1946

Electric circuit theorems.

Conrad J. Hemond
University of Massachusetts Amherst

ELECTRIC CIRCUIT THEOREMS

HEMOND, Jr. 1946

ELECTRIC CIRCUIT THEOREMS

by

CONRAD J. HEMOND JR.

Thesis Submitted For The Degree Of

Master Of Science At

Massachusetts State College, Amherst

June 1946

TABLE OF CONTENTS

SUBJECT	PAGE
Introduction	1
Thevenin-Helmholtz Theorem	6
The Delta Wye Theorem	19
Illustration of Helmholtz-Thevenin Theorem	22
A Reactance Theorem for a Resonator	24
Alternative Equivalent Circuit for the Thermionic Valve	31
Abstract Section	34
Bibliography	41
Acknowledgements	43

OUTLINE

This thesis is concerned with an attempt to gather under one heading several diverse theorems dealing with electric circuit problems the postulation of which are at present scattered through many technical journals and texts, with the aim of aiding instructional processes now in practice in courses dealing with this subject.

The Problems:- (1) Can these theorems be expressed in a more simplified and concise manner?

(2) Are any of these theorems, as now classified under diverse names, of a singular origin?

(3) Are any of these theorems adaptable to concise lecture demonstration exhibits?

Reason for Interest:- As a Professional Engineer interested in educational methods and as a college instructor with three years experience in teaching Physics and Mathematics, I have been interested in all projects which attempt to present complex material to the student in a more vivid and concise manner with little or no loss in accuracy. I am also interested in this problem as a means of increasing my educational background in the subject of electricity with which I have had little experience.

Limitations:- Because of the numerous theorems available I have limited myself to those more popular theorems dealing with direct current and simplified alternating current circuit theory.

Procedure:-(a) I have made a diligent search through the journals of the electrical engineering societies and the trade magazines of leaders in the electrical devices manufacturing field, as well as texts available in the surrounding college and municipal libraries.

(b) I have compiled statements of theorems dealing with this subject tracing the historical development in as many cases as possible.

(c) I have attempted to present a more concise and simple statement of these theorems from a college students' point of view.

(d) Under each of the circuit theorems which I have discussed, I have presented sample circuit problems as solved by the theorem.

(e) I have designed and constructed a piece of lecture-laboratory demonstration apparatus to show the application of two of the theorems discussed.

INTRODUCTION

The author of this thesis is making an attempt to clarify a situation existing in the field of electrical circuit problems wherein a number of theorems dealing with the solution of these problems have been postulated in many diverse and sundry manners. The difficulty often arises, that within the field of electrical circuit analysis, the mathematical solution of the problem often becomes so complex and involved that the original theory intended to be absorbed by the student is lost. In an attempt to remove some of this mathematical drudgery, research men have expounded theorems which may be used to solve difficult circuit problems with a minimum of labor, thus allowing a student more time for further research, if these theorems can be placed at his command in an adequate manner. It is at this point where the difficulty arises. Research men have as an outlet for their writings, many trade and technical journals within which to disseminate their theorems. For the student interested in the theorems, this means considerable library research merely to gain tools with which to ply his trade.

The student at this point is faced with another problem because when he consults the articles found in the journals or in some texts which have attempted to make "brief, concise statements" of the theorems he finds a variance in terminology - often unexplained and with symbols which are not standardized - as well as a variance in the statement of the theorem. Here again he has difficulty "seeing the tree because of the woods."

It was with these problems of the student as well as attendant problems of the professors conducting such courses in mind, that the author has made a search of leading technical journals and texts in the field of electric circuit theory and presents herein some of the more popular theorems. An attempt is made to clarify through schematic drawings and standard symbols the wording of the theorems presented. An attempt is also made to present in graphical manner the advantage of the theorem over the ordinary analytical or statistical methods of solution.

The author set out with the further purpose of adapting one or more of the theorems to concise lecture demonstration exhibits and succeeded to the point where he includes within this treatise schematic drawings of a device which will illustrate two of the more important theorems, i.e. the Delta Wye theorem and the Thevenin -

or Helmholtz Theorem. This device has been constructed by the author in the Physics laboratory at Massachusetts State College and can be constructed in most other college laboratories equipped for the instruction of courses in electrical circuit analysis.

Such exhibits as described herein will aid the lecturer in demonstrating to the student the adaptability of such theorems to ordinary problems thus aiding in presenting complex material to the student in a more vivid and concise manner. Such methods are in the belief of the author to be encouraged as aids to the educational development of the student.

Electric circuit theorems may be defined as statements which indicate definite relationships existing between the component parts of an electrical circuit if certain imposed conditions are fulfilled. Many textbooks dealing with the subject of electric circuits present at least four theorems which have been found to be of value to the student in the computation of quantities in various branches of the circuit. As most frequently found in the textbooks which the author has surveyed the following four theorems are used or quoted for the students reference.

(1) The Superposition Theorem:-

Each E.M.F., in a complex circuit acts independently of all others in producing network current.

Illustration: - (See sketch next page)

If current I_a flows in branch (a) due to e.m.f. (E_a), and an additional e.m.f. (E_b) is inserted in branch (b) producing a current I_b which is equal to the current which E_b would send through branch (a) in the absence of any other E, then the current in branch (a) is equal to $I_a + I_b$

(2) The Compensation Theorem:-

If any change is made in the resistance of a branch, the effect on all of the mesh currents is the same as if an E.M.F. equal to the negative of the product of the change in the resistance multiplied by the branch current, had been inserted.

Illustration:- (see sketch)

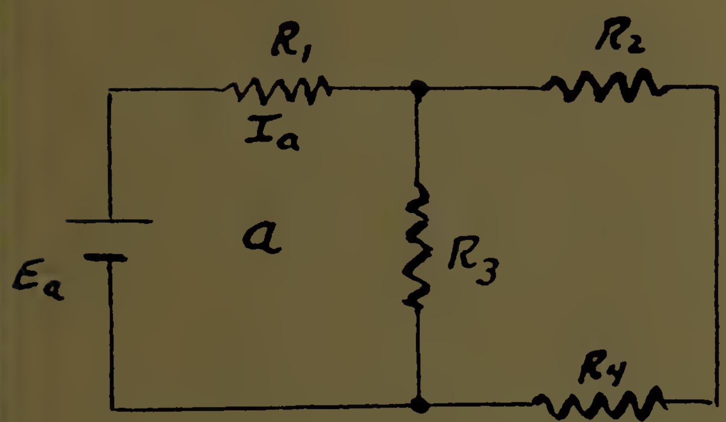
In circuit (a) we have a current i_1 equal to the E.M.F. (E) divided by the sum of the resistances.

In circuit (b) a resistance (dR) is added to (R_2) and the current is correspondingly reduced. However, if, as in circuit (c), an E.M.F. equal to the product of (i_1) and (dR) and of opposite sign is introduced into the circuit the total current flowing in the circuit remains at its original value (i_1).

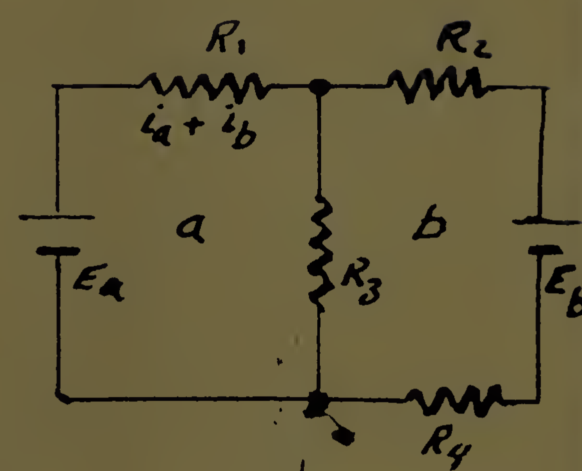
(3) The Reciprocity Theorem of Green:-

Where a resistance is common to two meshes the effect of the resistance from one mesh to another is reversible.

Superposition and Reciprocity Theorems

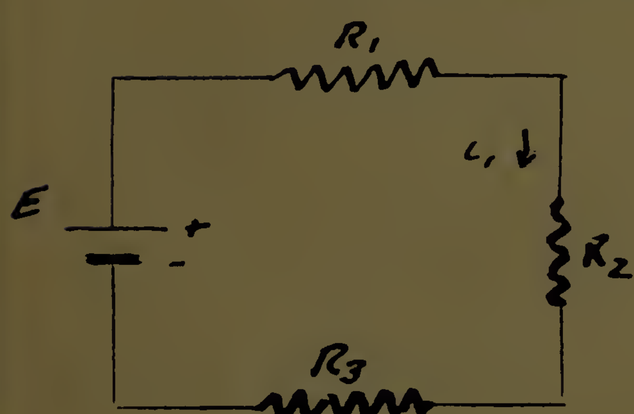


(1)

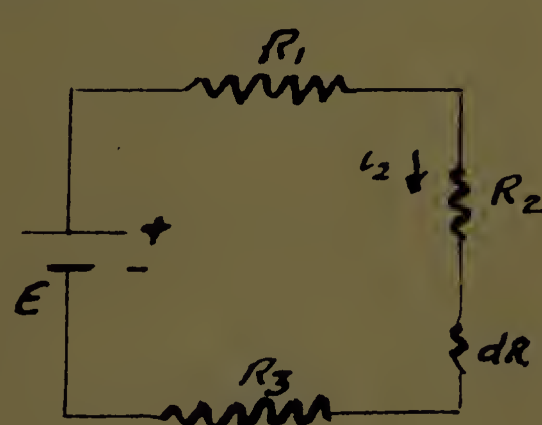


(2)

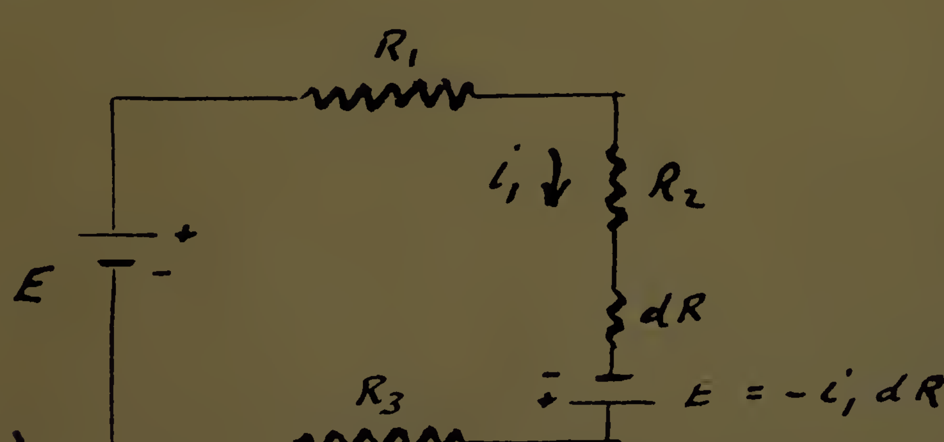
Compensation Theorem



(A)



(B)



(C)

Illustration:- (See Sketch used to illustrate Superposition Theorem)

The resistance (R_3) is common to both meshes (a) and (b), that is $R_{ab} = R_{ba}$. Because the determinant used to solve for the current is the same as effected by (R_3), the transfer conductance from mesh (a) to mesh (b) is the same as from mesh (b) to mesh (a) and an E.M.F. in branch (a) produces the same current in branch (b) as it would produce in branch (a) if placed in branch (b).

(4) The fourth theorem usually quoted is the Thevenin Theorem in one form or another. The author in this treatise begins at this point with an attempt at clarification of this Theorem and then presents additional theorems. The above theorems have been rather adequately discussed in most modern texts and therefore the author does not discuss them further but would refer the interested party to texts listed in the bibliography of this treatise.

As an aid to clarity and simplicity each theorem is discussed under a separate heading complete with summary and conclusions which can be drawn.

Since this treatise is intended as a collection of theorems as well as a discussion of their origin there has been included also a section of illustrative examples abstracted from leading authoritative texts and research papers. This section appears under a separate heading at end of treatise.

THE THEVENIN-HELMHOLTZ THEOREM

In the discussion of this theorem, the author has been forced to use within the title the name of L.Thevenin whose name through popular misinformation has become associated with a theorem originally expostulated by the German physicist, mathematician and philosopher H.von Helmholtz, who enunciated the theorem thirty years prior to the date when Monsieur Thevenin first published his statement apparently without knowledge of Helmholtz's research.

This theorem has been chosen as the first to be discussed also because of its popularity in present day analysis of communication circuits involving both direct and alternating current.

Before discussing the theorem itself, a brief historical background showing the confusion which exists regarding the origin of this particular theorem seems appropriate.

H.Von Helmholtz in a paper appearing on Page 21 in the "Annalen der Physik and Chemie" for the year 1853 entitled "Weber einige Gesetze der Vertheilung elektrisher Strome in korperlichen Leitem mit anivendung ans die thierisch-elektrishen Versuche" (On some laws of the distribution of electric currents and their application to experiments in

Animal Electricity) set forth the theorem which has come to be known as Thevenins' Theorem. Helmholtz first cites a law of Kirchhoff which states that if in any system of conductors, the Electromotive Forces exist at various points, the potential at every point in the system is the algebraic sum of the potentials which would be produced by each of the E.M.F's acting alone. He then applies this law to a linear network and sets forth the following statement of a theorem.

"If two points of a linear network are connected to other conductors, it behaves as a conductor of certain resistance, the magnitude of which can be calculated by the ordinary rules for branched networks, and of an E.M.F. equal to the Potential Difference that existed between the two points before they were connected by the other conductors."

In 1883, L. Thevenin published a note of little more than two pages in length in the "Comptes Rendus," Vol. 97, pg. 159-161" on a new theorem of dynamic electricity." The statement of Thevenin was a simple and plain statement of the Helmholtz theorem with which he was apparently unacq uainted. Thevenin's statement is as follows:-

"A current through any branch of a circuit is equal to the current through the same branch when an open circuited E.M.F. acts in the same branch and all other E.M.F.'s are replaced by their internal impedances."

As will be shown in later analysis of these two statements - essentially equal - as well as other statements of a more recent origin, there should be no doubt as to the origin of the theorem nor to the fact that the name of Thevenin applied to the theorem gives a wrong implication as to its founder.

Pleijel, an engineer of the Swedish Telegraphic Department, in an article appearing in the "Revve Gen.de l'Elect.", 16, Ap. 1919 written by Pomey, is described as offering a proof of "Thevenin Theorem" which in fact followed the theorem of Helmholtz.

Helmholtz's theorem was applied to alternating current networks in 1923 by Dr.F.T.Chapman and so reported by him in a paper appearing in Electrical Review (30th March) entitled "The Calculation of D.C. and A.C.networks."

The theorem was published under the name of Helmholtz in a paper by F.Wenner, who restated the theorem and applied it to a number of bridge problems, appearing in the "Pros.Phys.Soc." for 1927. Wenner does not even mention Thevenin.

The same theorem has appeared in other works under different names. V.Jenkin (Revue Gen.del'Elect for 1938) called the theorem a "method of superposition." Van den Meershe("Revue Gen.del'elect for 1935) restated the theorem in practically the original words of Helmholtz and called it "A theorem deduced from the generalized reciprocity theorem of Maxwell." Since Helmholtz, himself, attributed much of the basis for his theory to Kirchhoff and Green, there is no need to bring Maxwell into the discussion.

Applications of the Helmholtz theorem either under the name of Thevenin or Helmholtz are very numerous with some of the more recent being articles by Freeman in "Phil Mag." for Sept. 1942; by Wall in the "Elect.Rev." Nov.7,1941; by Ataka in the Phil. Mag, April 1938; by Wigge in "Arch. f.Elektrol", Nov. 1936; and by Lee and MacDonald in "Wir.Eng." Nov.1935. Most of these articles are extensions of the theorem to some particular network and, depending on the author, the theorem is attributed to either Thevenin or Helmholtz with no mention of the other. In Wigge's article in the "Archiv. for Elektrotechnik" for November 1936, for example, he referred to Helmholtz's theorem as applied to alternating current circuits and then proceeded to state a new theorem called the "Dual Theorem" in which he arrived at a formula for solution of these problems which is in reality a reciprocal formula of the original Helmholtz formula.

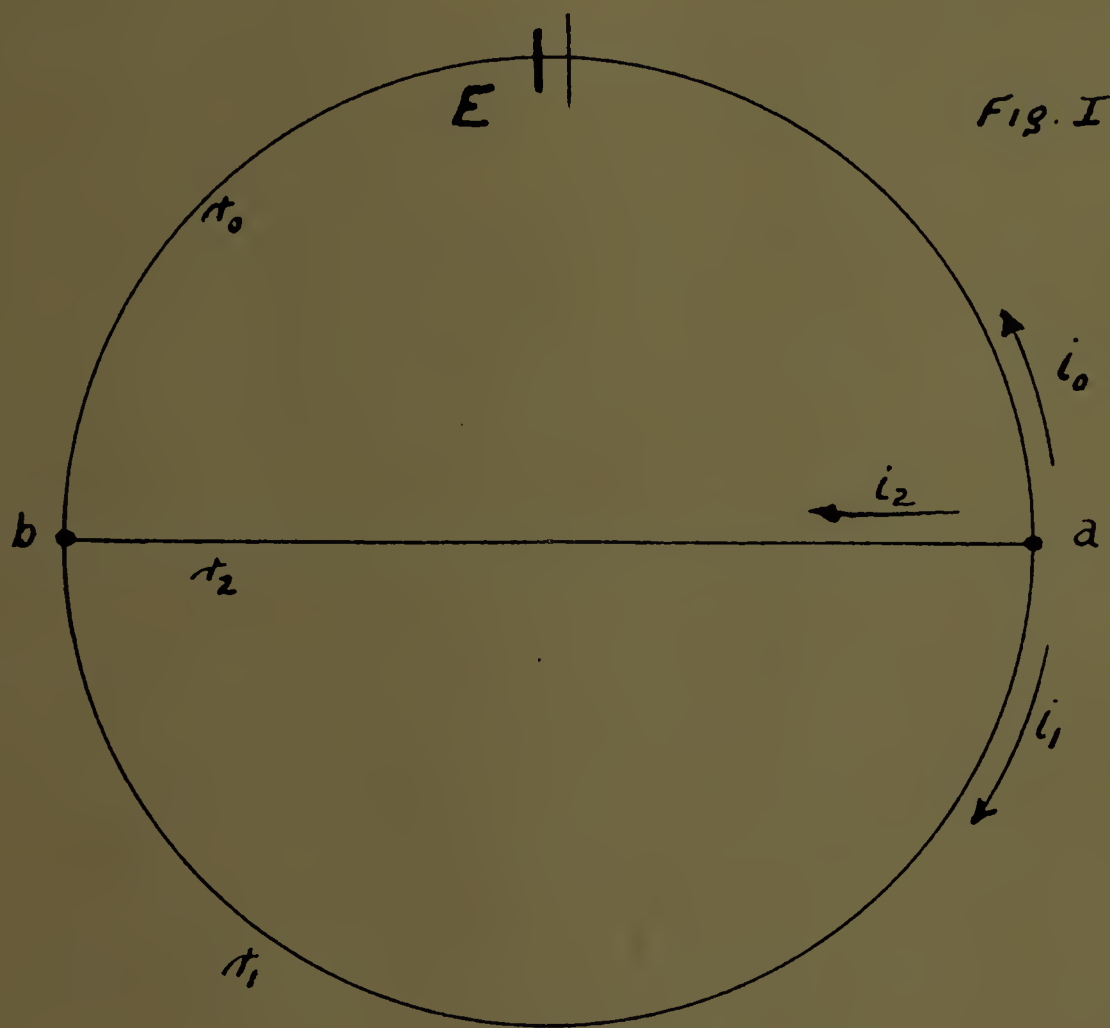
Little wonder is experienced after such a heterogeneous historical career that this theorem should come down to us, less than one hundred years later, with about as many forms in the statement thereof as there have been authors dealing with the subject. Therefore, let us consider no more than five selected statements of the theorem and examine them to note the similarity or differences. After examination let us recommend one statement and proceed to show its applications and values to various of the more prevalent problems.

First, returning to the statement of Helmholtz which as you may recall is that "if two points of a linear network are connected to other conductors, it behaves as a conductor of certain resistance, the magnitude of which can be calculated by the ordinary rules for branched networks, and of an E.M.F. equal to the Potential Difference that existed between two points before they were connected by the other conductors."

As the simplest possible example of this seemingly complex statement, we will examine a circuit illustration as presented by Helmholtz[4] (see sketch I on the next page). In this simple circuit, there is an electromotive force (E) and two points (a) and (b), which divide the circuit into two parts of resistance r_0 and r_1 , the former part containing the source of the electromotive force. The potential difference across a-b is then

$$V = \frac{E r_1}{r_0 + r_1}$$

Original Helmholtz Circuit



$$i_0 + i_1 + i_2 = 0 \qquad \text{(at a)}$$

(1) $$i_0 = -\frac{E}{r_0 + r_1} - \frac{V r_1}{r_0 r_1 + r_0 r_2 + r_1 r_2}$$
$$= -\frac{E(r_1 + r_2)}{r_0 r_1 + r_0 r_2 + r_1 r_2}$$

(2) $$i_1 = \frac{E}{r_0 + r_1} - \frac{V r_0}{r_0 r_1 + r_0 r_2 + r_1 r_2} = \frac{E r_2}{r_0 r_1 + r_0 r_2 + r_1 r_2}$$

(3) $$i_2 = \frac{V}{R + r_2} = \frac{E r_1}{r_0 r_1 + r_0 r_2 + r_1 r_2}$$

which becomes the electromotive force required for the equivalent system which is to be connected across a-b.

The resistance of this circuit as measured between points (a) and (b) becomes

$$R = \frac{r_0 r_1}{r_0 + r_1}$$

A resistance r_2 is now connected between these two points (see sketch I). Then Helmholtz states that according to his theorem the currents in each branch are as indicated in formulas 1, 2, and 3 shown accompanying sketch I.

Helmholtz obtains i_0 and i_1 by using a device (see Sketch III) of superimposing ficticious E.M.F's equal to V and -V in with r_2 . The current due to E and to -V is then the first term of the currents i_0 and i_1 since no current flows at that time through r_2. The second term in each equation is the current due to V acting alone.

The minus sign in formula (1) is due to Helmholtz taking the positive direction of i_0 in opposition to E so that at point (a) the sum of all of the currents equals zero, following the law of Kirchhoff.

NOTE:-

It should be noted that if the two points (a) and (b) are short-circuited, the current will obviously be equal to the open-circuit voltage divided by the resistance of the

Auxiliary Circuits for Helmholtz Theorem

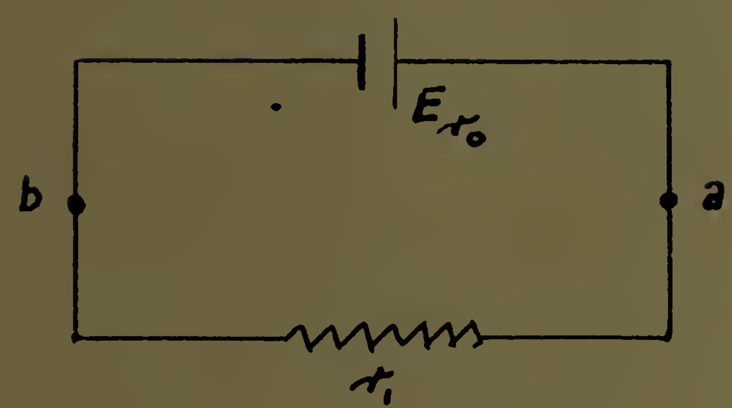


Fig II
(same as "I" minus resistance r_2)

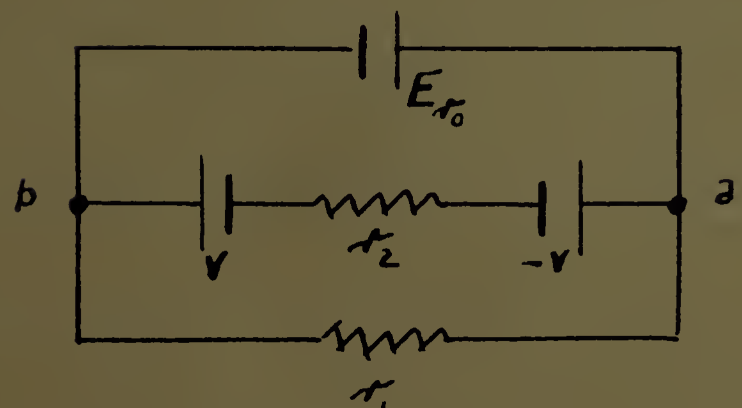


Fig III

(Demonstrating use of fictitious E.M.F.s) network as measured between (a) and (b). This is sometimes useful because the short-circuit current and the network resistance can sometimes be calculated quite simply and the open-circuit voltage thus calculated. Thus two other theorems which often appear under either the title "the short-circuit link theorem" or "the break or cutting point theorem" are seen to be in reality only slightly different statements of the Helmholtz - Thevenin Theorem.

Having presented Helmholtz's original work and thus established the priority of his theorem over that of Thevenin let us now consider four selected statements of the theorem appearing under the name of Thevenin taken from three authoritative textbooks on electric circuit analysis and from a paper published in the Philosophical Magazine and Journal of Science. These will demonstrate the diversity of statement as well as the complexity of terminology which has been introduced.

For the detailed proof as presented for each of these statements see the Abstract Section under the appropriate heading.

From the textbook entitled "Electrical Circuits", a volume of the Principles of Electrical Engineering Series by the electrical engineering staff of the Massachusetts Institute of Technology (See Bibliography) at page 146 we have the following statement.

"Any network of resistance elements and voltage sources if viewed from any two points in the network may be replaced by a voltage source and a resistance in series between the points."

From the textbook entitled "Electrical Circuits and Wave Filters" by A.T.Starr (See Bibliography) we have another statement.

"Any system with two accessible terminals may be replaced by an E.M.F. acting in series with an impedance. The E.M.F. is that between the terminals when they are unconnected externally and the impedance is that presented by the system to the terminals when all of the sources of E.M.F. in the system are replaced by their internal impedances."

From a paper by Hickosabura Ataka of the Heidi College of Technology in Tobaka, Japan published in the Philosophical Magazine and Journal of Science (See Bibliography) we have the following statement of the theorem.

"A current through any branch of a circuit is equal to the current through the same branch when an open circuited E.M.F. acts in the same branch and all other E.M.F's are replaced by their original impedances."

From the textbook entitled "Basic Electricity for Communications" by W.H.Timbie (See Bibliography) we have the following detailed "statement" of Thevenin's Theorem.

"Any two terminal network containing any number of D.C. sources and any number of resistances can be replaced by a single series circuit of one voltage source and one resistance. This equivalent series circuit will deliver to a given load the same power at the same voltage, and the same current that the original circuit will deliver to the same load.

(a) The voltage source of the equivalent circuit is equal to the open circuit voltage of the original circuit.

(b) To find the series resistance of the equivalent circuit:-

(1) Remove the voltage sources of the original circuit and replace them with short circuits.

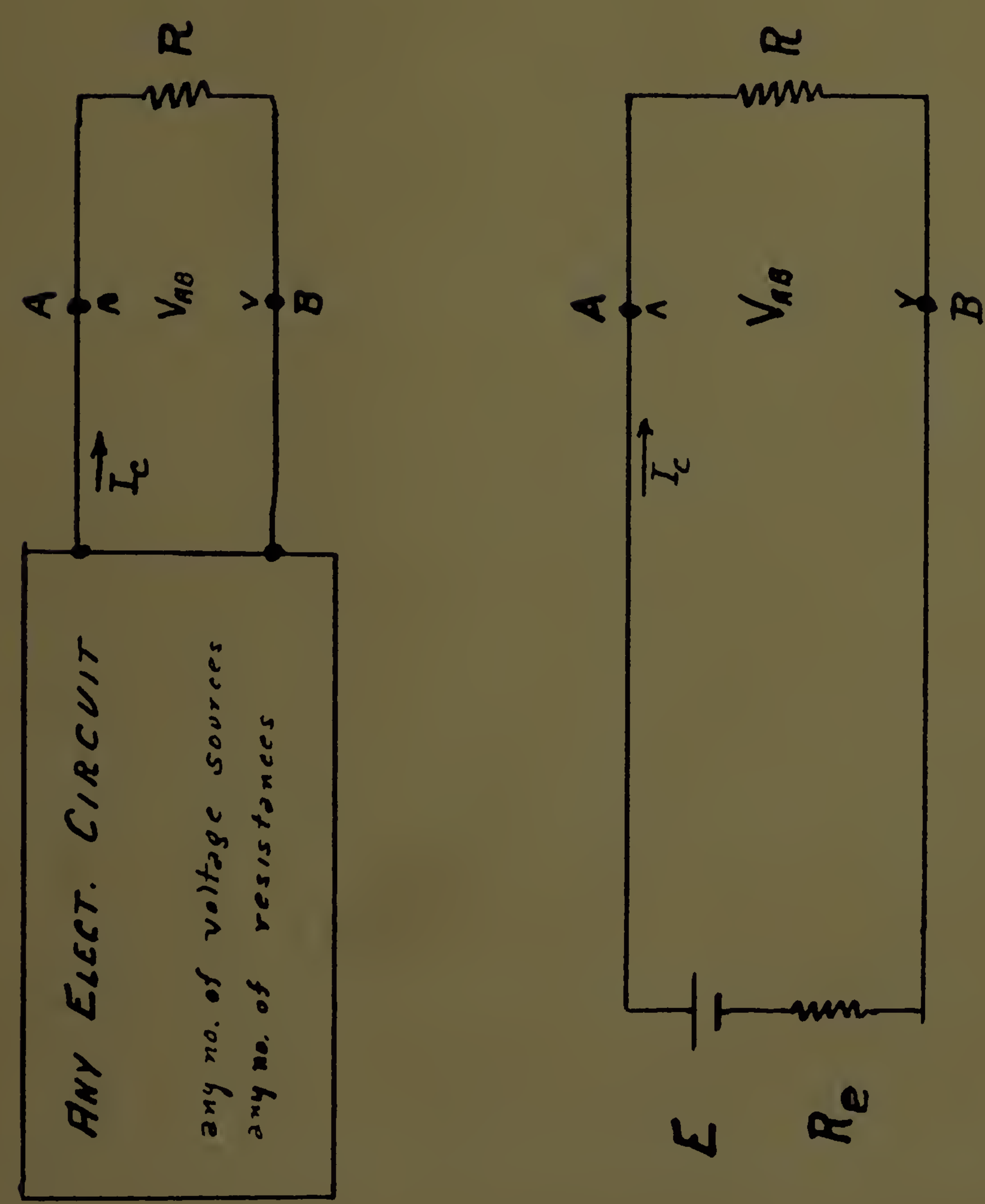
(2) Compute the resistance between the two open terminals of this modified circuit.

(c) The two circuits are equivalent only as far as their terminals are concerned. They do not necessarily consume the same power internally."

By inspecting the circuit diagramed on the next page it will be found that each of the above statements of the theorem will explain the transposition of current, power and voltage which can be supplied to a given resistance (R) by the method of replacing the original circuit with an electromotive force equivalent to that existing between the

HELMHOLTZ - THEVENIN THEOREM

General Circuit Diagram



terminals (a) and (B) in series with a resistance equivalent to that of the original circuit. It will be noted that the power used between (A and (B) remains constant. This theorem does not state that the power used by the equivalent circuit is the same as that of the original circuit, i.e. the Helmholtz-Thevenin Theorem is not an "equivalent power" theorem.

In each of the statements the principle is set forth that, where we have an electrical circuit of any number of voltage sources and any number of resistances and are interested in the electrical properties between any two terminals, we may replace the entire network exclusive of the two terminals by a simple series circuit of one voltage source and one resistance of a size calculated to deliver the same voltage, current and power to the two terminals as was delivered by the more complex circuit.

Thus referring again to the general circuit diagram, we see that in our simple series circuit, the electromotive force E must be equal to the voltage across the terminals under consideration. i.e. $E = V_{ab}$. The resistance of the simple circuit R_e must be such as to supply the same current I_c (the current supplied by the original circuit) to the load R between terminals A and B. Or by Ohm's Law

$$Re = \frac{Vab}{I_c}$$

where I_c may be calculated by Maxwell's equation or from the Wye-Delta Theorem as presented later in this treatise.

If I_c is not known then E may be calculated by computing the open circuit voltage drop V_{ab}, and R_e may be calculated from the series parallel relationships of the original circuit replacing the voltage sources of that circuit by short circuits. This replacing of the original E.M.F's by connections of zero resistance in order to determine R_e seems to be inherent in the Helmholtz theorem. This method will be at best, however, only a close approximation which neglects the internal resistance of the voltage sources, but it may be used to advantage where the resistance values of the circuit are large compared to the resistance of the voltage sources.

In the discussion of this theorem the author has used Direct Current terminology to avoid confusion, but the entire theorem can be applied as well to alternating current circuits with the substitution of the term "impedance" for "resistance". The theorem is applicable also for capacitance circuits.

Since to attempt to make a new statement of the theorem would in this author's opinion only add to rathen than subtract from the confusion existing by the present multitudinous array of "statements", the statement of the theorem as appearing in the text "Principles of Electrical Engineering" by the staff of M.I.T. is presented with a slight addition and a suggested change in title as the most concise and accurate statement of the theorem.

This author would use for a title:-

The Helmholtz - Thevenin Equivalent Circuit

and state the theorem thus:-

Any network of resistance or impedance elements and voltage sources, if viewed from any two points in the network, may be replaced by series circuit consisting of a single voltage source and a single resistance equivalent to the open circuit voltage and resistance across the points.

While it might be wise to present an illustrative problem at this point, the author, in the interest of greater clarity has deferred discussing this theorem further until he presents a second circuit theorem which will aid in the mathematical computation needed in the sample problem.

It is interesting to note, however, at this point that Ataka, in his paper on "An Extension of Thevenin's Theorem", applies the theorem to an alternating current circuit and sets up a new statement of the theorem in which he combines the Thevenin Theorem with the compensation theorem and shows that each of these theorems are merely special cases of his extended theorem. An abstract of Ataka's work is presented in the special section of this treatise reserved for that purpose.

In this abstract Ataka's special proof and his results are adequately portrayed for the use of a student particularly interested in this phase of the work.

THE DELTA- WYE THEOREM

The Delta-Wye theorem attributed to A.E.Kennelly in an article first published in 1899 in Volume 34 of "Electrical World" has not had the diverse expression which has been noted in connection with the Helmholtz-Thevenin theorem. The theorem has been known by such other names as the "Star-Mesh Transformation" and the "Delta-Tee Transformation" but the statement of the theorem is not subject to confusion.

Many times in the computation required to apply the Helmholtz-Thevenin theorem, conditions arise wherein the net resistance between two points in a circuit cannot be computed by series - parallel relationships. Where we have three resistances forming a triangular shaped figure (Delta), we may substitute a wye (Y) shaped group of resistances with absolute equality. Similarly a wye shaped circuit section may be replaced by a delta. To all intents and purposes this is the statement of the theorem which we will now discuss by means of an illustration.

Let us consider the Wheatstone Bridge circuit as illustrated in Sketch (1) on the next page. By Thevenin's theorem we might want to compute the galvanometer current and thus it will be necessary to compute the net resistance between X and Y. By considering the resistances above (sketch 2) and

WHEATSTONE BRIDGE
CIRCUIT

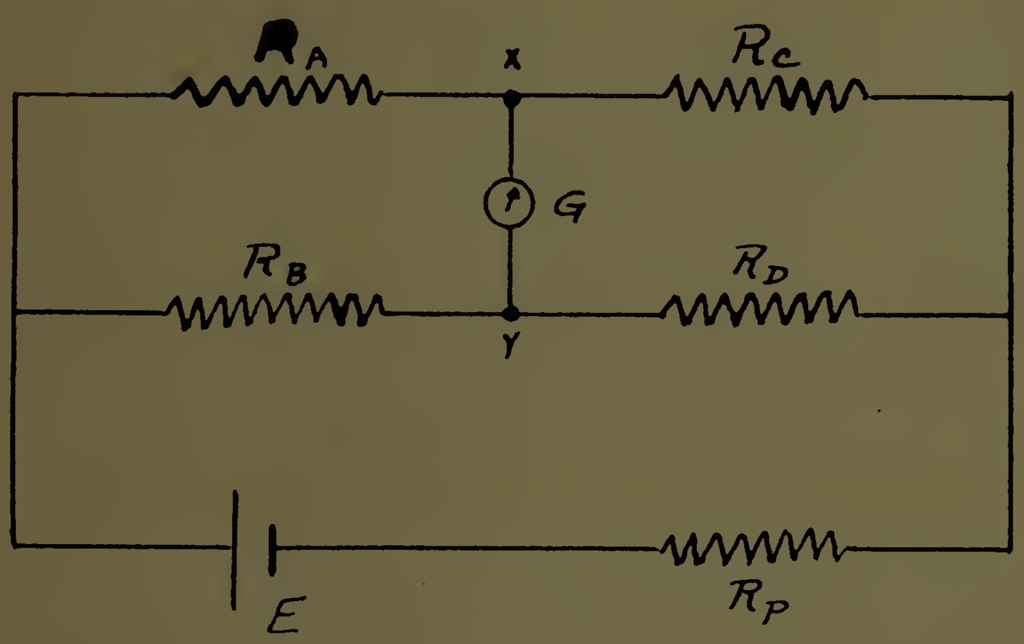


Fig (1)
(original circuit)

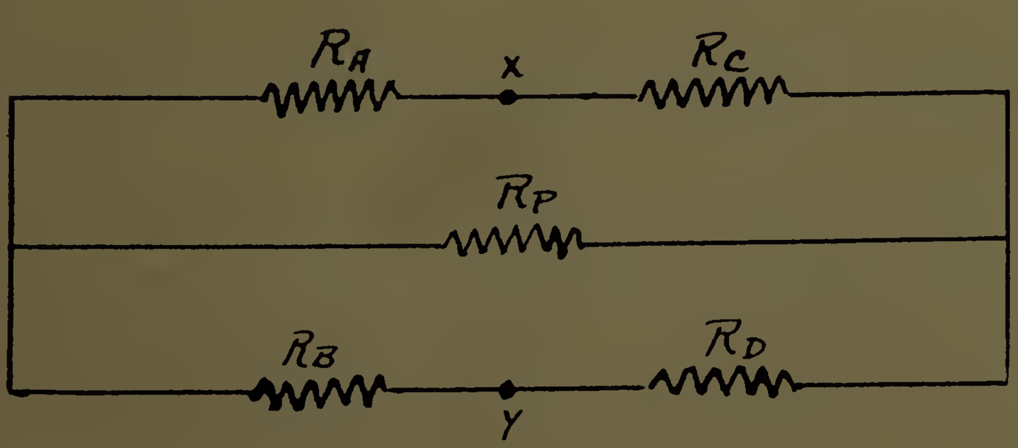


Fig (2)
(resis. alone - voltage removed)

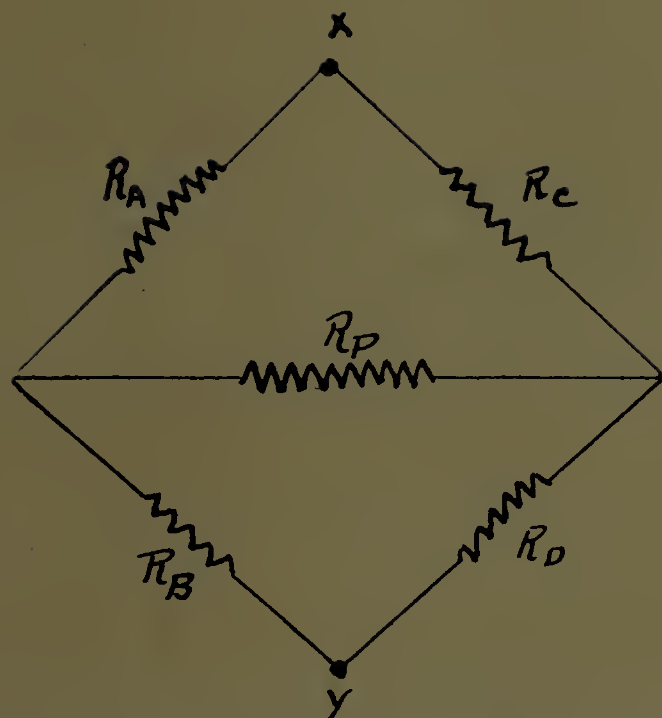
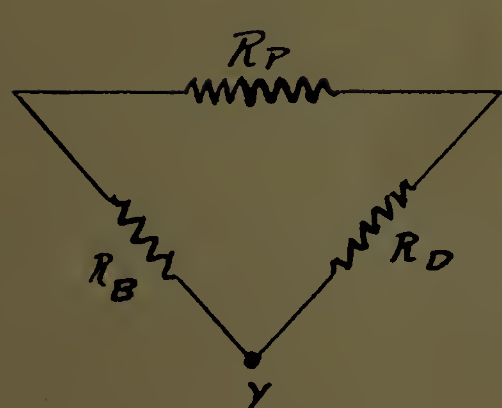


Fig (3)

(resis. in delta form)



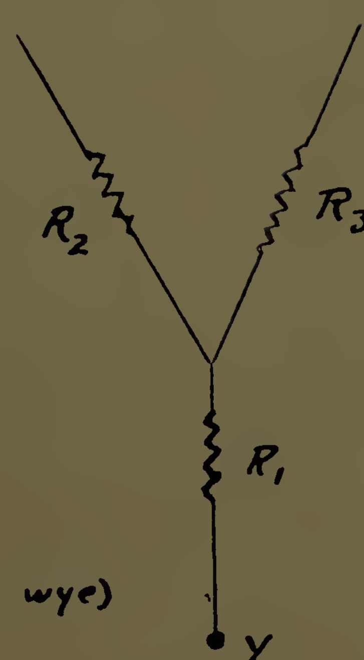


Fig (4)

(lower delta with equivalent wye)

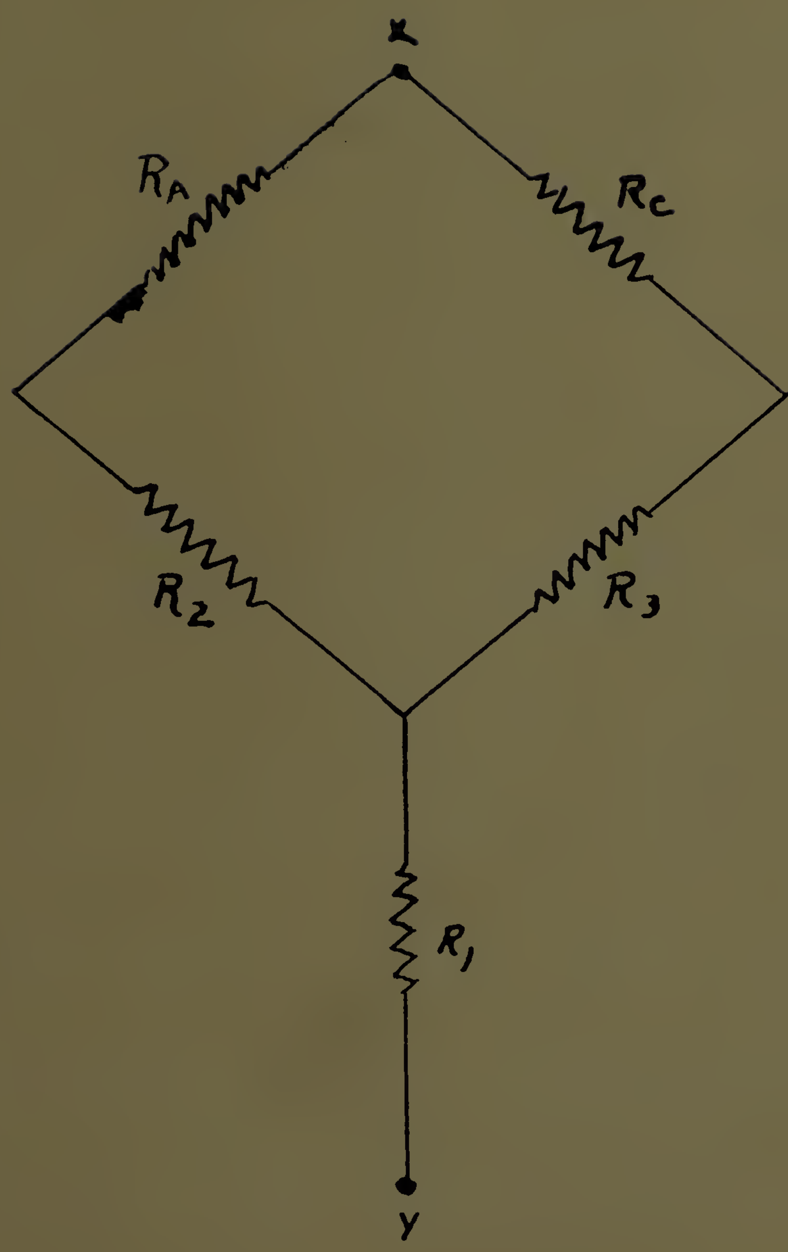


Fig (4)

Equivalent Series - Parrallel Circuit
for Wheatstone
Bridge

General Equations Expressing Relationship Between Original and Equivalent Resistances

$$R_1 + R_2 = \frac{(R_P + R_D)(R_B)}{R_P + R_D + R_B}$$

$$R_2 + R_3 = \frac{(R_B + R_D)(R_P)}{R_P + R_D + R_B}$$

$$R_3 + R_1 = \frac{(R_P + R_B)(R_D)}{R_P + R_D + R_B}$$

Solving the three equations above to get values which can be used in a transformation from a Delta to a Wye circuit, we have:-

$$R_1 = \frac{(R_D)(R_B)}{R_B + R_D + R_P}$$

$$R_2 = \frac{(R_P)(R_B)}{R_B + R_D + R_P}$$

and $$R_3 = \frac{(R_P)(R_D)}{R_B + R_D + R_P}$$

Solving the first three equations on the previous page for values to be used in a transformation from a Wye to a Delta circuit, we have :-

$$R_P = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

$$R_D = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_2}$$

and $$R_B = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_3}$$

rearranging as in sketch (3) we can see the delta arrangement from which the theorem derives its name. If one of these deltas can be replaced by a wye as in sketch (4),it would be possible to have a series - parallel circuit as in sketch (5) which would be relatively simple to compute.

In order for a wye to be equivalent to a delta, the resistance viewed from any pair of wye terminals must be the same as the resistance viewed from the corresponding pair of delta terminals. Thus as viewed in sketch (4) the series resistance totals of the wye ($R_1 + R_2$); $R_2 + R_3$); and $R_3 + R_1$) must equal the respective parallel resistances of the delta circuit.

Accompanying the circuit sketches, we have the formulas which are used in the respective transformations from a delta to a wye and from a wye to a delta circuit.

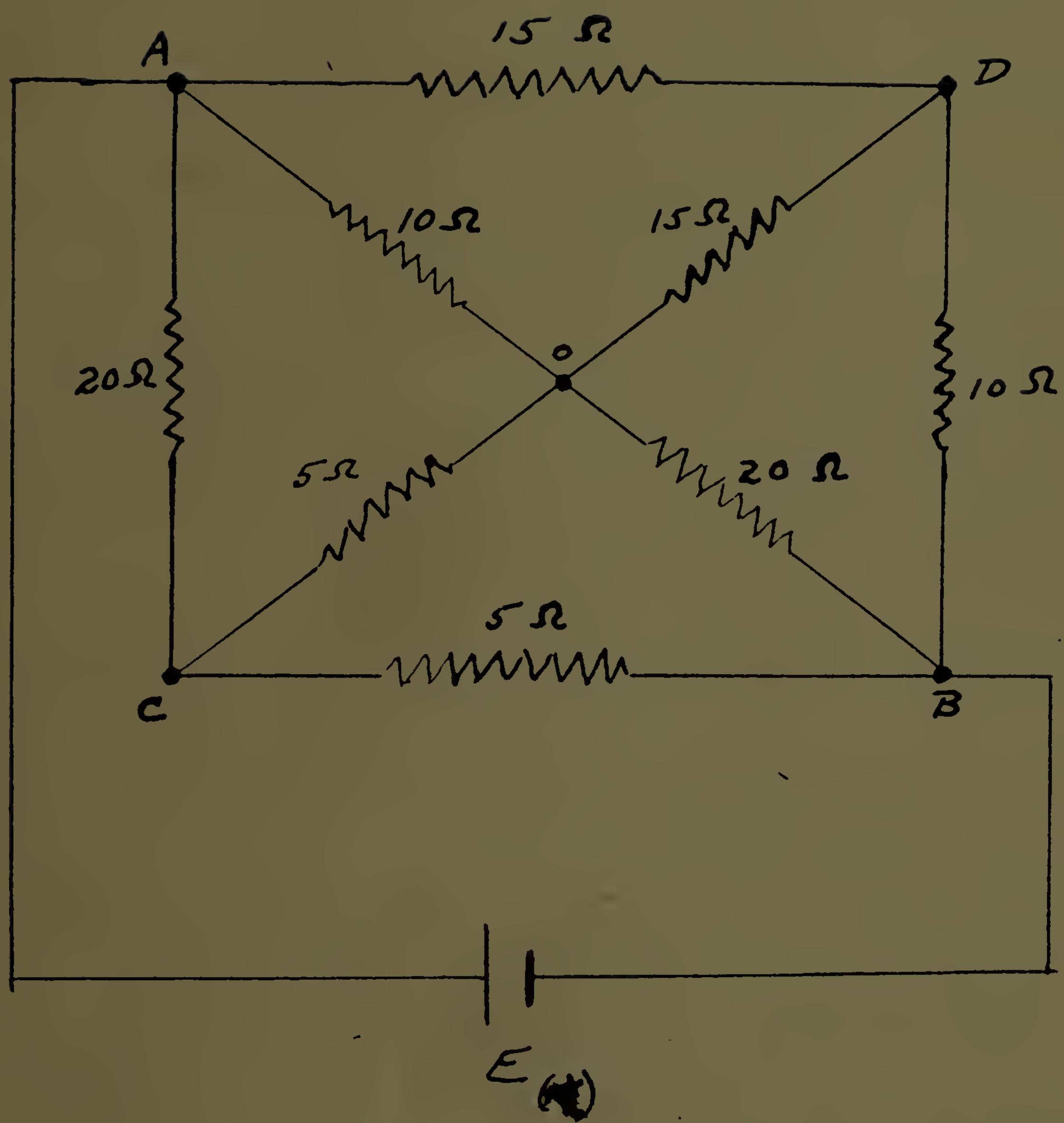
Thus we are now equipped with six equations which will enable us to solve any complex circuit where we can break down the original circuit into its component delta or wye forms and thus resolve the entire circuit into a series - parallel relationship which may be solved by use of the series and parallel rules for the sum of the resistances in a circuit.

Before making use of this theorem to aid in the demonstration of the application of the Helmholtz-Thevenin Theorem let us first by an illustrative problem show the value of the application of the delta-wye theorem to a complicated cincuit problem in Direct current.

The following plates in order of appearance show a circuit of eight resistances (See Plate I) in which the problem is to determine the current being drawn from the battery source. The Plates in order show first the solution of the problem by use of Maxwell's Equations involving a long tedious determinant solution which is susceptible of many errors.

Secondly a solution by means of the so-called "Doolittle" Statistical method of multiple corelation which is in reality a statistical solution of the same five simultaneous equations as are used in the first method.

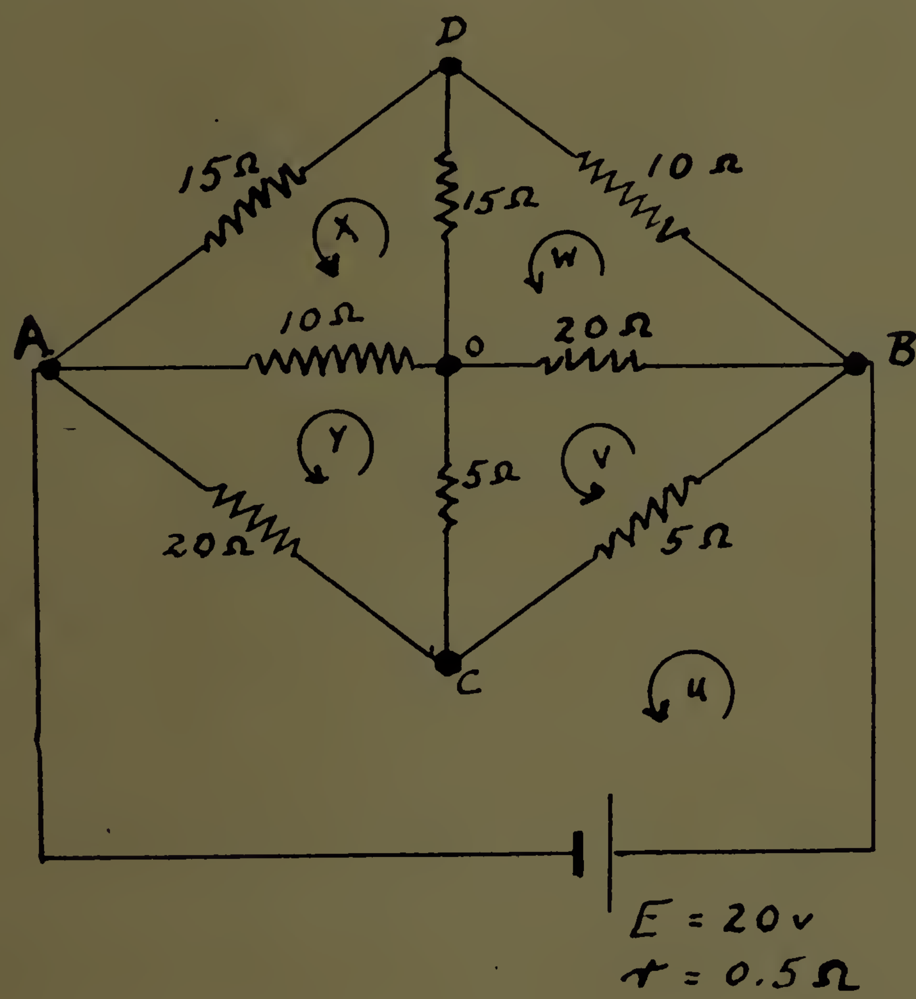
Third, the author has presented the solution by the application of the Delta-Wye theorem. An examination of each of these solutions will conclusively show the comparative ease with which a complicated circuit can be solved by means of this Delta-Wye Theorem.



SAMPLE CIRCUIT
TO BE SOLVED BY
DELTA - WYE COMBINATIONS

~ SAME ORIGINAL CIRCUIT ~

SOLUTION by Maxwell's EQUATIONS



MAXWELL EQUATIONS

(I) $25.5u - 5v - 20Y + 0 + 0 = +20$

(II) $-5u + 30v - 5Y - 20W + 0 = 0$

(III) $-20u - 5v + 35Y + 0 - 10X = 0$

(IV) $0 \quad - 20Y + 0 \quad + 45W - 15X = 0$

(V) $0 \quad + 0 - 10Y - 15W + 40X = 0$

Problem :- to solve for 'U' (current in battery circuit)

SOLUTION OF MAXWELL'S EQUATIONS BY DETERMINANTS

$$u = \frac{D_1}{D}$$

$$D = \begin{vmatrix} 25.5 & -5 & -20 & +0 & +0 \\ -5 & +30 & -5 & -20 & +0 \\ -20 & -5 & +35 & +0 & -10 \\ 0 & -20 & +0 & +45 & -15 \\ 0 & +0 & -10 & -15 & +40 \end{vmatrix} \qquad D_1 = \begin{vmatrix} 20 & -5 & -20 & +0 & +0 \\ 0 & +30 & -5 & -20 & +0 \\ 0 & -5 & +35 & +0 & -10 \\ 0 & -20 & +0 & +45 & -15 \\ 0 & +0 & -10 & -15 & +40 \end{vmatrix}$$

Solve for D

$$D = 25.5 \begin{vmatrix} 30 & -5 & -20 & +0 \\ -5 & +35 & +0 & -10 \\ -20 & +0 & +45 & -15 \\ +0 & -10 & -15 & +40 \end{vmatrix} - (-5) \begin{vmatrix} -5 & -20 & +0 & +0 \\ -5 & +35 & +0 & -10 \\ -20 & +0 & +45 & -15 \\ +0 & -10 & -15 & +40 \end{vmatrix} + (-20) \begin{vmatrix} -5 & -20 & +0 & +0 \\ +30 & -5 & -20 & +0 \\ -20 & +0 & +45 & -15 \\ +0 & -10 & -15 & +40 \end{vmatrix}$$

$$= (25.5)(30) \begin{vmatrix} +35 & +0 & -10 \\ 0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} - 25.5(-5) \begin{vmatrix} -5 & -20 & +0 \\ +0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} + 25.5(-20) \begin{vmatrix} -5 & -20 & +0 \\ +35 & +0 & -10 \\ -10 & -15 & +40 \end{vmatrix}$$

$$+ 5(-5) \begin{vmatrix} 35 & +0 & -10 \\ 0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} - 5(-5) \begin{vmatrix} -20 & +0 & +0 \\ +0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} + 5(-20) \begin{vmatrix} -20 & +0 & +0 \\ 35 & +0 & -10 \\ -10 & -15 & +40 \end{vmatrix}$$

$$- 20(-5) \begin{vmatrix} -5 & -20 & +0 \\ +0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} + 20(30) \begin{vmatrix} -20 & +0 & +0 \\ +0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} - 20(-20) \begin{vmatrix} -20 & +0 & +0 \\ -5 & -20 & +0 \\ -10 & -15 & +40 \end{vmatrix}$$

Now let

$$a = \begin{vmatrix} 35 & +0 & -10 \\ 0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} = 35 \begin{vmatrix} 45 & -15 \\ -15 & +40 \end{vmatrix} - 10 \begin{vmatrix} 0 & +10 \\ 45 & -15 \end{vmatrix} = 50,625$$

$$b = \begin{vmatrix} -5 & -20 & +0 \\ +0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} = -5 \begin{vmatrix} 45 & -15 \\ -15 & +40 \end{vmatrix} - 10 \begin{vmatrix} -20 & +0 \\ 45 & -15 \end{vmatrix} = -10,875$$

$$c = \begin{vmatrix} -20 & +0 & +0 \\ +0 & +45 & -15 \\ -10 & -15 & +40 \end{vmatrix} = -20 \begin{vmatrix} 45 & -15 \\ -15 & +40 \end{vmatrix} - 10 \begin{vmatrix} 0 & +0 \\ 45 & -15 \end{vmatrix} = -31,500$$

Solution by Determinants (cont.)

let

$$d = \begin{vmatrix} -5 & -20 & +0 \\ +35 & +0 & -10 \\ -10 & -15 & +40 \end{vmatrix} = -5\begin{vmatrix} 0 & -10 \\ -15 & +40 \end{vmatrix} -35\begin{vmatrix} -20 & +0 \\ -15 & +40 \end{vmatrix} -10\begin{vmatrix} -20 & +0 \\ +0 & -10 \end{vmatrix} = 26{,}750$$

$$e = \begin{vmatrix} -20 & +0 & +0 \\ +35 & +0 & -10 \\ -10 & -15 & +40 \end{vmatrix} = -20\begin{vmatrix} 0 & -10 \\ -15 & +40 \end{vmatrix} -35\begin{vmatrix} 0 & +0 \\ -15 & +40 \end{vmatrix} -10\begin{vmatrix} 0 & +0 \\ -20 & +0 \end{vmatrix} = 3{,}000$$

$$f = \begin{vmatrix} -20 & +0 & +0 \\ -5 & -20 & +0 \\ -10 & -15 & +40 \end{vmatrix} = -20\begin{vmatrix} -20 & +0 \\ -15 & +40 \end{vmatrix} +5\begin{vmatrix} 0 & +0 \\ -15 & +40 \end{vmatrix} -10\begin{vmatrix} 0 & +0 \\ -20 & +0 \end{vmatrix} = 16{,}000$$

$$\therefore D = 25.5\left[30(50625) + 5(-10875) - 20(26750)\right]$$

$$+ 5\left[-5(50625) + 5(-31500) - 20(3000)\right]$$

$$+20\left[5(-10875) + 30(-31500) + 20(16000)\right] =$$

$$= 7{,}758{,}437.5$$

$$D_1 = 20\begin{vmatrix} 30 & -5 & -20 & +0 \\ -5 & +35 & +0 & -10 \\ -20 & +0 & +45 & -15 \\ 0 & -10 & -15 & +40 \end{vmatrix} = 20(30)(35)\begin{vmatrix} 45 & -15 \\ -15 & +40 \end{vmatrix} + 20(30)(-10)\begin{vmatrix} 0 & -10 \\ 45 & -15 \end{vmatrix}$$

$$-20(-5)(-5)\begin{vmatrix} 45 & -15 \\ -15 & +40 \end{vmatrix} - 20(-5)(-10)\begin{vmatrix} -20 & +0 \\ 45 & -15 \end{vmatrix} + 20(-20)(-5)\begin{vmatrix} 0 & -10 \\ -15 & +40 \end{vmatrix}$$

$$-20(-20)(35)\begin{vmatrix} -20 & +0 \\ -15 & +40 \end{vmatrix} + 20(-20)(-10)\begin{vmatrix} -20 & +0 \\ 0 & -10 \end{vmatrix} =$$

$$= 18{,}587{,}500$$

$$\therefore\ u = \frac{18{,}587{,}500}{7{,}758{,}437.5} = 2.4^{\pm} \text{ amps}$$

Solution of Maxwell's equations
by Statistical Methods

(Note:- Eq. no. are same as in previous method)

		Sum
I	25.5 - 5 - 20 + 0 + 0 = 20	20.5
$\frac{I}{-25.5}$ = I'	-1 + 0.1961 + .7843 + 0 + 0 = -.7843	- .8039
II	-5 + 30 - 5 - 20 + 0 = 0	0
.1961 (I):	5 - 0.9804 - 3.9216 + 0 + 0 = 3.9216	4.0196
Σ_2	0 + 29.0196 - 8.9216 - 20 + 0 = 3.9216	4.0196
$\frac{\Sigma_2}{-29+}$ = II'	- 1 + 0.3074 + .6892 + 0 = .1351	.1385
III	-20 - 5 + 35 + 0 - 10 = 0	0
(.7843) I	20 - 3.9215 - 15.686 + 0 + 0 = 15.686	16.0785
(.3074) Σ_2	0 + 8.9215 - 2.7425 - 6.1480 + 0 = 1.2055	1.2365
Σ_3	0 + 0 + 16.5715 - 6.1480 - 10 - 16.8915	17.3150
$\frac{\Sigma_3}{-16+}$ = III'	- 1 + .3709 + .6034 - -1.0193	- 1.0450
IV	0 - 20 + 0 + 45 - 15 = 0	+10
(.6892) Σ_2	0 + 20 - 6.148 - 17.784 + 0 = 2.7022	-1.2298
(.3709) Σ_3	0 + 0 + 6.148 - 2.2803 - 3.7097 = 6.2651	6.4231
Σ_4	0 + 0 + 0 + 24.9351 - 18.7097 = 8.9683	15.1943
$\frac{\Sigma_4}{-24.9}$ = IV'	- 1 + .7502 = -.3798	-.6298
V	0 + 0 - 10 - 15 + 40 = 0	+ 15
(.6043) Σ_3	10 - 3.7094 - 6.034 - 10.1923	10.4489
(.7502) Σ_4	0 + 18.7094 - 14.036 = 6.7280	11.4014
	19.93 = 16.9103	36.8403
V'	-1 = -.8485	-1.8485

Solution by Stat. Methods (Cont.)

By Eq. V'

$X = 0.8485$ amps

By Eq IV'

$W = .3798 + .7502(.8485) = 1.0164$ amps

By Eq III'

$Y = 1.0193 + .3709(1.0164) + (.6039)(.8485) =$

$= 1.9083$ amps

By Eq II'

$V = -.1351 + .3709(1.9083) + (.6892)(1.0164) =$

$= 1.2731$ amps

By Eq I'

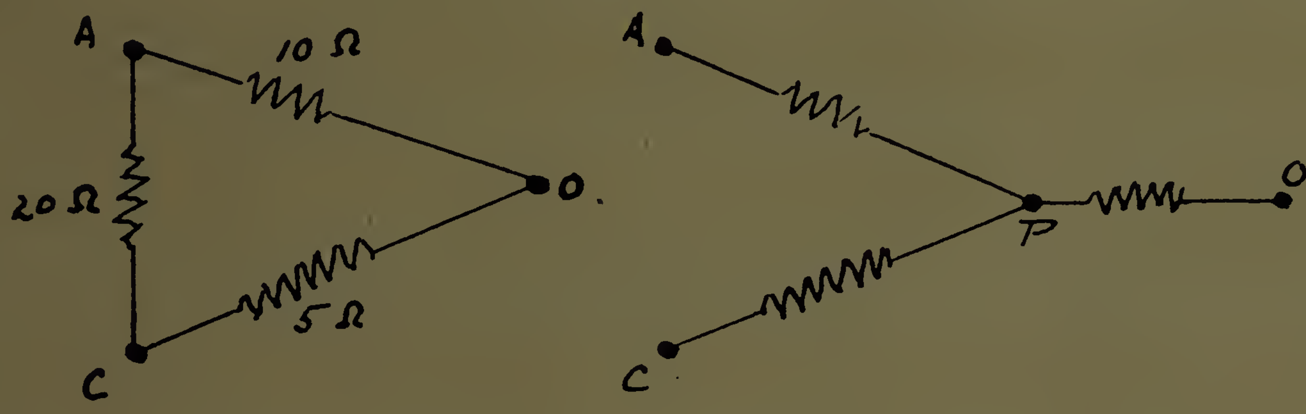
$u = .7843 + (.1961)(1.2731) + (.7843)(1.9083) =$

$= 2.5306$ amps

Note :- this is the value desired

Comparison of values obtained by three Methods

Method	1	—	2.4 ±	amps
"	2	—	2.4 ±	"
"	3	—	2.53	"

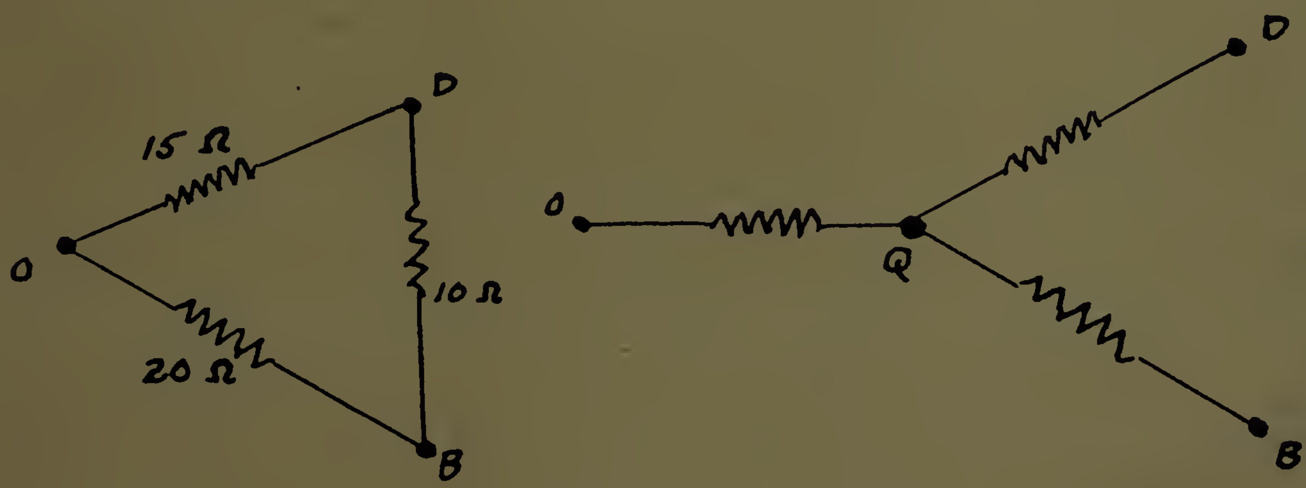


Resistance AP = $\frac{10 \times 20}{35}$ = 5.72 Ω

" CP = $\frac{5 \times 20}{35}$ = 2.86 Ω

" OP = $\frac{5 \times 10}{35}$ = 1.45 Ω

TRANSFORMATION OF DELTA ACO
TO EQUIVALENT WYE

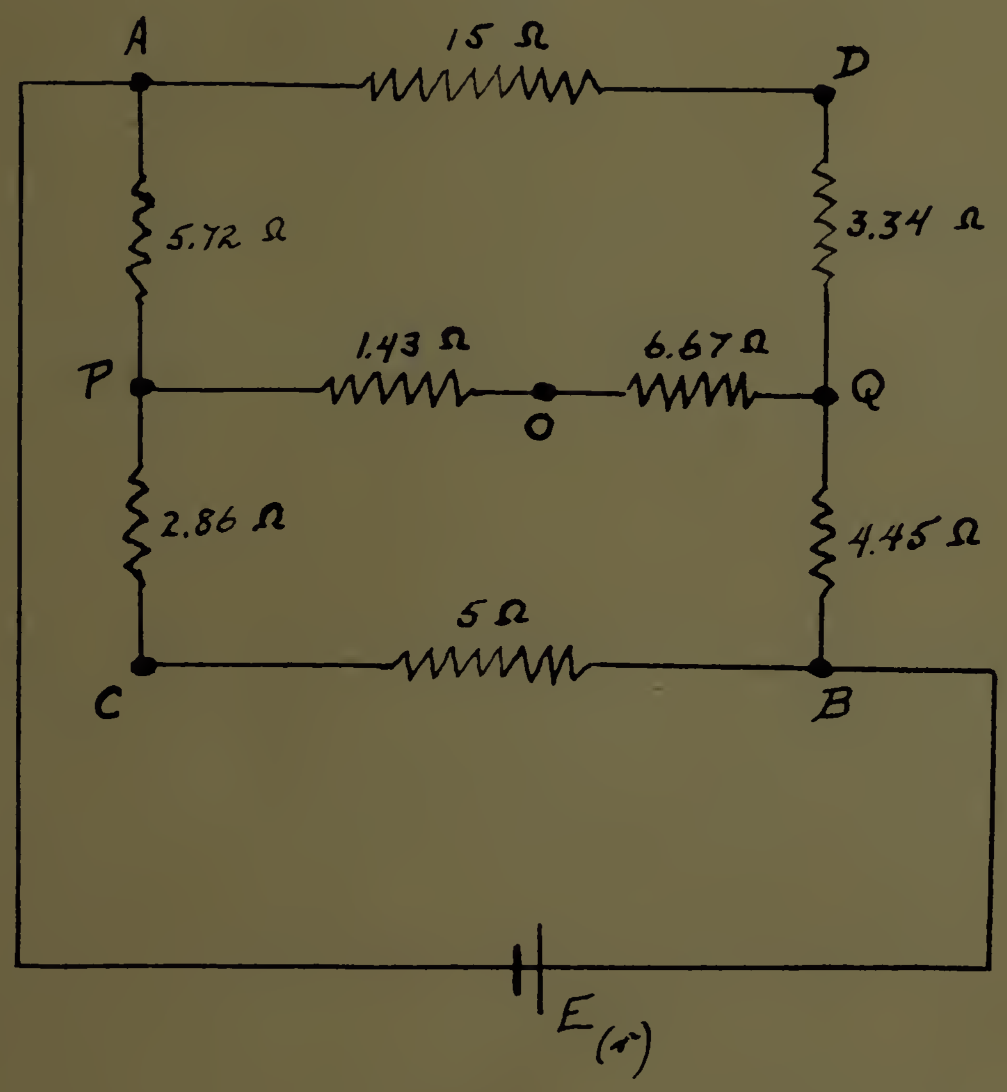


Resistance $OQ = \dfrac{15 \times 20}{45} = 6.67\ \Omega.$

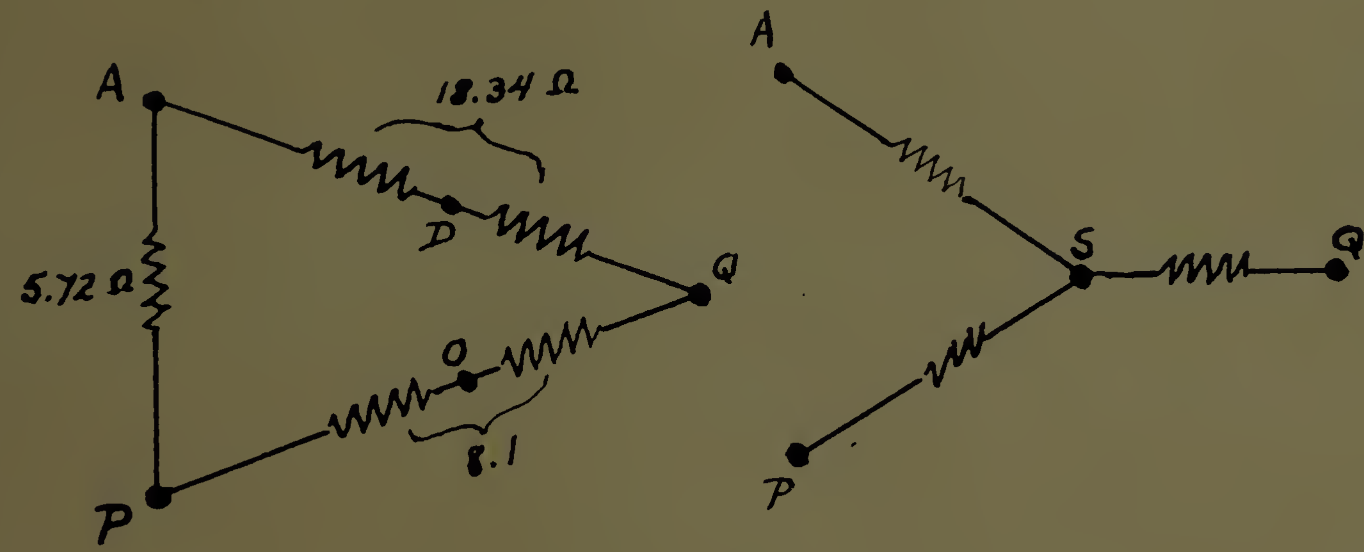
" $QD = \dfrac{15 \times 10}{45} = 3.34\ \Omega$

" $QB = \dfrac{20 \times 10}{45} = 4.45\ \Omega$

TRANSFORMATION OF DELTA ODB TO EQUIVALENT WYE



REVISED CIRCUIT AFTER
FIRST TWO TRANSFORMATIONS

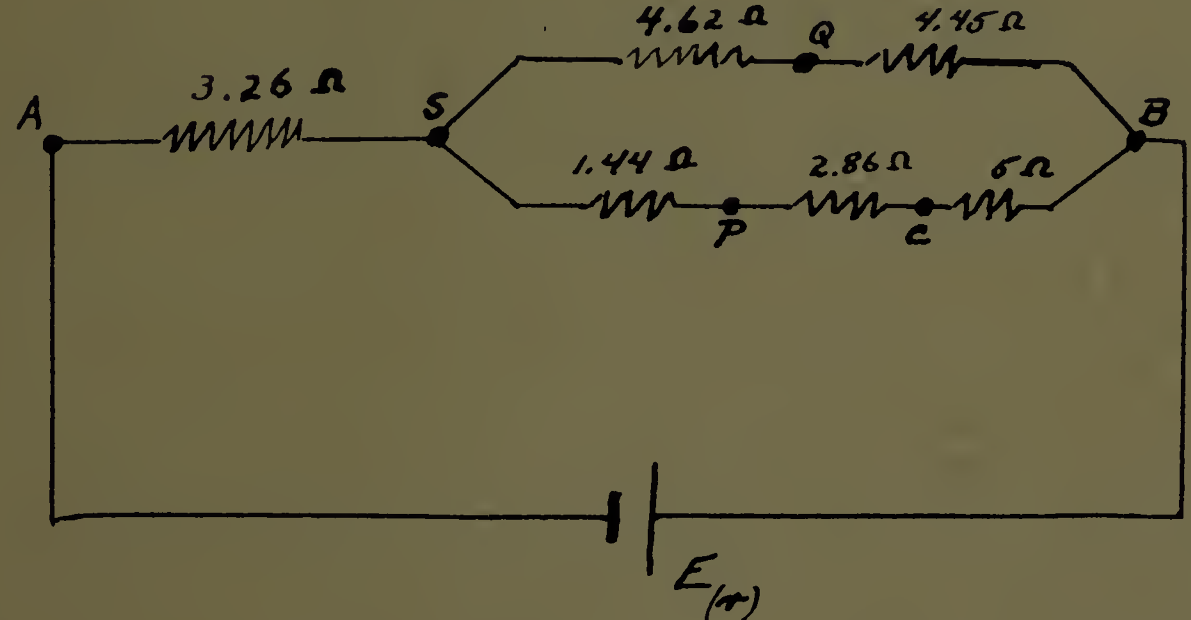


Resistance $\quad AS = \dfrac{(5.72)(18.34)}{32.16} = 3.26\ \Omega$

" $\quad PS = \dfrac{(5.72)(8.1)}{32.16} = 1.44\ \Omega$

" $\quad SQ = \dfrac{(8.1)(18.34)}{32.16} = 4.62\ \Omega$

TRANSFORMATION OF DELTA AQP
TO EQUIVALENT WYE



Resistance $AB = 3.26 + \frac{(9.07)(9.30)}{18.37}$

$= 7.85\ \Omega$

Note: if $E = 20V$, $r = 0.5\Omega$ } $I = \frac{20}{8.35} = 2.4 \pm$ amps

FINAL FORM OF EQUIVALENT SERIES-PARALLEL CIRCUIT

ILLUSTRATION OF THE HELMHOLTZ-THEVENIN THEOREM

To illustrate the Helmholtz-Thevenin theorem a bridge circuit has again been chosen. In this case the entire circuit problem has been based upon a piece of demonstration apparatus constructed by the author in the Physics Laboratory at Massachusetts State College. This apparatus was designed for the prime purpose of illustrating to a class or laboratory group the practicality of using both the Delta-Wye Theorem and the Helmholtz Theorem for equivalent circuits where knowledge of the properties of a particular portion of a given circuit are required.

Let us say that, in the bridge circuit as illustrated on the next page, we are interested in the properties of the circuit between terminals C and D or in other words we would like to know the current flowing through resistance R_5 which is representative of a galvanometer of 200 ohms resistance.

Since the resistance of equivalent circuit by the Helmholtz-Thevenin theorem must be such as to supply the same current to resistance R_5 as was supplied by the original circuit, it is necessary to compute the voltage drop across c - d. Instead of using the circuit formulas of Maxwell and the resultant determinant solution, we will first transform the delta ACD into its equivalent wye and derive a series -

parallel circuit which can be solved then for the voltage drop across c-d. This value obtained will be the required E for the Helmholtz circuit.

By replacing the voltage of the original source by a short circuit a - b and neglecting the internal battery resistance and the resistance between (c) and (d) we compute the equivalent resistance of the circuit. By use of Ohm's Law for the relationship between voltage, resistance and current, the calculation of the current through R_5 (or through a galvanometer) would then be the equivalent voltage found above divided by the resistance of the equivalent circuit as found above.

A calculation of the currents by the Maxwell equations plus a statistical solution of the three equations indicates that the current as found by the Helmholtz-Thevenin circuit is equal to the current calculated by the ordinary means.

An adaptation of this same bridge circuit where the characteristics of the circuit between points (B) and (D) are desired is also illustrated in the following pages.

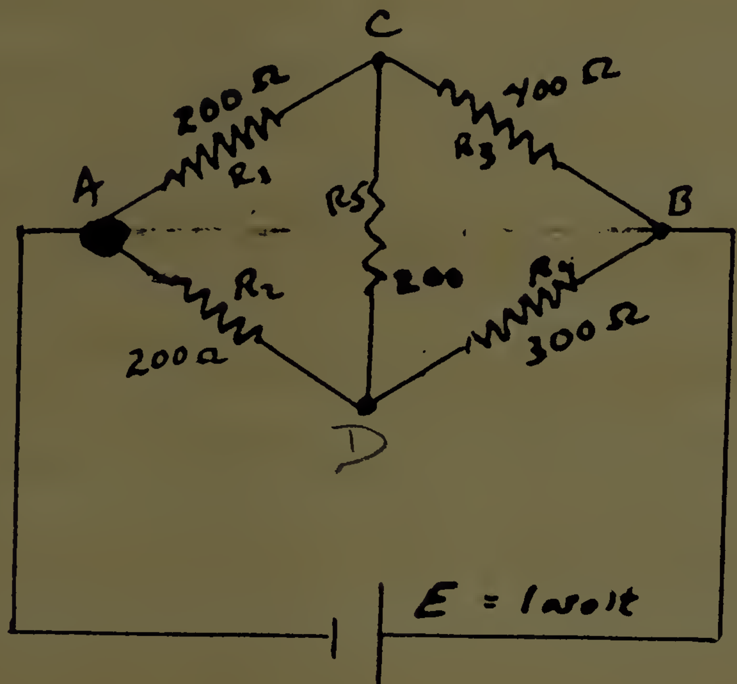


Fig (1)

Original circuit

Find current thru R_5 = ?

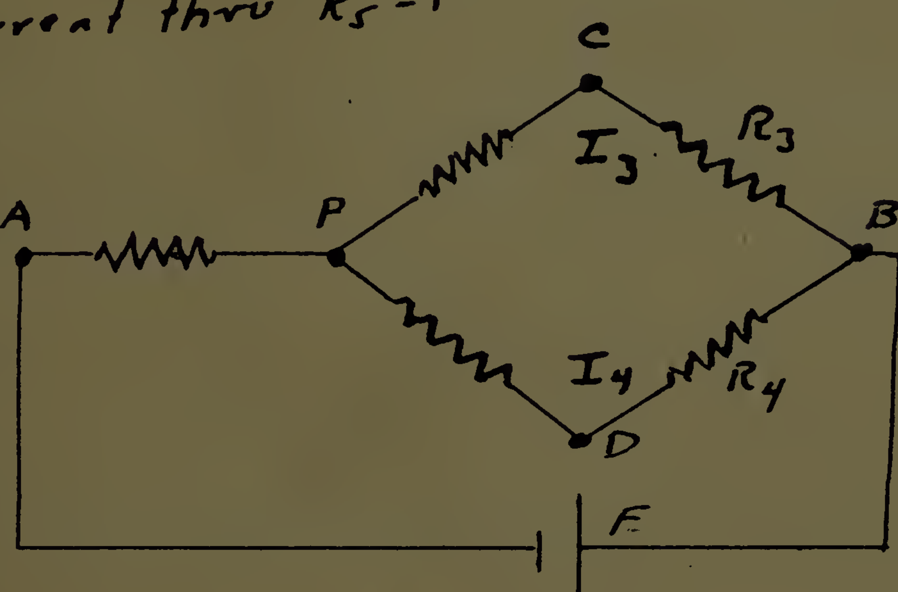


Fig (2)

$$AP = PC = PD = \frac{(AC)(AD)}{\Sigma R} = \frac{200 \times 200}{600} = 66.7\ \Omega$$

Series - Parallel Equivalent by Δ-Y
(Note R_5 is omitted

Thevenin Circuits

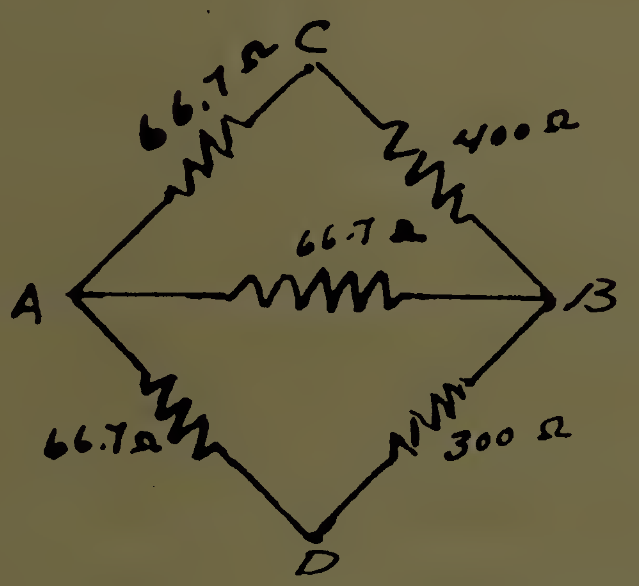


Fig (3)

Above circuit to det. Re

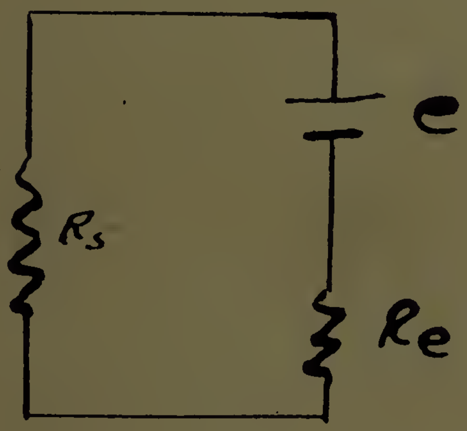


Fig (4)

Final transformed circuit

Computations

From Fig 2

$$V_{CD} = R_3 I_3 - R_4 I_4$$

where

$$I_3 = \frac{R_{PDB}}{R_{AP} R_{PCD} + R_{AP} R_{PDB} + R_{PCD} R_{PDB}} = .00162 \text{ amp}$$

and similarly $I_4 = .00205$ amps

$$\therefore V_{CD} = 400(.00162) - 300(.00205) = .031 \text{ volts}$$

From Fig 3

$$Re = \frac{(R_{ACB})(R_{ADB})(R_{AB})}{(R_{AD})(R_{ACD}) + (R_{AD})(R_{ADB}) + (R_{ACD})(R_{ADB})}$$

$$= 50.33 \ \Omega$$

From Fig 4

The current thru R_5

$$= I = \frac{e}{Re + R_5} = \frac{.031}{250.33} = .000124 \text{ amps}$$

Note:-

A check on this value is shown on the nextpage

Computations made on the basis of E.M.F = 1 volt
Using proper multiplier other E.M.F's may be used.

SOLUTION OF LAB. CIRCUIT BY "MAXWELL"

$$\left.\begin{aligned} 600u - 300V - 200W &= 1 \\ -300u + 900V - 200W &= 0 \\ -200u - 200V + 600W &= 0 \end{aligned}\right\}$$

I'	-1	$+.5$	$+.33$	$=$	$-.0017$
II	-300	$+900$	-200	$=$	0
$.5(I)$	300	-150	-100	$=$	$.5$
$\mathcal{E}_2$		750	-300	$=$	$.5$
II'		-1	$+.4$	$=$	$-.00067$
III	-200	-200	$+600$	$=$	0
$.33\,I$	200	-100	-67	$=$	$.33$
$.4\,\mathcal{E}_2$		300	-120	$=$	$.20$
$\mathcal{E}_3$			413	$=$	$.53$

$$\therefore \quad W = \frac{.53}{413} = .00128$$

$$V = 0.00118$$

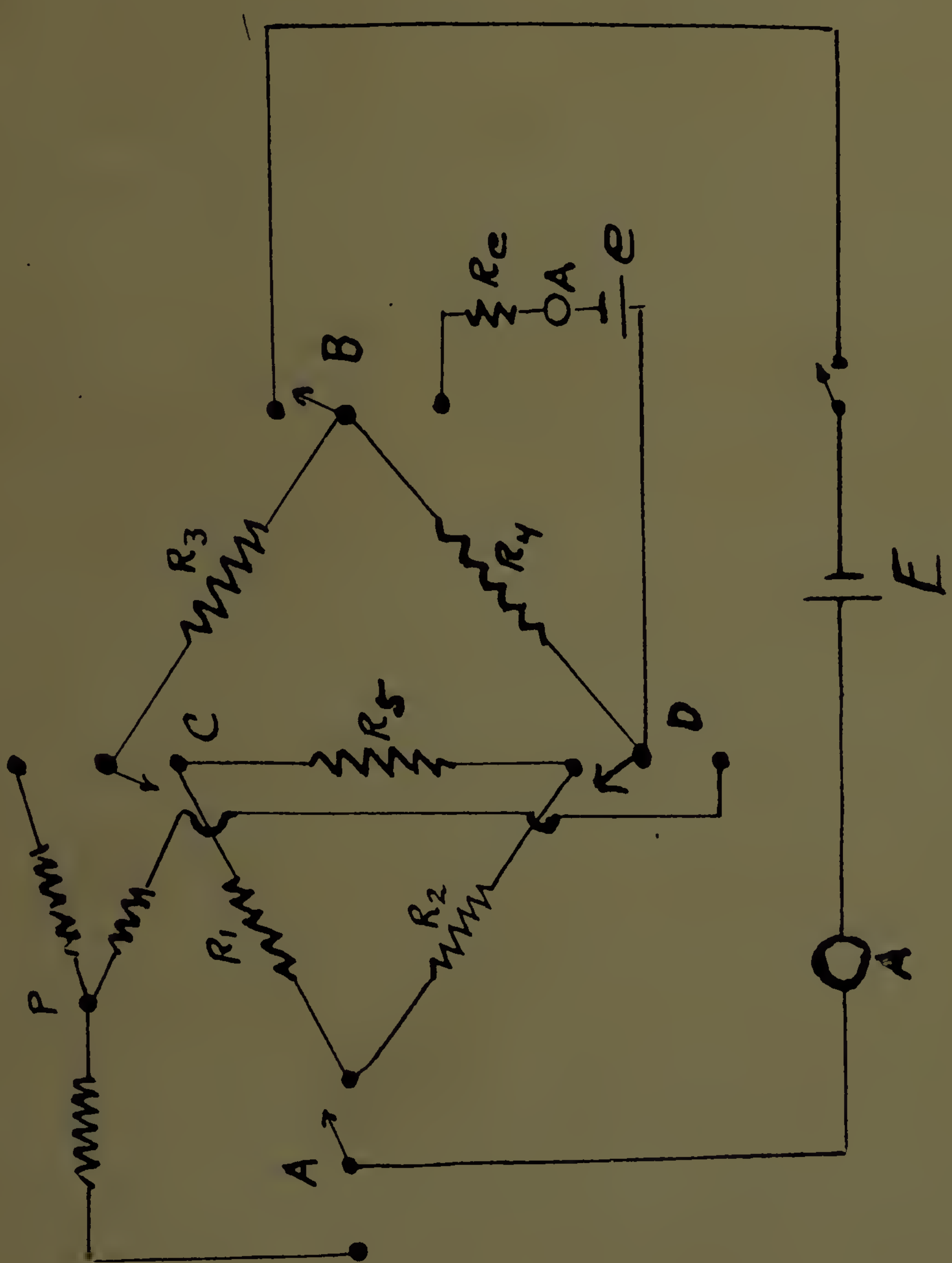
$$u = 0.00271$$

$$V_{CD} = u - V$$

$$= .00153 \text{ amps}$$

THIS VALUE CHECKS THE PREVIOUSLY OBTAINED VALUE AS WELL AS THE EXPERIMENTAL VALUE OF .0014 amps.

CIRCUIT DIAGRAM TO DEMONSTRATE Δ-Y & Thevenin THEOREMS



Thev. Circuit shown across R_4 — could be shown across any two terminals

O = Ammeter

Directions for Circuit Demonstration

(1) Close switches A, B, C, + D as indicated

(2) Reverse A, C + D to demonstrate equivalent Y (no change in meter reading)

(3) Open main line switch

(4) Reverse B to determine characteristics of R_4

A REACTANCE THEOREM FOR A RESONATOR

A theorem, which was first stated by G.A.Campbell in the Bell System Technical Journal for November 1922 and later restated by Ronald M.Foster in the Bell Journal,April 1924 and which has come to be known as Foster's Theorem gives the driving point impedance (- the ratio of an impressed electromotive force at a point in a branch of a network to the resulting current at that point -) of any network composed of a finite number of self-inductances, mutual inductances, an capacitances. It shows that the impedance is pure reactance with a number of resonant and anti-resonant frequencies which alternate with each other. It shows also how any such impedances may be physically realized by either a simple parallel-series or a simple series-parallel network of inductances and capacitances, provided the resistances can be made negligibly small.

Foster's Theorem in effect states that the driving point reactance of any non-dissipative network is an odd rational function of the frequency with an always positive slope. Foster worked out his proof based upon a solution of an analogous dynamical problem of the small oscillations of a system about a position of equilibrium with no frictional forces acting.

W. R. MacLean in a paper published in "Proc. I.R.E." for August 1945 has applied the reactance theorem of Foster to a Resonator and has in effect extended the theorem to any configuration, not necessarily with a finite number of degrees of freedom as in Foster's theorem, as long as the configuration fulfills the one requirement that it must have a driving point impedance. It is with this theorem of MacLean's that we will deal since it is a more general theorem than that of Foster although MacLean does call his discussion "an extension of the Foster reactance Theorem.

To have an impedance in the first place, it is necessary to have a definable voltage and current whose ratio can be taken. In the completely general case, since no scalar-potential function exists, there is no such thing as voltage and hence no impedance. Therefore the in-put to the system must be so arranged that a voltage exists at some point, or the theorem has no meaning. Also, if the system is to have no resistance component of input impedance, it must not only be non-dissipative but also non-radiative. Hence, it must be considered to be made of perfect conductors and to be completely surrounded with a perfectly conducting shield.

Suppose then, that one has such an enclosure whose internal configuration is any whatever, but which is fed through an attached concentric transmission line, or simply a grounded, shielded, and uniform line. If the frequency is below that required for the propagation of higher modes within the line, and if the line is of sufficient length and fed in any manner whatever at the far end, there will exist at any point which is some diameter distant from the resonator and also from the far end, a field pattern which is purely that of the principal mode. For this mode, the curl of the electric field lies in the transverse planes, and hence in these planes a scalar potential exists. This defines a difference of potential between the outer and the inner conductor which is the ordinary "voltage in the line."

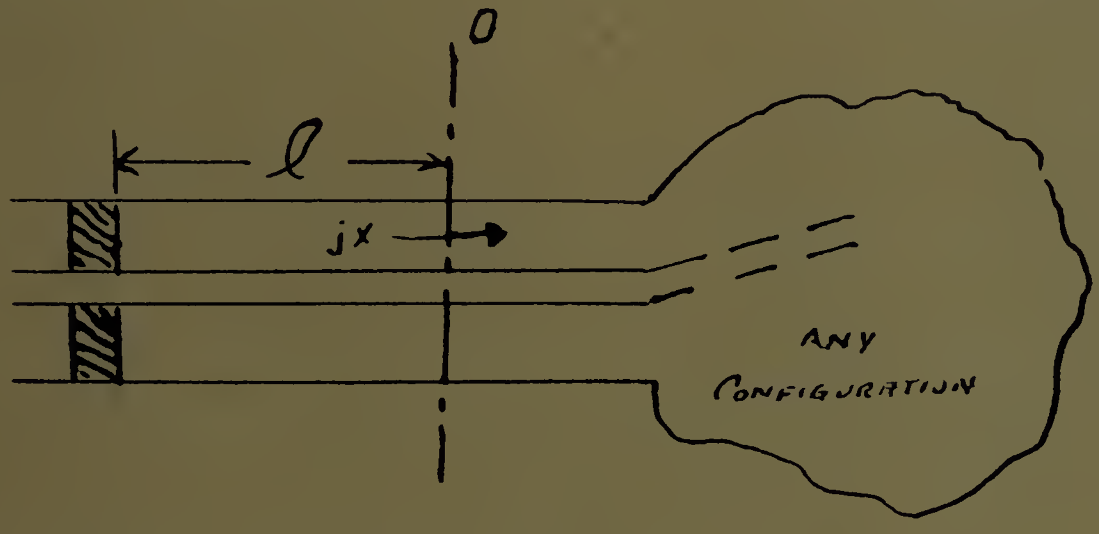
A value of current can be defined as the net flow through the center conductor toward the resonator that crosses the same transverse used for the voltage. The ratio of these two gives the input impedance to the resonator at a fixed point in the transmission line which needs only to be a few diameters distant from the resonator. This "definition" is valid from direct durrent up to frequencies whose wavelengths become comparable with the diameter of the transmission line, and coincides with the usual concept of impedance.

Since with a given apparatus there is a limit to the region of frequency in which the impedance is defined, we cannot hope to be able to evaluate the function by a knowledge of its poles and roots. But it is possible to show that, within the range of definition, the reactance function is odd, and has a positive slope.

To show that the reactance function is odd we may proceed in the following manner. Having decided on a certain connective feeder, and having picked a transverse plane (0) (See sketch on the next page) through this line at which the impedance is to be determined, one considers the cavity bounded by the perfectly conducting metal surfaces and by the annular ring cut out of (0) by the transmission line. Within this region, the theorem can be applied in the following way.

In formula (1), $(\bar{N})$ is the complex poynting vector, (T) is the total (time averaged) magnetic energy, and (U) is the total (time averaged) electric energy. The integral is taken over the bounding surface with outwardly directed normal. Due to the perfectly conducting metal, the integral vanishes except over the annular ring. If (w) is less than the value of (w_0) at which the impedance definition breaks down, the integral can be evaluated over the ring since the mode is known to be the principal one. A little computation leads to formula (2) where (V) and (I) are the complex voltage and current at (0). Instead we could write formulas (3) and (4) where (x) is the input reactance.

Substituting (- w) for (w) will produce no change in the energies since there is no physical change involved, therefore we derive formula (5) and we see that the function is odd.



(1) $$-\int_S \bar{N}\, dS = j\,2\omega\,(T-U)$$

(2) $$\bar{V}\,\bar{I} = j\,2\omega\,(T-U)$$

(3) $$Z = j\,2\omega\,\frac{(T-U)}{I^2}$$

or

(4) $$X = j\,2\omega\,\frac{(T-U)}{I^2}$$

(5) $$X(-\omega) = -X(\omega)$$

(6) $$X = -R_0 \tan\left(\frac{2\pi\nu l}{c}\right)$$

(7) $$\frac{dX}{d\nu} = -R_0 \sec^2\left(\frac{2\pi\nu l}{c}\right)\left(\frac{2\pi}{c}\right)\left(\frac{\nu\, dl}{d\nu} + l\right)$$

(8) $\delta(\nu W) = 0$

(8)a $\dfrac{\delta W}{W} = \dfrac{\delta V}{V}$

(9) $F = \omega$

(10) $-\dfrac{dW}{dl} = W$

(11) $\dfrac{1}{\nu}\left(\dfrac{d\nu}{dl}\right) = \dfrac{1}{W}\dfrac{dW}{dl} = -\dfrac{\omega}{W}$

(12) $\nu \dfrac{dl}{dV} = -\dfrac{W}{\omega}$

(13) $\dfrac{dx}{d\nu} = R_o \sec^2\left(\dfrac{2\pi\nu l}{c}\right)\left(\dfrac{2\pi}{c}\right)\left(\dfrac{W-\omega l}{\omega}\right)$

To demonstrate that the slope is always positive, we first imagine the concentric feeder extended indefinitely behind the plane (0) and finally closed with a frictionless plug at a distance (l) from (0). This is shown schematically on the sketch page. We than pick a starting frequency (v) as low as we please and determine the reactance (x) at (0) of the resonator for this frequency. (l) is then adjusted so that we have formula (6) where (R_o) is the characteristic resistance of the concentric line. As a result, there is a conjugate match at (0) at this frequency. Consequently the system, resonator, and line out to the plug has a natural mode at this frequency and if started in oscillation would continue indefinitely since there is no dissipation.

We now start it oscillating at this lowest frequency, and then move the plug with infinite, uniform slowness toward the resonator. This movement will take place against a radiation pressure. Hence it will increase the energy of the mode, but it will also change the frequency. The frequency will always be such that a conjugate match exists at every position on the line where an impedance exists. By this principle the impedance of the resonator at (0) will be given by formula (6) even when (v), (x), and (l) have been modified from their starting values.

As a result of the infinitely slow deformation (v) will be a function of (l), and (x) will be a function of either (v) or (l). Thus we determine the slope as shown in formula (7)

To prove the stated proposition, we must show that the right hand side of equation (7) is positive. All quantities are intrinsically positive except the negative sign and the final bracket. Information concerning the sign of the final bracket can be obtained by relating the change in the frequency to the change in energy of the mode.

Such a relation is given by the fact that the action of a resonator is an adiabatic invariant, i.e. if (W) is the energy and (r) the period, the product of (rW) cannot be changed by a deformation. We thus can write formulas (8) and (8_a).

Since one is putting energy in moving the plug towards the resonator, the frequency is thereby continually increased. (vl) is hence monotonic and l(v) is single valued.

Now, however, the radiation pressure gives information concerning the change in energy and leads to the desired result.

First it can be shown by a variety of means that the electromagnetic force (F) in the plug is equal to the linear density of the time averaged energy (w) along the line - see formulas (9) and (10).

Combining formulas (9) and (10) with(8_a) and then substituting in (12) we get formula (13) where(wl) is only the energy contained in part of the transmission line, whereas (W) is the entire energy of the oscillating cavity. Hence the deriative of (x) with respect to (v) is always positive while the plug moves from its original position up to (0). When the plug has reached (0), the frequenty has risen to a value say (v_1). We then stop the oscillating and move the plug back some large whole number of wave lengths of this new frequency. In this new position, the system has a mode of the same frequency since the conjugate match principle at (0) is maintained.

We create this mode and again push the plug toward the resonator. Hence it is seen that the slope of the reactance curve is always positive up to those frequencies for which the impedance is no longer fixed or defined.

ALTERNATIVE EQUIVALENT CIRCUIT FOR THE THERMIONIC VALVE

An important theorem dealing with the advantages of an equivalent circuit for the thermionic valve is stated in its most general form as follows:-

"It can be shown that any constant voltage generator of voltage (E), having an internal series resistance (R), is equivalent to a constant current generator of current equal to (E) divided by (R) with an internal shunt resistance equal to (R)."

The classic deriviation of an equivalent circuit for the anode circuit of a valve has been set forth at great length by H.W.Nichol in his article in the Physical Review (See Bibliography) and consists of a voltage (me) in series with a resistance (r), where (m) is the amplification factor of the valve, (e) is the applied grid voltage and (r) is the A.C. resistance of the valve.

Thus accepting this equivalent circuit, another can be obtained consisting of a constant current generator of current equal to (ke)with a shunt resistance (r). (k) is equal to the mutual conductance of the valve and can be found by dividing (m), the amplification factor, by (r).

Pointing out the advantages of this circuit, N.R.Bligh states that the valve capacities fall as usual between the points representing the grid, anode, and filament of the valve and he represents the two equivalent circuits as shown in Figures I and II on the next page.

The great advantage of the method of representation shown in Figure II is that all of the external impedances of the anode circuit are thrown in parallel with (r) and(C_3) and the combination of series and shunt impedances is avoided. If all the impedances are expressed as admittances, the case becomes still simpler.

For instance, in the case of a resistance coupled amplifier, the anode and grid coupling resistances and the valve resistance can all be considered in parallel over the range where the reactance of the coupling condenser is small. Thus it can be seen that it is advantageous to use a valve of high A.C. resistance, consistent with good mutual conductance, since coupling impedances are generally limited from the point of view of frequency characteristics.

Another example is afforded by the use of a tuned circuit as the coupling impedance. Using the constant current circuit the effect of the shunt resistance (r) on the resonance curve can easily be seen and for the greatest selectivity (r) should be large.

The input impedance of the valve can be obtained by using this circuit. If we call the impedance other than that

FIGURE 1

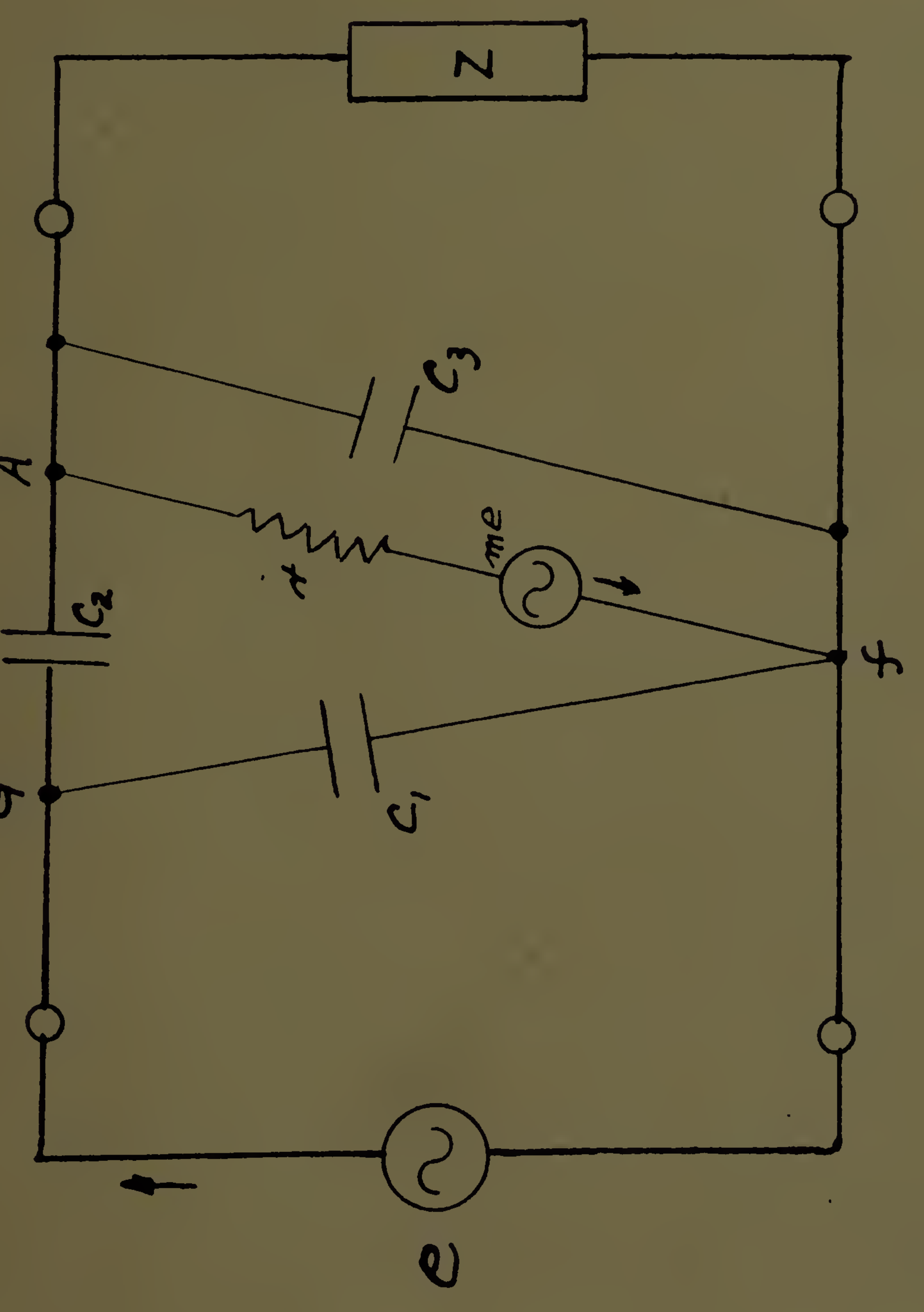
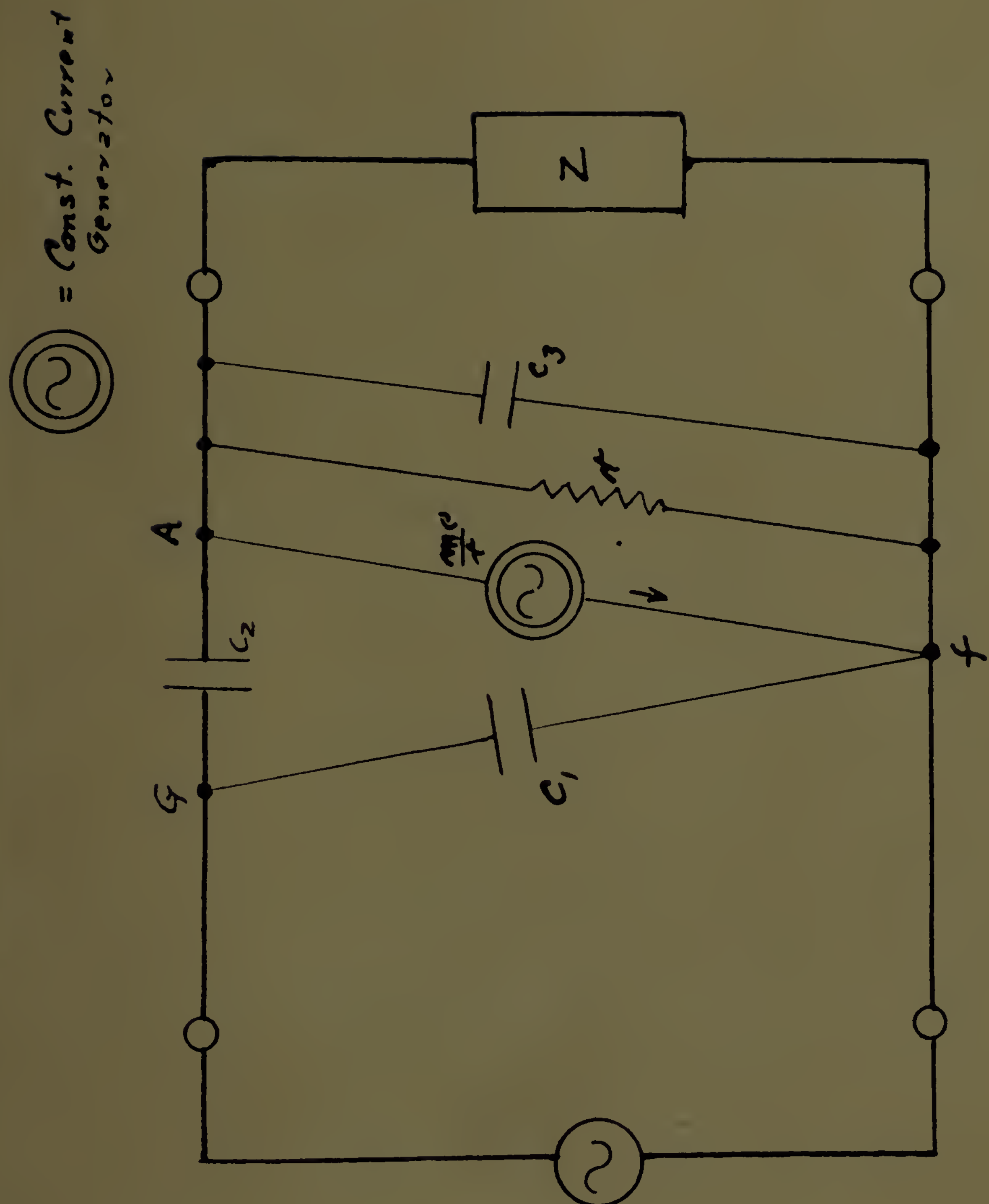


FIGURE 2



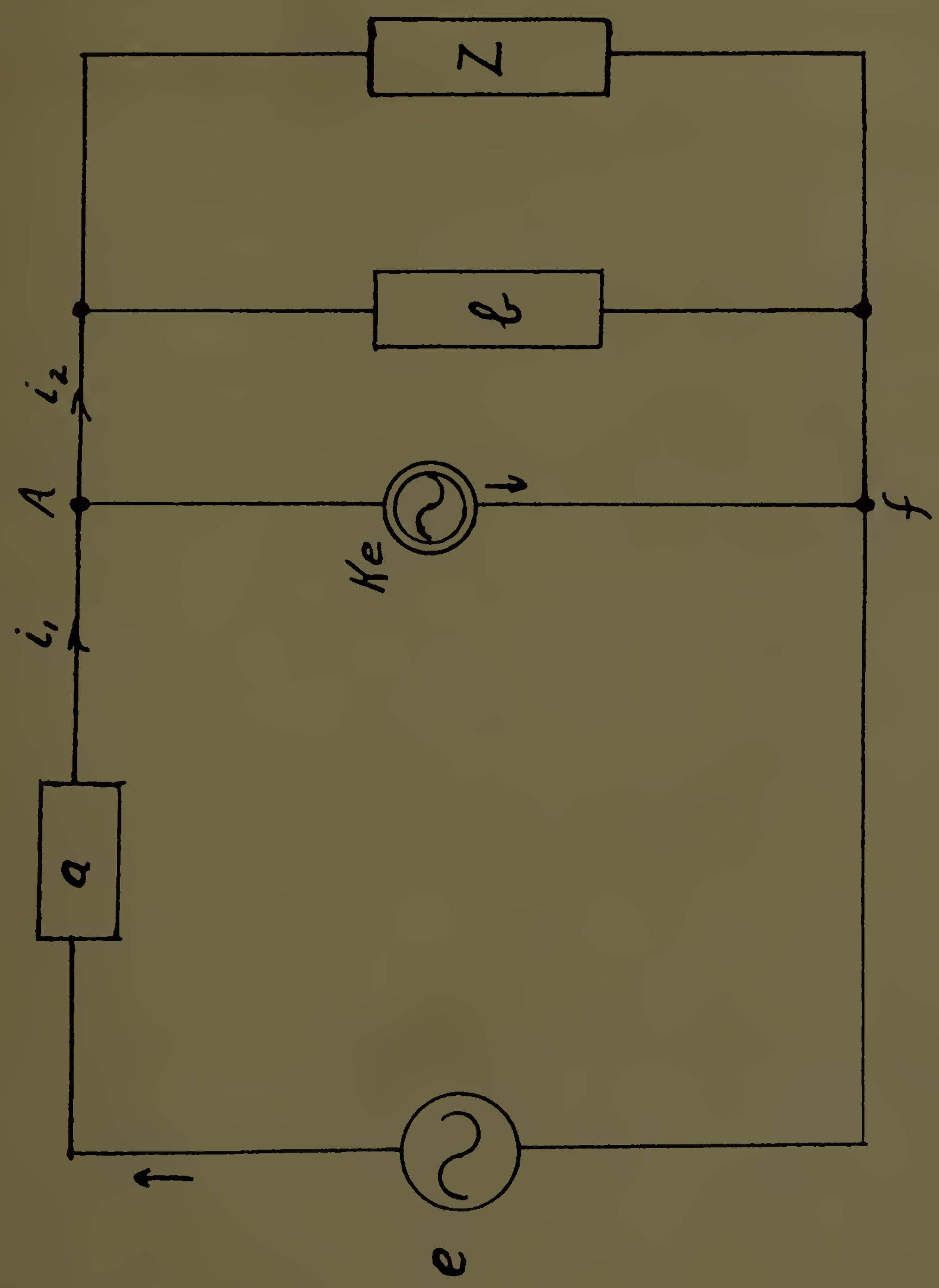
due to the grid filament capacity the Additional Input Impedance, this can be evaluated by using the circuit of Figure III.

The vector impedance of the grid anode capacity is shown as (a), the vector impedance of the anode filament capacity and (r) in parallel as (b), and (z) as the external load.

Then from Figure III, since (e) is equal to the sum of the voltages across (a) and across (b) and (z) in parallel and since the sum of the currents(by Kirchhoff) arriving at (A) is zero we determine equations (1) and (2) from which we can solve for the current in the first branch (Equation 3). The Additional Input Impedance is then determined by dividing (e) by this current (Equation 4).

The voltage amplification may be obtained by considering the combined impedance of (b) and (z) together with the current (L_2) (Equation 5) flowing into them. The voltage across (z) is shown by equation (6) and the resultant magnification by equation (7).

FIGURE 3



(1) $$e = a i_1 + i_2 \frac{bZ}{b+Z}$$

(2) $$i_1 - i_2 - Ke = 0$$

(3) $$i_1 = e \frac{Z(1+bK)+b}{Z(a+b)+ab}$$

(4) $$Z_1 = \frac{e}{i_1} = \frac{Z(a+b)+ab}{Z(1+bK)+b}$$

(5) $$i_2 = e \frac{(1-aK)(b+Z)}{Z(a+b)+ab}$$

(6) $$e_Z = \frac{i_2 Z b}{Z+b}$$

(7) $$\text{mag.} = \frac{Z(1-aK)b}{Z(a+b)+ab}$$

ON AN EXTENSION OF THEVENIN'S THEOREM

BY HIKOSABURO ATAKA

This note deals with some properties of electrical circuits, consisting of any number of linear impedances and any number of generators of the same frequency, connected in any manner, whatsoever.

Theorem:- A current Im through any branch (m) is equal to the sum of the current I_{mo} through the branch (m) when any branch (n) is open circuited, and a current I'_m through the branch (m) when an open circuited E.M.F., E_{no} acts in the branch (n) and all other E.M.F.'s in the circuit are replaced by their internal impedances.

The theorem is expressed by the formulas listed under (1) on the next page.

Proof:- The branches (m) and (n) are supposed to be drawn out of the circuit, then there will be formed an active four terminal circuit as represented in Figure I.

Let Z_m and Z_n be the impedances of the branches (m) and (n) respectively, and E_{no} be the open circuited E.M.F. of the branch n. Two E.M.F's are supposed to be placed oppositely in series in the branch (n). Then by the principle of superposition, the current I_m can be considered as being made up of two parts.

FIGURE I

$$I_m = I_{mo} + I'_m$$

$$I_m = I_{mo} + \frac{E_{no}}{Z_t} \qquad (1)$$

Z_t = transfer impedance between branch (m) and (n)

(A) a current I_{mo} when the branch (n) is open circuited and all E.M.F.'s in the circuit are present, and

(B) A current I'_{m} when the E.M.F., E_{no} is placed in the branch (n) and all other E.M.F.'s in the circuit are absent. See Figure II.

In the case (B) all E.M.F's are absent in the rectangular frame, and it may be taken to be a passive four terminal circuit.

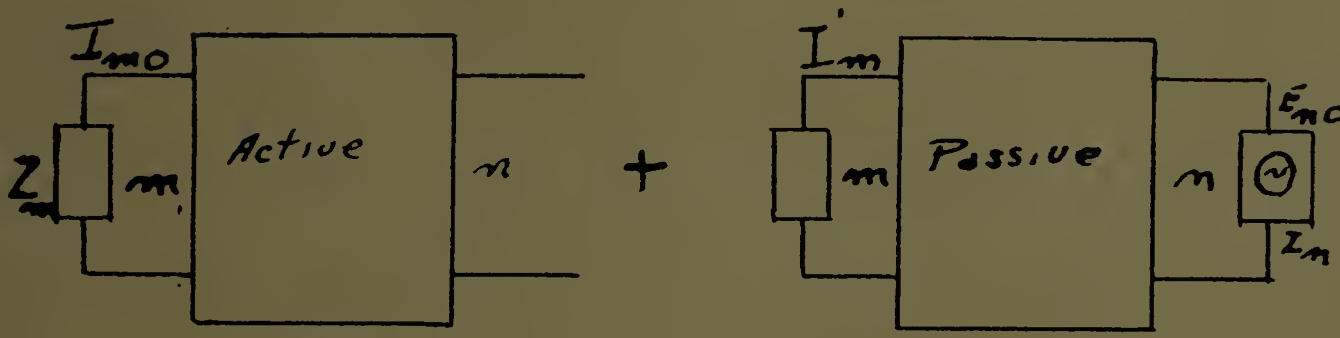
Denoting the constants of the four terminal circuit by A,B,C, and D, we have the formulas appearing under number (2).

The transfer impedance between branches (m) and (n) can then be calculated and we arrive at formulas (3) and (3A).

The theorem which is expressed by formula (3A) will be found to be an extension of Thevenin's theorem. Two special cases as applied to this formula will be of interest.

<u>Case I:</u>- In this case, we will let (m) equal (n), in which event the four terminal network degenerates into a two terminal circuit, the relationships of which are expressed under formulas (4). The values of the current in branches (m) and (n) will then be given by formula (5), which we can then restate as Thevenin's Theorem. "A current through any branch is equal to a current through the same branch when an open-circuited E.M.F. acts in the same branch and all other E.M.F.'s are replaced by their internal impedances."

FIGURE II



(A) (B)

$$\left.\begin{aligned} V_1 &= A V_2 + B I_2 \\ I_1 &= C V_2 + D I_2 \end{aligned}\right\} \quad (2)$$

$\left.\begin{matrix} V_1 \\ I_1 \end{matrix}\right\}$ voltage & current at branch (m)

$\left.\begin{matrix} V_2 \\ I_2 \end{matrix}\right\}$ " " " " (n)

$$Z_t = C Z_m Z_n + A Z_n + D Z_m + B \quad (3)$$

$$I_m = I_{mo} + \frac{E_{no}}{Z_t} \quad (3A)$$

$$\left.\begin{aligned} I_{mo} &= 0 \\ C Z_n + D &= 1 \\ Z_t = Z_m + A Z_n + B &= Z_m + Z_{mo} \end{aligned}\right\} \quad (4)$$

<u>Case II</u>:- Here we let the impedance of the branch (n) be varied by a small increment and thus we will have a variation in the current due to this change which will be a small increment of the original current.

We can restate formulas (1) by formula (6) and then the new current with its additional increment will be given by formula (7) and solve for the additional current increment as in formula (8). We can apply Thevenin's Theorem to get the value of I_m as in formula (9) and by substituting (9) in (8) we get an expression (formula 10) which can be stated as follows.

"If the circuit is altered by making a small change in the impedance of a branch, the change in current in the other branch is equal to that which will be produced by an equivalent E.M.F. acting in series with the modified branch."

Thus it will be found that "Thevenin Theorem" and the above "Compensation Theorem", which have been treated hitherto almost independently, can be wholly covered by an extension of Thevenin's Theorem.

$$I_m = I_n = \frac{E_{no}}{Z_m + Z_{mo}} \qquad (5)$$

$$I_m = I_{mo} + \frac{E_{mo}}{(CZ_m + A)Z_n + (DZ_m + B)} \qquad (6)$$

$$I_m + \delta I_m = I_{mo} + \frac{E_{mo}}{(CZ_m + A)(Z_n + dZ_n) + (DZ_m + B)} \qquad (7)$$

$$\delta I_m = \frac{-E_{no}(CZ_m + A)\,dZ_n}{\left[(CZ_m + A)(Z_n + dZ_n) + DZ_n + B\right]\left[(CZ_m + A)Z_n + DZ_m + B\right]} \qquad (8)$$

$$I_n = \frac{E_{no}}{Z_n + \frac{DZ_m + B}{CZ + A}}$$

$$= \frac{E_{no}(CZ_m + A)}{(CZ_m + A)Z_n + DZ_m + B} \qquad (9)$$

$$I_m = \frac{-I_n\,dZ_n}{(CZ_m + A)(Z_n + \delta Z_n) + (DZ_m + B)}$$

$$= \frac{-I_n\,dZ_n}{Z_t'} \qquad (10)$$

(where $Z_t' = (CZ_m + A)(Z_n + \delta Z_n) + (DZ_m + B)$)

THEVENIN'S THEOREM
From
"ELECTRICAL CIRCUITS" - M.I.T.
(see citation in Bibliography)

Theorem:- Any network of resistance elements and voltage sources if viewed from any two points in the network may be replaced by a voltage source and a resistance in series between these two points.

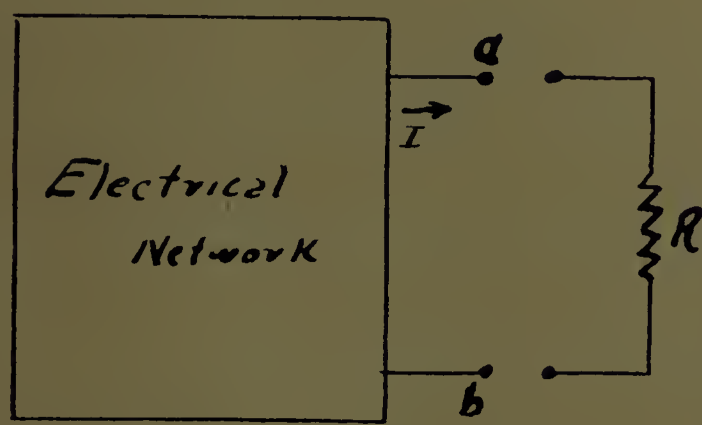
In figure (A) any network is represented, and (a) and (b) are any two points in it. Figure (B) represents the equivalent of the network as viewed from (a) and (b)

In order to use the equivalent, R_n and E must be determined, which requires two conditions. The conditions of open and short-circuit across (a) and (b) serve as well as any.

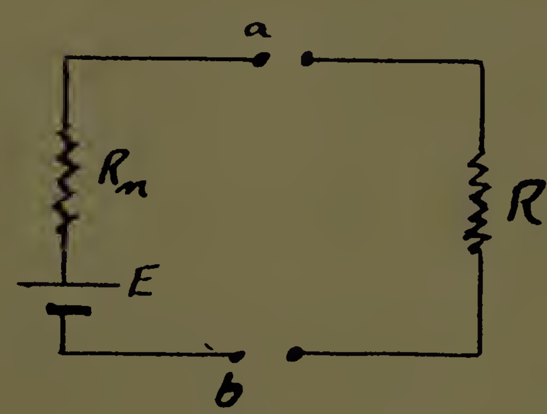
For the open circuit, (E) is equal to the voltage drop across (a) and (b) of the original network (V_{ab}).

For the short-circuit (I) is given by formula (1), where R_n is the resistance measured at the terminals of the original network with all of its voltage sources replaced by connections of zero resistance. If a resistance (R) is connected across a-b the current in it is then given by formula (2).

It must be remembered that V_{ab} is the voltage across a - b when (R) is absent. If the resistance (R) is in series with a voltage source E' the current then is found by



(A)



(B)

(1) $$I = \frac{E}{R_n}$$

(2) $$I_r = \frac{V_{ab}}{R_n + R}$$

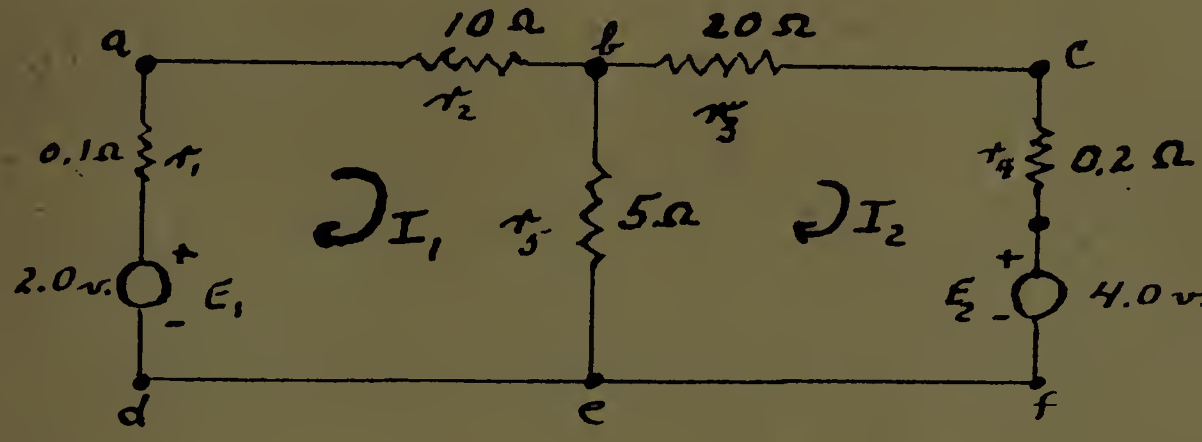
(3) $$I_R = \frac{V_{ab} \pm E'}{R_n + R}$$

(4) $$I = \frac{V_{ab} \pm V_{a'b'}}{R_n + R'_n}$$

formula (3) or if the two networks are inter-connected, the interchange current is then shown in formula (4).

An illustrative example is presented on the following pages to show the advantage of the theorem when the current in one element of a network is particularly desired or when the current in an additional element is desired.

Illustrative Problem



Find current I_{be}

In the absence of r_5 :-

$$V_{be} = E_2 - \left(\frac{E_2 - E_1}{r_1 + r_2 + r_3 + r_4}\right)(r_3 + r_4)$$

$$= 4 - \left(\frac{4 - 2}{30.3}\right)(20.2) = 2.67 \text{ volts}$$

Net Rest. of cir. =

$$R_n = \frac{(r_1 + r_2)(r_3 + r_4)}{r_1 + r_2 + r_3 + r_4} = \frac{(10.1)(20.2)}{30.3} = 6.73 \text{ ohms}$$

whence

$$I_{be} = \frac{V_{be}}{R_n + r_5} = \frac{2.67}{6.73 + 5.0} = 0.228 \text{ amps}$$

THEVENIN'S THEOREM
From
"ELECTRIC CIRCUITS" BY A. T. STARR
(See citation in Bibliography)

Application to an Aerial Circuit

If we wish to know how the input impedance of a radio receiver affects the magnitude of the received signal, we can replace the signal and aerial systems as shown in the accompanying Figure (1) by an E.M.F. equal to (V) and impedance (Z), which are independent of any frequency, in a straight forward manner.

The receiver signal is then given by the formula (1) and its variation with Z_r can be computed.

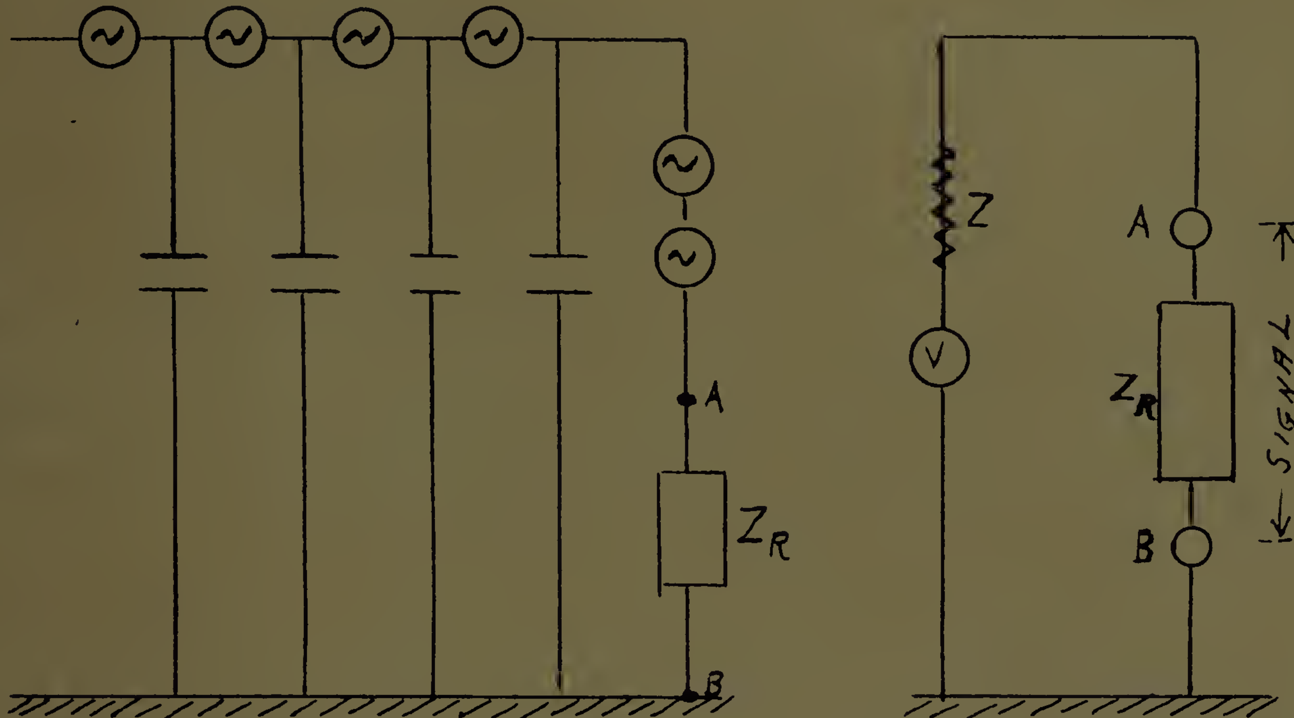
Example 2:-

In figure (2) we have an E.M.F. acting in series with a condenser (C_1) and a resistance (R) in parallel (representing an aerial, say) with a tuned circuit (L,r,C). The voltage across (C) which is (v), is put on the grid circuit of the receiving circuit. It is required to see how (v) varies with (C), (C_1) and (R).

The arrangements to the left of AB may be replaced as in figure (3) by (c') in series with the parallel combination of (C), (C_1), and (R). (c')can then be calculated by means of formula (2).

If we are considering a reasonably narrow band of

FIGURE (1)



Aerial Circuit

Network Equivalent

(1) Received Signal $= \dfrac{V Z_R}{Z + Z_n}$

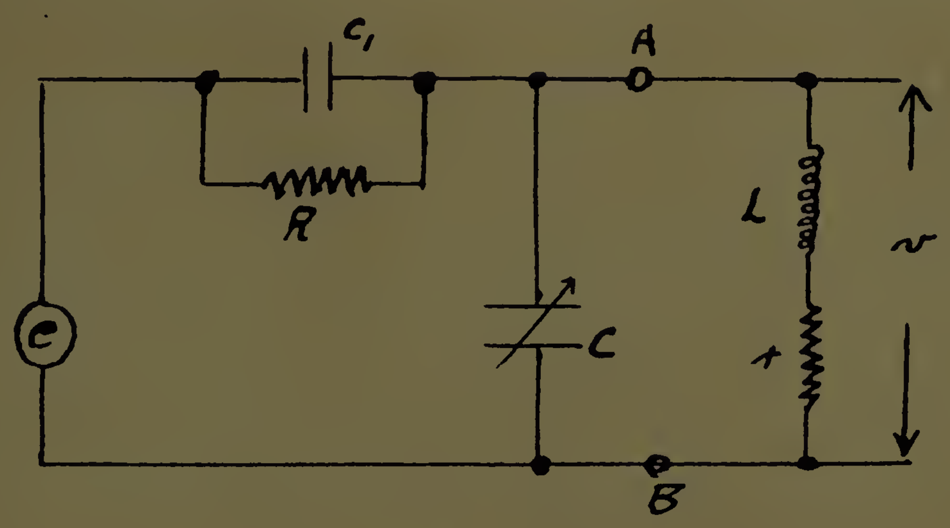


FIGURE (2)

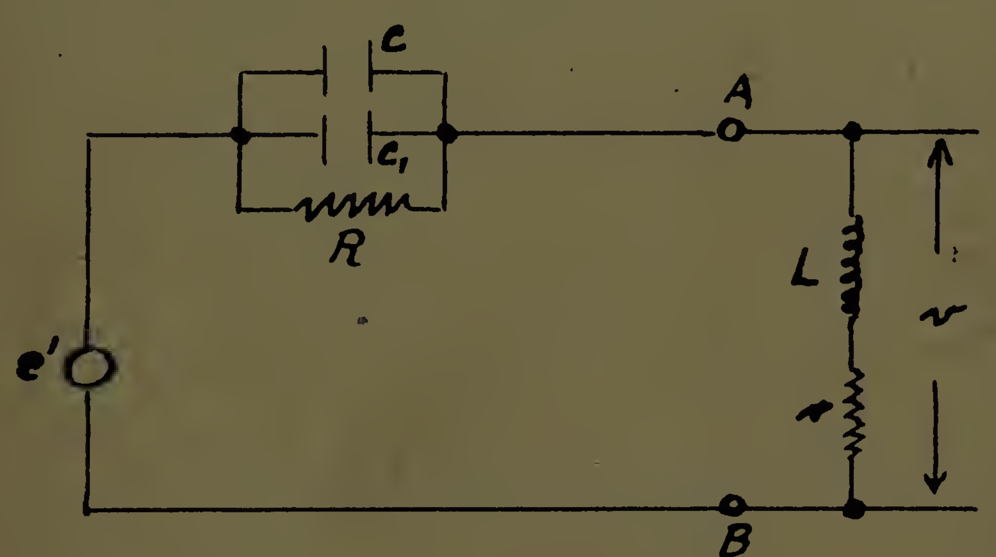


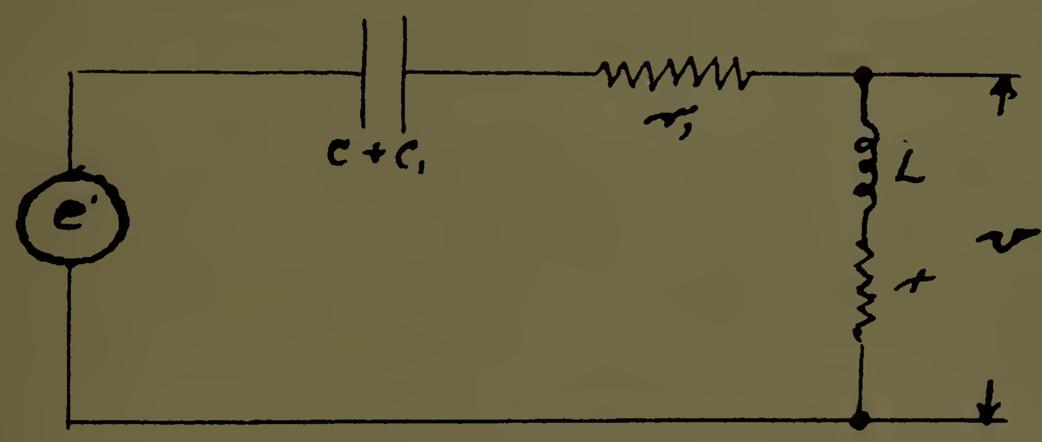
FIGURE (3)

frequencies the (C) (C_1) and (R) may be replaced by a capacitance and resistance of values shown in formula (3) resulting in a circuit as pictured in Figure(4).

The voltage (v) is a maximum when the reactance (l) neutralizes the reactance of (C) plus (C_1). When this happens the current and voltage are approximately equal to the values as shown in formulas (4) and (5).

The influence of the aerial resistance and capacitance on the amplitude of the received signal as well as the effect on selectivity can thus be observed.

FIGURE (4)



$$(2)\quad e' = \frac{e\left(\frac{1}{j\omega C}\right)}{\frac{1}{j\omega C} + \frac{1}{j\omega C + \frac{1}{R}}}$$

$$= 1 + \frac{\frac{e}{C}}{C_1 + \frac{1}{j\omega R}}$$

$$\doteq \frac{e}{1 + \frac{e}{C_1}}$$

(since R is much greater than $\frac{1}{j\omega C_1}$)

$$(3)\quad C_2 = C + C_1$$

$$r_1 = \frac{1}{\omega^2 (C+C)^2 R}$$

(4) $$i = \frac{e'}{r_1 + r}$$

(5) $$v \doteqdot \frac{j\omega L e'}{r + r_1}$$

$$= \frac{e i \omega L}{\left(1 + \frac{c}{c_1}\right)\left[r + \frac{1}{\omega^2 (c + c_1)^2 R}\right]}$$

Symbols

$j = \sqrt{-1}$

$\doteqdot$ = "nearly equal to"

ω = Pulsatance

= 2π (frequency)

Two Basic Circuits for an Extended Employment of Thevenin

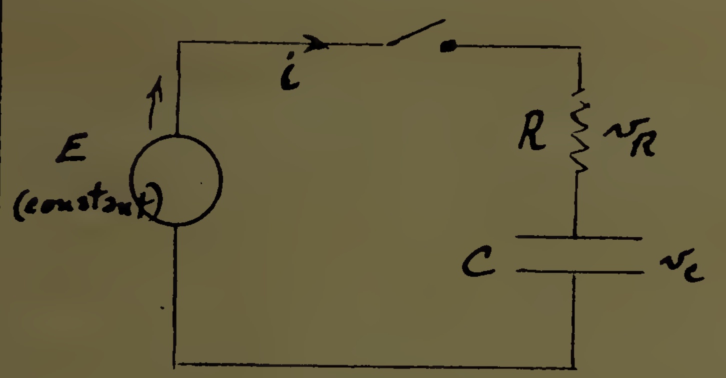


Fig. (1)

if $t = 0$

and $T = RC$

$$\therefore\ i = \frac{E}{R} e^{-\frac{t}{T}}$$

$$v_R = E e^{-\frac{t}{T}}$$

$$v_c = E\left(1 - e^{-\frac{t}{T}}\right)$$

Note:- these two circuits give the basic relationships needed for circuit computations in Fig 3, 4, & 5

Fig (2)

if $t = 0$ $T = RC$

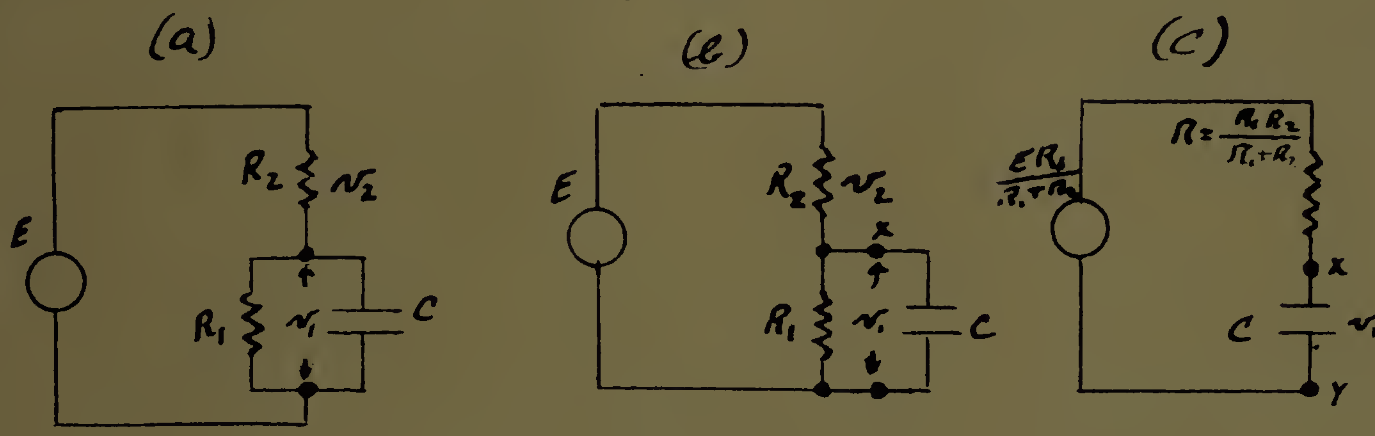
$$v = IR\left(1 - e^{-t/T}\right)$$

$$i_c = I e^{-t/T}$$

$$i_R = I\left(1 - e^{-t/T}\right)$$

Extended Employment of Thevenin

FIG (3)



(b) + (c) are applications of Thevenin's Theorem to (a)

In (c)

$$v_1 = \frac{ER_1}{R_1 + R_2}\left(1 - e^{-t/RC}\right) = \frac{ER}{R_2}\left(1 - e^{-t/T}\right)$$

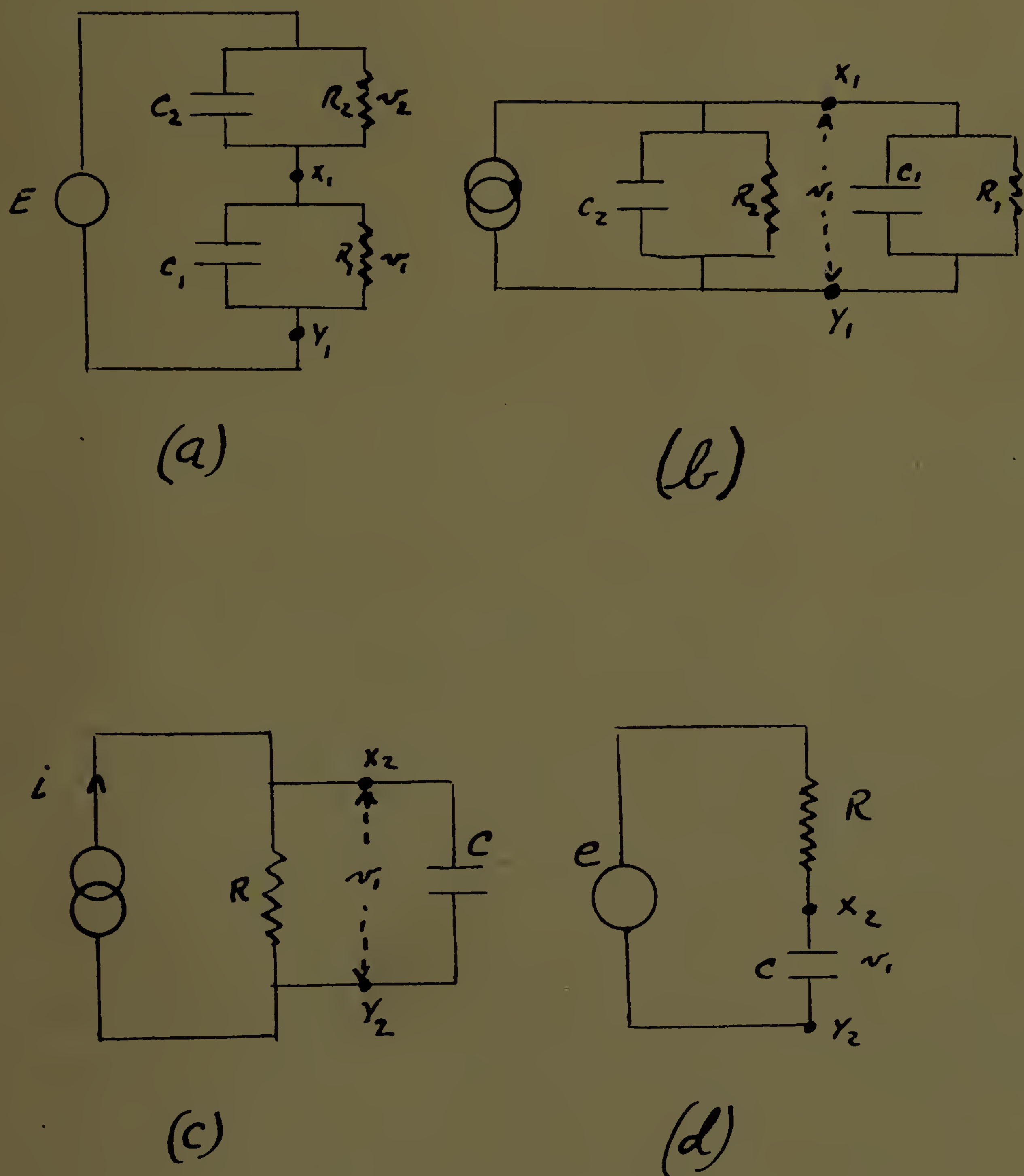
where $T = RC = \frac{R_1 R_2}{R_1 + R_2} C$

$$\therefore \quad v_2 = E - v_1$$

$$= E\left(1 - \frac{R_1}{R_1+R_2} + \frac{R_1}{R_1+R_2} e^{-t/RC}\right)$$

$$= E\left(\frac{R}{R_1} + \frac{R}{R_2} e^{-t/RC}\right)$$

FIGURE (4)



For equations for solution of Problem Circuit appearing in (a) see next page

Employing the constant current theorem we arrive at circuit (b) in which

$$i \equiv E\left(\frac{1}{R_2} + C_2 D\right) \qquad \left(\begin{array}{l}\text{where } D = \text{process} \\ \text{of time differentiation}\end{array}\right)$$

This circuit (b) reduces to (c)

where

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

and $C = C_1 + C_2$

Transforming (c) to (d) we have

$$e = iR - E R\left(\frac{1}{R_2} + C_2 D\right)$$

Thus

$$v_1 = e(1 - e^{-t/T}) \equiv \frac{ER}{R_2}\left(1 - e^{-t/T}\right) + \frac{ERC_2 D}{R_2}\left(1 - e^{-t/T}\right)$$

By differentiating

$$v_1 = \frac{ER_1}{R_1 + R_2}\left(1 - e^{-t/T}\right) + \left(\frac{EC_2 R_1}{R_1 + R_2}\right)\left(\frac{R_2}{RC}\right) e^{-t/T}$$

$$= \frac{ER_1}{R_1 + R_2}\left(1 - \left(1 - \frac{T_2}{T}\right) e^{-t/T}\right)$$

where $T_2 = R_2 C_2$

BIBLIOGRAPHY

Ataka, Hikosabura, "On an extension of Thevenin's Theorem," Phil.Mag.&Jour. of Sc. Seventh Series, No 169 VOl 25, April 1938, pp 663 - 666.

Bligh, N. R., "A Note on an Alternative Equivalent Circuit for the Thermionic Valve"; Exp. Wireless & Wireless Engineer, Vol VII, No. 80 London, Sept. 1930, pg. 480

"Dah-You-Maa", "A general reactance theorem for electrical, mechanical and acoustical systems", Proc. I.R.E. Vol 31, July 1943, pg 365

Foster, R.M. , "A reactance theorem", Bell Sys.Tech. Jour. Vol 3, April 1924, pg 259.

Harnwell, G.P. , Principles of Electricity and Electro-Magnetism, New York, McGraw-Hill 1945, pp 108-118

Helmholtz, H. von, "Ueber einige Gesetye der Vertheilung elektrisher Strome in korperlichen Leitern mit anivendung aus die thierisch-elektrischen Versuche" Annalen der Physik und Chemie, 1853 pg 211

Helmholtz, H. von , "The Make and Break Network Theorem of Helmholtz," Wireless Engineer, July 1943 pg. 319

Kennelly, A.E. , "The Equivalence of Triangle and Three-pointed Stars in Conducting Networks" Electrical World, 1899, Vol 34, Pg. 413

Koenigsberger, Leo , "The Investigations of Hermann von Helmholtz on the Fundamental Principles of Mathematics and Mechanics." Smithsonian Institution Report, 1896, pp 93-124

Lee, A , "The Extended Employment of Thevenin's Theorem", Wir. Eng. Nov. 1945, pg 534

MacLean, W.R. , "The Reactance Theorem for a Resonator", Proc. I.R.E. Aug. 1945, Vol 33, No.8, pg 539

M.I.T. Electrical Circuits ; Prin.of Elect. Eng. Series, New York, John Wiley & Sons Inc., The Technology Press, M.I.T.pg 138-146.

Nichols, H.W. "Equivalent circuit for the anode circuit of a valve", Physical Review Ser.2 1919, Vol 13, pp 404-414

Starr, A.T. Electric Circuits and Wave Filters New York, Pitman Pub. Co. 1938 pp 78-80

Thevenin, M.L. "Sur un nouveau theoreme d'electricite dynamique," Comptes rendus, Vol 47, 1883, pp 159-161

Timbie, W.H. Basic Electricity for Communications New York, John Wiley & Sons, Inc. 1944, pp 387-433

Wigge, H. "Some Deductions from Thevenin's Theorem" Arch. f. Elektrot. 21st Nov.1936, Vol. 30, No.11, pp 754-759

ACKNOWLEDGMENTS

The author of this thesis wishes to acknowledge his indebtedness to the several persons without whose assistance and encouragement the work presented herein would have been tremendously difficult.

I am greatly indebted,

To the three members of my Thesis Committee Professor Wallace F.Powers, Professor Albert W.Purvis and Professor George A. Marston who have given freely of their time and advice at times.

To the librarians at Massachusetts State College and Amherst College for the assistance rendered during the search for my material.

To my parents for their constant encouragement with regard to the completion of this project.

To my excellent typist Mrs. William Wolfe without whose services, this work would have been impossible.

To my students at Amherst College who, I fear have had to bear with a sometimes disgruntled and tired lecturer.

Approved by:

Albert W. Purvis

G. A. Marston

W. F. Powers

Thesis Committee

Date June, 1946