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A relation between Mirkovic-Vilonen cycles and modules over preprojective algebra of Dynkin quiver of type ADE

Zhijie Dong

A RELATION BETWEEN MIRKOVIC-VILONEN CYCLES AND MODULES OVER PREPROJECTIVE ALGEBRA OF DYNKIN QUIVER OF TYPE ADE

A Dissertation Presented

by

ZHIJIE DONG

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Dedication

To my parents

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Special thanks to my advisor Ivan Mirkovic. His style of doing math and greatly impacts me. He is very patient teaching me, answering questions, and tolerant to my sometimes being stupid. He gave me the thesis problem and inspired me all the time. He helped me revising the paper very carefully. Thanks to Tom Braden. He suggested one key step (to use equivariant homology tools) to me and this really helped me to move on. Thanks to Paul Hacking, Jenia Tevelev, Paul Gunnells, Eric Sommers, Eyal Markman for teaching me at different stages of my study in Amherst.

ABSTRACT

A RELATION BETWEEN MIRKOVIC-VILONEN CYCLES AND MODULES OVER PREPROJECTIVE ALGEBRA OF DYNKIN QUIVER OF TYPE ADE

SEPTEMBER 2018

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The irreducible components of the variety of all modules over the preprojective algebra and MV cycles both index bases of the universal enveloping algebra of the positive part of a semisimple Lie algebra canonically. To relate these two objects Baumann and Kamnitzer associate a cycle in the affine Grassmannian to a given module. It is conjectured that the ring of functions of the T-fixed point subscheme of the associated cycle is isomorphic to the cohomology ring of the quiver Grassmannian of the module. I give a proof of part of this conjecture. The relation between this conjecture and the reduceness conjecture in [KMW16] is explained at the end.

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CHAPTER 1

INTRODUCTION

Let $\mathfrak{g}$ be simply-laced semisimple finite dimensional complex Lie algebra. There are several modern constructions of irreducible representation of $\mathfrak{g}$. In this paper we consider two models which realize the crystal of the positive part $U(\mathfrak{n})$ of $U(\mathfrak{g})$. One is by Mirkovic-Vilonen (MV) cycles and the other is by the irreducible compotents of Lusztig's nilpotent variety Λ of the preprojective algebra of the quiver Q corresponding to $\mathfrak{g}$.

Baumann and Kamnitzer [BK12] studied the relations between Λ and MV polytopes. They associate an MV polytope $P(M)$ to a generic module M and construct a bijection between the set of irreducible components of Λ and MV polytopes compatible with respect to crystal structures. Since MV polytopes are in bijection with MV cycles, Kamnitzer and Knutson launched a program towards geometric construction of the MV cycle $X(M)$ in terms of a module M over the preprojective algebra.

Here we consider a version by Kamnitzer, Knutson and Mirkovic: conjecturally, the ring of functions $\mathcal{O}(X(M)^T)$ on the T fixed point subscheme of the cycle $X(M)$ associated to M, is isomorphic to $H^*(Gr^{\Pi}(M))$, the cohomology ring of the quiver Grassmannian of M. In this paper I will construct a map from $\mathcal{O}(X(M)^T)$ to $H^*(Gr^{\Pi}(M))$ and prove it is isomorphism for the case when M is a representation of Q.

In chapter two I recall the basics of loop Grassmannian.

In chapter three I recall quivers, preprojective algebra and Lusztig's nilpotent variety and state the conjecture precisely.

In chapter four I describe the ring of functions on the T fixed point subscheme of the intersection of closures of certain semi-infinite orbits (which is called "cycle" in this paper). A particular case of these intersections is a scheme theoretic version of MV cycles. We realize these cycles as the loop Grassmannian with a certain condition Y.

In chapter five I construct the map Ψ from $\mathcal{O}(X(M)^T)$ to $H^*(Gr^{\Pi}(M))$. Here, Ψ maps certain generators of $\mathcal{O}(X(M)^T)$ to Chern classes of tautological bundles over $Gr^{\Pi}(M)$. So we need to check that the Chern classes satisfy the relations of generators of $\mathcal{O}(X(M)^T)$. We reduce this problem to a simple SL_3 case. In this case we have a torus action on $Gr^{\Pi}(M)$ so we could use localization in equivariant cohomology theory (GKM theory).

In chapter six I will prove Ψ is an isomorphism in the case when M is a representation of Q of type A.

In chapter seven I will state some consequences given the conjecture (one of which is the reduceness conjecture).

1.1 Notation

Let G be a complex semisimple algebraic group unless stated otherwise. Let I be the set of vertices in the Dynkin diagram of G. In this paper I will work over base field $k = \mathbb{C}$. We fix a Cartan subgroup T of G and a Borel subgroup $B \subset G$. Denote by N the unipotent radical of B. Let $\varpi_i, i \in I$ be the fundamental weights.

Let X_*, X^* be the cocharacter, character lattice and $\langle\ ,\ \rangle$ be the pairing between them. Let W be the Weyl group. Let e and w_0 be the unit and the longest element in W. Let α_i and $\check{\alpha}_i$ be simple roots and coroots. Let $\Gamma = \{w\varpi_i, w \in W, i \in I\}$. Γ is called the set of chamber weights.

Let d be the formal disc and d^* be the punctured formal disc. The ring of formal Taylor series is the ring of functions on the formal disc, $\mathcal{O} = \{\sum_{n\geq 0} a_n t^n\}$. The ring of formal Laurent series is the ring of functions on the punctured formal disc, $\mathcal{K} = \{\sum_{n\geq n_0} a_n t^n\}$.

For X a variety, let $\mathrm{Irr}(X)$ be the set of irreducible components of X.

CHAPTER 2

BASICS ABOUT LOOP GRASSMANNIAN

2.1 Definition of loop Grassmannian

One of the origin for this object is to understand the Satake isomorphism. Let G be a reductive algebraic group. The classical Satake isomorphism says the Hecke algebra $\mathcal{H}G(\mathcal{K}), G(\mathcal{O})$ is isomorphic to the representation ring $\mathrm{Rep}(\check{G})$, where $\check{G}$ is the Langlands dual group. Geometric Satake upgraded this to an between two tensor categories.

One side is the category of representations of the Langlands dual group $G^\vee$ of G with the tensor functor being tensor product. The other side is the category of $G(\mathcal{O})$ equivariant perverse sheaves on the loop Grassmannian of G. Here $\mathcal{O}$ is the ring of formal Taylor series and we denote by $\mathcal{K}$ the ring of formal Laurent series. The tensor functor is given by convolution. The functor giving the equivalence is the (hyper)cohomology functor. For a group G, the loop Grassmannian $\mathcal{G}(G)$ is an infinite dimensional geometric object naturally constructed from G. If we work over complex numbers, the two categories are semisimple so we can just consider the simple objects. The intersection homology of Gr^λ, the $G(\mathcal{O})$ orbit closure of the point indexed by λ, is isomorphic to $L(\lambda)$(for $G^\vee$) as $G^\vee$ representation.

There are several ways to understand our $\mathcal{G}(G)$. We first review how to give it an ind-scheme structure.

We will assume G to be GL_n for the rest of this section. For general affine algebraic group G, we could use an embedding $G \hookrightarrow GL_n$. For details, see [Zhu16].

Let V be a n-dimensional vector space. we call an $\mathcal{O}$-submodule L in $V\hat{\otimes}\mathcal{K} \cong \mathbb{C}((t))$ a lattice. Denote $V\hat{\otimes}\mathcal{O} \cong \mathbb{C}[[t]]$ by L_0. The loop group $G(\mathcal{K})$ acts on the set of all lattices. The orbit of L_0 is the subset of all rank n lattices. Recall the rank of a module over a PID is by definition the dimension of the module tensor with the fractional field. In this case, this is equivalent to say $L\hat{\otimes}\mathcal{K} \cong V\hat{\otimes}\mathcal{K}$.

Lemma 1. *The stabilizer of L_0 is $G(\mathcal{O})$. Hence we have a bijection between the quotient $G(\mathcal{K})/G(\mathcal{O})$ and the set of all rank n lattices in $V\hat{\otimes}\mathcal{K}$, which we will denote by F.*

Proof. The action of $G(\mathcal{K})$ on $V\hat{\otimes}\mathcal{K}$ is just the action induced from the natural action of GL_n on $V \cong \mathbb{C}^n$ by left-multiplication. It preserves rank since $G(\mathcal{K})$ is invertible. To see the stabilizer, one could choose a basis of V, say $e_1, e_2, , , e_n$. For $g \in G(\mathcal{K})$, since $gL_0 = L_0$, the set $ge_1, ge_2, , , ge_n$ is also a basis of L_0, so the entries of g is in $\mathcal{O}$ and there exists g' with entries also $\in G(\mathcal{O})$ such that $e_1, e_2, , , e_n = g'ge_1, , , ge_n = (g'g)e_1, , , e_n$ hence $g \in G(\mathcal{O})$.

□

Note that "to endow" an algebraic structure means to find an ind-scheme (in this case, union of schemes) such that the set of all its $\mathbb{C}$ points is $G(\mathcal{K})/G(\mathcal{O})$.

To get the desired scheme, we have to filter the set of all lattices in $V\hat{\otimes}\mathcal{K}$.

Lemma 2. *Let F_n be the set of lattices L such that $t^{-n}L_0 \subset L \subset t^n L_0$. we have $F = \cup F_n$.*

Proof. For $L \in F$, by previous lemma, we have $g \in G(\mathcal{K})$, such that $L = gL_0$. Denote by ord the discrete valuation on $\mathcal{K}$. Now we could take n to be

$$\max(\max(\operatorname{ord}(g_{ij})), \max(-\operatorname{ord}(g_{ij})))$$

, note that n is no less than $\max(\operatorname{ord}(g_i j))$ is to guarantee $t^n L_0 \subset gL_0$ and n is less than $max(-ord(g_i j))$ is to guarantee $gL_0 \subset t^{-n} L_0$.

□

Lemma 3. *F_n is in bijection with the set of all subspace $V \subset t^{-n}L_0/t^n L_0$ such that $tV \subset V$. Here t is the operator acting on the vector space $t^{-n}L_0/t^n L_0$ from the its $\mathcal{O}$-module structure.*

Proof. We construct a map τ between the two sets. For $L \supset t^n L_0$, $\tau(L) = L/t^n L_0$. For $V \subset t^{-n}L_0/t^n L_0$, we define the inverse of $\tau : \tau^{-1}(V) = V \oplus t^n L_0$. □

Lemma 4. *We can endow the set of all subspaces $V \subset t^{-n}L_0/t^n L_0$ such that $tV \subset V$ with a scheme structure.*

Define a functor F_n such that $F_n(R)$ is the set of all projective R-modules M which is a quotient of $(t^{-n}L_0 \otimes R)/(t^n L_0 \otimes R)$ such that $tM \subset M$.

Proof. If we do not put the condition $tM \subset M$, this is just the usual functor defining the Grassmannian in the vector space $t^{-n}L_0/t^n L_0$. Then we see the condition $tM \subset M$ is a closed condition since t is a nilpotent operator. □

The scheme F_n represents need not be reduced. In general, we have

Proposition 1. *$\mathcal{G}(G)$ is reduced if and only if $Hom(G, G_m) = 0$. In particular, if G is semisimple, $\mathcal{G}(G)$ is reduced.*

We refer to prop 4.6 in [LS95] for the proof.

There is a natural embedding from F_n to F_{n+1}, hence we have defined an ind-projective scheme. We notice that we have different choices of choosing the filtration and our $\mathcal{G}(G)$ should not be dependent on it. Actually, we could write the functor in a more canonical way. The definition is in 1.1 of [Zhu16], for the sake of completeness, I included this in my thesis.

Definition 1. *Let R be a k-algebra. An R-family of lattices in $k((t))^n$ is a finitely generated projective $R[[t]]$-submodule Λ of $R((t))^n$ such that $\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n$.*

Definition 2. *The affine Grassmannian $\mathcal{G}(GL_n)$ is the presheaf that assigns every k-algebra R the set of R-families of lattices in $k((t))^n$.*

Lemma 1.1.5 in [Zhu16] shows that the functor we defined is the same as this.

Remark 1. *The lattice description turns out to be useful for computations in examples when n is small.*

The fact that the set of all k points of $\mathcal{G}(G)$ is $G(\mathcal{K})/G(\mathcal{O})$ is also very useful.

The set of all R points of $\mathcal{G}(G)$ is not easy to describe. In fact, we have the following theory.

Theorem 1. *Define the functor $L(G)$ assigning R to $G(R((t)))$ and the functor $L^+(G)$ assigning R to $G(R[[t]])$. The loop Grassmannian $\mathcal{G}(G)$ as in the definition 2 is the fpqc quotient of $L(G)$ and $L^+(G)$.*

Proof. see [Zhu16]. This is a rather formal thing. We want to define a quotient functor $L(G)/L^+(G)$. The naive definition $L(G)/L^+(G)(R) = L(G)(R)/L^+(G)(R)$ is not a sheaf. However, it is a presheaf w.r.t fpqc topology. We have a canonical way to sheafify this presheaf. To check $L(G)/L^+(G)$ is $\mathcal{G}(G)$, we only need to check they coincide as presheaves. This means R is local ring and R-mod behaves like vector space.

$\square$

2.2 $G(\mathcal{O})$-orbit in $\mathcal{G}(G)$

We make an analog to the finite case. For flag variety G/B, we could view it as a quotient and also we could view it as functors as R-family of all Borel subgroups in $G(R)$. Like we consider B-orbit in G/B and get Bruhat decomposition for flag variety, we could consider $G(\mathcal{O})$-orbit in $\mathcal{G}(G)$. We also have Bruhat decomposition in this case:

Theorem 2. *We have the decomposition*

$$G(\mathcal{K}) = \bigsqcup_{\lambda \in X_*^+(T)} G(\mathcal{O}) t^\lambda G(\mathcal{O})$$

Note that for a cocharacter λ, we could define a point $t^\lambda \in G(\mathcal{K})$. Abstractly, λ defines a map from multiplicative group G_m to Cartan T, the point t^λ is just the composition of the injection of the formal punctured disc to the group G_m and λ and $T \hookrightarrow G$. In the case when G is GL_n, we could write it very easily, in this case $\lambda = diag(\lambda_1, , , \lambda_n)$ and $t^\lambda = diag(t^{\lambda_1,,,\lambda_n})$. In general when doing calculation, when we write G as a matrix group, we could always write t^λ easily since the Cartan T could be realized concretely in G. We denote by L_λ the set of all k points of $\mathcal{G}(G)$ corresponding to the coset $t^\lambda G(\mathcal{O})$.

Proof. There is no R-point here. The statement is purely set-theoretic. When G is GL_n, this is just Smith normal form theorem over PID. When G is a general group, we refer to [PS86] section 8.1 for proof.

$\square$

Corollary 1. *The loop Grassmannian $\mathcal{G}(G)$ has a stratification by $G(\mathcal{O})$ orbit.*

Proof. The above union is disjoint. $\square$

We briefly explain how the orbit is defined. We say a group scheme G acting on X, if for any R, $G(R)$ acts on $X(R)$ and the action is compatible with the map $R \to R'$, or in other word, there exists $G \times X \xrightarrow{a} X$ such that certain diagram (for it to be a group action) commutes. For a closed point $x \in X$, we could define the map from G to X by fixing the second factor of $G \times X$ to be x. We define the orbit of $x \in X$ as follows: first we take underlying topological space to be the set-theoretic image. We can show that the image is locally closed in X (closed in an open subspace of X). Then we put the induced close reduced scheme structure on it. Actually in our case the group acting on $\mathcal{G}(G)$ ($G(\mathcal{O})$ and $N(\mathcal{K})$) will be reduced. In this case, the orbit map $G \to X$ factors through X_{red} so when we talk about orbit, we could just think of $\mathcal{G}(G)$ as it is reduced and in practice, we could think in the variety level. Denote by Gr_λ the $G(\mathcal{O})$ orbit of λ and we could assume λ is dominant since $W \subset G(\mathcal{O})$. We now restrict G to be GL_n and compute some examples of Gr_λ. Since we are dealing with Gr_λ, we could just consider k-point. We use lattice description of $\mathcal{G}(G)$ so a k-point of $\mathcal{G}(G)$ is a lattice in the sense of lemma1. Let $\lambda = (\lambda_1, , , \lambda_n)$.

Lemma 5. *If $L \in Gr_\lambda$, we have the containment relation $t^{-\lambda_1} L_0 \subset L \subset t^{-\lambda_n} L_0$.*

Proof. L_λ satisfies the relation and $G(\mathcal{O})$ preserve $t^{-\lambda_1} L_0$ and $t^{-\lambda_n} L_0$. $\square$

This lemma could be generalized to certain dimension equalities.

Lemma 6. *If $L \in Gr_\lambda$, we have for any $1 \leq i \leq n$:*

$$(L \cap t^{-\lambda_i} L_0)/t^{-\lambda_n} L_0 = (L_\lambda \cap t^{-\lambda_i} L_0)/t^{-\lambda_n} L_0$$

.

Proof. L_λ satisfies the equality and $G(\mathcal{O})$ preserve $t^{-\lambda_i}L_0$ for any $i = 1, , , n$. □

Actually these dimension equalities determine which orbit L lies.

Lemma 7. *The inverse of lemma6 holds.*

Proof. By theorem1 and its corollary. □

We formulate this two lemmas in a theorem.

Theorem 3. *The $G(\mathcal{O})$ orbit of L_λ consists of all lattices L such that*

$$(L \cap t^{-\lambda_i}L_0)/t^{-\lambda_n}L_0 = (L_\lambda \cap t^{-\lambda_i}L_0)/t^{-\lambda_n}L_0.$$

Under the bijection in lemma3, the $G(\mathcal{O})$ orbit of L_λ consists all t-invariant subspaces V in $t^{-\lambda_0}L_0)/t^{-\lambda_n}L_0$ such that

$$dim(V \cap (t^{-\lambda_i}L_0/t^{-\lambda_n}L_0)) = dim((L_\lambda \cap (t^{-\lambda_i}L_0)/t^{-\lambda_n}L_0)).$$

Example 1. *When $\lambda = (1, 1, , , 1, 0, , , 0)$, we have Gr_λ is the Grassmannian of all k dimensional linear subspaces of V, denoted by $Gr_k(V)$. We can see this by the above description: there is only one dimension equality and all subspace is t invariant. The irreducible representation L_λ is k^{th} wedge product of k^n, $\Lambda^k(k^n)$. Each weight space is one dimensional and indexed by k-elements subsets in n-elements set. In the Geometric Satake, this corresponds to the $IH(\overline{Gr_\lambda})$. In this case Gr_λ is closed and smooth so $IH(\overline{Gr_\lambda})$ is the ordinary homology group. The Grassmannian $Gr_k(\mathbb{C}^n)$ has an affine paving also indexed by k-elements subset in n-elements set.*

Example 2. *We compute a examples when $G = GL_2$, and $\lambda = (2, 0)$. By the above description, $Gr_{(2,0)}$ contains $L \subset span(t^{-1}V, V)$ such that $tL \subset L$, $dimL = 2$ and $dim(L \cap V) = 1$. Since $dim(L \cap V) = 1$, we have some v_1 both in V and*

L. We extend v_1 to a basis v_1, v_2 of V. Since dimL=2, we have another vector $v \in span(t^{-1}V, V)$ such that v_1, v span L. Suppose $v = at^{-1}v_1 + bt^{-1}v_2 + cv_2$, since $tL \subset L$, we have $tv = t(at^{-1}v_1 + bt^{-1}v_2 + cv_2) = av_1 + bv_2 \in L$, so b=0, otherwise $dimL > 2$. Also a is not 0, otherwise $dim(L \cap V) = 2$. So we can let a be 0 and L is span$(v_1, t^{-1}v_1 + bv_2)$, where $b \in k$. Abstractly we have constructed a map from $Gr_{(2,0)}$ to $Gr_1(V)$ by sending L to $L \cap V$. We showed that it is subjective and the fiber is an affine line A^1. We can further identify the fiber over the line $l \subset V$ with $Hom(l, V/l)$. The map between fibers is mapping an element $B \in Hom(l, V/l)$ to span$(v_1, t^{-1}v_1 + bv_2)$ where $Bv_1 = b\overline{v_2}$, here v_1 is the basis for l and $\overline{v_2}$ for V/l . We notice this is actually independent of the choice of basis we choose and write it in a more intrinsic way. The fiber over l is the set of all spaces L such that $l \subset L \subset V \oplus t^{-1}l, L \neq V$, which is isomorphic to $Gr_1(t^{-1}l \oplus V/l \setminus V/l) \cong$ $Graph((l \oplus V/l) \cong Hom(l, V/l)$. Denote the tautological line subbundle over P^1 by T and the quotient bundle by Q, we see that $Gr_{(2,0)}$ is isomorphic to $Hom(T, Q)$, which isomorphic to the tangent bundle over P^1. Now we try to describe the closure of $Gr_{(2,0)}$. Let b go to ∞ [1], we will get $L_{(1,1)}$ and this point actually lies in every fiber. So the closure of $Gr_{(2,0)}$ contains $Gr_{(1,1)}$. Define the Functor $F_{2,0}$ assigning R to the all rank 2 projective R-module M in $(V \oplus t^{-1}V) \otimes R$ such that $tM \subset M$. From the above description of $Gr_{(2,0)}$ and $Gr_{(1,1)}$, $F_{(2,0)}$ coincides with $Gr_{(2,0)} \cup Gr_{(1,1)}$ on the level of k point. Also $F_{(2,0)}$ is closed and the reduced part of $F_{(2,0)}$ is the closure of $Gr_{(2,0)}$, which we will denote by $\overline{Gr_{(2,0)}}$.

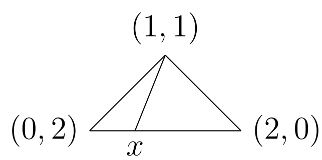
We learn two facts in this example: $Gr_{(2,0)}$ is a line bundle over P^1. The closure of $Gr_{(2,0)}$ is the union of $Gr_{(2,0)}$ and $Gr_{(1,1)}$. These can be generalized to the following general properties:

[1] *here we use the analytic topology*

Proposition 2. *Gr_λ is a vector bundle over G/P_λ. The closure of Gr_λ is the union of all Gr_μ where $\mu \leq \lambda$* [2]*. We denote the closure of Gr_λ by $\overline{Gr_\lambda}$.*

2.3 Relation to nilpotent orbit and transverse slice

Let me first explain the above example $Gr_{(2,0)}$ again. We could draw a picture from the above description.



The base line from $(0,2)$ to $(2,0)$ represents P^1. The fiber over any $x \in P^1$ is a line and and all fibers goes to the point $L_{(1,1)}$. We will see if we remove the base line, we get a nilcone of sl_2.

Lemma 8. *There is an embedding from $\mathcal{N}_2$ to $Gr_{(2,0)}$.*

Proof. There is a very short proof from the point of view of quotient description. Here I am giving a proof in the lattice description point of view. We construct the embedding as follows. Given a matrix A, we think of it as an operator from V to V. We have natural maps $Map(V, V) \cong Map(V, t^{-1}V) \cong Graph(V \oplus t^{-1}V) \subset (V \oplus t^{-1}V)$. Under these maps, A goes to $(V, At^{-1}V)$, which has to be t invariant to be in $\overline{Gr_{(2,0)}}$. Hence we have $t(V, At^{-1}V) \subset (W, At^{-1}W)$ and since $t(V, At^{-1}V) = (AV, 0)$ so $W = AV, At^{-1}W = 0$. Plug in $AV = W$ to the second equation we get $A^2 = 0$.

□

This could be generalized for arbitrary GL_n.

We can see the relation $\overline{Gr_\lambda} = \cup_{\mu \leq \lambda} Gr_\mu$ is similar to the case in the nilcone where $\overline{\mathcal{O}_\lambda} = \cup_{\mu \leq \lambda} \mathcal{O}_\mu$. The former is projective but the latter is affine. Lusztig first

[2] *remember that we assume μ is dominant.*

observe the relation between $\overline{Gr_\lambda}$ and $\mathcal{O}_\lambda$. One would like to study the geometry near one fixed point. For example, how singular is the point? One way is to construct a resolution and the fiber characterize some property. Here we already have G_m action so make use of the symmetry , we could consider the schematic fixed point. How "fatness" is the point in some sense characterizes the how singular it is. This is a local behaviour so would like to consider inside an affine chart, where locally have only one fixed point. Also we should not take account of the smooth part near the point. The two requirements could be axiomized into a definition of transverse slice.

Definition 3. *Let G act on a variety X. We say S_x is a normal slice to the orbit $G \cdot x$ if*

- *The tangent space of S_x is a direct summand: $T_x(S_x) \oplus T_x(G \cdot x) = T_x(X)$*
- *There exits an action of G_m on X that preserves S_x and $G \cdot x$ and it contracts S_x to the point x.*

In the nilcone case, one example is Slodowy slice. In the loop Grassmannian case, we consider the slice Gr_μ at point L_μ in $\mathcal{G}(G)$ being the K^{-1} orbit of L_μ, where we denote by K^{-1} the kernel of the $G(k[t^{-1}])$ to G. [3] In type A, we have [MV07b]:

Theorem 4. *There is a slice T_λ in gl_n, such that $T_\lambda \cap \overline{\mathcal{O}_\mu} \cong K^{-1}L_\mu \cap Gr_\lambda$*

Proof. The proof is similar to the case n=2 we showed and could be found again in [MV07b]. □

[3] One has to check this satisfies the above definition, we omit this.

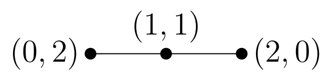
2.4 T-action and fixed point

There is a torus action on $\mathcal{G}(G)$ by left multiplication.

Lemma 9. *The action restricts to $G(\mathcal{O})$ orbit Gr_λ and its closure.*

This is a crucial tool to understand the geometry of $\overline{Gr_\lambda}$.

Lemma 10. *The set theoretic fix points are $L_{w\mu}, \mu \leq \lambda, w \in W$. We think in the smooth topology over $\mathbb{C}$. The action is a Hamiltonian action. The moment map image is the convex cone of all the weight indexing the fix points: $conv(w\lambda)$ in $X_*(T) \otimes \mathbb{R}$.*

Example 3. *The polytope for $\overline{Gr_{(2,0)}}$ is:* 

2.5 BB decomposition

For a multiplicative group G_m acting on a smooth variety X , we could consider its attracting set of each component of G_m fixed variety. When the fixed variety is isolated points, the attracting sets gives a cell decomposition of X, hence we can read off the homology group from this. We already see an example in Example 1. In our case although $\overline{Gr_\lambda}$ is not smooth[4], we can still study the attracting set for each fixed point and it could be described as intersections with $N(\mathcal{K})$ orbits. We first need a embedding form G_m to T such that the induced action of G_m is generic in the sense that the fixed point is the same as T. Different choices of the embedding will give different attracting sets. In fact, we have:

[4]the orbit closure Gr_λ is not smooth when λ is not minuscule.

Lemma 11. *The choice of $w \in W$ gives a map $w \cdot \rho^\vee : G_m \to T$.*

$$S_\mu^w = \{L \in \mathcal{G}(G) : \lim_{s \to \infty} L \cdot (w \cdot \rho^\vee)(s) = L_\mu.\}$$

We restrict to $\overline{Gr_\lambda}$. It turns out the set $\overline{Gr_\lambda} \cap S_\mu$ gives the information about its intersection homology group just like attracting cells gives the information about its usual homology groups in the smooth case.

Lemma 12. *The global cohomology functor decomposes into weight functor F_μ. We apply the weight functor F_μ to IC sheaf on $\overline{Gr_\lambda}$. We have the isomorphism $F_\mu(IC(Gr_\lambda)) \cong Irr(\overline{Gr_\lambda} \cap S_\mu)$.*

Proof. This is prop 3.10 in [MV07a]. □

By geometric Satake, we have $F_\mu(IC(Gr_\lambda)) \cong (L_\lambda)_\mu$, so $Irr(\overline{Gr_\lambda} \cap S_\mu) \cong (L_\lambda)_\mu$. There is a combinatorial way to phrase this.

Proposition 3. *The union of $Irr(Gr_\lambda \cap S_\mu)$ over all μ has a crystal structure of $B(\lambda)$.*

For the definition of crystal, we refer to [BG$^+$01] for details. One can think of $Irr(Gr_\lambda \cap S_\mu)$ index basis of the μ weight space of L_λ in a way that behaves nicely with respect to tensor product. In [BG$^+$01], the proof uses some geometric argument. One way to construct the irreducible module $L(\lambda)$ is to define it as the maximal quotient of the Verma module $W(\lambda)$. As vector spaces, $W(\lambda)$ is isomorphic to $U(n)$. $W(\lambda)$ could be thought as putting the vector space $U(n)$ an $\mathfrak{g}$-module structure depending on λ. So in some sense $U(n)$ is some universal object dominating all $L(\lambda)$. For instance, the crystal $B(\lambda)$ is always part of the crystal $B(\infty)$. It turns out that the union of $Irr(S_0^+ \cap S_\mu^-)$ over μ forms the crystal $B(\infty)$.

Theorem 5. *$Irr(S_0^+ \cap S_\mu^-)$ over μ forms the crystal $B(\infty)$.*

The origin of this is the loop Grassmannian construction of $U(n^+)$. For details, see [Mir97]. The proof was simplified by Kamnitzer after he found a explicit description of MV cycles. To explain the result, we introduce the following.

2.6 Weyl polytope and Pseudo-Weyl polytope

For λ dominant, $W_\lambda = conv(W \cdot \lambda)$ is called the λ-**Weyl polytope**.

The Weyl polytope W_λ can be described in three different ways. It is the convex hull of the orbit of λ, it is the intersection of translated and reflected cones, and it is the intersection of half spaces. Define the w-reflected cone $C_w^\mu := \{\alpha : \alpha \geq_w \mu\} = \{\alpha : \langle \alpha, w \cdot \varpi_i \rangle \geq \langle \mu, w \cdot \varpi_i \rangle$ for all $i\}$.

In particular,

$$W_\lambda = \bigcap_w C_w^{w\cdot\lambda} = \{\alpha : \langle \alpha, w \cdot \varpi_i \rangle \geq \langle w_0 \cdot \lambda, \varpi_i \rangle \text{ for all } w \in W \text{ and } i \in I\}.$$

We call a weight $w \cdot \varpi_i$ a **chamber weight** of level i. So the chamber weights $\Gamma := \bigcup_{w\in W, i\in I} w \cdot \varpi_i$ are dual to the hyperplanes defining any Weyl polytope.

We see that the moment map image of $\overline{Gr_\lambda}$ is λ-Weyl polytope. The interpretation has a geometric meaning.

Lemma 13. *The moment graph of $\overline{S_\lambda^w}$ is C_w^λ.*

Conjecture 1. *The intersection $\cap_{w\in W} S_w^{w\cdot\lambda}$ is $\overline{Gr_\lambda}$.*

This is the reduceness conjecture.

One could break the W-symmetry of Weyl polytope and we will get Pseudo-Weyl polytope. The data defining a Pseudo-Weyl polytope is W-collection of coweights λ_w such that $\mu_v \geq_w \mu_w$ for all v, w. The condition guarantees that the shape of the polytope is the same as the Weyl polytope. The corresponding geometric object is then $\cap_{w \in W} \overline{S^w_{\lambda_w}}$. We will eventually use this version, but for the description of MV cycle a variety, we think the the corresponding geometric object as $\overline{\cap_{w \in W} S^w_{\lambda_w}}$.

Also we have Pseudo-Weyl polytopes which also admit a description in terms of intersecting half spaces.

Let $\underline{M} = \left(M_\gamma\right)_{\gamma \in \Gamma}$ be a collection of integers, one for each chamber weight. Given such a collection, we can form $P(\underline{M}) := \{\alpha : \langle \alpha, \gamma \rangle \geq M_\gamma \text{ for all } \gamma \in \Gamma\}$. This is the polytope made by translating the hyperplanes defining the Weyl polytopes to distances M_γ from the origin.

Proposition 4. *Let $\underline{\mu} = \left(\mu_w\right)_{w \in W}$ be a collection of coweights such that $\mu_v \geq_w \mu_w$ for all v, w. Then the set of vertices of $P(\underline{\mu})$ is the collection $\underline{\mu}$ (which may have repetition).*

A pseudo-Weyl polytope has defining hyperplanes dual to the chamber weights. In particular, if P is a pseudo-Weyl polytope with vertices $\underline{\mu}$, then $P = P(\underline{M})$ where

$$M_{w \cdot \varpi_i} = \langle \mu_w, w \cdot \varpi_i \rangle. \tag{2.1}$$

Moreover, the $\underline{M}$ satisfy the following condition which we call the ***edge inequalities****. For each $w \in W$ and $i \in I$, we have:*

$$M_{ws_i \cdot \varpi_i} + M_{w \cdot \varpi_i} + \sum_{j \neq i} a_{ji} M_{w \cdot \varpi_j} \leq 0 \tag{2.2}$$

Conversely, suppose that a collection of integers $(M_\gamma)_{\gamma \in \Gamma}$ satisfies the edge inequalities. Then the polytope $P(\underline{M})$ is pseudo-Weyl polytope with vertices given by

$$\mu_w = \sum_i M_{w \cdot \varpi_i} w \cdot \alpha_i^\vee. \tag{2.3}$$

2.7 Lusztig walk and Kostant partition function

Theorem 6. *The dimension of μ weight space of $U(n^+)$ is the number of partitions of μ into positive coroots.*

Definition 4. *We fix a reduced decomposition $w_0 = s_{i_1} \cdots s_{i_n}$ of the longest element $w_0 \in W$. For each partition, we can associate a path from 0 to μ as follows: Fix a reduced word $\boldsymbol{i} = (i_1, \ldots, i_p)$ for an element $w \in W$. The word $\boldsymbol{i}$ determines a sequence of distinct Weyl group elements $w_k^{\boldsymbol{i}} := s_{i_1} \cdots s_{i_k}$ and distinct positive coroots $\beta_k^{\mathbf{i}} := w_{k-1}^{\boldsymbol{i}} \cdot \alpha_{i_k}^{\vee}$, $k = 1 \ldots p$.*

In particular, when $w = w_0$, we get all the positive coroots this way. A reduced word determines a distinguished path $w_0^i = e, w_1^i = s_{i_1}, w_2^i, \ldots, w_m^i = w$ through the 1-skeleton of Σ.

The kth leg of this path is the vector $w_{k-1}^i \cdot \rho - w_k^i \cdot \rho = \beta_k^i$. The i-chamber weights are exactly those dual to hyperplanes incident to the vertices along this path.

if $\mu = \sum_k n_k \beta_k^i$, Lusztig walk is the distinguished path $\mu_e, \mu_{s_{i_1}}, \mu_{s_{i_1} s_{i_2}}, \ldots, \mu_{w_0}$, such that the difference of the adjacent vertices is $n_i \beta_i$.

Lemma 14. *Each Lusztig walk determines $\cap S^i_{\mu_{s_{i_s}}}$,which is irreducible. Hence all MV cycles are of this kind since the number of MV cycles is the dimension of μ weight space of $U(n^+)$, which is the numbers of all Lusztig walks.*

The key of the proof is the observation that S_λ is the joint level set of some constructible functions D_γ. The functions are not independent, they satisfy certain relations generically (BZ transformation). A Lusztig walk will determine a Pseudo-Weyl polytope,i.e, the data of vertices on the 1-skeleton determine all vertices, such that

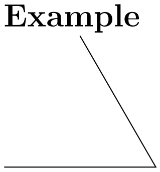
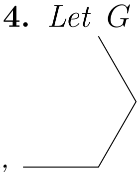
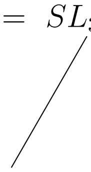
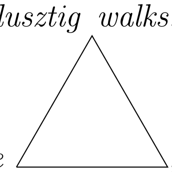
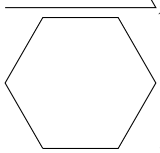
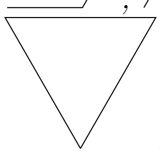
$$\overline{\cap S^i_{\mu_{s_{i_s}}}} = \overline{\cap_{w \in W} S^w_{\lambda_w}}.$$

2.8 MV cycles and polytopes

Theorem 7 ([Kam05]). *Given a collection of integers $(M_\gamma)_{\gamma\in\Gamma}$, if it satisfies edge inequalities, and certain tropical relations, put $\lambda_w = \sum_i M_{w\varpi_i} w\check{\alpha}_i$.*
Then $\overline{\bigcap_{w\in W} S^w_{\lambda_w}}$ is an MV cycle, and each MV cycle arises from this way for the unique data $(M_\gamma)_{\gamma\in\Gamma}$.

The data $(M_\gamma)_{\gamma\in\Gamma}$ determines a pseudo-Weyl polytope. It is called an MV polytope if the corresponding cycle $\overline{\bigcap_{w\in W} S^w_{\lambda_w}}$ is an MV cycle. MV polytopes are in bijection with MV cycles. Using this description, Kamnitzer [Kam07] reconstruct the crystal structure for MV cycles.

Proposition 5 ([Kam07]). *MV polytopes have a crystal structure isomorphic to $B(\infty)$.*

Example 4. *Let $G = SL_3$ Begin from 0 to i+j we have three lusztig walks.* , ,  *The polytope corresponding to them are* , , .

CHAPTER 3

OBJECTS ON THE QUIVER SIDE

Let $Q = \{I, E\}$ be a Dynkin quiver of type ADE, where I is the set of vertices and E is the set of edges. We double the edge set E by adding all the opposite edges. Let $E^* = \{a^* | a \in E\}$ where for $a : i \to j, a^* = j \to i$, also we define $s(a) = i, t(a) = j$. Define $\epsilon(a) = 1$ when $a \in E$, $\epsilon(a) = -1$, when $a \in E^*$. Let $H = E \bigsqcup E^*$ and $\overline{Q} = \{I, H\}$. The preprojective algebra Π of Q is defined as quotient of the path algebra by a certain ideal:

$$\Pi_Q = k\overline{Q} / < \sum_{a \in H} \epsilon(a) a a^* > .$$[1]

A Π_Q-module is the data of an I graded vector space $\bigoplus_{i \in I} M_i$ and linear maps $\phi_a : M_{s(a)} \to M_{t(a)}$ for each $a \in H$ satisfying the preprojective relations $\sum_{a \in H, t(a) = i} \epsilon(a) \phi_a \phi_{a*} = 0$.

Given a dimension vector $d \in \mathbb{N}^I$, define $\Lambda(d)$ to be the variety of all representations of Π on M for $M_i = k^{d_i}$.

Proposition 6 ([Lus90], [BK12]). *Irr*(Λ) *has a crystal structure isomorphic to* $B(\infty)$.

[1]Since most time we fix Q so Q is omitted when there is no confusion.

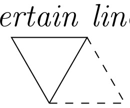
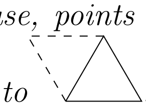
Example 5. *For the case sl_3, we have two vertices. Let $d_1 = d_2 = 1$, we have the nilpotent variety of all pair of numbers (ϕ_1, ϕ_2 such that $\phi_1 \cdot \phi_2 = 0$), which is two affine lines interseting in a point. The number of irreducible compotents is two.*

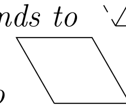
CHAPTER 4

A CONJECTURAL RELATION BETWEEN MV CYCLES AND MODULES OVER THE PREPROJECTIVE ALGEBRA

Baumann and Kamnitzer found an isomorphism between the crystal structure of $\mathrm{Irr}(\Lambda)$ and MV polytopes. For each $\gamma \in \Gamma$, they define a constructible funtion $D_\gamma : \Lambda(d) \to \mathbb{Z}_{\geq 0}$[1]. For any $M \in \Lambda(d)$, the collection $(D_\gamma)_{\gamma\in\Gamma}$ satisfies certain edge inequalities hence determines a polytope which we denote by $P(M)$.

Example 6. *Continue with the above problem. In sl_3, we have six chamber weights. We could describe the functors D_γ explicity.*

$D_\gamma = 0$ when γ is antidominant. $D_{\varpi_i} = d_i$ and ,$D_{\varpi_1-\varpi_2} = dimker(\phi_1)$, $D_{\varpi_2-\varpi_1} = dimker(\phi_2)$. We will generalize this description of D_γ in term of dimension of certain linear maps in Lemma 16 . In this case, points $(a,0), a \neq 0$ correspond to  *and points $(0,a), a \neq 0$ corresponds to* .

The intersection point is (0,0), which corresponds to  *.*

Theorem 8 ([BK12])**.** *When M is generic, $P(M)$ is an MV-polytope and for $d = (d_i)_{i\in I}$ this gives a map from $Irr(\Lambda(d))$ to the set of MV polytopes of weight*

[1] $\Lambda(d)$ and D_γ do not depend on the direction of the edges in E.

$\sum_{i \in I} d_i \alpha_i$. *This map is a bijection compatible with the crystal structures.*

Example 7. *Continue with the above example. We see that the generic points correspond to MV cycles. The intersection does not.*

We have MV-cycles (in bijection with MV-polytopes) as the geometric object on the loop Grassmannian side. In order to upgrade the relations geometrically, Kamnitzer-Knutson consider the quiver Grassmannian on the quiver side.

The quiver Grassmannian $Gr^{\Pi}(M)$ of a Π-module M is defined as the moduli of submodules of M.

It is a subscheme of the moduli of k-vector subspaces of M which is product of usual Grassmannian $\prod_{i \in I} Gr(M_i)$. Here we will only consider $Gr^{\Pi}(M)$ with its reduced structure, and actually just as a topological space. As the case of usual Grassmannian, the quiver Grassmannian $Gr^{\Pi}(M)$ is disjoint union of Grassmannians of different dimension vectors. Denote $Gr^{\Pi}_e(M)$ by the moduli of submodule N of M of dimension vector e, we have $Gr^{\Pi}_e(M) \subset \prod_{i \in I} Gr_{e_i}(M_i)$.

Given a module $M \in \Lambda(d)$, form the subscheme[2] $X(M) = \bigcap_{w \in W} \overline{S^w_{\lambda_w}}$, where $\lambda_w = \sum_{i \in I} -D_{-w\varpi_i}(M) w\check{\alpha}_i$. The torus T acts on $S^w_{\lambda_w}$ by multiplication, hence it also acts on the closure and the intersection $X(M)$.

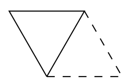
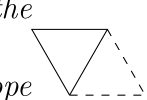
Conjecture 2. *The ring of functions on the T-fixed point subscheme of $X(M)$ is isomorphic to the cohomology ring of the quiver Grassmannian of M*

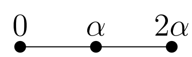
$$\mathcal{O}(X(M)^T) \underset{\sim}{\xrightarrow{\Psi}} H^*(Gr^{\Pi}(M)).$$

[2]We will call it cycle in this paper.

More precisely, $X(M)^T$ is disjoint union of finite schemes $X(M)^T_\nu$ supported at L_ν, $\nu \in X_*(T)$ and we can further identify two sides for each connnected component

$$\mathcal{O}(X(M)^T_\nu) \underset{\sim}{\xrightarrow{\Psi}} H^*(Gr^{\Pi}_{e}(M)), \text{ where } e_i = (\nu, \varpi_i).$$

Example 8. *We again use the above example. In the case where the module* $k \underset{0}{\overset{a}{\rightleftarrows}} k$ *corresponds to* . *The above correspondence is that: the three fixed points are all reduced and this corresponds to the fact that quiver Grassmannian of dimension (1,0), (1,1),(0,0) are all single point. Moreover, the quiver Grassmannian of dimension (0,1) is empty, which corresponds to the polytope*  *does not contain the point $i = (1, -1, 0)$.*[3]

Example 9. *This is more trivial in terms of correspondence but have content in the isomorphism. Let G be SL_2 so there is one vertex. In this case all chamber weights are fundamental weights. MV cycles are just $S_0 \cap S^{w_0}_{d\alpha}$. Let d=2, the module is just* $\overset{k^2}{\bullet}$ *and the corresponding MV polytope is* 

. The cohomology ring of $Gr_1(k^2) = P^1$ is $k[x]/x^2$, which corresponds to the T-fixed point is a double point. In this case, $S_0 \cap S^{w_0}_{d\alpha} = Gr_{2,0}$ and the T-fixed point is T-fixed point supported at the singular point in the nilcone $\mathcal{N}_2$.

Remark: we define $X(M)$ as a scheme theoretic intersection of closures while MV-cycles have been defined as varieties (closure of intersections). We lack a moduli description in the variety level which is essential used to understand the T fixed point of a cycle. We notice that $X(M)$ may be reducible even when $P(M)$ is an MV-polytope. The former certainly contains the latter and a further hope is to relate the latter to some subvariety of the quiver Grassmannian.

[3] *this is not contained in the above conjecture since I only said for all μ that is in the support of $X(M)^T$. In fact, we could prove that the corresponding quiver Grassmannian is empty when μ is not in the support of $X(M)^T$.*

CHAPTER 5

THE T FIXED POINT SUBSCHEME OF THE CYCLE

We introduce some notation first. It is known that the T-fixed point subscheme of the loop Grassmannian of a reductive group G is the loop Grassmannian of the Cartan T of G, i.e., $\mathcal{G}(G)^T = \mathcal{G}(T)$. We identify T with I copies of the multiplicative group by $T \xrightarrow[\sim]{\prod \varpi_i} G_m^I$ and this gives $\mathcal{G}(T) \xrightarrow[\sim]{\prod \varpi_i} \mathcal{G}(G_m)^I$.

For $\mathcal{G}(G_m)$, we have

$$\mathcal{G}(G_m) = G_m(\mathcal{O}) \setminus G_m(\mathcal{K}) \tag{5.1}$$

$$= \{\text{unit} \in \mathcal{O}\} \setminus \{\text{unit} \in \mathcal{K}\} \tag{5.2}$$

$$= t^{\mathbb{Z}} \cdot K_- \tag{5.3}$$

where K_- is called the negative congruence subgroup (of G_m). The R-points of K_- can be described as:

$$K_-(R) = \{a = (1 + a_1 t^{-1} + ... + a_m t^{-m}) \mid a_i \text{ is nilpotent in } R\}.$$

We define the degree function from $K_-(R)$ to $\mathbb{Z}_{\geq}$: $\deg(a) = m$ if $a_m \neq 0$.

Then $(\bigcap \overline{S^w_{\lambda_w}})^T$ is a subscheme of $\mathcal{G}(G)^T \cong (t^{\mathbb{Z}} \cdot K_-)^{|I|}$.

Theorem 9. *Let $(\lambda_w)_{w\in W}$ be a collection of cocharacters such that $\lambda_v \geq_w \lambda_w$[1] for all $w \in W$ in which case we know ([Kam05]) that $(\lambda_w)_{w\in W}$ determines a pseudo-Weyl polytope. The integers $A_{w\varpi_i}$ are well defined by $A_{w\varpi_i} = (\lambda_w, w\varpi_i)$. The*

[1] *This notation is used in [Kam05], $\lambda_v \geq_w \lambda_w$ whenever $w^{-1}\lambda_v \geq w^{-1}\lambda_w$.*

R-points of $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$ *is the subset of R-point of* $(t^{\mathbb{Z}} \cdot K_-)^{|I|}$ *containing elements* $(t^{(\nu,\varpi_i)}a_i) \in \prod (t^{\mathbb{N}} \cdot K_-)^{|I|}$ *subject to the degree relations:*

$$deg(\Pi_{i \in I} a_i^{(\gamma,\check{\alpha}_i)}) \leq -A_\gamma + \sum(\gamma,\nu) \textit{ for all } \gamma \in \Gamma.$$

Proof. We define loop Grassmannian with a condition Y and list the facts we need. For details, see [Mir17]. Let G acts on scheme Y and y be a point in Y. Denote the stack quotient by Y/G. Then $\mathcal{G}(G,Y)$ is the moduli of maps of pairs from (d,d^*) to $(Y/G,y)$. When Y is a point we recover $\mathcal{G}(G)$. We will omit writing y when there is a natural choice. In general, $\mathcal{G}(G,Y)$ is the subfunctor of $\mathcal{G}(G)$ subject to a certain extension condition:

$$\mathcal{G}(G,Y) = G_{\mathcal{O}} \setminus \{g \in G_{\mathcal{K}} \mid d^* \xrightarrow{g} G \xrightarrow{o} Y \text{ extends to } d\}, \text{ where } o(g) = gy.$$

We can realize semi-infinite orbits and their closures as follows:

- $\mathcal{G}(G,G/N) = S_0$, where G acts G/N by left multiplication and $y = N$.
- $\mathcal{G}(G,(G/N)^{\text{aff}}) = \overline{S_0}$, where "aff" means affinization.
- $\mathcal{G}(G \times T,(G/N)^{\text{aff}})_{red} = \bigsqcup \overline{S_\lambda}$, where "red" means the reduced subscheme. Here T acts on G/N by left multiplication with the inverse and this extends to an action on $(G/N)^{\text{aff}}$.
- $$\mathcal{G}\left(G \times \prod_{w \in W} T_w, \prod_{w \in W} (G/N^w)^{\text{aff}}\right) = \bigsqcup_{((\lambda_w)_{w \in W}} \left(\bigcap_{w \in W} \overline{S^w_{\lambda_w}}\right),$$
  We denote a copy of T corresponding to $w \in W$ by T_w.

A single cycle $\bigcap_{w \in W} \overline{S^w_{\lambda_w}}$ can be written as the fiber product:

$$\bigcap_{w \in W} \overline{S^w_{\lambda_w}} = \mathcal{G}\left(G \times \prod_{w \in W} T_w, \prod (G/N^w)^{\text{aff}}\right) \times_{\mathcal{G}(\prod_{w \in W} T_w)} (t^{\lambda_w})_{w \in W}.$$

In this fiber product, the morphism for the first factor is the second projection and the morphism for the second factor is the inclusion of the single point $t^{\underline{\lambda}} = (t^{\lambda_w})_{w\in W}$.

For a reductive group G, we have $\mathcal{G}(G,Y)^T = \mathcal{G}(T,Y)$, where T is the Cartan of G.

So, the T fixed point subscheme is

$$(\bigcap_{w\in W} \overline{S^w_{\lambda_w}})^T = \mathcal{G}(T \times \prod_{w\in W} T_w, \prod (G/N^w)^{aff}) \times_{\mathcal{G}(\prod_{w\in W} T_w)} t^{\underline{\lambda}}.$$

In terms of the above extension condition, this fiber product is:

$$(\bigcap_{w\in W} \overline{S^w_{\lambda_w}})^T = T(\mathcal{O})\backslash\{g \in T_{\mathcal{K}}, \text{ such that } d^* \xrightarrow{g,t^{\underline{\lambda}}} T \times T^W \to \prod (G/N^w)^{aff} \text{ extends to } d\}$$

This is the $T(\mathcal{O})$ quotient of the set of all $g \in T_{\mathcal{K}}$, such that

$$d^* \xrightarrow{g,t^{\lambda_w}} T \times T_w \to (G/N^w)^{aff} \text{ extends to } d \text{ for all } w \in W.$$

For $\gamma \in W \cdot \varpi_i \subset \Gamma$, we fix weight vectors v_γ in the weight space $(V_{\varpi_i})_\gamma$ of V_{ϖ_i}. For each $w \in W$, we embed G/N^w into $\bigoplus_{i\in I} V_{\varpi_i}$ by $g \mapsto (g \cdot v_{w\varpi_i})_{i\in I}$. Under this embedding, $(G/N^w)^{aff}$ is a closed subscheme in $\bigoplus_{i\in I} V_{\varpi_i}$.

For $g \in T_{\mathcal{K}}$, $w \in W$, the composition $y_w(g)$ of the map :

$$d^* \xrightarrow{g,t^{\lambda_w}} T \times T_w \to G/N^w \hookrightarrow \bigoplus V_{\varpi_i}$$

is

$$y_w(g) = (g \cdot (t^{\lambda_w})^{-1}) \sum_{i\in I} v_{w\varpi_i} = \sum_{i\in I} (w\varpi_i(g \cdot t^{-\lambda_w}))v_{w\varpi_i}.$$

This map extends to d when for each $i \in I$, the coefficient of $v_{w\varpi_i}$ is in $\mathcal{O}$. The coefficient of $v_{w\varpi_i}$ is

$$w\varpi_i(g\cdot t^{-\lambda_w}) = w\varpi_i(g)\cdot w\varpi_i(t^{-\lambda_w}) = w\varpi_i(g)\cdot t^{-(w\varpi_i,\ \lambda_w)}$$
$$= w\varpi_i(g)t^{-A_{w\varpi_i}} = \gamma(g)z^{-A_\gamma} \text{ where } \gamma = w\varpi_i.$$

It follows that

$$(\bigcap_{w\in W} \overline{S^w_{\lambda_w}})^T = T(\mathcal{O})\setminus\{g\in T(\mathcal{K}); \gamma(g)t^{-A_\gamma}\in\mathcal{O} \text{ for all } \gamma\in\Gamma\}.$$

and the description of the R-points of $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$ in the theorem follows when we identify $\mathcal{G}(T) \xrightarrow{\Pi\varpi_i} \mathcal{G}(G_m)^I = (t^{\mathbb{Z}}\cdot K_-)^I$.

□

5.1 Ring of functions on $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$

For an R-point $(t^{(\nu,\varpi_i)}a_i)_{i\in I}$ of $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$, let us write $a_i = 1+a_{i1}t^{-1}+\cdots+a_{im}t^{-m}$. When $\gamma = \varpi_i$, the degree inequality is $\deg(a_i) \le (\varpi_i,\nu) - A_{\varpi_i}$. We can take the coefficients a_{ij} to be the coordinate functions on $(\bigcap \overline{S^w_{\lambda_w}})^T_\nu$. Since $\deg(a_i) \le (\varpi_i,\nu) - A_{\varpi_i}$, there are finitely many a_{ij}s which generate the ring of functions on $\mathcal{O}((\bigcap \overline{S^w_{\lambda_w}})^T_\nu)$.

When we take inverse of a_i, it is computed in K_- as $a_i^{-1} = 1+\sum_{s\ge 0}(-1)^i(a_{i1}t^{-1}+\cdots+a_{im}t^{-m})^s$ and then expands in the form $\sum_i b_{ik}t^{-k}$, where b_{ik} is the coefficient of t^{-k} in a_i^{-1}.

$$deg(\Pi_{i\in I}a_i^{(\gamma,\check{\alpha_i})}) \le -A_\gamma + \sum(\gamma,\nu) \text{ for all } \gamma\in\Gamma.$$

is equivalent to the condition that the coefficient of the term t^{-1} to the power $-A_\gamma+\sum(\gamma,\nu)+1$ in $(\Pi_{i\in I}a_i^{(\gamma,\check{\alpha_i})})$ is 0. These coefficients are polynomials of a_{ij}s.

Set $b_i = 1 + \sum_k b_{ik} t^{-k} = a_i^{-1}$, add b_{ij}'s as generaters and also add the relations $a_i b_i = 1$ for $i \in I$ which eliminate all b_{ij}'s. For $\gamma \in \gamma$, let $\gamma_i = (\gamma, \check{\alpha}_i)$. Denote by I_γ^+ the subset of I containing all i such that γ_i is positive and by I_γ^- containing all i γ_i is negative. Set $\gamma_i^+ = \gamma_i$ when γ_i is positive and $\gamma_i^- = -\gamma_i$ when γ_i negative.

Corollary 2. *The ring of functions on $\mathcal{O}((\bigcap \overline{S^w_{\lambda_w}})^T_\nu)$ is generated by a_{ij}'s and b_{ik}'s, for $i \in I$. The relations are degree conditions:*

$$deg(\prod_{i \in I_\gamma^+} a_i^{\gamma_i{}^+} \prod_{i \in I_\gamma^-} b_i^{\gamma_i{}^-}) \leq (\gamma, \nu) - A_\gamma$$

for each $\gamma \in \Gamma$ and conditions $a_i b_i = 1$ for each i in I.

CHAPTER 6

CONSTRUCTION OF THE MAP Ψ FROM FUNCTIONS TO COHOMOLOGY

6.1 Map Ψ

For $M \in \Lambda(d)$, to apply corollary 1 to $X(M)$, we set $A_\gamma = -D_{-\gamma}(M)$. Then

$$\mathcal{O}(X(M)_\nu^T) = k[a_{ij}, b_{ik}]/I(M)$$

where $I(M)$ is the ideal generated by the degree conditions:

$$deg(\prod_{i \in I_\gamma^+} (a_i)^{\gamma_i^+} \prod_{i \in I_\gamma^-} (b_i)^{\gamma_i^-}) \leq (\gamma, \nu) + D_{-\gamma}(M)$$

for each $\gamma \in \Gamma$ and the conditions $a_i b_i = 1$ for each i in I.

The conjecture $\mathcal{O}(X(M)_\nu^T) \cong H^*(Gr_e^\Pi(M))$, where $e_i = (\nu, \varpi_i)$, is now equivalent to

$$k[a_{ij}, b_{ik}]/I(M) \cong H^*(Gr_e^\Pi(M)).$$

The quiver Grassmannian $Gr_e^\Pi(M))$ is a subvariety of $\prod_{i \in I} Gr_{e_i}(M_i)$ and we have on each $Gr_{e_i}(M_i)$ the tautological subbundle S_i and quotient bundle Q_i. We pull back S_i and Q_i to $\prod_{i \in I} Gr_{e_i}(M_i)$ and denote their restrictions on $Gr_e^\Pi(M))$

still by S_i and Q_i by abusing notion. For a rank n bundle E, denote the Chern class by $c(E)$ and the i^{th} Chern class $c_i(E)$, where $c(E) = 1 + c_1(E) + \cdots + c_n(E)$. We want to define the map

$$\Psi : \mathcal{O}(X(M)^T_\nu) \to H^*(Gr^\Pi_e(M)), \text{ where } e_i = (\nu, \varpi_i),$$

by mapping the generators a_{ij} to $c_j(S_i)$ and b_{ij} to $c_j(Q_i)$.

Theorem 10. *The map Ψ described above is well defined.*

6.2 Two lemmas

For the proof, we need two lemmas. Lemma 15 is the special case of theorem 10 when Q is the quiver $1 \to 2$ and M is a kQ-module.

Lemma 15. *Let Q be the quiver $1 \to 2$ and M be $\mathbb{C}^{d_1} \xrightarrow{\phi} \mathbb{C}^{d_2}$.On $X = Gr^\Pi_e(M)$, we have $c_i(S_2 \oplus Q_1) = 0$ when $i > e_2 - e_1 + dim(ker\phi)$.*

Let $\phi_{ij} : M_i \to M_j$ be the composition of ϕ_a where a travels over the unique no going-back path which links i and j. Let $M_\gamma = \bigoplus_{i \in I^-_\gamma} M_i^{\gamma^-} \xrightarrow{\phi_\gamma = \oplus \phi_{ij}} \bigoplus_{i \in I^+_\gamma} M_i^{\gamma^+}$ be the module over $k(1 \to 2)$.

Lemma 16. *For a Π-module M and any chamber weight γ, we have*

$$dim(ker\phi_\gamma) = D_{-\gamma}(M).$$

Lemma 16 is a property of D_γ and will be proved in the appendix.

6.3 Proof of theorem 4 from lemmas in § 4.2

Proof of theorem 4. We prove that the theorem can be reduced to lemma 15.

For each $\gamma \in \Gamma$,we have to prove the degree inequalities carry over to Chern classes:

$$\deg\left(\Psi(\prod_{i\in I_\gamma^+} t_i^{\gamma_i^+}\prod_{i\in I_\gamma^-} s_i^{\gamma_i^-})\right) = \deg\left(\prod_{i\in I_\gamma^+} c(S_i)^{\gamma_i^+}\prod_{i\in I_\gamma^-} c(Q_i)^{\gamma_i^-}\right) \leq D_{w_0\gamma}(M) + (\nu, \gamma).$$

Define a map Φ from $Gr^{\Pi}(M)$ to $Gr^{k(1\longrightarrow 2)}(M_\gamma)$: for $N \in Gr^{\Pi}(M)$, $\Phi(N) = \oplus_{i\in I_\gamma^-} N_i \xrightarrow{\phi_\gamma} \oplus_{i\in I_\gamma^+} N_i$. We have

$$\Phi^*(c(S_2)c(Q_1)) = c(\Phi^*(S_2))c(\Phi^*(Q_1)) = c(\oplus_{i\in I_\gamma^+} S_i^{\gamma_i^+})c(\oplus_{i\in I_\gamma^-} Q_i^{\gamma_i^-})$$

$$= \prod_{i\in I_\gamma^+} c(S_i)^{\gamma_i^+}\prod_{i\in I_\gamma^-} c(Q_i)^{\gamma_i^-}.$$

Apply lemma15 to M_γ we have

$$\deg(c(Q_1)c(S_2)) \leq \dim ker(\phi_\gamma) + \sum_{i\in I_\gamma^+} \gamma_i e_i - \sum_{i\in I_\gamma^-} \gamma_i e_i$$

$$= \dim ker(\phi_\gamma) + \sum_{i\in I} \gamma_i e_i = \dim ker(\phi_\gamma) + (\gamma, \nu).$$

Then the theorem follows by lemma 2. $\square$

Chern class vanishes in certain degree when the bundle contains a trivial bundle of certain degree but the desired trivial bundle in $Q_1 \oplus S_2$ does not exist. The idea is to pass to T-equivariant cohomology. Over X^T which is just a union of isolated points we will decompose $Q_1 \oplus S_2$ into the sum of the other two bundles E_1 and E_2 pointwisely, where E_1 will play the role of trivial bundle. Although there is no bundle over X whose restriction is E_2, there exist T-equivariant cohomology class in $H_T^*(X)$ whose restriction on X^T is the T-equivariant Chern class of E_2.

6.4 Recollection of GKM theory

We first recall some facts in T-equivariant cohomology theory.

We follow the paper [Tym05]. Denote a n-dimensional torus by T, topologically T is homotopic to $(S^1)^n$. Take ET to be a contractible space with a free T-action. Define BT to be the quotient ET/T. The diagonal action of T on $X \times ET$ is free, since the action on ET is free. Define $X \times_T ET$ to be the quotient $(X \times ET)/T$. We define the equivariant cohomology of X to be

$$H_T^*(X) = H^*(X \times_T ET).$$

When X is a point and $T = G_m$,

$$H_T^*(X) = H^*(\mathrm{pt} \times_T ET) = H^*(ET/T) = H^*(BT) = H^*(\mathbb{CP}^\infty) \cong k[t].$$

When $T = (S^1)^n$,

$$H_T^*(pt) = k[t_1, \cdots, t_n] \cong S(\mathfrak{t}^*). \tag{6.1}$$

So we can identify any class in $H_T^*(pt)$ as a function on the Lie algebra $\mathfrak{t}$ of T. The map $X \to pt$ allows us to pull back each class in $H_T^*(pt)$ to $H_T^*(X)$, so $H_T^*(X)$ is a module over $H_T^*(pt)$.

Fix a projective variety X with an action of T. We say that X is equivariantly formal with respect to this T-action if $E^2 = E^\infty$ in the spectral sequence associated to the fibration $X \times_T ET \longrightarrow BT$.

When X is equivariantly formal with respect to T, the ordinary cohomology of X can be reconstructed from its equivariant cohomology. Fix an inclusion map $i : X \to X \times_T ET$, we have the pull back map of cohomologies: $H^*(X \times_T ET \xrightarrow{i^*} H^*(X)$. The kernel of i^* is $\sum_{s=1}^n t_s \cdot H_T^*(X)$, where t_s is the generator of $H_T^*(pt)$

(see (4)) and we view it as an element in $H^*_T(X)$ by pulling back the map $X \to pt$. Also i^* is surjective so $H^*(X) = H^*_T(X)/ker(i^*)$.

If in addition X has finitely many fixed points and finitely many one-dimensional orbits, Goresky, Kottwitz, and MacPherson show that the combinatorial data encoded in the graph of fixed points and one-dimensional orbits of T in X implies a particular algebraic characterization of $H^*_T(X)$.

Theorem 11 (GKM, see [Tym05], [GKM97]). *Let X be an algebraic variety with a T-action with respect to which X is equivariantly formal, and which has finitely many fixed points and finitely many one-dimensional orbits. Denote the one-dimensional orbits $O_1, \ldots, O_m$. For each i, denote the the T-fixed points of O_i by N_i and S_i and denote the stabilizer of a point in O_i by T_i. Then the map $H^*_T(X) \xrightarrow{l} H^*_T(X^T) = \oplus_{p_i \in X^T} H^*_T(p_i)$ is injective and its image is*

$$\left\{ f = (f_{p_1}, \ldots, f_{p_m}) \in \bigoplus_{\text{fixed pts}} S(t^*) : f_{N_i}|_{\mathfrak{t}_i} = f_{S_i}|_{\mathfrak{t}_i} \text{ for each } i = 1, \ldots, m \right\}.$$

Here $\mathfrak{t}_i$ is the lie algebra of T_i.

6.5 Affine paving of $Gr_e^{\Pi}(M)$ when M is a representation of Q of type A

Definition 5 ([Tym07] 2.2). *We say a space X is paved by affines if X has an order partition into disjoint $X_1, X_2, \cdots$ such that each finite union $\bigcup_{i=1}^{j} X_i$ is closed in X and each X_i is an affine space.*

A space with an affine paving has odd cohomology vanishing.

Proposition 7 ([Tym07], 2.3)**.** *Let $X = \bigcup X_i$ be a paving by a finite number of affines with each X_i homeomorphic to $\mathbb{C}^{d_i}$. The cohomology groups of X are given by $H^{2k}(X) = \bigoplus_{\{i \in I \ | \ d_i = k\}} \mathbb{Z}$.*

The main observation is the following lemma.

Lemma 17. *Let M be a representation of Q, where Q is of type A with all edges in E pointing to the right. Then the quiver Grassmannian $Gr_e^{\Pi}(M)$ is paved by affines for any dimension vector e.*

We need a sublemma first.

Sublemma 1. *Suppose X is paved by X_i's. Let $Y \subset X$ be a subspace. If for each i, $Y_i = X_i \bigcap Y$ is $\emptyset$ or affine then $Y = \bigcup Y_i$ is an affine paving.*

Proof. $\bigcup_{i \leq j} Y_i = \bigcup_{i \leq j}(X_i \bigcap Y) = (\bigcup_{i \leq j} X_i) \bigcap Y$ is closed in Y since $\bigcup_{i \leq j} X_i$ is closed in X. □

Before we give the proof of lemma17, let us introduce some notations about Schubert decomposition of Grassmannian, following [Shi85]. let V be an n-dimensional vector space over a field k. Let d be an integer such that $1 \leq d < n$. Let V^d be the direct sum of d copies of V and $\wedge^d V$ be the d-th alternating product of V. Let

$$\pi : V^d \to \wedge^d V$$

be the morphism defined by $(v_1, \cdots, v_d) \to v_1 \wedge \cdots \wedge v_d$. Fix a basis $\{e_1, \cdots, e_n\}$ of V. Then we can identify V^d with the set of all $d \times n$ matrices over k by

$$(v_1, \cdots, v_d) \mapsto (x_i(j))_{1 \leq i \leq d, 1 \leq j \leq n},$$

where $v_i = \Sigma_{1 \le j \le n} x_i(j) e_j, x_i(j) \in k$. Let $\mathbf{P}(\wedge^d V)$ be the projection. We have $\mathrm{Grass}(d, V) = p(\pi V^d - \{0\})$. Put

$$I = \{\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}^d; 1 \le \alpha_1 < \cdots < \alpha_d \le n\}.$$

For $\alpha = (\alpha_1, \cdots, \alpha_d)$ in I, put

$$D_\alpha = \{(x_i(j)) \in V^d; x_i(j) = 0 \mathrm{for} j < \alpha_i (1 \le i \le d)\},$$

$$C_\alpha = \{(x_i(j)) \in D_\alpha; x_i(\alpha_j) = \delta_{ij} (1 \le i, j \le d)\}.$$

Then $S_\alpha = p\pi C_\alpha$ is the Schubert cell and $p(\pi D_\alpha - \{0\})$ is the Schubert variety, which is the closure of S_α. We will consider the subvariety of Grassmannian $Gr_d(k^n)^\phi$ that is invariant under a nilpotent operator ϕ. Let λ be an ordered partition of n, i.e, an ordered sequence $(\lambda_1, \cdots, \lambda_r)$ of positive integers such that $\lambda_1 + \cdots + \lambda_r = n$. We represent λ by a Young diagram with rows consisting of $\lambda_1, \cdots, \lambda_r$ squares respectively.

Definition 6. *Fix a Young diagram λ with n squares. Let d be an integer such that $1 \le d < n$. A d-tableau is a Young diagram of type λ whose d squares are distinguished by* $\blacksquare$.

Fill in the squares of λ with the numbers $1, 2, \cdots, n$ in the order from the left column to the right and for each column from up to down. For example, if $\lambda = (4, 3, 2)$, we have $\begin{array}{|c|c|c|c|}\hline 7 & 4 & 2 & 1 \\ \hline 8 & 5 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 9 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$. Now we could identify $\alpha = (\alpha_1, \cdots, \alpha_d)$ in I with the d-tableau of type λ whose $\alpha_1, \cdots, \alpha_d$-th squares are $\blacksquare$. An nilpotent operator can be represented by a Young diagram using its Jordan normal form. We could choose our basis $\{e_1, \cdots, e_n\}$ to be a Jordan normal basis. Explicitly, for a Young diagram numbered above, we define ϕ corresponding to λ by $\phi e_i = e_j$ if λ contains $\boxed{i}\boxed{j}$ and $\phi e_i = 0$ if $\boxed{i}$ lies on the most left column.

Definition 7. *A d-tableau is said to be semistandard if every square on the left position to* $\blacksquare$ *on the same row is* $\blacksquare$.

Lemma 18. *([Shi85] lemma1.8, and see the proof there also) Let* S_α^ϕ *be the subspace invariant under* ϕ*, i.e*

$$S_\alpha^\phi = \{W \in S_\alpha ; \phi W \subset W\},$$

where S_α *is the Schubert cell corresponding to* α*. Then* S_α^ϕ *is nonempty if and only if* α *is semistandard.*

Now we want to understand S_α^ϕ when α is semistandard. For that purpose, we introduce a notion called initial number.

Definition 8. *For a semistandard d-tableau* $\alpha = (\alpha_1, \cdots, \alpha_i, \cdots, \alpha_d)$ *of type* λ*, we say* i *is an initial number of* α *if the square on the right side of* $\boxed{\alpha_i}$ *is not* $\blacksquare$.

Lemma 19. *For a semistandard d-tableau* α*, put*

$C_\alpha^\phi = \{v_1, \cdots, v_d) \in D_\alpha ; \phi v_i = v_j$ *if* α *contains* $\boxed{\alpha_i}\boxed{\alpha_j}$, $(1 \leq i < j < d)$, *when* i *is initial* $x_i(\alpha_j) = \delta_{ij}$.

Then the morphism $p\phi : D_\alpha \cong S_\alpha$ *induces an isomorphism*

$$C_\alpha^\phi \cong S_\alpha^\phi.$$

Proof. The lemma is given in [Shi85] lemma1.10 but I found there seems have a mistake. The original proof says: take an element $p(w_1 \wedge \cdots \wedge w_d) \in S_\alpha^\phi$, where $(w_1, \cdots, w_d) \in C_\alpha$. Let $(v_1, \cdots, v_d)$ be an element in C_α^ϕ such that $\{(v_1, \cdots, v_d\} = \{\phi^h w_i : i's$ are the initial numbers of α and $h \geq 0\} - \{0\}$. However, this element is not always in C_α. For example, when α is $\begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & \square \\ \hline \blacksquare & \square & \\ \cline{1-2}\end{array}$, (remember the number indexing basis is $\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array}$) let $v_2 = e_2 + e_3$, then $v_4 = e_4 + e_5$, violating $x_5(\alpha_4) = 0$. So we have to make a change to the statement to relax the condition that $(v_1, \cdots, v_d) \in C_\alpha$ to D_α. After this change, the proof goes word by word. $\square$

Now we can give the proof of lemma17.

Proof of lemma 17. Let $V = \oplus M_i$ be the underlying vector space and $\phi = \oplus_{a\in H}\phi_a$ be the nilpotent operator on V.

Let $n = dimV$ and $d = \sum e_i$. From the above discussion, we know that $Gr_d(k^n)^\phi = \bigsqcup S_\alpha^\phi$. We want to show $S_\alpha^\phi \bigcap Gr_e^\Pi(M)$ is affine. Take $x \in S_\alpha^\phi \bigcap Gr_e^\Pi(M)$, from lemma19, $x = \wedge(\phi^h w_i : i's$ are the initial numbers of α and $\phi^h w_i \neq 0)$.

We now show that for $x = v_1 \wedge \cdots \wedge v_d \in Gr_e^\Pi(M)$, where $v_i = e_i + \sum_{j\neq i} x_i(j)e_j$, if $e_i \in M_t$, we have $v_i \in M_t$ and conversely, if for each i, there exists t such that $v_i \in M_t$, $x \in Gr_e^\Pi(M)$. We denote this t determined uniquely by i as t(i).

Since $span(v_1, ..., v_d)$ is a direct sum of some $N_i \subset M_i$, we have $Pr_t(v_i) \in span(v_1, ..., v_d)$, where Pr_t is the projection from V to M_t, and so $Pr_t(v_i) = \sum a_p v_p$. Comparing the coefficient of e_p, by the definition of D_α, we have $a_p = 0$ for $p \neq i$. So we have $Pr_t(v_i) = v_i$, which implies $v_i \in M_t$.

Note that $\{v_1, \cdots, v_d\}$ is determined by w_i where i is an initial number(and vice versa).

We have $w_i \in M_{t(i)}$. Denote $l(i)$ be the number on the left of i in the d-tableaus. If i is the leftmost, set $l(i)$ to be $\emptyset$, and set $e_\emptyset = 0$. Write $w_i = e_i + \sum x_{ij}e_j$, where $e_j \in M_{t_i}$, we have $\phi^r(w_i) = e_{l^r(i)} + \sum x_{ij}e_{l^r(j)}$. Since M is kQ-module, we have $l^r(i) = l^r(j)$ hence $v_{i_r} \in M_{t(i_r)}$ and $x \in Gr_e^\Pi(M)$. So we have $S_\alpha^\phi \bigcap Gr_e^\Pi(M)$ is affine. Apply lemma 17, we are done.

□

6.6 Proof of lemma 15

Proof. For M given by $\mathbb{C}^{d_1} \xrightarrow{\phi} \mathbb{C}^{d_2}$ and a choice of $e = (e_1, e_2)$, denote $X = Gr_e^{\Pi}(M)$. First, we define a torus action on X. Let $I = ker\phi$. Choose a basis $e_1, e_2, \cdots, e_s$[1] of I and extend it to a basis $e_1, \cdots, e_s, e_{s+1}, \cdots, e_t$ of M_1. Let J be span$\{e_{s+1}, \cdots, e_t\}$ so the image of J is span $\{f_{s+1}, \cdots, f_t\}$. We extend the basis $\{f_i = \phi(e_i)\}$ of the image of J to a basis $(f_{s+1}, \cdots, f_t, f_{t+1}, \cdots, f_r)$ of M_2. Let K=span$\{f_{t+1}, ...f_r\}$. we have $M_1 = I \oplus J$ and $M_2 = \phi(J) \oplus K$.

Let $\mathcal{I} = \{1, \cdots, s\}$, $\mathcal{J} = \{s+1, \cdots, t\}$ and $\mathcal{K} = \{t+1, \cdots, r\}$. Let tori $T_I = G_m^{\mathcal{I}}, T_J = G_m^{\mathcal{J}}, T_K = G_m^{\mathcal{L}}$ act on $I, J \cong \phi(J)$, K by compotentwise multiplication (For instance, T_I acts on I by $(t_1, \cdots, t_s) \sum a_i e_i = \sum a_i t_i e_i$ and on J, K trivially). Hence they act on $M_1 = I \oplus J$ and $M_2 = \phi(J) \oplus K$. This induces an action of $T = T_I \times T_J \times T_K$ on $Gr_e^{\Pi}(M)$. By lemma 3, $Gr_e^{\Pi}(M)$ is paved by affines so by proposition 3 it has odd cohomology vanishing therefore the spectral sequence associated to the fibration $X \times_T ET \longrightarrow BT$ converges at E^2 and X is equivariantly formal.

Denote by f the forgetful map $H_T^*(X) \xrightarrow{f} H^*(X)$. From §4.4 we have $ker(f) = \sum_1^{dimT} t_s H_T^*(X)$. Since $c^i(S_2 \oplus Q_1) = f(c_T^i(S_2 \oplus Q_1))$, it suffices to prove $c_T^i(S_2 \oplus Q_1)) \in ker(f)$ when $i > e_2 - e_1 + dimI$.

To use GKM theorem, we need to know the one dimensional orbits and T-fixed points of X.

First, we see what X^T is. For a point $p = (V_1, V_2)$ in X, in order to be fixed by T, V_1 and V_2 need to be spanned by some of basis vectors e_i and f_i. For a subset S of $\mathcal{I}$ (resp. $\mathcal{J}$), we denote by e_S (resp. f_S) the span $\{e_i | i \in S\}$ (resp.

[1] I also use letter e for dimension vector, but it should be clear which I mean.

span$\{f_i|i \in S\}$). The T-fixed points in X consist of all $V = (V_1, V_2)$, such that $V_1 = e_{A\bigcup B}, V_2 = f_{C\bigcup D}$, for some $A \subset I, B \subset C \subset J$ and $D \subset K$.

For any point $p = (V_1, V_2)$ in X^T, let $V_1 = e_{A\bigcup B}, V_2 = f_{C\bigcup D}$. Over p, $Q_1 = (I \oplus J)/e_{A\bigcup B}$ is isomorphic to $e_{(\mathcal{I}\setminus A)\oplus(\mathcal{J}\setminus B)}$ (The restriction of a T-equivariant bundle to a T-fixed point is just a T-module). So over X^T, we can decompose $S_2 \oplus Q_1$ as follows:

$$S_2 \oplus Q_1 \cong e_{(\mathcal{I}\setminus A)\oplus(\mathcal{J}\setminus B)} \oplus f_{(C\bigcup D)} = (e_{(\mathcal{I}\setminus A)\oplus(C\setminus B)} \oplus f_D) \oplus (e_{\mathcal{J}\setminus C} \oplus f_C).$$

Denote the bundle over X^T whose fiber over each point p is $e_{(\mathcal{I}\setminus A)\oplus(C\setminus B)} \oplus f_D)$ by E_1 and the bundle over X^T whose fiber over p is $e_{(\mathcal{J}\setminus C)} \oplus f_C$ by E_2.

We now use localization. Denote by l the map $H_T^*(X) \xrightarrow{l} H_T^*(X^T) = \oplus_{p\in X^T} H^*(p)$. From GKM theory l is injective, so the condition $c_T^i(S_2 \oplus Q_1)) \in ker(f)$ is equivalent to $l(c_T^i(S_2 \oplus Q_1)) \in l(ker(f))$. We have

$$l(ker(f)) = l(\sum_{s=1}^{dimT} t_s H_T^*(X)) = \sum_{s=1}^{dimT} \underbrace{(t_s, \cdots, t_s)}_{\text{the number of T-fixed points in X}} l(H_T^*(X)). \quad (6.2)$$

By functorality of Chern class, $l(c_T^i(S_2 \oplus Q_1)) = c_T^i(S_2|_{X^T} \oplus Q_1|_{X^T}))$.

We compute the T-equivariant Chern class over X^T. For[2] each p,

$$c_T^p(S_2 \oplus Q_1) = c_T^p(E_2 \oplus E_1) = \sum_i c_T^{p-i}(E_1)c_T^i(E_2) = \sum_{i\geq 1} c_T^{p-i}(E_1)c_T^i(E_2).$$

The last equality holds since $c_T^p(E_2) = 0$ when $p > dimE_2 = dimI + e_2 - e_1$. Now to show $c_T^p(S_2 \oplus Q_1) \in l(ker(f))$, It suffices to show that $c_T^{p-i}(E_1)c_T^i(E_2) \in l(ker(f))$, for any i. The action of T on E_2 is actually the same on each T-fixed point. And at each point, $c_T^i(E_2)$ is the i^{th} elementary symmetric polynomial of $t_s, 1 \leq s \leq dimT$. So by (5), it suffices to show that $c_T^i(E_1) \in l(H_T^*(X))$.

[2]We always denote S and Q but indicate over which space we are considering.

Now we will see what 1-dimensional orbits are. Take an orbit O_i, in order to be 1 dimensional its closure must contain two fixed points. Let $\overline{O_i} = O_i \bigcup \{N_i\} \bigcup \{S_i\}$, where $N_i = (e_{A \bigcup B}, f_{C \bigcup D})$ and $S_i = (e_{A' \bigcup B'}, f_{C' \bigcup D'})$ are the fixed points. O_i is one dimensional whenever either $A \bigcup B$ and $A' \bigcup B'$ differ by one element with $C \bigcup D = C' \bigcup D'$ or $C \bigcup D$ and $C' \bigcup D'$ differ by one element with $A \bigcup B = A' \bigcup B'$. In the first case, we have some $s \in A \bigcup B$ and $s' \in A' \bigcup B'$, such that $A \bigcup B \setminus s = A' \bigcup B' \setminus s'$.

Notice that the annihilator for the lie algebra $\mathfrak{t}_i$ in $S(t^*)$ is generated by $t_s - t_{s'}$, so by theorem 5, the condition along O_i for an element $h \in H_T^*(X^T)$ to be in $im(l)$ is

$$(t_s - t_{s'}) \mid (h_{N_i} - h_{S_i}).$$

But we have

$$c_T(E_1)|_{N_i} - c_T(E_1)|_{S_i} = (1+t_{s\prime}) \prod_{i \in \mathcal{I} \bigcup C \setminus (A \bigcup B) \setminus \{s'\}} (1+t_i) - (1+t_s) \prod_{i \in \mathcal{I} \bigcup C \setminus (A' \bigcup B') \setminus \{s\}} (1+t_i).$$

Note that $\mathcal{I} \bigcup C \setminus (A \bigcup B) \setminus \{s'\} = \mathcal{I} \bigcup C \setminus (A' \bigcup B') \setminus \{s\}$, so $t_s - t_{s'}$ divides $c_T(E_1)|_{N_i} - c_T(E_1)|_{S_i}$. We conclude that $c_T^i(E_1) \in l(H_T^*(X))$.

The other case is similar.

$\square$

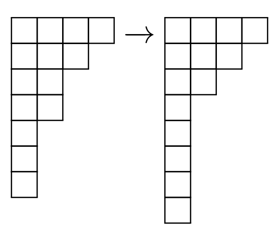
CHAPTER 7

PROOF OF ISOMORPHISM WHEN M IS A REPRESENTATION OF Q OF TYPE A

We first prove that Ψ is surjective.

Lemma 20. *(a) Denote $Y = \prod Gr_{e_i}(k^{d_i})$ and $X = Gr_e^{\Pi}(M)$. Then $Y \setminus X$ is paved by affines.*
(b) Ψ is surjective.

Proof. (a). Let a be the number where the Young diagram of ϕ_Y has a^{th} row as the first row from the bottom that does not have one block. For example, in the left diagram, a=4.



Define ϕ' be the operator of V that corresponds to the diagram by moving the left most block A of the a^{th} row to the bottom in the diagram of ϕ_Y. Let M' be the corresponding module and X' be $Gr_e M'$.
We claim that $X' \setminus X$ is paved by affines.
By lemma 18, we have $X = \bigsqcup_{\alpha \in I} C_\alpha$, where I is the set of all semi-standard young

tableau in λ. Also we have $Y' = \bigsqcup_{\alpha \in I'} C'_\alpha$, where I' is the set of all semi-standard young tableau in λ'. If α contains block A, α is s.s in λ implies α' is s.s in λ'. If α does not contain block A , α also does not contain any block in that row, so α is still s.s in λ'. So $I \subset I'$.

For α that contains block A, there are two types. Let E be the set of α that contains block A and some other block in the row of A. Let F be the set of α that contains block A but no other block in the row of A. Let G be the set of α that does not contain block A. So we have $I = E \bigsqcup F \bigsqcup G = F \bigsqcup (E \bigsqcup G)$.

Take $\alpha \in F$, in λ, the block A in α is not initial so the vector indexed by A is determined by the initial vector. In λ', A is the last block so the vector indexed by A is the basis vector indexed by block A. In both case the vector indexed by A has been determined, so $C_\alpha = C'_\alpha$ when $\alpha \in F$.

For $\alpha \in E \bigcup G$, let $s(\alpha)$ be the tableau of the same relative position in λ' as α in λ. Then $C_\alpha = C'_{s(\alpha)}$. Since s is a bijection between $E \bigsqcup G$ and $E' \bigsqcup G'$ we have $\bigsqcup_{\alpha \in E \bigsqcup G} C_\alpha = \bigsqcup_{\alpha \in E' \bigsqcup G'} C'_\alpha$.

Then we have $X' \setminus X = \bigsqcup_{\alpha \in I' \setminus I} C'_\alpha$ is paved by affines. We can do this procedure step by step until X' becomes Y, so we are done.

(b). By lemma 1, X is paved. With part(a) , we have the homology map from X to Y is injective hence Ψ is surjective since it is the dual of homology map in complex coefficient. (see 2.2 in[Tym07]). $\square$

We want to prove the two sides of Ψ have the same dimension as k-vector spaces and actually we will prove it for a more general setting.

Definition 9. *For a Π-mod M Let $V = \oplus M_i$ be the underlying vector space and $\phi = \oplus_{a \in H} \phi_a$ be the nilpotent operator on V. We say M is I-compatible if there is a Jordan basis $\{v_{ij}\}$ of V such that each v_{ij} is contained in some M_r.*

Definition 10. *For a Π-module M, if the Young diagram of the associated operator ϕ has one row, we call M one direction module.*

Proposition 8. *If M is I-compatible, it is a direct sum of one direction module with multiplicities.*

Lemma 21. *If M is I-compatible, the dimension of two sides of Ψ have the inequality: dim $k[a_{ij}, b_{ik}]/I(M, e)) \leq \chi(Gr_e^{\Pi}(M))$, where χ is the Euler characteristic. We denote the ring on the left by $R(M, e)$.*

First we state a lemma due to Caldero and Chapoton.

Lemma 22 (see prop 3.6 in [CC04])**.** *For Π-module M, N, we have*

$$\chi(Gr_g(M \oplus N) = \sum_{d+e=g} \chi(Gr_d(M))\chi(Gr_e(N)).$$

The following two lemmas are proved after the proof of lemma 19.

Lemma 23. $R(M, e)/ < b_{1(d_1-e_1)} > \cong R(M', e)$.

Lemma 24. $b_{1(d_1-e_1)}R(M, e)$ *is a module over* $R(M'', e - \sum_{i \in I} \alpha_i)$.

Proof of lemma 19. We index the basis vector according to the Young diagram of ϕ as before but slightly different: e_{ij} corresponds to the block of i^{th} row (from up to down) and j^{th} column (from right to left, which is the difference from before and this will cause the problem that two blocks in the same column but different row have different j but we will fix an i sooner so will not be of trouble). so $\phi(e_{ij}) = e_{i(j-1)}$.

The basis vector in M_1 appears in the first column or in the last column. If it lies in the last, we can take the dual to make it in the first. So, there exist i such that $e_{i1} \in M_1$.

Let λ' be the young diagram removing the block of e_{i1} from the original one and M' be the corresponding module. Let λ'' be the young diagram removing i^{th} row and M'' be the corresponding module. Apply lemma 20, we have

$$\chi(Gr_e^{\Pi}(M)) = \chi(Gr_e(M')) + \chi(Gr_{e-\sum\alpha_i}(M'')).$$

We count the dim of $R(M,e)$ by dividing it into two parts.

$$dimR(M,e) = dimb_{1(d_1-e_1)}R(M,e) + dimR(M,e)/b_{1(d_1-e_1)}R(M,e).$$

By lemma 21 and 8, since $b_{1(d_1-e_1)}R(M,e)$ is acyclic,

$$dimb_{1(d_1-e_1)}R(M,e) \leq dimR(M'', e - \sum_{i\in I}\alpha_i).$$

So $dimR(M,e) = dimb_{1(d_1-e_1)}R(M,e)) + dimR(M,e)/b_{1(d_1-e_1)}R(M,e)$
$\leq dimR(M'', e-\sum_{i\in I}\alpha_i) + dimR(M',e) = \chi(Gr_e(M') + \chi(Gr_{(e-\sum_{i\in I}\alpha_i)}(M'')) =$
$\chi(Gr_e^{\Pi}(M))$. □

Now we prove lemma 21 and 22.[1]

Proof of lemma 21. Recall $I(M) = \deg\prod_{i\in I^+}(t_i)^{\gamma_i}\prod_{i\in I^-}(s_i)^{\gamma_i} \leq (\gamma,\nu) + D_{-\gamma}(M)$ We denote $v(M,\gamma,e) = (\gamma,\nu) + D_{w_0\gamma}(M)$. The difference between $I(M)$ and $I(M)'$ only occurs when $\gamma = -\varpi_1$. In this case $v(M', -\varpi_1, e) = v(M, -\varpi_1, e) - 1$. The degree of s_1 goes down by by 1, meaning we have one more vanishing condition which is $b_{1(d_1-e_1)} = 0$. □

Proof of lemma 22. In order to define a module structure on $b_{1(d_1-e_1)}R(M,e)$, we lift the element in $R(M'', e-\sum_{i\in I}\alpha_i)$ to $R(M,e)$ (since the former is a quotient of

[1] The following proof is originally written for type A case in general, but later the author found for general type A, the proof of existence of affine paving does not work so here we only consider the special case where the module M is just a kQ-module. In this, the proof is much simpler.

the latter) and let it act on $b_{1(d_1-e_1)}R(M,e)$ by multiplication. We denote J to be the degree $v(M,\gamma,e)$ part of $< \prod_{i\in I+}(t_i)^{\gamma_i}\prod_{i\in I-}(s_i)^{\gamma_i}, \gamma\in\Gamma >$.

We need to check it is independent of the choice of the lift:

$$b_{1(d_1-e_1)}J \subset I(M,e), \text{ where } I(M,e) = J \oplus I(M'', e - \sum_{i\in I}\alpha_i).$$

Denote the module corresponding to i^{th} row P. We have $M = M'' \oplus P$. Then $v(M,\gamma,e) - v(M'',\gamma,\sum_{i\in I}\alpha_i) = v(P,\gamma,\sum_{i\in I}\alpha_i)$.
We claim that $v(P,\gamma,\sum_{i\in I}\alpha_i) = 0 \textit{ or } 1$ and is 0 when $\gamma_1 = 1$.
This is a direct calculation.

When $\gamma_1 = -1$, t_{1j} appears in each summand $\prod_{i\in I+}(t_i)^{\gamma_i}\prod_{i\in I-}(s_i)^{\gamma_i}, \gamma\in\Gamma$.
Let one of the summand be $t_{1j}k$.
$s_{1(d_1-e_1)}t_{1j}k = -\sum_{p+q=d_1-e_1+j} s_{1p}t_{1q}k = -s_{1(d_1-e_1+j-q)}\sum_{q>j}t_{1q}k$. We have $\sum_{q>j}t_{1q}k$ is in $I(M,e)$ since this is of degree larger than $v(M,\gamma,e)$.
When $\gamma_1 = 0$,
we claim that when $v(P,\gamma,\sum_{i\in I}\alpha_i)$ is 1 , we have $\gamma - \varpi_1 \in \Gamma$.
Then we want to show $s_{1(d_1-e_1)}\prod_{i\in I+}(t_i)^{\gamma_i}\prod_{i\in I-}(s_i)^{\gamma_i}$ is in $I(M,e)$. Let k is a summand of degree $v(M,\gamma,e)$ part of $I(M,e)$. We want to show $s_{1(d_1-e_1)}k$ is of degree $v(M,\gamma-\varpi_1,e)+1$. So we need $v(M,\gamma,e)+d_1-e_1 \geq v(M,\gamma-\varpi_1,e)+1$. By the dimension description, the image of $\Phi_{\gamma-\varpi_1}$ is at least 1 dimensional bigger than the image of Φ_γ since $\phi_{1m}(e_{i1}) = e_{im}$ is in the image but for $\gamma-\varpi_1$ (since ϕ_{1m} is not a summand of ϕ) the projection of $img(\phi)$ on V_m is zero, where m is the smallest number in Γ_+. □

Theorem 12. *Ψ is an isomorphism when M is a kQ-module.*

Proof. by lemma 18, Ψ is surjective so dim $k[a_{ij},b_{ik}]/I(M,e)) \geq \chi(Gr_e^{\Pi}(M))$ and by lemma 19 this is an equality so the theorem follows. □

CHAPTER 8

A CONSEQUENCE OF THIS CONJECTURE

In section 3, we defined $\mathcal{G}(G,Y)$ as moduli of maps of between pairs form (d,d^*) to $(G/Y,pt)$. This is actually a local version of (fiber at a closed point c) the global loop Grassmannian with a condition Y to a curve C, $\mathcal{G}^C(G,Y)$. To a curve C, define $\mathcal{G}^C(G,Y)$ over the ran space $\mathcal{R}_C$ with the fiber at $E \in \mathcal{R}_C$:

$$\mathcal{G}^C(G,Y)_E =^{def} map[(C, C-E),(G/Y,pt)].$$

Denote the map from $\mathcal{G}^C(G,Y)$ to $\mathcal{R}_C$ remembering the singularities by π.

One can ask if π is (ind) flat for any G and (Y,pt). The case we are concerned is when $G' = G \times \prod_w T_w$ and $Y = \prod_w (G/N^w)^{aff}$. Let $c \in C, \underline{\lambda_w}, \underline{\mu_w} \in X_*(T)^W$. In particular, we restrict $\mathcal{G}^C(G',Y)$ to $C \times c$ and denote the image under projection from $\mathcal{G}(G')$ to $\mathcal{G}(G)$ by X. We have X is a closed subscheme of $Gr_{G,X\times c}$. Explicitly, an R-point of $Gr_{G,X\times c}$ consists of the following data

- $x : specR \to C$. Let Γ_x be the graph of x. Let Γ_c be the graph of the constant map taking value c.
- β a G-bundle on spec$R \times C$.
- A trivialization $\eta : \beta_0 \xrightarrow{\eta} \beta$ defined on $specR \times C - (\Gamma_x \bigcup \Gamma_c)$.

An R-point of X over $C \times c$ consists of an R-point of $Gr_{G,X\times c}$ subject to the condition: For every $i \in I$, the composition

$$\eta_i : \beta_0 \times^G V(\varpi_i) \to \beta \times^G V(\varpi_i) \to \beta \times^G V(\varpi_i) \otimes \mathcal{O}(\langle \gamma, \lambda_w \rangle \cdot \Gamma_x + \langle \gamma, \mu_w \rangle \cdot \Gamma_c).$$

is regular on all of $specR \times C$.

We can show the fiber over a closed point other than c is $\bigcap \overline{S^w_{\lambda_w}} \times \bigcap \overline{S^w_{\mu_w}}$ and the fiber over c is $\bigcap \overline{S^w_{\lambda_w+\mu_w}}$.

Corollary 3 (Given the conjecture)**.** *The T-fixed point subscheme of this family is flat.*

Proof. $dim\mathcal{O}((\overline{\bigcap S^w_{\lambda_w+\mu_w}}^T)_\nu) = dimH^*(Gr^\Pi_e(M)) =$[1] $\sum_{e_1+e_2=e} dimH^*(Gr^\Pi_{e_1}(M_1)dimH^*(Gr^\Pi_{e_2}(M_2) =$ $\sum_{\nu_1+\nu_2=\nu} \mathcal{O}(\overline{\bigcap S^w_{\lambda_w \nu_1}}^T) \cdot \mathcal{O}(\overline{\bigcap S^w_{\mu_w \nu_2}}^T) = dim\mathcal{O}(\bigsqcup_{\nu_1+\nu_2=\nu} \overline{\bigcap S^w_{\lambda_w \nu_1}}^T \times \overline{\bigcap S^w_{\mu_w \nu_2}}^T = dim(\overline{\bigcup S^w_{\lambda_w}}^T \times \overline{\bigcap S^w_{\lambda_w}}^T)_\nu$. □

Conjecture 3. *T-fixed subschemes flatness imply flatness.*

We take $\lambda_w = -w_0\lambda + w\lambda$, then $\overline{\bigcap S^w_{\lambda_w}} = \overline{Y^\lambda}$. In this case the conjecture is proved to be true. This flatness is mentioned in [KMW16] remark 4.3 and will reduce the proof of reduceness of $\overline{Y^\lambda}$ to the case when λ is ϖ_i for each $i \in I$.

[1] By lemma 20, and given the conjecture, Euler character is the same as total cohomology.

APPENDIX

PROOF OF LEMMA 16

For an expression $s_{i_m} \cdots s_{i_1}$ of an element w in W, we say it is j-admissible if $\langle \alpha_{i_a}, s_{i_{a-1}} \cdots s_{i_1} \varpi_j \geq 0$ for any $a \leq m$.

Lemma 25. *For any element $w \in W$, any reduced expression of w is j-admissible.(since we will fix an j, we will omit j and just say admissible).*

Proof. Since we are in the ADE case,

$$s_i \varpi_i = -\varpi_i + \sum_{h \text{ is adjacent to i}} \varpi_h. \tag{A.1}$$

$$s_i \varpi_h = \varpi_h, \text{ for } h \neq i. \tag{A.2}$$

We use induction on the length of w. Suppose lemma holds when $l(w) \leq m$. Take a reduced expression of $w \in W$ with length $m+1: w = s_{i_{m+1}} \cdots s_{i_1}$. Suppose this expression is not admissible, we have $\langle \alpha_{i_{m+1}}, s_{i_m} \cdots s_{i_1} \varpi_j \rangle \leq 0$. Since $\langle \alpha_{i_{m+1}}, \varpi_j \rangle \geq 0$, and by (6),(7)

$$\langle \alpha_{i_{m+1}}, s_t \gamma \rangle \geq \langle \alpha_{i_{m+1}}, \gamma \rangle.$$

unless $t = i_{m+1}$, there must exists k such that $i_k = i_{m+1}$.Let k be the biggest number such that $i_k = i_{m+1}$.

In the case there is no element in the set $\{i_m, \cdots, i_{k+1}\}$ is adjacent to i_{m+1} in the

Coxeter diagram, $s_{i_{m+1}}$ commutes with $s_{i_m}\cdots s_{i_{k+1}}$. Therefore $s_{i_{m+1}}s_{i_m}\cdots s_{i_{k+1}}s_{i_{m+1}} = s_{i_{m+1}}s_{i_{m+1}}s_{i_m}\cdots s_{i_{k+1}} = s_{i_m}\cdots s_{i_{k+1}}$ so the $w = s_{i_{m+1}}\cdots s_{i_1}$ is not reduced, contradiction.

In the case where for some t, i_t is adjacent to i_{m+1}, we will show we can reduce to the case we have only one such t. Suppose we have at least two elements $i_{t_1}, i_{t_2}\cdots, i_{t_h}$ such that they are all adjacent to i_{m+1}. Since $\langle \alpha_{i_{m+1}}, s_{i_{m+1}}\cdots s_{i_k}\cdots s_{i_1}\varpi_j\rangle \leq 0$ and $h > 1$, by (6), (7), we must have some i_{u_1}, i_{u_2} such that they are adjacent to i_{m+1}. Since one point at most has 3 adjacent points we must have some $i_{u_x} = i_{t_y}$. Let $p = i_{u_x} = i_{t_y}$. Using $s_p s_{i_{m+1}} s_p = s_{i_{m+1}} s_p s_{i_{m+1}}$ we can move $s_{i_{m+1}}$ in front of s_{t_y} so the number h is reduced by 1. We could do this procedure until h=1. In this case we can rewrite the sequence before $s_{i_{k-1}}$ using the braid relation between i_{m+1} and i_t:

$$s_{i_m+1}\cdots s_{i_t}\cdots s_{i_k} = s_{i_m+1}\cdots s_{i_t}\cdots s_{i_m+1} = \cdots s_{i_{m+1}}s_{i_t}s_{i_{m+1}}\cdots = \cdots s_{i_t}s_{i_{m+1}}s_{i_t}\cdots .$$

Set $\beta = s_{i_{k-1}}\cdots s_{i_1}\varpi_j$. By induction hypothesis, $s_{i_{m+1}}s_{i_t}s_{i_{m+1}}\cdots$ and $s_{i_t}s_{i_{m+1}}s_{i_t}\cdots$ are admissible. So $\langle \alpha_{i_t}, \beta\rangle \geq 0$ and $\langle \alpha_{i_{m+1}}, \beta\rangle \geq 0$. Again using (6) and (7) we have $\langle s_{i_t}s_{i_{m+1}}\beta, \alpha_{i_{m+1}}\rangle \geq 0$. Then $\langle s_{i_{m+1}}\cdots s_{i_1}\varpi_j, \alpha_{i_{m+1}}\rangle = \langle s_{i_t}s_{i_{m+1}}\beta, \alpha_{i_{m+1}}\rangle \geq 0$, contradicts with $s_{i_{m+1}}\cdots s_{i_1}$ is not admissible.

□

Proof of lemma 16. Set $F_0 = \{\varpi_j\}$. Let F_m be the set which contains all $w\varpi_j$, where $l(w) \leq m$. We use induction. Suppose lemma 2 holds for $\gamma \in F_m$, we will prove lemma holds when $\gamma \in F_{m+1}$. For any $\gamma = w\varpi_j \in F_{m+1}$, by lemma 1, w has a reduced admissible expression: $w = s_{i_{m+1}}\cdots s_{i_1}$. Denote i_{m+1} by i and β by $s_{i_m}\cdots s_{i_1}\varpi_j$. So $\gamma = s_i\beta$, $\beta \in F_m$. Since $s_{i_{m+1}}\cdots s_{i_1}$ is admissible, $\langle \beta, \alpha_i\rangle \geq 0$. Therefore $\langle \gamma, \alpha_i\rangle = \langle s_i\beta, \alpha_i\rangle = -\langle \beta, \alpha_i\rangle \leq 0$ and we can apply prop 4.1 in [BK12]. Then $D_\gamma(M) = D_{s_i(s_i\gamma)}(M) = D_{s_i\gamma}(\Sigma_i M)$, where Σ_i is the reflection functor defined in section 2.2 in [BK12].

Let $A = \{j \mid j$ is adjacent to i, $j \in I\}$ and $M_A = \oplus_{s\in A} M_s$. The i^{th} component of $\Sigma_i M$ is the kernel of the map ξ (Still see section 2.2 in [BK12] for the definition of ξ) from M_A to M_i. Since $\beta \in F_m$, by induction hypothesis, we can apply this lemma to the case where γ is taken to be β and the module M is $\Sigma_i M$. Recall we denote by I_γ^+ the subset of I containing all i such that $\langle \gamma, \check{\alpha}_i \rangle$ is positive and by I_γ^- containing all i $\langle \gamma, \check{\alpha}_i \rangle$ is negative.

Denote $A_+ = \{j \mid j$ is adjacent to i, $j \in I_\gamma^+\}$ and $A_- = A \setminus A_+$. For a multiset S, let $M_S = \oplus M_s^{m(s)}$. Regarding I_γ^- as a multiset by setting $m(i) = \gamma_i^-$, we can rewrite $\oplus_{i\in I_\gamma^-} M_i^{\gamma^-}$ as $M_{I_\gamma^-}$, similarly $\oplus_{i\in I_\gamma^+} M_i^{\gamma^+}$ as $M_{I_\gamma^+}$.

Consider the case when $\langle \gamma, \alpha_i \rangle = -1$. We have $I_\beta^+ = I_{s_i\gamma}^+ = (I_\gamma^+ \setminus A_+) \bigcup \{i\}$ and $I_\beta^- = I_{s_i\gamma}^- = (I_\gamma^- \setminus \{i\}) \bigcup A_-$ as multisets. Therefore $D_{s_i\gamma}(\Sigma_i M)$ is the dimension of the kernel the natural map (which is ϕ_β) from $M_{I_\gamma^+ \setminus A_+} \oplus ker(M_A \xrightarrow{\xi} M_i)$ to $M_{I_\gamma^- \setminus \{i\}} \oplus M_{A_-}$. This is equal to the dimension of the kernel of the natural map from $M_{I_\gamma^+ \setminus A_+} \oplus ker(M_{A_+} \xrightarrow{\xi} M_i)$ to $M_{I_\gamma^+ \setminus \{i\}}$, which is just $ker(M_{I_\gamma}^+ \xrightarrow{\phi_\gamma} M_{I_\gamma^+})$. The case when $\langle \gamma, \alpha_i \rangle = -2$ is similar.

$\square$

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