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An external control study of diagram drawing skills for the solution of algebra word problems by novice problem solvers.

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AN EXTERNAL CONTROL STUDY OF DIAGRAM DRAWING SKILLS FOR THE SOLUTION OF ALGEBRA WORD PROBLEMS BY NOVICE PROBLEM SOLVERS

A Dissertation Presented

By

MARTIN A. SIMON

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I dedicate my dissertation to the memory of my grandfathers, Abe Ronis and Joseph Simon.

PREFACE

In order to avoid the use of gender biased pronouns and to avoid the awkwardness of "he/she," "himself/herself," etc., the author has attempted to balance the use of male and female pronouns in this document.

ABSTRACT

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Diagram drawing is generally accepted as an important heuristic strategy for solving mathematical problems. However, novice problem solvers do not frequently choose to use this strategy. Further, when asked to draw a diagram, their attempts often do not result in a useful representation of the problem.

The exploratory study, which used individual interviews with remedial mathematics students at the University of Massachusetts, identified five factors that influence whether a diagram is used and whether its use is successful:

1. Understanding of the mathematics involved in the problem and of basic arithmetic concepts (i.e. fractions, ratio)
2. Diagram drawing skills and experience
3. Conceptions of mathematics
4. Self-concept in mathematics

5. Motivation to solve the problem correctly

The interviews also generated a set of diagram drawing subskills.

The main study focused on factor two. It attempted to experimentally verify the importance of the subskills identified in the exploratory study. The list of subskills was translated into a series of external control suggestions for guiding the subjects' work during individual interviews. Subjects were precalculus students at the University of Massachusetts. These suggestions were provided by the experimenter as appropriate. Subjects who received these suggestions drew significantly higher quality diagrams than did subjects in the control group. The enhanced quality was particularly apparent in the area of completeness of the diagram. In addition, the study indicated several important metacognitive skills necessary for successful diagram drawing as well as a number of specific difficulties encountered by the subjects.

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CHAPTER I

THE PROBLEM

Introduction

Within the field of mathematics education, in recent years, no skill or topic has received as much attention as problem solving. The National Council of Supervisors of Mathematics (1977) stated that problem solving is a basic skill of mathematics and that "learning to solve problems is the principal reason for studying mathematics." (page 2)

The National Council of Teachers of Mathematics has made problem solving one of its priority items in its Agenda for Action (1980) and emphasized that "Problem Solving must be the focus of school mathematics in the 1980's."

Problem solving has become a priority for education because American industries are suffering from scarcity of students with well developed problem solving skills and because tests of American school children reveal substantial weaknesses in this area. The National Assessment of Educational Progress (1979) demonstrated that students in the United States are skilled at basic computational algorithms and the solution of word problems that require simply the selection of one arithmetic algorithm. However, these same students are very weak in dealing with problems containing irrelevant information, problems with insufficient information, problems which require two or more steps, and non-routine problems.

Definition of Terms

In this study, problem refers to non-routine mathematical problems, problems for which the subject, at the outset, has no known methods with which to solve the problem. In contrast, a mathematical task for which the subject has only to practice a known method or algorithm is referred to as an exercise.

Metacognition, also referred to as control knowledge or managerial skills, is the ability to use the knowledge and problem solving strategies that one posesses. It involves knowing the limitations and domain of particular strategies, thinking to use strategies when they are appropriate, monitoring work using the strategy, and evaluating the results produced.

A diagram refers to a spatial representation of the problem situation. It includes, but is not limited to, area diagrams, number lines, and graphs. In the literature, diagrams are also referred to as figures and pictures.

Spatial visualization refers to the ability to "see" and manipulate mental images of two and three dimensional relationships.

A _geometric context_ (McKee 1983) is a problem which involves distance, height: quantities that are "instinctively" represented by a diagram. An _algebraic context_ such as age, amount of money, the amount of work done in a particular amount of time, is less likely to suggest a diagram and requires the ability to take a non-spatial quantity and model it spatially.

Background of the Problem

The drawing of diagrams has traditionally been used by individuals as an aid in solving mathematics and science problems. Polya's _How to Solve It_ (1945), which classified "draw a figure" as a heuristic strategy for problem solving, focused attention on the use of diagrams by experienced problem solvers. Although mathematics education researchers have not been able to consistently show a link between the use of diagrams and improved problem solving (see Chapter II), there is evidence (McKee 1983) of a link between the ability to draw high quality diagrams and the successful solution of problems. A number of studies have looked at whether high spatial visualization abilities contribute to a greater tendency to draw diagrams (Landau 1984). However, here also the results are less than clearcut. Chapter II looks at some of the difficulties in investigating the link between diagram drawing and problem solving performance and between spatial abilities and tendency to draw diagrams.

Instructional interventions, which attempted to improve students' diagram drawing abilities have been largely unimpressive. Such studies have been of short duration, one to six weeks, and have lacked a theory of what skills, knowledge, beliefs and affective factors contribute to the successful use of diagrams in problem solving.

Statement of the Problem

Mathematics educators have described the multiple advantages of diagram drawing in the problem solving process. However, novice problem solvers, although encouraged at times by their instructors to draw diagrams, seem to use diagrams infrequently to solve mathematical problems and with little success. In order to assist students in becoming effective problem solvers, teachers must understand the processes of learning to use diagrams and of choosing to use diagrams in a problem situation.

Research Questions Remedial mathematics students at the University of Massachusetts, despite frequently being encouraged and at times required to draw diagrams, seem to consistently choose not to draw diagrams in problem situations for which diagrams would be appropriate.

Observations of this phenomenon led to the following questions which motivated this project:

1. What factors affect whether a student chooses to draw a diagram when a diagram could be helpful?

2. What skills and knowledge are required to draw useful diagrams for solving mathematical problems?

Research questions 1 and 2 lead to exploratory investigations. They were open questions, not constrained by particular hypotheses, which were best answered by observing novice problem solvers solving problems and drawing diagrams and by questioning them on their choices, beliefs, feelings, and difficulties.

The exploratory study lead to the development of two models (described in Chapter III) which were created to organize the preliminary findings relevant to questions 1 and 2. Model One specified the factors which influence the choice to draw a diagram and the usefulness of the resulting diagram. Model Two added detail to one of those factors, subskills of diagram drawing.

The main study focused in on these diagram drawing subskills. The research questions were refocused as follows:

A. Are the subskills identified in the exploratory study (Model Two in Chapter Three) important in the creation of useful diagrams?

B. How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

C. What effect does the problem context (geometric versus algebraic) have on the quality of the diagrams that are drawn?

D. What important skills and knowledge were not identified during the exploratory study?

E. What are the difficulties which prevent successful diagram drawing?

Questions A, B and C motivated an experimental design and the following research hypotheses.

H1. The subskills identified in the exploratory study lead to improved diagram drawing.

H2. An important factor in the successful implementation of the diagram subskills outlined in Model Two is the metacognitive ability to decide when to use each skill.

H3. Higher quality diagrams are created for problems with geometric contexts than for problems with algebraic contexts.

These research hypotheses are stated in the null form in Chapter III.

Questions D and E were investigated by including in the main study the type of open-ended analysis of videotapes which had been so informative in the exploratory study.

<u>Assumptions on Which the Study is Based</u> The study is based on the following assumptions which will need to be verified in future research. These assumptions do not conflict with the current diagram drawing literature.

1. Diagram drawing is a useful strategy in solving a wide range of mathematical problems (although not the majority of problems).

2. All college students, who have no relevant handicaps, regardless of previous mathematical experience can learn to use diagrams effectively.

3. Learning to use diagrams to represent mathematical problems is beneficial for all students even if they do not tend to be predominantly visual learners.

<u>Significance of the Study</u> If we believe that the ability to draw a diagram to represent the mathematical structure of a problem is important, then teachers must be prepared to teach diagram drawing. In order to do so effectively, they must understand the prerequisite skills and understandings, the subskills which make up the larger skills, and the affective variables and beliefs that affect diagram drawing choice and success. They also must be aware of many of the difficulties that students encounter when they attempt to draw diagrams. This study was

designed to begin the process of providing information in this relatively unexplored area.

Overview of the Study

Exploratory Study An exploratory study was conducted to investigate the two research questions. A clinical interview approach was used in order to investigate, not only the problem solving and diagram work of the subjects, but also the subjects' explanations for their work and their feelings and attitudes about mathematics, problem solving, and the use of diagrams. Observations from the exploratory study generated a model of diagram drawing subskills as well as a model of the factors which influence the use of diagrams. These are presented in Chapter III.

Main Study The main study was designed to answer five research questions:

A. Are the subskills identified in the exploratory study (Model Two in Chapter Three) important in the creation of useful diagrams?

B. How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

C. What effect does the problem context (geometric versus algebraic) have on the quality of the diagrams that are drawn?

D. What important skills and knowledge were not identified during the exploratory study?

E. What are the difficulties which prevent successful diagram drawing?

The main study was composed of two parts: an experimental design and an analysis of videotaped diagram drawing interviews. The experimental component was a three group design which tested the effect of the subskills identified in the exploratory study on diagram quality (research questions A and B) and investigated the effect of problem context on diagram quality (research question C).

Initially the external control paradigm of Heller and Reif (1984) was selected to begin to check out whether the model that had been created is a useful description of diagram drawing skills. Heller and Reif had developed a prescriptive model for the development of "theoretical problem descriptions" (representations) for mechanics problems in physics. They considered the development of these representations, which involved diagrams, to be a key step in the problem solving process, a step in which specific knowledge of mechanics is brought to bear on the problem. Their study assumed that students who had successfully completed a first course in basic physics had the necessary knowledge to solve the mechanics problems, but were often unable to apply and exploit that knowledge in problem solving.

Heller and Reif"s experimental model was an attempt at describing an effective process for applying knowledge in mechanics to create useful theoretical descriptions. It was not an attempt to model the performance of experts who seem to be able to do much of what is necessary automatically. No attempt was made to teach the subjects. The model, which was translated into a set of external control directions, a set of directions that guided the subject through the problem solving process, was tested to see if it indeed specified procedures and control knowledge which were necessary and sufficient for creating useful representations. Control knowledge or metacognition (defined above) refers to knowing when to use particular strategies or knowledge, thinking to use them when appropriate, and monitoring their correctness and usefulness.

Preliminary trials of the main study revealed that the Heller and Reif experimental paradigm would not be applicable without some modifications. The preliminary trials indicated lack of discrete and ordered steps in the development of diagrams for algebraic problems. Thus, the use of step by step directions as in the Heller and Reif design was not appropriate. In this study, therefore, rather than using the directions to direct a sequence of steps, these directions were given as needed, without regard to order of use.

The main study was designed to examine whether the control knowledge and skills, identified during the exploratory study, significantly improve the quality of student-drawn diagrams created to solve algebra word problems. In order to do so, a set of external control directions were created which "suggest" that the subject carry out particular behaviors deemed helpful in creating useful diagrams. Rather than presenting these directions, then, in a step-by-step fashion, the experimenter read these directions in response to particular behaviors of the subject.

Subjects were asked not to solve the problems, only to create the diagramatic representation. This allowed the study to focus on just that part of the problem solving process and reduced the pressure on the subjects to get the "right" answer.

In addition to the experimental component of the main study, which focused on questions A,B and C, video tapes of the problem solving sessions were analyzed by the experimenter to continue the exploratory nature of the study and to focus on questions D and E.

Delimitations of the Study

1. The study focused on the solution of typical algebra problems only.

2. Subjects were remedial and precalculus students at the University of Massachusetts which represented the lower level mathematics students.

3. Interviews all involved individual subjects and the experimenter only.

4. The main study took place during a six week period.

5. The main study was not an instructional intervention; it focused on the benefits of using the subskills. No assumption was made that because a subskill was used during the study that it had been learned by the subject.

A section, "Limitations of the Study," is included in Chapter III.

Outline of the Dissertation

Chapter II offers a review of the literature on diagram drawing which serves as a background for this study. It includes the advantages and disadvantages of diagram drawing, research relating diagram drawing to problem solving and to spatial abilities, and research involving instructional interventions. The chapter concludes with a discussion of some of the inherent difficulties in investigating diagram drawing and the relationship of the literature to the research questions posed in this study.

Chapter III describes the design of the exploratory study and then the results of that study since the main study is based on those results. The chapter than describes the design of the main study, the data analysis, and the limitations of the study. It also includes operational statements of the hypotheses.

Chapter IV examines the results and interpretations of the results with respect to research questions A through E.

Chapter V begins with a summary of the first four chapters. It than focuses on the conclusions that can be drawn from the findings and recommendations for future research.

CHAPTER II

REVIEW OF RELATED LITERATURE

Overview of the Chapter

This review of the literature begins with a look at the advantages and disadvantages of diagram drawing, the basis for the assumption (stated in Chapter I) that diagram drawing is a useful strategy in mathematical problem solving. This section is followed by a description of mathematics researchers' efforts to characterize the different types of diagrams that are drawn by students. Work reviewed in this section implies some of the skills involved in diagram drawing. The following section looks at previous experimental attempts to study diagram drawing. The author closes the chapter with a discussion of conclusions that can be drawn from the literature and implications for further research, focusing particularly on connections between the literature and the research questions which motivate this study.

Advantages and Disadvantages of Diagram Drawing

Advantages in Problem Solving Since Polya's work is generally accepted as a cornerstone of modern problem solving education, it seems appropriate to begin with Polya's widely quoted four steps (1945). Polya divided the process of problem solving into four steps or stages through which the problem solver procedes sequentially.

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

It is common, however, that the information generated in a particular step sends the problem solver back to one of the earlier steps. For example, in "carrying out the plan" (step 3), results may be generated which cast a new light on the solver's understanding of the problem. The solver then goes back to step one and procedes sequentially. This cycling back through the steps may occur many times at different stages of solving the problem.

The first step, understanding the problem, an essential step, has received little attention in traditional mathematics teaching (O'Regan 1984). Routine textbook "problems" are usually exercises for practicing algorithmic skills. They provide a vehicle for the numbers which the student must plug into the learned algorithm. In such a process involving known computational methods and one or two step problems, skill in understanding the problem is not challenged or developed. In addition, the misconception that problem solving is nothing more than choosing and using the appropriate algorithm is reinforced.

Hayes (1981) and Mason (1984b) contend that an important part of understanding the problem is the creation of an internal representation of the problem. Thus, the learner develops a mental picture of the problem. Hayes and Mason emphasize that it is often helpful to make an external representation (diagram, model, etc.) to capture the internal representation.

Mason insists that making sense of a problem or concept requires manipulation of objects. These objects may be diagrams, symbols, or images, as well as physical objects, if the learner is confident of "these things as objects." This relationship of internal representation leading to external representation can sometimes be reversed. Building an external representation by diagraming the problem information can help to generate an internal representation of the problem as the diagram takes shape.

In step one, then, drawing a diagram can help in understanding the problem (Tanaka 1982, Reif, Larkin & Brackett 1976). Bell (1981) and Lester (1977) pointed out that the diagram reduces the dependence on words and gives a concise translation of the problem. Kinsella (1970, cited in McKee 1983), Mayer and Revlin (1978), and Simon (1975) have pointed out that students' greatest difficulty in solving a problem is the selection of useful representations. Newell and Simon (1972) noted that understanding is tied to the construction of effective representations. They pointed out that representations have not been well studied. Greeno (1983) stated, "it seems very likely that students success in solving word problems could be improved by instruction focused on the process of representing problems." He observed that such instruction currently is, at most, an implicit part of the educational process.

Difficulties in problem solving are, in part, a result of the gap that students perceive between their concrete experience of the world and the abstract nature of mathematics. Diagrams help bridge this gap (Botsmanova 1972a). A diagram may provide a concrete representation of the problem situation which clearly portrays the relevant relationships in the problem. These relationships can be connected to the necessary mathematical abstractions (Herring 1980, Tanaka 1982, Hooper 1981).

The diagram can further contribute to "understanding the problem" by providing a context for estimation of the answer (Bell 1981) or, in more complex problems, an opportunity to characterize the answer or determine how to recognize when the problem has been solved.

Herring (1980) noted that the process of creating the pictorial representation of a problem demands certain aspects of understanding. The solver must:

1. eliminate distracting details
2. clarify her thoughts
3. identify relevant attributes of the problem
4. identify relationships in the problem situation.

Diagrams also serve as extensions of memory (McKee 1983, Newell and Simon 1972). Mayer (1976) asserts that diagrams improve performance when they replace complex verbal representations. The result is better access to the problem's information.

In addition to pencil and paper diagrams, computer graphics has provided a more dynamic, mutable medium for representing problems. Luerhman (1982) observed that students who explore science problems through interactive computer graphics obtain a richer understanding of the problem's dynamic properties. He concluded, "The ability to change the picture and see how it looks when you change your premises enables the student to perform at a higher cognitive level." (p. 3)

In addition to the value of diagrams for "understanding the problem", diagrams are significant in step two, "devising a plan". Seeing the pictorial representation of a problem and its key relationships, leads to strategies for solving the problem (Bell 1981).

Larkin (1983) specified that problem solving expertise involves searching the problem space effectively, and that the creation of a representation reduces the size of the problem space to be searched.

Landau (1984) pointed out that creating an image of a problem "permits a conceptual (i.e. 'how should I think about this problem?') rather than a procedural (i.e. 'what should I do next?') approach" (p. 6). She refers to evidence from the Applied Problem Solving Project (Lesh et al. 1983) which suggested that problem solvers who take a conceptual approach are more often successful than the problem solvers who take a procedural approach.

Besides facilitating the logical/sequential mode of thinking, valued in mathematics problem solving, diagams also lend themselves, better than verbal descriptions or mathematical symbols, to engaging the intuition (Hooper 1981). Although largely absent from the problem solving literature, intuitive thinking plays a key role in the solving of complex mathematics problems. Intuition is observed in the "intuitive leaps" made by expert problem solvers and are often the source of creative approaches to non-routine problems. Elliott, (1980, p. 218) noted that "creative thinking and problem solving in mathematics are just as much unconscious and intuitive as they are logical and formal." (See the discussion of "Visual Thinking" below.)

In some problems, the diagram can be manipulated directly to obtain an answer, thus functioning in Polya's step three. See Figure 2.1 (O'Regan 1984).

Polya's step four can also be enhanced by fitting answers obtained back into the diagram to check the reasonableness of the answers.

<u>Visual Thinking</u> One of the areas which has been linked to pictorial solutions of problems is the area of visual thinking. Many educators insist that there are two distinct but complementary types of thought that go on in the human brain, (i.e. logical/sequential and wholistic/intuitive) and that they are both essential to maximize problem solving potential. Much of the work on brain-hemisphere specialization supports this notion (Ornstein 1972, Hendricks and Wills 1975, Levy 1983).

Moses (1982) describes visual thinking as a non-analytic and non-algorithmic process. It is a "wholistic" process, referring to the fact that it involves a perception of the whole rather than a sequential look at the parts. In this process, creative insights emerge as mental images. Often, these images are then recorded as drawings which allows the images to be examined, analyzed and manipulated. Moses suggested the need for instruction designed to help students develop their abilities for mental imagery.

Fig. 2.1

Answer Determined Directly from Diagram

WE SEE THAT 3/5 OF THE CHILDREN IN A ROOM ARE GIRLS. WE OBSERVE THAT IF WE DOUBLE THE NUMBER OF BOYS IN THE ROOM AND INCLUDE 6 MORE GIRLS, THEN THERE WILL BE AN EQUAL NUMBER OF BOYS AND GIRLS. HOW MANY CHILDREN ARE IN THE ROOM NOW?

I BUILD A MODEL OF THE ROOM

B B	B B	G G G	G
B B	B B	G G G	G
..	..	..	G
..	..	..	G
..	..	..	G
?	?	?	G
I DOUBLE THE NUMBER OF BOYS. I STILL CAN'T COUNT THE ROWS.	I'LL BE FINISHED WHEN I CAN FIGURE OUT HOW MANY ROWS THERE ARE IN THE ROOM. SO THE MODEL OF THE ROOM, WHEN COMPLETED, WILL EXPRESS THE ANSWER.		I INCLUDE THE SIX GIRLS. NOW I CAN COUNT THE ROWS.

Mason (1984b) wrote that attempting to get students to draw diagrams before they have learned to create mental pictures is useless. Instruction is too often focused on the external behaviors of students rather than on the internal construction of images and knowledge. Mason contended that diagrams are a recording of mental imagery which may not be pictorial until the diagram is created.

De Groot (1966, cited in Herring 1980) has shown that expert chess players remember a large number of chess board situations. Egan and Schwartz (1979, cited in Herring 1980) found that electronic technicians had a similar memory for schematic diagrams of electrical components. This memory of meaningful "chunks" may be evidence for the existence of a visual or wholistic memory.

Hooper (1981) suggests that diagram drawing helps to engage the problem solver's intuition. Maier (1983) offers the following description of physicist, Richard Feynman:

> Dick just wrote down the solutions out of his head without ever writing down the equations. He had a physical picture of the way things happen, and the picture gave him the solution directly, with a minimum of calculations (p. 2).

Maier advocated the developmet in our students of a balance of visual and analytical thinking and quotes Robert Sommers, University of California at Davis, "New math failed because of its bias towards abstraction and its devaluation of imagery." (p. 5)

<u>Disadvantages of Diagram Drawing</u> Although the literature strongly supports the value of drawing diagrams for solving mathematical problems, there are disadvantages, too, in the use of diagrams. A diagram that represents the solvers preconceptions of the problem can fix a particular inappropriate image in the mind of the solver and inhibit his flexibility in creating alternative representations (Wicker, Weinstien, Yelick, and Brooks 1978). Sherill (1973) and Webb and Sherrill (1974) showed that inaccurate diagrams in the problem presentation resulted in poorer problem solving than for the case where no diagram was presented.

The section below, "Spatial Abilities and Diagram Drawing," describes evidence that requiring students to draw diagrams may interfere with problem solving performance depending on the spatial abilities of the problem solver.

Although diagram drawing may not be a learned or a preferred mode of problem representation for many students, thus not advantageous, this lack of advantage is different than a disadvantage.

<u>Benefits of Diagram Drawing in the Mathematics Classroom</u> McKee (1983, p.6) described the importance to the teacher of diagram drawing.

> Drawing a figure not only serves as a helpful strategy [for problem solving], but can show that

> a student understands the problem (Cooney, Davis, and Henderson 1975 p. 248) since it requires identifying the structure of the problem (Johnson and Rising, 1967 p. 124).

McKee observed that a student who draws literal representations of problems is not at the same level of mathematical development as the student who uses "highly abstract, schematized figures." Vest and Congleton (1978) advocated the teaching of diagram drawing as a way to help students learn to build mathematical models. Involving students in diagram drawing encourages them to work in a medium that demands thought and understanding as well as creativity. This is in contrast to most algorithmic work which requires only the imitation of learned procedures.

The teaching of diagram drawing may have affective payoffs as well:

> 'poor problem solvers do not strongly believe that persistent analysis is an effective way (in fact the only way) to deal with academic reasoning problems.' (Whimbey and Lochhead, 1980 p. 29) Thus these 'one-shot' thinkers are less limited by their capabilities than by their habits and beliefs. (Lochhead 1981, p. 20)

However, the teaching of diagram drawing may result in "working on" the problem (Mason 1984b); that is in increased activity by the student as he <u>sees how it works</u> and <u>tries things</u> when a solution is not readily apparent. The manipulation of diagrams may not only cause the student to be more active in problem solving, but may also contribute to a shift in the student's beliefs about mathematics and about himself as a problem solver.

Advantages Versus Disadvantages The literature clearly describes many more advantages than disadvantages for the use of diagrams. The assumption, therefore, that diagram drawing is a useful general problem solving strategy seems reasonable. What remains to be answered, however, is what are the component skills which contribute to a student's ability to use diagram drawing successfully and what are the factors which determine whether students make use of this strategy.

Characteristics of Diagrams

McKee (1983) chose to investigate four characteristics of drawn diagrams:

1. type the literalness versus the abstractness of the diagram
2. completeness: how much of the relevant information is represented in the diagram, and whether it is done in one integrated diagram rather than several separate diagrams
3. labeling: extent to which the parts of the diagram are appropriately labeled
4. accuracy: correctness of the representation of problem information.

Although it is sometimes difficult to judge individual diagrams, the general criteria for completeness, labeling, and accuracy are easily agreed on by expert observers (McKee 1983). Even with respect to "type", the classification of the diagrams into categories, where various criteria could be used, there seems to be fairly close agreement. Botsmanova (1972a) classified student diagrams similarly to McKee:

1. Object illustrative refers to a diagram of the objects and or setting of the problems. Such diagrams do not reflect the mathematical structure of the problem.

2. Object analytical refers to a diagram of the objects that uses a spatial arrangement to represent relationships. Such diagrams do reflect the mathematical structure and the essential data of the problem.

3. Abstract spatial refers to a diagram that reflects only the relevant mathematical relationships of the data (schematic representation).

Larkin's (1983) observations were consistent with the classification schemes above. She stated that experts tend to represent physical and mathematical relationships while novices represent objects as described. She suggested that this distinction may be the major cause of observed differences between novices and experts.

Schultz (1983) also focused on the literalness of representations, calling them:

1. meaningful (i.e. pictures of coins for a coin problem)

2. indirect meaningful (i.e. rectangles to represent beds)

3. non-meaningful (i.e. circles for odd and even numbers).

With regard to completeness, Paige and Simon (1966, cited in McKee 1983) noted the importance of representing all the problem information in one "integrated" diagram as opposed to a series of diagrams, each showing only part of the problem situation. See Figures 2.2 and 2.3 below which were drawn for the following problem:

> A rabbit is eighty of her own leaps ahead of a dog. She takes three leaps for every two that the dog takes, but he covers as much ground in one leap as she does in two. How many leaps will the rabbit have taken before she is caught?

FIG. 2.2

Non-integrated Diagram

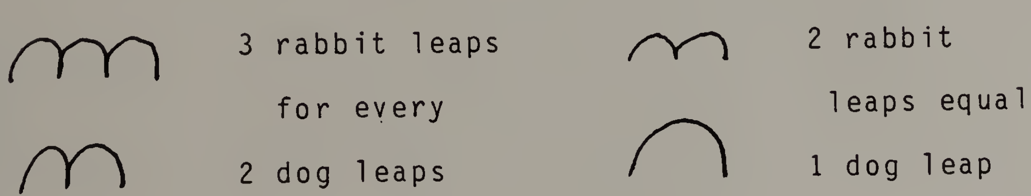
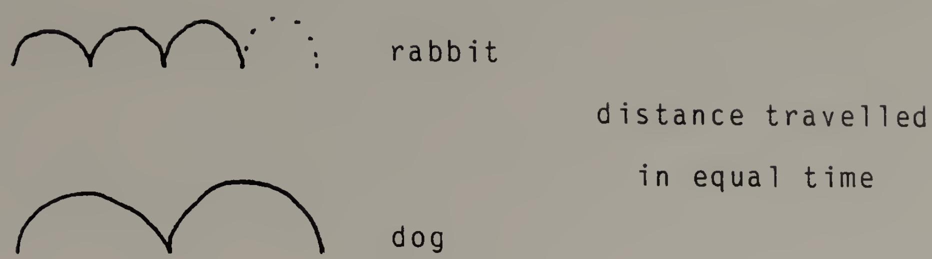


Fig. 2.3

Integrated Diagram



The characteristics of diagrams that have been described provide a first step in the investigation of the subskills of diagram drawing. An element common to the characterization schemes of the researchers above is the ability to extract the mathematical structure of the problem and to represent it schematically. In addition, McKee's work, which provides a basis for examining diagram quality, focuses on the subskill of labeling effectively.

Research on Diagrams

Observational Research on Diagram Drawing The most central question in diagram drawing research is whether the drawing of diagrams substantially improves problem solving. Although many experts are convinced of its value, (Polya 1945, Simon 1972, Botsmanova 1972b, Larkin 1983, Schoenfeld 1980, Charles and Lester 1982) there is little solid research evidence to support this point of view.

Webb (1979) found some improvement in problem solving when students used visual representations. Swart (1970) found that students who were drawing diagrams to aid in problem solving out-performed those who were taught to use an analytic/abstract symbol approach.

On the other hand, Kilpatrick (1967, cited in Landau 1984) observed above average eighth graders and found that drawing diagrams was not related to success in problem solving. Lean and Clements (1981), testing engineering students in New Guinea, found that students who used a verbal-analytical approach to problem solving outperformed the students who took a visual approach.

McKee (1983) found that the tendency to draw diagrams was not significantly related to problem solving performance. She observed, however, that student diagrams were generally of low quality on all four criterea; type, completeness, labeling, and accuracy. Her measure of diagram drawing ability, which reflected the quality of the diagrams drawn, was significantly correlated with problem solving performance. This may indicate that the drawing of diagrams is only helpful if the diagrams are of high quality, or that students with more problem solving ability are able to draw better diagrams.

McKee suggested that:

> The lack of association between drawing a figure and getting the problem correct might be attributed to the difficulty of the problem, the

> low quality and number of figures drawn, or a combination of all three. (p. 106)

McKee's results were consistent with those of Schwartz (1971) and Schonberger (1976) who found that a correlation did exist between the drawing of higher quality diagrams and solving the problems correctly.

Reasons for the seemingly contradictory nature of these results are discussed later in the section, "Conclusions Drawn from the Literature".

Diagrams in Problem Presentation The inclusion of diagrams in the problem presentation seems to increase solution success. Research has shown that these diagrams must be accurate and represent the mathematical structure of the problem to be effective. Sherrill (1973) presented problems to tenth grade students with accurate diagrams, inaccurate diagrams, and no diagrams. Accurate diagrams improved performance over no diagrams, while inaccurate diagrams resulted in worse performance. Sherrill and Webb (1974) repeated these results with pre-service elementary teachers.

The National Assessment of Educational Progress (1979) demonstrated that diagrams were an aid in problem solving. Threadgill-Sowder and Sowder (1982) found that diagrams in the presentation of problems led to significantly higher rates of successful solution.

Ehr (1980) made diagrams, hints, facts, and formulas available to students and found that students most often selected diagrams.

Botsmanova (1972a) found that good students selected diagrams that showed the mathematical, rather than the physical, characteristics of the problem. Their use of these diagrams resulted in shorter solution times.

Improved problem solving success seems to have been more convincingly linked to the use of diagrams in the problem presentation than to the drawing of diagrams by the problem solvers. The lack of clear relationship in the latter case may be a function of the poor diagram drawing skills of the populations being studied. Drawing poor diagrams does not aid and may detract from problem solving. The more clearcut evidence of the improvement in problem solving that results from the use of diagrams in the problem presentation may give us a peek at the potential benefits that students might derive if they could create effective diagrams.

This potential suggests the importance of learning how to teach diagram drawing and motivates a study such as this one which can contribute foundational information for such teaching.

Research Studies Involving Instructional Interventions A number of researchers have attempted to improve diagram drawing through instruction. The results are inconclusive, leading to the observation that diagram drawing is a complex skill which likely requires a lengthy developmental process.

Nelson (1974) provided sixth grade students with eight hours of instruction which included instruction in diagraming word problems and in translating diagram-posed problems to word form. He found no significant improvement by the total group of students receiving this instruction. He did observe, however, that this group of students (who had been instructed in diagraming) drew more diagrams for problems which lend themselves to diagrams. He also found that when he looked at those students who actually benefited from the instruction in diagraming (those students who used diagrams to solve problems), he observed that they did significantly better than students who did not diagram.

Schultz (1983) provided a brief instruction period followed by structured practice which encouraged the use of concrete manipulatives and computer graphics as well as diagrams. She found that average students used such models more frequently than the above or below average students and that increased use led to increased problem solving success.

Threadgill-Sowder and Juilfs (1980) created two instruction groups; one which focused on manipulative models for problem solving, and a second which focused on symbolic solutions. They found that those that scored low on math concepts and problem solving pretests did significantly better in the manipulative models group, while the high scorers on the pre-tests did better with symbolic solutions.

Botsmanova (1972b) provided ten lessons over a three month period to third graders in the Soviet Union. He found that comparison of a "sub-analytical" diagram and a graphic diagram was an effective instructional technique. Students were able to focus on mathemtical relationships in the problem which had been hidden from them before. The group receiving the instruction in graphic representation did significantly better than the control group. He and his colleagues also identified three stages in the use of diagrams for problem solving.

> 1. The stage of unanalyzed reflection of the problem's subject situation: As a rule, this broad reflection of the situation, general in an undifferentiated way, is accompanied by the isolation of one or two essential elements.
> 2. The stage of specification [is characterized by] the isolation of all or almost all of the basic elements and relationships, without a final synthesis.
> 3. The stage of an adequate diagram [is] based on a complete analysis and synthesis of the problem's situation. (Botsmanova 1972b, p. 121)

He observed that the use of diagrams involves analysis of the problem which is closely related to abstraction. As one analyzes the partially drawn diagram, one sees new mathematical relations in the problem text.

Spatial Abilities and Diagram Drawing A number of studies (Schonberger 1976, Guay and Mc Daniel 1977, Moses 1978) have linked spatial abilities to problem solving success. Educational theorists have speculated that diagram drawing may be the link between these two areas (Landau 1984).

Several investigators have tried to determine whether students with a high level of spatial reasoning skills are better able to draw diagrams and/or more likely to do so in problem solving. The results have been inconsistent and seemingly contradictory from one study to the next.

Khoury and Behr (1982) found that high spatial visualizers did significantly better than low spatial visualizers on pictorial modes of representing problems, while they showed no significant advantage in symbolic and mixed modes (pictorial and symbolic together).

Schonberger (1976) found a positive correlation between problem solving performance and visual spatial abilities. She concluded that high spatial ability is a better predictor of the correctness of diagramatic representations than whether a diagram is drawn. She concluded that more spatial training is needed in schools.

Moses (1978) found no correlation between visual approaches to problem solving and problem solving performance. She concluded that students with high spatial ability frequently are able to represent the problem and manipulate it mentally, thus showing no written diagrams.

Landau (1984) found significant correlation between problem solving and spatial abilities. She created four groups. The first group was asked to assess whether a diagram would be helpful before solving each problem. The second group was instructed to draw a diagram for each problem. The third group was presented with two diagrams for each problem and was asked to work with one of them to solve the problem. The fourth group, the control, was given the same problems to solve with no special instructions. The results of these four conditions on both high and low spatial ability students did not lead Landau to a simple explanation of the relationship of spatial abilities and diagram drawing. She found that encouraging low spatial ability students to draw diagrams resulted in worse performance in problem solving, but presenting them with diagrams improved their problem solving.

High spatial ability students were hampered by Landau's experimental condition instructing them to draw diagrams. She concluded, similarly to Moses, that the high spatial ability students would have manipulated internal images and were hampered by having to externalize them.

Conclusions Drawn from the Literature

Difficulties of Diagram Drawing Research The inconclusiveness and often contradictory nature of most of the research on diagram drawing suggests that a number of difficulties may be inherent in this work. Many of these difficulties are present in the field of problem solving research in general. Some of the difficulties in diagram drawing research are discussed below.

1. Great variability and lack of standardization of problem solving tasks: Researchers in problem solving work with a great variety of problems while attempting to study the same phenomena. Their problems range from standard textbook word problems to real world problems and non-routine problems. These problems also vary in difficulty, number of possible solutions, and amount of insight required. In addition, research problems range from problems requiring no mathematical knowledge to those that require a sophisticated mathematical background. Much of the variability of the results is more a function of the tasks selected than of the experimental conditions that have been created.

In diagram drawing research, an additional variable is introduced since certain types of problems lend themselves more to diagram drawing then others. In an attempt to look at this issue in her research, McKee (1983) used the work of Goldin and McClintock (1979) and Caldwell and Goldin (1979) to create two sets of parallel problems. One set of problems had a "geometric" context (i.e. distance, area) while the second set had an "algebraic" context (i.e. money, age). The problems were matched with respect to nine other characteristics. McKee found that students drew more and higher quality diagrams for the geometric problems than for the algebraic ones, but students showed no significant difference in their ability to solve problems from the two sets. Schonberger (1976) obtained similar results.

Differences in problem solving tasks could also partially explain the variability of research results with respect to spatial abilities. Schwartz (1971), conceding the possibility that high spatial ability students were creating mental images but no diagrams, suggested that tasks be created in which the memory load is too great to permit successful solution with internal representations only.

2. Visual strategies are only applicable for problems of a certain difficulty: Researchers have been unable to show conclusively that the drawing of diagrams leads to improved problem solving performance. One of the factors that clouds these results is the level of problem difficulty. For a given problem, some students will find it routine; that is they know immediately how to go about

solving it. They have either solved similar problems before or they possess an appropriate algorithm or method for solving such a problem. These students have no need to draw a diagram since they can procede directly to an answer.

Other students will find the problem beyond their abilities and fail to solve the problem even if they draw diagrams. This leaves a narrow range of students who can solve the problem, but find the problem to be non-routine and challenging enough to warrant the drawing of a diagram. Therefore, the majority of students, on any given problem will either not draw a diagram, but solve it correctly, or draw a diagram and fail to solve it, which results in a decreased chance of obtaining statistically significant correlations when problem solving performance is related to diagram drawing.

3. The populations being studied are unskilled in diagram drawing: It is difficult to assess the advantages of diagram drawing if the students lack the skill to draw effective diagrams. Most populations being studied are products of educational systems that have not valued or taught diagram drawing for problem solving. Therefore, even if these students draw diagrams, the diagrams very often lack the quality to be of real benefit. McKee (1983) who rated high school algebra and geometry students' diagrams on type, completeness, labeling, and accuracy

(discussed above under "Characteristics of Diagrams"), pointed out that the lack of correlation between drawing a diagram and solving the problem correctly, might be explained by the fact that the diagrams drawn were of such poor quality. She characterized the diagrams as lacking

> ...information essential to solving the problems; there was evidence of misunderstanding the problems, and the figures were not labeled as well as they might have been. Few figures were schematic, more were illustrative, and most tended to be somewhere in between. (p. 100)

Because of the mathematical experience and instructional backgrounds (devoid of diagram drawing) of most students, researchers are much more likely to see successful problem solving when students use symbol manipulation approaches than when they use diagramatic approaches. These results, however, tell us little of the potential benefits of long term, quality instruction in diagram drawing.

Clement, Lochhead, and Monk's (1981) work with translation difficulties pointed out that being able to create an accurate diagram is not always sufficient. Students must also develop the ability to translate from diagrams to algebraic symbols to make full use of diagram drawing skills.

4. Instructional interventions require substantial duration to be successful: Many of the attempts at improving diagram drawing have included only one to twelve

hours of instruction (Nelson 1975, Shoecraft 1972, Frandsen and Holder 1969 and Heseman 1977). Diagram drawing is far too complex a skill to be influenced significantly in such a short period of instruction. McKee (1983) concluded,

> As with many problem solving skills, diagram drawing needs to be promoted and encouraged over a period of time in order for students to adopt it as part of their repertoire of strategies and to be skilled in the use of a diagram. (p. 25)

Schoenfeld (1979) listed three prerequisites for using a heuristic strategy:

1. Know how to use it.
2. Understand the problem sufficiently to apply the heuristic correctly.
3. Think to apply the heuristic.

To draw effective diagrams the student must learn all three of these skills. Learning to, not only draw effective representations, but to adapt these skills to a wide variety of problems, requires considerable experience with diagrams.

Schoenfeld's second prerequisite opens up a whole other area of concern. Students, who do not have a conceptual understanding of the mathematical ideas which are being manipulated, cannot represent them pictorially in an accurate and usable manner. The inability to draw an effective diagram, therefore, may be only the tip of an iceberg that lies deep in the past mathematics education of the student, a mathematics education that has stressed procedural knowledge far more than conceptual knowledge.

5. Students have conceptions of mathematics which are antagonistic to the idea of drawing diagrams: Peck (1984), describing an above average student, wrote:

> This student (lacking proper conceptual referents for the symbols and operations in fractions) perceives math as a collection of rules whose attachment to reality is vague, at best, and that such an attachment is unimportant. Furthermore, the student does not perceive a necessary underlying logic for the rules -- they just are. (p. 166)

He listed five counterproductive perceptions which he has repeatedly encountered in students over the years:

1. Mathematics is a collection of rules which are chained together to provide answers in narrowly specified circumstances.
2. Mathematics is not helpful in solving real problems.
3. Mathematics was invented by geniuses. Most ordinary people cannot be expected to understand it.
4. Right or wrong cannot be decided by the learner, but is the province of the answer key or the teacher.
5. The learner's role is to be told specifically what to do, then follow instructions precisely.

Schoenfeld (1983) has identified a similar list. Students who have the conceptions of mathematics, described above, will not see diagram drawing as even potentially useful. In fact, the suggestion "draw a diagram" only creates an additional problem which they feel unequipped to handle. In order to see diagram drawing as a useful activity, students must see mathematics as connected to the "real world", feel that they can create mathematical understandings and feel that they can invent problem solutions. These conceptions of mathematics need not be developed prior to instruction in diagram drawing. In fact, such instruction can contribute significantly to the development of these conceptions. However, the effect of such conceptions and their opposites cannot be ignored.

6. Both high spatial ability and low spatial ability students may not draw diagrams: High spatial ability students may not draw diagrams because they are working from an internal image (Schwartz 1971). On the other hand, low spatial ability students may not possess sufficient spatial skills to become good diagramers unless they are given an opportunity to enhance their spatial skills.

Implications for Further Research Even with all the difficulties inherent in diagram drawing research, two preliminary conclusions seem to merit further investigation. 1. The drawing of high quality diagrams for problems which lend themselves to diagram drawing improves problem solving. 2. Diagram drawing ability and diagram drawing tendency can be improved through instruction.

Large scale paper-and pencil correlational studies, which attempt to relate diagram drawing frequency and quality to problem solving success, will probably fail to provide much additional information for the reasons noted above. Short term instructional interventions also have little hope of significant impact (see discussion above).

Diagram drawing research can possibly be better served, at the present time, by clinical interview examinations of diagram drawing behaviors and attitudes, and by longer term, intensive instructional interventions for the teaching of diagram drawing.

Clinical interviews can focus more directly on the individual, working on problems of relevant difficulty (see discussion of problem difficulty above). The interview can focus on the skills and attitudes of the problem solver and attempt to identify sub-skills and prerequisites of diagram drawing proficiency. In addition, the clinical approach can allow for characterization of the student who works more readily and more effectively with a visual approach to problems.

Longer term, instructional interventions, of one semester to several years, hold the key to learning more about improving diagram drawing ability. If successful, these instructional programs will give us new populations of students to study who have developed their diagram drawing abilities.

Implications for Present Study The research on diagrams in problem presentations, more so than the research examining the effect of student diagram drawing on problem solving performance, has indicated the potential benefits of the drawing of high quality diagrams. Instructional interventions which attempted to improve students abilities in diagram drawing have been uninformed as to the components of such abilities and to appropriate instructional methodologies. As the two initial research questions, set forth in Chapter I, are reexamined in light of the research reviewed in this chapter, it becomes clear that few studies have addressed these questions even indirectly.

1. What factors affect whether a student chooses to draw a diagram when a diagram could be helpful? Although this question is not addressed directly, McKee's study (1983) found evidence that students created poor quality diagrams. This might suggest that their inability to create useful diagrams may restrict their making a choice to draw a diagram when a diagram is indicated. Whimbey (Whimbey & Lochhead 1980), Schoenfeld (1983) and Peck's (1984) discussions of student conceptions of mathematics suggest another area to examine in investigating factors that affect student choices.

2. What skills and knowledge are required to draw useful diagrams for solving mathematical problems? This question has also not been addressed directly. McKee (1983), Schultz (1983), Botsmanova (1972a) and Larkin (1983) in their descriptions of characteristics of effective diagrams imply that students must be able to identify the mathematical relationships in a problem and

then to represent them schematically. This does not go very far in breaking down such abilities.

McKee also focuses attention on the skill of labeling the diagram effectively.

Schoenfeld (1979), in writing about heuristic strategies in general, of which diagrm drawing is one, identified three components of being able to use a strategy:

1. Know how to use it.
2. Understand the problem sufficiently to apply the heuristic correctly.
3. Think to apply the heuristic.

His work also suggests a focus on the importance of metacognitive skills in diagram drawing.

CHAPTER III

METHODOLOGY

As seen in Chapter II, the literature review found little past work which bears directly on the two research questions which motivated this study. Therefore, it was important, at the outset of this study, to explore the domain in an open-ended manner. The research consisted of two phases; an exploratory study, involving individual interviews, followed by a main study which featured a three group experimental design.

The Exploratory Study

Purpose The purpose of the exploratory study was to investigate the following research questions

1. What factors affect whether a student chooses to draw a diagram when a diagram could be helpful?

2. What skills and knowledge are required to draw useful diagrams for solving mathematical problems?

It had been a consensus observation of the instructors and researchers in the Cognitive Processes Research Group at the University of Massachusetts that students in the remedial Math 010 course made extremely infrequent use of diagrams to help them solve problems. If diagram drawing is considered a useful strategy for problem solving, than it is important to know to what extent students are choosing not to make use of this strategy and to what extent they are unable (i.e. do not have the skills) to use the strategy as well as what other factors contribute to their choice.

Past studies had attempted to relate diagram drawing to spatial abilities, to mathematical abilities, and to problem solving success. However, research had not looked at the specific skills that are necessary to represent mathematical problems spatially.

The two research questions were open questions, not constrained by particular hypotheses, which were best answered by observing novice problem solvers solving problems and drawing diagrams and by questioning them on their choices, beliefs, feelings, and difficulties.

<u>Subjects</u> Eleven student volunteers from Math 010 at the University of Massachusetts were paid to participate in the study. The Math 010 course is the lowest level mathematics course taught at the University. Its emphases are the development of problem solving skills, the remediation of arithmetic concepts and skills, and the improvement of study skills. The course carries no graduation credit.

Students were told that the study had to do with problem solving and that they would be required to attend two sessions; the first, a two hour session involving written problem solving and the second, a one-hour video-taped interview. No level of competence was required. We were only interested in how they approached and thought about the problems.

<u>Procedure</u> Each student, who knew only that the research would relate to problem solving, was given two sets of problems to be solved as paper and pencil tasks. The first set asked the studet to show all work, while the second set asked that all problems be solved by drawing a diagram. The second set was administered only after the first set had been collected. The student had an hour to do each set. The student, then, returned on another day for a one hour video-taped interview.

In the videotaped interview, students were asked to explain their written work and were asked to draw diagrams for problems from the first set which had not been previously solved using diagrams. They were asked to explain their choices to use or not to use diagrams. Attention was paid to affective factors and student-reported effects of past mathematics instruction. Affective factors included mathematics confidence or anxiety and motivation in problem solving.

Instrumentation

Problem Sets: Problems used in the exploratory study (see Appendix A) had been collected from various sources and could all be solved by manipulating a diagram or by manipulating a diagram and doing some routine calculations which were generated by the diagram. Several of the problems were taken from those used by McKee (1983).

Interview Questions: Interview questions were basically free-form. Students were asked to explain their choices (to draw or not draw a diagram), think out loud, and to describe the difficulties that they encountered. When a subject's work seemed blocked, the experimenter tried out suggestions that seemed appropriate.

Questions on affect and beliefs included:

1. How do you feel about mathematics? Why do you think you feel this way?

2. What makes someone a good mathematics student?

3. Do you tend to draw diagrams when you solve mathematics problems (why or why not)?

4. Describe your experiences of drawing diagrams in this study?

Results The analysis of students' written work, as well as videotapes of the interviews resulted in the creation of two descriptive models of diagram drawing for problem solving.

Model One: Five characteristics of problem solvers seem to affect whether they choose to draw a diagram and whether they draw useful diagrams. They are:

1. Conceptual understanding of the mathematics involved in the problem: In order for students to recognize and represent schematically the mathematical structure of the problem, they must have an understanding of the mathematics involved. Rote algorithms, which can often be applied even when such understanding is lacking, are generally not helpful in creating a diagrammatic representation. (E.g. Fraction operations can be computed with little understanding, however, an understanding of basic fraction concepts is often required in diagram drawing.)

2. Diagram drawing skills and experience: There are specific skills which are important to successfully create and manipulate a diagram. Some of these skills are also applicable when using other problem solving strategies while others are particular to diagram drawing. Particular skills are identified in Model Two, below.

In addition to diagram drawing skills, the experience that the student has with diagram drawing gives the student a sense of how diagram drawing works and what its benefits are. This experience leads to the "metacognitive" skill (Schoenfeld 1983) of thinking to draw a diagram. For example, experienced geometry problem solvers, when faced with problems that ask them to find the size of an angle, procede to label all of the known angles with numbers or variables, as well as all congruent line segments in hopes of "seeing" the value of the angle in question or a key relationship to it. Without the expectations of how a diagram might help, they would be unlikely to procede in this way.

3. Conceptions of mathematics: The beliefs that students have about mathematics strongly influence the likelihood of their using a diagram. For some students, attempting to draw a pictorial representation makes no sense. If students believe that mathematics is unrelated to the real world, that it is a "black box" which can never be understood, and that there is one correct way to do a math problem, then they are unlikely to try to represent the problem diagrammatically in order to <u>explore</u> solution possibilities.

4. Self-concept in mathematics: Strongly tied to the students' beliefs about mathematics are their beliefs about themselves as mathematics students. Key to choosing to use diagrams is a belief that "I can figure out math problems and understand each step of the solution". To the extent that the student feels a sense of personal power, a sense of control in mathematics, she is more likely to try to represent the problem in a diagram. If the student feels

that the power lies outside of herself, "the teacher hasn't shown me how to do that", or helpless in solving problems, "I'm just not good at word problem", she is unlikely to see any value in drawing a diagram. In fact, the challenge to draw a diagram becomes another frustrating problem to be avoided.

5. Motivation to solve the problem correctly: Seemingly obvious, but important to mention, is the student's motivation to solve the problem correctly. For the low ability student, who is anxious when solving a math problem, the motivation may be to finish with the problem (ending the anxiety) and move on, rather than to figure out how to solve it correctly. For this student diagram drawing appears to be an unnecessary, time consuming step which only increases the struggle with the problem. Finding a neat algorithm, even if it yields incorrect solutions, more directly serves the goal of finishing with the problem.

Model Two: The following are skills and procedures which seem to contribute to successful diagram drawing.

1. Represent all relevant information.
   a. Determine relevant aspects of the problem situation (relevant concepts as well as given information).

b. Represent everything spatially if possible. (Avoid the use of arithmetic symbols in the diagram (+ - x / =)).

c. Draw unknown quantities into the diagram keeping track of the unknown (arbitrary) aspects of the diagram.

2. Create one integrated diagram with related parts.

   a. Avoid creating several unrelated diagrams for different aspects of the information.

   b. Operate on the diagram so that the diagram reflects the changes in the problem situation.

   c. As each new piece of information is represented, relate it as much as possible to the already represented information.

3. Label completely.

   a. Label each part of the diagram descriptively (what does it represent).

   b. Label all known quantities (include those which become known as you label others).

   c. Create and label equal parts wherever possible.

4. Draw multiple representations.

   a. Create alternative representations

      (1) when unsure how to represent the information.

(If unable to decide on an appropriate representation, draw something.)

(2) when the figure introduces a specificity that has not yet been determined (e.g. do two areas overlap?)

b. The introduction of new information may necessitate a new diagram.

c. The diagram evolves

(1) to make a more effective/helpful diagram.

(2) to give the diagram more accurate proportions.

5. Verbalize about what is represented in the diagram and what needs to be represented.

6. Check the accuracy of your diagram.

Main Study

Purpose The purpose of the main study was to verify and extend findings that were produced in the exploratory study. It focused on factor #2 of Model One, the subskills of diagram drawing. The choice to focus on just one of the five factors permitted more in depth work in that area. The other four factors of Model One deserve to be studied, as well, in future studies. The main study was designed to investigate the following, modified set of research questions:

A. Are the subskills identified in the exploratory study (Model Two above) important in the creation of useful diagrams?

B. How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

C. What effect does the problem context, geometric versus algebraic, (defined in Chapter One) have on the quality of the diagrams that are drawn?

D. What important skills and knowledge were not identified during the exploratory study?

E. What are the difficulties which prevent successful diagram drawing?

The research hypotheses which correspond to these questions, are presented in the section following the description of the "Procedure."

<u>Subjects</u> Eighteen volunteers from precalculus classes at the University of Massachusetts were paid to participate in this study. The remedial mathematics population used in the exploratory study was not used because some diagram drawing is done in the remedial classes which could interfere with the results of this study.

Procedure As in the exploratory study, standard algebra problems which can be solved with a diagram approach were used. (See Appendix B.)

The structure of the Heller and Reif (1984) paradigm, which had been introduced in the area of physics mechanics problems, provided the basis for the methodology employed. The Heller/Reif design, began with a pretest of each subject in an individual interview format. The subjects were randomly assigned to one of three groups: an experimental group, a modified experimental group, and a control group. The control group repeated the pretest procedure, using a parallel form of the test, one week later while the other two groups received experimental treatments.

Table 3.1

Schedule of the Experiment

SESSION ONE (pretest)

	Exper. I	Mod. Exp. II	Control III
1 hour	Pretest (3 prob.)	Pretest (3 prob.)	pretest (3 prob.)
20 min.	practice treatment proc. I	practice treatment proc. II	additional problem (unscored)

-----one week elapsed time -----

SESSION TWO (treatment)

	Exper. I	Mod. Exp. II	Control III
1 hour	oral suggest. (3 prob.)	list of suggest. (3 prob.)	same as pretest (3 prob.)

Following the pretest, the experimental and modified experimental groups received a twenty minute practice session to familiarize them with the treatment procedures. The control group worked one additional problem (which did not count in the scoring) to make the amount of time spent on diagram drawing, prior to the experimental treatment, more nearly equal to Groups I and II. This last feature was a modification of the Heller and Reif design.

All subjects returned one week later for treatment sessions which were conducted using an individual interview format, parallel to that used in the pretest. The purpose of the pretest was to identify any initial difference in skill among the three groups and to permit a measurement of improvement for each group. The three groups could than be compared to look for different effects of the experimental treatments on diagram drawing.

During the pretest and treatment sessions, all subjects were given a set of three problems for which they were asked to draw the most complete and useful diagrams that they could. Subjects were asked to "think out loud" and reminded to do so during the interviews.

As mentioned in chapter one, Heller and Reif used external control directions to guide the subjects' work in a step by step fashion. Since successful diagram drawing for algebra problems does not seem to follow a particular sequence, the step by step directions were replaced with "suggestions" that were made at what the experimenter deemed appropriate times. The list of external control suggestions was created based on Model Two (see Appendix C).

During the treatment sessions, Group I was given the above control suggestions orally by the experimenter. In so doing the experimenter made use of his own control knowledge, knowledge of when the particular suggestions might be appropriate. His interventions were limited to those listed in Appendix C.

Group II subjects received a written list of the suggestions and were reminded, during their work on each problem, to refer to the list for help. They received the same suggestions as Group I but did not benefit from the control knowledge of the experimenter who made decisions of when to use each suggestion for Group I.

Group III, the control group, received no interventions or assistance of any kind. Their treatment interview was identical to the pretest interviews.

Subjects were not asked to get an answer to the problem. Emphasis was always on creating complete and useful representations of the problem. The lack of focus on the answer was designed to take the pressure off the subjects that exists when work is either "right" or "wrong" and to allow subjects to focus on the task of drawing an effective diagram. Pressure to obtain an answer might have increased subjects resistance to giving up the more familiar algebraic algorithms and mental calculations to comply with requests that they attempt a less familiar approach, diagrams.

Experimental Hypotheses In this section, research questions A, B, and C are listed with the hypotheses which correspond to each question. From these general statements of the hypotheses, null hypotheses were created as a basis for statistical analyses. Research questions D and E are not included in this section since they led to open ended videotape analysis rather than specific hypotheses.

Investigation of research questions D and E are discussed in the section below, "Analysis of Treatment Interviews."

1. Research Question A: Are the subskills identified in the exploratory study (Model Two above) important in the creation of useful diagrams?

H1. The subskills identified in the exploratory study lead to improved diagram drawing.

According to hypothesis 1, subjects in treatment Group I should show greater improvement from pretest problems to treatment problems, than subjects in the control group (Group III). This result is based on the predicted benefit of the oral suggestions which encourage the use of subskills from Model Two. The null hypothesis, therefore is:

> Ho1: Group I and Group III will show no differences in improvement (treatment scores minus pretest scores) for the measures of diagram quality: type, completeness, labeling, and accuracy and the scores representing the total of the four measures.

2. Research Question B: How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

H2. An important factor in the successful implementation of the diagram subskills outlined in Model Two is the metacognitive ability to decide when to use each skill.

Experimental hypothesis 2 suggests that improvement in diagram drawing performance requires both the skills outlined in Model Two and the metacognitive ability to decide when to use each skill. Based on this hypothesis, Group I, who received not only the oral suggestions but also the metacognitive decisions of the experimenter, should draw higher quality diagrams than Group II, which had access to the suggestions (the written list) but not the decisions of the experimenter as to when to use each skill. The corresponding null hypothesis is:

> Ho2: Group I and Group II will show no differences in improvement (treatment scores minus pretest scores) for the measures of diagram quality: type, completeness, labeling, and accuracy and the scores representing the total of the four measures.

3. Research Question C: What effect does the problem context (geometric versus algebraic) have on the quality of the diagrams that are drawn?

H3. Higher quality diagrams are created for problems with geometric contexts than for problems with algebraic contexts.

Based on McKee's work (1983), it was predicted that subjects would draw better quality diagrams for problems with geometric formulations than for problems with algebraic formulations. Statistical analysis was based on the null hypothesis:

> Ho3: Measures of type, completeness, labeling, and accuracy of diagrams, as well as the total score for these four criteria, are not affected by whether the problem is formulated in an algebraic or geometric context.

Instrumentation

Pretest and Treatment Problem Sets: The six pretest and treatment problems (see Appendix B) were selected from McKee's study (1983). McKee's problems were used for the following reasons:

1. Data existed on the relative difficulty of the problems (from McKee's research) which was useful in creating two parallel forms of the problem sets.

2. McKee had developed criteria for evaluation of the quality of diagrams drawn for these diagrams.

3. McKee had developed "geometric" and "algebraic" formulations for each problem (see Chapter Two) which allowed for further analysis of the effect of these formulations.

All of these problems were standard algebra problems, which subjects taking college precalculus could be expected to solve algebraically. Two matched sets of problems were created (sets A and B) based on problem difficulty according to McKee's data, and controling for algebraic versus geometric problem formulations. Half the subjects in each group received set A for the pretest and set B for the treatment. The other half of the subjects received the sets in the reverse order.

Criteria for Evaluation of Diagram Quality: The scoring criteria were based on McKee's four categories. However, two of her categories, completeness and labeling were scored on a 0 to 4 scale rather than McKee's 1 to 3 scale to allow for more sensitive scoring in these areas. Samples of diagrams for the six problems were collected and analyzed by the experimenter in order to specify particular criteria for awarding points for each category for each problem. These criteria are described in Appendix D.

External Control Suggestions: The list of external control suggestions (Appendix C) was created by the experimenter from Model Two which resulted from the exploratory study.

Methods of Data Analysis

Analysis of Experimental Study (Questions A,B, and C) The quality of each diagram was scored by the experimenter and another graduate student using a modification of the point system developed by McKee (1983). (See Appendix D for a detailed description of the scoring.) A correlation of the scores of the two scorers was computed to determine the reliability of the scoring system. Scoring resulted in scores for each subject on both the pretest and treatment problems for each measure on each problem. Pretest scores were compared to determine whether the groups were equivalent at the outset.

Difference scores (treatment score - pretest score) were computed for each group on each measure, reflecting improvement from the first session to the second. The differences for each group were compared on each measure and the total score. Analysis of variance (ANOVA) was used to compare the three groups. Where significant F values were obtained, Student t-tests were used to compare the groups two at a time.

Analysis of Treatment Interviews (Questions D and E)

The second part of the analysis was a study of the videotapes from the treatment sessions to determine the frequency of the various suggestions given to Group I and the types of difficulties which existed despite the assistance provided by the external control suggestions (Appendix C). Such difficulties reveal weaknesses of the experimental model and/or highlight other important factors (i.e. affective factors and the effects of past experience) that contribute to diagram drawing success.

The open-ended investigation, begun in the exploratory study, was continued in the main study in response to questions D and E. This investigation was not based on particular hypotheses.

Research question D: What important skills and knowledge were not identified during the exploratory study?

Research question E: What are the difficulties which prevent successful diagram drawing?

The investigation consisted of analysis of the videotapes of the treatment sessions from the experimental part of the study. Analysis was done by the experimenter and focused on the critical points of each diagram attempt. A critical point was when a subject came up against an obstacle in creating his representation and either overcame it, failed to negotiate the obstacle, or created an inaccurate diagram as a result of attempting to negotiate the obstacle. These points focused the experimenter on skills used as well as the difficulties encountered by the subjects. These observations were then examined for skills and difficulties that were common to more than one subject.

<u>Methodological Assumptions</u> The research design was based on the following assumptions:

1. Improvement due to the oral suggestions indicates a verification of at least some of the subskills contained in Model Two.

2. Differences in scores between treatment Groups I and II are due to the metacognitive information provided by the experimenter.

3. Subjects' performance on the tasks given accurately reflect their diagram drawing abilities.

Limitations of the Study The following limitations were inherent in the design of the study:

1. All interviews were analyzed by the experimenter only, thus observations were restricted by the experimenters perceptions and influenced by his biases.

2. The number of subjects in each group was limited to permit the use of pre and treatment interviews and the examination of three treatment groups. As a result, the statistical results are less sensitive than they would have been with a larger sample.

3. The questions asked during the treatment interviews were considerably restricted. In depth probing into the thought processes of the subject could have positively affected the clarity of the subject's thinking, thus interfering with the experimental (statistical) results.

4. The experimenter's oral suggestions to Group I may have had some additional affects. Subjects may have spent more time on each problem as a result of the suggestions. They also may have felt more supported and therefore more confident. Suggestions towards the end of their work on a given problem may have signaled them that they could still improve the diagram.

Correspondingly, to the extent that subjects in the control group (III) experienced the pretest as anxiety provoking, they had no external help on which to base hope that their second diagram session would be more successful.

5. Subjects had only limited time to become familiar with diagram drawing, the pretest and the treatment session. For most of them this was the first time that they were engaged in diagram drawing tasks.

CHAPTER IV

DATA ANALYSES AND RESULTS

This chapter focuses on the results of the main study and interpretations of those results in light of the five research questions which motivated the study. The results of the experimental study are treated first followed by the results of the analysis of the videotaped interviews.

Results of Statistical Analyses

Statistical analyses were done using BMDP Statistical Software (1983).

Preliminary Analyses

Equivalence of Test Forms: Although possible differences in the difficulty of Form A and Form B problem sets were controled for in the design of the study by staggering their use within each group, the scores on the two forms were compared for all subjects using t-tests (see Table 4.1). No significant difference in student performance was found on any of the four measures of diagram quality nor on the total score, suggesting that Form A and Form B were indeed closely matched.

Interscorer Reliability: The consistency of scores given by the two scorers was checked using a Pearson-r. The mean of correlations for all variables was .868

(standard deviation of .110 and range of .661 to 1.000) indicating that scores assigned by both the experimenter and the second scorer were very close (see Appendix E for r-values). All scores reported in this chapter are based on the mean score for the two scorers.

Table 4.1

Comparison of Test Form A and Form B

Measure	p-values
M (total)	.695
M1 (type)	.119
M2 (completeness)	.643
M3 (labeling)	.153
M4 (accuracy)	.213

Equivalence of Groups: Table 4.2 gives the mean scores and range of possible scores for each treatment group for the total (M) and separately for each measure of diagram quality.

An analysis of variance was used to evaluate the null hypothesis that there was no difference between groups on the pretest. This hypothesis was rejected on the basis of an F-score of 6.32 ($p = .008$). A two-tailed t-test was used to find where those differences occured. Group II, compared to each of the other groups, was significantly different (I vs.II -- $p < .01$, II vs. III -- $p < .05$). There was no significant difference between Group I and Group III. This indicated that differences in treatment scores or in improvement scores for Group II could not be meaningfully compared to those for the other two groups. The analysis therefore reflects only a two group comparison, the experimental (Group I) versus the control

TABLE 4.2

Mean Scores, Standard Deviations and Range of Possible Scores

PRETEST SCORES

Measure	Group I	Group II	Group III	Possible
M (total)	9.02 (0.86)	11.00 (1.29)	9.60 (1.03)	2-14
M1	2.64 (0.50)	2.90 (0.25)	2.76 (0.45)	1-3
M2	2.17 (0.55)	2.83 (0.71)	2.29 (0.48)	0-4
M3	2.07 (0.43)	2.52 (0.63)	2.05 (0.42)	0-4
M4	2.14 (0.31)	2.73 (0.42)	2.50 (0.30)	1-3
TREATMENT SCORES				
M	12.43 (1.15)	12.17 (1.59)	11.05 (0.94)	2-14
M1	3.00 (0.00)	3.00 (0.00)	2.79 (0.39)	1-3
M2	3.57 (0.32)	3.62 (0.34)	2.83 (0.55)	0-4
M3	3.26 (0.64)	2.79 (1.06)	2.79 (0.39)	0-4
M4	2.60 (0.27)	2.76 (0.32)	2.64 (0.31)	1-3

group (Group III).

Research Question A Are the subskills identified in the exploratory study (Model Two in Chapter Three) important in the creation of useful diagrams?

> Ho1: Group I and Group III will show no differences in improvement (treatment scores minus pretest scores) for the measures of diagram quality: type, completeness, labeling, and accuracy and the scores representing the total of the four measures.

The effect of the experimental treatment was analyzed by comparing the improvement scores (treatment score minus the pretest score) for the experimental and control groups. Table 4.3 shows the comparison of total score and scores on each measure (Groups I vs. III) using a one-tailed t-test.

Table 4.3 indicates a significantly greater improvement for the experimental group (I) than for the control group (III), based on total scores ($p < .01$). Looking at individual measures, we find that the only significant difference exists in the category of diagram completeness ($p < .01$). Therefore, the null hypothesis is rejected for the total score and for diagram completeness.

TABLE 4.3

Group I Versus Group III

Measure	Group I	Group III	p-value	signif.
M (total)	3.40 (1.35)	1.45 (1.04)	.005	**
M1	.38 (.50)	.02 (.06)	.054	
M2	1.40 (.74)	.55 (.30)	.008	**
M3	1.19 (.94)	.74 (.71)	.166	
M4	.45 (.33)	.14 (.35)	.058	

* = significant, $p < .05$
** = highly significant, $p < .01$

Research Question B How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

Ho2: Group I and Group II will show no differences in improvement (treatment scores minus pretest scores) for the measures of diagram quality: type, completeness, labeling, and accuracy and the scores representing the total of the four measures.

Null hypothesis 2 (Ho2) could not be evaluated reliably due to the lack of equivalent groups at the outset

(Group I versus Group II). The greater improvement of Group I is not meaningful since they were weaker than Group II on the pretest (see M total in Table 4.2 above).

Research Question C What effect does the problem context (geometric versus algebraic) have on the quality of the diagrams that are drawn?

> Ho3: Measures of type, completeness, labeling, and accuracy of diagrams, as well as the total score for these four criteria, are not affected by whether the problem is formulated in an algebraic or geometric context.

T-tests were used to compare the measures of diagram quality for geometric context problems with those for algebraic context problems. Only the two problems in each problem set which had parallel problems in the other set were used in the analysis (Form A problems 1 and 2 and Form B problems 2 and 3).

Form A

1. The sum of the measures of the sides of a triangle is 35 inches. One of the sides is 4 times longer than the second side and 1 inch longer than the third side. What are the lengths of the sides?

Form B

2. The sum of the ages of three children is 26. One of the children is 3 times older than the second child and 2 years older than the third child. What are the ages of the children?

2. Sam has four times as much money as his sister segment
Eileen. If Sam's money is 30
decreased by 39 cents and Eileen's money is increased by 39 cents, then Eileen and Sam have the same amount. How much money did Sam and Eileen have at the start?

3. Line segment AB is six times as long as line

CD. If AB is decreased by

centimeters and CD is increased by 30 centimeters, then AB and CD are the same length. What are the original lengths of AB and CD?

The pair of problems used from each form was made up of one algebraic and one geometric context problem. The results are listed in Table 4.4.

Table 4.4.

Comparison of Performance on Geometric Versus Algebraic Problem Contexts

Measure	Mean-G	Mean-A	G-A	p	Signif.
M (total score)	2.86	2.64	0.22	.004	**
M1 (type)	2.99	2.69	0.30	.008	**
M2 (completeness)	3.13	2.80	0.33	.029	*
M3 (labeling)	2.74	2.62	0.12	.154	--
M4 (accuracy)	2.58	2.46	0.12	.106	--

* = significant, $p < .05$
** = highly significant, $p < .01$

The results for the total score reveal that subjects did considerably better on the geometrically posed problems than on the algebraic ($p < .01$). This result is consistent with McKee's (1983) findings. Further analysis indicates that the difference is mainly a result of differences of diagram type ($p < .01$) and completeness ($p < .05$). Therefore the null hypothesis is rejected for the total measure, diagram type and diagram completeness.

Observations from Videotapes

Research Question D What important skills and knowledge were not identified during the exploratory study?

This question is not addressed in this section since answers to it cannot be directly observed. Important skills and knowledge must be inferred from the observations. Therefore, it is treated in the next major section "Interpretations of Results."

Research Question E What are the difficulties which prevent successful diagram drawing?

1. Subjects in Group I continued to have the following difficulties with labeling which were common to the other groups as well as all groups in the pretest situation:

a. When encouraged to label parts descriptively, they often focused on numerical labels instead.

b. Descriptive labels, when used, were often incomplete (e.g. "books" instead of "Jack's books" or "Barb's age" instead of "Barb's age now."

c. They omitted labels for parts of the diagram that were either not contained in the given or that were not the part(s) to be found. Figures 4.1 and 4.2 show examples of

Fig. 4.1

Unlabeled Parts
(Problem 3, Form A)

Mrs. Brown is 38 years old and her daughter Barbara is 8 years old. If Mrs. Brown and Barbara both have birthdays on the same day, when will Mrs. Brown be three times as old as Barbara?

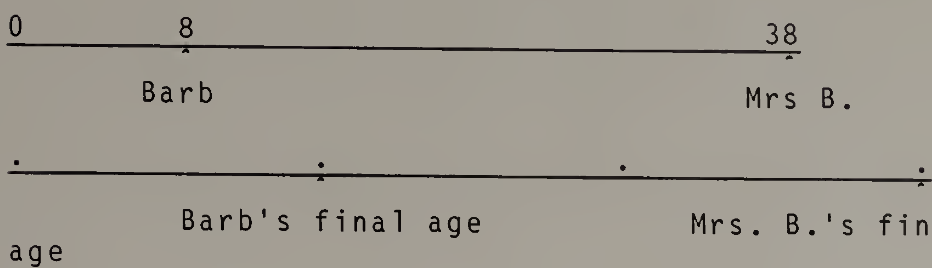
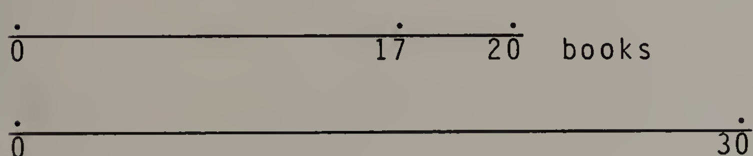


Fig. 4.2

Unlabeled Parts
(Problem 1, Form B

The sum of the number of books Jack and Jill have is 20. If Jill lost 3 of her books and Jack doubled the number he has, they would then have a total of 30 books. How many books does each have?



diagrams by subjects that were limited by this type of omission.

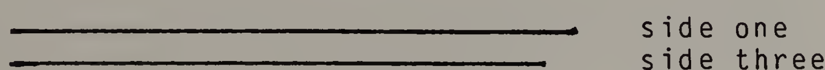
The subject who drew the diagram in Figure 4.1 neglected to label the difference between Barb's age and Mrs. Brown's age, which might have provided the key to a solution. Likewise, the subject who drew the diagram in Figure 4.2 might have benefited by labeling the difference between the 17 books that Jack and Jill had (after Jill lost three) and the 30 books that they had in the end.

2. Subjects represented information implicitly rather than explicitly.

Fig. 4.3

Implicit Rather than Explicit Representation
(Problem 1, Form A)

The sum of the measures of the sides of a triangle is 35 inches. One of the sides is 4 times longer than the second side and 1 inch longer than the third side. What are the lengths of the sides?



The 1" is represented by a difference in length but does not explicitly appear in the diagram. This seemed to reduce the likelyhood that the subject would make use of the 1" as the problem solution proceded.

3. The representing of unknown quantities was difficult for many of the subjects. They often wanted to try particular

numbers for the unknown quantity when they were unable to represent it in a more general way. Some of the subjects in Group I seemed to be helped by suggestion 3,

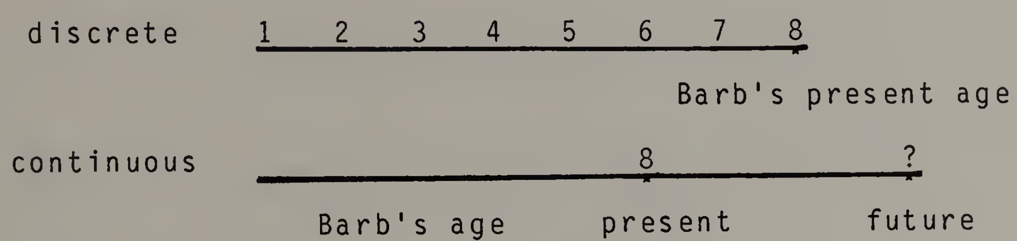
> If the size of the fraction to be represented is unknown, mark off a space remembering that its size is arbitrary. Try to avoid drawing it to look equal in size to parts that may not be equal to it.

4. Subjects often used discrete and continuous diagrams without apparent awareness of having made a choice or that it was having an effect on their attempts to represent all of the information in the problem. Most commonly the choice of a discrete diagram contributed to their inability to represent an unknown quantity.

Fig. 4.4

Discrete Versus Continuous Diagrams
(Problem 3, Form A)

Mrs. Brown is 38 years old and her daughter Barbara is 8 years old. If Mrs. Brown and Barbara both have birthdays on the same day, when will Mrs. Brown be three times as old as Barbara?



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5. Subjects sometimes confused additive relationships with multiplicative relationships (see Figure 4.5) and showed

evidence of difficultiy with fraction concepts (see Figure 4.6).

In Figure 4.5 the subject was drawing a diagram to show that the first child was three times older than the second child and two years older than the third child.

The multiplicative relationship between the first child and the second child is represented by three segments and one segment. However, when she attempts to show that the first child is two years older than the third child,

Fig. 4.5

Additive Versus Multiplicative Relationships
(Problem 2, Form B)

The sum of the ages of three children is 26. One of the children is 3 times older than the second child and 2 years older than the third child. What are the ages of the children?

First child: .____.____.____.
Second child: .____.
Third child: .____.

---

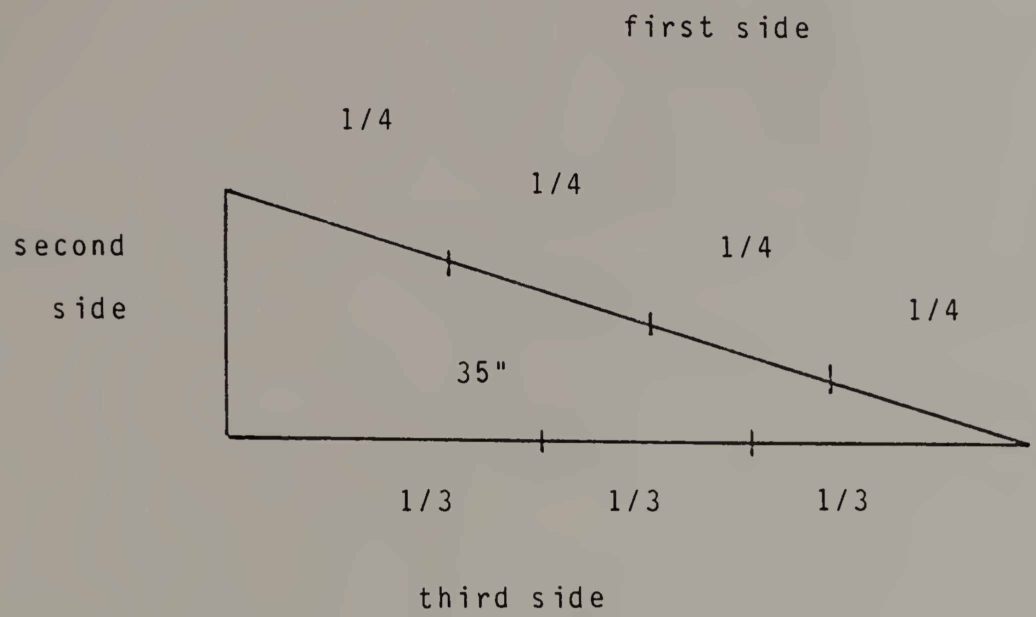
she draws the third child's age as one segment (two segments less than the first child's). The resulting equivalent representations for the second child's and third child's ages did not seem to be particularly significant to the subject.

Figure 4.6 shows an example of the types of difficulties with fractions that hampered subjects' abilities to draw useful diagrams. Note that the spatial representation is accurate, however, the labeling of fractions does not show an apreciation that "one whole" must be identified and kept constant in order for the fractions to be meaningful.

Fig. 4.6

Difficulties with Fraction Concepts
(Problem 1, Form A)

The sum of the measures of the sides of a triangle is 35 inches. One of the sides is 4 times longer than the second side and 1 inch longer than the third side. What are the lengths of the sides?



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6. Subjects reported at times that they were unclear about whether it is important that the diagram look accurate. In

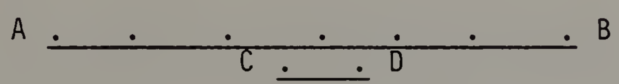
addition some subjects expressed a concern that the diagram would be of limited value if it were not measured and drawn to scale. At times subjects would label two different numbers of equal parts as representing equal parts without seeing a contradiction.

One subject sketched the diagram shown in Figure 4.7.

Fig. 4.7

Scale Drawing Versus Sketch
(Problem 3, Form B)

Line segment AB is six times as long as line segment CD. If AB is decreased by 30 centimeters and CD is increased by 30 centimeters, then AB and CD are the same length. What are the original lengths of AB and CD?



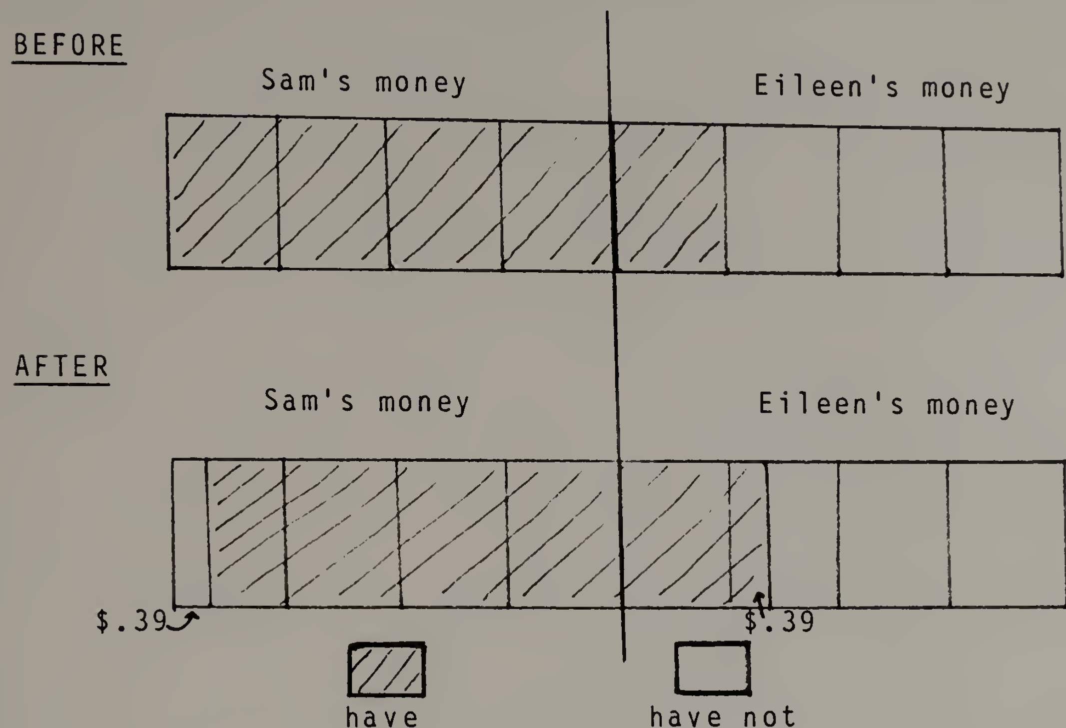
She then, in order to show that the same length was removed from segment AB as was added to CD, measured, using another piece of paper, the amount she was marking off of AB in order to add exactly that length to CD.

In Figure 4.8, the subject created a representation of the new amounts of money that Sam and Eileen had. He then announced that their two amounts were equal. The unequal diagram (3+ boxes versus 1+ box) did not seem to make him question his representation.

Fig. 4.8

Unequal Number of Equal Parts
(Problem 2, Form A)

Sam has four times as much money as his sister Eileen. If Sam's money is decreased by 39 cents and Eileen's money is increased by 39 cents, then Eileen and Sam have the same amount. How much money did Sam and Eileen have at the start?



7. Subects failed to make use of information about age that they could be expected to know. Specifically:

a. Two people's ages increase by the same number of years during any given time period.

b. The difference between two people's ages remains constant as they get older.

When as a follow-up question at the end of the second interview, some subjects were asked to write equations for problem 3 of Form A, many were unable to do so.

> Mrs. Brown is 38 years old and her daughter Barbara is 8 years old. If Mrs. Brown and Barbara both have birthdays on the same day, when will Mrs. Brown be three times as old as Barbara?

Commonly they wrote:

x = Mrs. Brown's age
y = Barb's age

$x = 3y$

At this point they would try to substitute in either 8 for y or 38 for x or both. The subjects rejected the results of these substitutions but proved unable to identify the source of their difficulties.

8. A number of the subjects came into the study with no sense of how to represent the four basic arithmetic operations spatially. For example they did not realize that they could show the sum of two line segments by putting them together to form one longer segment or that the difference between two quantities could be represented by putting them both on the same segment starting from a common origin and labeling the lack of overlap as the difference.

9. Subjects at times repeated strategies that were not working for them, seemingly unaware that they were making a

choice, that it was causing difficulty and that an alternative might work better.

10. Subjects seemed unaware of what relationships would lead to the desired information. For example, a subject who had six equal segments, of which he wanted to know the length of one segment, seemed to not be aware that if he could find the length of the six segments together that he would be able to then determine the length of one segment.

<u>Other Observations</u>

1. The suggestions given most often by the experimenter to Group I subjects encouraged them to label numerically (7a,b), label descriptively (6a), check that all information has been represented (11d), and to check that the problem has been accurately represented (11e). See Table 4.5.

TABLE 4.5

Frequency of Use of Individual Suggestions

Suggestion #	Frequency
1	2
2	6
3	6
4a	10
4b	7
5	7
6a	13
6b	6
7a	21
7b	17
7c	8
8	5
9	5
10	1
11a	* see 6a
11b	* see 7a & 7b
11c	* see 4a
11d	12
11e	12

* 11a and 6a were considered identical and counted under 6a, likewise 11b and 7a,b, also 11c and 4a.

See Appendix C for the list of suggestions

2. Subjects reported that the suggestion 4b,

> If one part is a multiple of the other (and the number values are unknown for these parts), subdivide the larger to make parts equal to the smaller part. Label these new equal parts clearly,

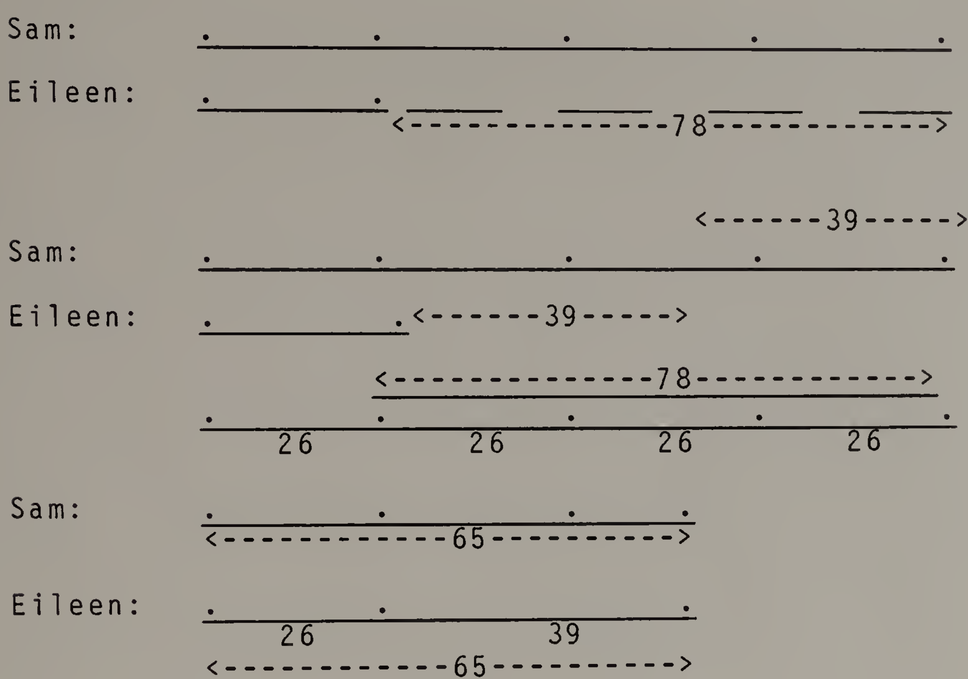
was most helpful even though it was not given by the experimenter very often.

3. Subjects in Groups I and II frequently reminded themselves of suggestions 1,2, and 4b.

Fig. 4.9

New Diagram for Each Step
(Problem 2, Form A)

Sam has four times as much money as his sister Eileen. If Sam's money is decreased by 39 cents and Eileen's money is increased by 39 cents, then Eileen and Sam have the same amount. How much money did Sam and Eileen have at the start?



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4. Some subjects created a new diagram for each additional piece of information rather than modifying the existing one. Figure 4.9 shows one subject who was successful yet created a new diagram for each step. Suggestion #2, to create one diagram, was not helpful to some of the subjects who proceded to explain why that could not be done in the particular problem.

5. Subjects expressed that the suggestion to avoid the use of arithmetic signs in the diagrams (#1) was a hindrance not a help. They showed no indication that they understood the purpose for its inclusion (to require that relationships be represented spatially rather than in a label).

6. Subjects frequently reported that their attempts to represent these problems diagramatically required a mental effort that surpassed what was required of them in mathematics classes. ("This strains my brain.")

7. Subjects lacked effective strategies for cutting a segment into six equal parts. They divided it from left to right, often winding up with very unequal looking segments. They seemed to lack the awareness that cutting the segment into two or into three segments would be a helpful first step.

8. At times language use seemed to affect the subject's ability to model the world (operate on the diagram). Abstract language connected with algorithms seemed to make diagram drawing more difficult (i.e. subtraction, addition) while non-mathematical, active language seemed to aid modeling by diagram (i.e. "takes from one and adds to the other").

9. Subjects, representing the triangle problem (Form A problem 1), generally drew an equilateral triangle first and then rejected it in favor of a right triangle.

10. Subjects consistently reported that using diagrams, in the way that the study required them to, was a task for which they had little or no previous experience.

Interpretations of Results

Subskills of Diagram Drawing: Model Two

Research Question A: Are the subskills identified in the exploratory study (Model Two above) important in the creation of useful diagrams?

The oral suggestions, presented to the subjects (Group I), resulted in significantly greater improvement in the quality of drawings than in those created by subjects in the control group who received no assistance. This suggests that at least some of the subskills contained in the model are lacking in the experimental population and contribute to the creation of useful diagrams.

Looking at the four separate measures (type, completeness, labeling, and accuracy), we can get a more precise idea of the effects of the experimental treatment. The most clearcut effect was in completeness of the diagram (M2) ($p=.008$). The direction to "create one diagram...instead of several separate ones" (#2) contributed to the integratedness of the diagram which earned one point out of the four offered for completeness. The instruction to represent unknown quantities by using an arbitrary size (#3) and the instruction to represent ratio relationships by drawing equal parts seemed to offer some of the subjects a way to represent important information when they might otherwise have failed to do so or to do so explicitly. The reminder to check that the "relevant information" has been represented (#11d) also contributed to the completeness of certain drawings.

The measures for type of diagram (M1) and accuracy of diagram, while not significant were close to significant (p=.054 and p=.058 respectively), indicating that a difference may reveal itself in a study with a larger number of subjects. Type, the ability to draw a schematic representation of the mathematical quantities involved rather than an illustration of the problem setting, did not prove to be a significant difficulty for most of the subjects. Possible differences for type (M1) were likely depressed due to the ceiling effect of scores on both the pretest and treatment problems (Group I--pretest=2.64, treatment=3.00 and Group III--pretest=2.76, treatment= 2.79 out of a possible 3 points).

While none of the suggestions directly addressed diagram type, questions related to completeness may have encouraged some subjects to abandon an illustrative diagram in favor of a schematic diagram. If a real improvement did take place, it also may have been due to the effect of working the sample problems (at the end of the pretest session to familiarize subjects with the list of suggestions). The sample problems provided subjects the opportunity to see and work with schematic diagrams.

Experimental subjects showed no significant difference over control subjects on labeling. This may suggest that the process of labeling effectively may be a more complex process which is not generally improved by a simple intervention of making external suggestions. This is discussed below under "Metacognition." The final point in the scoring of this measure was awarded for the labeling of a derived quantity. The earning of that point indicated substantial progress towards solution of the problem.

Research Question B: How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

It is important to note that subjects in the experimental group received not only the content of the suggestions but the benefit of the experimenter's judgement as to when each suggestion was appropriate. Although subjects often were unable to make use of the suggestions and often offered explanations as to why the particular suggestion was inappropriate in that situation, it is likely that the metacognitive skills of the experimenter, in deciding when to try each suggestion, were helpful and compensated for metacognitive skills that were not well developed in this population.

Due to the inequality of subjects assigned to Group II as opposed to the other groups, it was not possible to derive conclusive evidence from this study as to the relative importance of knowing when to use each suggestion. The question, however, will remain an important one as one looks to develop diagram drawing abilities in students.

Metacognitive aspects of diagram drawing are discussed further, below, as part of the discussion of skills not covered by Model Two.

Algebraic Versus Geometric Problem Contexts

Research Question C: What effect does the problem context (geometric versus algebraic) have on the quality of the diagrams that are drawn?

Consistent with, McKee's work with high school students, subjects drew significantly higher quality diagrams for problems with geometric contexts than for problems with algebraic contexts.

The highly significant difference on diagram type (M1) points out an added difficulty of algebraic context problems. Subjects are less likely to represent an algebraic context schematically, tending to draw a more illustrative "picture" rather than an abstract diagram of the mathematical structure of the problem. This difficulty is considerably less for geometric contexts since an attempt to draw a "picture" may result in a schematic representation. This was certainly the case for the two geometrically posed problems in this study, one involved a triangle and the other involved line segments. Drawing a picture of the triangle and of the line segments resulted, as well, in schematic representations of the lengths of the sides of the triangle and of the segments. The corresponding algebraic contexts involved two people's money and three children's ages. For these algebraic contexts, subjects, less clear about "what to draw", sometimes drew representations of the people rather than spatial representations of the key quantities.

The drawing of schematic diagrams for the geometric contexts probably permitted the drawing of more complete diagrams (M2), which may explain much of the difference in that measure. A diagram which is drawn schematically, representing the mathematical structure of the problem, tends to be a better vehicle for representing all of the important information and relationships.

Labeling (M3) and accuracy (M4) did not seem to be affected appreciably by whether the problem is stated in a geometric or an algebraic context. McKee, on the other hand, found significant differences, as well, for labeling and accuracy. This can be explained, perhaps, by the fact that her study was done by large scale paper and pencil measures. Subjects who drew illustrative diagrams may have realized, at some level, that the diagram was not very useful and failed to labor further over labeling and accuracy. Subjects in this study, who were constantly observed by the experimenter and asked if they had drawn the most complete and useful diagram that they can, probably felt some pressure to label their diagrams and make them accurate.

Other Important Skills

> Research Question D: What important skills and knowledge were not identified during the exploratory study?

The Need for Metacognitive Skills: Subjects frequently reported that the diagram work required more thinking than was generally demanded of them in mathematics classes. Rather than simply explaining this by the fact that diagram drawing is unfamiliar to these subjects, it is worth looking at this phenomenon in more detail. Besides the fact that the subjects were not used to thinking spatially, they were also not used to solving problems for which they had no algorithm. In other words, they were used to solving exercises rather than non-routine problems.

> When you do it algebraically, you're not thinking about how the algebra is working, you're just plugging stuff in. When you do it this way [using diagrams] you understand how it all fits together. [Subject #20]

One of the most important and underdeveloped components of solving nonroutine problems is metacognition (Schoenfeld 1983). An examination of the difficulties that subjects showed as they attempted to draw diagrams, points out some of the metacognitive skills that were needed.

In particular, although the suggestion to label all parts of the diagram descriptively and numerically were given more frequently than any other suggestions, the experimental subjects often did not successfully label all of the parts in the diagram. They seemed to lack several aspects of metacognition which were not addressed by the experimental model:

1. an understanding of the importance of labeling all parts of the diagram, not just those that represent the given information or those that represent the quantities being sought in the problem:

The experienced diagram drawer, when faced with a problem whose solution is not readily apparent, knows that the labeling of all parts may result in a convergence of information which can lead to a breakthrough in solving the problem. For example in a geometry problem that asks for the measure of a particular angle, she is likely to label all known angles, sides and congruent parts, hoping to find two algebraic expressions for the same angle which would permit the determiniation of its specific value (i.e. an angle which is 2x + 90 and also 180 - x). It is only through experience of this type that the student develops expectations of the power of thorough labeling of diagrams.

It is possible that the modeling of competent use of diagram labeling to solve a problem may be helpful to the novice.

2. the ability to perceive parts or combinations of parts which have not been labeled:

Often subjects, in response to suggestions to make sure that all parts have been labeled, looked over the diagram carefully and reported that all parts were labeled even though important parts still remained unlabeled. They seemed to be unable to identify parts which potentially could be labeled.

3. the ability to judge whether a label is sufficiently detailed [e.g. "books" rather than "Jack's books"]:

This may, in part, be due to the way labeling of answers to word problems is taught frequently in schools. The student often learns that some "unit" is necessary after the numerical answer (for the purpose of satisfying the "picky" teacher). As a result an answer such as "4 books" is accepted when "4 books per student" more accurately describes that quantity.

It should be mentioned that skilled diagram drawers do not always label everything that they might. However, the less that is labeled the greater the mental demand to keep track of the parts of the problem. One of the benefits of drawing a diagram is to reduce the mental load in a problem (McKee 1983, Newell and Simon 1972). For the subjects in this study, not labeling parts of the problem often caused them to excede their abilities to keep track of all the parts. Perhaps experienced problem solvers develop an additional metacognitive skill which allows them to monitor the mental demands of the problem so that they do not excede their abilities to keep information in their heads.

As is the case with problem solving heuristics, metacognitive skills seem to include both general skills and subject specific skills. Thus, some of the metacognitive demands of diagram drawing are specific to diagram drawing while some are applicable in problem solving in general.

The discussion above of metacognitive skills involved in labeling diagrams revealed both diagram specific skills (the understanding of the importance of labeling all parts of a diagram and the ability to identify unlabeled parts) and general skills (the ability to judge whether a label is sufficiently detailed). Below are additional metacognitive abilities whose importance for effective diagram drawing seems to be indicated by the observations made of the videotapes.

Specific Metacognitions: These metacognitive abilities are specific to diagram drawing and are not required for other problem solving strategies.

1. The knowledge of what aspects of the diagram are important:

For some subjects, there seemed to be confusion between the deductive aspects of diagram solutions and information that can be gleaned from visual inspection of the diagram. For example, if one knows that the parts are equal, one can see by looking at the diagram how many equal parts make up the larger quantity. On the other hand one cannot conclude that the parts are equal by looking to see if they look equal.

Subjects at times reported that their diagram solutions were limited by the fact that the diagrams were not measured with a ruler and drawn to scale. This seemed to show a lack of understanding of how diagrams are used to advance a solution. Subjects frequently created equal parts in their diagrams but then failed to look for equal numbers of those parts when attempting to divide the whole quantity in half. Uncertainty about what information in the diagram is useable seems to be a source of confusion to novice diagram drawers.

2. The knowledge of when to use a particular strategy:

This is made up of thinking to use the strategy and knowing the domain of utility for the strategy. Subjects often reminded themselves to avoid arithmetic signs (#1), to draw one diagram (#2), and to show ratio relationships by creating equal parts (#4b). The "trigger" for thinking to use these suggestions seemed to be straight forward and learned quickly by the subjects. When a subject began to use an arithmetic sign in the diagram, he often stopped, realizing that it was suggested that he not do so. Likewise when confronted with "Sam has four times as much money as his sister Eileen," subjects seemed able to select the recently learned strategy of drawing five equal parts, four to represent the amount of Sam's money and one to represent the amount of Eileen's money.

Subjects from Group II, who had drawn several diagrams to represent the information, often when reading over the list, realized that the suggestion to create one diagram with all the information was relevant. However, they lacked the metacognitive skill to distinguish between two diagrams which needed to be integrated and one integrated diagram with two parts. Therefore they sometimes tried to apply the suggestion where it was not appropriate and at other times decided that it was not appropriate when it would have, in fact, provided the relationships between non-integrated diagrams. The latter suggests that they also lacked the knowledge of how to apply the suggestion in a broad range of situations.

3. The knowledge of when to use a discrete and when to use a continuous diagram:

As mentioned in Chapter IV, subjects frequently had difficulties representing unknown quantities in part because they had created discrete rather than continuous diagrams. Where as number-lines or quantitative graphs are often useful diagrams, such discrete diagrams proved to be disadvantageous for a number of these problems. Subjects seemed automatically to use a discrete diagram based on the type of information that was presented first, rather than making a conscious and knowledgeable decision. As a result, they often never became aware of the source of the difficulty.

General Metacognitions:

1. The ability to monitor one's solution attempts:

Important in all problem solving is the ability to monitor one's progress, to decide whether a strategy is working, at what point to abandon it and to be able to glean information from the abandoned strategy. This involves an ability to be conscious of one's decisions so that the unsuccessful ones can be reversed or replaced by alternatives.

This reflection on one's own process seemed to be lacking for many of the subjects. The unconscious use of discrete diagrams (discussed in the preceding section) is one example of the inability to monitor one's work. Another example was seen in problem 2 Form B:

> The sum of the ages of three children is 26. One of the children is 3 times older than the second child and 2 years older than the third child. What are the ages of the children?

Subjects repeated the same sequence of steps (drawing the 26 year total first) even though that sequence was making it difficult to successfully represent the information in the problem.

2. The ability to evaluate one's solution:

Subjects frequently asked the experimenter whether they had correctly solved the problem. They appeared to be unable to take the answers which they had generated and evaluate them in terms of whether they fit for the requirements of the problem. A solution such as Sam had $1.17 and Eileen had $.39 would have been rejected ("Sam has four times as much money as his sister Eileen") if the subject had had the ability to evaluate his answer.

3. The ability to monitor the memory demands of the task:

It was postulated above that the expert problem solver may have the ability to monitor the amount of information that she must keep in memory. She likely has an approximate sense of the limit of her ability to do so effectively and uses external memory (i.e. a labeled diagram, a key for algebraic or calculus expressions) well before that limit is reached.

In contrast, novice problem solvers in this study frequently failed to make use of available information which was not explicitly recorded in the diagram.

Difficulties in Diagram Drawing

Research Question E: What are the difficulties which prevent successful diagram drawing?

Modeling Real World Events with Diagrams Diagram drawing has been proposed as the necessary bridge to span the gap that exists between the student's real world experience and his abstract mathematical work (Botsmanova 1972b). Subjects in this study often demonstrated their inability to model familiar situations in either their diagrams or in the follow-up requests by the experimenter to show algebraic equations.

This lack of ability to model real world events seems to be due to a limited approach to algebraic problem solving. These subjects have learned that to solve a problem one translates the given information into algebraic symbols and then manipulates the symbols according to some learned procedures which produce an answer. Since this limited view of algebra is familiar to these subjects while diagram drawing is not, they tend to try to apply it to their diagram drawing attempts as well.

Several observed behaviors support the notion that their attempts to draw diagrams is based on their experience with algebra rather than on a sense of how the real world phenomena might be represented spatially.

1. In problem 3 of Form A subjects consistently failed to model the equivalent increase in Barb and Mrs. Brown's ages, even though they could solve the problem informally.

> Mrs. Brown is 38 years old and her daughter Barbara is 8 years old. If Mrs. Brown and Barbara both have birthdays on the same day, when will Mrs. Brown be three times as old as Barbara?

Informally they would say "When Barb is 9, Mrs. Brown is 39, etc. However, when they attempted to draw the diagram they failed to even refer to the amount of increase in the ages, a failure that paralleled their unsuccessful attempts to write algebraic equations at the end of the interview. The algebraic approach was limited to the representation of explicitly described quantities in the problem. The informal approach was not limited in this way and, consequently, used more of the available information.

2. Rather than <u>operating on</u> the diagram to model what happened to the original quantities, the subject created a new diagram, showing only the result of the last event. Such a sequence of diagrams seemed to be more characteristic of the recording of an algebraic solution than of a visual solution. The language that is used to describe the phenomenon to be modeled can contribute to its

being thought of in more concrete relationships or in more abstract algorithms.

3. Subjects frequently showed not only that they had no familiar spatial models to represent basic arithmetic operations (models that might demonstrate what physically happens when we add, subtract, multiply, or divide), but they often had difficulty generating such diagrams. Subjects were stumped by how to show the sum of two line segments, not thinking to combine them into one longer segment. Several students mistakenly interchanged diagrams representing the sum of two quantities with diagrams showing the difference of the quantities. Such difficulties may be the result of trying to draw the abstract relationship ("how do I draw a plus sign?") rather than attempting to draw the physical situation ("If this segment represents the amount of money one child has, and this one represents the amount of money the second child has, then to show how much money they both have...").

Partial protocols from two subjects show two contrasting approaches (see Appendix F). Note that subject #2 attempts to pull out the quantities in the problem and represent them without regard for order. Student #1, on the other hand, used the sequence of events in the problem to organize the creation of the diagram (focuses more on modeling the events of the problem).

The notion, mentioned earlier, that diagrams can bridge the gap from real world experience to mathematical abstraction, is an extension of the popular theory (Hooper 1981) that learning generally procedes from the more concrete to the more abstract. The learner could procede from the concrete event, through an intermediate diagram step, finally to the abstract mathematical representation (see Figure 4.10). The idea here is that creating a diagram requires less of a jump in the level of abstraction than going directly from the concrete situation to the algebraic expression. The learner is not faced with the tasks of representing the information and of translating it into abstract symbols at the same time. The skill of representing the information in a diagram should therefore be more easily acquired than the skill of algebraic symbolization.

Fig. 4.10

Drawing from the Concrete

Concrete ---> Diagram ---> Algebraic expression

However, the subjects in this study, who have reached the level of precalculus with virtually no experience with diagrams, are used to solving problems algebraically, although not always with competence and understanding.

Whereas the medium of the diagram is more closely related to the real world event (less of a jump in abstraction), these subjects try to draw the mathematical abstractions that they would normally represent algebraically (see Figure 4.11).

Fig. 4.11

Drawing from the Abstract

Concrete ------> Algebraic expression ---------> Diagram

Working from the algebraic abstraction back to a diagram representation may be the most difficult task of all. Spatial modeling of the abstraction may be a more difficult problem than the original algebra problem. The subject is less likely to be successful with that task than with basing the diagram on the original concrete event.

Difficulties with Mathematical Concepts Subjects seemed to be limited by poorly developed arithmetic concepts. In particular they revealed weaknesses in their understandings of fractions and they confused additive relationships with multiplicative relationships. Some of these difficulties might be observable in their algebraic work as well. However, demanding that they draw diagrams puts them in a situation devoid of familiar algorithms, where success seems to be more dependent on understanding of concepts. Observations of the effect on diagram drawing of poorly understood mathematical concepts emphasize the importance of this factor which was described in Model One (see the exploratory study, Chapter Three).

Indications of Lack of Preparation for Algebra A number of the difficulties that subjects' had with the tasks in this study suggest weaknesses which reduce their effectiveness in algebra as well. In the last section, subjects' difficulties in representing arithmetic relationships found in real world problems was discussed. Earlier, difficiencies in labeling and their lack of general metacognitive skills were described. There are other difficulties which also point to weaknesses in the subjects' foundations for algebra.

1. Subjects showed a lack of appreciation for the importance of generality. Subjects, when faced with problem 1 of Form A, frequently drew the triangle initially as an equilateral triangle. When it turned out to not fit for the information that they were representing, they generally drew right triangles. This seemed to reflect a lack of appreciation for the importance of using the least specific representation. While this is of greater concern in geometry, it may also reflect learning that is important in describing algebraic generalizations.

2. Subjects had difficulties representing unknown quantities. While the simplest explanation is that they were unfamiliar with doing so using diagrams, it seems worth pointing out that their inability to devise ways to represent unknowns in diagrams may reflect a narrow or incomplete concept of unknowns in algebra. The skill of representing variables spatially certainly seems to be related to the situation of "take any point, (x,y), in the plane...."

3. Subjects frequently seemed unaware of the relationships that might lead to the desired information. Subjects' efforts often seemed unsystematic, lacking the direction that might have resulted from searching for ways to determine and express particular relationships. Subjects who created non-integrated drawings seemed to be unaware of the importance of seeing relationships between the information in the two drawings.

CHAPTER V

SUMMARY, CONCLUSIONS, RECOMMENDATIONS

Summary

Background The heuristic "draw a diagram" has consistently been included in lists of general strategies for problem solving (Polya 1945, Schoenfeld 1980, Charles and Lester 1982). Research in the area of diagram drawing has focused primarily in the following areas:

1. correlation of problem solving performance with use of diagrams

2. correlation of use of diagrams with spatial abilities

3. effect of diagrams in the problem presentation on problem solving performance

4. effect of diagram drawing instruction on problem solving performance

Research findings have not been consistent in linking diagram drawing with problem solving performance (Webb 1979, Swart 1970, Kilpatrick 1967, Lean and Clements 1981) or with spatial abilities (Moses 1978, Landau 1984, Khoury and Behr 1982). Several studies however have found a significant relationship between problem solving performance and the drawing of high quality diagrams (McKee 1983, Schonberger 1976, Schwartz 1971).

This relationship seems to be consistent with the findings of the research on diagrams in problem presentation. They indicate that providing accurate diagrams results in improved problem solving while providing inaccurate diagrams results in worse problem solving performance (Sherrill 1973, Threadgill-Sowder and Sowder 1982).

Instructional interventions have basically been short-term, one week to three months, and have not yielded impressive improvement in problem solving performance. Such interventions were likely too short and lacked appropriate methodology to have significant impact. The student populations that were studied were mostly naive to diagram drawing prior to the studies.

The research supports the assumptions that the ability to draw high quality diagrams is desirable and that this ability should be taught. However, two questions that are left largely unanswered by the research literature are:

1. What factors affect whether a student chooses to draw a diagram when a diagram could be helpful?

2. What skills and knowledge are required to draw useful diagrams for solving mathematical problems?

These questions motivated this study.

<u>The Exploratory Study</u> Students from remedial mathematics classes at the University were given two written tests

followed by an individual interview a week later. The first written test required subjects to show their work while the second required them to draw diagrams to solve the problems. All problems could be solved directly using diagrams. In the interviews, subjects were asked to explain previous work and to draw diagrams for problems for which they had not previously attempted diagrams.

The experimenter's analysis of the written work and the videotapes of the interviews resulted in the postulating of five factors affecting diagram drawing choice and performance, Model One:

1. Understanding of the mathematics involved in the problem and of basic arithmetic concepts (i.e. fractions, ratio)
2. Diagram drawing skills and experience
3. Conceptions of mathematics
4. Self-concept in mathematics
5. Motivation to solve the problem correctly

The analysis also focused in more depth on factor #2 to generate a list of diagram drawing subskills, Model Two. (See the list in Chapter III).

<u>The Main Study</u> The research questions were modified, as follows, to focus the main study:

A. Are the subskills identified in the exploratory study (Model Two above) important in the creation of useful diagrams?

B. How important are control (metacognitive) skills to the creation of high quality diagrams, particularly the ability to think to use the various subskills and to choose appropriately among available subskills?

C. What effect does the problem context (geometric versus algebraic) have on the quality of the diagrams that are drawn?

D. What important skills and knowledge were not identified during the exploratory study?

E. What are the difficulties which prevent successful diagram drawing?

Students from precalculus classes at the University were divided randomly into three treatment groups. Each group had the same pretest interview conditions and returned one week after for treatment interviews. Problems, typical of those used in first year algebra, were used for all interviews (see Appendix B). Subjects were asked to draw complete and useful diagrams and to refrain from using algebra.

Model Two was converted into a list of external control suggestions. During treatment interviews, subjects in Group I received these suggestions as deemed appropriate by the experimenter. Subjects in Group II were given the printed list of suggestions and encouraged to refer often to it. Group III, the control group, repeated the pretest conditions; no assistance was provided.

The quality of the diagrams were evaluated and compared for the three groups based on measures of type, completeness, labeling, and accuracy, as well as the total of the four measures.

Group II data was not comparable to the data for the other groups because of their significantly higher performance on the pretest. This was attributable to the number of students in each group (7), which was kept low to permit the experimenter to conduct two individual interviews with each subject. As a result, the value of the metacognitive contribution made by the experimenter (in deciding when to offer each suggestion to Group I subjects) could not be assessed reliably from the data.

Group I subjects scored significantly higher than the control group (III) on the total score for diagram quality as well as on the specific measure of diagram completeness. These results indicate that some of the subskills in Model Two do contribute to improved diagram drawing performance. Suggestions related to drawing integrated diagrams, to representing unknown quantities, and to representing ratio relationships seemed to be particularly helpful in improving diagram completeness. Differences in scores for diagram type and accuracy were close to significant, indicating the need for further study.

Geometric problem contexts resulted in better scores for overall diagram quality than did algebraic problem contexts. This was consistent with the findings of McKee (1983). Examination of the effect of context on the individual measures of diagram quality showed significant effects on diagram type and completeness but no effect on labeling or accuracy of diagrams. This suggests that students, when attempting to draw diagrams for problems with algebraic contexts, face an additional difficulty of deciding how to represent the important information spatially.

Analysis of treatment interviews revealed a number of metacognitive skills which were not addressed by the oral suggestions. Some of these metacognitive skills are skills necessary for problem solving in general. These include judging the completeness of the label given to a particular quantity, monitoring one's solution strategy, evaluating the answer obtained, and monitoring the memory demands of the task. Other metacognitive skills were specific to diagram drawing including: identifying all parts of the diagram that should be labeled, knowing what information gleaned from the diagram can be used, and knowing when to use a discrete and when to use a continuous diagram.

Analysis of the treatment interviews revealed also specific difficulties encountered by the subjects. Some of these difficulties were the result of gaps in the subjects' understandings of mathematical concepts, particularly fractions and ratio concepts. This finding supported a similar finding of the exploratory study. Another critical difficulty was the inability to model real world problem situations. Not only were many subjects unable to do so using diagrams, but follow up questions revealed weaknesses in their ability to do so using algebra, a domain with which they had had extensive experience.

Conclusions and Recommendations

Conclusions and Implications for Mathematics Instruction

Both the exploratory study and the main study have contributed to the conclusion that diagram drawing is a complex ability which is dependent upon the student's understanding of mathematical concepts, his self-concept in mathematics, and his beliefs about mathematics, as well as a host of general and specific skills and metacognitions. While high levels of functioning on these factors are important for diagram drawing success, it is also possible, and worth investigating, that effective instruction in diagram drawing, conversely, can contribute to general improvement in mathematical understanding, confidence and beliefs about mathematics as well as to problem solving skills and metacognition. Thus, diagram drawing may be a useful vehicle through which students' experiences with mathematics can change in a positive direction.

While the study did not explore methods of teaching diagram drawing, Model Two (see Chapter III) provides some indications of component skills that must be focused on as instructional interventions are developed. In addition to these component skills, the study focuses attention on the importance of building the related metacognitions. Students must learn to monitor their work, think to use particular skills or knowledge, and to understand the utility and limitations of particular strategies.

The study also focuses attention on a larger issue, the inabiltiy of students to model real world problem situations. Diagram drawing is a potential bridge between the physical world and the abstraction of mathematical symbolization. This bridge requires that elementary and secondary mathematics curricula reflect a commitment to an ongoing development of students' abilities to represent mathematical relationships spatially. Short instructional interventions are unlikely to have significant impact for most students.

Improvement in diagram abilities will not take place, however, in a vacuum. Effective instruction in this area will necessitate and contribute to some important changes in mathematics instruction in general, including:

1. a greater focus on the process of mathematical problem solving with a concommitant decrease in the importance placed on the answers produced

2. a greater attention to metacognitive skills

3. more use of non-routine problems and a reduction in the time spent on routine exercises

4. an appreciation of divergent thinking (Diagram drawing does not lend itself to one "right" way.)

5. the development of a "debugging" view of mathematical modeling (One creates a model and then continues to change it and improve it until it is useful for the task for which it was designed.)

Instruction in algebra in particular may benefit from preparatory work with diagrams. Algebraic competence with respect to modeling real world problems and using variables to represent unknown quantities might be improved if students had the opportunity to develop these skills first with diagrams. Algebra affords us the opportunity to manipulate expressions, often with little need to understand the relationships in depth. The procedures of algebra provide a short cut for relating partial representations automatically. It is important, however, that students learn to think about the overall relationships in a problem, not just to identify specific parts. Diagram drawing may provide a vehicle for such learning.

Suggestions for Future Research This research represents a first attempt to specify the factors which influence the decision to use a diagram and determine the success in using a diagram. It also marks a beginning of breaking diagram drawing competence into its component subskills. Several areas of further study are indicated. Future studies should focus on:

1. diagram drawing skills (using similar problems) of students who have not as yet been given instruction in algebra.

The study used subjects who had a background in algebra but prohibited them from using algebra. This probably caused some interference due to subjects' confusion over what was permissible and what was not.

2. diagram drawing skills of expert problem solvers.

3. instructional interventions which are based on findings from this study.

4. the importance of the metacognitive components of diagram drawing.

5. correlational studies which relate diagram drawing to the factors specified in Model One.

Diagram drawing provides researchers in mathematics education a potentially rich medium for studying problem solving and higher order thinking due to its independence from rote algorithms. As a research topic it offers both the advantages and disadvantages of being a complex skill, learned over an extended period of time.

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Appendix A

Problems Used in Exploratory Study

SET 1: Show all your work as you solve these problems:

1) Abby buys a 6 ft. long board. She cuts it into 3/4 ft. sections. How many 3/4 ft. sections did she make?

2) Chan has 3/4 of a gallon of ice cream. He gives 2/3 of what he has to Barry. How much ice cream does he have left?

3) There are several colors of dogs in a pen. 1/5 of the dogs are black, 10 dogs are spotted, and the remaining 2/3 of the dogs are brown. How many dogs are in the pen?

4) Alex buys his car at a "2/7 off" sale. He pays $3500. What was the original list price of the car? (DO NOT USE ALGEBRA)

5) If six people shake hands such that each one shakes hands with each other person, how many handshakes will there be?

6) The label on a tin can extends from one end to the other. It wraps completely around the can with the ends of the lable overlapping 1 inch. The can is 6 inches tall and its radius is 2 inches. What is the area of the label?

7) Dave decided to walk to the local gas station. After he walked 1 mile, he decided to walk half the remaining distance before resting. After he reached his resting point, he still had 1/3 of the distance of the trip plus 1 mile to walk. How long was Dave's trip?

8) A cougar spots a fawn 200 feet away. The cougar starts toward the fawn at 50 ft. per second. At the same instant, the fawn starts running away at 30 ft. per second. How long will it take the cougar to catch the fawn?

9) There are eight points on a circle. Each point is connected to every other point by a line segment. How many line segments are there?

SET 2: Solve these by drawing a diagram:

1) 5 X 3 =

2) 2/3 of 3/5 =

3) 4 divided by 3/4=

4) The sum of the number of books Jack and Jill have is 20. If Jill lost 3 of her books and Jack doubled the number he has, they would then have a total of 30 books. How many books does each have?

5) Line segment AB is 5 times as long as line segment CD. If AB is decreased by 18 centimeters and CD is increased by 18 centimeters, then AB and CD are the same length. What are the original lengths of AB and CD?

6) The sum of the measures of the sides of a triangle is 35 inches. One of the sides is 4 times longer than the second side and 1 inch longer than the third side. What are the lengths of the sides?

7) The government wants to contact all druggists, all gun store owners, and all parents in a town. How many people must be contacted, using these statistics?

Druggists..........................10
Gun store owners...................5
Parents..........................3000
Druggists who own gun stores.......0
Druggists who are parents..........7
Gun store owners who are parents...3

8) We see that 3/5 of the children in the room are girls. We also note that if we double the number of boys and then add six more girls to the class, then there will be an equal number of boys and girls. How many children are in the room at the beginning?

9) Mrs. Brown is 38 years old and her daughter Barbara is 8 years old. If Mrs. Brown and Barbara both have birthdays on the same day, when will Mrs. Brown be three times as old as Barbara?

10) The sum of the measures of two line segments is 24 inches. If one segment was 4 inches shorter and the other segment was doubled in length, the sum of the measures would be 30 inches. How long are the originals?

11) Sam has four times as much money as his sister Eileen. If Sam's money is decreased by 15 cents and Eileen's money

is increased by 15 cents, then Eileen and Sam have the same amount. How much money did Sam and Eileen have at the start?

Appendix B

Problems Used in Main Study

Form A

1. The sum of the measures of the sides of a triangle is 35 inches. One of the sides is 4 times longer than the second side and 1 inch longer than the third side. What are the lengths of the sides?

2. Sam has four times as much money as his sister Eileen. If Sam's money is decreased by 39 cents and Eileen's money is increased by 39 cents, then Eileen and Sam have the same amount. How much money did Sam and Eileen have at the start?

3. Mrs. Brown is 38 years old and her daughter Barbara is 8 years old. If Mrs. Brown and Barbara both have birthdays on the same day, when will Mrs. Brown be three times as old as Barbara?

Form B

1. The sum of the number of books Jack and Jill have is 20. If Jill lost 3 of her books and Jack doubled the number he has, they would then have a total of 30 books. How many books does each have?

2. The sum of the ages of three children is 26. One of the children is 3 times older than the second child and 2 years older than the third child. What are the ages of the children?

3. Line segment AB is six times as long as line segment CD. If AB is decreased by 30 centimeters and CD is increased by 30 centimeters, then AB and CD are the same length. What are the original lengths of AB and CD?

Extra problem worked by Group III at the end of the pretest (did not figure in scoring).

Chan has 3/4 of a gallon of ice cream. He gives 2/3 of what he has to Barry. How much ice cream does he have left?

Appendix C

Suggestions

1. Avoid using arithmetic signs in your diagram, such as + - x / =.

2. Create one diagram that has all the information in it instead of several separate diagrams.

3. If the size of the fraction to be represented is unknown, mark off a space remembering that its size is arbitrary. Try to avoid drawing it to look equal in size to parts that may not be equal to it.

4. a)Are there any equal parts in the picture? Label the equal parts so they are easily recognized as such.
   b)If one part is a multiple of the other (and the number values are unknown for these parts), subdivide the larger to make parts equal to the smaller part. Label these new equal parts clearly.

5. If it would be helpful now to redraw the picture, do so.

6. a) Label what you have drawn, naming the part or parts that you have created. If you have drawn part of a whole and labeled it, you may also be able to label the remaining part of the whole.
   b) What does this represent? [referring to an unlabled space]

7. a) Label all parts numerically for which you now know the appropriate numbers.
   b) Be aware of parts of your diagram for which you can now figure out the number value by combining numbers that are already in the diagram, looking for differences or working with equal parts.
   c) Do you know the numerical value of this space? [referring to an unlabled space]

8. Try drawing a representation of the final or goal state of the problem. See if you can work from there.

9. Can you draw that information? [referring to information that is written in as a label rather than drawn into the diagram]

10. If two parts are equal and one part is subdivided, make the same subdivisions in the other equal part.

11. Review and check over certain steps.

a) Check to see that all parts are labeled descriptively, that the name of what that space represents is included.

b) Check to see that all parts are labeled with number values. Look to see if there are any other parts you now can determine the number value of.

c) Are all equal parts clearly indicated?

d) Has all of the relevant information been represented?

e) Reread the problem to make sure that you have accurately represented the problem.

Appendix D

Evaluation of Diagrams and Sample Diagrams

Evaluation Scale: quality of diagrams

Note: Sample diagrams are provided for reference on page 4 of Appendix D.

Type: 3 points--schematic, represents math structure
2 points--part schematic and part illustrative
1 point--illustrative, no math structure

Completeness: 4 points possible
Note: completeness only deals with what is represented spatially and not with what is described in labels.
1 point subtracted for non-integrated diagram (two or more unrelated diagrams)
1 point subtracted for each piece of relevant information not represented spatially (see criteria for each problem, below)

Labeling: 4 points possible--includes numerical labels and descriptive labels (see criteria for each problem below)

Accuracy: 3 points--shows correct understanding of problem
2 points--some inaccuracy
1--not at all accurate or not enough information represented to evaluate accuracy

Criteria for completeness

Form A

1. -Ratio of sides
   -Total length of the three sides
   -1 inch difference between long side and mid-length side

2. -Ratio relationship
   -Increase in Eileen's money is equal to the decrease in Sam's money.
   -Resulting amounts are equal

3. -Ratio of future ages
   -Current ages
   -Future ages or increment (current ages and future ages can be replaced by a representation of the difference in their future ages).

Form B

1. -Sum of their books at the outset
   -The lost books
   -The doubling of Jack's books

2. -Ratio of children's ages
   -Total of the children's ages
   -Two year difference between first and third child's ages

3. -Ratio relationship between the segments
   -Increase in CD is equal to the decrease in AB.
   -Resulting lengths are equal

Criteria for Labeling

Form A

1. -Designation of the three sides
   -35" total
   -1" difference
   -36" (one inch added on) or 4" per segment

2. -Sam's money before and Eileen's money before
   -Sam's money after and Eileen's money after or amount increased for Eileen and the amount decreased for Sam

-$.39
-$.26 per segment or $.135 per half segment or $S_o = \$1.04$ or $E_f = S_f \$.65$

3. -Barb's current age and Mrs. Brown's current age
-increase in their age
-8 years and 38 years
-increase = 7 years or Barb's future age = 15 years or Mrs. Brown's future age is 45 years
OR
-Barb's future age
-Mrs. Brown's future age
-30 years difference in their ages
-increase = 7 years or Barb's future age = 15 years or Mrs. Brown's future age is 45 years

Form B

1. -Jill's books and Jack's books (at the outset), 20 books
-3 books lost and Jack's number of books doubled
-30 books total
-17 books total after Jill lost three or 13 books Jack gained or 7 books Jill had at the outset

2. -Three children's ages
-26 years total
-2 years difference in age between first and third child
-28 years (two years added on) or 4 years per segment

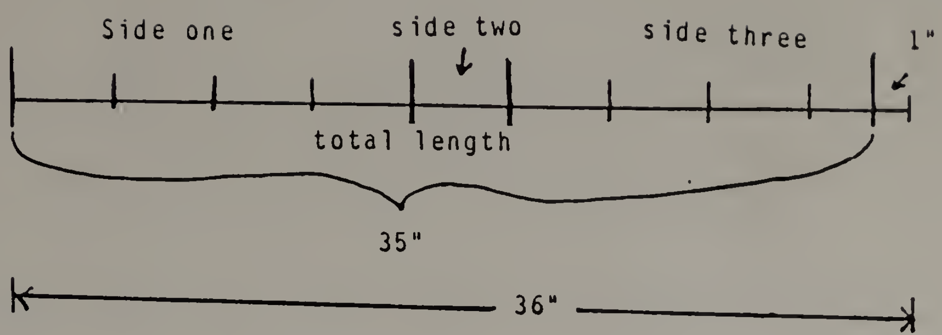
3. -AB_o and CD_o
-AB_f and CD_f or the increase in CD and the decrease in AB
-30 cm
-12 cm per segment or $AB_o = 72$ cm or $AB_f = CD_f = 42$ cm

Fig. D.1

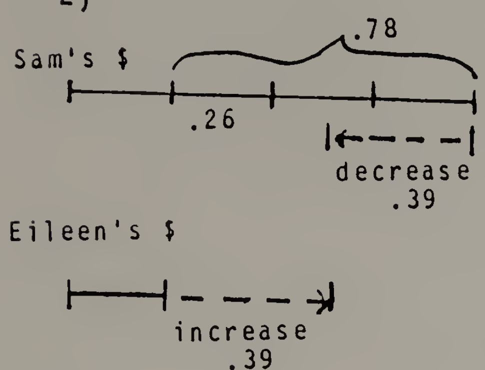
Sample Diagrams

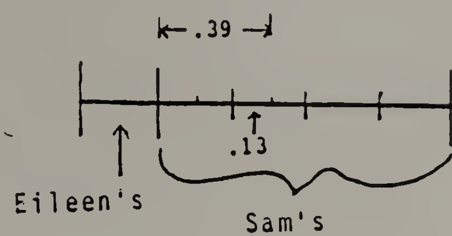
FORM A

1)

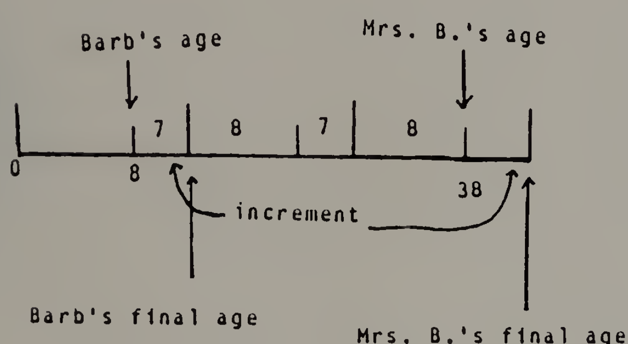


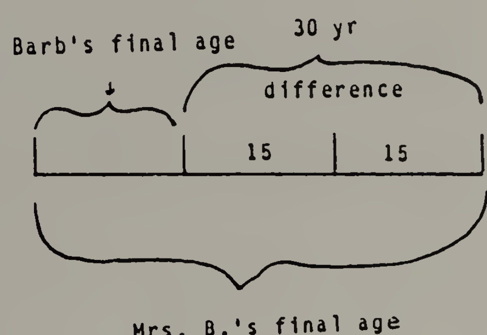
2)





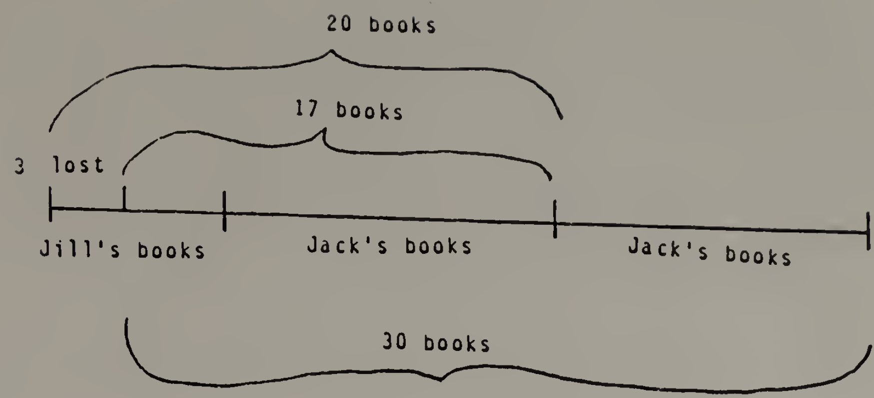
3)



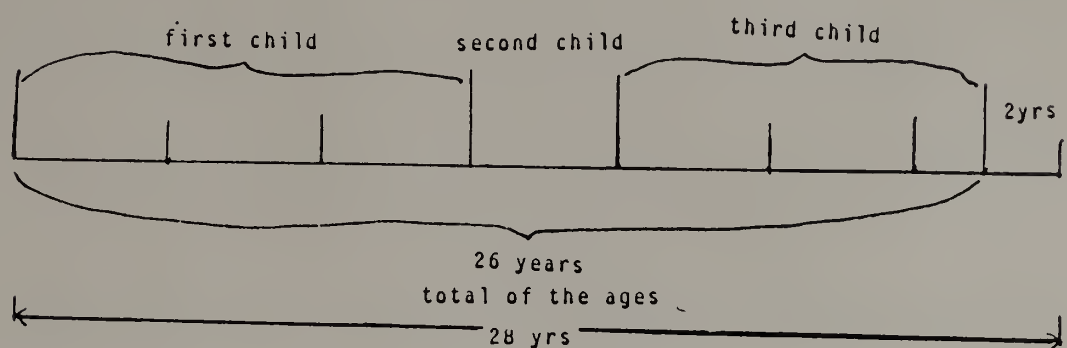


FORM B

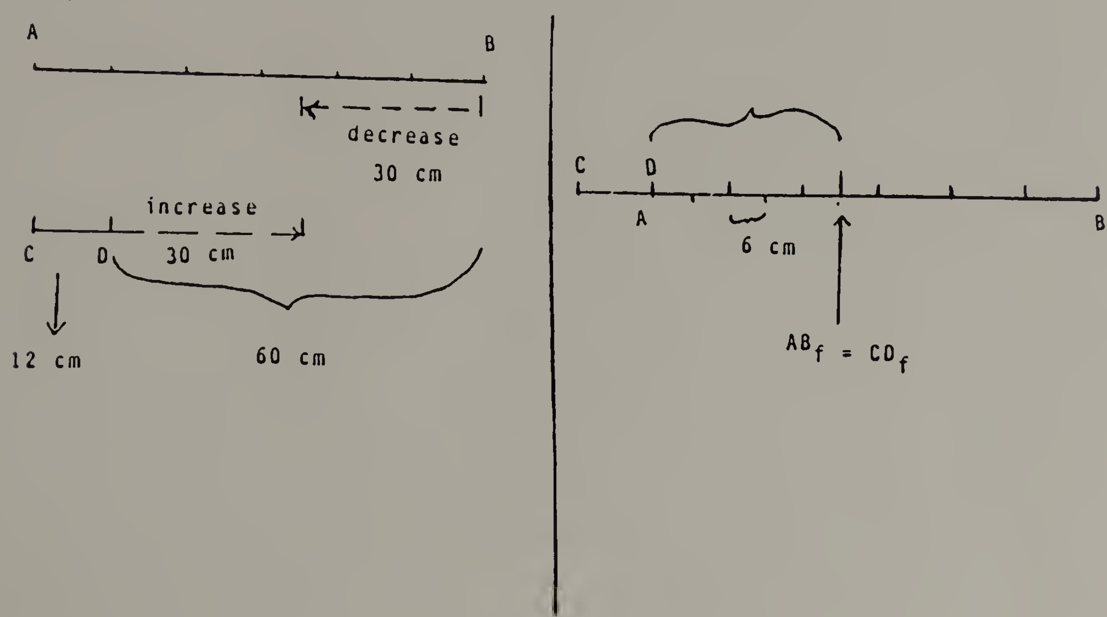
1)


20 books
17 books
3 lost
Jill's books
Jack's books
Jack's books
30 books

2)


first child
second child
third child
2yrs
26 years
total of the ages
28 yrs

3)


A
B
decrease
30 cm
increase
C
D
30 cm
12 cm
60 cm
C
D
A
B
6 cm
AB$_f$ = CD$_f$

APPENDIX E

Table E.1

Correlations of Scores by the Two Scorers
(columns are by problem, rows are by measure)

	A1	A2	A3	B1	B2	B3
M1	1.000	.750	1.000	.671	.932	1.000
M2	.916	.976	.846	.853	.906	.976
M3	..962	1.000	.791	.907	.836	.828
M4	.795	.718	1.000	.661	.720	.791

* "A1" indicates form A, problem #1.

APPENDIX F

Sample Protocols

<u>Subject One:</u>

Subject: This here is the sum of Jack and Jill's books. It's equal to 20 books total.

```
.______________________________. Jack & Jill's books
-----------20------------------
```

Jill lost three books (draws) and Jack doubles his then they have thirty

```
                           3
.____________________.____. Jack & Jill's books
-----------20-------------------
                         17
```

Jill lost three then they had to have 17 in all (labels 17). Then Jack doubled his then they'd have up to 30. (draws second figure)

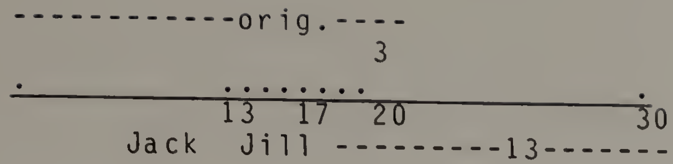
```
------------orig.----
                 3
.______________.____.____________.
              17   20            30
```

[pauses]
So they have 17 books totaled since Jill lost 3. And then Jack doubles his. Jack doubles his. Let's see. That would be 13.

If Jack doubled his total amount of books, they would have 30. So... [pauses]

Exper: Please think out loud.

Subject: Okay so, they had 17 and Jack doubled his books. He added on 13 books so he must have originally had 13 books because the difference between 30 and 17. He must have started off with 13 and he doubled 13 to get that. And then there's 4 difference so Jill must have 4 books. No ...oh yeah. He had 13. That's if Jill lost 3. So if she didn't lose 3 than she had 7...

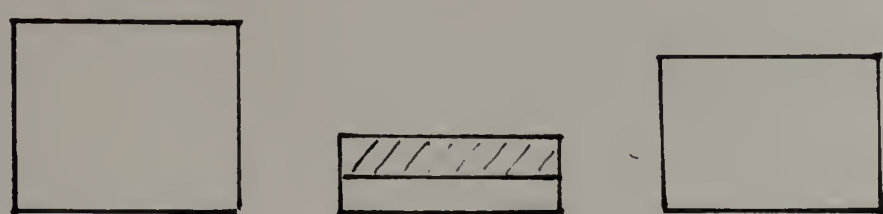


<u>Subject Two</u>

Subject: (Draws a box) 20 books. Jill lost 3 of her books. Okay, arbitrary... (draws)

Exper: You just remembered one of the suggestions?

Subject: Yes. Arbitrary. I remembered not to make it 1/2. I can relate... it was confusing last time because sometimes if you make it similar or equal to 1/2.... This is Jill. Now Jack... another arbitrary (draws). If this is 20 books we know that half of that is 10 and this is 30 (draws).



Jack + Jill
20

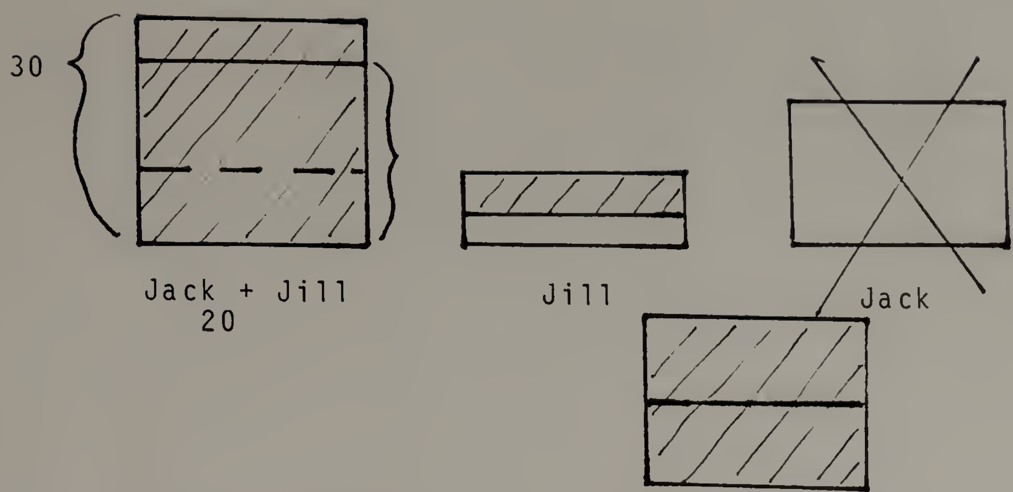
Jill

Jack

Jack has more books than Jill, it looks like. I guess I need to make an association. If I double what Jack had, it almost makes it look equal to the 30. That's not taking into association what Jill has. So I guess I need to make a picture that better represents what Jack has.

Let's see. Leave out that. (crosses out a box)

That's what Jack has. (draws another box)



The sum of these two will equal 20 and if you subtract 3 and double that you know you get 30.

Okay. Since I don't really know what Jill has, I can't really even begin to figure out what fraction is the 3 books. (draws a box to show the doubled quantity)

So this being what Jill has left over. And again that's just a rough sketch.

I know at least that Jack has more books left over than Jill.

That's a fair representation. Iknow that half of that and you tack on another half equals 30. It's the only ground base thing that I have that I know is right. This is too arbitrary really. This is the biggest unknown. It makes it too difficult without using algebra.