1), & \text{for } t \in [\frac{1}{2},1]. \end{cases}$ A path f is said to be based at a point $x_1 \in X$ if $f(0) = f(1) = x_1$, and a path based at x_1 is called a loop based at x_1.

If f and g are paths in X such that $f(0) = g(0)$ and $f(1) = g(1)$ then we define f and g to be equivalent if and only if there exists a continuous function $F: [0,1] \times [0,1] \to X$ such that $F(t,0) = f(t)$, $F(t,1) = g(t)$, $F(0,t) = f(0) = g(0)$, and $F(1,t) = f(1) = g(1)$. This relation is an equivalence relation.

To form a group we will consider equivalence classes of loops based at a fixed point $x_1 \in X$. Each equivalence class is called a loop class, and the loop class determined by the loop f is denoted by [f].

We define the product of two loop classes [f] and [g] to be $[f\cdot g]$. The set of all loop classes, with the above product is a group called the fundamental group of X based at x_1 and is denoted by $\pi(X,x_1)$ [4].

Suppose ϕ is a continuous map of X into Y, and let f and g be two equivalent loops in X. Since f and g are equivalent there exists a continuous map $F: [0,1] \times [0,1] \to X$ such that

$$F(t,0) = f(t),\ F(t,1) = g(t).$$

$$F(0,t) = f(0) = g(0),$$

and $$F(1,t) = f(1) = g(1), \text{ for } t \in [0,1].$$

Consider the composition of F with ϕ, $\phi F: [0,1] \times [0,1] \to Y$. Now ϕF is continuous and $(\phi F)(t,0) = (\phi f)(t)$, $(\phi F)(t,1) = (\phi g)(t)$,

$(\phi F)(0,t) = (\phi f)(0) = (\phi g)(0)$,

and $(\phi F)(1,t) = (\phi f)(1) = (\phi g)(1)$, for $t \varepsilon [0,1]$. Hence ϕf and ϕg are equivalent loops in Y, with base point $\phi(x_1)$.

Theorem 1. If X and Y are topological spaces and ϕ is a continuous map of X into Y, then $\pi(X,x_1)$ and $\pi(Y,\phi(x_1))$ are homomorphic.

Proof: Let [f] be a loop class in X, and let $f',f'' \varepsilon [f]$. Then the composed maps $\phi f'$ and $\phi f''$ are equivalent loops in Y. Hence we use $\phi_*([f])$ to denote the loop class in Y containing $\phi f'$ and $\phi f''$. We will now show ϕ_* is a homomorphism. Thus let [f] and [g] be two loop classes in $\pi(X,x_1)$. Now $\phi(f \cdot g)$ is the composition of $f \cdot g$ with ϕ; hence $\phi(f\cdot g)(t) =$

$$\begin{cases}\phi(f(2t)) \\ \phi(g(2t-1))\end{cases} = \begin{cases}\phi f(2t) \\ \phi g(2t-1)\end{cases} = (\phi f \cdot \phi g)(t)$$

It follows that $\phi_*([f]\cdot[g]) = \phi_*([f])\cdot\phi_*([g])$.

The function ϕ_*, defined in the proof of Theorem 1, is called the homomorphism induced by $\phi: X \to Y$. If ψ is a continuous map of Y onto Z, then $(\psi\,\phi)_* = \psi_*\,\phi_*$.

Theorem 2. If X and Y are topological spaces and ϕ is a homeomorphism of X onto Y, then $\pi(X,x)$ and $\pi(Y,\phi(x))$ are isomorphic.

Proof: Let the kernel of induced homomorphism ϕ_* be K, and let $[f] \varepsilon K$. Let $[e_x]$ and $[e_{\phi(x)}]$ be the identity elements of $\pi(X,x)$ and $\pi(Y,\phi(x))$, respectively. Then $\phi_*([f]) = [e_{\phi(x)}]$. Since ϕ_* is a homomorphism $\phi_*([e_x]) = [e_{\phi(x)}]$. Now ϕ^{-1} is continuous and ϕ is a bijection, hence $[e_x] = [f]$. Therefore $K = \{[e_x]\}$, and ϕ_* is an isomorphism.

Two maps f and g : $X \to Y$ are called homotopic if and only if there exists a continuous map $F: X \times [0,1] \to Y$ such that, for $x \varepsilon X$, $F(x,0) = f(x)$

and $F(x,1) = g(x)$. Two maps f and $g: X \to Y$ are homotopic relative to the subset A of X if and only if there exists a continuous map $F: X \times [0,1] \to Y$ such that, for $x \in X$,

$$F(x,0) = f(x)$$
$$F(x,1) = g(x)$$

and

$$F(a,t) = f(a) = g(a), \text{ for } a \in A,\ t \in [0,1].$$

The relation of "homotopic to" induces an equivalence relation on the set of continuous maps of X into Y.

A subspace A of X is called a retract of X if there exists a continuous map $r: X \to A$ such that $r(a) = a$, for each $a \in A$. In this setting r is called a retraction. If a retraction $r: X \to A$ is homotopic relative to A to the identity map $i_X: X \to X$ then we call A a deformation retract of X.

Theorem 3. If A is a deformation retract of X, then the inclusion map $i: A \to X$ induces an isomorphism of $\pi(A,a)$ onto $\pi(X,a)$, for any a in A.

Proof: Let r be a retraction of X onto A, let a_1 be any point in A, and let $[f]$ be any loop class in $\pi(X,a_1)$. As before let $i_*: \pi(A,a_1) \to \pi(X,a_1)$ be defined by $i_*([\alpha]) = [i\alpha]$. To show i_* is an epimorphism we prove that f and rf are equivalent. Thus consider the map $\phi: [0,1] \times [0,1] \to X \times [0,1]$ given by $\phi(t_1,t_2) = (f(t_1), t_2)$; it is clear that ϕ is continuous.

Since A is a deformation retract there is a continuous map $F: X \times [0,1] \to X$ such that $F(x,0) = r(x)$, $F(x,1) = i_X(x) = x$, for $x \in X$, and $F(a,t) = r(a) = a$, for $a \in A$ and $t \in [0,1]$. Consider the composition of ϕ with F, $F\phi: [0,1] \times [0,1] \to X$. Now $F\phi$ is continuous and

$$F\phi(t,0) = F[(f(t),0)] = r(f(t)) = rf(t),$$
$$F\phi(t,1) = F[(f(t),1)] = f(t),$$

$$F\phi(0,t) = F[(f(0),t)] = rf(0) = a_1 = f(0),$$

$$F\phi(1,t) = F[(f(1),t)] = rf(1) = a_1 = f(1).$$

Therefore, f and rf are equivalent loops in X. Hence $i_*([rf]) = [rf] = [f]$ and i_* is an epimorphism.

We will next show i_* is a monomorphism. Let [f] and [g] be two distinct loop classes in $\pi(A,a_1)$. If if and ig are equivalent, then from the remarks preceeding Theorem 1, r(if) and r(ig) are equivalent. Since ri is the identity map on A it follows that f and g are equivalent and hence that[f] and [g] are not distinct. This is a contradiction, hence i_* is a monomorphism. It follows that i_* is an isomorphism.

Recall that a path X is a continuous map $f:[0,1] \rightarrow X$. We define the inverse of a path f to be $f^{-1}(t) = f(1-t)$, $t \varepsilon [0,1]$. If [f] is a path class, then $[f]^{-1} = [f^{-1}]$

Theorem 4. If X is arcwise connected, then $\pi(X,x)$ and $\pi(X,y)$ are isomorphic, for any x, $y \varepsilon X$.

Proof: Let x and y be any two points in X. Since X is arcwise connected, there is a path class [f] with f(0) = x and f(1) = y. Define $\phi: \pi(X,x) \rightarrow \pi(X,y)$ by $\phi([g]) = [f^{-1} \cdot g \cdot f]$ and $\psi: \pi(X,y) \rightarrow \pi(X,x)$ by $\psi([h]) = [f \cdot h \cdot f^{-1}]$. If $[\ell] \varepsilon\ \pi(X,y)$ then $[f \cdot \ell \cdot f^{-1}] \varepsilon\ \pi(X,x)$. Therefore $\phi([f \cdot \ell \cdot f^{-1}]) = [\ell]$. Now $\phi([g] \cdot [h]) = \phi([g \cdot h]) = [f^{-1} \cdot g \cdot h \cdot f] = [f^{-1} \cdot g \cdot f \cdot f^{-1} \cdot h \cdot f] = [f^{-1} \cdot g \cdot f] \cdot [f^{-1} \cdot h \cdot f] = \phi([g]) \cdot \phi([h])$. Thus ϕ is an epimorphism.

Suppose $[\ell] \varepsilon\ \pi(X,x)$. Then $\psi\phi([\ell]) = \psi([f^{-1} \cdot \ell \cdot f]) = [f \cdot f^{-1} \cdot \ell \cdot f \cdot f^{-1}] = [\ell]$. Hence $\psi\phi$ is the identity on $\pi(X,x)$. If $[\ell_1]$ and $[\ell_2]$ are two loop classes in $\pi(X,x)$ such that $\phi([\ell_1]) = \phi([\ell_2])$, then $\psi\phi([\ell_1]) = \psi\phi([\ell_2])$, and since $\psi\phi$ is the identity on $\pi(X,x)$ it follows that $[\ell_1] =$

$[\ell_2]$. Thus ϕ is a monomorphism. Therefore ϕ is an isomorphism.

Theorem 5. The fundamental group of the product space, $\pi(X \times Y,(x,y))$ is isomorphic to the direct product $\pi_1(X,x) \otimes \pi_1(Y,y)$ of the fundamental groups.

Proof: Let P_{X*} and P_{Y*} be the homomorphisms induced by the projection maps P_X and P_Y, respectively. Define $\phi: \pi(X \times Y, (x,y)) \rightarrow \pi(X,x) \otimes \pi(Y,y)$ by $\phi([f]) = (P_{X*}([f]), P_{Y*}([f]))$, where $[f]$ is a loop class in $\pi(X \times Y,(x,y))$.

To show ϕ is bijective we show $f,g \in [f]$ if and only if $P_X f$, $P_X g$ are equivalent and $P_Y f$, $P_Y g$ are equivalent. Let $f,g \in [f]$. Hence there exists a continuous map $F: [0,1] \times [0,1] \rightarrow X \times Y$ such that

$$F(t,0) = f(t) \quad F(t,1) = g(t),$$

$$F(0,t) = f(0) = g(0),$$

and $$F(1,t) = f(1) = g(1), \text{ for } t \in [0,1].$$

Now $P_X F$ is a continuous map from $[0,1] \times [0,1]$ into X and clearly

$$(P_X F)(t,0) = (P_X f)(t),$$

$$(P_X F)(t,1) = (P_X g)(t),$$

$$(P_X F)(0,t) = (P_X f)(0) = (P_X g)(0),$$

and $$(P_X F)(1,t) = (P_X f)(1) = (P_X g)(1), \text{ for } t \in [0,1].$$

Hence $P_X f$ and $P_X g$ are equivalent. A similar proof shows $P_Y f$ and $P_Y g$ are equivalent.

Conversely, suppose $P_X f$ and $P_X g$ are equivalent and $P_Y f$, $P_Y g$ are equivalent. Then there exists continuous maps F_1 and $F_2: [0,1] \times [0,1] \rightarrow X \times Y$ such that

$$F_1(t,0) = (P_X f)(t), \ F_1(t,1) = (P_X g)(t),$$

$$F_1(0,t) = (P_X f)(0) = (P_X g)(0),$$

and $F_1(1,t) = (P_X f)(1) = (P_X g)(1)$, for $t \in [0,1]$.

Similarly,

$$F_2(t,0) = (P_Y f)(t),\ F_2(t,1) = (P_Y g)(t),$$

$$F_2(0,t) = (p_Y f)(0) = (P_Y g)(0),$$

$$F_2(1,t) = (P_Y f)(1) = (P_Y\ g)(1),\ \text{for } t \in [0,1].$$

The continuous map $F = (F_1, F_2) : [0,1] \times [0,1] \to X \times Y$ gives us the equivalence between f and g.

Since P_{X*} and P_{Y*} are homomorphisms it follows immediately that ϕ preserves the product operation. Hence ϕ is an isomorphism.

Theorem 6. The fundamental group $\pi(S^1,(1,0))$ is the infinite cyclic group generated by the loop ℓ defined by $\ell(t) = (\cos 2\pi t, \sin 2\pi t)$ [4].

II. TORUS KNOTS

In this section we will describe the fundamental group of a torus and determine which loop classes, if any, contain a loop whose image is a simple closed curve. By definition a torus T^2 is $S^1 \times S^1$; however, in this section we consider the torus T^2 pictured in Figure 1 and containing $J_1 \times J_2$, where $J_2 = \{(x,z) \mid (x-2)^2 + z^2 = 1\}$ and $J_1 = \{(x,y) \mid x^2 + y^2 = 9\}$.

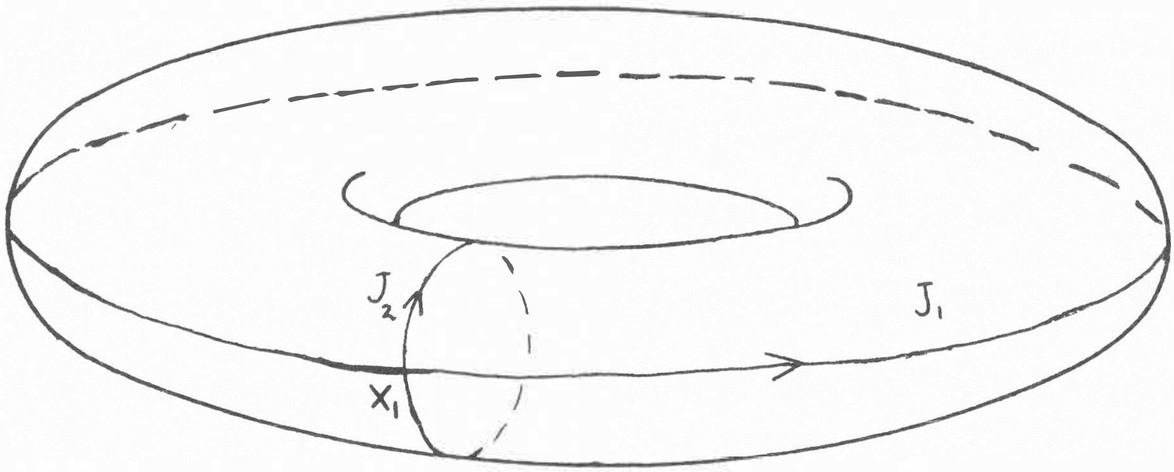


Figure 1

From Theorems 2 and 6 in Section I it follows that $\pi(J_1,x_1)$ and $\pi(J_2,x_1)$ are each isomorphic to the group Z of integers under addition.

Now $\pi(J_i,x_1)$ is generated by $[J_i]$ which we may assume to be identified with the element 1 in Z under the isomorphism, where J_i has the orientation indicated in Figure 1.

Since the torus is arcwise connected it follows from Theorem 4, Section I, that the fundamental group of the torus is independent of the choice of base point. Hence we let $J_1 \cap J_2 = \{x_1\}$ be the base point and will refer to $\pi(T^2,x_1)$ as $\pi(T^2)$. Now $T^2 = J_1 x J_2$; therefore, $\pi(T^2) = \pi(J_1,x_1) \otimes \pi(J_2,x_1) \cong Z \otimes Z$. Thus we see that $\pi(T^2)$ can be thought of as a set of ordered pairs of integers. The pair (1,0) determines the loop class containing J_2; hence, [(1,0)] contains a loop whose image is a simple closed curve. Similarly we think of [(0,1)] as the loop class containing a loop whose image is J_1. A simple closed curve imbedded in E^3 is called a <u>knot</u>. We say that two knots K_1 and K_2 are <u>equivalent</u> if there exists a homeomorphism h of E^3 onto itself such that $h(K_1) = K_2$. Equivalent knots are said to be of the same <u>knot type.</u> Those knots equivalent to the unknotted circle $\{(x,y) | x^2 + y^2 = 1\}$ are called <u>trivial.</u> We shall restrict our discussion to tame knots; a knot is called <u>tame</u> if its type has a polygonal representative. We define a <u>torus knot</u> to be a knot on the torus $T^2 = J_1 x J_2$. In general a torus knot is a knot on a torus in E^3 which is imbedded in E^3 just as J_1 x J_2 is imbedded (see Figure 2).

With the aid of the following lemmas we will prove that the loop class [(p,q)] contains a loop whose image in T^2 is a torus knot if and only if p and q are relatively prime.

<u>Lemma 1.</u> Let X be a compact connected subset of E^2, and let L be a straight line interval of length ℓ in E^2 with both endpoints of L on X. Let a and b be two positive numbers such that $a+b = \ell$. Then there exists a straight line interval L' parallel to L in E^2 with endpoints on X such that L' has length either a or b.

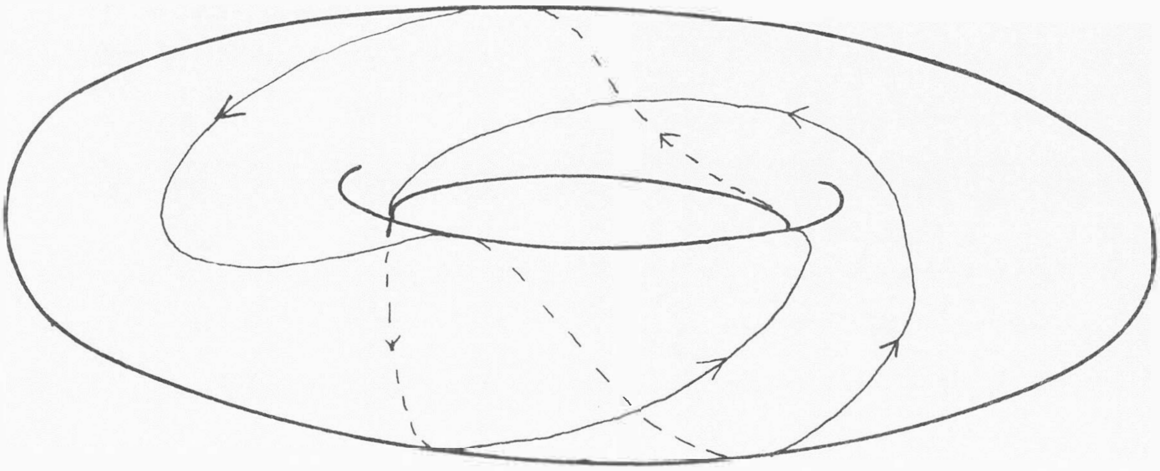


Figure 2

Proof: We begin by assuming that L is horizontal, and that there exists a and b such that $a+b = \ell \ni$ the conclusion of the lemma does not hold. Translate X the distance a to the right of the left endpoint of L. Call the image under this translation X_a. We define X_b and X_{a+b} similarly.

Now $X_a \cap X = \emptyset$. For, if not, let $(x_a,y) \in X_a \cap X$. Then corresponding to (x_a,y) is a point (x,y) such that $x = x_a - a$ and $(x,y) \in X$, but then the line segment $\overline{(x,y)(x_a,y)}$ is parallel to L and of length a, contrary to our assumption.

Similarly we see that $X_b \cap X = \emptyset$. However $X_{a+b} \cap X \neq \emptyset$ since the left endpoint of L is translated the distance a+b to the right endpoint of L, which by assumption is in X. Furthermore, $X_a \cap X_{a+b} = \emptyset$. For, if $(x,y) \in X_a \cap X_{a+b}$, then $(x-a,y) \in X$ and $(x-a,y) \in X_b$, contrary to the fact that $X_b \cap X = \emptyset$.

Now X_a is compact; hence, there exists points (x_1,M) and (x_2,m)

such that if $(x,y) \varepsilon X_a$, then $m \leq y \leq M$. Let R_m be the vertical ray extending from (x_1,M) to $+\infty$ and R_m be the vertical ray extending from (x_2,m) to $-\infty$.

I claim that $B = R_m \cup R_M \cup X_a$ separates the plane into at least two disjoint sets; one containing X and one containing X_{a+b}. To prove this we suppose B does not separate E^2. Let p_1 and p_2 be two points whose coordinates are $(x_1 -1, M+1)$ and $(x_1 +1, M+1)$ respectively. Since each connected open subset of E^2 is arcwise connected there exists an arc A in E^2-B with endpoints p_1 and p_2. Now A and the line segment $\overline{p_1(x_1,M+1)}$ are compact, hence $A \cap \overline{p_1(x_1,M+1)}$ is compact. It follows that there is a point $p_1' = (x',M+1)$ in $A \cap \overline{p_1(x_1,M+1)}$ such that $x' \geq x$, for all $(x,M+1)$ in $A \cap \overline{p_1(x_1,M+1)}$. Similarly, there is a point $p_2' = (x'',M+1)$ such that $x'' \leq x$, for all $(x,M+1)$ in $A \cap \overline{(x_1,M+1)\,p_2}$.

Consider the simple closed curve $A \cup \overline{p_1'p_2'}$. Since $\overline{p_1'p_2'}$ perpendicularly bisects R_m, there is a point of R_m either above or below $(x_1,M+1)$ in the interior of the simple closed curve $A \cup \overline{p_1'p_2'}$. Otherwise A would intersect R_m (see Figure 3).

If the interior of the simple closed curve $A \cup \overline{p_1'p_2'}$ contains a point of R_m above $(x_1,M+1)$, then all of the connected set $R_M-(\overline{(x_1,M)(x_1,M+1)})$ must lie in the interior of $A \cup \overline{p_1'p_2'}$, since $A \cup \overline{p_1'\ p_2'}$ separates E^2 into disjoint sets. Thus we obtain a contradiction. Similarly, we arrive at a contradiction if the interior of $A \cup \overline{p_1'p_2'}$ contains a point of R_M below $(x_1,M+1)$ because $R_m \cup X_a \cup (\overline{(x_1,M)\ (x_1,M+1)})$ is connected.

Therefore p_1' and p_2' lie in different components of E^2-B. Let R be the horizontal ray with right endpoint p_1', and let R' be the vertical ray extending from the point (x_1-a,M) in X to positive infinity.

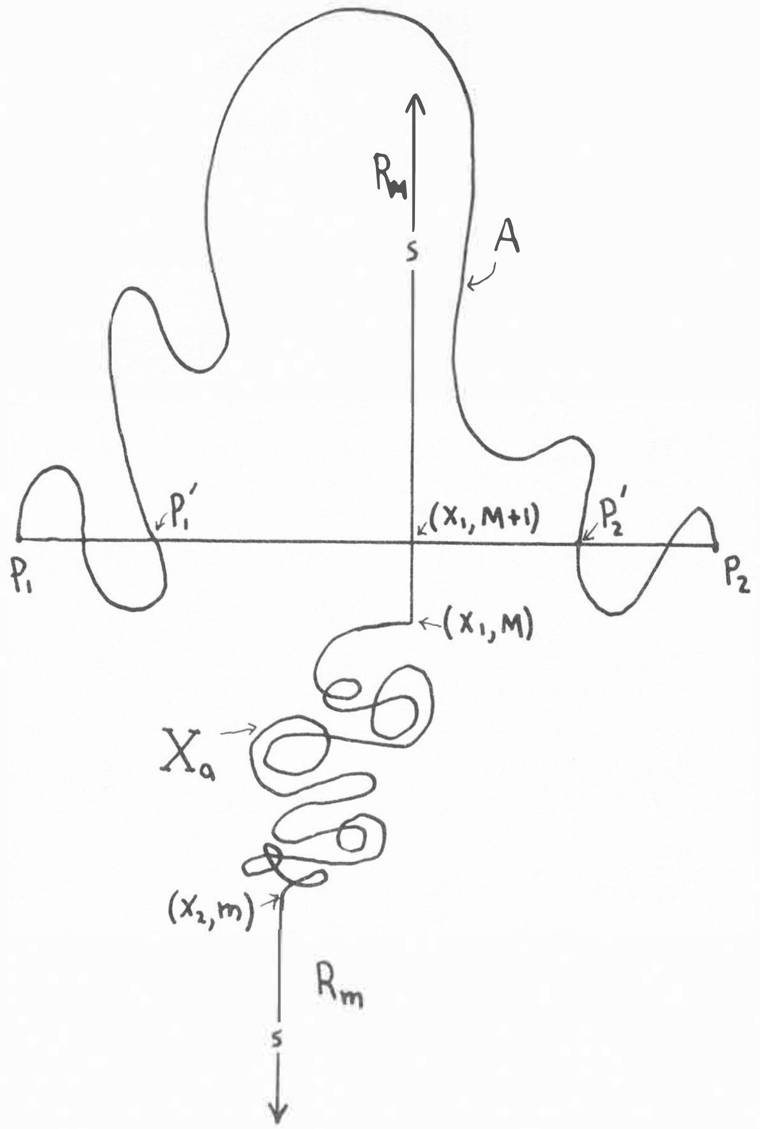


Figure 3

From the definition of p_1' it follows that $R' \cap R \neq \emptyset$; hence we conclude that X is in the same component of E^2-B as p_1'. A similar proof shows us that X_{a+b} is in the same component of E^2-B as p_2'. It follows that X and X_{a+b} lie in distinct components, contrary to the fact that $X \cap X_{a+b} \neq \emptyset$.

Therefore, we conclude that there exists a straight line interval

L' parallel to L in E^2 with endpoints on X such that L' has length either a or b.

A <u>covering space</u> of a topological space X is a pair $(\overline{X}, \phi)$ consisting of a space $\overline{X}$ and a continuous surjection $\phi\colon \overline{X} \to X$ such that the following condition holds: Each point $x \in X$ has an arcwise connected neighborhood U (elementary neighborhood) such that each component of $\phi^{-1}(U)$ is mapped homeomorphically onto U by ϕ.

<u>Lemma 2.</u> Let $(\overline{X},\phi)$ be a covering space of X, let $\overline{x}_o \in \overline{X}$, and let $x_o = \phi(\overline{x}_o)$. Then for any path in X with initial point x_o, there exists a path g in $\overline{X}$ with initial point $\overline{x}_o$ such that $\phi g = f$.

Proof: If the path $f\colon I \to X$ has an image f(I) contained in an elementary neighborhood U, then each component of $\phi^{-1}(U)$ would contain a subset A homeomorphic to f(I). Let $h = \phi|A$. Then h is a homeomorphism of A onto f(I) and $\phi(h^{-1}f) = f$. Thus $h^{-1}f$ is the desired path.

If f(I) is not in an elementary neighborhood, then we let $\{U_\alpha\}$ be a covering of X by elementary neighborhoods. Hence the collection $\{f^{-1}(U_\alpha)\}$ is an open cover of $I = [0,1]$. Choose n so that $\frac{1}{n}$ is a Lebesgue number for this cover. Divide the interval I into n subintervals $[0,\frac{1}{n}], [\frac{1}{n},\frac{2}{n}], \ldots, [\frac{n-1}{n},1]$. Now f maps each of these subintervals into an elementary neighborhood. Let $f|[\frac{i-1}{n}, \frac{i}{n}] = f_i$.

As mentioned above, the path $f_i\colon [\frac{i-1}{n},\frac{i}{n}] \to X$ lifts to a corresponding $g_i\colon [\frac{i-1}{n},\frac{i}{n}] \to \overline{X}$ such that $\phi g_i = f_i$.

Consider the path $g\colon [0,1] \to \overline{X}$ defined by

$$g(t) = \begin{cases} g_1(t), & \text{if } t\varepsilon\ [0,\frac{1}{n}], \\ g_2(t), & \text{if } t\varepsilon\ [\frac{1}{n},\frac{2}{n}], \\ \vdots & \\ g_n(t), & \text{if } t\varepsilon\ [\frac{n-1}{n},1] \end{cases}$$

Hence

$$\phi g(t) = \begin{cases} \phi \overline{g_1}(t), & \text{if } t\varepsilon\ [0,\frac{1}{n}], \\ \phi g_2(t), & \text{if } t\varepsilon\ [\frac{1}{n},\frac{2}{n}], \\ \vdots & \\ \phi g_n(t), & \text{if } t\varepsilon\ [\frac{n-1}{n},1] \end{cases}$$

$$= \begin{cases} f_1(t), & \text{if } t\varepsilon\ [0,\frac{1}{n}], \\ f_2(t), & \text{if } t\varepsilon\ [\frac{1}{n},\frac{2}{n}], \\ \vdots & \\ f_n(t), & \text{if } t\varepsilon\ [\frac{n-1}{n},1] \end{cases}$$

$= f(t)$. It follows that g is the desired path.

We will now exhibit a map ϕ such that (E^2,ϕ) is a covering space of T^2. Consider the disk $I^2 = [0,1] \times [0,1]$. We can obtain T^2 by identifying opposite sides of I^2 as pictured below.

We divide the plane into square disks whose vertices have integral coordinates and we let $\phi : E^2 \to T^2$ map each disk onto T^2 as described above.

More precisely, let ϕ be a continuous map which sends points with integral coordinates to $J_2 \cap J_1$, wraps horizontal lines with integral ordinates on J_1, wraps vertical lines with integral abscissa on J_2, and such that $\phi(x_1,y_1) = \phi(x_2,y_2)$ if and only if

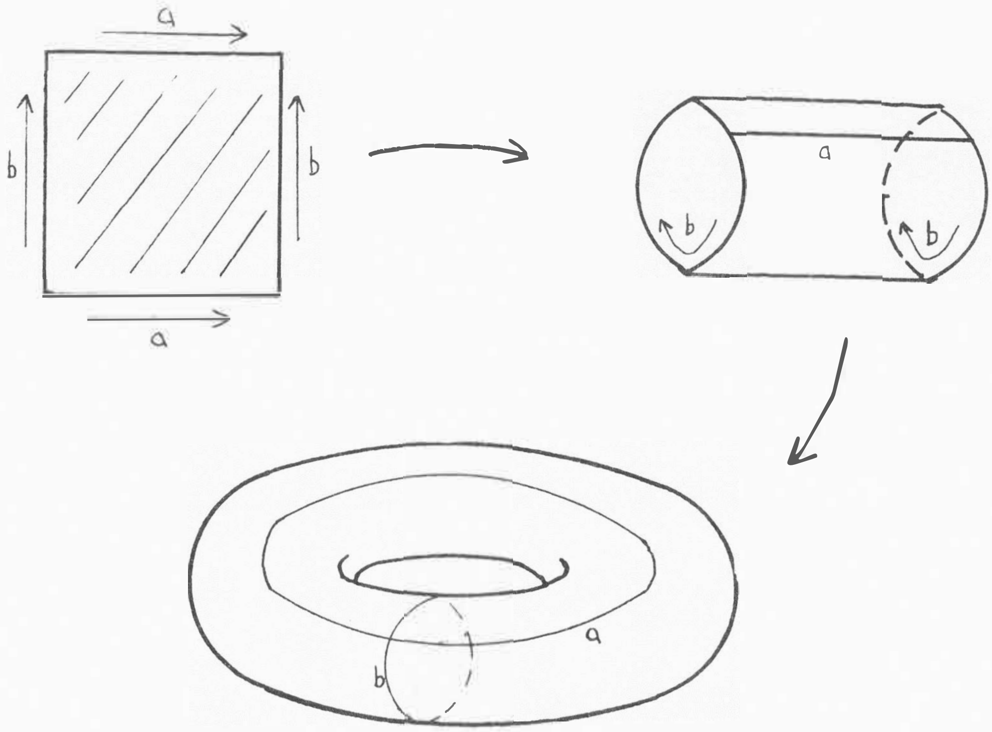


Figure 4

there exists positive integers N and M such that $x_1 = x_2 + N$ and $y_1 = y_2 + M$.

We now describe the elementary neighborhoods for various locations of x in T^2. First we consider the case in which x is in $T^2 - (J_1 \cap J_2)$. In this case there is an open disk U containing x such that $U \subset T^2 - (J_1 \cup J_2)$ and $\phi^{-1}(U)$ is the union of disjoint open disks, one in each square disk of E^2. See Figure 5 which illustrates this case. Next we consider the case in which x is on $J_1(J_2)$. In this case there is an open disk U_1 containing x such that U_1 is a subset of $T^2 - J_2(T^2 - J_1)$. Now $\phi^{-1}(U_1)$ is the union of disjoint open discs overlapping the solid squares on the horizontal(vertical) lines. Finally we examine the case in which

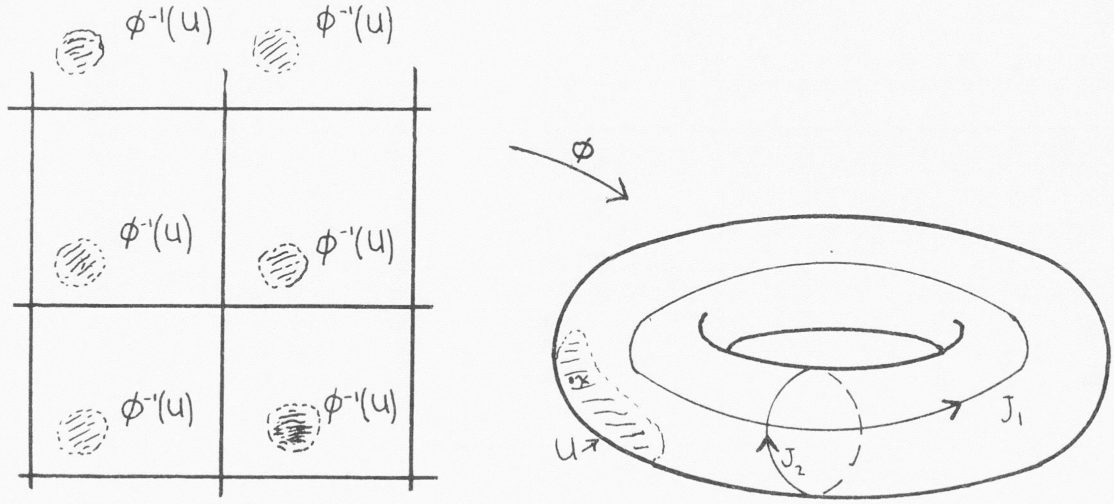


Figure 5

the point x is in $J_1 \cap J_2$. Again there is an open disk U_2 containing x such that $\phi^{-1}(U_2)$ is the union of disjoint disks each of which contains the common corner of four solid squares. Each such corner lies in exactly one $\phi^{-1}(U_2)$.

In all three cases the components of $\phi^{-1}(U)$, $\phi^{-1}(U_1)$ and $\phi^{-1}(U_2)$ are mapped by ϕ homeomorphically onto U, U_1, and U_2, respectively.

Recall that a loop f is a continuous map of I into a topological space X, however we shall sometimes think of the image of f, f(I), as a loop. We remarked earlier that we would prove that the loop class [(p,q)] contains a loop whose image in T^2 is a simple closed curve if and only if p and q are relatively prime. To say that the loop class [(p,q)] contains a simple closed curve means that [(p,q)] contains a loop $f:I \to T^2$ such that the image f(I) is a simple closed curve. It is quite often convenient to use this notation and we will do so in the statement and proof of the following theorem.

Theorem 1. The loop class $[(p,q)]$ contains a simple closed curve if and only if p and q are relatively prime.

Proof: We will assume that p and q are positive since the proof is similar in case one or both are negative. Let (E^2, ϕ) be a covering space for T^2, with ϕ defined as in the preceding remarks.

($\Longrightarrow$). Let f be a loop in the class $[(p,q)]$ whose image is a simple closed curve in T^2. Suppose p and q are not relatively prime. Then there exists an integer $k \geq 2$ and integers r and s such that $p = kr$ and $q = ks$. Lift f to a path α in E^2 from (0,0) to (p,q) in accordance with Lemma 2 such that $\phi\alpha = f$. We will ultimately reach a contradiction by exhibiting two distinct points in $\alpha(I)$, one of which is neither (0,0) nor (p,q), such that their images under ϕ are equal.

Let L be the straight line segment from (0,0) to (p,q). Now L has length $\sqrt{p^2 + q^2}$. It follows that $\sqrt{p^2 + q^2} = \sqrt{k^2r^2 + k^2s^2} = \sqrt{r^2 + s^2} + (k - 1)\sqrt{r^2 + s^2}$. Hence, by Lemma 1, there exists a straight line interval L' with endpoints in $\alpha(I)$ parallel to L such that L' has length either $\sqrt{r^2 + s^2}$ or $(k - 1)\sqrt{r^2 + s^2}$.

If L' has length $\sqrt{r^2 + s^2}$, then there exists points (x,y) and $(x + r, y + s)$ such that $d((x,y), (x + r, y + s)) = \sqrt{r^2 + s^2}$, and (x,y) and $(x + r, y + s)$ are in $\alpha(I)$. It follows that $\phi(x,y) = \phi(x + r, y + s)$. Therefore $\phi(\alpha(I))$ is not a simple closed curve contrary to the assumption that $f(I) = \phi(\alpha(I))$ is a simple closed curve. The proof is similar if L' has length $(k-1)\sqrt{r^2 + s^2}$.

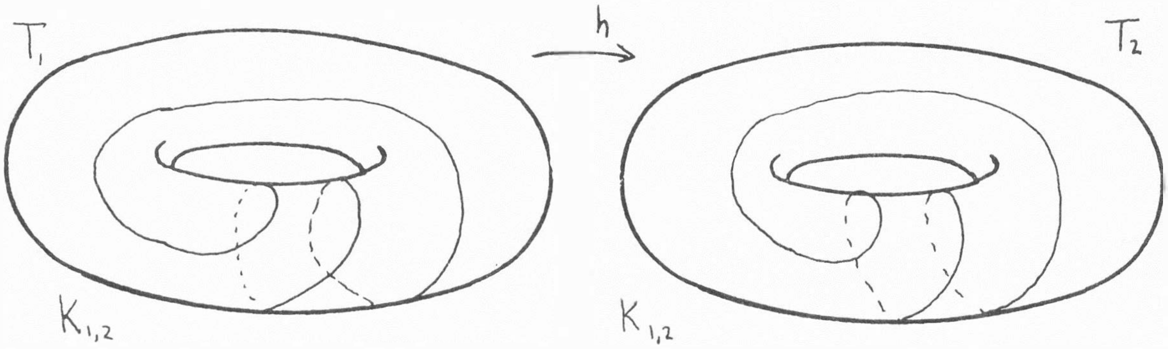
($\Leftarrow$). Suppose p and q are relatively prime. Let L be the straight line segment from (0,0) to (p,q). We know there exists a homeomorphism h of I = [0,1] onto L such that h(0) = (0,0) and h(1) = (p,q), thus it will suffice to show that $\phi(L) = \phi h(I)$ is a simple closed curve and that ϕh is in the class [(p,q)].

Suppose there are points p_1 and p_2, different from (0,0) and (p,q), such that $p_1 \neq p_2$, but $\phi(p_1) = \phi(p_2)$. If p_1 is the point (x,y); then it follows, from the definition of ϕ, that there exists integers M<p and N<q such that $p_2 = (x+M, y+N)$.

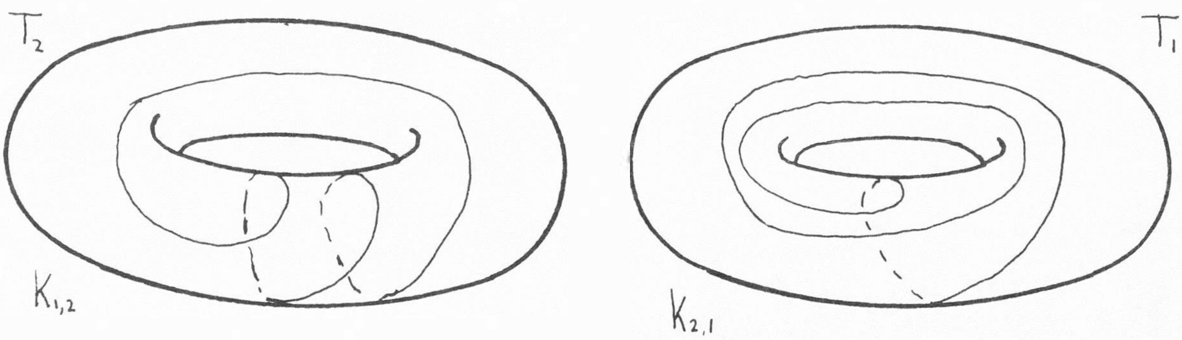
Now, L has slope $\frac{N}{M} = \frac{p}{q}$. Therefore pN = Mq and p divides Mq. Since p and q are relatively prime there exists integers r and s such that pr + qs = 1. Hence Mpr + Mqs = M. Since p divides Mpr and Mq it follows that p divides M. Since M<p we have obtained a contradiction.

Therefore $\phi(L)$ is a simple closed curve. Let A be the union of the straight line segment $\overline{(0,0)(p,0)}$ with the straight line segment $\overline{(p,0)(p,q)}$. Now, there is a homeomorphism h' of I onto A such that h'(0) = (0,0) and h'(1) = (p,q). Clearly the paths h and h' are equivalent in E^2. Since $\phi(\overline{(0,0)(p,0)}) \in [J_1]^p$ and $\phi(\overline{(p,0)(p,q)}) \in [J_2]^q$ it follows that $\phi h'$ is a loop in the class [(p,q)]. Because continuous maps preserve equivalence it follows that ϕh is in the loop class [(p,q)]. Now $\phi h(I) = \phi(L)$, and thus the loop class [(p,q)] contains a loop whose image is a simple closed curve.

A knot in the class [(p,q)] is called a <u>torus knot of type</u> (p,q) and is denoted by $K_{p,q}$. We will now show that the knots $K_{p,q}$ and $K_{q,p}$ are equivalent knots. We will consider S^3 as the identification space obtained by identifying points on the boundaries of the two disjoint solid tori T_1 and T_2 (for our purposes we insist that T_1 and T_2 be unknotted). Let $K_{p,q}$ be on the boundary of T_1. Let h be a homeomorphism of S^3 onto itself such that $h(T_1) = T_2$. The image of $K_{p,q}$ under h is a knot of type (p,q) on the boundary of T_2, since continuous maps preserve equivalence. Figure 6-a illustrates this in the case (p,q) = (1,2).



(a)



(b)

Figure 6

Under the identification map, a knot $K_{p,q}$ on the boundary of T_2 is a knot $K_{q,p}$ on T_1. Hence $h(K_{p,q}) = K_{q,p}$ on T_1, and it follows that $K_{p,q}$ and $K_{q,p}$ are equivalent knots.(see Figure 6-b).

Now we will examine the knots $K_{p,q}$ and $K_{p,-q}$ with p and q positive. The map $f: E^3 \to E^3$ defined by $f(x,y,z) = (x,y,-z)$ is a homeomorphism, and we now show that f takes $K_{p,q}$ onto $K_{p,-q}$. If the original coordinate system is considered to be right handed, we see that the image under f is a left handed system. In fact the orientation of J_2 is reversed by f. Now $f(K_{p,q})$ is a knot of type (p,q) relative to $f(T^2)$. However, since J_2's orientation is reversed by f it follows that $f(K_{p,q}) = K_{p,-q}$ relative to the original torus T^2. Figure 7 illustrates this for (p,-q) = (1,-1). The proof that $K_{p,q} \sim K_{-p,q}$ is similar, with f taking the

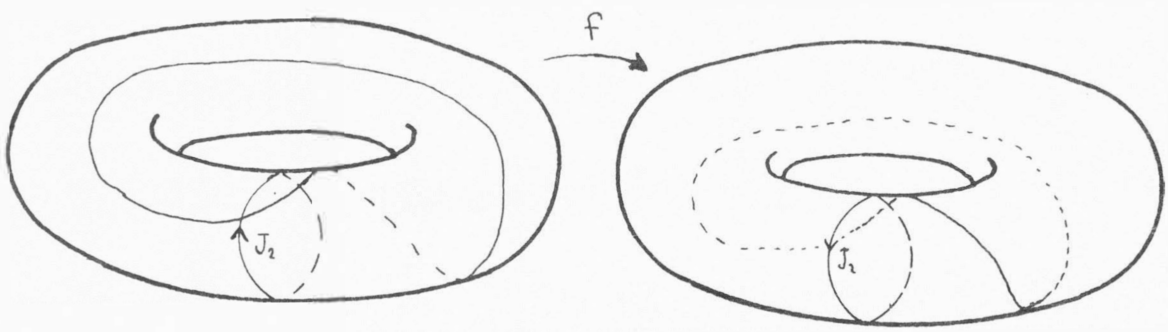


Figure 7

point (x,y,z) to (-x,y,z). In this case f reverses the orientation of J_1 and $f(K_{p,q}) = K_{-p,q}$ relative to the original torus T^2.

Next we consider the knots $K_{p,q}$ and $K_{-p,-q}$ where p and q are both positive. The argument is similar to that given above; the difference being that f takes (x,y,z) to (-x,y,-z). Here the orientations of J_1 and J_2 are both reversed by f and $f(K_{p,q}) = K_{-p.-q}$.

We have thus shown that $K_{p,q} \sim K_{q,p}$, $K_{p,q} \sim K_{-p,q}$, $K_{p,q} \sim K_{p,-q}$, and that $K_{p,q} \sim K_{-p,-q}$. These results allow us to assume $q>p>0$ whenever we so desire.

III. AN ALGORITHM FOR PICTURING A TORUS KNOT

It is often useful to have a picture of a representative knot of type (p,q) at our disposal. The following is an algorithm for picturing a knot $K_{p,q}$ on T^2. Now, the intersection of T^2 with the xy-plane is the union of two concentric circles centered at the origin. We begin by assuming $q > p > 0$ and labeling q points on the inner circle of the torus; 0, 1, ..., q-1. On the outer circle we label the diametrically opposed points 0', 1', ..., (q-1)', respectively.

Beginning at 0' traverse T^2 counter-clockwise to the point labeled p. From p, traverse under T^2 to the point p', diametrically opposed to p. We repeat this process with an arc from 1' to (1+p)(mod q) and under to ((1+p)(mod q))'. This process is continued until returning to the point labeled 0'. Of course we are careful not to cross any arcs already drawn.

To show that the knot K constructed is really in the class [(p,q)], we refer to the proof of Theorem 5, Section 1. There we showed that two loops f and g are equivalent in $T^2 = J_1 x J_2$ if and only if the projections of f and g are equivalent in $P_1(T^2) = J_1$ and $P_2(T^2) = J_2$, respectively. That is, the loop K is in the class [(p,q)] if and only if $P_1(K) \varepsilon [J_1^p]$ in $\pi(J_1)$ and $P_2(K) \varepsilon [J_2^q]$ in $\pi(J_2)$. Clearly, from the construction of K, this is the case. The algorithm is illustrated in Figure 8 with (p,q) = (2,3). We may "remove" the torus to obtain a clear picture of the knot $K_{2,3}$, and we see that $K_{2,3}$ is the trefoil knot.

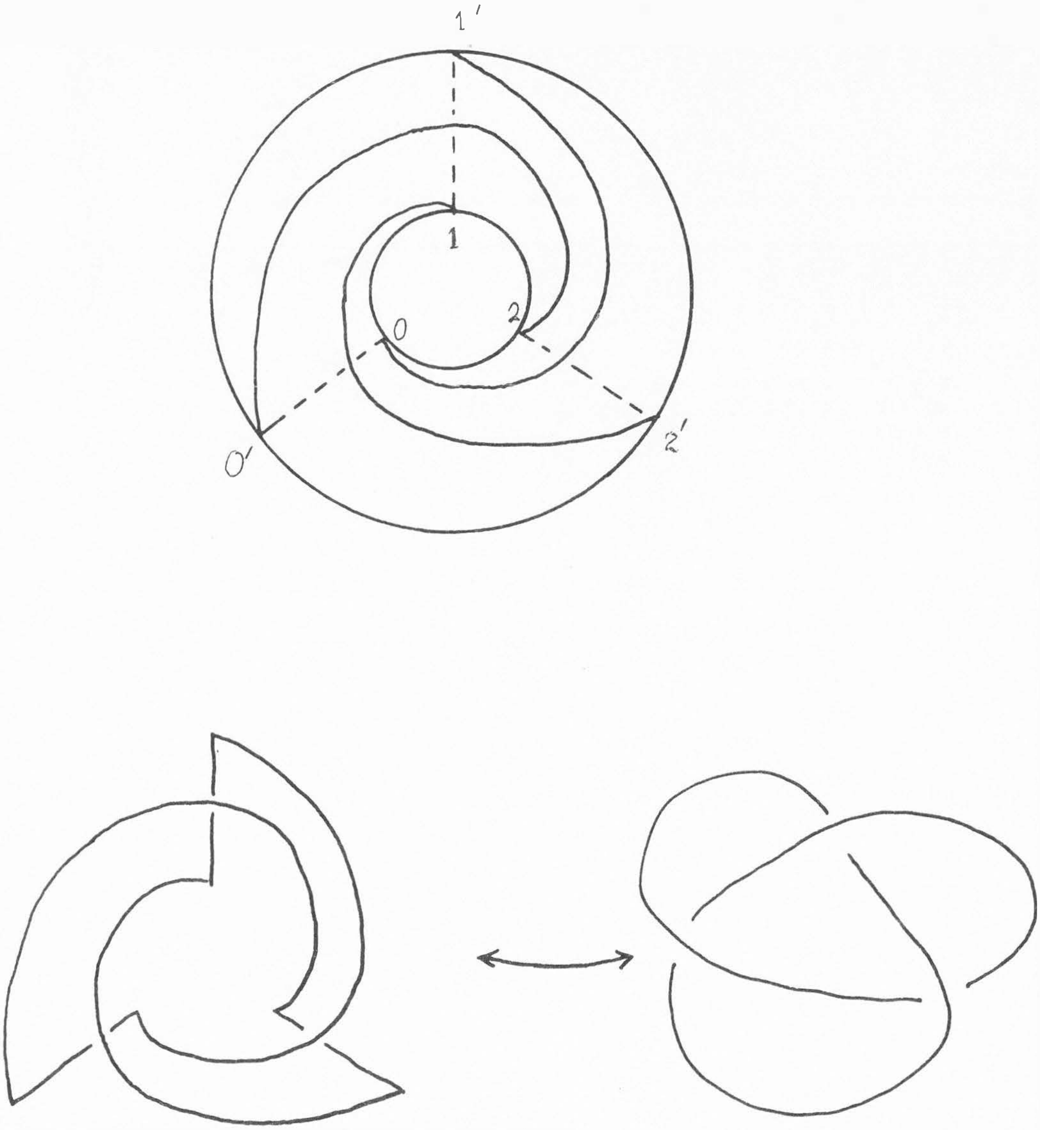
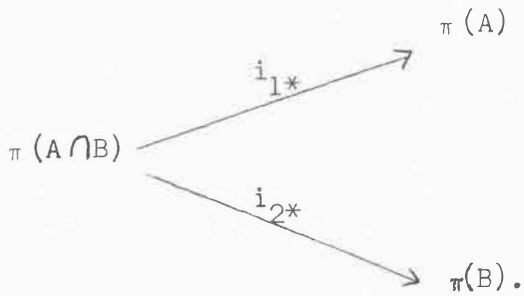


Figure 8

IV. THE FUNDAMENTAL GROUP OF $E^3 - K_{p,q}$

We define a knot group to be the fundamental group of the complement of the knot in E^3. In this section we will give a presentation for the knot group $\pi(E^3 - K_{p,q})$. To do this we will use the following form of Van Kampen's Theorem.

We take X to be a topological space which is the union of two open subsets A and B such that $A \cap B$ is arcwise connected, and we let p be in $A \cap B$. Suppose it is known that $\{z_1, z_2, \ldots, z_t\}$ generates $\pi(A \cap B, p)$, that $\pi(A,p)$ has a presentation $\{x_1, x_2, \ldots, x_n \mid r_1, \ldots, r_m\}$, and that $\pi(B,p)$ has a presentation $\{y_1, \ldots, y_r \mid s_1, \ldots, s_k\}$. The inclusion maps $i_1: A \cap B \to A$ and $i_2: A \cap B \to B$ induce homomorphisms as pictured:



Van Kampen's Theorem. Under the conditions stated above, $\pi(X) = \pi(A \cup B) = \{x_1, \ldots, x_n, y_1, \ldots, y_r \mid r_1, \ldots, r_m, s_1, \ldots, s_k, i_{1*}(z_j) = i_{2*}(z_j) \text{ for } j = 1, 2, \ldots, t\}$ [4].

Theorem 1. The knot group $\pi(E^3 - K_{p,q})$ has a presentation $\{a,b \mid a^p = b^q\}$.

Proof: Let T be the solid torus bounded by T^2, and let $K_{p,q}$ be in T^2. Let N_ε be a tubular ε- neighborhood of $K_{p,q}$ whose closure is denoted by $\overline{N}_\varepsilon$. Let S be the interior of a solid torus containing T, obtained by uniformly expanding the interior of T by ε.

We define A to be $S - \overline{N}_\varepsilon$ and B to be $E^3 - T - \overline{N}_\varepsilon$. Now A and B are open and $E^3 - \overline{N}_\varepsilon = A \cup B$. The center circle of S is a deformation retract of A and hence, from Theorems 3 and 6, Section 1, $\pi(A)$ is infinite cyclic. We also see that looped through the hole of T is a simple closed curve that generates the fundamental group of B and hence $\pi(B)$ is infinite cyclic.

Now $A \cap B$ is a thickened open annulus obtained by removing $\overline{N}_\varepsilon$ from a thickened torus. Hence $A \cap B$ has a knot of type (p,q) as a deformation retract. Thus $\pi(A \cap B)$ is also infinte cyclic. It follows that $\pi(A)$, $\pi(B)$ and $\pi(A \cap B)$ have presentations $\{a | -\}$, $\{b | -\}$, and $\{z | -\}$ respectively.

Now $i_{1*}(z) = a^p$ and $i_{2*}(z) = b^q$; thus, by Van Kampen's Theorem, $\pi(E^3 - \overline{N}_\varepsilon) = \pi(A \cup B) = \{a,b | a^p = b^q\}$. Expand $\overline{N}_\varepsilon$ slightly to obtain a neighborhood N'_ε such that $E^3 - N'_\varepsilon$ is a deformation retract of $E^3 - \overline{N}_\varepsilon$. Now it follows that $E^3 - N'_\varepsilon$ is also a deformation retract of $E^3 - K_{p,q}$. From Theorem 3, Section 1, we see that $\pi(E^3 - K_{p,q}) \cong \pi(E^3 - \overline{N}_\varepsilon) \cong \pi(E^3 - N'_\varepsilon)$ and hence that $\pi(E^3 - K_{p,q})$ has a presentation $\{a,b | a^p = b^q\}$.

We now have a presentation for $\pi(E^3 - K_{p,q})$ written in terms of the integers p and q. With the aid of the following definitions and lemmas we will show that there are infinitely many groups which appear as the fundamental group of the complement of a torus knot. We will denote $\{a,b | a^p = b^q\}$ by $G_{p,q}$.

Let G and H be groups having the presentations $\{g_1, g_2, g_3, \ldots | A, B, C, \ldots\}$ and $\{h_1, h_2, h_3, \ldots | P, Q, R, \ldots\}$, respectively.

We define the free product $G*H$ of G and H to be the group with the presentation $\{g_1, g_2, g_3, \ldots, h_1, h_2, h_3, \ldots \mid A, B, C, \ldots, P, Q, R, \ldots\}$. The groups G and H are called the free factors of $G*H$. Note that the free factors G and H are isomorphic to subgroups $\overline{G}$ and $\overline{H}$ of $G*H$ (under the obvious isomorphism) such that $\overline{G} \cap \overline{H} = \{1\}$. It is customary to identify $\overline{G}$ with G and $\overline{H}$ with H.

In the following discussion we will be concerned with finite cyclic groups of order n having the presentation $G_n = \{g \mid g^n = 1\}$. The free product of two cyclic groups G_n and G_m is $G_n*G_m = \{a,b \mid a^n = 1 = b^m\}$. It can be shown that the abelianization $(G_n*G_m)'$ of G_n*G_m has the presentation $\{a,b \mid a^n = 1 = b^m, ab = ba\}$ [3]. In fact, if the integers n and m are relatively prime, then $(G_n*G_m)'$ is isomorphic to the cyclic group $G_{nm} = \{c \mid c^{nm} = 1\}$. We will show this by making a series of transformations that ultimately transform the presentation $\{a,b \mid a^n = 1 = b^m, ab = ba\}$ into $\{c \mid c^{nm} = 1\}$. Since n and m are relatively prime there exists integers x and y such that $xn + ym = 1$, from which it follows that $(ab)^{xn} = b$ and $(ab)^{ym} = a$. Let $c = ab$. Then the relations $c^{xn} = b$, $c^{ym} = a$, and $c^{nm} = 1$ are consequences of the given relations and may be added to the presentation of $(G_n*G_m)'$. Thus $(G_n*G_m)'$ has the equivalent presentation $\{a,b,c \mid a^n = b^m = 1, ab = ba, c = ab, c^{nm} = 1, a = c^{ym}, b = c^{xn}\}$. Similarly, we may delete the relations $a^n = 1$, $b^m = 1$, and $ab = ba$ since they are merely consequences of the other relations. The generaters a and b are powers of c, hence they may be dropped and we obtain the presentation $\{c \mid c^{nm} = 1\}$. It follows that $(G_n*G_m)'$ has order nm.

Suppose x is an element in G_n*G_m. Since G_n*G_m is generated by a and b, x is a product $a^{\varepsilon 1}b^{\varepsilon 2}a^{\varepsilon 3}b^{\varepsilon 4}\ldots a^{\varepsilon n-1}b^{\varepsilon n}$, where ε_j and ε_i are integers.

Lemma 1. The only elements of $G_n * G_n$ having finite order are the elements of G_n and G_m and their conjugates.

Proof: Suppose $x \varepsilon G_n * G_m$ and $x^p = 1$. It follows that x is a product $g_1 g_2 \ldots g_k$, where each g_i is a power of either a or b and g_i and g_{i+1} are not powers of the same element. We will prove the lemma by induction on k. Thus, let $S = \{ k |$ If $x \varepsilon G_n * G_m$, $x = g_1 g_2 \ldots g_k$ where each g_i is a power of a or b and g_i and g_{i+1} are not powers of the same element, and x has finite order, then x is an element of G_n or G_m or a conjugate of an element of G_n or $G_m \}$. Clearly $1 \varepsilon S$.

Suppose $k \varepsilon S$ and x has finite order p, where $x = g_1 g_2 \ldots g_{k+1}$. If g_1 and g_{k+1} are in different free factors, it follows that x is not of finite order; consequently g_1 and g_{k+1} are in the same free factor. Since g_1 and g_{k+1} are in the same free factor we express their product h as a power of either a or b. Thus, $g_1^{-1} x g_1 = g_2 \ldots g_k g_{k+1} g_1 = g_2 \ldots g_k h$. It follows from the inductive hypothesis that $g_1^{-1} x g_1 = g_2 \ldots g_k h$ is an element of G_n or G_m or a conjugate of an element of G_n or G_m. Thus x is an element of G_n or G_m or a conjugate of an element of G_n or G_m.

An immediate consequence of this is the following:

Lemma 2. The maximum order of any element of $G_n * G_m$ of finite order is the maximum of n and m.

Lemma 3. If $(p,q) = 1$, $(r,s) = 1$, and $G_p * G_q$ is isomorphic to $G_r * G_s$, then $p = r$ and $q = s$, or $p = s$ and $q = r$.

Proof: We begin by observing that the abelianizations $(G_p * G_q)'$ and $(G_r * G_s)'$ are isomorphic whenever $G_p * G_q$ and $G_r * G_s$ are isomorphic. Thus pq and rs are equal if $G_r * G_s$ and $G_p * G_q$ are isomorphic since pq and rs are the orders of their respective abelianizations.

Suppose $p \neq r$ and $p \neq s$. Then either p or q is greater than both r and s; or one of r and s is greater than both p and q. We may assume p>r and p>s. Thus $p > \max\{r,s\}$ and, from Lemma 2, the maximum order of any element of $G_r * G_s$, of finite order, is equal to $\max\{r,s\}$. However, the generator a of G_p has order p in $G_p * G_q$ and its image under any isomorphism must be of order p. Thus we obtain a contradiction, and p = r or p = s. If follows that p = r and q = s, or p = s and q = r.

<u>Theorem 2.</u> If torus knots $K_{p,q}$ and $K_{r,s}$ are equivalent (of the same knot type), and if p,q,r and s are all greater that 1, then either p = r and q = s, or p = s and q = r.

Proof: Let N be the subgroup generated by a^p in the group $G_{p,q} = \{a,b \mid a^p = b^q\}$. Note that $a^p a = a a^p$ and that $a^p b = b^q b = bb^q = ba^p$. It follows that a^p commutes with every element in $G_{p,q}$, and hence that N is normal. Let $\bar{a}$ and $\bar{b}$ be the cosets of a and b; respectively, relative to N in $G_{p,q}$. It is clear that $G_{p,q}/N$ is generated by $\bar{a}$ and $\bar{b}$. We also see that $\bar{a}^p = a^p N = N = \bar{1}$, and similarly that $\bar{b}^q = \bar{1}$. It follows that $G_{p,q}/N$ has the presentation $\{\bar{a},\bar{b} \mid \bar{a}^p = \bar{b}^q = \bar{1}\}$. From this presentation we see that $G_{p,q}/N$ is the free product $G_p * G_q$. Similarly, $G_{r,s}/\langle g^r \rangle$ is $G_r * G_s$. Since a^p commutes with every element in $G_{p,q}$ we see that N lies in the center Z of $G_{p,q}$. Suppose x is in the center Z' of $G_{p,q}/N = G_p * G_q$. Then $x\bar{a} = \bar{a}x$ and $x\bar{b} = \bar{b}x$, which implies that x is in the center of each free factor G_p and G_q. Since $G_p \cap G_q = \{\bar{1}\}$ it follows that $Z' = \{\bar{1}\}$. Let η be the homomorphism of $G_{p,q}$ onto $G_{p,q}/N$ given by $\eta(x) = xN$. Then $\eta(Z) \subset Z' = \{\bar{1}\}$. Since N is the kernel of η, $Z \subset N$. It follows that N is the entire center Z of $G_{p,q}$. Thus the quotient of $G_{p,q}$ by its center is $G_p * G_q$. Similarly, the quotient of $G_{r,s}$ by its center is $G_r * G_s$.

Now, if $K_{p,q}$ is equivalent to $K_{r,s}$, then it follows from Theorem 3 of Section I that $G_{p,q}$ is isomorphic to $G_{r,s}$. Thus their quotient groups by their centers, $G_p * G_q$ and $G_r * G_s$, are isomorphic; and p = r and q = s, or p = s and q = r follows from Lemma 1.

Suppose p = 1. Then a presentation for $\pi(E^3 - K_{1,q})$ is $\{a,b \mid a = b^q\}$. Since a is a power of b, a may be deleted from the presentation; hence $\pi(E^3 - K_{1,q})$ has the presentation $\{b \mid - \}$. Similarly, if q = 1, then $\pi(E^3 - K_{p,1})$ is infinite cyclic. The following theorem tells us that the torus knots of type (p,q) with p or q equal to 1 are all equivalent; in fact, they are all trivial. For this reason many authors restrict their definition of torus knots to those types having $p>1$ and $q>1$.

<u>Theorem 3.</u> A knot K is trivial if and only if $\pi(E^3 - K)$ is infinite cyclic [8].

<u>Theorem 4.</u> A torus knot $K_{p,q}$ is non-trivial if and only if $p>1$ and $q>1$.

Proof: $(\Rightarrow)$. Suppose p is 0 or 1. It follows that $\pi(E^3 - K_{p,q})$ is infinite cyclic. Thus $K_{p,q}$ is trivial. The proof is similar if q is 0 or 1.

$(\Leftarrow)$. Suppose $p>1$ and $q>1$. Then $G_{p,q}/Z = G_p * G_q = \{a,b \mid a^p = 1 = b^q\}$ where Z is the center of $G_{p,q}$. Now, the abelianization of $G_p * G_q$ is a finite group of order $pq > 1$. If $K_{p,q}$ is the trivial knot, then the center of the group $G_{p,q}$ is the entire group; hence $G_{p,q}/Z = \{1\}$. It follows that $K_{p,q}$ is non-trivial.

Suppose $K_{p,q}$ and $K_{r,s}$ are two non-trivial torus knots and $E^3-K_{p,q}$ is homeomorphic to $E^3-K_{r,s}$. Then $G_{p,q}$ is isomorphic to $G_{r,s}$, from which it follows that $p = r$ and $q = s$, or $p = s$ and $q = r$. In the closing remarks of Section II we showed that this relationship between (p,q) and (r,s) implies $K_{p,q}$ and $K_{r,s}$ are of the same knot type. We now summarize these results.

Theorem 5. If $K_{p,q}$ and $K_{r,s}$ are non-trivial torus knots, then $K_{p,q}$ and $K_{r,s}$ are of the same knot type if and only if $p = r$ and $q = s$, or $p = s$ and $q = r$.

It follows from Theorem 5 that there are infinitely many non-equivalent torus knots. Another interesting corollary (to the proof of Theorem 5) provides a partial solution to an unsolved problem in knot theory.

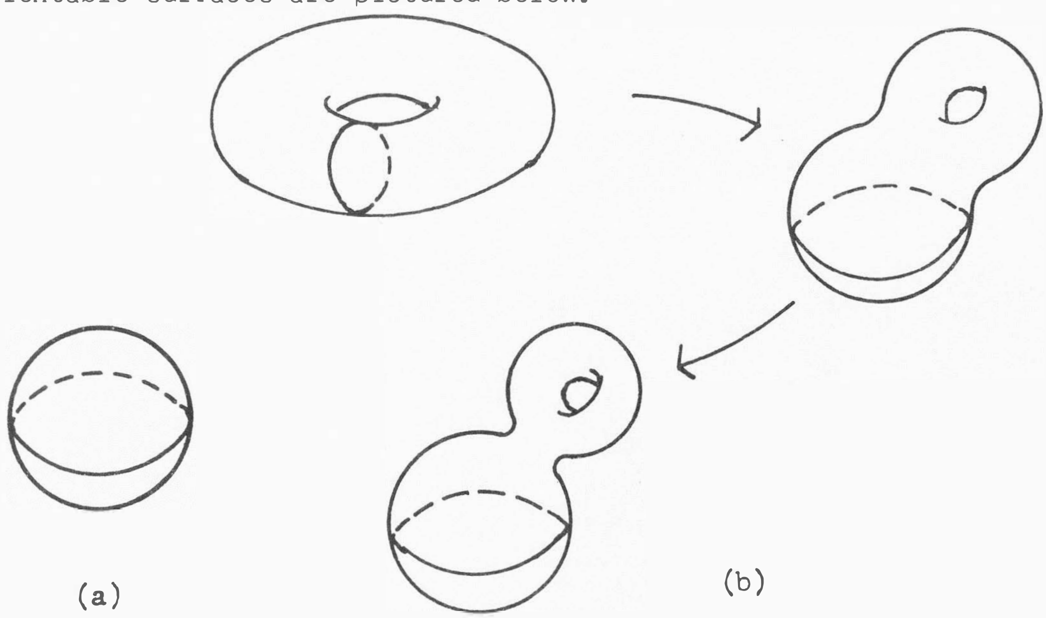
Problem. Suppose K_1 and K_2 are two knots in E^3 such that E^3-K_1 is homeomorphic to E^3-K_2. Are K_1 and K_2 of the same knot type? If K_1 and K_2 are torus knots, the answer is in the affirmative.

In the proof that $\pi(E^3-K_{p,q}) = \{a,b \mid a^p = b^q\}$ we constructed open sets A and B such that $\pi(A)$, $\pi(B)$, and $\pi(A \cap B)$ had the presentations $\{a \mid -\}$, $\{b \mid -\}$, and $\{z \mid -\}$ respectively. Then we used Van Kampen's Theorem to obtain $\{a,b \mid a^p = b^q\}$. Hence, in the group $\{a -\}$, $a^p = 1$ implies $p = 0$. Previously, in the proof of Theorem 2, we showed that the center of $G_{p,q} = \{a,b \mid a^p = b^q\}$ is $Z = \langle a^p \rangle$. It follows that the center of $G_{p,q}$ is non-trivial. For if $p = 0$, then $G_{p,q}$ is infinite cyclic and $Z \neq \{1\}$. Furthermore, if $p \geq 1$, then $a^p \neq 1$. In fact, it has been conjectured that torus knots are the only knots whose knot groups have non-trivial centers [7].

V. THE GENUS OF A KNOT

The genus of a knot K is a nonnegative integer associated with K in a particular way. It is defined in such a manner as to be invariant under space homeomorphisms. We will show that there exist torus knots of arbitraty genus. Although we have already proven that there are infinitely many nonequivalent torus knots, we point out that the work in this section gives us the same result. It is also a consequence of the work to be done here that there are infinitely many knots that are not torus knots.

We begin by giving some standard terminology. A Hausdorff space X is said to be a 2-manifold if each point in X is in an open set homeomorphic to the open disk $D = \{(x,y) \mid x^2 + y^2 < 1\}$ in E^2. In this paper all manifolds considered will be metric spaces. A connected 2-manifold is called a surface. We define an orientable space to be a space that does not contain a Möbius Strip. Some orientable surfaces are pictured below.



(a) (b)

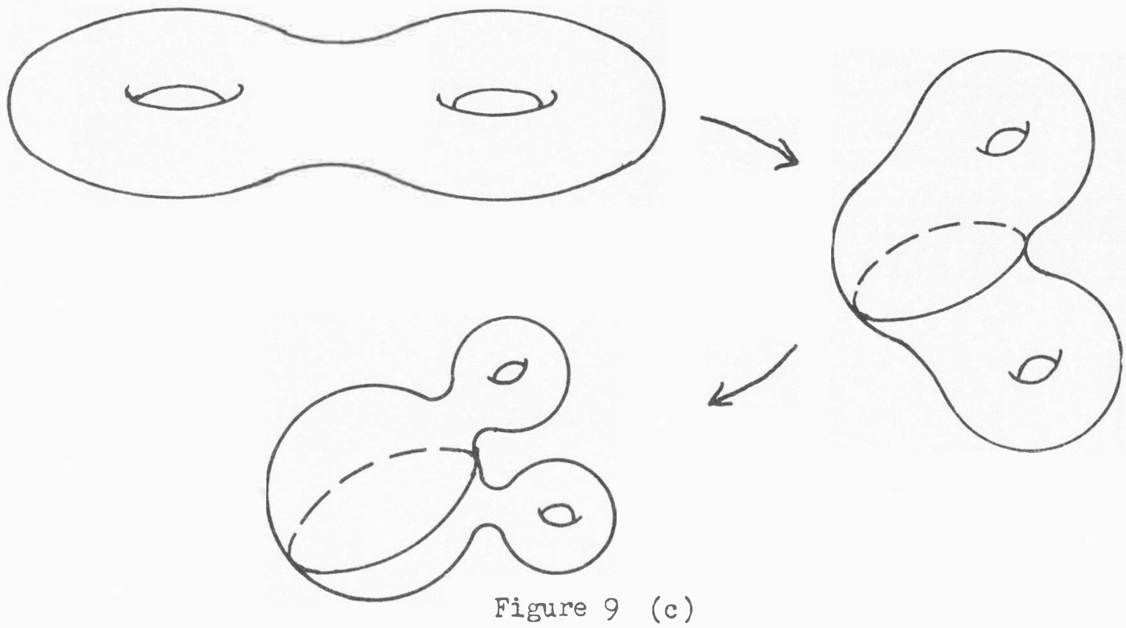


Figure 9 (c)

Illustration (b) shows how to think of a torus as a sphere-with-one-handle. From figure (c) we can see that a double torus may be thought of as a sphere-with-two-handles. A sphere with two handles is called the connected sum of two tori. Similarly, a sphere with n handles is called the connected sum of n tori. It can be shown that any compact orientable surface is homeomorphic to a sphere or to the connected sum of finitely many tori [4].

We define the genus of a compact orientable surface to be zero if the surface is homeomorphic to a sphere and the genus is n if the surface is homeomorphic to the connected sum of n tori.

A bordered 2-manifold is a Hausdorff space X such that each point in X is in an open set homeomorphic either to the open disk $D = \{(x,y) \mid x^2+y^2 < 1\}$ or the subspace $\{(x,y) \mid x \geq 0\}$ of E^2. As above, a connected bordered 2-manifold is called a bordered surface. The subset of a bordered 2-manifold X consisting of all points that lie only in open sets homeomorphic to the subspace $\{(x,y) \mid x \geq 0\}$ of E^2 is called the boundary of X.

Suppose the boundary of a compact bordered surface X has n components. It can be shown that each boundary component is locally like the line, and it follows from the compactness of X that each boundary component is a simple closed curve. We can obtain a compact surface X' by taking n closed disks and sewing the boundary of the i^{th} disk to the i^{th} boundary component of X. It is clear that the orientability of X' depends only on the orientability of X. If X is a compact orientable bordered surface, we define the genus of X to be the genus of the compact orientable surface X' obtained by "capping the boundary components with disks" as described in the preceding sentences. We see that by sewing the boundary of a disk onto the boundary of a disk we obtain a sphere. Thus the genus of a disk is zero. The bordered surfaces pictured below are all homeomorphic. The space shown in (a) is obtained by removing an open disk from the torus and is called a disk-with-one-handle. By stretching the hole (with a homeomorphism) as indicated in (b), we obtain the spaces in (c). Finally, the junction of the two intersecting bands of the figure in (c) is enlarged to obtain a disk with two bands having one boundary component. As noted, these spaces are homeomorphic and hence the construction is reversible. It follows that the genus of each of the bordered surfaces above is 1. We will now show that the genus of an orientable disk with 2n bands having one boundary component is n. A band is simply a disk but the word "band" is ordinarily used to denote a "long skinny" disk. We say D is a disk with 2n bands if D is the union of a disk D' with 2n disjoint bands $\{B_1, B_2, \ldots, B_{2n}\}$ such that the intersection of D' with any

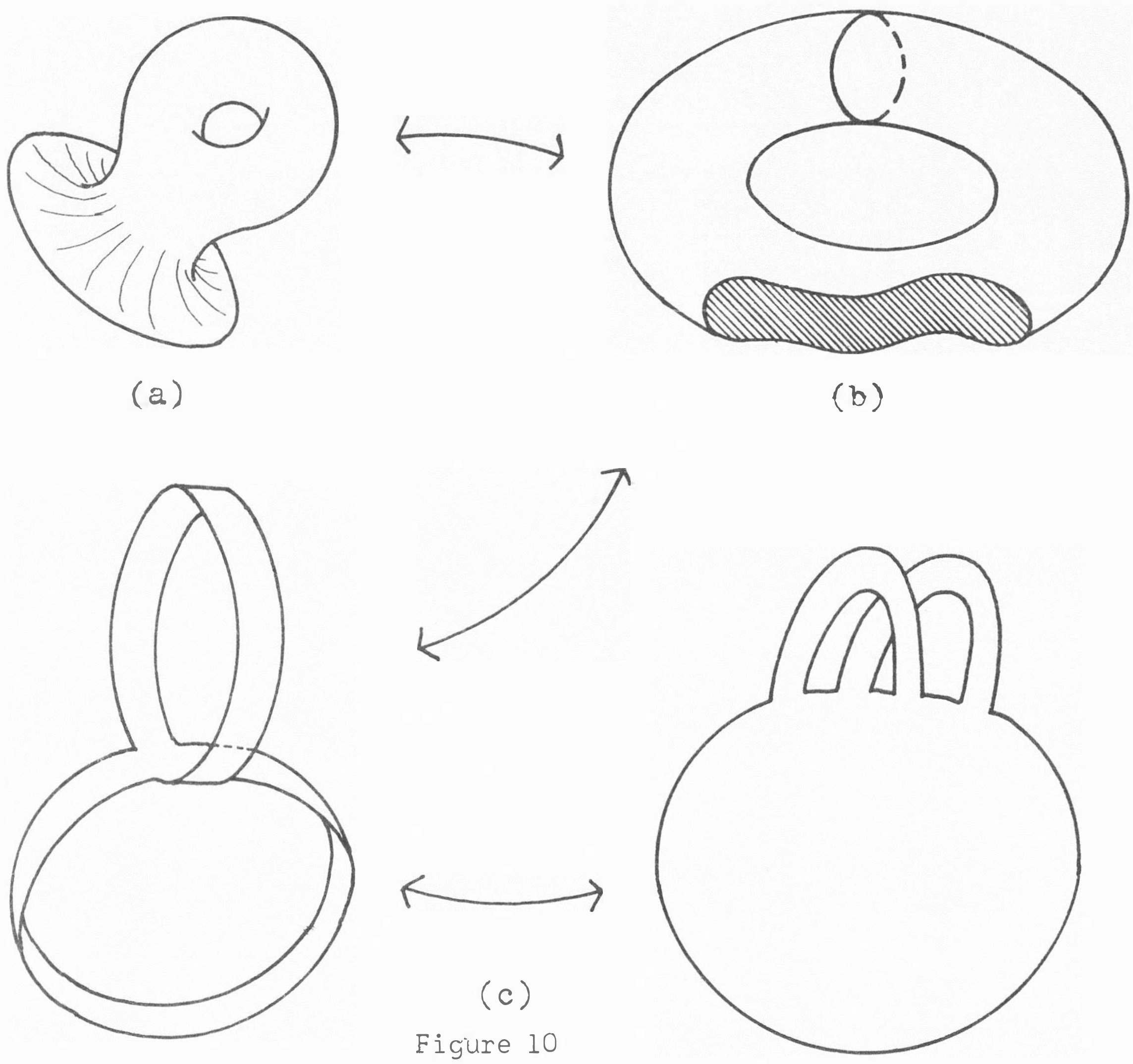


Figure 10

band Bi, i = 1, ..., 2n, is the union of two disjoint arcs. We will assume that no band has an odd number of twists to guarantee that the manifold is orientable (see Figure 11). Suppose B is one of the 2n bands in D. Let a_1 and a_2 be the two arcs whose union is $B \cap D'$. Since D has an even number of bands there exists another band B' disjoint from B. In fact B' has the property that the intersection of B' with each component of $D' - (a_1 \cup a_2)$ is nonempty. For if not, then D would have more that one boundary component. We call two such bands a band pair. We show that no loss in generality is introduced by assuming the band pairs to be separated from one another by arcs spanning D'.

The following sequence of figures shows how to obtain a space homeomorphism that helps "untangle" the band pairs. We illustrate the procedure where n = 2. The shaded band is the one we wish to move. We begin by "walking the band" around band B in the direction of the arrows. This homeomorphism is the identity map outside the shaded disk (see Figure a). Similarly the space in (c) is obtained from the one in (b). Finally, the bands are separated as shown in (d) by walking the shaded band over band B in the direction of the arrows in figure (c). (The bands may be "knotted" or "linked" about each other, but separation of the band pairs in the disk D is all we require and can be accomplished as described above.) Thus we may assume that in a disk D with 2n bands there exist n-1 arcs such that band pairs lie in different components of D minus the arcs (see Figure 11-d). Next we make n-1 cuts along the separating arcs in the disk D so as to obtain n disks, each of which has two bands. Now each of these disks-with-two-bands is homeomorphic to a disk with one handle (see Figure 10 on page 35). The boundary of each disk with a handle contains an arc along which the cut was originally made. When we sew these back together along the original cut we obtain a disk with n handles. It follows that the genus of D is n. The proof is pictured in Figure 12 for n = 3.

We will next show that a tame knot in E^3 bounds a compact orientable bordered surface in E^3. Once this is accomplished we will define the genus of a knot and obtain a formula for the genus of a torus knot of type (2,q).

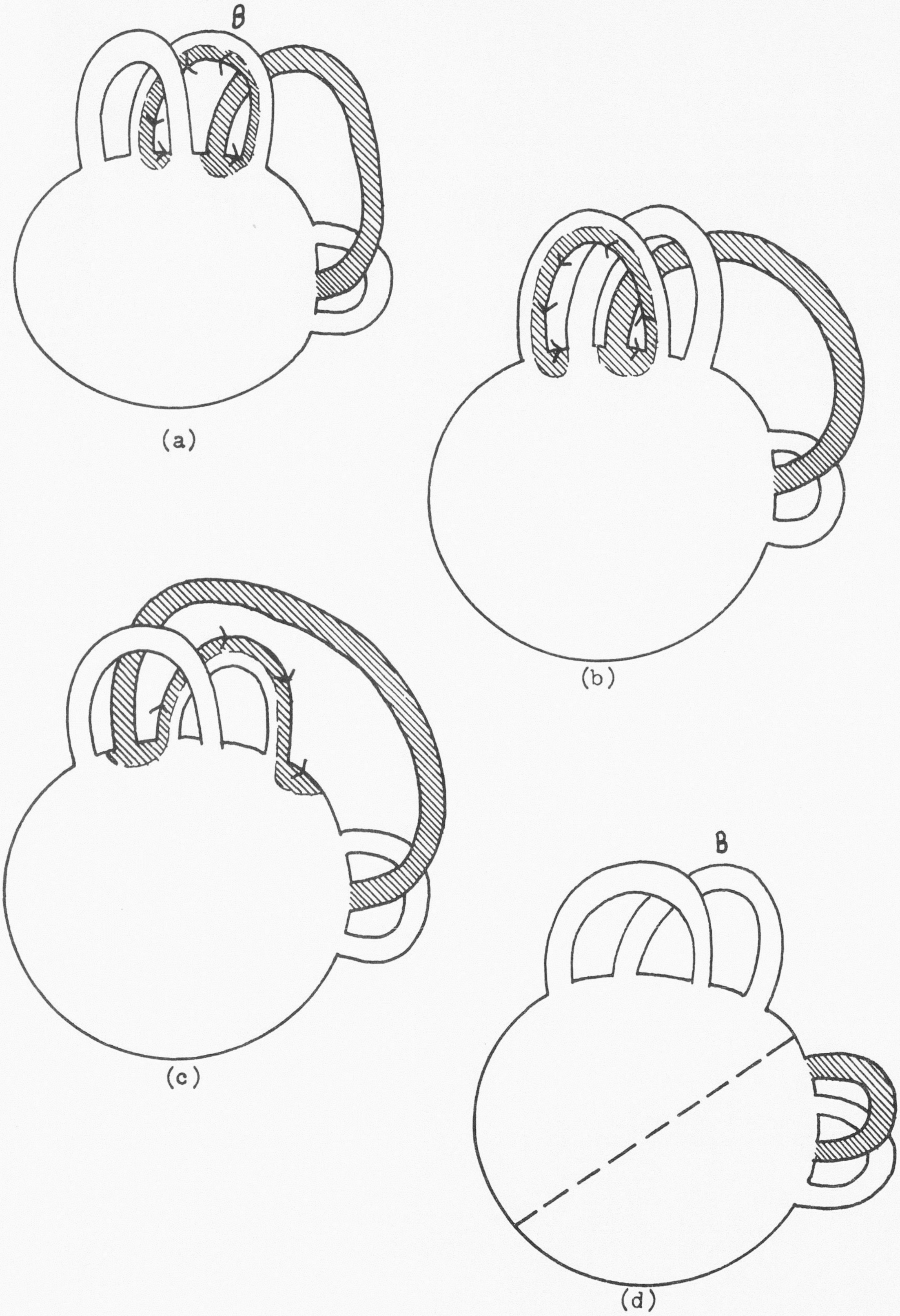


Figure 11

Since we are concerned only with tame knots, we will begin by assuming that we are given a polygonal knot in E^3. Choose a plane such that the

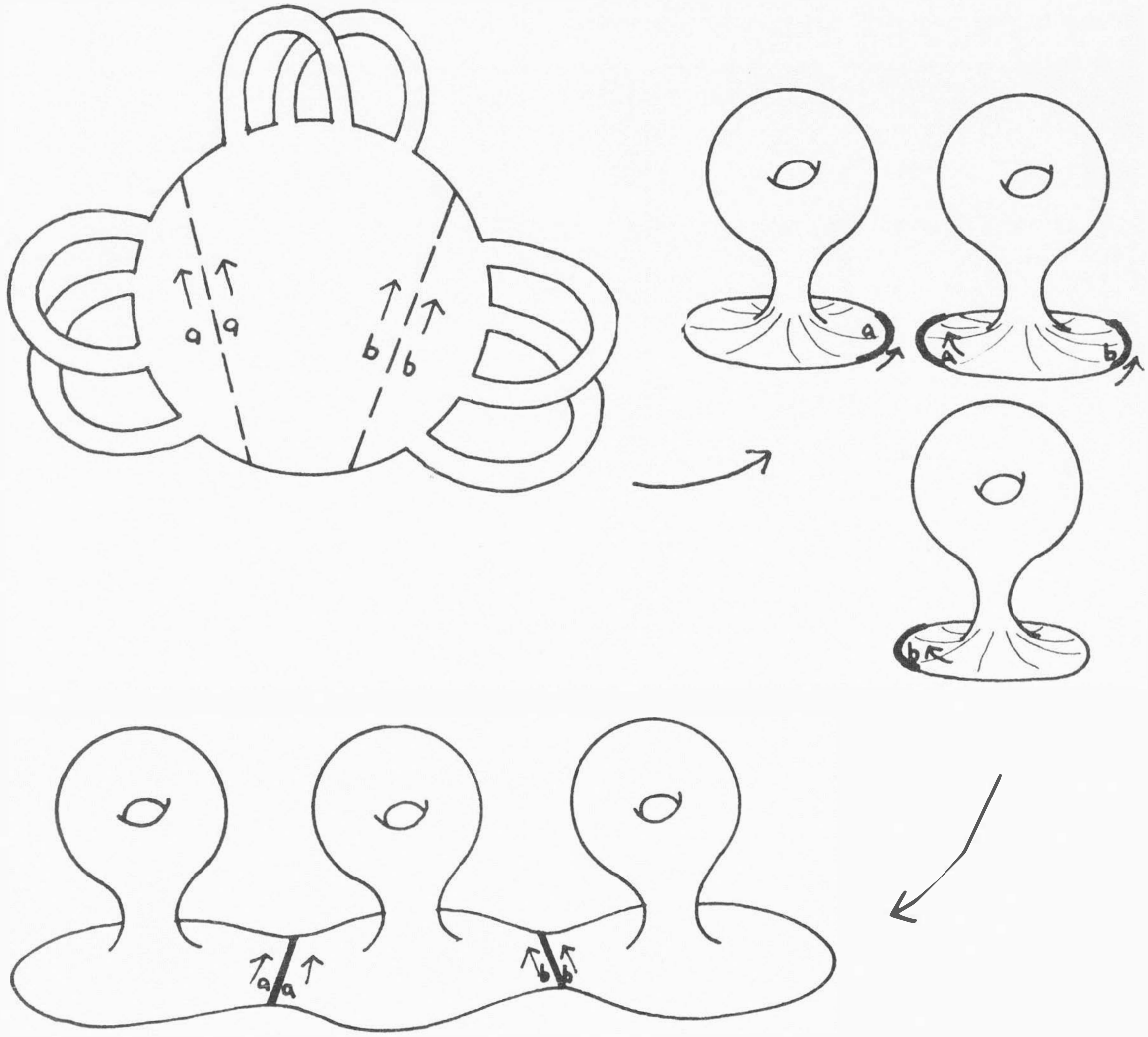


Figure 12

knot can be projected into the plane with only a finite number of singular points, and such that each of these points has exactly two points in its pre-image, both of which are interior points of the straight line segments making up the polygonal knot [1]. We may assume that the plane P is horizontal, and we adjust the projection a little to remove singularity at each singular point p_i. One of the two pre-images of p_i is higher than the other, and we lift a small neighborhood of the higher point (the neighborhood is on the straight line segment) out of the plane (see Figure 13). We call this new curve C, and we note that C is now almost entirely in P and is homeomorphic,

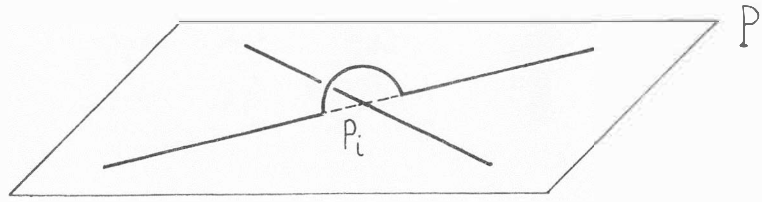


Figure 13

by a space homeomorphism h, to the original curve. At each singular point p_i choose a round neighborhood N_i in E^3 such that the lifting, discribed above, at p_i was done in the interior of N_i, $\overline{N_i} \cap \overline{N_j} = \emptyset$, for $i \neq j$, and the intersection of N_i with the original projection is homeomorphic to the letter X. One such neighborhood is pictured below.

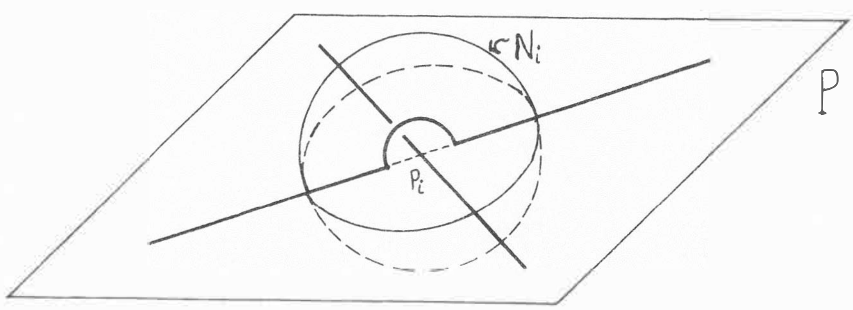


Figure 14

We now assign an orientation to the curve C and we will now describe how to choose a collection of mutually exclusive simple closed curves J_k (called Siefert circles) which covers C except for the part of C which lies in $\bigcup_1^n N_i$. We start at a point of $C - (\bigcup_1^n \overline{N_i})$ and move along C in the direction of C's orientation, until a point of some BdN_i is reached. Observe that BdN_i is pierced four times by C; twice inward and twice outward. We move along $BdN_i \cap P$ to an outward piercing point, choosing the direction so that we remain in the plane and so that we do not meet either of the other two piercing points. We continue moving along C, in the direction of C's orientation, repeating the process at each N_i until the original starting point is reached. Call this simple closed curve J_1. If J_1 does not cover $C - (\bigcup_1^n \text{int } N_i)$, choose another starting point, not on J_1, and repeat the process.

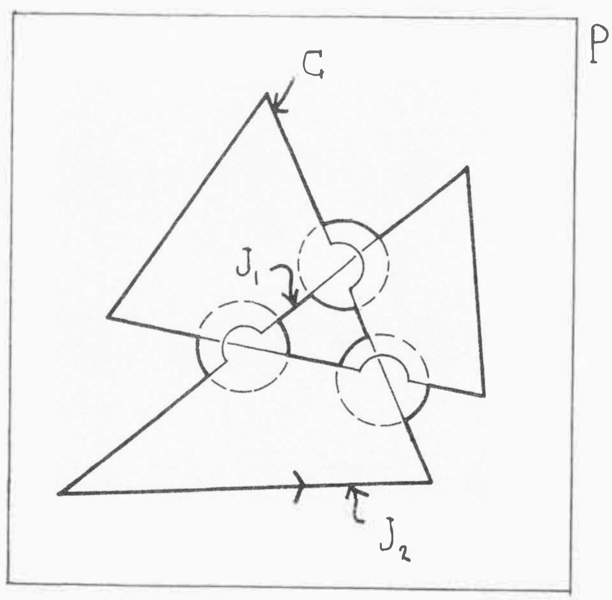


Figure 15

It is clear that this process gives us the collection of disjoint simple closed curves described above. Each J_k bounds a disk in P. If these disks are disjoint, then we take them as they are. However, they probably won't be. If the disks bounded by the J_k's are not disjoint, then we replace them with disjoint disks obtained by pushing the flat disks down below P. In this way we obtain a collection of disks $\{D_k\}$ such that the D_k's are pairwise disjoint, $BdD_k = J_k$, and the interior of each D_k lies below P.

There is also a collection of simple closed curves K_i, one for each singular point p_i. The curve K_i is the union of the two arcs of C which lie in N_i with the two arcs on the boundary of N_i which are contained in the union of the J_k's. This curve bounds a disk E_i in $\overline{N_i}$ such that no point of E_i lies below the plane P. Hence the E_i's are pairwise disjoint, and $[(\cup D_k) \cap (\bigcup_1^n E_i)] = [(\cup J_k) \cap (\bigcup_1^n BdN_i)]$ (see Figure 16).

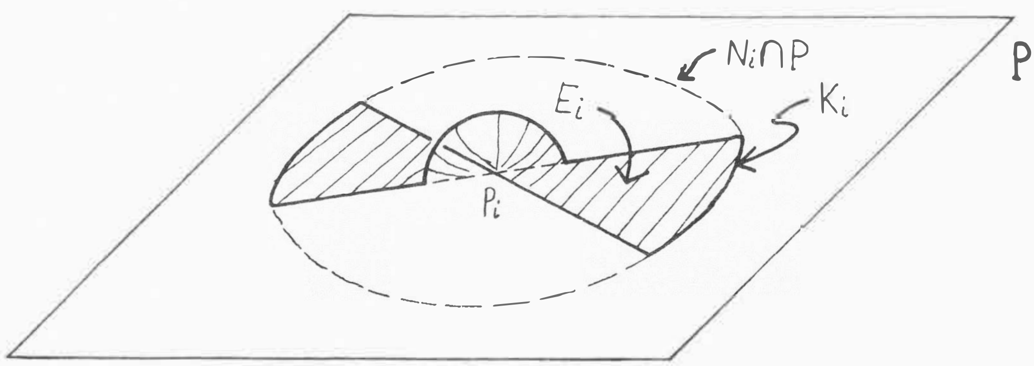


Figure 16

Notice that the only boundary component of the resulting bordered surface $M' = [(\cup D_k) \cup (\bigcup_1^n E_i)]$ is the simple closed curve C. Then $M = h^{-1}(M')$ is the desired bordered surface whose boundary is the original knot $h^{-1}(C)$.

We claim that M' is orientable. The proof depends on results not yet mentioned in this paper. With the aid of the following definitions and observations we will ultimately show that M' cannot contain a Möbius Strip.

We make the assumption that any compact bordered surface S can be triangulated [4]; that is, there exists a finite cover $\{T_1, \ldots, T_n\}$ of S consisting of n sets each homeomorphic to a triangle in E^2, having the property that any two distinct sets T_i and T_j are either disjoint, have a vertex in common, or have an entire edge in common. We will call the elements of $\{T_1, \ldots, T_n\}$ triangles. Suppose a space S is triangulated and each triangle has an orientation on its boundary which is pictured as an arrow. If for each pair of triangles sharing an edge the arrows on the common edge have opposite directions, then we say S has an orientation preserving triangulation. For example the arrows in the triangles of the disk below show that the disk has an orientation preserving triangulation. Observe that an orientation of the boundary simple closed curve of a disk induces an orientation on the triangulation of the disk. Thus, in the manifold M', the orientation assigned to the curve C induces an orientation preserving triangulation of each disk D_k and each band E_i. Let D_i and D_m be two disks connected by a band E_n. With the aid of Figure 18 we see that the orientation induced by the orientation of C on triangulations

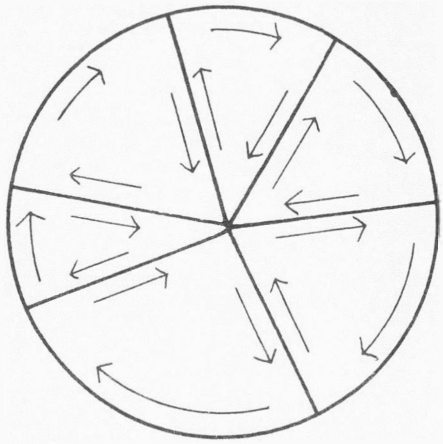


Figure 17

of D_i, D_m and E_n are compatible in the sense that the arrows conflict at edges in the intersection.

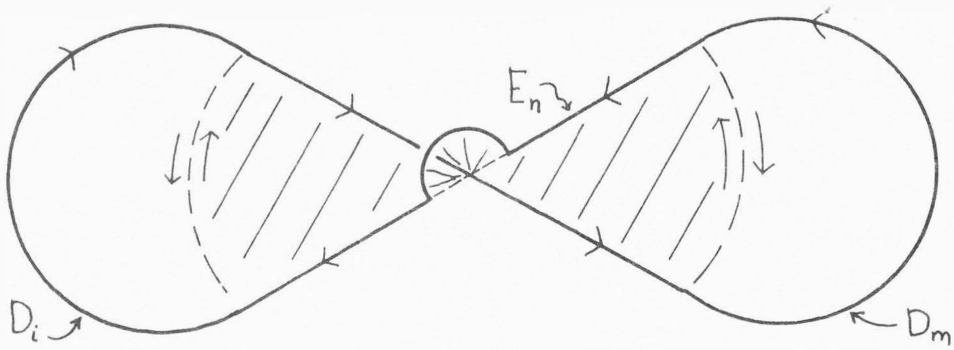


Figure 18

We will next show that a Möbius Strip S cannot have an orientation preserving triangulation. The boundary of S is a simple closed curve, and it is clear that if S has an orientation preserving triangulation it induces an orientation on the boundary. Consider S as the identification space obtained by identifying opposite sides of I^2 as pictured below. Suppose S has a triangulation, and let T_1 be a triangle with one edge on the top of I^2. Further, suppose T_1 has a clockwise orientation. It follows that there

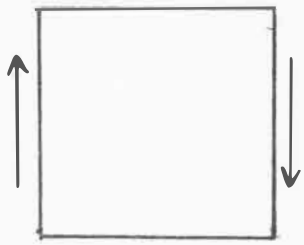


Figure 19

is a chain of K triangles extending from top to bottom of I^2 all having a clockwise orientation. Now T_1 induces a direction from left to right on the top of I^2 (note that the top of I^2 is part of the boundary of S), which in turn induces a direction from left to right on the bottom of I^2. Thus the orientations conflict on the bottom of I^2.

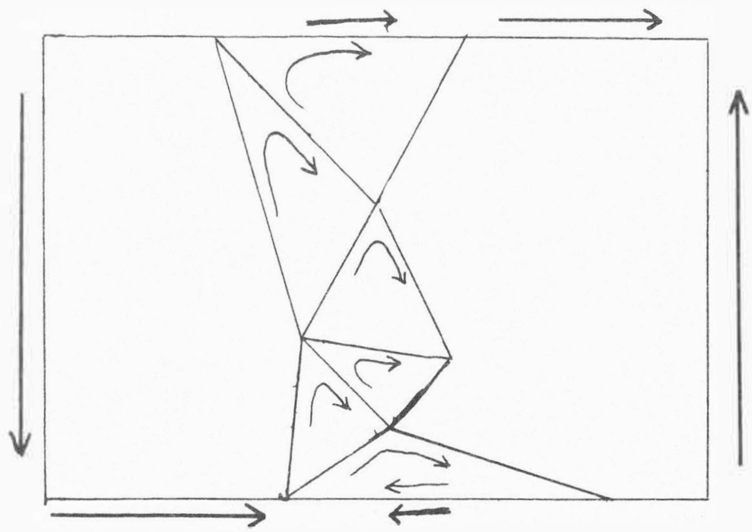
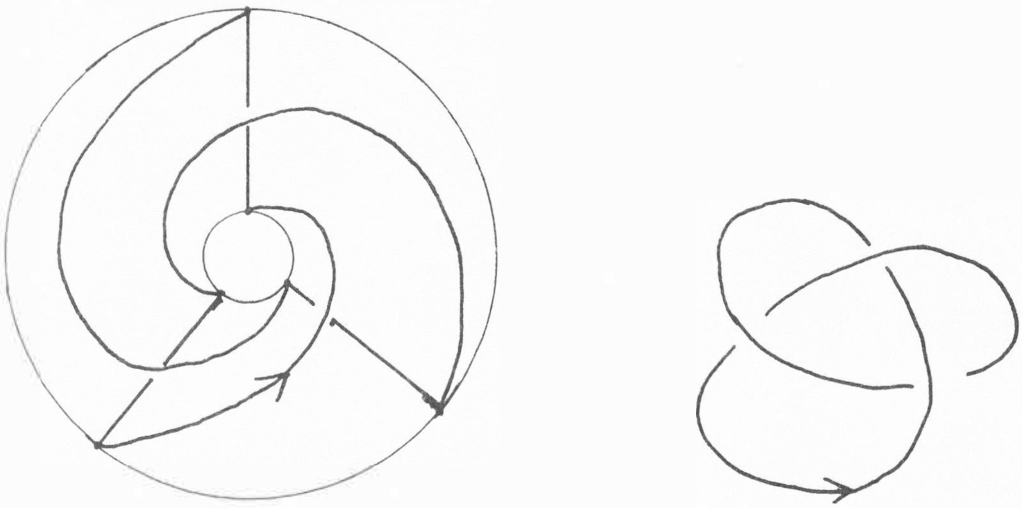


Figure 20

If M' contained a Möbius Strip S, then S would lie in the union of some subcollection of the D_k's and E_i's. However, from what we said above we see that the orientation on the disks D_k and E_i would induce an oreintation preserving triangulation on S, which is impossible.

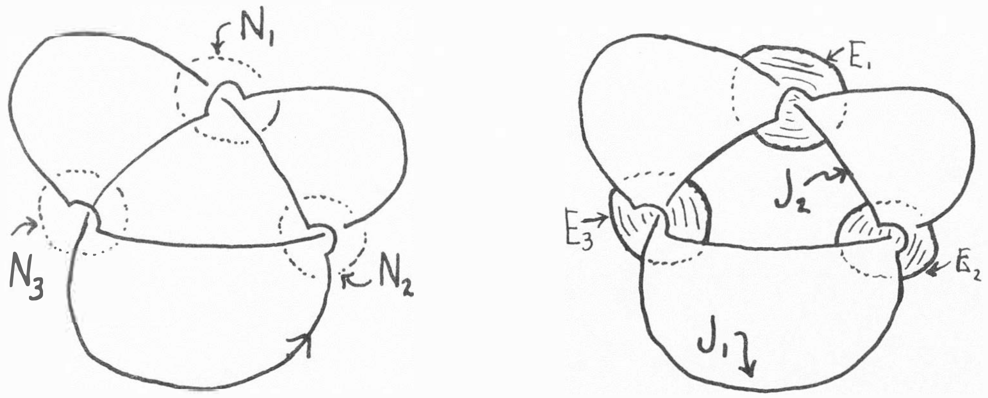
We define the genus g(K) of a knot K to be the minimum genus of all bordered surfaces bounded by K. A polygonal knot is alternating if it can be projected into a plane with only a finite number of singular points such that each of these points has exactly two points in its pre-image both of which are interior points of the straight line segments making up the knot, and the overcrossings and undercrossings alternate around the projection of the knot. A knot type is alternating if it has an alternating representative. It can be shown that for alternating knots the above algorithm produces the bordered surface of minimum genus [5]. Thus the genus of an alternating knot is just the genus of the bordered surface obtained above.

The genus g(K) of an alternating knot K is determined as follows: Construct an orientable bordered surface M bounded by K in accordance with the algorithm. Recall that we obtain a collection of r disjoint disks (the D_k of the algorithm) connected by n disjoint bands (the E_i of the algorithm). Since each disk D_j must be connected to another of the D_k's by at least one band, it follows that $r - 1 \leq n$. The union of the r disks with a particular set of (r - 1) of the bands is a topological disk. The number n - (r - 1) of remaining bands is nonnegative since $r - 1 \leq n$. Furthermore if n - (r - 1) were odd we would have an orientable bordered surface M homeomorphic to a disk with an odd number of bands attached. Then we could separate one of the bands (as described on page 37) and see that M would have at least two boundary components. However, we have only one boundary component, the knot itself. Thus n - (r - 1) is even and the bordered surface obtained is a disk with n - (r - 1) = 2m bands and one boundary component. It follows that $g(K) = m = \frac{n - (r - 1)}{2}$. We illustrate the procedure in the following picture, where we see that $g(K_{2,3}) = 1$.



(a) (b)

Alternating projection obtained by the algorithm of Section III.



(c) (d)

Remove singularities. Construct 2 Siefert circles.

Figure 21

Thus $g(K_{2,3}) = \frac{3 - (2 - 1)}{2} = 1$, according to the formula derived in the preceding paragraph. Similarly, the figure 8 knot has genus 1.

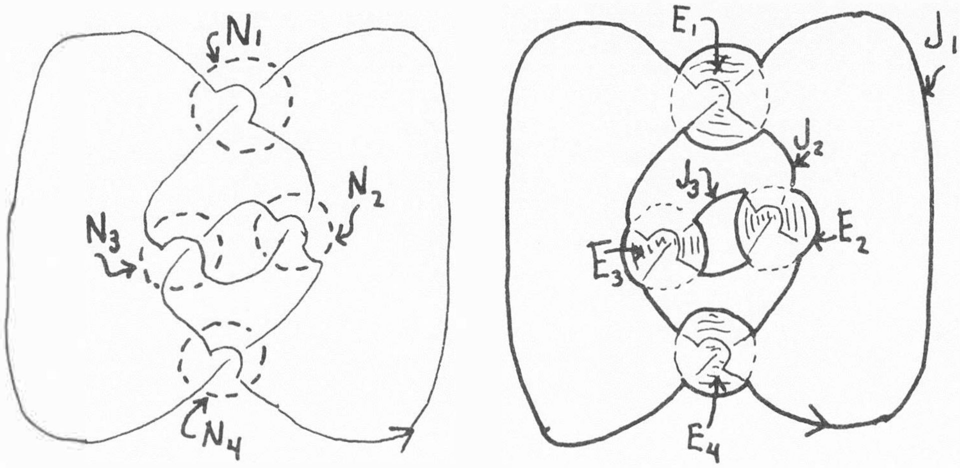


Figure 22

It is clear, from the algorithm for picturing a torus knot, that any torus knot of type (2,q) is alternating. In fact, an alternating projection is given by the algorithm and we will use it to compute $g(K_{2,q})$. It is clear that for arbitrary q the number of Siefert circles is 2 and that there are q crossings. By virtue of these results and the formula for the genus of an alternating knot it follws that $g(K_{2,q}) = \frac{q - 1}{2}$.

Now, if two alternating knots K_1 and K_2 are equivalent under a space homeomorphism h and K_1 bounds the bordered surface M, then K_2 bounds h(M) and g(M) = g(h(M)). Thus $g(K_1) = g(K_2)$. Therefore, if q is allowed to vary over the positive odd integers, we obtain a countable infinite collection of nonequivalent torus knots. Furthermore, we see, by letting q = 2n + 1, that for each nonnegative integer n there exists a torus knot $K_{2,q}$ having genus n. We point out the fact that torus knots of type (p,q) with $p > 2$ and $q > 2$ have no alternating projection [6]. However, an interesting relation was given by R. H. Fox concerning the genus of a knot of type (p,q) and the genus of the bordered surface bounded by the knot obtained in accordance with the above algorithm [2]. If we use the projection of $K_{p,q}$ described in Section III, we see that there are q arcs "under" the torus, each of which is crossed over p - 1 times. It follows that there are q(p - 1) bands. It can be shown that there are p nested Siefert circles. Thus, the knot $K_{p,q}$ bounds a disk with $q(p - 1) - (p - 1) = (q - 1)(p - 1)$ bands. It follows that the genus of the bordered surface is $\frac{(q - 1)(p - 1)}{2}$. We remark that this number may not be $g(K_{p,q})$; nevertheless, this proves that $g(K_{p,q}) \leq \frac{(p-1)(q-1)}{2}$, and it is well known that equality holds if $K_{p,q}$ is alternating [2].

We now construct an infinite collection of knots, each having genus 1, none of which is a torus knot. Consider the torus T^2 and the knot K_o inside of T^2 as shown in Figure 23 below.

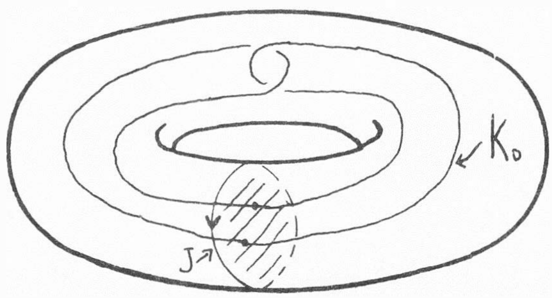


Figure 23

K_o is called a trivial double-knot (or a doubled trivial knot). The knot obtained from K_o by slicing the solid torus vertically through J, making one 360^o twist of the right hand side of the torus in the direction of the arrow, and then connecting the knot back together where the original cut was made is called a 1 - twist knot. Similarly, a twist of $2\pi n$ in the direction of the arrow converts K_o to a knot K_n called an n - twist knot. The knots K_1 and K_2 are shown in Figure 24.

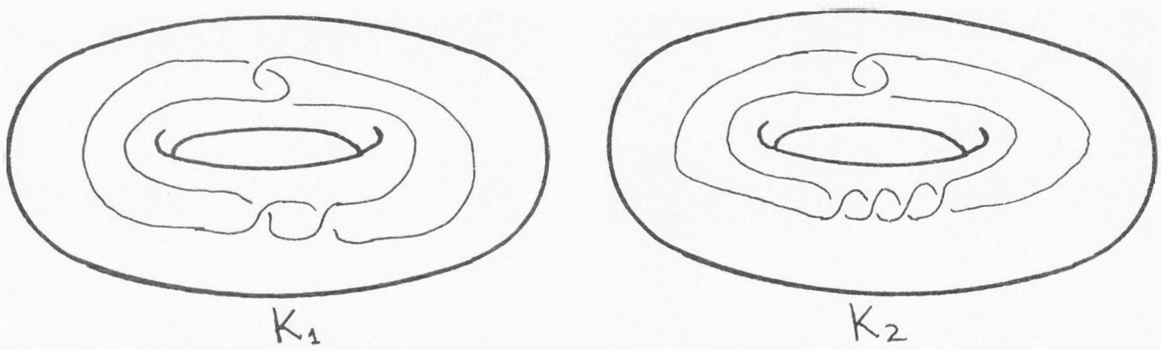


Figure 24

It is clear that these knots are alternating. Thus we may use the algorithm to determine $g(K_n)$. We see that K_o is the trivial knot and hence $g(K_o) = 0$. Now, each twist knot has the original two crossings as in K_o and two more crossings for each twist; therefore the number of bands in the bordered surface bounded by K_n is $2n + 2$. It can be shown by induction on n that the number of Siefert circles is $2n + 1$. Hence, from the formula given on page 46, it follows that $g(K_n) = \frac{(2n + 2) - [(2n + 1) - 1]}{2} = \frac{2}{2} = 1$.

Since the twist knots are alternating and $g(K_n) = 1$, $n \geq 1$, the only torus knot that could be a twist knot is the trefoil. It can be shown via another knot type invariant, called the Alexander polynomial, that the collection of twist knots is infinite and that no twist knot is of the same type as the trefoil [1]. We point out, however, that the knot obtained by giving J one twist in the opposite direction is the trefoil.

REFERENCES

1. Crowell, Richard H., and Ralph H. Fox. Introduction to Knot Theory. Ginn and Co., Boston, Massachusetts. 1965.

2. Fox, R. H. "A Quick Trip Through Knot Theory," Topology of 3-manifolds. Prentice-Hall, Englewood Cliffs, New Jersey. 1962.

3. Magnus, Wilhelm, Abraham Karrass, and David Solitar. Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations. Interscience Publishers, New York. 1966.

4. Massey, William S. Algebraic Topology: An Introduction. Harcourt, Brace, and World Inc., New York. 1967.

5. Murasugi, Kunio. "On the Genus of the Alternating Knot II," Journal of Math. Soc. of Japan 10 (1958), 235-248.

6. Neuwirth, Lee. "A Note On Torus Knots And Links Determined by Their Groups," Duke Math. Journal 28 (1961), 545-551. M. R. 24-A, 683.

7. Neuwirth, Lee. "A Remark On Knot Groups With a Center," Proc. Amer. Math. Soc. 14 (1963), 378-379. M. R., 26, 1057.

8. Papakyriakopolous, C. D. "Some Problems On 3-Dimensional Manifolds," Bull. Amer. Math. Soc. 64 (1958), 317-335.

VITA

David S. Bradley

Candidate for the Degree of

Master of Science

Report: Torus Knots

Major Field: Mathematics

Biographical Information:

Personal Data: Born at Hawthorne, California, January 15, 1948, son of Gaylen S. and Katherine Bradley; married Laura Anderson, November 6, 1965; one child--Gina Gay.

Education: Graduate of Manti High School, Manti, Utah in June 1965. Associate in Science from Snow College; Ephraim, Utah in June 1967. B. S. In Mathematics Education from Utah State University; Logan, Utah in June 1969.