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# Sequence Design via Semidefinite Programming Relaxation and Randomized Projection 

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# SEQUENCE DESIGN VIA SEMIDEFINITE PROGRAMMING RELAXATION AND RANDOMIZED PROJECTION 

A Dissertation Presented<br>by<br>DIAN MO

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

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Electrical and Computer Engineering
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# SEQUENCE DESIGN VIA SEMIDEFINITE PROGRAMMING RELAXATION AND RANDOMIZED PROJECTION 

A Dissertation Presented
by
DIAN MO

Approved as to style and content by:

Marco F. Duarte, Chair

Patrick A. Kelly, Member

Dennis L. Goeckel, Member

Arya Mazumdar, Member

Christopher V. Hollot, Department Chair
Electrical and Computer Engineering

## ABSTRACT

# SEQUENCE DESIGN VIA SEMIDEFINITE PROGRAMMING RELAXATION AND RANDOMIZED PROJECTION 

FEBRUARY 2019

DIAN MO
B.Sc., BEIHANG UNIVERSITY
M.Sc., UNIVERSITY OF MASSACHUSETTS AMHERST

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Marco F. Duarte

Wideband is a booming technology in the field of wireless communications. The receivers in wideband communication systems are expected to cover a very wide spectrum and adaptively extract the parts of interest. The literature has focused on mixing the input spectrum to baseband using a pseudorandom sequence modulation and recovering the received signals from linearly independent measurements by parallel branches to mitigate the pressures from required extreme high sampling frequency. However, a pseudorandom sequence provides no rejection for the strong interferers received together with weak signals from distant sources. The interferers cause significant distortion due to the nonlinearity of the subsequent amplifier and mask the weak signals.

In this dissertation, we optimize the modulation sequences with a specific spectrum shape to mitigate interferers while preserving messages; the sequences have binary entries to simplify hardware implementation. Though the resulting sequence design problems are NP-hard, we solve them approximately by semidefinite relaxation and randomized projection.

First, we formulate the design algorithm for a single spectrally shaped binary sequence base on a randomized convex optimization method. We analyze the performance of the algorithm in obtaining binary sequences and show its advantages compared with method available in the literature. And, we show a comparison between the proposed sequence design method with the exhaustive approaches when feasible. Additionally, we propose several custom sequence scoring functions that allow for an improved selection of binary sequences for message preservation and interference rejection.

Second, we propose an algorithm to design a multi-branch set of binary sequences one by one by introducing the constrains on the orthogonality between pairs of sequences. Numerical results show the proposed algorithm obtains sequences with a small search size compared with the exhaustive search.

Finally, we extend the randomized method to multi-branch sequence design. In order to avoid the unstable performance and high complexity of designing multibranch sequence iteratively, the whole branch sequences will be obtained directly via matrix randomized projection from the relaxed problems.

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## CHAPTER 1

## INTRODUCTION

Receivers for emerging wireless communication systems are expected to deal with a very wide spectrum and adaptively choose which parts of it to extract. The intense demand on the available spectrum for commercial users force many devices to share the spectrum [1, 2]. As a result, wideband communication systems have caused interference for applications using overlapping regions [3, 4]. Thus, a major issue for wideband communication receivers is to process spectra having very weak signals from a distant source mixed with strong signals from nearby sources. Practical nonlinearities in receiver circuits make the separation of such signals a key barrier for wideband communication systems since unfiltered interferers are large enough to cause distortion that can mask weaker signals.

### 1.1 Scalable Receivers

Recently, random sequences for wideband signal modulation have been employed in the realization of communication system receivers [5-8], where a prominent example is the wideband modulated converter (MWC). In essence, the signal after random modulation contains a baseband spectrum that is the linear combination of all frequency components of the input signal. Thus, using a large enough number of modulation branches allows for the successful recovery of the wideband signal, where the number of necessary branches is determined by the occupancy of the spectrum. The resulting multi-branch modulation system can be abstracted as an all-pass filter that preserves all frequency components of interest from the input signal.

In cases when the locations of one or more interferers are known, the modulation to null out the interferer band is desirable to reduce the distortion due to nonlinearities. Therefore, it is promising to replace the pseudo-random sequence with a spectrally shaped sequence that effectively implements a notch filter to suppress interferers. In addition to the strong interferer case described earlier, a similar problem arises in dynamic spectrum management (DSM), an approach that allows for flexibility in spectrum use. DSM attempts to determine the frequencies being used by previous or licensed applications and selects an optimal subset from the remaining frequencies for new or unlicensed applications. The signals for unlicensed applications should be optimized to minimize the interference with licensed signals while keeping their own capacities. More specifically, for Direct Sequence Spread Spectrum, a designed spreading code with particular spectral characteristic has been used to shape the unlicensed power spectra [9]. Another similar problem arises in active sensing, which obtains valuable information of targets or the propagation medium by sending probing waveforms toward an area of interest [10-12]. A well-designed waveform is crucial to the performance of active sensing.

### 1.2 Sequence Design

Since the spectrally shaped sequences work as a bandstop filter for the interferer, it is intuitive to design the spectrally shaped sequences by filter design with spectral constraints. While such filter designs are usually expressed as an optimization problem that maximizes stopband attenuation subject to a given bounded passband ripple, they are, in general, not convex and global optimal designs are not guaranteed since the magnitude response is not linear in the filter coefficients. It is shown in [13] that the filter design with spectral constraints can be written in term of the spectral power density and the Fourier transform of the autocorrelation of the filter coefficients. The resulting filter designs are convex optimization problems in the spectral
power density. However, the subsequent spectral factorization that determines the filter coefficients from the autocorrelation coefficients prevents extending the method to other sequence design where additional constraints on the filter coefficients are needed.

To maximize the power efficiency of the designed probing waveforms in active sensing, it is a standard requirement that the modulating waveform is constrained to be unimodular. A common performance criterion gives preference to low autocorrelation sidelobes in the time domain or a flat spectrum in the frequency domain since the autocorrelation function and the power spectral density form a Fourier transform pair. Such a choice improves the target detection performance [14]. Based on this relationship, an iterative method has been developed to design unimodular waveforms with the flat spectrum and impulse-like autocorrelation function by alternatively determining the waveform and an auxiliary phase [15, 16]. In [17], a majorization-minimization method implements the waveform design by minimizing the integrated sidelobe level.

Additionally, it is desirable that the waveform possess some specific spectrum magnitude to have spectral nulls in specific bands to keep the mutual interference within acceptable levels. Based on the iterative method of [15, 16], the SHAPE algorithm designs sequences that simultaneously approach the desired spectrum magnitude and satisfy the envelope constraint [2, 12]. A sequence is obtained by alternatively searching the frequency and time domains to minimize the estimation error while meeting the two aforementioned criteria. In [18], a Lagrange programming neural network (LPNN) algorithm [19], based on nonlinear constrained optimization, is applied to design waveforms with unit modulus and spectral constrains. Recently, an alternating direction method of multipliers (ADMM) solver was used for waveform design subject to constant modulus and spectral constraints [20]. While it is possible to modify some of the algorithms in the literature to switch from a unimodular constraint to a binary constraint, which is also required in the random modulation receivers so that
the resulting binary sequence can be easily implemented by a standard shift register with high rate [8]. Our numerical simulations in the sequel show that such changes result in significant performance losses in modulation.

Recently, the waveform design problems have been formulated as a quadratically constrained quadratic program (QCQP) [21-25]. In [21, 22], [23] and [25], a sequence is selected to maximize the Doppler estimation accuracy while the similarity with a prefixed sequence is upper bounded, minimize the signal to noise ratio (SNR) while the peak to average power ratio is upper bounded, and minimize the SNR and minimize the power in stopband and the similarity with a prefixed sequence, respectively. All of these waveform design can be written as a QCQP with different constraints and solved by randomized projections.

As will be shown in Chapter 2, the design of spectrally shaped binary sequences can be formulated as a QCQP with both equality constraints (for binary quantization) and inequality constraints (for interferer mitigation). Though there are some similarities between the mentioned waveform design and the proposed binary sequence design, some gaps make it hard to apply the analysis in waveform design to binary sequence design: first, the sequences required in waveform design have complex-valued entries, which allows more freedom than the binary sequences needed in our problems; second, the QCQP in waveform design contains either no inequality constraint or only the inequality constraints whose characteristic matrices are diagonal.

### 1.3 Outline

This thesis is organized as follows.
Chapter 2 introduces notations and overviews the concepts in MWC including the system description and frequency analysis. The following content about QCQPs covers existing QCQP variations under different kinds of constraints, methods to approximately solve the QCQP with semidefinite program (SDP) relaxation and ran-
domized projection, and the analysis of approximation ratio to measure the quality of the approximate solution. Additionally, we give a brief review of the Slepian basis and the difference between the Slepian basis and the Fourier basis.

Chapter $\mathbf{3}$ presents our approach of the design of a single spectrally shaped binary sequences that can be used to replace a random sequence in any branch of MWC. We formulate the sequence design using a QCQP to maximize the sequence power for the message band while keeping the sequence power for the interferer band as low as possible. Additionally, we provide a theoretical analysis of the feasibility probability and approximation ratio for the obtained sequences from the solution of the SDP relaxation by randomized projection. We also present a set of more suitable metrics for sequence selection after randomized projection so that the resulting sequence allows for better interferer rejection.

Chapter 4 provides the detail on our iterative sequence set design for multibranch modulation. The sequences are selected iteratively to have the smallest sequence power for the interferer band but be close to orthogonal to the sequences in the set that have been previously obtained. Additionally, we include an analysis of the connection between the condition number of the measurement operator matrix and the tolerance of the sequence orthogonality to show the difficulty of searching orthogonal sequence sets. Furthermore, we show the necessity of oversampling in the multi-branch sequence design in order to obtain sequences set with stable invertibility.

Chapter 5 presents our approach for sequence set design based on matrix optimization. The sequence set design is formulated as a matrix optimization after extending the metrics of power and orthogonality from a single sequence to a sequence set. We also derive the convex relaxation of the matrix optimization by rewriting the binary constraint in matrix form. Inspired by fact that the randomized projection returns a rank-one approximation in the single sequence design, we develop a method
to obtain a low rank approximation, whose columns correspond to the sequences after binary quantization.

Chapter 6 presents a simulation example to show the performance of the designed sequence sets when they are used by a modulated wideband converter with nonlinearity and noise. Additionally, we provide a discussion on the bases that are used to express the signals bandlimited to the message or interferer bands in the sequence set design.

Finally, we conclude with a summary of our findings and a discussion of ongoing work in Chapter 7.

## CHAPTER 2

## BACKGROUND

### 2.1 Notation

We denote vectors (as columns by default) by bold lower case letters s, where the entries are listed as $s_{1}, s_{2}, \ldots, s_{N}$. Matrices are denoted by bold upper case letters $\mathbf{S}$, where the entry in row $i$ and column $j$ is denoted by $S_{i, j}$ and the column vectors are indexed as $\mathbf{s}_{j}$. Upper case letters $(N, R, C)$ and lower case letters $(i, j, k)$ are used to represent scalar quantities. Most calligraphic letters denote sets except $\mathcal{F}$, which is a special complex exponential vector. Two commonly used operators $(\cdot)^{T}$ and $(\cdot)^{H}$ return the transpose and conjugate transpose of a vector or a matrix.

### 2.2 Modulated Wideband Converter

As discussed in Chapter 1, signals for wideband communications consist of some narrow occupied bands spread over a wideband spectrum. Due to the wide spectrum, the sampling rates of the signals could be too high and exceed the specifications of most analog-to-digital converters. Therefore, there is a needed for an approach to acquire such signals with a sampling rate lower than the Nyquist rate of the signals.

When the location of each message band is known, it is common to demodulate a signal with the center frequency of the message band such that the message band is centered at the baseband [26, 27]. All other bands are rejected by a low-pass filter. Then conversion can be performed at a sampling rate proportioned to the actual frequency occupancy of the message band.

When the location of message bands are unknown, the task is to design a receiver which can operate at a sub-Nyquist rate. The output should contain sufficient information so that the input could be fully reconstructed. In [6, 28, 29], a multi-coset sampling method was proposed to acquire signals at low sampling rate. Instead of sampling the entire signal uniformly, the multi-coset sampling method samples the signal uniformly in blocks but the sampling of each block is shifted by a different delay. The output of the multi-coset sampling contains a baseband consisting of a linear combination of all pieces of the input spectrum, where the coefficients are determined by the delays. Recently, a new architecture acquires signals at a low sampling rate by random modulation [5, 7]. The signals are mixed with a high-rate pseudorandom sequence to alias the small bands of input spectra into the baseband of outputs. Based on the random modulation method, MWC is developed to be composed of a set of modulators and low-pass filters [8].

### 2.2.1 System Description

As shown in Figure 2.1, the system of MWC consists of a set of channels, also known as branches. More specifically, the signal $x(t)$ is processed by $M$ channels simultaneously. In the $m^{\text {th }}$ channel, the signal $x(t)$ is modulated by mixing with the sequence signal $p_{m}(t)$, which has periodic $T_{p}$. After modulation, the signal $\hat{x}_{m}(t)$ passes through a low pass filter with cut off frequency $1 / 2 T_{s}$ to prevent the alias. Then the filtered signal is sampled at rate $1 / T_{s}$.

While other periodic waveforms are possible, it is common to select a square-wave waveform to modulate the signal for simplicity. The sequence signal is a piecewise constant waveform that alternates between the levels of 1 and -1 for each of $N$ equal-size time intervals. Formally, we can express the sequence signal as

$$
\begin{equation*}
p_{m}(t)=S_{m, n}, \frac{n-1}{N} T_{p} \leq t \leq \frac{n}{N} T_{P}, n=1,2, \ldots, N \tag{2.1}
\end{equation*}
$$



Figure 2.1. System diagram of modulated wideband converter. Signals are mixed with pseudorandom sequence and sampled at each channel.
where $S_{m, n} \in\{-1,1\}$. Thus, a sequence signal is totally characterized by the vector $\mathbf{s}_{m}=\left[s_{m, 1}, s_{m, 2}, \ldots, s_{m, n}\right]^{T}$.

Although the signal is obtained at sampling rate $1 / T_{s}$, which can be low enough so that existing analog-to-digital converters can achieve the sampling task, the modulation and filtering are still operating at the condition of high rates. Fortunately, the sign alternating waveform $p_{m}(t)$ can be implemented by a standard shift register, which allows the alternation rates to reach as high as 23 GHz or even higher [30, 31].

### 2.2.2 System Analysis

We now derive the relationship between the sampled output $y_{m}[l]$ and the signal $x(t)$. This analysis not only explains the reconstruction scheme, but also motivates us to replace the pseudorandom sequence with a spectrally shaped sequence.

Consider the $m^{\text {th }}$ channel. Since $p_{m}(t)$ is $T_{p}$ periodic, it can be written as a linear combination of the Fourier series

$$
\begin{equation*}
p_{m}(t)=\sum_{k=-\infty}^{\infty} c_{m, k} e^{j 2 \pi k \frac{t}{T_{P}}} \tag{2.2}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{m, k}=\frac{1}{T_{p}} \int_{0}^{T_{p}} p_{m}(t) e^{-j 2 \pi k \frac{t}{T_{p}}} \mathrm{~d} t \tag{2.3}
\end{equation*}
$$

The modulated signal is evaluated as $\hat{x}_{m}(t)=x(t) p_{m}(t)$. Then the Fourier transform of the modulated signal is

$$
\begin{align*}
\hat{X}_{m}(f) & =\int_{-\infty}^{\infty} \hat{x}_{m}(t) e^{-j 2 \pi f t} \mathrm{~d} t \\
& =\int_{-\infty}^{\infty} x(t) p_{m}(t) e^{-j 2 \pi f t} \mathrm{~d} t \\
& =\int_{-\infty}^{\infty} x(t) \sum_{k=-\infty}^{\infty} c_{m, k} e^{j 2 \pi k \frac{t}{T_{P}}} e^{-j 2 \pi f t} \mathrm{~d} t \\
& =\sum_{k=-\infty}^{\infty} c_{m, k} \int_{-\infty}^{\infty} x(t) e^{-j 2 \pi\left(f-\frac{k}{T_{p}}\right) t} \mathrm{~d} t \\
& =\sum_{k=-\infty}^{\infty} c_{m, k} X\left(f-\frac{k}{T_{p}}\right) . \tag{2.4}
\end{align*}
$$

Therefore, the spectrum of the modulated signal $\hat{x}_{m}(t)$ is a linear combination of shifted copies of the spectrum of the signal $x(t)$.

When the low pass filter is an ideal filter whose magnitude response is a rectangle function, only the spectrum of $\hat{x}_{m}(t)$ in the baseband $\left[-1 / 2 T_{s}, 1 / 2 T_{s}\right]$ is preserved in the output $y_{m}[l]$. Thus, the discrete-time Fourier transform of $y_{m}[l]$ is expressed as

$$
\begin{align*}
Y_{m}\left(e^{j 2 \pi f T_{s}}\right) & =\sum_{l=-\infty}^{\infty} y_{m}[l] e^{-j 2 \pi f l T_{p}} \\
& =\sum_{k=-K}^{K} c_{m, k} X\left(f-\frac{k}{T_{p}}\right), f \in\left[-\frac{1}{2 T_{s}}, \frac{1}{2 T_{s}}\right], \tag{2.5}
\end{align*}
$$

where $K$ is the smallest integer such that all nonzero contributions of $X(f)$ are contained in the baseband. When the single $x(t)$ is bandlimited to the band $[-W / 2, W / 2]$, the exact value of $K$ can be calculated by

$$
\begin{align*}
& -\frac{W}{2}+\frac{K}{T_{p}}<\frac{1}{2 T_{S}} \leq-\frac{W}{2}+\frac{K+1}{T_{p}} \\
\Leftrightarrow & \frac{1}{2} W T_{p}+\frac{T_{p}}{2 T_{s}}-1 \leq K<\frac{1}{2} W T_{p}+\frac{T_{p}}{2 T_{s}} \\
\Leftrightarrow & K=\left\lceil\frac{1}{2} W T_{p}+\frac{T_{p}}{2 T_{s}}\right\rceil-1 . \tag{2.6}
\end{align*}
$$

Due to the ideal lowpass filter, the $k^{\text {th }}$ contribution $X\left(f-k / T_{p}\right)$ only covers the small band $\left[k / T_{p}-1 / 2 T_{s}, k / T_{p}+1 / 2 T_{s}\right]$, which has width $1 / T_{s}$. Usually, $T_{s} \gg T_{p}$. Thus, the width of the small band is much smaller than the distance between the adjacent center frequencies, which is $1 / T_{p}$. There is no pair of contributions in the baseband that come from the same frequency of $X(f)$.

In other words, after modulation and low pass filtering, the spectrum of the signal $X(f)$ is partitioned into small bands so that all the bands cover different parts of the spectrum. All small bands are aliased into the baseband and each small band appears only once in the baseband.

From (2.4), the small bands of $X(f)$ are mixed in the baseband with coefficients $c_{m, k}$, the coefficients of the Fourier expansion of $p_{m}(t)$. When $p_{m}(t)$ are waveform alternating at the rate of $1 / N T_{p}$, as defined in (2.1), the coefficients can be evaluated as

$$
\begin{align*}
c_{m, k} & =\frac{1}{T_{p}} \int_{0}^{T_{p}} p_{m}(t) e^{-j 2 \pi k \frac{t}{T_{p}}} \mathrm{~d} t \\
& =\frac{1}{T_{p}} \sum_{n=1}^{N} \int_{\frac{n-1}{N} T_{p}}^{\frac{n}{N} T_{p}} p_{m}(t) e^{-j 2 \pi k \frac{t}{T_{p}}} \mathrm{~d} t \\
& =\frac{1}{T_{p}} \sum_{n=1}^{N} \int_{\frac{n-1}{N} T_{p}}^{\frac{n}{N} T_{p}} s_{m, n} e^{-j 2 \pi k \frac{t}{T_{p}}} \mathrm{~d} t \\
& =\frac{1}{T_{p}} \sum_{n=1}^{N} s_{m, n} \frac{e^{-j 2 \pi k n / N}-e^{-j 2 \pi k(n-1) / N}}{-j 2 \pi k / T_{p}} \\
& =\frac{1}{N} \sum_{n=1}^{N} s_{m, n} e^{-j 2 \pi k \frac{n-1}{N}} \frac{1-e^{-j 2 \pi k / N}}{j 2 \pi k / N} . \tag{2.7}
\end{align*}
$$

By Taylor series, we have the approximation $1-e^{-j 2 \pi k / N} \approx j 2 \pi k / N$. Thus, the coefficients is approximated by

$$
\begin{equation*}
c_{m, k} \approx \frac{1}{N} \sum_{n=1}^{N} s_{m, n} e^{-j 2 \pi k \frac{n-1}{N}} \tag{2.8}
\end{equation*}
$$

which is the discrete Fourier transform (DFT) of the sequence $\mathbf{s}_{m}$.
An important conclusion from (2.8) is that the magnitudes of the small bands in the baseband of $Y\left(e^{j 2 p i f T_{s}}\right)$ are directly proportional to the corresponding magnitude response of the sequences $\mathbf{s}_{m}$. If the $\mathbf{s}_{m}$ are pseudorandom sequences, all magnitudes of the small bands will be almost the same since pseudorandom sequences have flat spectra. The modulation with pseudorandom sequences is equivalent to an all pass filter that preserves all parts of the signal $x(t)$.

Combining (2.4) and (2.8) also provides us with the possibility to change the magnitudes of the input components in the output: for those components that are not of interest, we can use the sequences that have small magnitude responses at the corresponding frequencies to reduce their contribution in the output. As we mentioned in Chapter 1, this is very important to wideband communication when some bands of spectrum are shared by many devices and then the received signals
contain some interferers. We can use some specially designed sequences to reduce the interferers before the processing in analog-to-digital converters.

### 2.3 Quadratically Constrained Quadratic Programming

In this section, we summarize QCQPs, including the related work and applications, and provide available analytical frameworks for the approximation performance of solving QCQPs by randomized projection. The approaches described here originate from the seminal paper [32], which has been extended to several related problems [3339].

Goemans and Williamson [32] proposed a randomized projection and binary quantization method to provide improved approximations for the Maximum Cut problem, which sparked the rapid development of similar approximations for related optimization problems. In [32], SDP relaxation and randomized projection was shown to provide an approximation solution with the accuracy of no worse than 0.8756 for the Maximum Cut problem, which is NP hard. Since then, several results in the literature have approximately solved many similar optimization problems under different settings, greatly improving the understanding of the capabilities of this method [33-39]. Even when the objective function satisfies the condition that there exists a binary solution such that the objective function is non-negative, SDP relaxation and randomized projection provides an exact optimal solution to the original problems [38, 40, 41].

In the fields of signal processing and communications, many practical applications have already proved that SDP relaxation and randomized projection provide accurate approximations. For example, SDP relaxation and randomized projection are now known as an efficient high performance approach in MIMO detection [4244], beamforming [45], and sensor network localization [46-48]. More recently, the randomized projection has been used for the waveform design problem in radar and provide the theoretical approximation accuracy [21-25].

### 2.3.1 Problem Formulation

In general, a QCQP problem can be written as

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \max _{\mathbf{s} \in \mathbb{F}^{N}} & \mathbf{s}^{T} \mathbf{A} \mathbf{s} \\
\text { s.t. } & \mathbf{s}^{T} \mathbf{B}_{i} \mathbf{s} \leq \alpha_{i}, \quad i \in \mathcal{I}, \\
& \mathbf{s}^{T} \mathbf{C}_{j} \mathbf{s}=\beta_{j}, \quad j \in \mathcal{E}, \tag{2.9}
\end{align*}
$$

where $\mathbf{A}, \mathbf{B}_{i}$ and $\mathbf{C}_{j}$ are characteristic matrices for the objective function $f=\mathbf{s}^{T} \mathbf{A s}$, the inequality constraint function $g_{i}=\mathbf{s}^{T} \mathbf{B}_{i} \mathbf{s}$ and the equality constraint function $h_{j}=\mathbf{s}^{T} \mathbf{C}_{j} \mathbf{s}$, respectively. $\mathbb{F}$ can be either the real valued space $\mathbb{R}$ or the complex valued space $\mathbb{C}$. Furthermore, $\mathbf{A}, \mathbf{B}_{i}$ and $\mathbf{C}_{j}$ can be either real symmetric or complex Hermitian. The class of non-convex QCQP (2.9) captures many problems that are of interest in the signal processing and communications. Specific instances of this QCQP, placing different conditions in the involved matrices, have been studied in the literature, as follows:
(I) There are no inequality constraints (i.e., $|\mathcal{I}|=0$ ), $|\mathcal{E}|=N$, and $\mathbf{C}_{k}=\mathbf{e}_{k} \mathbf{e}_{k}^{T}[22$, $24,32]$. Then the general QCQP reduces to the following boolean quadratic program:

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \max _{\mathbf{s} \in \mathbb{F}^{N}} & \mathbf{s}^{T} \mathbf{A} \mathbf{s} \\
\text { s.t. } & s_{k}^{2}=\beta_{k}, \quad k \in \mathcal{E} . \tag{2.10}
\end{align*}
$$

When all $\beta_{k}=1$ and $\mathbf{s} \in \mathbb{C}^{N}$, the solutions of (2.10) have unit modulus, which is commonly required in waveform design for radar. When all $\beta_{k}=1$ and $\mathbf{s} \in \mathbb{R}^{N}$, the solutions take values of either 1 or -1 . The most important application in this class is the Maximum Cut problem, where $\mathbf{A}$ is additionally assumed to be positive semidefinite and all of its off-diagonal entries are non-positive.


Figure 2.2. Illustration of problem (2.11) in $\mathbb{R}^{2}$
(II) There are no equality constraints (i.e., $|\mathcal{E}|=0$ ) and $\mathbf{B}_{k}$ are all positive semidefinite $[25,35,37,39]$. Then the general QCQP reduces to the following quadratic program:

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \max _{\mathbf{s} \in \mathbb{F}^{N}} & \mathbf{s}^{T} \mathbf{A s} \\
\text { s.t. } & \mathbf{s}^{T} \mathbf{B}_{k} \mathbf{s} \leq \alpha_{k}, \quad k \in \mathcal{I} . \tag{2.11}
\end{align*}
$$

An illustration of problem (2.11) is presented in Figure 2, where the colored dashed lines are the contours of the objective function with different values, the white ellipses are the feasible set for the individual constraints, and the black lines are the boundaries of constrains. Since $\mathbf{s}^{T} \mathbf{B}_{k} \mathbf{S}=\beta$ represents a highdimensional ellipsoid, the feasible set $\left\{\mathbf{s} \mid \mathbf{s}^{T} \mathbf{B}_{k} \mathbf{s} \leq \alpha_{k}, k \in \mathcal{I}\right\}$ is an intersection of ellipsoids with common center, which is neither convex nor concave and makes the problem difficult.
(III) There is only one inequality constraint and the characteristic matrix $\mathbf{B}$ is diagonal, $|\mathcal{E}|=N$, and $\mathbf{C}_{k}=\mathbf{e}_{k} \mathbf{e}_{k}^{T}[23,36,38]$. As shown in the sequel, this problem is similar to the sequence design problem except that the matrix $\mathbf{B}$ is not diagonal in the latter. This class is equivalent to the first class when the
feasible set is not empty, due to the fact that any solution that satisfies the equality constraints and thus belong to a vertex of a high-dimensional rectangle will always lie inside the high-dimensional ball described by the inequality constraint.

### 2.3.2 Semidefinite Programming Relaxation

Solving a QCQP is NP-hard [39]. Most optimization methods for QCQP are based on a relaxation of the problems where an upper bound of the optimal objective function value is computed. The SDP relaxation has been an attractive approach due to its potential to find a good approximate solution for many QCQPs, including the specific classes mentioned above.

By lifting $\mathbf{s}$ to a symmetric matrix $\mathbf{T}=\mathbf{s s}^{T} \in \mathbb{F}^{N \times N}$ with $\operatorname{Rank}(\mathbf{T})=1$, the objective function $f$ in (2.9), the inequality constraint functions $g_{k}$, and the equality constraint functions $h_{k}$ have linear representations with respect to $\mathbf{T}$ :

$$
\begin{gather*}
f(\mathbf{s})=\mathbf{s}^{T} \mathbf{A} \mathbf{s}=\operatorname{Trace}\left(\mathbf{A} \mathbf{s s}^{T}\right)=\operatorname{Trace}(\mathbf{A T})=f(\mathbf{T})  \tag{2.12}\\
g_{k}(\mathbf{s})=\mathbf{s}^{T} \mathbf{B}_{k} \mathbf{s}=\operatorname{Trace}\left(\mathbf{B}_{k} \mathbf{s s}^{T}\right)=\operatorname{Trace}\left(\mathbf{B}_{k} \mathbf{T}\right)=g_{k}(\mathbf{T})  \tag{2.13}\\
h_{k}(\mathbf{s})=\mathbf{s}^{T} \mathbf{C}_{k} \mathbf{s}=\operatorname{Trace}\left(\mathbf{C}_{k} \mathbf{s s}^{T}\right)=\operatorname{Trace}\left(\mathbf{C}_{k} \mathbf{T}\right)=h_{k}(\mathbf{T}) \tag{2.14}
\end{gather*}
$$

Therefore, the QCQP in (2.9) can be expressed equivalently as

$$
\begin{align*}
\widehat{\mathbf{T}}=\arg \max _{\mathbf{T} \in \mathbb{S}^{N}} & \operatorname{Trace}(\mathbf{A T}) \\
\text { s.t. } & \operatorname{Trace}\left(\mathbf{B}_{k} \mathbf{T}\right) \leq \alpha_{k}, \quad k \in \mathcal{I} \\
& \operatorname{Trace}\left(\mathbf{C}_{k} \mathbf{T}\right)=\beta_{k}, \quad k \in \mathcal{E} \\
& \operatorname{Rank}(\mathbf{T})=1, \tag{2.15}
\end{align*}
$$

where $\mathbb{S}^{N}$ represents the set of all $N$-dimensional positive semidefinite matrices. Given that such matrices are positive semidefinite and rank-one, any feasible solution $\mathbf{T}$ to
(2.15) can be factorized as $\mathbf{s s}^{T}$ such that $\mathbf{s}$ is a feasible solution to (2.9). On the other hand, for any solution $\mathbf{s}$ to (2.9), one can always obtain an feasible solution $\mathbf{T}=\mathbf{s s}^{T}$ to (2.15).

Though (2.15) is as difficult to solve as (2.9), it indicates that the only non-convex constraint is the rank constraint and that the objective function and all other constraints are convex with respect to $\mathbf{T}$ when $\mathbf{A}, \mathbf{B}_{k}$, and $\mathbf{C}_{k}$ are all positive semidefinite. Thus the SDP relaxation of (2.9) is obtained by ignoring the rank constraint:

$$
\begin{align*}
\widehat{\mathbf{T}}=\arg \max _{\mathbf{T} \in \mathbb{S}^{N}} & \operatorname{Trace}(\mathbf{A T}) \\
\text { s.t. } & \operatorname{Trace}\left(\mathbf{B}_{k} \mathbf{T}\right) \leq \alpha_{k}, \quad k \in \mathcal{I}, \\
& \operatorname{Trace}\left(\mathbf{C}_{k} \mathbf{T}\right)=\beta_{k}, \quad k \in \mathcal{E} \tag{2.16}
\end{align*}
$$

The resulting convex problem (2.16) can be efficiently solved, e.g., by interior-point methods $[49,50]$. In the worst case, the $\operatorname{SDP}(2.16)$ can be solved with a complexity of $\mathcal{O}\left(N^{4.5} \log (1 / \epsilon)\right)$, given a solution accuracy $\epsilon>0$. Some SDPs with special structure can be solved with more efficient customized interior-point algorithms. For instance, with a special interior-point algorithm from [50], the complexity of the corresponding SDP relaxation for problem (2.10) reduces to $\mathcal{O}\left(N^{3.5} \log (1 / \epsilon)\right)$.

### 2.3.3 Randomized Projection

After solving the SDP relaxation, the next important step is to extract a feasible solution $\tilde{\text { s }}$ to (2.9) from the optimal solution $\widehat{\mathbf{T}}$ resulting from (2.16). If $\widehat{\mathbf{T}}$ is rank-one, then one can obtain $\tilde{\mathbf{s}}$ by factorizing $\widehat{\mathbf{T}}=\tilde{\mathbf{s}} \tilde{\mathbf{S}}^{T}$ and $\tilde{\mathbf{s}}$ will be the feasible and optimal solution to (2.9). Otherwise, if the rank of $\widehat{\mathbf{T}}$ is larger than one, then we need to obtain a $\tilde{\mathbf{s}}$ such that the multiplication $\tilde{\mathbf{s}}^{T}$ is close to $\widehat{\mathbf{T}}$ while remaining feasible to (2.9). However, in general, the obtained feasible solution $\tilde{\mathbf{s}}$ will not be the optimal solution $\hat{\mathbf{s}}$.

It is natural to use the principal eigenvector of $\widehat{\mathbf{T}}$, the eigenvector corresponding to the eigenvalue with largest magnitude, to build the rank-one approximation. Specifically, when $\operatorname{Rank}(\widehat{\mathbf{T}})=r$, then $\widehat{\mathbf{T}}$ has $r$ eigenvalues $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq$ $\lambda_{r}>0$ and eigenvectors $\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{r} \in \mathbb{R}^{N}$ such that the eigendecomposition is $\widehat{\mathbf{T}}=\sum_{k=1}^{r} \lambda_{k} \mathbf{u}_{k} \mathbf{u}_{k}^{T}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{T}$, where $\mathbf{U}=\left[\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{r}\right]$ and $\boldsymbol{\Lambda}$ is the diagonal matrix with $\operatorname{Diag}(\boldsymbol{\Lambda})=\left[\lambda_{1}, \lambda_{2}, \ldots, \lambda_{r}\right]^{T}$. Since $\lambda_{1} \mathbf{u}_{1} \mathbf{u}_{1}^{T}$ is the best rank one approximation of $\widehat{\mathbf{T}}$ in the Frobenius norm sense, $\mathbf{w}=\sqrt{\lambda_{1}} \mathbf{u}_{1}$ can be a candidate solution to problem (2.9), provided that it remains feasible.

Randomization is another way to perform the rank-one approximation. Assume that $\mathbf{v} \in \mathbb{R}^{r}$ is a random vector whose entries are drawn independently and identically according to the standard Gaussian distribution, i.e., $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, where $\mathbf{I}$ is the identity matrix. Let $\mathbf{w}=\mathbf{U} \boldsymbol{\Lambda}^{1 / 2} \mathbf{v}$, where $\boldsymbol{\Lambda}^{1 / 2}$ is the element-wise square root of $\boldsymbol{\Lambda}$. A simple calculation indicates that $\widehat{\mathbf{T}}=\mathbb{E}\left(\mathbf{w w}^{T}\right)$, where $\mathbb{E}(\cdot)$ returns the elementwise expectation, i.e., $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \widehat{\mathbf{T}})$. Furthermore, we have

$$
\begin{align*}
& \mathbb{E}\left(\mathbf{w}^{T} \mathbf{A} \mathbf{w}\right)=\mathbb{E}\left(\operatorname{Trace}\left(\mathbf{A} \mathbf{w} \mathbf{w}^{T}\right)\right)=\operatorname{Trace}(\mathbf{A} \widehat{\mathbf{T}})  \tag{2.17}\\
& \mathbb{E}\left(\mathbf{w}^{T} \mathbf{B}_{i} \mathbf{w}\right)=\mathbb{E}\left(\operatorname{Trace}\left(\mathbf{B}_{k} \mathbf{w} \mathbf{w}^{T}\right)\right) \operatorname{Trace}\left(\mathbf{B}_{k} \widehat{\mathbf{T}}\right)  \tag{2.18}\\
& \mathbb{E}\left(\mathbf{w}^{T} \mathbf{C}_{j} \mathbf{w}\right)=\mathbb{E}\left(\operatorname{Trace}\left(\mathbf{C}_{k} \mathbf{w} \mathbf{w}^{T}\right)\right) \operatorname{Trace}\left(\mathbf{C}_{k} \widehat{\mathbf{T}}\right) \tag{2.19}
\end{align*}
$$

Thus $\mathbf{w}$ maximizes the expected value of the objective function in (2.9) and satisfies the corresponding constraints in expectation. In other words, the SDP relaxation in (2.16) is equivalent to the following stochastic QCQP:

$$
\begin{array}{rl}
\widehat{\mathbf{T}}=\arg \max _{\mathbf{T} \in \mathbb{S}^{N}} & \mathbb{E}\left(\mathbf{w}^{T} \mathbf{A} \mathbf{w}\right) \\
\text { s.t. } & \mathbb{E}\left(\mathbf{w}^{T} \mathbf{B}_{k} \mathbf{w}\right) \leq \alpha_{k}, \quad k \in \mathcal{I}, \\
& \mathbb{E}\left(\mathbf{w}^{T} \mathbf{C}_{k} \mathbf{w}\right)=\beta_{k}, \quad k \in \mathcal{E}, \\
& \mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{T}) . \tag{2.20}
\end{array}
$$

Such a stochastic interpretation of the SDP relaxation provides an alternative way to generate the rank-one approximated solution to (2.16).

However, both approximated solutions $\mathbf{w}$ from the eigen-decomposition and the randomized projection are not guaranteed to be feasible for the original problem (2.9). A feasible solution $\tilde{\mathbf{s}}$ can be obtained by projecting the approximated solution $\mathbf{w}$ onto the feasible solution set such that $\tilde{\mathbf{s}}$ is the nearest feasible solution to $\mathbf{w}$. There is no general method to convert the approximation solution $\mathbf{w}$ to a feasible solution $\tilde{\mathbf{s}}$. The mentioned specific classes have different procedures to obtain feasible solutions.

The feasible solutions for QCQP classes I and III are obtained by the element-wise multiplication

$$
\begin{equation*}
\tilde{\mathbf{s}}=\operatorname{Sign}(\mathbf{w}) \otimes \boldsymbol{\beta}^{1 / 2} \tag{2.21}
\end{equation*}
$$

where $\operatorname{Sign}(\mathbf{w})$ returns the signs of all entries of $\mathbf{w}$ and $\otimes$ represents element-wise multiplication [36, 38]. It is easy to check that $\tilde{s}_{k}^{2}=\beta_{k}$ for all $k \in \mathcal{E}$; thus, $\tilde{\mathbf{s}}$ satisfies the equality constraints for classes I and III and is a feasible solution to class I. Additionally, when $\mathbf{A}_{k}$ is diagonal,

$$
\begin{align*}
\operatorname{Trace}\left(\mathbf{A}_{k} \widehat{\mathbf{T}}\right) & =\operatorname{Diag}\left(\mathbf{A}_{k}\right) \otimes \operatorname{Diag}(\widehat{\mathbf{T}}) \\
& =\operatorname{Diag}\left(\mathbf{A}_{k}\right) \otimes \operatorname{Diag}\left(\tilde{\mathbf{s}} \tilde{\mathbf{s}}^{T}\right) \\
& =\operatorname{Trace}\left(\mathbf{A}_{k} \tilde{\mathbf{s}}^{T}\right) \\
& =\tilde{\mathbf{s}}^{T} \mathbf{A}_{k} \tilde{\mathbf{s}} . \tag{2.22}
\end{align*}
$$

In other words, $\tilde{\mathbf{s}}$ is also a feasible solution to class III. As a special case of QCQP class I, the feasible solutions for QCQPs under binary constraints (i.e., all $\beta_{k}=1$ ) are obtained via binary quantization $\tilde{\mathbf{s}}=\operatorname{Sign}(\mathbf{w})$ [32], which is known as binary quantization.

Alternatively, for class II, a feasible solution to (2.11) can be obtained by

$$
\begin{equation*}
\tilde{\mathbf{s}}=\mathbf{w} \min _{i \in \mathcal{I}} \sqrt{\frac{\alpha_{i}}{\mathbf{w}^{T} \mathbf{B}_{i} \mathbf{w}}} . \tag{2.23}
\end{equation*}
$$

The solution $\tilde{\mathbf{s}}$ is feasible because for each $k \in \mathcal{I}$,

$$
\begin{equation*}
\tilde{\mathbf{s}}^{T} \mathbf{B}_{k} \tilde{\mathbf{s}}=\min _{i \in \mathcal{I}} \frac{\alpha_{i}}{\mathbf{w}^{T} \mathbf{B}_{i} \mathbf{w}} \mathbf{w}^{T} \mathbf{B}_{k} \mathbf{w} \leq \frac{\alpha_{k}}{\mathbf{w}^{T} \mathbf{B}_{k} \mathbf{w}} \mathbf{s}^{T} \mathbf{B}_{k} \mathbf{w} \leq \alpha_{k} . \tag{2.24}
\end{equation*}
$$

### 2.3.4 Approximation Ratio

The goal of the SDP relaxation (2.16) is to obtain the candidate solution $\tilde{\mathbf{s}}$ for problem (2.9) that is as close to the optimal solution $\hat{\mathbf{s}}$ as possible. Since any optimal solution $\hat{\mathbf{s}}$ to the QCQP (2.9) can produce a feasible solution $\hat{\mathbf{s}} \hat{\mathbf{S}}^{T}$ to the SDP relaxation (2.16), we have $f(\hat{\mathbf{s}}) \leq f(\widehat{\mathbf{T}})$. Additionally, $f(\tilde{\mathbf{s}}) \leq f(\hat{\mathbf{s}})$ due to the fact that the solution $\tilde{\mathbf{s}}$ resulting from the relaxation solution $\widehat{\mathbf{T}}$ by any method should be feasible to the original problem (2.9). Based on these relationships, if $\gamma=f(\tilde{\mathbf{s}}) / f(\widehat{\mathbf{T}})$ is the ratio between the objective function for a feasible solution $\tilde{\mathbf{s}}$ obtained by a rankone approximation method and the objective function value for the QCQP optimal solution $\widehat{\mathbf{T}}$, then this performance ratio is no smaller than that for $\hat{\mathbf{s}}$ with the same factor:

$$
\begin{equation*}
\gamma=\frac{f(\tilde{\mathbf{s}})}{f(\widehat{\mathbf{T}})} \leq \frac{f(\widehat{\mathbf{s}})}{f(\widehat{\mathbf{T}})} \leq 1 \tag{2.25}
\end{equation*}
$$

The factor $\gamma$ measures not only how good the approximation method is but also how close the resulting solution is to the optimal solution in terms of the objective function's value.

The SDP relaxation with randomized projection provides guaranteed approximation ratios for many QCQP problems. In the work of Goemans and Willianson [32], it was shown that the expected approximation ratio $\mathbb{E}(\gamma)$, which satisfies

$$
\begin{equation*}
\mathbb{E}(\gamma) f(\widehat{\mathbf{T}}) \leq \mathbb{E}(f(\tilde{\mathbf{s}})) \leq f(\hat{\mathbf{s}}) \leq f(\widehat{\mathbf{T}}) \tag{2.26}
\end{equation*}
$$

is no less than 0.88 . For the problem (2.10) with the only assumption that $\mathbf{A}$ is positive semidefinite, the expected approximation ratio satisfies $\mathbb{E}(\gamma) \geq 2 / \pi[34,38]$. If the problem (2.10) is set in complex space, such as waveform design in radar, it is proved to have $\mathbb{E}(\gamma) \geq \pi / 4>2 / \pi$, which indicates that the QCQP with unit modulus constraints has more freedom than that with binary constraints.

### 2.4 Slepian Transform

In this section, we introduce the Slepian transform, the Slepian basis, and the fast Slepian transform. Additionally, we contrast the representations of spectrally compact signals in the Fourier and Slepian bases.

We define an $N$-dimensional complex exponential vector as

$$
\begin{equation*}
\mathcal{F}(f)=\frac{1}{\sqrt{N}}\left[1, e^{j 2 \pi f}, \ldots, e^{j 2 \pi(N-1) f}\right]^{T} \tag{2.27}
\end{equation*}
$$

where $f \in \mathcal{M}=[0,1]$ is the corresponding normalized frequency. The elements of the Fourier basis $\mathbf{f}_{m}=\mathcal{F}\left(f_{m}\right)(m=1,2, \ldots, N)$ sample the normalized frequency range $\Omega$ uniformly with the sampled frequencies $f_{m}=(m-1) / N \in \Omega$, which we also refer to as on-grid frequencies, while we refer to all other frequencies $f \in \mathcal{M}$ as off-grid frequencies.

### 2.4.1 Discrete Prolate Spheroidal Sequences

It is well-known that any bandlimited signal must be infinite in the time domain and no signal with finite length in the time domain can be bandlimited. In [51, 52],

Slepian provided a remarkable representation for bandlimited, approximately finitelength discrete-time signals using discrete prolate spheroidal sequences (DPSSs).

Given a length $N$ and a half-bandwidth $W \in(0,0.5)$, the DPSSs are a collection of $N$ discrete infinite-length signals that are strictly bandlimited to the frequency range $[-W, W]$ but highly concentrated in their first $N$ entries. DPSSs are defined as the eigenvectors of a procedure that suppresses all entries of an infinite-length signal except for the first $N$ entries and then filters out all components of the signal outside the frequency range $[-W, W]$.

We denote by $\mathbb{T}$ the operator that keeps the first $N$ entries of an infinite-length discrete signal and sets all other entries to zero. Additionally, we use $\mathbb{B}$ to represent the operator that implements a perfect low pass filter for the frequency range $[-W, W]$. Then the DPSSs is defined to be the set of $N$ real-valued infinite-length vectors $\mathbf{x}_{1}, \mathbf{x}_{2}, \ldots, \mathbf{x}_{N}$ that satisfy

$$
\begin{equation*}
\mathbb{B}\left(\mathbb{T}\left(\mathbf{x}_{i}\right)\right)=\lambda_{i} \mathbf{x}_{i}, i=1,2, \ldots, N \tag{2.28}
\end{equation*}
$$

where the eigenvalues $1 \geq \lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{N} \geq 0$, and the DPSSs are normalized such that

$$
\begin{equation*}
\left\|\mathbb{T}\left(\mathbf{x}_{i}\right)\right\|_{2}=1 \tag{2.29}
\end{equation*}
$$

for all $i=1,2, \ldots, N$.
In [51], one of the center contributions indicates that the first $2 N W$ eigenvalues are extremely close to 1 , while the remaining eigenvalues are extremely close to 0 , which is precisely shown in the following lemma.

Lemma 1 [51] For arbitrary fixed $W$ and $\rho \in(0,1)$, there exist constants $C_{1}$ and $N_{1}$ such that

$$
\begin{equation*}
\lambda_{i} \geq 1-e^{-C_{1} N} \tag{2.30}
\end{equation*}
$$

for all $i \leq 2 N W(1-\rho)$ and all $N \geq N_{1}$, and there exist constants $C_{0}$ and $N_{0}$ such that

$$
\begin{equation*}
\lambda_{i} \leq e^{-C_{0} N} \tag{2.31}
\end{equation*}
$$

for all $i \geq 2 N W(1+\rho)$ and all $N \geq N_{0}$.

Lemma 1 shows that the effective dimension of the two-step procedure $\mathbb{B}(\mathbb{T})$ is about $2 N W$. Additionally, we can approximate the eigenvalues $\lambda_{i}$ by either 1 or 0 except for those with indices around $2 N W$.

### 2.4.2 Slepian Basis

Due to their infinite length, DPSSs have limited impact in practice. It is important for computational convenience to have finite-length vectors that capture as much energy of DPSSs as possible. From [51], it was shown that

$$
\begin{equation*}
\left\|\mathbb{B}\left(\mathbb{T}\left(\mathbf{x}_{i}\right)\right)\right\|_{2}=\sqrt{\lambda_{i}} \tag{2.32}
\end{equation*}
$$

for all $i=1,2, \ldots, N$. By comparing (2.29) and (2.32), for those sequences $\mathbf{x}_{i}$ with corresponding eigenvalues $\lambda_{i} \approx 1$, nearly all the energy in $\mathbb{T}\left(\mathbf{x}_{i}\right)$ is contained in the frequency range $[-W, W]$. In other word, while any DPSS is perfectly bandlimited, the corresponding time-limited DPSS will have their spectrum almost completely concentrated in this bandwidth for the first $2 N W$ DPSS elements. Thus, the first $2 N W$ time-limited DPSSs can be used to approximate the entire infinite-length DPSSs.

Additionally, the time-limited DPSSs are orthogonal:

$$
\begin{equation*}
\left\langle\mathbb{T}\left(\mathbf{x}_{i}\right), \mathbb{T}\left(\mathbf{x}_{j}\right)\right\rangle=0 \tag{2.33}
\end{equation*}
$$

for any $i, j \in 1,2, \ldots, N$ and $i \neq j$. With the unit norm defined in (2.29), the time-limited DPSSs provide an orthonormal basis for time-limited signals.

Finally, the time-limited DPSSs have a similar eigenvalue relationship with the time-limiting and band-limiting operators to the DPSSs. By applying the timelimiting operator $\mathbb{T}$ on both side of (2.28), we have

$$
\begin{equation*}
\mathbb{T}\left(\mathbb{B}\left(\mathbb{T}\left(\mathbf{x}_{i}\right)\right)\right)=\lambda_{i} \mathbb{T}\left(\mathbf{x}_{i}\right) \tag{2.34}
\end{equation*}
$$

So the time-limited DPSSs $\mathbb{T}\left(\mathbf{x}_{i}\right)$ are actually the eigenvectors of the procedure of first band-limiting and then time-limiting the sequence.

All these properties motivate the definition of the Slepian basis to be the timelimited DPSSs:

$$
\begin{equation*}
\mathbf{g}_{i}=\mathbb{T}\left(\mathbf{x}_{i}\right), i=1,2, \ldots, N \tag{2.35}
\end{equation*}
$$

All $N$ elments indeed provide an orthonormal basis for $\mathbb{C}^{N}$.
The first $2 N W$ elements of the Slepian basis are usually sufficient to represent the $N$-length samples of any signal bandlimited to the frequency range [ $-W, W$ [53, 54]. By modulating the baseband Slepian basis with an element of the Fourier basis, one can obtain a subspace approximation of signals restricted to any frequency subset of $[0,1]$. For example, the modulated Slepian basis $\mathcal{F}(f) \circ \mathbf{g}_{1}, \mathcal{F}(f) \ldots, \mathcal{F}(f) \circ \mathbf{g}_{N}$ can be used to compactly represent signals bandlimited to the range $[f-W, f+W]$, where - denotes an element-wise product. We can also concatenate such bases for different frequency ranges to obtain a frame that can successfully approximate multi-band signals of interest.

### 2.4.3 Slepian versus Fourier

Both Slepian and Fourier bases can be used to form an orthonormal basis for a subspace approximation to the set of signals band-limited to $[-W, W]$. There are some similarities between the Fourier and Slepian bases.

From (2.34), the Slepian basis elements are eigenvectors of the operator of timelimiting and band-limiting. It can be shown that an alternative way to derive the Slepian basis $\mathbf{g}$ is to consider the eigenvectors of the prolate matrix $\mathbf{Q}$ [55], which is the matrix with entries given by

$$
\begin{equation*}
Q_{m, n}=2 W \operatorname{sinc}(2 W \pi(m-n))=\frac{\sin (2 W \pi(m-n))}{\pi(m-n)} . \tag{2.36}
\end{equation*}
$$

In fact, $\mathbf{G}$ can be interpreted as the finite truncation of the infinite matrix representation of $\mathbb{B}(\mathbb{T}(\cdot))$. Assume that $y(\omega)$ is the discrete-time Fourier transform of $\mathbb{T}(\mathrm{x}):$

$$
\begin{equation*}
y(\omega)=\sum_{n=1}^{N} x_{n} e^{-j \omega n} . \tag{2.37}
\end{equation*}
$$

The entries of $\mathbb{B}(\mathbb{T}(\mathbf{x}))$ can be obtained as

$$
\begin{align*}
\mathbb{B}(\mathbb{T}(\mathbf{x}))_{m} & =\frac{1}{2 \pi} \int_{-2 \pi W}^{2 \pi W} y(\omega) e^{j \omega m} d \omega \\
& =\sum_{n=1}^{N} x_{n} \frac{1}{2 \pi} \int_{-2 \pi W}^{2 \pi W} e^{j \omega(m-n)} d \omega \\
& =\sum_{n=1}^{N} x_{n} \frac{e^{j 2 \pi W(m-n)}-e^{-j 2 \pi W(m-n)}}{j 2 \pi(m-n)} \\
& =\sum_{n=1}^{N} x_{n} \frac{\sin (2 \pi W(m-n))}{\pi(m-n)} . \tag{2.38}
\end{align*}
$$

Let $\mathbf{P}$ be the matrix to projects a signal to the subspace spanned by the elements of the Fourier basis with frequencies lying in the range $[-W, W]$, which is also the

Gram matrix of those elements. Then the entries of $\mathbf{P}$ are given by

$$
\begin{align*}
P_{m, n} & =\frac{1}{N} \sum_{k=-N W+\frac{1}{2}}^{N W-\frac{1}{2}} e^{j 2 \pi(m-n) k / N} \\
& =\frac{e^{j 2 \pi(m-n)\left(-N W+\frac{1}{2}\right) / N}-e^{j 2 \pi(m-n)\left(N W-\frac{1}{2}\right) / N}}{N\left(1-e^{j 2 \pi(m-n) / N}\right)} \\
& =\frac{\sin (2 \pi W(m-n))}{N \sin (\pi(m-n) / N)} \\
& =\frac{\sin (2 W N \pi(m-n) / N)}{N \sin (\pi(m-n) / N)} . \tag{2.39}
\end{align*}
$$

Comparing (2.36) with (2.39), it is easy to see that $\mathbf{Q}$ and $\mathbf{P}$ share the similar structure: $\mathbf{Q}$ is a Toeplitz matrix with rows and columns given by the sinc function; $\mathbf{P}$ is a circulant matrix with rows and columns given by the digital sinc or Dirichlet function. Actually, it has been shown in [53] that the difference between these two matrices has low rank, which is formally stated in the following theorem.

Theorem 1 [53] Let $N \in \mathbb{N}$ and $W \in\left(0, \frac{1}{2}\right)$ be fixed. For any $\epsilon \in\left(0, \frac{1}{2}\right)$, there exist matrices $\mathbf{L}_{1}, \mathbf{L}_{2} \in \mathbb{R}^{N \times r}$ and a matrix $\mathbf{E} \in \mathbb{R}^{N \times N}$ such that

$$
\begin{equation*}
\mathbf{Q}=\mathbf{P}+\mathbf{L}_{1} \mathbf{L}_{2}^{T}+\mathbf{E}, \tag{2.40}
\end{equation*}
$$

where $r \leq\left(\frac{4}{\pi^{2}} \log (8 N)+6\right) \log \frac{15}{\epsilon}$ and $\|\mathbf{E}\| \leq \epsilon$

The most significant difference between the Fourier and Slepian representations appears for signals containing off-grid frequencies. For example, when $W=1 / N$, a complex exponential $\mathbf{x}=\mathcal{F}(f)$ for $f \in\left[f_{m-1}, f_{m+1}\right]$ can be approximated as a linear combination of three elements of either the Fourier basis $\mathbf{M}=\left[\mathbf{f}_{m-1}, \mathbf{f}_{m}, \mathbf{f}_{m+1}\right]$ or the three modulated elements of the Slepian basis $\mathbf{M}=\left[\mathbf{f}_{m} \circ \mathbf{g}_{1}, \mathbf{f}_{m} \circ \mathbf{g}_{2}, \mathbf{f}_{m} \circ \mathbf{g}_{3}\right]$ with coefficients $\mathbf{c}=\mathbf{M}^{H} \mathbf{x}$. Figure 2.3 shows the energy of the coefficients for a complex exponential signal under both bases as a function of its frequency, with


Figure 2.3. Coefficient vector power of single-tone signal under the Slepian and Fourier baes as a function of the component frequency. The Fourier basis captures all of the signal energy only for on-grid frequencies, while some energy is lost when the signal contains off-grid frequencies. The Slepian basis preserves almost all energy at most frequencies.
$m=4$. Although the Fourier basis compacts the signal energy to a single coefficient when the frequency is on-grid (i.e., $f \in\left\{f_{m-1}, f_{m}, f_{m+1}\right\}$ ), some energy is leaked to other coefficients when the frequency is off-grid. In contrast, the top three coefficients of the signal in the Slepian basis capture almost all of the energy of a signal at all values of the frequency within the band of interest. Nonetheless, the Fourier basis has better rejection than the Slepian basis for signals with on-grid frequencies outside the bandwidth of interest, which also affects its suitability to model signals restricted to a bandwidth within our design approach.

## CHAPTER 3

## SINGLE SEQUENCE DESIGN VIA SDP RELAXATION AND RANDOMIZED PROJECTION

In this chapter, ${ }^{1}$ we present an algorithm to design binary sequences targeted to meet a specific spectrum shape. The algorithm is based on an SDP relaxation of a QCQP followed by a randomized projection and binary quantization, an approach that is inspired by [32]. Our main contributions can be detailed as follows. First, we propose a spectrally shaped binary sequence design approach based on optimization via a QCQP. Second, we extend the randomized projection and binary quantization method of [32] to our QCQP, which features both equality and inequality constraints. Third, we provide analytical and numerical results that study the feasibility of the sequences obtained from the proposed randomization, as well as the quality of the approximation achieved by the proposed algorithm. Fourth, we propose several custom score functions for the sequences obtained from randomization that allow for an improved selection of binary sequences that achieve both message preservation and interference rejection. Finally, we present numerical simulations that perform a comparison between an exhaustive search and the proposed sequence design method when the sizes that are sufficiently small to make the exhaustive search feasible. Numerical results verify that our proposed method finds the optimal binary sequences. We also provide numerical simulations that show the advantages of the proposed algorithm against algorithms from the literature that have been modified when necessary to provide binary sequence designs.

[^0]
### 3.1 Related Work

We summarize in this section existing approaches for the problem of sequence design with unit modulus. Furthermore, we introduce some proposed modifications to these algorithms that change the constraint in the sequences from unimodularity to binary quantization, for the purpose of later comparison with our proposed approach.

### 3.1.1 SHAPE Algorithm

The SHAPE algorithm aims to find an unimodular sequence $\mathbf{s}$ whose normalized spectrum $\mathbf{x} \in \mathbb{C}^{N}$ has magnitude that meets both an upper bound $\mathbf{u} \in \mathbb{R}^{N}$ and a lower bound $\mathbf{l} \in \mathbb{R}^{N}$, repectively $[2,12]$ :

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \min _{\mathbf{s}, \mathbf{x} \in \mathbb{C}^{N}, \alpha \in \mathbb{C}} & \left\|\mathbf{F}^{H} \mathbf{s}-\alpha \mathbf{x}\right\|_{2}^{2} \\
\text { s.t. } & \left|s_{k}\right|^{2}=1, k=1,2, \ldots, N \\
& \left|x_{k}\right| \leq u_{k}, \quad k=1,2, \ldots, N \\
& \left|x_{k}\right| \geq l_{k}, \quad k=1,2, \ldots, N, \tag{3.1}
\end{align*}
$$

where $\mathbf{F}$ collects all elements of a discrete Fourier transform basis and $\alpha$ is a scalar factor accounting for the possible energy mismatch between the sequence and the constraints. The SHAPE algorithm solves (3.1) using an iterative approach with the following three main steps.

1. Given $\mathbf{s}$ and $\alpha$, find the spectrum $\mathbf{x}$ :

$$
\begin{align*}
\hat{\mathbf{x}}=\arg \min _{\mathbf{x} \in \mathbb{C}^{N}} & \left\|\mathbf{F}^{H} \mathbf{s}-\alpha \mathbf{x}\right\|_{2}^{2} \\
\text { s.t. } & \left|x_{k}\right| \leq u_{k}, \quad k=1,2, \ldots, N, \\
& \left|x_{k}\right| \geq l_{k}, \quad k=1,2, \ldots, N . \tag{3.2}
\end{align*}
$$

The solution is given by

$$
\hat{x}_{k}=\left\{\begin{array}{lc}
u_{k} \frac{\mathbf{f}_{k}^{H} \mathbf{s} / \alpha}{\left|\mathbf{f}_{k}^{H} \mathbf{s} / \alpha\right|}, & \left|\mathbf{f}_{k}^{H} \mathbf{s} / \alpha\right| \geq u_{k}  \tag{3.3}\\
l_{k} \frac{\mathbf{f}_{k}^{H} \mathbf{s} / \alpha}{\left|\mathbf{f}_{k}^{H} \mathbf{s} / \alpha\right|}, & \left|\mathbf{f}_{k}^{H} \mathbf{s} / \alpha\right| \leq l_{k}, \quad k=1,2, \ldots, N \\
\mathbf{f}_{k}^{H} \mathbf{s} / \alpha, & \text { otherwise }
\end{array}\right.
$$

where $\mathbf{f}_{k}$ denotes the $k^{\text {th }}$ column of $\mathbf{F}$.
2. Given $\mathbf{s}$ and $\mathbf{x}$, find the factor $\alpha$ :

$$
\begin{equation*}
\hat{\alpha}=\arg \min _{\alpha \in \mathbb{C}}\left\|\mathbf{F}^{H} \mathbf{s}-\alpha \mathbf{x}\right\|_{2}^{2} \tag{3.4}
\end{equation*}
$$

The solution is given by

$$
\begin{equation*}
\hat{\alpha}=\frac{\mathbf{x}^{H} \mathbf{F}^{H} \mathbf{s}}{\|\mathbf{x}\|_{2}^{2}} \tag{3.5}
\end{equation*}
$$

3. Given $\alpha$ and $\mathbf{x}$, find the sequence $\mathbf{s}$ :

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \min _{\mathbf{s} \in \mathbb{C}^{N}} & \left\|\mathbf{F}^{H} \mathbf{s}-\alpha \mathbf{x}\right\|_{2}^{2} \\
\text { s.t. } & \left|s_{k}\right|^{2}=1, \quad k=1,2, \ldots, N . \tag{3.6}
\end{align*}
$$

The solution is given by

$$
\begin{equation*}
\hat{s}_{i}=\frac{\alpha\left\langle\mathbf{f}_{k}, \mathbf{x}\right\rangle}{\left|\alpha\left\langle\mathbf{f}_{k}, \mathbf{x}\right\rangle\right|}, \quad k=1,2, \ldots, N \tag{3.7}
\end{equation*}
$$

A straightforward change to the SHAPE algorithm to provide binary sequences instead of unimodular ones is to replace the optimization (3.6) with the binary constraint problem

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \min _{\mathbf{s} \in \mathbb{R}^{N}} & \left\|\mathbf{F}^{H} \mathbf{s}-\alpha \mathbf{x}\right\|_{2}^{2} \\
\text { s.t. } & \left|s_{k}\right|^{2}=1, \quad k=1,2, \ldots, N . \tag{3.8}
\end{align*}
$$

The resulting binary sequence can be obtained as

$$
\begin{equation*}
\hat{s}_{k}=\operatorname{Sign}\left(\operatorname{Real}\left(\alpha\left\langle\mathbf{f}_{k}, \mathbf{x}\right\rangle\right)\right), \tag{3.9}
\end{equation*}
$$

where Real $(\cdot)$ denotes the real part of a complex number and $\operatorname{Sign}(\cdot)$ denotes the sign of a real number.

### 3.1.2 LPNN Algorithm

In [18], a Lagrange programming neural network (LPNN) for unimodular sequence design with target spectrum $\mathbf{x}$ is formulated as follows:

$$
\begin{align*}
\hat{\mathbf{s}}=\arg \min _{\mathbf{s} \in \mathbb{C}^{N}, \alpha \in \mathbb{R}} & \sum_{i=1}^{N} w_{i}\left(\left|\mathbf{f}_{i}^{H} \mathbf{s}\right|^{2}-\alpha x_{k}\right)^{2}+c_{0} \sum_{i=1}^{N}\left(\left|\mathbf{e}_{i}^{H} \mathbf{s}\right|^{2}-1\right)^{2} \\
\text { s.t. } & \left|\mathbf{e}_{i}^{T} \mathbf{s}\right|^{2}=1, \quad i=1,2, \ldots, N . \tag{3.10}
\end{align*}
$$

Here $\mathbf{e}_{i}$ denotes the canonical basis whose $i^{\text {th }}$ entry is 1 and others are 0 , and $w_{i}$ are weights for each frequency component. The second term in the objective function is the augmented term to improve the convexity and stability. By separating the real and imaginary parts of the matrices and vectors in the equation as

$$
\begin{align*}
\overline{\mathbf{s}} & =\left[\begin{array}{c}
\operatorname{Real}(\mathbf{s}) \\
\operatorname{Imag}(\mathbf{s})
\end{array}\right], \\
\overline{\mathbf{F}}_{i} & =\left[\begin{array}{cc}
\operatorname{Real}\left(\mathbf{f}_{i}\right) & -\operatorname{Imag}\left(\mathbf{f}_{i}\right) \\
\operatorname{Real}\left(\mathbf{f}_{i}\right) & \operatorname{Imag}\left(\mathbf{f}_{i}\right)
\end{array}\right], \\
\overline{\mathbf{E}}_{i} & =\left[\begin{array}{ll}
\mathbf{e}_{i} & \mathbf{0} \\
\mathbf{0} & \mathbf{e}_{i}
\end{array}\right] \tag{3.11}
\end{align*}
$$

where $\operatorname{Imag}(\cdot)$ denotes the imaginary part of a complex number, the complex-valued optimization (3.10) is transformed into the real-valued optimization

$$
\begin{align*}
\min _{\overline{\mathbf{s}} \in \mathbb{R}^{2 N}, \alpha \in \mathbb{R}} & \sum_{i=1}^{N} w_{i}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{F}}_{i} \overline{\mathbf{F}}_{i}^{T} \overline{\mathbf{s}}-\alpha x_{i}\right)^{2}+c_{0} \sum_{i=1}^{N}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}}-1\right)^{2} \\
\text { s.t. } & \overline{\mathbf{s}}^{T} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}}=1, \quad i=1,2, \ldots, N . \tag{3.12}
\end{align*}
$$

The Lagrangian function for this problem is set up as

$$
\begin{align*}
l(\overline{\mathbf{s}}, \alpha, \boldsymbol{\mu})= & \sum_{i=1}^{N} w_{i}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{F}}_{i} \overline{\mathbf{F}}_{i}^{T} \overline{\mathbf{s}}-\alpha x_{i}\right)^{2}+c_{0} \sum_{i=1}^{N}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}}-1\right)^{2} \\
& +\sum_{i=1}^{N} \mu_{i}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}}-1\right) \tag{3.13}
\end{align*}
$$

where $\boldsymbol{\mu}$ is the Lagrange multiplier. The LPNN then computes increments for the parameters and solution of this problem as follows:

$$
\begin{align*}
\Delta \overline{\mathbf{s}}= & -\frac{\partial l}{\partial \overline{\mathbf{s}}} \\
= & -4 \sum_{i=1}^{N} w_{i}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{F}}_{i} \overline{\mathbf{F}}_{i}^{T} \overline{\mathbf{s}}-\alpha x_{i}\right) \overline{\mathbf{F}}_{i} \overline{\mathbf{F}}_{i}^{T} \overline{\mathbf{s}}-4 c_{0} \sum_{i=1}^{N}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}}-1\right) \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}} \\
& -2 \sum_{i=1}^{N} \mu_{i} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \bar{s}  \tag{3.14}\\
\Delta \alpha= & -\frac{\partial l}{\partial \alpha}=2 \sum_{i=1}^{N} w_{i}\left(\overline{\mathbf{s}}^{T} \overline{\mathbf{F}}_{i} \overline{\mathbf{F}}_{i}^{T} \overline{\mathbf{s}}-\alpha x_{i}\right) x_{i}  \tag{3.15}\\
\Delta \mu_{i}= & -\frac{\partial l}{\partial \mu_{i}}=\overline{\mathbf{s}}^{T} \overline{\mathbf{E}}_{i} \overline{\mathbf{E}}_{i}^{T} \overline{\mathbf{s}}-1 \tag{3.16}
\end{align*}
$$

The LPNN algorithm initializes the so-called neurons $\bar{s}, \alpha, \boldsymbol{\mu}$ randomly. The neurons are updated using the increments above at each iteration $k$ :

$$
\begin{align*}
\overline{\mathbf{s}}^{k+1} & =\overline{\mathbf{s}}^{k}+\rho \Delta \overline{\mathbf{s}}  \tag{3.17}\\
\alpha^{k+1} & =\alpha^{k}+\rho \Delta \alpha,  \tag{3.18}\\
\boldsymbol{\mu}^{k+1} & =\boldsymbol{\mu}^{k}+\rho \Delta \boldsymbol{\mu} \tag{3.19}
\end{align*}
$$

Finally, the unimodular sequence $\hat{\mathbf{s}}$ is constructed by taking the first and last $N$ entries of $\overline{\mathbf{s}}$ as its real and imaginary parts.

The LPNN algorithm can be modified to provide binary sequences by changing the $\mathbf{s} \in \mathbb{C}^{N}$ constraint in (3.10) to $\mathbf{s} \in \mathbb{R}^{N}$, making (3.10) a real-valued optimization. Thus, we can directly obtain the dynamics for the neurons $\mathbf{s}, \alpha$, and $\boldsymbol{\mu}$ by replacing $\overline{\mathbf{s}}$ to $\mathbf{s}, \overline{\mathbf{F}}$ to $\mathbf{F}$, and $\overline{\mathbf{E}}_{i}$ to $\mathbf{e}_{i}$ in (3.15-3.17).

### 3.2 Binary Sequence Design via QCQP

In this section, we develop an efficient method to generate a binary sequence that is based on the SDP relaxation and randomized projection introduced in Section 2.3. A filter implemented to have such a sequence as its impulse response provides a frequency response with a bandpass and a notch for the message and interferer bands,
respectively. We also provide a theoretical analysis of the algorithm to show its approximation ratio and the likelihood of feasibility for the randomized sequences obtained. To improve the performance of the algorithm, we end the section with a discussion on possible additional criteria to select among the multiple sequences obtained via the proposed randomization.

### 3.2.1 Design Algorithm

We desire for the spectrally shaped sequence to provide a passband and notch for the pre-determined message and interferer bands, respectively. We denote by $\mathbf{F}_{\mathcal{P}}$ and $\mathbf{F}_{\mathcal{S}}$ the collection of all discrete Fourier transform basis elements corresponding to the message band $\mathcal{P} \subseteq\{1,2, \ldots, N\}$ and interferer band $\mathcal{S} \subseteq\{1,2, \ldots, N\}$, respectively. We also assume that $\mathcal{P} \cap \mathcal{S}=\emptyset$, but we do not place any other restrictions on the message and interferer bands. A least-squares fitting approach for designing an $N$-points binary sequence can be written as the QCQP

$$
\begin{array}{rl}
\hat{\mathbf{s}}=\arg \max _{\mathbf{s} \in \mathbb{R}^{N}} & f(\mathbf{s})=\left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}\right\|_{2}^{2} \\
\text { s.t. } & g(\mathbf{s})=\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right\|_{2}^{2} \leq \alpha \\
& h_{k}(\mathbf{s})=s_{k}^{2}=1, \quad k=1,2, \ldots, N \tag{3.20}
\end{array}
$$

for some interferer tolerance $\alpha>0$, where $s_{k}$ denotes the $k^{\text {th }}$ entry of $\mathbf{s}$. As mentioned in Section 2.3, such an integer optimization problem is NP-hard. Though it is possible to use an exhaustive method that searches over all possible binary sequences to return the optimal sequence when the sequence length is very small, it is too inefficient and even impossible to use the exhaustive method when the sequence length is relatively large.

Following the framework prescribed in Section 2.3, the SDP relaxation for the QCQP (3.20) can be obtained by noting that $\left\|\mathbf{F}_{\mathcal{P}} \mathbf{s}\right\|_{2}^{2}=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{s s}^{T}\right)$ and
$\left\|\mathbf{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{s s}^{T}\right)$, providing us with the optimization

$$
\begin{array}{rl}
\widehat{\mathbf{T}}=\arg \max _{\mathbf{T} \in \mathbb{S}^{N}} & f(\mathbf{T})=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{T}\right) \\
\text { s.t. } & g(\mathbf{T})=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{T}\right) \leq \alpha / 2, \\
h_{k}(\mathbf{T})=T_{k, k}=1, \quad k=1,2, \ldots, N, \tag{3.21}
\end{array}
$$

where $T_{k, k}$ denotes the $k^{\text {th }}$ diagonal entry of $\mathbf{T}$. Note that we omit the redundant operations that take the real part of $f(\mathbf{T})$ and $g(\mathbf{T})$ since both $\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H}$ and $\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H}$ are Hermitian and these quadratic functions will always be real-valued. Note also that the inequality constraint bound has been halved, which will be justified in Theorem 2.

Our proposed SDP approximation and randomization for the QCQP is detailed in Algorithm 1. After obtaining and decomposing the optimal solution $\widehat{\mathbf{T}}$ for the SDP relaxation, a randomly generated vector $\mathbf{v} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}_{N}\right)$ is used to project $\widehat{\mathbf{T}}$ from a high dimensional space to a low dimensional space and obtain the approximation vector $\mathbf{w}_{\ell}$. A candidate binary sequence $\tilde{\mathbf{s}}_{\ell}$ is then obtained by quantizing the approximation vector $\mathbf{w}_{\ell}$. The algorithm repeats the random projection $L$ times to provide a set of candidate sequences and finally outputs the sequence that maximizes the message band power while meeting the requested upper bound for the interferer band power.

### 3.2.2 Approximation Performance

As mentioned in Section 2.3.4, the key to the performance analysis of the spectrally shaped binary sequence design (i.e., the performance of using a candidate sequence $\tilde{\mathbf{s}}$ as the approximation of optimal sequence $\hat{\mathbf{s}}$ ) is to evaluate the approximation ratio $\gamma$ such that any $\tilde{\mathbf{s}}$ generated in step 7 of Algorithm 1 satisfies $f(\tilde{\mathbf{s}}) \geq \gamma f(\hat{\mathbf{s}})$. The larger that approximation ratio $\gamma$ is, the closer that candidate sequence $\tilde{\mathbf{s}}$ could be to the optimal sequence $\hat{\mathbf{s}}$ in the sense of objective function value.

```
Algorithm 1 Binary Sequence Design
Input: message band \(\mathcal{P}\), interferer band \(\mathcal{S}\), interferer tolerance \(\alpha\), random search size
    \(L\)
Output: binary sequence \(\hat{\mathbf{s}}\)
    generate bases \(\mathbf{F}_{\mathcal{P}}, \mathbf{F}_{\mathcal{S}}\) for message and interferer bands
    obtain optimal solution \(\widehat{\mathbf{T}}\) to SDP relaxation (3.21)
    compute eigendecomposition for \(\widehat{\mathbf{T}}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{T}\)
    for \(\ell=1,2, \ldots, L\) do
        generate random vector \(\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})\)
        obtain approximation by projecting \(\mathbf{w}_{\ell}=\mathbf{U} \boldsymbol{\Lambda}^{1 / 2} \mathbf{v}\)
        obtain candidate by quantization \(\tilde{\mathbf{s}}_{\ell}=\operatorname{Sign}\left(\mathbf{w}_{\ell}\right)\)
    end for
    select best binary sequence
```

$$
\hat{\mathbf{s}}=\arg \max _{\tilde{\mathbf{s}}_{\ell}: 1 \leq \ell \leq L}\left\{f\left(\tilde{\mathbf{s}}_{\ell}\right): g\left(\tilde{\mathbf{s}}_{\ell}\right) \leq \alpha\right\}
$$

Our binary sequence design has a very similar form as the QCQP class III (cf. Section 2.3): both contain equality constraints and inequality constraints, and the characteristic matrices for the inequality constraints can be factorized as the multiplication of a canonical vector and its transpose. Those similarities inspire us to use the binary quantization $\tilde{\mathbf{s}}=\operatorname{Sign}(\mathbf{w})$ after the randomized projection in sequence design.

However, while the characteristic matrices for inequality constraints in the QCQP class III are diagonal, those in sequence design are rarely diagonal. It is impossible for $\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H}$, the characteristic matrix for the inequality constraint in (3.21), to be diagonal except for the uninteresting case when $\mathcal{S}=\{1,2, \ldots, N\}$ and $\mathcal{P}=\emptyset$, i.e., the interferer band covers the whole spectrum. This causes the result discrepancy between our proposed approach and the QCQP class III relaxation: while the characteristic matrix $\mathbf{B}$ for the inequality constraints of a class III QCQP is diagonal, the inequality constraint function for the candidate solution $g(\tilde{\mathbf{s}})=\tilde{\mathbf{s}}^{T} \mathbf{B} \tilde{\mathbf{s}}=\operatorname{Trace}\left(\mathbf{B} \tilde{\mathbf{s}} \tilde{\mathbf{s}}^{T}\right)$ is equal to the inequality constraint function for the SDP relaxation solution $g(\widehat{\mathbf{T}})=\operatorname{Trace}(\mathbf{B} \widehat{\mathbf{T}})$, since the diagonal entries of $\tilde{\mathbf{s}} \tilde{\mathbf{s}}^{T}$ and $\widehat{\mathbf{T}}$ are the same. In contrast, in our pro-



Figure 3.1. Illustration of feasible sets in (Left) the QCQP class III and (Right) sequence design. Red dots represent the possible binary vectors. Black ellipses represent the bounds of inequality constraints.
posed sequence design algorithm, given that $\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H}$ is not diagonal, we have that $g(\tilde{\mathbf{s}})=\tilde{\mathbf{s}}^{T} \mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \tilde{\mathbf{s}}$ is not equal to $g(\widehat{\mathbf{T}})=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \hat{S}\right)$, even when the diagonal entries of $\tilde{\mathbf{s}} \tilde{\mathbf{s}}^{T}$ and $\widehat{\mathbf{T}}$ are still the same.

There is also some geometric intuition behind this difference. Any binary vector obtained via randomized projection and binary quantization is one of the vertices of a hypercube. To be a feasible solution, the binary vector must lie inside the set defined by the inequality constraints. Both $g(\mathbf{s})$ in (2.9) and (3.20) are quadratic functions and both characteristic matrices $\mathbf{B}$ and $\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H}$ are positive semidefinite, so each inequality constraint defines a set bounded by an ellipsoid in a high dimensional space. The eigenvectors for $\mathbf{B}$ and $\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H}$ are the principal axes of the two ellipsoids. Since $\mathbf{B}$ is diagonal, the eigenvectors are the canonical vectors and the ellipsoid is symmetric with respect to any axes. If a binary vector lies inside the ellipsoid, then all binary vectors also lie inside the ellipsoid. In contrast, the eigenvectors for $\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H}$ are rarely the canonical vectors, so it is possible for some binary vectors to lie outside the ellipsoid even when others lie inside. Figure 3.1 shows the difference in an example two-dimensional space.

In summary, binary sequences $\tilde{\mathbf{s}}$ resulting from $\widehat{\mathbf{T}}$ via randomized projection and binary quantization may not be feasible to the inequality constraint, i.e., $\left\|\mathbf{F}_{\mathcal{S}} \tilde{\mathbf{s}}\right\|_{2}^{2} \geq$ $\alpha$. Analyzing the performance of $\tilde{\mathbf{s}}$ consists of evaluating the feasibility probability
and approximation ratio: the former describes how often $\tilde{\mathbf{s}}$ satisfies the inequality constraints and the latter measures how good $\tilde{\mathbf{s}}$ is provided that it is feasible.

Intuitively, the feasibility probability of $\tilde{\mathbf{s}}$ highly depends on $\alpha$ and the $\operatorname{rank}$ of $\mathbf{F}_{\mathcal{S}}$, which is also the width of the interferer band. As can be seen in Figure 3.1, decreasing $\alpha$ shrinks the ellipsoid defined by the inequality constraints and therefore fewer binary sequences are contained in the ellipsoid, which causes a reduced feasibility probability. Furthermore, a wider interferer band put more strict constraints on the sequences, which makes it harder for the sequences to be feasible. These can be shown in the following theorem, proven in Appendix 7.

Theorem 2 Assume that $\widehat{\mathbf{T}}$ is a solution for the SDP relaxation (3.21) and $\mathbf{s}$ is a binary vector obtained via randomized projection and binary quantization from $\widehat{\mathbf{T}}$. Define the ratio

$$
\begin{equation*}
\beta=\frac{\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \arcsin \widehat{\mathbf{T}}\right)}{\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}}^{H} \mathbf{F}_{\mathcal{S}} \widehat{\mathbf{T}}\right)} \tag{3.22}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\operatorname{Prob}\left\{\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right\|_{2}^{2} \geq \frac{1}{\pi}(\beta+1) \alpha\right\} \leq \exp \left(-C \frac{\alpha^{2}}{K^{2}}\right) \tag{3.23}
\end{equation*}
$$

where $C$ is a constant and $K$ is the number of columns of $\mathbf{F}_{\mathcal{S}}$.

It is worth noting that the ratio $\beta$ depends on the particular solution $\widehat{\mathbf{T}}$. Furthermore, it is not possible to obtain a probability tail bound that is independent of $\mathbf{T}$. To see this, consider the case when all columns of $\widehat{\mathbf{T}}$ lie in the null space of $\mathbf{F}_{\mathcal{S}}$, which would cause $\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}} \widehat{\widehat{\mathbf{T}}}\right)=0$. The ratio $\beta$ will be infinite even if $\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \arcsin \widehat{\mathbf{T}}\right)$ is very small but not zero. We also evaluate this dependence numerically: Figure 3.2 shows the empirical probability of the ratio $\beta$ over $10^{6}$


Figure 3.2. Empirical probability of the ratio $\beta$ between $\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}}^{H} \mathbf{F}_{\mathcal{S}} \arcsin \widehat{\mathbf{T}}\right)$ and Trace $\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \widehat{\mathbf{T}}\right)$ in different setting of sequence length $N$, interferer width $K$, and rank $R$ of $\widehat{\mathbf{T}}$.
randomly generated positive semidefinite matrices $\widehat{\mathbf{T}}$ for several choices of sequence design problems. Virtually all instances of the ratio $\beta$ are below $\pi-1 \approx 2.14$. When this bound on $\beta$ holds, the result above is reduced to

$$
\begin{equation*}
\operatorname{Prob}\left\{\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right\|_{2}^{2} \geq \alpha\right\} \leq \exp \left(-C \frac{\alpha^{2}}{K^{2}}\right) . \tag{3.24}
\end{equation*}
$$

We note that the reduction of the feasibility bound in (3.21) from $\alpha$ to $\alpha / 2$ is necessary to obtain the result above, given the values of $\beta$ that are observed in practice.

Numerical simulations in the sequel serve as further validation of Theorem 2, and confirm the conclusion that the larger that $\alpha$ is, and the narrower that the interferer band is, the more likely that the sequence $\tilde{\mathbf{s}}$ will meet the interferer band power constraint. Additionally, Theorem 2 implies that it is necessary to generate a sufficiently large number of candidate sequences to meet the feasibility constraints, as described in Algorithm 1.

When $\tilde{\mathbf{s}}$ is feasible to all constraints, it is possible to calculate the approximation ratio. We claim the approximation ratio by the following conjecture. Such result matches the results of other QCQPs proved repeatedly in the literature, e.g., [58,

Corollary 2.1] and [36, Proposition 1]. Though we have found it difficult to establish a theoretical proof of the following statement, we will verify the conjecture numerically in the sequel.

Conjecture 1 Consider a binary sequence $\tilde{\mathbf{s}}$ obtained via randomized projection and binary quantization from $\widehat{\mathbf{T}}$, which is the solution to (3.21). Given that $\tilde{\mathbf{s}}$ meets the inequality constraints, i.e., $\left\|\mathbf{F}_{\mathcal{S}}^{H} \tilde{\mathbf{s}}\right\|_{2}^{2} \leq \alpha$, the approximation ratio

$$
\gamma=\frac{\left\|\mathbf{F}_{\mathcal{P}}^{H} \tilde{\mathbf{s}}\right\|_{2}^{2}}{\operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \widehat{\mathbf{T}}\right)}
$$

satisfies $\gamma \geq \pi / 2-1$.

Theorem 2 and Conjecture 1 together establish that it is possible to use the randomized projection and binary quantization to generate feasible binary sequences with high probability for which the message band power is no less than $\pi / 2-1$ of the optimal power among arbitrary sequences. These two theorems are the theoretical foundation for our proposed binary sequence design method.

### 3.2.3 Sequence Selection

In Algorithm 1, the final sequence selection step not only excludes the candidate sequences that fail the interferer constraints but also finds a sequence that has an objective function value as close to that of the optimal sequence as possible. Intuitively, one would choose the feasible sequence that maximizes the objective function of (3.20), which corresponds to the sequence with maximal energy in the message band.

However, the sequence with the largest message power is not necessarily the best suited sequence for the problem of interest. As shown in Figure 3.3, the sequence selected according to the message band power maximization often has a large magnitude dynamic range (i.e., the ratio between the largest magnitude and smallest


Figure 3.3. Spectra of example binary sequences that maximize (Left) the message power and (Right) the interferer rejection ratio (3.25). Green and red markers denote the interferer and message bands.
magnitude in the sequence spectrum), in both the message band and the interferer band. Additionally, the sequence fails to attenuate the interferer with respect to the message since some magnitudes in the message band are even smaller than some in the interferer band, which potentially does not allow for successful interference rejection.

To ensure the necessary attenuation, we propose the use of the interferer rejection ratio as the metric for sequence selection after randomization. This metric is defined as the ratio between the minimum magnitude of the spectrum in the message band and the maximum magnitude in the interferer band, i.e.,

$$
\begin{equation*}
\rho(\mathbf{s}):=\frac{\min \left|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}\right|}{\max \left|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right|}, \tag{3.25}
\end{equation*}
$$

where the absolute value is taken in an element-wise fashion and the minimum and maximum are evaluated over the entries of the corresponding vectors. We find that a sequence selection driven by this criterion provides more amenable spectra for the applications of interest, as shown in Figure 3.3. Furthermore, we also find in Figure 3.3 that the dynamic range of the spectra in the bands of interest is reduced as well.

### 3.3 Numerical Experiments

To test our proposed binary sequence design algorithm, we present two groups of experiments: the first group provides experimental validation to the two theorems; the second group studies the performance of the obtained sequences in comparison to existing approaches, including the modifications listed in Section 3.1 and the exhaustive search when feasible. In all experiments, the SDP optimization (3.20) is implemented using the CVX package [59, 60].

In the first experiment, we illustrate the probability that the candidate sequences $\tilde{\mathbf{s}}$, obtained according to Algorithm 1, satisfy the interferer constraint. To validate the theorems, we set the sequence length to $N=128$, and draw $L=10^{6}$ candidate sequences to evaluate the statistical behavior of the algorithm. Figure 3.4 shows the feasibility probability as a function of the interferer tolerance $\alpha \in[0.5,10]$ when the message and interferer bands include the frequencies $\mathcal{P}=\{25,26, \ldots, 30,40,41, \ldots, 45\}$ and $\mathcal{S}=\{10,11, \ldots, 15,50,51, \ldots, 55\}$, respectively, and the feasibility probability, when the message band is $\mathcal{P}=\{1,2, \ldots, 10,50,51, \ldots, 60\}$ and the interferer tolerance is $\alpha=3$, as a function of the interferer width $|\mathcal{S}| \in[1,20]$ such that the interferer band includes frequencies with indices $\mathcal{S}=\{20,21, \ldots, 20+|\mathcal{S}|\}$. Both validate the exponential relationships predicted by Theorem 2. "Random sequence" in the figures corresponds to sequences drawn uniformly at random from $\{-1,1\}^{N}$. The random sequences have much lower probability to satisfy the interferer constraints than the candidate sequences. This indicates that it is beneficial to use the combination of an SDP relaxation and randomized projection to find the feasible sequence.

In the second experiment, we illustrate the distribution of the approximation ratio of the candidate sequence resulting from the randomized projection and binary quantization. The approximation ratio corresponds to the ratio of the values of the objective function $f(\mathbf{s})$ from (3.20) for the solution $\tilde{\mathbf{s}}$ obtained from Algorithm 1 to the objective function $f(\mathbf{T})$ from (3.21) for the solution $\widehat{\mathbf{T}}$. The setting is the same


Figure 3.4. Probability of a candidate sequence satisfying the interferer constraint as a function of (Left) interferer tolerance and (Right) interferer bandwidth.
as in the previous experiments. We also compare to $R$ random binary sequences with entries drawn from a uniform Rademacher distribution. Figure 3.5 shows that all feasible sequences generated by Algorithm 1 have approximation ratio $\gamma \geq \pi / 2-$ 1, which is marked by the red dotted line; the figure also shows that Algorithm 1 consistently outperforms random sequence designs, as expected from the spectral shaping. This numerically proves that the sequences obtained from Algorithm 1 meets the approximation ratio $\pi / 2-1$, as detailed in Conjecture 1. Additionally, the results motivate the use of random projections rather than the only eigendecomposition to obtain the candidate sequences, given that some feasible solutions are able to achieve a higher approximation ratio than the quantized principal eigenvector of $\widehat{\mathbf{T}}$, whose approximation ratio is marked by the black dashed line in Figure 3.5. These numerical results show that the candidate sequences obtained from Algorithm 1 have a high probability of satisfying the interferer constraints and large message band power, which we can interpret as successful spectrally shaped binary sequence design.

In the third experiment, we compare the performance of the sequences obtained from Algorithm 1 versus the optimal sequences obtained by the exhaustive search. By setting the sequence length $N=16$, we can feasibly perform an exhaustive search over all the $2^{16}=65536$ possible binary sequences. Both the message and interferer


Figure 3.5. Distribution of approximation ratio $\gamma$ of candidate sequences that are feasible to the interferer constraints. The red dotted and black dashed lines represent the bound predicted by Conjecture $1(\gamma \geq \pi / 2-1)$ and the approximation ratio corresponding to the quantized principal eigenvector of $\widehat{\mathbf{T}}$, respectively.
band contain only two frequency bins, and so there are $\binom{8}{2} \times\binom{ 6}{2}=420$ different choices to set the message and interferer band in the spectrum accordingly, given that message and interferer bands share no common frequency and the spectrum of a binary sequence is symmetric. Figure 3.6 shows the ratio of the performance metrics for the sequences obtained by Algorithm 1 over those for the optimal sequences from an exhaustive search as a function of the size of the random search size $L$ (i.e., the number of candidate sequences generated in Algorithm 1) after being normalized by the size of the exhaustive search space. We use three measures of performance averaged over all 420 choices: the message band power, the interference rejection ratio (3.25) and reciprocal message dynamic range, which is defined as

$$
\begin{equation*}
\chi(s)=\frac{\min \left|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}\right|}{\max \left|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}\right|} \tag{3.26}
\end{equation*}
$$

to measure the dynamic range in message band. Noting that each of these metrics can be correspondingly used as the score function in the selection step of Algorithm 1. The message power of the sequences obtained from Algorithm 1 matches that obtained from exhaustive search, even if $L$, the random search size in Algorithm 1, is


Figure 3.6. Average ratio of message power and interferer rejection ratio and reciprocal message dynamic range between the sequences obtained from Algorithm 1 and the optimal sequences from an exhaustive search.
much smaller than the exhaustive search size. Though the proposed method could not find the sequence with the largest interferer rejection ratio, it provides a good approximation with low complexity.

In the fourth experiment, we compare the performance of the sequences obtained from Algorithm 1 versus both unimodular and binary sequences from SHAPE and LPNN algorithms (cf. Section 3.1) over 100 randomly drawn message and interferer configurations. The sequence length and the message bandwidth are fixed to be $N=128$ with $|\mathcal{P}|=10$ and $|\mathcal{S}|$ varying between 1 and 10 . The proposed algorithm chooses the best sequence from $L=10^{5}$ candidate sequences, while the maximum iteration for SHAPE and LPNN algorithms is $10^{4}$. Figure 3.7 shows the average rejection ratio and computation time for all tested algorithms. Our proposed algorithm shows the ability to obtain a binary sequence with a clear distinction in the magnitude of the message and interferer bands. The performance of our proposed algorithm decreases as the interferer bandwidth becomes larger. Although it can be expected that the unimodular sequences obtained from both the SHAPE and the LPNN algorithms provide better interference rejection, surprisingly, our proposed method achieves performance similar to the unimodular sequences found by existing methods. Furthermore, our


Figure 3.7. Average interferer rejection ratio (Left) and computation time (Right) of SHAPE and LPNN for unimodular sequence and binary sequence and Algorithm 1 for binary sequence.
proposed method outperforms their binary-constrained versions, which is indicative of the difficulty of this more severely constrained problem. Note also that the quantized principal eigenvectors have much worse performance than the designed sequences, which is evidence of the benefit provided by the randomized projection search included in Algorithm 1. We finally note that the computation time for each algorithm is roughly constant over the interferer widths chosen: our proposed algorithm takes 356 seconds on average while both versions of SHAPE take 0.5 seconds on average, and the two versions of LPNN take 606 and 181 seconds on average, respectively.

## CHAPTER 4

## ITERATIVE SEQUENCE SET DESIGN FOR MULTI-BRANCH MODULATION

In Chapter 3, we presented an algorithm to design a single binary sequence targeted to meet a specific spectrum shape. Such sequences provide a passband and notch for the message and interferer bands, respectively. These previous results can be leveraged in the multi-branch modulation by mixing the signal with the same designed sequence but with different delays in different branches. The resulting linear measurement operator can be represented by a Toeplitz matrix of which each row is a circularly shifted version of the sequence. However, there are no guarantees that such modulation can provide stable recovery, as shown in the numerical experiments in this chapter.

In this chapter, ${ }^{1}$ we propose an algorithm to design multi-branch binary sequences that are capable of mitigating strong narrowband interferers. The algorithm is based on our previously proposed single sequence design, which is used repeatedly to obtain spectrally shaped sequences for all branches iteratively. Furthermore, in order to provide stable recovery performance, an additional constraint is included to require the expected sequence in each branch to be approximately orthogonal to all previously designed sequences for other branches. Our main contributions can be detailed as follows. First, we propose a multi-branch sequence design approach based on the single sequence design. Second, we introduce the inner product constraints to obtain sequences that are as pairwise mutually orthogonal as possible. Third, we provide an

[^1]analysis of the connection between the condition number of the measurement operator matrix, which serves as a measure of the stability of the recovery to noise, and the tolerance of the sequence orthogonality, and show that the number of sequences that can be obtained meeting the constraints is dependent on the sequence length. Fourth, we show the necessity of oversampling in the multi-branch sequence design in order to obtain sequences with stable invertibility performance. Finally, we present numerical results to show the advantages of the sequences obtained from the proposed algorithm in interferer mitigation and recovery stability against the pseudorandom sequence and the sequence from the single sequence design in Chapter 3.

### 4.1 Iterative Sequence Set Design

In this section, we present the details of our proposed approach to the multibranch sequence design. Our approach is based on our prior work for single sequence design from Chapter 3, which is used iteratively to obtain sequences for different branches, and an approximate orthogonality constraint is included. We also analyze the relationship between the condition number and the tolerance value. Our analysis shows that it is difficult to obtain a set of mutual orthogonal sequences of size equal to the sequence length in order to provide stable recovery, and so it is necessary to introduce oversampling in the sequence design.

### 4.1.1 Problem Formulation

We seek a set of binary sequences $\mathbf{s}_{1}, \mathbf{s}_{2}, \ldots, \mathbf{s}_{N} \in\{-1,1\}^{N}$ that are used to modulate the received signals in a multi-branch modulation architecture. In the modulation, the interferer band should be suppressed as much as possible; after the modulation, the message band should be recovered from the multiple modulations of the signal. To mitigate the interferer, the binary sequence modulation should work as a band-stop filter that provides a notch at the interferer band. Therefore,
the sequences shall have small power in the interferer band. In order to obtain a stable reconstruction of the message, the modulation system involving the sequence set should be well-conditioned to prevent large distortion in the output due to noise. As we will show in the next subsection, the requirement for the sequences to be as close to mutually orthogonal as possible provides a guarantee that the sequence set has small condition number.

We denote by $\mathbf{F}_{\mathcal{S}}$ and $\mathbf{F}_{\mathcal{P}}$ the collection of basis elements for the interferer and message bands, respectively. We also assume that $\mathcal{S}$ and $\mathcal{P}$ are disjoint, i.e., there is no overlap between the message and interferer bands. The power of a sequence $\mathbf{s}_{k}(k=$ $1,2, \ldots, N)$ in the interferer band can be measured by $\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}_{k}\right\|_{2}^{2}$. The orthogonality between a pair of sequences $\mathbf{s}_{i}$ and $\mathbf{s}_{j}$ can be measured by the normalized inner product $\left\langle\mathbf{s}_{i}, \mathbf{s}_{j}\right\rangle /\left(\left\|\mathbf{s}_{i}\right\|_{2}\left\|\mathbf{s}_{j}\right\|_{2}\right)=\mathbf{s}_{i}^{T} \mathbf{s}_{j} / N$ due to every binary sequence satisfying $\left\|\mathbf{s}_{i}\right\|_{2}=\sqrt{N}$.

The sequence design problem for each channel is then to find a binary sequence $\mathbf{s}$ such that the corresponding sequence power in the interferer band is minimized while the inner product between the sequence and each previously designed sequence for other channels is sufficiently small. Thus an approach for designing the sequence for $k^{\text {th }}$ channel $(k=1,2, \ldots, N)$ can be written as the QCQP

$$
\begin{align*}
\hat{\mathbf{s}}_{k}=\arg \min _{\mathbf{s} \in \mathbb{R}^{N}} & \left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right\|_{2}^{2} \\
\text { s.t. } & \left|\hat{\mathbf{s}}_{i}^{T} \mathbf{s}\right|^{2} \leq \alpha N, i=1,2, \ldots, k-1, \\
& s_{n}^{2}=1, n=1, \ldots, N \tag{4.1}
\end{align*}
$$

where $\alpha$ is the orthogonality tolerance, and $\hat{\mathbf{s}}_{i}(i=1, \ldots, k-1)$ is the obtained sequence for the $i^{\text {th }}$ channel.

It is easy to verify that (4.1) is a QCQP due to the fact that $\left|\hat{\mathbf{s}}_{i}^{T} \mathbf{s}_{k}\right|^{2}=\mathbf{s}_{k}^{T} \hat{\mathbf{s}}_{i} \hat{\mathbf{s}}_{i}^{T} \mathbf{s}_{k}=$ Trace $\left(\hat{\mathbf{s}}_{i} \hat{\mathbf{S}}_{i}^{T} \mathbf{s}_{k} \mathbf{s}_{k}^{T}\right)$. Following the framework prescribed in Section 2.3, the sequence design can be approximately solved by the SDP relaxation and randomized projection.

Before we present the details about solving (4.1), some discussion about the condition number for the equivalent modulation operator matrix of the obtained sequence set is necessary.

### 4.1.2 Condition Number

From [5-7], the modulation of the input signal $\mathbf{x}$ can be simplified as $\mathbf{y}=\mathbf{S x}$ when noise and nonlinearity are ignored, where $\mathbf{S}=\left[\hat{\mathbf{s}}_{1}, \ldots, \hat{\mathbf{s}}_{N}\right]^{T}$ represents the sequence set and $\mathbf{y}$ is a vector containing the modulation samples after integration for the different channels. When $\mathbf{S}$ is invertible, the original sequence can be recovered by $\hat{\mathbf{x}}=\mathbf{S}^{-1} \mathbf{y}$.

When there is noise or distortion e added to the receiver samples y before recovery, the error in the output will be $\mathbf{S}^{-1} \mathbf{e}$. Thus, the recovery performance can be measured by the proportion of the signal-to-noise ratios (SNRs) before and after recovery:

$$
\begin{equation*}
\frac{\|\mathbf{y}\|_{2} /\|\mathbf{e}\|_{2}}{\left\|\mathbf{S}^{-1} \mathbf{y}\right\|_{2} /\left\|\mathbf{S}^{-1} \mathbf{e}\right\|_{2}}=\frac{\left\|\mathbf{S}^{-1} \mathbf{e}\right\|_{2}}{\|\mathbf{e}\|_{2}} \frac{\|\mathbf{y}\|_{2}}{\left\|\mathbf{S}^{-1} \mathbf{y}\right\|_{2}}=\frac{\left\|\mathbf{S}^{-1} \mathbf{e}\right\|_{2}}{\|\mathbf{e}\|_{2}} \frac{\|\mathbf{S} \mathbf{x}\|_{2}}{\|\mathbf{x}\|_{2}} \tag{4.2}
\end{equation*}
$$

Smaller values for this ratio indicate better recovery performance. The condition number of $\mathbf{S}$ can be similarly defined as the maximum possible value of the ratio product

$$
\begin{equation*}
\kappa=\max _{\mathbf{e} \neq \mathbf{0}} \frac{\left\|\mathbf{S}^{-1} \mathbf{e}\right\|_{2}}{\|\mathbf{e}\|_{2}} \max _{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{S} \mathbf{x}\|_{2}}{\|\mathbf{x}\|_{2}} \tag{4.3}
\end{equation*}
$$

which shows the maximum possible error occurring in the recovery. The larger that the condition number is, the worse that the recovery can potentially be.

We denote the singular value decomposition of $\mathbf{S}=\mathbf{U} \boldsymbol{\Sigma} \mathbf{V}^{T}$, where $\mathbf{U}=\left[\mathbf{u}_{1}, \ldots, \mathbf{u}_{N}\right]$ and $\mathbf{V}=\left[\mathbf{v}_{1}, \ldots, \mathbf{v}_{N}\right]$ are both unitary matrices and $\boldsymbol{\Sigma}$ is a diagonal matrix whose diagonal entries are the singular values $\sigma_{1} \geq \cdots \geq \sigma_{N}>0$. Then

$$
\begin{equation*}
\max _{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{S} \mathbf{x}\|_{2}}{\|\mathbf{x}\|_{2}}=\sigma_{1}, \tag{4.4}
\end{equation*}
$$

where the equality is satisfied if and only if $\mathbf{v}_{1}^{T} \mathbf{x}=1$ and $\mathbf{v}_{i}^{T} \mathbf{x}=0$ for $i=2, \ldots, N$, i.e., $\mathbf{x}=\mathbf{v}_{1}$.

Since both $\mathbf{U}$ and $\mathbf{V}$ are unitary, $\mathbf{S}^{-1}=(\mathbf{U} \boldsymbol{\Sigma} \mathbf{V})^{-1}=\mathbf{V}^{T} \boldsymbol{\Sigma}^{-1} \mathbf{U}$, where $\boldsymbol{\Sigma}^{-1}$ is a diagonal matrix whose diagonal entries are $1 / \sigma_{N} \geq \cdots \geq 1 / \sigma_{1}>0$. Similarly,

$$
\begin{equation*}
\max _{\mathbf{e} \neq \mathbf{0}} \frac{\left\|\mathbf{S}^{-1} \mathbf{e}\right\|_{2}}{\|\mathbf{e}\|_{2}}=\frac{1}{\sigma_{N}} . \tag{4.5}
\end{equation*}
$$

Thus the condition number is also equal to

$$
\begin{equation*}
\kappa=\frac{\sigma_{1}}{\sigma_{N}} \tag{4.6}
\end{equation*}
$$

which indicates that the condition number is also defined as the ratio between the largest and smallest singular values. When the condition number is infinity, $\sigma_{N}=0$ and $\mathbf{S}$ is not invertible. When $\sigma_{N}$ is close to zero rather than strictly equal to zero, the condition number is extremely large. The error after recovery is very large even if the error before recovery is very small; such recovery is not stable. The closer that the minimum singular value is to zero, the worse that the recovery is. To guarantee the stable recovery, it is required that the minimum singular value $\mathbf{S}$ is far away from zero.

We can leverage the relationship between the minimum singular value of $\mathbf{S}$ or the minimum eigenvalue of the Gram matrix $\mathbf{Q}=\mathbf{S S}^{T}$, whose diagonal entries $Q_{i, i}=$ $\hat{\mathbf{s}}_{i}^{T} \hat{\mathbf{s}}_{i}=N$ due to the binary entries of $\hat{\mathbf{s}}_{i}$, and all off-diagonal entries $Q_{i, j}=\hat{\mathbf{s}}_{i}^{T} \hat{\mathbf{s}}_{j}$ are upper bounded in the sequence design. In [63], Gershgorin proved the following theorem to reveal the relationship between the eigenvalues and entries of a matrix.

Theorem 3 (Gershgorin Circle Theorem) For a square matrix $\mathbf{Q} \in \mathbb{C}^{N \times N}$, let $R_{i}=\sum_{j \neq i}\left|Q_{i, j}\right|(n=1,2, \ldots, N)$ be the sum of the absolute values of the off-diagonal
entries in the $i^{\text {th }}$ row, and $\mathcal{D}\left(Q_{i, i}, R_{i}\right) \subset \mathbb{C}$ be a closed disc centered at $Q_{i, i}$ with radius $R_{i}$, which is called a Gershgorin disc. Then each eigenvalue of $\mathbf{Q}$ lies within at least one of the Gershgorin discs $\mathcal{D}\left(Q_{i, i}, R_{i}\right)$.

In other words, there exists at least one index $i=1, \ldots, N$ such that the minimum singular value satisfies $\left|\sigma_{N}^{2}-Q_{i, i}\right| \leq \sum_{j \neq i}\left|Q_{i, j}\right|$. If there exists $i \in\{1, \ldots, N\}$ such that $\sum_{j \neq i}\left|Q_{i, j}\right| \geq Q_{i, i}=N$, which implies that the corresponding Gershgorin discs of $\mathbf{Q}$ contain the origin, then the minimum singular value of $\mathbf{Q}$ could be arbitrarily close to zero, and so the condition number could be arbitrarily large. In order to have a stable recovery, the value of the off-diagonal entries should be as small as possible, which indicates that sequences should be as close to mutually orthogonal as possible.

For a pair of binary sequences $\mathbf{s}_{i}, \mathbf{s}_{j} \in\{-1,1\}^{N},\left\langle\mathbf{s}_{i}, \mathbf{s}_{j}\right\rangle=\sum_{n=1}^{N} s_{i, n} s_{j, n}$, where $s_{i, n}$ is the $n^{\text {th }}$ entry of $\mathbf{s}_{i}$. Since the product term $s_{i, n} s_{j, n}$ also takes a binary value, the pair of binary sequences are orthogonal, i.e., $\left\langle\mathbf{s}_{i}, \mathbf{s}_{j}\right\rangle=0$ if and only if $N / 2$ pairs of entries of $\mathbf{s}_{i}$ and $\mathbf{s}_{j}$ at the same indices have the same values, and the other pairs of entries of $\mathbf{s}_{i}$ have the opposite values. If there are more or less than $N / 2$ matching pairs of entries, then the numbers of positive and negative terms do not match.

When $N$ is odd, it is obviously impossible for a pair of binary sequences $\mathbf{s}_{i}$ and $\mathbf{s}_{j}$ to be be orthogonal; additionally, $\left|\left\langle\mathbf{s}_{i}, \mathbf{s}_{j}\right\rangle\right| \geq 1$. Thus, the sum of absolute values of all off-diagonal entries $\sum_{i \neq j}\left|Q_{i, j}\right| \geq N-1$ for all rows. According to Theorem 3, the minimum eigenvalue may be as small as 1 and the maximum eigenvalue may be as large as $2 N-1$, making the condition number of $\mathbf{Q}$ as large as $2 N-1$ even in this case where all sequences are as orthogonal to each other as possible.

There are special cases of orthogonal binary sequence sets when $N$ is a power of 2: a well-known example is the Walsh-Hadamard codes. These codes are constructed from the elementary matrix

$$
\mathbf{H}_{2}=\left[\begin{array}{cc}
1 & 1  \tag{4.7}\\
1 & -1
\end{array}\right]
$$

This so-called Hadamard matrix contains the two codewords in the 2-dimensional Walsh-Hadamard codes. Higher-dimensional Walsh-Hadamard codes can be constructed using the Hadamard matrix as follows:

$$
\mathbf{H}_{n}=\left[\begin{array}{cc}
\mathbf{H}_{n / 2} & \mathbf{H}_{n / 2}  \tag{4.8}\\
\mathbf{H}_{n / 2} & -\mathbf{H}_{n / 2}
\end{array}\right]
$$

The matrix $\mathbf{H}_{n}$ contains $n$ orthogonal binary codewords of length $n$. Although the Walsh-Hadamard codes in multi-branch modulation provides a perfect recovery, their construction is binary only when $N$ is a power of 2 . Additionally, the WalshHadamard codes are fixed and its interferer mitigation cannot be tailored to prior knowledge of the interferer band. In fact, it is straightforward to verify that the Hadamard codes for $N$-dimensional space contains the sequences $[1,1, \ldots, 1]^{T}$ and $[1,-1, \ldots,-1]^{T}$. Those two sequences have non-zero spectra only at normalized frequencies 0 and $1 / 2$, respectively. Therefore, the Walsh-Hadamard codes mitigation performance suffers when the interferer bands contain those frequencies.

Although it is difficult to provide an analytical proof of the difficulty of the design of approximately mutually orthogonal binary sequences, we will numerically explore the feasibility of the QCQP (4.1) that aims to find $N$ binary sequences of length $N$ that are approximately mutually orthogonal.

### 4.1.3 Oversampling

The analysis above shows that it is hard to obtain $N$ binary sequences in $N$ dimensional space that are approximately orthogonal such that the Gershgorin discs are far away from the origin. Intuitively, it is easier to find $N$ binary sequences that
are approximately mutually orthogonal in a higher dimensional space. Assume that each sequence has length $R N$, which can be used to modulate signals oversampled by a factor of $R$.

As described in Section 2.3, we obtain candidate sequences $\mathbf{s}_{\ell}$ from the solution of a SDP relaxation $\widehat{\mathbf{T}}$ via randomized projection and binary quantization. In summary, we generate random vectors $\mathbf{w}_{\ell}$ as independent samples from a multivariate Gaussian distribution and the candidate binary sequences are obtained via the quantization $\mathbf{s}_{\ell}=\operatorname{Sign}\left(\mathbf{w}_{\ell}\right)$. While the presence of $\widehat{\mathbf{T}}$ in the design of the sequences introduces correlations, we study the simpler case in which the sequence entries are independent, i.e., when $\widehat{\mathbf{T}}=\mathbf{I}$ and $\mathbf{w}_{\ell} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. In this case, the entries of $s_{\ell, n}=\operatorname{Sign}\left(w_{\ell, n}\right)$ $(n=1,2, \ldots, R N)$ are random variables drawn independently and identically from a Rademacher distribution, i.e., $\mathbb{P}\left(s_{\ell, n}=1\right)=\mathbb{P}\left(s_{\ell, n}=-1\right)=1 / 2$, where $\mathbb{P}(\cdot)$ returns the probability of an event. Additionally, the pairwise entry products involved in the computation of the inner products of $\mathbf{s}_{i}$ and $\mathbf{s}_{j}$ also follow a Rademacher distribution, i.e., $\mathbb{P}\left(s_{i, n} s_{j, n}=1\right)=\mathbb{P}\left(s_{i, n} s_{j, n}=-1\right)=1 / 2$ for any $i \neq j$. The following theorem shows an upper for a so-called Rademacher sum.

Theorem 4 ([64]) Assume that $s_{1}, s_{2}, \ldots, s_{N}$ is a sequence of random variables following a Rademacher distribution and $x_{1}, x_{2}, \ldots, x_{N}$ is a set of real numbers. Then $\mathbb{P}\left(\sum_{n=1}^{N} x_{n} s_{n} \geq t \sqrt{\sum_{n=1}^{N} x_{n}^{2}}\right) \leq e^{-t^{2} / 2}$ for any $t \geq 0$.

To apply Theorem 4 to the study of the inner product $\mathbf{s}_{i}$ and $\mathbf{s}_{j}$, we set $t=\alpha \sqrt{R N}$ and $x_{n}=1$ for $n=1, \ldots, R N$ to obtain the probabilities

$$
\begin{align*}
\mathbb{P}\left(\sum_{n=1}^{R N} s_{i, n} s_{j, n} \geq \alpha R N\right) & \leq e^{-\alpha^{2} R N / 2}  \tag{4.9}\\
\mathbb{P}\left(\sum_{n=1}^{R N} s_{i, n} s_{j, n} \leq-\alpha R N\right) & \leq e^{-\alpha^{2} R N / 2} \tag{4.10}
\end{align*}
$$

where the latter statement is obtained by symmetry. Thus,

$$
\begin{equation*}
\mathbb{P}\left(\left|\mathbf{s}_{i}^{T} \mathbf{s}_{j}\right| \geq \alpha R N\right)=\mathbb{P}\left(\left|\sum_{n=1}^{R N} s_{i, n} s_{j, n}\right| \geq \alpha R N\right) \leq 2 e^{-\alpha^{2} R N / 2} \tag{4.11}
\end{equation*}
$$

Equation (4.11) shows that the probability that the pair of binary sequences $\mathbf{s}_{i}$ and $\mathbf{s}_{j}$ is not approximately orthogonal is inverse proportional to the oversampling rate $R$. With a higher oversampling rate, we are more likely to find approximately orthogonal sequences. Although it is difficult to derive a similar result to (4.11) when the sequences are drawn according to the solution of the SDP relaxation, the numerical results in the sequel confirm the conclusion that increasing the oversampling helps to obtain sequences with better interferer performance. Nonetheless, the signals cannot be recovered when the oversampled sequences are used to modulate the signals, given that the columns of the resulting modulation matrix operator $\mathbf{S} \in\{-1,1\}^{N \times R N}$ are not linearly independent.

We redefine the complex exponential vector for normalized frequency $f \in \mathcal{M}$ in the oversampled space as

$$
\begin{equation*}
\mathcal{F}(f)=\frac{1}{\sqrt{R N}}\left[1, e^{j 2 \pi f}, \ldots, e^{j 2 \pi(R N-1) f}\right] . \tag{4.12}
\end{equation*}
$$

The Fourier basis elements $\mathbf{f}_{m}(m=1,2, \ldots, R N)$ corresponding to the complex exponential vectors with the on-grid frequencies $f_{m}=(m-1) / R N \in \mathcal{M}$ that sample the normalized frequency range more finely than those in original space. Due to the oversampling, The original frequency range $[0,1]$ for N -dimensional signals is mapped to the frequency range $[0,1 / R]$ for RN-dimensional signals. Thus, while discussing the oversampled signal representations, we focus on signals that lie in the normalized frequency range $[0,1 / R]$, which contains the on-grid frequencies $f_{1}, f_{2}, \ldots, f_{N}$.

Note that the signal can be expressed as the linear combination of the basis elements for message band and interferer bands, i.e., $\mathbf{s}=\mathbf{F}_{\mathcal{P}} \mathbf{c}_{\mathcal{P}}+\mathbf{F}_{\mathcal{S}} \mathbf{c}_{\mathcal{S}}$, where $\mathbf{c}_{\mathcal{S}}$ and $\mathbf{c}_{\mathcal{P}}$ are the corresponding basis coefficients. Then the observations $\mathbf{y}=\mathbf{S x}=$
$\mathbf{S F}_{\mathcal{P}} \mathbf{c}_{\mathcal{P}}+\mathbf{S F} \mathbf{S}_{\mathcal{S}} \mathbf{c}_{\mathcal{S}}$. When the message band does not cover the whole spectrum, i.e., $|\mathcal{P}|<N, \mathbf{S F}_{\mathcal{P}}$ has linearly independent columns, and so it is possible to recover the coefficients via the pseudoinverse $\hat{\mathbf{c}}_{\mathcal{P}}=\left(\mathbf{S F}_{\mathcal{P}}\right)^{\dagger} \mathbf{y}=\left(\left(\mathbf{S F}_{\mathcal{P}}\right)^{T}\left(\mathbf{S F}_{\mathcal{P}}\right)\right)^{-1}\left(\mathbf{S F}_{\mathcal{P}}\right)^{T} \mathbf{y}$.

In the observations, the error with respect to the message, denoted by $\mathbf{e}^{\prime}=\mathbf{e}+$ $\mathrm{SF}_{\mathcal{S}} \mathbf{c}_{\mathcal{S}}$, consists of both an additional error $\mathbf{e}$ from noise and nonlinearity and the interferer. Following an analysis similar to (4.2), the recovery performance for the message coefficients can be measured by

$$
\frac{\left\|\mathbf{S F}_{\mathcal{P}} \mathbf{c}_{\mathcal{P}}\right\|_{2} /\left\|\mathbf{e}^{\prime}\right\|_{2}}{\left\|\left(\mathbf{S F}_{\mathcal{P}}\right)^{\dagger} \mathbf{y}\right\|_{2} /\left\|\left(\mathbf{S F}_{\mathcal{P}}\right)^{\dagger} \mathbf{e}^{\prime}\right\|_{2}}=\frac{\left\|\left(\mathbf{S F}_{\mathcal{P}}\right)^{\dagger} \mathbf{e}^{\prime}\right\|_{2}}{\left\|\mathbf{e}^{\prime}\right\|_{2}} \frac{\left\|\mathbf{S F}_{\mathcal{P}} \mathbf{c}_{\mathcal{P}}\right\|_{2}}{\left\|\mathbf{c}_{\mathcal{P}}\right\|_{2}}
$$

Thus, the recovery performance can be measured by the condition number of $\mathbf{S F}_{\mathcal{P}}$, i.e., the ratio between the maximum and minimum nonzero singular value of $\mathbf{S F}_{\mathcal{P}}$. This indicates that the sequence projections onto the message band are required to be as close to be orthogonal to each other as possible.

### 4.1.4 Design Algorithm

Based on the analysis in the previous section, the sequence design of each channel finds a binary sequence $\mathbf{s}$ such that the corresponding sequence power in the interferer band is minimized while the inner products between the projections of each pair of sequences on the message subspace are sufficiently small. Thus, it can be written as the QCQP

$$
\begin{align*}
\hat{\mathbf{s}}_{k}=\arg \min _{\mathbf{s} \in \mathbb{R}^{N}} & \left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right\|_{2}^{2} \\
\text { s.t. } & \left|\left\langle\mathbf{F}_{\mathcal{P}}^{H} \hat{\mathbf{s}}_{i}, \mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}\right\rangle\right|^{2} \leq \alpha R N, i=1,2, \ldots, k-1, \\
& s[n]^{2}=1, n=1, \ldots, R N . \tag{4.13}
\end{align*}
$$

The SDP relaxation for (4.13) can be obtained by noting that $\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}\right\|_{2}^{2}=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{s s}^{T}\right)$ and $\left|\left\langle\mathbf{F}_{\mathcal{P}}^{H} \hat{\mathbf{s}}_{i}, \mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}\right\rangle\right|^{2}=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \hat{\mathbf{s}}_{i} \hat{\mathbf{s}}_{i}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{s s}^{T}\right)$. By lifting $\mathbf{s}$ to $\mathbf{T}=\mathbf{s} \mathbf{s}^{T}$, the SDP
relaxation is

$$
\begin{align*}
\widehat{\mathbf{T}}_{k}=\arg \min _{\mathbf{T} \in \mathbb{S}^{R N}} & \operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{T}\right) \\
\text { s.t. } & \operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \hat{\mathbf{S}}_{i} \hat{\mathbf{S}}_{i}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{T}\right) \leq \alpha R N, \\
& i=1,2, \ldots, k-1, \\
& T_{n, n}=1, \quad n=1,2, \ldots, R N . \tag{4.14}
\end{align*}
$$

After obtaining each $\widehat{\mathbf{T}}_{k}$, we use the randomized projection and binary quantization to extract a binary sequence $\hat{\mathbf{s}}_{k}$ for each channel. As shown in Algorithm 2, we repeatedly generate a random vector $\mathbf{v}$ to project $\widehat{\mathbf{T}}_{k}$ from a matrix space to a vector space and obtain the approximation vector $\mathbf{w}_{\ell}$. A candidate binary sequence $\tilde{\mathbf{s}}_{\ell}$ is then obtained by applying binary quantizion on the approximation vector $\tilde{\mathbf{s}}_{\ell}=\operatorname{Sign}\left(\mathbf{w}_{\ell}\right)$. The algorithm repeats the random projection $L$ times to provide a set of candidate sequences and finally outputs the sequence $\hat{\mathbf{s}}_{k}$ that minimizes the objective function and satisfies the constraints in the QCQP (4.1). All sequences are obtained iteratively by following the same process.

It is important and necessary to generate a sufficiently large number of candidate sequences to meet the constraints and return the best sequence for each channel. The results in the following section show that the size of the proposed randomized search is far smaller than that of an exhaustive search.

### 4.2 Numerical Experiments

We conduct several experiments to test the performance of the proposed algorithm for the design of multi-branch binary sequences. In the following experiments, we fix both the sequence length and the number of channels to be $N=15$. The oversampling rate varies in the range $R \in\{1,2, \ldots, 10\}$. The half bandwidth of the interferer band is $W=1 / R N$ such that the band covers the frequency range $\mathcal{M}_{\mathcal{S}}=$

```
Algorithm 2 Multi-Branch Sequences Set Design
Input: interferer band basis \(\mathbf{F}_{\mathcal{S}}\), message band basis \(\mathbf{F}_{\mathcal{P}}\), coherence tolerance \(\alpha\),
    sequence length \(N\), oversampling factor \(R\), number of randomized projections \(L\)
Output: sequences \(\hat{\mathbf{s}}_{1}, \hat{\mathbf{s}}_{2}, \ldots, \hat{\mathbf{s}}_{N}\)
    for \(\mathrm{k}=1,2, \ldots, \mathrm{~N}\) do
        obtain optimal solution \(\widehat{\mathbf{T}}_{k}\) to SDP relaxation (4.14)
        compute eigendecompostion for \(\widehat{\mathbf{T}}_{k}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{T}\)
        for \(\ell=1,2, \ldots, L\) do
            generate random vector \(\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})\)
            obtain approximation by projecting \(\mathbf{w}_{\ell}=\mathbf{U} \boldsymbol{\Lambda}^{1 / 2} \mathbf{v}\)
            obtain candidate by quantization \(\tilde{\mathbf{s}}_{\ell}=\operatorname{Sign}\left(\mathbf{w}_{\ell}\right)\)
        end for
        select best binary sequence
```

$$
\hat{\mathbf{s}}_{k}=\arg \max _{\tilde{\mathbf{s}}_{\ell}: 1 \leq \ell \leq L}\left\{\left\|\mathbf{F}_{\mathcal{S}}^{H} \tilde{\mathbf{s}}_{\ell}\right\|_{2}^{2}:\left|\hat{\mathbf{s}}_{i}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \tilde{\mathbf{s}}_{\ell}\right| \leq \alpha R N\right\}
$$

end for
$\left[f_{c-1}, f_{c+1}\right] \subset \mathcal{M}$, where the on-grid frequency $f_{c}=(c-1) / R N(c=2,3, \ldots, N-1)$ is its center frequency. The message band covers the rest of spectrum, i.e., $\mathcal{M}_{\mathcal{P}}=$ $\left(f_{1}, f_{c-1}\right) \cup\left(f_{c+1}, f_{N}\right) \subset \mathcal{M}$. We denote the indices of the on-grid frequencies that fall into the message and interferer band by $\mathcal{P}=\left\{i: f_{i} \in \mathcal{M}_{\mathcal{P}}, i=1,2, \ldots, N\right\}=$ $\{1, \ldots, c-2, c+2, \ldots, N\}$ and $\mathcal{S}=\left\{i: f_{i} \in \mathcal{M}_{\mathcal{P}}, i=1,2, \ldots, N\right\}=\{c-1, c, c+1\}$, respectively.

We use two metrics for the performance of the obtained sequence set $\mathbf{S}=\left[\hat{\mathbf{s}}_{1}, \ldots, \hat{\mathbf{s}}_{N}\right]^{T}$. To measure the interferer mitigation, we use the normalized sequence power in the interferer band, i.e. $\left\|\mathbf{S F}_{\mathcal{S}}\right\|_{F}^{2} / R N=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}}^{H} \mathbf{S}^{T} \mathbf{S F}_{\mathcal{S}}\right) / R N$, where $R N$ represents the total power of the sequence set. To measure the recovery stability, we use the condition number of $\mathbf{S F}_{\mathcal{P}}$, the projection of the sequence set onto the message space. Additionally, when no sequence set candidate meeting the constraints of (4.13) is found, we set the interferer power and condition number to be infinity.

In the first experiment, we fix the number of the randomized projections $L=10^{5}$ and coherence tolerance $\alpha=0.4$. Figure 4.1 shows the average interferer power and condition number of 100 independently generated multi-branch sequence sets when


Figure 4.1. Average interferer power (Left) and condition number (Right) as a function of the oversampling rate $R$. When there is no oversampling, both the interferer power and the condition number are outside the plotted range as no feasible sequence set was found. When oversampling is included, the interferer power decreases and the condition number increases as the oversampling rate increases.
the oversampling rate $R$ varies among [1, 8]. Values of the interferer power above 0 dB and of the condition number above 20 are not shown in the figures. As we mentioned, when $R=1$, i.e., there is no oversampling, it is hard to obtain $N$ binary sequences that are approximately orthogonal to each other. Thus, the condition number is very large. When $R>1$, i.e., oversampling is included in the sequence design, the interferer power decreases as the oversampling rate increases at the cost of an increasing condition number.

In the second experiment, we vary the coherence tolerance in the range $[0.1,1]$. Figure 4.2 shows the corresponding average interferer power and condition number, and demonstrates that there is a trade-off between the interferer power and condition number: relaxing the coherence tolerance helps to obtain sequences with better interferer mitigation performance; however, the recovery performance decreases in turn. Note that when $R=2$ and $\alpha$ is small, the condition number falls outside the plotted range. This again confirms the conclusion that it is hard to obtain a binary sequence set that is approximately mutually orthogonal in a low-dimensional space.


Figure 4.2. Average interferer power (Left) and condition number (Right) as a function of the coherence tolerance $\alpha$. When $\mathrm{R}=2$, setting the coherence tolerance to a small value makes it difficult to search for a feasible solution. For all oversampling rates, increasing the coherence tolerance improves the interferer mitigation performance but also increases the condition number.

In the third experiment, we fix the coherence tolerance $\alpha=0.4$. Figure 4.3 shows the average interferer power and condition number when the number of randomized projections $L$ varies among $\left[10^{0}, 10^{6}\right]$. As shown in the figure, when the randomized projection number is not sufficiently large, the sequence obtained in at least one iteration is suboptimal or even not feasible, which results in high values in the interferer power and condition number. Additionally, the necessary number of randomized projections decreases as the oversampling rate increases. When $L$ increases, we have decreasing interferer power, since the larger size of randomized search provides better interferer mitigation performance, as well as more stable condition numbers, which are bounded by the constraints.


Figure 4.3. Average interferer power (Left) and condition number (Right) as a function of the number of randomized projections $L$. The interferer power decreases as the number of randomized projections increases, while the condition number remain stable. The necessary number of randomized projections to obtain a feasible sequence set is inversely proportion to the oversampling rates.

## CHAPTER 5

## SEQUENCE SET DESIGN BASED ON MATRIX OPTIMIZATION

In Chapter 3, we have developed an algorithm to design a single binary spectrally shaped sequence via SDP relaxation and randomized projection. In Chapter 4, we proposed an extension of the single sequence design to a sequence set design for multi-branch modulation. Based on the single sequence design, the sequence set design iteratively obtains the sequences for all branches. An additional constraint is included in the sequence set design to enforce that all sequences in the obtained set are approximately mutually orthogonal.

Unfortunately, the iterative approach of designing a sequence set for multi-branch modulation suffers some disadvantages. First, the iterative method repeats the single sequence design multiple times to find the sequence set. Redundant work is introduced in the sequence set design that increases its complexity unnecessarily. Second, since the single sequence design obtains only an approximately optimal sequence, the iterative method can propagate the error from one sequence to another sequence due to the constraints. Third, the coherence constraints in the iterative method require every pair of the sequences in the set to be as orthogonal as possible. Such constraints may be far more strict than what is necessary for stable recovery. As shown in Chapter 4, it is difficult to find a feasible sequence set without the help of oversampling.

In this chapter, we present a promising approach to design sequence sets for multibranch modulation based on a matrix optimization. The proposed method obtains
the entire sequence set simultaneously by a single optimization to prevent redundant work and cumulative approximation error caused by obtaining sequences iteratively. The main contributions of this chapter are summarized as follows. First, we propose a spectrally shaped sequence set design based on a matrix optimization after extending the metrics of power and orthogonality from sequences to a sequence set. Second, we derive a novel approach to obtain the convex relaxation of the formulated matrix optimization. Third, we build a new randomized projection for a sequence set based on the one that is used to obtain the approximation for a single sequence. Finally, we present numerical experiments that compare the performance of the sequence set design algorithms proposed in this chapter and Chapter 4.

### 5.1 Sequence Design

In this section, we present the details of the sequence set design based on matrix optimization. We start with the problem formulation for sequence set design to highlight the difference between the method introduced here and the one described in Chapter 4 . We then continue by presenting an approximation to the resulting optimization that consists of a convex relaxation and a modified randomized projection.

### 5.1.1 Problem Formulation

Recall that the goal of sequence set design for multi-branch modulation is to find multiple sequences which provide good rejection for the interferer and stable recovery for the message. While there are many measurements that can be used to evaluate those conditions, we used the sequence power in the interferer band and the maximum normalize inner products of the sequences as the metrics in the sequence set design in the iterative method presented in Chapter 4.

As in previous chapters, the bases for the message and interferer bands are denoted by $\mathbf{F}_{\mathcal{P}}$ and $\mathbf{F}_{\mathcal{S}}$, respectively. For each binary sequence $\mathbf{s}_{k}(m=1,2, \ldots, N)$ in the set,
the sequence power in the interferer band is measured by $\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}_{k}\right\|_{2}^{2}$. Therefore, the sequence set power in the interferer band can be measured by the sum of the interferer power of the different sequences, i.e., $\sum_{k=1}^{N}\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}_{k}\right\|_{2}^{2}$. By denoting the sequence set by $\mathbf{S}=\left[\mathbf{s}_{1}, \mathbf{s}_{2}, \ldots, \mathbf{s}_{N}\right]$, which collects all sequences as the columns, we have the relationship $\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{S}\right\|_{F}^{2}=\sum_{k=1}^{N}\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{s}_{k}\right\|_{2}^{2}$. Therefore, we can use the Frobenius norm $\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{S}\right\|_{F}^{2}$ to measure the sequence set power for the interferer bands. In order to have good rejection for the interferer band, it is necessary to obtain the sequence set $\mathbf{S}$ that minimizes this norm.

In order to provide stable recovery, all sequences in the set should be approximately orthogonal to each other. In the iterative approaches, this condition is achieved by the constraints $\left|\left\langle\mathbf{s}_{i}, \mathbf{s}_{j}\right\rangle\right|=\mathbf{s}_{i}^{T} \mathbf{s}_{j} \leq \alpha N$ for any $i \neq j$, where $\alpha$ is the coherence tolerance. Due to its binary entries, each sequence satisfies $\left\langle\mathbf{s}_{i}, \mathbf{s}_{i}\right\rangle=\left\|\mathbf{s}_{i}\right\|_{2}^{2}=N$. Under these constraints, we have

$$
\begin{equation*}
\left\|\mathbf{S}^{T} \mathbf{S}-N \mathbf{I}\right\|_{F}^{2}=\sum_{i=1}^{N}\left|\mathbf{s}_{i}^{T} \mathbf{s}_{i}-N\right|^{2}+\sum_{i=0}^{N} \sum_{j \neq i}\left|\mathbf{s}_{i}^{T} \mathbf{s}_{j}\right|^{2} \leq \alpha(N-1) N^{2} \tag{5.1}
\end{equation*}
$$

Therefore, we can use the constraint $\left\|\mathbf{S}^{T} \mathbf{S}-N \mathbf{I}\right\|_{F}^{2} \leq \alpha(N-1) N^{2}$ to search sequence set with approximate orthogonality.

There are many similarities between this matrix constraint and the vector constraints $\left|\left\langle\mathbf{s}_{i}, \mathbf{s}_{j}\right\rangle\right| \leq \alpha N$. However, the vector constraints require that every pair of sequences should have small coherence, which is far more strict than the matrix constraints. In fact, any sequence set satisfying the vector constraints must be feasible to the matrix constraints. As discussed in Chapter 4, it is difficult to find sequence sets that are feasible to the vector constraints. Therefore, the relaxed constraint in the proposed method may make it easier to find a sufficient number of orthogonal sequences, but, at the expensive of worse recovery stability. Even when the norm $\left\|\mathbf{S}^{T} \mathbf{S}-N \mathbf{I}\right\|_{F}^{2}$ is small, it is not guaranteed that the sum of absolute values of off-
diagonal entries in every row of $\mathbf{S}^{T} \mathbf{S}$ is small. When there are some rows containing off-diagonal entries with a large value, the corresponding Gershgorin discs cover the origin and the minimum eigenvalue of $\mathbf{S}^{T} \mathbf{S}$ can potentially very close to 0 according to the Gershgorin circle theorem. Therefore, the condition number, which is inversely proportioned to the minimum eigenvalue, can be very large.

Formally, we propose the sequence set design based on a single matrix optimization problem

$$
\begin{align*}
\widehat{\mathbf{S}}=\arg \min _{\mathbf{S} \in \mathbb{R}^{N} \times N} & \left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}\right\|_{F}^{2} \\
\text { s.t. } & \left\|\mathbf{S}^{T} \mathbf{S}-N \mathbf{I}\right\|_{F}^{2} \leq \alpha(N-1) N^{2} \\
& S_{i, j}^{2}=1, i=1,2, \ldots, N, j=1,2, \ldots, N \tag{5.2}
\end{align*}
$$

It is worth noting that the matrix optimization (5.2) is not a QCQP, since the inequality constraint cannot be expressed as a quadratic function with respect to $\mathbf{S}$. Therefore, we cannot directly extend the method described Section 2.3 to obtain a SDP relaxation for (5.2).

### 5.1.2 Oversampling

As mentioned in Section 4.1.2, it is hard to obtain $N$ binary sequences that are approximately mutually orthogonal in $N$-dimensional space. Therefore, we have to introduce the oversampling into sequence set design, in a similar way to the iterative method described in Section 4.1.3. We have to modify the constraint in (5.2) to require the projections of sequences onto the subspace spanned by message band are mutually orthogonal, i.e., $\left|\left\langle\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{i}, \mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{j}\right\rangle\right|$ for any $i \neq j$ is small enough.

Though the value of $\left|\left\langle\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{i}, \mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{i}\right\rangle\right|=\left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{i}\right\|_{2}^{2}$ depends on $\mathbf{F}_{\mathcal{P}}$ and $\mathbf{s}_{i}$, we could obtain an upper bound as

$$
\begin{equation*}
\left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{i}\right\|_{2}^{2} \leq\left\|\mathbf{F}_{\mathcal{P}}^{H}\right\|_{2}^{2}\left\|\mathbf{s}_{i}\right\|_{2}^{2}=N \sigma_{1}^{2}=N \tag{5.3}
\end{equation*}
$$

where $\sigma_{1}=1$ is the maximum singular value of $\mathbf{F}_{\mathcal{P}}^{H}$. Additionally, it is necessary that $\left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{s}_{i}\right\|_{2}^{2}$ are as close to $N$ as possible for every $i=0,1, \ldots, N$ so that the majority of the message power in the received signal is preserved. Finally, we use the constraint $\left\|\mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}-N \mathbf{I}\right\|_{F}^{2} \leq \alpha(N-1) N^{2}$ in the sequence set design to obtain sequence sets with stable recovery.

### 5.1.3 Convex Relaxation

In order to implement the convex relaxation and randomized projection to solve the matrix optimization problem (5.2) approximately, it is necessary to rewrite both the objective function and constraints as functions with respect to $\mathbf{S S}^{T}$.

Similarly to (2.16), the objective function in (5.2) can be expressed as $\left\|\mathbf{F}_{\mathcal{S}}^{H} \mathbf{S}\right\|_{F}^{2}=$ $\operatorname{Trace}\left(\mathbf{S}^{T} \mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{S}\right)=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{S S}^{T}\right)=\operatorname{Trace}\left(\mathbf{F}_{\mathcal{S}} \mathbf{F}_{\mathcal{S}}^{H} \mathbf{T}\right)$ by lifting $\mathbf{S}$ to $\mathbf{T}=\mathbf{S S}^{T}$. In other words, the objective function is also a linear function with respect to the matrix variable $\mathbf{T}$.

For the inequality constraint, we have the following relationship:

$$
\begin{align*}
\left\|\mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}-N \mathbf{I}\right\|_{F}^{2} & =\operatorname{Trace}\left(\left(\mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}-N \mathbf{I}\right)^{T}\left(\mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}-N \mathbf{I}\right)\right) \\
& =\operatorname{Trace}\left(\mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S} \mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}\right)-2 N \operatorname{Trace}\left(\mathbf{S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S}\right)+N^{3} \\
& =\operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}}^{H} \mathbf{S S}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{S S}^{T} \mathbf{F}_{\mathcal{P}}\right)-2 N \operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}}^{H} \mathbf{S S}^{T} \mathbf{F}_{\mathcal{P}}\right)+N^{3} \\
& =\operatorname{Trace}\left(\left(\mathbf{F}_{\mathcal{P}}^{H} \mathbf{S S}^{T} \mathbf{F}_{\mathcal{P}}-N \mathbf{I}\right)^{T}\left(\mathbf{F}_{\mathcal{P}} \mathbf{S S}^{T} \mathbf{F}_{\mathcal{P}}-N \mathbf{I}\right)\right)+N^{2}(N-|\mathcal{P}|) \\
& =\left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{S S}^{T} \mathbf{F}_{\mathcal{P}}-N \mathbf{I}\right\|_{F}^{2}+N^{2}(N-|\mathcal{P}|) \tag{5.4}
\end{align*}
$$

Thus we can replace the constraint in (5.2) by the equivalent formulation

$$
\begin{equation*}
\left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{T F}_{\mathcal{P}}-N \mathbf{I}\right\|_{F}^{2} \leq \alpha(N-1) N^{2}-N^{2}(N-|\mathcal{P}|) . \tag{5.5}
\end{equation*}
$$

Such constraint is not a linear function with respect to $\mathbf{T}$, but it is still convex due to the Frobenious norm.

For the equality constraint, when the condition $S_{i, j}^{2}=1$ is satisfied for any $i=$ $1,2, \ldots, N$ and $j=1,2, \ldots, N$, we have $T_{i, i}=\operatorname{Trace}\left(\mathbf{s}_{i} \mathbf{s}_{i}^{T}\right)=\left\|\mathbf{s}_{i}\right\|_{2}^{2}=N$. However, the converse is not necessarily true: it is impossible to obtain $S_{i, j}^{2}=1$ given only $T_{i, i}=N$. In other words, $T_{i, i}=N$ is the relaxed constraint to $S_{i, j}^{2}=1$ and any binary sequence set $\mathbf{S}$ can be used to build a matrix $\mathbf{S S}^{T}$ that is feasible to the former constraint.

Therefore, the convex relaxation of (5.2) can be restated as

$$
\begin{align*}
\widehat{\mathbf{T}}=\arg \min _{\mathbf{T} \in \mathbb{S}^{N}} & \operatorname{Trace}\left(\mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \mathbf{T}\right) \\
\text { s.t. } & \left\|\mathbf{F}_{\mathcal{P}}^{H} \mathbf{T} \mathbf{F}_{\mathcal{P}}-N \mathbf{I}\right\|_{F}^{2} \leq \alpha(N-1) N^{2}-N^{2}(N-|\mathcal{P}|) \\
& T_{i, i}=N, i=1,2, \ldots, N . \tag{5.6}
\end{align*}
$$

Again, we emphasize that the relaxation (5.6) is not a SDP since the inequality constraint is not a linear function with respect to the matrix variable T. Nonetheless, it is still convex and can be solved the many algorithms, including the interior-point methods.

The biggest difference between the relaxation problems of (5.6) and (3.21), which is used for single sequence design, is that the latter is derived by dropping the rank constraint $\operatorname{Rank}(\mathbf{T})=1$. In the sequence set design based on matrix optimization, the rank constraint is not necessary. The solution $\widehat{\mathbf{T}}$ should have sufficient rank if the coherence tolerance is sufficiently small such that all obtained sequences are approximately orthogonal to each other.

### 5.1.4 Randomized Projection

After obtaining the solution $\widehat{\mathbf{T}}$ for the relaxation (5.6), we still need to use the randomized projection to obtain approximation solution. Recall from Section 2.3.3 given the eigendecomposition $\widehat{\mathbf{T}}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{T}$, a randomly generated vector $\mathbf{v}$ drawn
from the standard Gaussian distribution $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ is used to obtain the projection $\mathbf{w}=\mathbf{U} \boldsymbol{\Lambda}^{1 / 2} \mathbf{v}$, which then produces a candidate sequence $\tilde{\mathbf{s}}=\operatorname{Sign}(\mathbf{w})$ by binary quantization.

To adapt this approach to our sequence set search, we need to generate multiple random vectors. Each vector is used to obtain one sequence of the entire set. More specifically, we generate a set of random vectors $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{N}$, where each $\mathbf{v}_{i}$ are independently and identically drawn from the standard Gaussian distribution, i.e., $\mathbf{v}_{i} \stackrel{\text { i.i.d }}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{I})$. Denote by $\mathbf{V}$ the random matrix that collects all random vectors $\mathbf{v}_{i}$ in columns. Then the random matrix $\mathbf{V}$ is used to generate the projections $\mathbf{W}=$ $\mathbf{U} \boldsymbol{\Lambda}^{1 / 2} \mathbf{V} / \sqrt{N}$.

In Section 2.3.3, it is mentioned that a single projection satisfies $\mathbb{E}\left(\mathbf{w w}^{T}\right)=$ $\widehat{\mathbf{T}}$. Such property is preserved for the projection matrix $\mathbf{W}$ when the randomized projection described above is used. Since $\mathbf{v}_{i} \stackrel{\text { i.i.d }}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{I}), \mathbb{E}\left(\mathbf{v}_{i} \mathbf{v}_{j}^{T}\right)=\mathbf{0}$ for any $i \neq j$ and $\mathbb{E}\left(\mathbf{v}_{i} \mathbf{v}_{i}^{T}\right)=\mathbf{I}$, we have

$$
\begin{equation*}
\mathbb{E}\left(\mathbf{V} \mathbf{V}^{T}\right)=\mathbb{E}\left(\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{v}_{i} \mathbf{v}_{j}^{T}\right)=\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}\left(\mathbf{v}_{i} \mathbf{v}_{j}^{T}\right)=\sum_{i=1}^{N} \mathbb{E}\left(\mathbf{v}_{i} \mathbf{v}_{j}^{T}\right)=N \mathbf{I} . \tag{5.7}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
\mathbb{E}\left(\mathbf{W} \mathbf{W}^{T}\right) & =\frac{1}{N} \mathbb{E}\left(\mathbf{U} \boldsymbol{\Sigma}^{1 / 2} \mathbf{V} \mathbf{V}^{T} \boldsymbol{\Sigma}^{1 / 2} \mathbf{U}^{T}\right) \\
& =\frac{1}{N} \mathbf{U} \boldsymbol{\Sigma}^{1 / 2} \mathbb{E}\left(\mathbf{V} \mathbf{V}^{T}\right) \boldsymbol{\Sigma}^{1 / 2} \mathbf{U}^{T} \\
& =\mathbf{U} \boldsymbol{\Sigma} \mathbf{U}^{T} \\
& =\widehat{\mathbf{T}} . \tag{5.8}
\end{align*}
$$

In other words, the projection $\mathbf{W}$ minimizes the expectation of the objective function while also satisfying the constraint in expectation.

In Chapter 3, we show that the principal eigenvector of $\widehat{\mathbf{T}}$, the one corresponding to the eigenvalue with largest absolute value, is a suboptimal approximation vector
since the resulting $\widehat{\mathbf{T}}$ in the single sequence design always has rank greater than 1 . In order to obtain a single approximation vector, we have to use a random vector to project all eigenvectors from high dimensional space to low dimensional space. As we mentioned in Section 5.1.3, the resulting $\widehat{\mathbf{T}}$ from the matrix optimization (5.6) should have full rank. Then the numbers of the eigenvectors of $\widehat{\mathbf{T}}$ and the designed sequences are matched. In fact, when the matrix $\mathbf{V}$ is selected to be the matrix $\mathbf{V}=\sqrt{N} \mathbf{I}$, each column of the projection $\mathbf{W}=\mathbf{U} \mathbf{\Lambda} \mathbf{V} / \sqrt{N}$ is a scaled eigenvector of $\widehat{\mathbf{T}}$. This provides an alternative way to obtain the candidate sequence with smaller complexity. It has less complexity than the randomized projection method since the repeated optimization and random projection in the latter approach can be avoided.

### 5.1.5 Design Algorithm

Based on the analysis above, the sequence set design based on matrix optimization is formulated as Algorithm 3. In a similar way to Algorithm 1, Algorithm 3 starts with the formulation of a convex relaxation to obtain the optimal solution $\widehat{\mathbf{T}}$. After eigendecomposition, $\widehat{\mathbf{T}}$ is projected onto the space spanned by each random vector in the set $\mathbf{V}$ to obtain an approximation of the sequence set $\mathbf{W}_{\ell}$. The subsequent binary quantization produces a candidate sequence set $\tilde{\mathbf{S}}_{\ell}$ by preserving only the sign of all entries. The algorithm repeats the randomized projection and binary quantization $L$ times to return the optimal sequence set with minimum power for the interferer band while satisfying the constraints in (5.2).

### 5.2 Numerical Experiments

We conduct several experiments to test the performance of the proposed algorithm on designing multi-branch binary sequences. In the following experiments, we fix both the sequence length and the number of channels to be $N=15$. The oversampling rate varies in the range $R \in\{1,2, \ldots, 10\}$. The half bandwidth of the interferer band

```
Algorithm 3 Sequence Set Design Based on Matrix Optimization
Input: Fourier basis \(\mathbf{F}_{\mathcal{P}}\) for the message band, Fourier basis \(\mathbf{F}_{\mathcal{S}}\) for the interferer
    band, coherence tolerance \(\alpha\), random search size \(L\)
Output: binary sequence set \(\widehat{\mathbf{S}}\)
    obtain optimal solution \(\widehat{\mathbf{T}}\) to SDP relaxation (5.6)
    compute SVD for \(\widehat{\mathbf{T}}=\mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^{T}\)
    for \(\ell=1,2, \ldots, L\) do
        generate random vectors \(\mathbf{v}_{i} \stackrel{\text { i.i.d }}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{I})\)
        obtain approximations by projecting \(\mathbf{W}_{\ell}=\mathbf{U} \boldsymbol{\Lambda}^{1 / 2} \mathbf{V}\)
        obtain candidate by quantization \(\tilde{\mathbf{S}}_{\ell}=\operatorname{Sign}\left(\mathbf{W}_{\ell}\right)\)
    end for
    select best binary sequence
```

        \(\widehat{\mathbf{S}}=\arg \max _{\tilde{\mathbf{S}}_{\ell}: 1 \leq \ell \leq L}\left\{\left\|\mathbf{F}_{\mathcal{P}}^{H} \tilde{\mathbf{S}}\right\|_{F}^{2}:\left\|\tilde{\mathbf{S}}^{T} \mathbf{F}_{\mathcal{P}} \mathbf{F}_{\mathcal{P}}^{H} \tilde{\mathbf{S}}-N \mathbf{I}\right\|_{F}^{2} \leq \alpha(N-1) N^{2}\right\}\)
    is $W=1 / R N$ so that the band covers the frequency range $\mathcal{M}_{\mathcal{S}}=\left[f_{c-1}, f_{c+1}\right] \subset \mathcal{M}$, where the on-grid frequency $f_{c}=(c-1) / R N(c=2,3, \ldots, N-1)$ is its center frequency. The message band covers the rest of spectrum, i.e., $\mathcal{M}_{\mathcal{P}}=\left(f_{1}, f_{c-1}\right) \cup$ $\left(f_{c+1}, f_{N}\right) \subset \mathcal{M}$. We denote the indices of the on-grid frequencies that fall into the message and interferer band by $\mathcal{P}=\left\{i: f_{i} \in \mathcal{M}_{\mathcal{P}}\right\}=\{1, \ldots, c-2, c+2, \ldots, N\}$ and $\mathcal{S}=\left\{i: f_{i} \in \mathcal{M}_{\mathcal{P}}\right\}=\{c-1, c, c+1\}$, respectively.

We use two metrics for the performance of the obtained sequence set $\mathbf{S}=\left[\hat{\mathbf{s}}_{1}, \ldots, \hat{\mathbf{s}}_{N}\right]^{T}$. To measure the interferer mitigation, we use the normalized sequence power in the interferer band, i.e. $\left\|\mathbf{S F}_{\mathcal{S}}\right\|_{F}^{2} / R N$, where $R N$ represents the total power of the sequence set. To measure the recovery stability, we use the condition number of $\mathbf{S F}_{\mathcal{P}}$, the projection of the sequence set onto the message space. Additionally, when no sequence set candidate meeting the constraints of (5.2) is found, we set the interferer power and condition number to be infinity. Values of the interferer power above 0dB and of the condition number above 20 are not shown in the following figures.

In the first experiment, we fix the number of randomized projection $L=10^{5}$ and coherence tolerance $\alpha=0.1$. Figure 5.1 shows the average interferer power and condition number of 100 independently generated multi-branch sequence sets


Figure 5.1. Average interferer power (Left) and condition number (Right) as a function of the oversampling rate $R$. When there is no oversampling, both the interferer power and the condition number are outside the plotted range as no feasible sequence set was found. When oversampling is included, the interferer power decreases and the condition number increases as the oversampling rate increases.
when the oversampling rate $R$ varies among $[1,10]$. As we mentioned, when $R=$ 1, i.e., there is no oversampling, it is hard to obtain $N$ binary sequences that are approximately orthogonal to each other. Thus, the condition number is very large. When $R>1$, i.e., oversampling is included in the sequence design, the interferer power decreases as the oversampling rate increases at the cost of an increasing condition number.

Compared to the iterative method shown in Figure 4.1, the matrix optimization method has worse performance in interferer mitigation and stable recovery, even when we use the smaller coherence tolerance. The bad performance may be due to the modified random projection used in Algorithm 3. The modified random projection provides us the benefit of obtaining sequence set in smaller complexity but at the cost of worse performance.

In the second experiment, we vary the coherence tolerance in the range $\left[10^{-3}, 10^{0}\right]$. Figure 5.2 shows the corresponding average interferer power and condition number, and demonstrates the benefit from the oversampling: increasing coherence tolerance improves the interferer mitigation while the condition number increases slightly when


Figure 5.2. Average interferer power (Left) and condition number (Right) as a function of the coherence tolerance $\alpha$. Small coherence values make the search for a feasible solution difficult. The necessary coherence tolerance decreases as the oversampling rate increases.
the sequence set design is feasible. Note that the interferer power and condition number are not sensitive to the choice of tolerance after tolerance valuer is sufficiently large. Additionally, the necessary level of coherence tolerance to obtain a feasible sequence set is inversely proportional to the coherence tolerance.

In the third experiment, we fix the coherence tolerance $\alpha=0.1$. Figure 5.3 shows the average interferer power and condition number when the number of randomized projections $L$ varies among $\left[10^{0}, 10^{6}\right]$. As shown in the figure, when the randomized projection number is not sufficiently large, the sequence obtained in each iteration is suboptimal or even not feasible, which results in high values in the interferer power and condition number. When $L$ increases, we have decreasing interferer power, since the oversampling provides better interferer mitigation performance, and almost stable condition number, which is bounded by the constraints.


Figure 5.3. Average interferer power (Left) and condition number (Right) as a function of the number of randomized projections $L$. The interferer power decreases as the number of randomized projections increases, while the condition number remains stable. The necessary number of randomized projections is inversely proportion to the oversampling rates.

## CHAPTER 6 MODULATED WIDEBAND CONVERTER SIMULATION

In previous chapters, we have presented several algorithms to design three different kinds of binary spectrally shaped sequence sets: Chapter 3 shows the design of single binary sequences which provide a pass and notch for the message and interferer bands, respectively; in Chapter 4, we design binary sequence sets that minimize the sequence set power for the interferer band while keeping all sequences mutually orthogonal; and we propose a second extension of the single sequence design to sequence set design in Chapter 5 , where a matrix optimization is used to obtain all sequences simultaneously. The optimization problems in all three design approaches are solved approximately by convex relaxation and randomized projection.

In each chapter, we included experiments to test the performance of the designed sequences to mitigate or suppress the interferer, which is measured by the sequence power for interferer band. Additionally, we use the condition number to measure the recovery stability for the designed sequence sets. However, all numerical experiments in previous chapters show the performance of sequence sets under the assumption of no noise and nonlinearity. In this chapter, we will present some results of a simulation model for the modulated wideband converter (MWC) architecture to receive and recover the signals. Such results present the performance of the designed sequence sets when noise and nonlinearity are included.

We start with the description of the simulation model for the MWC. Then we provide some alternative bases that can be used to express signal models. We also show the performance comparison among sequence sets obtained using those bases. Finally,


Figure 6.1. System Diagram for Simulation.
we present numerical results under different values of the parameters, including the noise level and nonlinearity level.

### 6.1 Simulation Description

Recall from Section 2.2.1 that the MWC system consists of multiple channels. A signal is modulated by different sequences, and the modulated signal of each channel is filted by an identical low-pass filter. The output of each filter is sampled at a low rate simultaneously for all channels.

In this simulation, we use the model described by the system diagram shown in Figure 6.1. In detail, at the $k^{\text {th }}(k=1,2, \ldots, N)$ channel, the input signal $\mathbf{x}$ is modulated by a binary sequence $\mathbf{s}_{k}$ with period $N$, i.e., $s_{k, n}=s_{k, n+l N}$ for any integer $\ell \in \mathbf{Z}$. Then the modulated signal is

$$
\begin{equation*}
\hat{x}_{k, n}=s_{k, n} x_{n} . \tag{6.1}
\end{equation*}
$$

The low-pass filter is performed by the integration block, which, when combined with the low rate samping, produces the observations

$$
\begin{equation*}
y_{k, m}=\frac{1}{N} \sum_{n=(m-1) N+1}^{m N} \hat{x}_{k, n}=\frac{1}{N} \sum_{n=(m-1) N+1}^{m N} s_{k, n} x_{n} . \tag{6.2}
\end{equation*}
$$

For our purposes, it is convenient to write the observation as

$$
\begin{equation*}
\mathbf{y}_{m}=\frac{1}{N} \mathbf{S} \mathbf{x}_{m} \tag{6.3}
\end{equation*}
$$

where $\mathbf{y}_{m}=\left[y_{1, m}, y_{2, m}, \ldots, y_{N, m}\right]^{T}$ and $\mathbf{x}_{m}=\left[x_{m N+1}, x_{m N+2}, \ldots, x_{(m+1) N}\right]^{T}$ when we ignore the noise and nonlinearity. We assume that the noise is white Gaussian at the baseband with standard deviation $\sigma$ and the nonlinearity is third-order with nonlinearity coefficient $\mu$. For the sake of simplicity, the observation with noise and nonlinearity is written as

$$
\begin{equation*}
\mathbf{y}_{m}=\frac{1}{N} \mathbf{S} \mathbf{x}_{m}-\mu\left(\frac{1}{N} \mathbf{S} \mathbf{x}_{m}\right)^{3}+\mathbf{n}_{m} \tag{6.4}
\end{equation*}
$$

where $\mathbf{n}_{m} \stackrel{\text { i.i.d }}{\sim} \mathcal{N}\left(\mathbf{0}, \sigma^{2} \mathbf{I}\right)$.
When $\mathbf{S}$ provides stable recovery, i.e., the condition number is small, we can directly obtained the recovered signal as $\hat{\mathbf{x}}_{m}=N \mathbf{S}^{\dagger} \mathbf{y}_{m}$. When oversampling is used, we can only recovery the message signal via the Fourier coefficients for the message band, i.e., $\hat{\mathbf{x}}_{m}=R N \mathbf{F}_{\mathcal{P}}\left(\mathbf{S F}_{\mathcal{P}}\right)^{\dagger} \mathbf{y}_{m}$.

### 6.2 Basis Choice

In the sequence design or sequence set design, the message and interferer in the signals are always assumed to lie in the space spanned by the Fourier basis elements for the message and interferer bands, respectively. When there is no overlap between the message and interferer bands, all elements in the basis for the message band are orthogonal to those in the basis for the interferer band. If the signals contain components with only on-grid frequencies, the frequencies sampled by the Fourier
basis, there is no distortion caused by the interferer in recovering the message when the system operates in the linear regime.

As mentioned in Section 2.4.3, when a signal contains components with frequencies not belonging to the on-grid frequency set, the representation of the signal with the Fourier basis is not perfect and not all energy of the signal is captured by the basis projections. If the interferer in the signal has components with off-grid frequencies, then some distortion appears in the recovered message even if a nonlinearity is not present. Therefore, the sequence set designed using the Fourier basis has sub-optimal performance when the interferer band covers all frequencies in a narrow band.

The Slepian basis has been advertised as a suitable representation for any signal with frequencies that lie in a small range containing both on-grid and off-grid frequencies. However, the Slepian basis for a frequency range that is outside of baseband relies on modulation with a complex exponential component whose frequency is the center frequency of the range. When the complex exponential components for the message and interferer bands are not orthogonal, the interferer may cause some distortion during the message recovery.

We test the interferer mitigation performance when signals containing off-grid frequencies are modulated by different sequences, including pseudorandom sequences. We also include the sequences obtained from our single sequence design shown in Algorithm 1 based on Fourier and Slepian bases, which we called Single Fourier and Single Slepian sequences, respectively, the sequence sets obtained from our iterative method shown in Algorithm 2, which are denoted by Multiple Fourier and Multiple Slepian, and the sequence sets obtained from our matrix optimization shown in Algorithm 3, which are denoted by Matrix Fourier and Matrix Slepian.

All design sequences are designed to block the frequency range $\left[f_{c}-1 / R N, f_{c}+1 / R N\right]$, which is centered at the on-grid frequency $f_{c}$. The input signal represents a single interferer expressed by the complex exponential vector


Figure 6.2. Average modulation gain as a function of the frequency offset $d$. The sequences designed based on the Fourier basis have the small gain at on-grid frequencies, but large gain at frequencies far away to any on-grid frequency. The sequences designed based on the Slepian basis have almost the same gain across most frequencies.

$$
\begin{equation*}
\mathcal{F}\left(f_{c}+\frac{d}{R N}\right)=\frac{1}{\sqrt{R N}}\left[1, e^{j 2 \pi\left(f_{c}+d / N\right)}, \ldots, e^{j 2 \pi(R N-1)\left(f_{c}+d / R N\right)}\right]^{T} \tag{6.5}
\end{equation*}
$$

where $d$ denotes the frequency offset to the center frequency $f_{c}$ of the interferer band. The output performance is then measured by the modulation gain, which is defined as the power of the modulated signals normalized by the sequence power, i.e., $\left\|\mathbf{S} \mathcal{F}\left(f_{c}+d / R N\right)\right\|_{2}^{2} / R N$.

Figure 6.2 shows the average modulation gain over 100 independently generated sequence sets as a function of the frequency offset $d$. Since the pseudorandom sequences have a flat spectrum, the modulation gain of the pseudorandom sequences is almost 0 dB at all frequency offsets. The sequences from Algorithm 2 when the Fourier basis is used have a good rejection for the on-grid frequencies (i.e., the frequency offset is 0 or 1 ). However, the rejection performance decreases as the interferer frequency moves farther from the on-grid frequencies. By contrast, the sequences based on the Slepian basis have good performance in interferer mitigation for most frequencies, both on-grid and off-grid, except for those that are close to the edge of the interferer bands.

### 6.3 Simulation Experiments

We conduct several experiments to test the performance of all sequence sets, including pseudorandom sequences, Multiple Fourier, Multiple Slepian, Matrix Fourier and Matrix Slepian described in Section 6.2, in the simulation described in Section 6.1.

The input signal $\mathbf{x}=\mathbf{x}_{\mathcal{P}}+\mathbf{x}_{\mathcal{S}}$ consists of two single-tone components: one lies in the message band and another one lies in the interferer band. More specifically, the interferer band covers all frequencies in the normalized frequency range $\left[f_{c}-1 / R N, f_{c}+1 / R N\right]$, i.e.,

$$
\begin{equation*}
\mathbf{x}_{\mathcal{S}}=A_{\mathcal{S}}\left[1, \cos \left(2 \pi\left(f_{c}+d / R N\right)\right), \ldots, \cos \left(2 \pi\left(f_{c}+d / R N\right) L\right)\right]^{T} \tag{6.6}
\end{equation*}
$$

where $A_{\mathcal{S}}$ is the amplitude of the interferer, $d \in[-0.5,0.5]$ is the frequency offset, and $L=1000 R N$. The message band contains only the on-grid frequencies in the range $\left(0, f_{c}-1 / R N\right) \cup\left(f_{c}+1 / R N, 1 / R\right)$, i.e.,

$$
\begin{equation*}
\mathbf{x}_{\mathcal{P}}=A_{\mathcal{P}}\left[1, \cos \left(2 \pi f_{\mathcal{P}}\right), \ldots, \cos \left(2 \pi f_{\mathcal{P}} L\right)\right]^{T} \tag{6.7}
\end{equation*}
$$

where $A_{\mathcal{P}}$ and $f_{\mathcal{P}}$ are the message amplitude and frequency, respectively. The simulation performance is measured by the recovered message SNR defined as

$$
\begin{equation*}
S N R=20 \log _{10} \frac{\left\|\mathbf{x}_{\mathcal{P}}\right\|_{2}}{\left\|\mathbf{x}_{\mathcal{P}}-\hat{\mathbf{x}}_{\mathcal{P}}\right\|_{2}} \tag{6.8}
\end{equation*}
$$

All results shown in this section are the average message SNR over 100 randomly selected message and interferer frequencies.

In the first experiment, we fix the standard deviation of the noise $\sigma=10^{-6}$, the message amplitude 0.01 , the interferer amplitude 0.1 and the frequency offset $d=0$ (i.e. the interferer frequency is also on-grid). Figure 6.3 shows the average recovered SNR as a function of the nonlinearity coefficient $\mu$. The SNRs for all sequence sets


Figure 6.3. Average message SNR as a function of the nonlinearity coefficients $\mu$. All SNR decreases as the nonlinearity level increases. All designed sequence sets have better performance than the pseudorandom sequences due to their ability to mitigate the interferer.
decay as the nonlinearity coefficient increases. All designed sequence sets have better recovery SNR than the pseudorandom sequences, which indicate their ability to reduce the interferers. However, under the case with noise and nonlinearity, the gaps between all designed sequence sets and the pseudorandom are not as significant as shown in Figure 6.2, where no noise and nonlinearity is assumed.

In the second experiment, we vary the noise and fix the nonlinearity coefficient $\mu=1$; the other parameters are the same as in the first experiment. Figure 6.4 shows the average recovered SNR as a function of the standard derivate of the noise. When the noise level is large, the pseudorandom sequences have better performance than the designed sequence sets. The pseudorandom sequences have the smallest condition number. As we discussed in Section 4.1.2, a smaller condition number of the sequence set indicates better recovery performance. Additionally, when the noise level is small enough such that the nonlinearity dominates the recovery error, the designed sequence sets show better performance again.

In the third experiment, we test the performance of the designed sequence sets in MWC for off-grid frequencies when the nonlinearity coefficient and the noise standard deviation are $\mu=1$ and $\sigma=10^{-6}$, respectively. Figure 6.5 shows the average recov-


Figure 6.4. Average message SNR as a function of the noise standard deviation $\sigma$. All SNR decreases as the noise level increases. When the noise level is large, all designed sequence sets have worse performance than the pseudorandom sequences due to their larger condition number.
ered SNR as a function of the interferer frequency offset $d$, defined in (6.5). Though all sequences have decreasing performance as the interferer frequency moves away from the on-grid frequencies, the performance decay of the sequences sets obtained based on the Slepian basis is slower than those based on the Fourier basis. This again confirms that the Slepian basis is a more suited basis to represent signals with an off-grid component.


Figure 6.5. Average recovered SNR as a function of the frequency offset $d$. The designed sequence sets based on the Slepian basis have more stable performance than those based on the Fourier basis.

## CHAPTER 7 CONCLUSION AND FUTURE WORK

### 7.1 Conclusion

In this thesis, we have proposed and studied three kinds of design for spectrally shaped binary sequence sets. The sequence set obtained by all design algorithms can be leveraged for multi-branch modulation to mitigate the interferer and thus reduce the distortion caused by nonlinearity and noise present in receive systems.

In Chapter 3, we proposed an algorithm to design a spectrally shaped binary sequence that provides a passband and a notch for a pair of pre-determined message and interferer bands, respectively. We first pose the sequence design problem as a QCQP problem, and combine it with a randomized projection of the solution of an SDP relaxation (a common convex relaxation) to obtain an approximation to the optimal sequence in a statistical sense. The candidate sequences obtained by this method are shown to satisfy the interferer constraints with a probability that depends on the interferer tolerance and the interferer bandwidth. We numerically show that the candidate sequences are better approximations (in terms of the objective function value) than sequences obtained by quantizing the principal eigenvector and than randomly generated binary sequences. Our method also outperforms existing approaches for unimodular sequence design that are modified to meet the required binary quantization constraint. Our experiments show that for small sequence lengths the proposed method is able to obtain the same optimal sequences as the exhaustive search at a fraction of the search cost, which shows promise for the use of our randomized method in spectrally shaped binary sequence design featuring larger lengths.

In Chapter 4, we proposed an algorithm to design a set of binary sequences for multi-branch modulation that provides a notch for interferer bands while the message can be recovered from the modulated signals. The sequence design for each branch can be written as a QCQP to minimize the sequence power in the interferer band with the constraints that enforce approximate mutual orthogonality among the sequences in the set. We provide an analysis of the difficulty of orthogonal binary sequence set search for stable recovery and highlight the necessity of oversampling in sequence design. We numerically showed that the performance of the designed sequence sets increases as the number of randomized projections increases.

In Chapter 5, we presented another algorithm to design sequence set for multibranch modulation. By formulating the sequence set design problem as a matrix optimization, we could obtain all sequences in the set simultaneously. We modified the randomized projection, which is used to obtain single sequences so that it returns the approximations to sequence sets. We numerically showed that the performance of the matrix optimization method is slightly worse than the iterative method.

In Chapter 6, we described a simulation model for the MWC system which we can use to test the obtain sequence sets from the mentioned design algorithms. We also provided a discussion on the basis choice to compare the advantage and disadvantage between the Fourier and Slepian Bases. With the simulation model, we tested the performance of the designed sequence set when nonlinearity and noise exist. Numerical results confirmed our work in sequence set design: designed sequence sets have better interferer mitigation due to their small power for the interferer band but larger noise degradation since their larger condition number.

### 7.2 Future Work

Many questions remain open both on the analysis and possible refinements of our analysis and algorithms.

For single sequence design, the theoretical proof of the approximation ratio is still missing to complete the analysis of sequence design. It is necessary to provide a good statistic model for the obtained binary sequence after randomized projection and quantization.

For iterative sequence set design, one could consider changes to the objective function and the constraints (e.g., switching the two) such that the optimization searches for sequence sets with minimal orthogonality while the sequence power in the interferer band is bounded. This may be beneficial by allowing for sequence sets with better recovery performance.

For sequence set design based on matrix optimization, lots of significant work is needed to improve the performance of obtained sequence sets. We need to seek a more tight metric for the orthogonality constraints since not all sequences of an obtained set are approximately orthogonal when the set is feasible to the constraints with the Frobenius norm. Such metric should allow us to formulate the convex relaxation so that it is possible to solve the sequence set design approximately via randomized projection.

## APPENDIX

## PROOF OF THEOREM 2

We will use the following results in our proof.
Theorem 5 (McDiarmid's Inequality [65]) Let $\mathbf{x}=\left[x_{1}, x_{2}, \ldots, x_{N}\right]^{T}$ be a family of random variables with $x_{i}$ taking values in a set $\mathcal{X}_{i}$ for each $i \in \mathbf{I}=\{1,2, \ldots, N\}$. Assume the function $g: \prod_{i \in \mathbf{I}} \mathcal{X}_{i} \rightarrow \mathbb{R}$ satisfies $|g(\mathbf{x})-g(\overline{\mathbf{x}})| \leq c_{n}$ whenever $\mathbf{x}, \overline{\mathbf{x}} \in$ $\prod_{i \in \mathbf{I}} \mathcal{X}_{i}$ differ only in their $n^{\text {th }}$ entries for some $n \in \mathbf{I}$. For any $\zeta>0$, we have

$$
\begin{equation*}
\operatorname{Prob}\{g(\mathbf{x})>\mathbb{E}(g(\mathbf{x}))+\zeta\} \leq \exp \left(-\frac{2 \zeta^{2}}{\sum_{i \in \mathbf{I}} c_{i}^{2}}\right) \tag{7.1}
\end{equation*}
$$

To use Theorem 5 to prove Theorem 1, we need to present some additional results.

Lemma 2 ([32, Lemma 3.2]) If $\mathbf{s}$ is a binary vector obtained via randomized projection and binary quantization from $\mathbf{T}$, then for any indices $i, j \in \mathbf{I}$,

$$
\begin{equation*}
\operatorname{Prob}\left\{s_{i} \neq s_{j}\right\}=\frac{1}{\pi} \arccos \left(\frac{t_{i, j}}{\sqrt{t_{i, i} t_{j, j}}}\right) . \tag{7.2}
\end{equation*}
$$

This lemma provides an important connection between the original binary sequence design and its SDP relaxation, and allows us to prove the following result.

Lemma 3 If $\mathbf{s}$ is a binary vector obtained via randomized projection and binary quantization from $\mathbf{T}$, then

$$
\begin{equation*}
\mathbb{E}\left(s_{i} s_{j}\right)=\frac{2}{\pi} \arcsin t_{i, j}, \tag{7.3}
\end{equation*}
$$

for any indices $i, j \in \mathbf{I}$.

Proof. Since $\mathbf{T}$ is a solution for the SDP relaxation (3.21), $t_{i, i}=1$ for each $i \in \mathbf{I}$, and so $\mathbb{E}\left(s_{i}^{2}\right)=1=\frac{2}{\pi} \arcsin t_{i, i}$. When $i \neq j$,

$$
\begin{equation*}
\mathbb{E}\left(s_{i} s_{j}\right)=1-2 \operatorname{Prob}\left\{s_{i} \neq s_{j}\right\}=\frac{2}{\pi}\left(\frac{\pi}{2}-\arccos t_{i, j}\right)=\frac{2}{\pi} \arcsin t_{i, j}, \tag{7.4}
\end{equation*}
$$

where the second equality is due to Lemma 2
To use McDiarmid's Inequality, we need to prove the following conditions for binary sequences.

Lemma 4 If $\mathbf{s}$ is a binary vector obtained via randomized projection and binary quantization from $\mathbf{T}$, then $\left|\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}-\left\|\mathcal{F}_{\mathcal{S}} \overline{\mathbf{s}}\right\|_{2}^{2}\right| \leq 4|\mathcal{S}|$ whenever $\mathbf{s}, \overline{\mathbf{s}} \in\{-1,1\}^{N}$ differ only in the $n^{\text {th }}$ entries for any $n \in \mathbf{I}$.

Proof. We can express the entries of $\mathcal{F}_{\mathcal{S}}$ as $a_{k, i}=\frac{1}{\sqrt{N}} e^{(2 \pi(k-1)(i-1) / N)}(k \in \mathcal{S}, i \in \mathbf{I})$. Since $\mathcal{F}_{\mathcal{S}}$ is a row submatrix of the Fourier orthonormal basis matrix, $\sum_{i \in \mathbf{I}}\left|a_{k, i}\right|^{2}=1$. Additionally,

$$
\begin{align*}
\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}= & \operatorname{Trace}\left(\mathcal{F}_{\mathcal{S}}^{H} \mathcal{F}_{\mathcal{S}} \mathbf{S s}^{T}\right) \\
= & \sum_{i \in \mathbf{I}} \sum_{j \in \mathbf{I}} \sum_{k \in \mathcal{S}} a_{k, i}^{*} a_{k, j} s_{i} s_{j} \\
= & \sum_{i \neq n} \sum_{j \neq n} \sum_{k \in \mathcal{S}} a_{k, i}^{*} a_{k, j} s_{i} s_{j}+\sum_{i \neq n} \sum_{k \in \mathcal{S}} a_{k, i}^{*} a_{k, n} s_{i} s_{n} \\
& \quad+\sum_{j \neq n} \sum_{k \in \mathcal{S}} a_{k, n}^{*} a_{k, j} s_{n} s_{j}+\sum_{k \in \mathcal{S}} a_{k, n}^{*} a_{k, n} s_{n}^{2} . \tag{7.5}
\end{align*}
$$

Since $\mathbf{s}, \overline{\mathbf{s}} \in\{-1,1\}^{N}$ differ only in the $n^{\text {th }}$ entries, $s_{i}=\bar{s}_{i}$ if $i \neq n$ and $s_{n}^{2}=\bar{s}_{n}^{2}=1$. The first and fourth terms in the right hand side of (7.5) for $\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}$ and $\left\|\mathcal{F}_{\mathcal{S}} \overline{\mathbf{s}}\right\|_{2}^{2}$ are the same. Therefore,

$$
\begin{align*}
\left|\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}-\left\|\mathcal{F}_{\mathcal{S}} \overline{\mathbf{s}}\right\|_{2}^{2}\right| & =\left|\sum_{i \neq n} \sum_{k \in \mathcal{S}} a_{k, i}^{*} a_{k, n} s_{i}\left(s_{n}-\bar{s}_{n}\right)+\sum_{j \neq n} \sum_{k \in \mathcal{S}} a_{k, n}^{*} a_{k, j} s_{j}\left(s_{n}-\bar{s}_{n}\right)\right| \\
& \leq 2\left|\sum_{i \neq n} \sum_{k \in \mathcal{S}} a_{k, i}^{*} a_{k, n} s_{i}\right|\left|s_{n}-\bar{s}_{n}\right| \\
& \leq 4 \sqrt{\sum_{k \in \mathcal{S}}\left|a_{k, n}\right|^{2} \sum_{k \in \mathcal{S}}\left|\sum_{i \neq n} a_{k, i}^{*} s_{i}\right|^{2}} \\
& \leq 4 \sqrt{\sum_{k \in \mathcal{S}}\left|a_{k, n}\right|^{2}} \sqrt{\sum_{k \in \mathcal{S}} \sum_{i \neq n}\left|a_{k, i}\right|^{2} \sum_{i \neq n} s_{i}^{2}} \\
& \leq 4 \sqrt{\frac{|\mathcal{S}|}{N}} \sqrt{|\mathcal{S}| \frac{N-1}{N}(N-1)} \leq 4|\mathcal{S}| \tag{7.6}
\end{align*}
$$

where the second and third inequalities result from Cauchy-Schwarz inequality.
Now, we are ready to prove Theorem 1. From (7.3), we have

$$
\begin{align*}
\mathbb{E}\left(\left\|\mathcal{F}_{\mathcal{S}}\right\|_{2}^{2}\right) & =\mathbb{E}\left(\mathbf{s}^{T} \mathcal{F}_{\mathcal{S}}^{H} \mathcal{F}_{\mathcal{S}} \mathbf{s}\right) \\
& =\mathbb{E}\left(\operatorname{Trace}\left(\mathcal{F}_{\mathcal{S}}^{H} \mathcal{F}_{\mathcal{S}} \mathbf{s s}^{T}\right)\right) \\
& =\operatorname{Trace}\left(\mathcal{F}_{\mathcal{S}}^{H} \mathcal{F}_{\mathcal{S}} \mathbb{E}\left(\mathbf{s s}^{T}\right)\right) \\
& =\frac{2}{\pi} \operatorname{Trace}\left(\mathcal{F}_{\mathcal{S}}^{H} \mathcal{F}_{\mathcal{S}} \arcsin \mathbf{T}\right) \\
& \leq \frac{1}{\pi} \beta \alpha \tag{7.7}
\end{align*}
$$

where $\beta$ is defined in (3.22). By picking $\zeta=\frac{1}{\pi} \alpha>0$ and applying McDiarmid's inequality for $g(\mathbf{s})=\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}$ with Lemma 4 and (7.7), we finally obtain

$$
\begin{align*}
\operatorname{Prob}\left\{\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2} \geq \frac{1}{\pi}(\beta+1) \alpha\right\} & =\operatorname{Prob}\left\{g(\mathbf{s}) \geq \mathbb{E}\left(\left\|\mathcal{F}_{\mathcal{S}} \mathbf{s}\right\|_{2}^{2}\right)+\frac{1}{\pi} \alpha\right\} \\
& \leq \exp \left(-\frac{2\left(\frac{1}{\pi} \alpha\right)^{2}}{N(4|\mathcal{S}|)^{2}}\right) \\
& \leq \exp \left(-\frac{1}{8 N \pi^{2}} \frac{\alpha^{2}}{|\mathcal{S}|^{2}}\right) \tag{7.8}
\end{align*}
$$

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[^0]:    ${ }^{1}$ This work is in collaboration with Marco F. Duarte [56, 57]

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