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Coverings of Graphs and Tiered Trees

Sam Glennon

COVERINGS OF GRAPHS AND TIERED TREES

A Dissertation Presented

by

SAM GLENNON

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Approved as to style and content by:

Paul Gunnells, Chair

Farshid Hajir, Member

Tom Braden, Member

David Mix-Barrington, Member

Farshid Hajir, Department Head
Mathematics

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ABSTRACT

COVERINGS OF GRAPHS AND TIERED TREES

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SAM GLENNON

B.A., BRANDEIS UNIVERSITY

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Paul Gunnells

This dissertation will cover two separate topics. The first of these topics will be coverings of graphs. We will discuss a recent paper by Marcus, Spielman, and Srivastava proving the existence of infinite families of bipartite Ramanujan graphs for all regularities [11]. The proof works by showing that for any d-regular Ramanujan graph, there exists an infinite tower of bipartite Ramanujan graphs in which each graph is a twofold covering of the previous one. Since twofold coverings of a graph correspond to ways of labeling the edges of the graph with elements of a group of order 2, we will generalize the content of [11] by discussing what happens when we label the edges of a graph by larger groups. We will give a version of their proof using threefold coverings instead of twofold coverings. We will also examine ways of reducing the size of the set of twofold coverings that we must consider when we follow the proof in [11].

The other topic that will be covered in this dissertation will be alternating trees and tiered trees. We will define a new generalization of alternating trees, which we will

call *tiered trees*. We will also define a generalized weight system on these tiered trees. We will prove some enumerative results about tiered trees that demonstrate how they can be viewed as being obtained by applying certain procedures to certain types of alternating trees. We also provide a bijection between the set of permutations in S_n and the set of weight 0 alternating trees with $n+1$ vertices. We use this bijection to define a new statistic of permutations called the *weight* of a permutation, and use this weight to define a new q-Eulerian polynomial.

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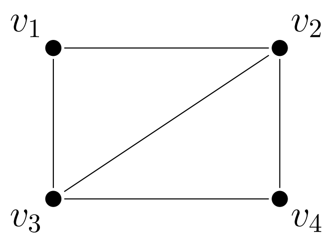
CHAPTER 1

INTRODUCTION

This dissertation will cover two separate topics. The first of these topics will be ways of generalizing the content of a recent paper by Marcus, Spielman, and Srivastava involving signings and coverings of graphs [11]. The second topic will be alternating trees and a generalization thereof called *tiered trees*. All material appearing in chapters 2 and 3 of this dissertation outside of the background sections (Sections 2.1 and 3.1) is original unless otherwise stated.

The well-connectedness of a graph G is deeply related to the spectrum of its adjacency matrix. This matrix is formed by indexing the rows and columns by the vertices of G and letting the uv entry be the number of edges in G that connect the vertices u and v (See Figure 1.1). We say that G is *d-regular* if each vertex is an endpoint of exactly d edges. If G is d-regular, then the adjacency matrix of G will have d as one of its eigenvalues, and if G is bipartite, then it will also have the eigenvalue $-d$ (We sometimes refer to the eigenvalues of the adjacency matrix of G as simply the eigenvalues of G). We call d and $-d$ the *trivial eigenvalues* of G. If all nontrivial eigenvalues of G are between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$, then G is called a *Ramanujan graph*. This bound of $2\sqrt{d-1}$ on the absolute value of the nontrivial eigenvalues becomes sharp as the number of vertices grows in the following sense: for any $\lambda < 2\sqrt{d-1}$, there is no infinite family of d-regular graphs (with the number of vertices growing to infinity) for which all nontrivial eigenvalues are bounded in absolute value by λ.



$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Figure 1.1. An example of a graph and its adjacency matrix

Ramanujan graphs are, in a sense, maximally well-connected, because a graph's spectral gap (the difference between the absolute values of its two largest eigenvalues) and the speed with which a random walk on the graph converges to the uniform distribution are directly related [1, 3]. Prior to [11], the existence of infinite families of d-regular bipartite Ramanujan graphs had only been proven for certain values of d using deep results from number theory, namely the Eichler-Deligne bound on the Fourier coefficients of cuspidal holomorphic modular forms. These families were discovered by Lubotzky, Phillips, and Sarnak, and are known as LPS Graphs [10]. For such graphs and closely related ones due to Morgenstern [12], d must have the form $p^k + 1$ for a prime p.

In a recent paper, Marcus, Spielman, and Srivastava showed that infinite families of d-regular bipartite Ramanujan graphs exist for all degrees d [11]. To prove this, they considered the set of all ways of signing the edges of a graph with either the number 1 or the number -1 and showed that these signings corresponded to twofold coverings of the graph. The eigenvalues of a twofold covering are given by the eigenvalues of a *signed adjacency matrix* determined by the corresponding signing. Marcus, Spielman, and Srivastava showed that the characteristic polynomials of these signed adjacency matrices form an *interlacing family*, which allowed them to conclude that one of these polynomials has roots that are bounded by those of the average over all of these polynomials. Based on results from earlier papers, the roots of this average are known to be within the Ramanujan bounds, so this allowed them to conclude that there is a Ramanujan twofold covering for any bipartite Ramanujan graph.

In Chapter 2 of this dissertation, we discuss ways of generalizing the content of [11]. In Section 2.2, we prove a statement similar to the main result of [11] using threefold coverings instead of twofold coverings. In Section 2.3, we define *resonance conditions* that determine whether a generalized version of Theorem 3.6 in [11] holds. In Section 2.4, we discuss how to reduce the size of the set in which we know a Ramanujan covering of a graph exists. We also define an *extended matching polynomial* that interpolates between the matching polynomial of a graph and the characteristic polynomial of its adjacency matrix. In the next two sections, we consider ways of labeling the edges of a finite graph with a more general group and defining an analog of the signed adjacency matrix. In Section 2.5, we do this by constructing a matrix whose entries are determined by applying a character of the group to the labels of the graph's edges. In Section 2.6 we do this by using a representation of the group instead. We define *extended multi-matchings* on a graph in order to prove a theorem that allows us to apply our analysis from Section 2.4 to the results of a recent paper by Hall, Puder, and Sawin [7] that generalizes [11] by using group representations.

In Chapter 3, we will discuss a type of labeled tree fist discussed by Postnikov called *alternating trees* or *minmax trees* [13]. An alternating tree is a tree whose vertices are labeled with positive integers in such a way that the label of each vertex is either higher than that of all of its neighbors or lower than that of all of its neighbors. These trees have been connected to several other types of combinatorial objects. These include regions of the Linial Arrangement (the affine arrangement on $\mathbb{R}^n$ defined by equations of the form $x_i - x_j = 1$ with $j > i$), local binary search trees (labeled rooted plane binary trees in which every left child of a vertex is smaller than its parent and every right child is greater than its parent) and semiacyclic tournaments (directed complete labeled graphs where every directed cycle has more descending edges than ascending edges) [4, 14]. We work with a weight system on these alternating trees that arises from examining the Tutte polynomials of graphs. The problem of counting weighted alternating trees has been connected to various other counting problems. For example, work of Gunnells, Letellier, and Villegas connects them to Kac polynomials of dandelion quivers and enumeration of torus orbits on homogeneous varieties [6]. We explain the latter connection here. Let S be the maximal torus in PGL_n acting on the Grassmanian $X = Gr(m, n)$. Gunnells, Letellier, and Villegas showed that the number of S-orbits with trivial stabilizer in $X(\mathbb{F}_q)$ is equal to $\sum_T q^{w(t)}$, where the sum is taken over all alternating trees with n vertices and m maxima (this polynomial can also be viewed as a sum of Tutte polynomials of alternating graphs). For example, when $X = Gr(2, 4)$, there are $q + 4$ S-orbits with trivial stabilizer in $X(\mathbb{F}_q)$, and of the 5 alternating trees on 4 vertices with 2 maxima, four of them have weight 0 and one has weight 1.

In Section 3.2, we define a generalization of alternating trees called *tiered trees* that allows vertices to lie in an intermediate "tier" rather than just being local maxima or minima. We also extend our weight system to these tiered trees. In Section 3.3, we show that every weight 0 tiered tree with three tiers can be obtained by moving vertices in an alternating tree into an intermediate tier. This allows us to write formulas for the numbers of weight 0 3-tiered trees where one of the tiers contains only one or two vertices. In Section 3.4, we describe a bijection between the symmetric group S_n and the set of weight 0 alternating trees on $n + 1$ vertices. We then use this bijection to define a statistic of permutations called the *weight* of a permutation. We do this by extending the domain of the map that produces permutations from weight 0 trees to the set of all alternating trees of any weight. We use weights of permutations to define a refinement of the Eulerian polynomial called a q-Eulerian polynomial. The weight of a permutation appears not to be equivalent to any other statistic of permutations that has been formulated before. Similarly, the resulting q-

Eulerian polynomial does not seem to match any other q-Eulerian polynomial that has been studied in the past, so this statistic of permutations can be viewed as revealing new information about the permutations.

CHAPTER 2

CYCLIC RAMANUJAN 3-LIFTS

2.1 Background on 2-lifts

The argument made by Marcus, Spielman, and Srivastava works by showing that there exists some degree 2 covering of any bipartite Ramanujan graph whose eigenvalues fall within the Ramanujan bounds.

Definition 2.1.1. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *2-lift* of G is a graph H with vertex set $\{v_i \mid v \in V(G), i \in \{0,1\}\}$ and edge set $\{e_i \mid e \in E(G), i \in \{0,1\}\}$ with the following properties:
(1) For every edge e connecting vertices v and u in G, each e_i connects some v_j to some u_k.
(2) For every edge e connecting vertices v and u in G, exactly one of the e_i has v_0 as an endpoint and the other has v_1 as an endpoint.
We call the two vertices $\{v_0, v_1\}$ in H corresponding to $v \in v(G)$ the *fibre* of v.

Every edge (v, u) in G corresponds to two edges in the 2-lift. These two edges connect vertices in the fibre of v to vertices in the fibre of u, and this can be done in two ways: The edge pair can be $\{(v_0, u_0), (v_1, u_1)\}$ or $\{(v_0, u_1), (v_1, u_0)\}$. Therefore, 2-lifts of a graph G are in bijection with functions $s\colon E(G) \to \{\pm 1\}$ assigning the value 1 to an edge if the corresponding edge pair in the 2-lift is of the first type, and assigning the value -1 if the corresponding edge pair is of the second type.

In [11], such functions are called *signings*. For example, if $s(u, v) = 1$ for all edges in $E(G)$, the corresponding 2-lift is a disjoint union of two copies of G. Bilu and Linial

[2] proved that the eigenvalues of the adjacency matrix of the 2-lift corresponding to a signing s of a graph G are the union of those of the adjacency matrix A of G, and those of the *signed adjacency matrix* A_s. By definition, A_s is obtained by replacing each 1 in the uv entry of A with $s(u, v)$. Therefore, if G is a d-regular Ramanujan graph, and it has a signing s for which the absolute values of the eigenvalues of A_s are less than $2\sqrt{d-1}$, the corresponding 2-lift is also Ramanujan.

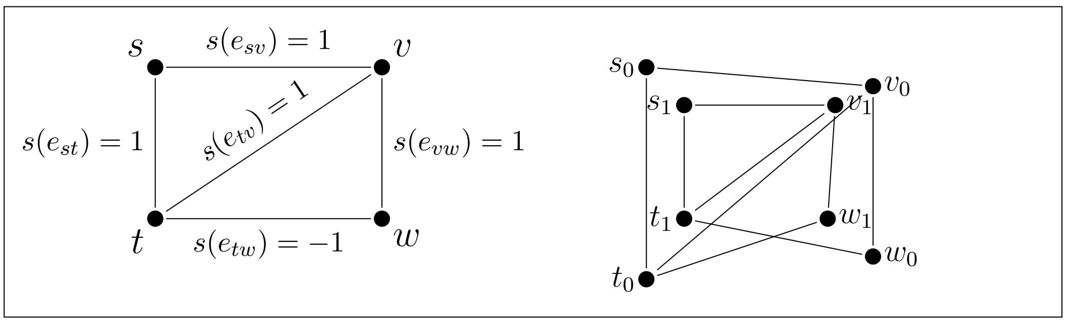


Figure 2.1. A signing of a graph and the correspnding 2-lift

An *i-matching* on a graph G is defined as a disjoint collection of i edges in G. If G has n vertices, we call an $\frac{n}{2}$-matching on G a *perfect matching*. Taking m_i to be the number of i-matchings on G (and $m_0 = 1$), the *matching polynomial* of G is defined to be $\mu_G(x) = \sum_{i \geq 0} x^{n-2i}(-1)^i m_i$. For example, if G is the complete graph on 4 vertices, $\mu_G(x) = x^4 - 6x^2 + 3$, because it has 6 edges and the 3 perfect matchings pictured in Figure 2.2.

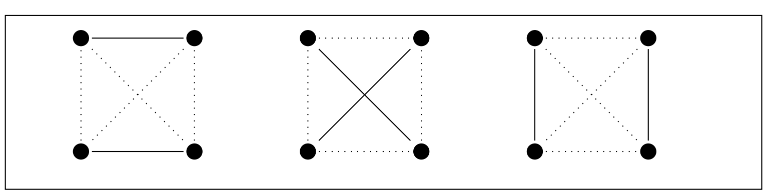


Figure 2.2. The 3 perfect matchings on K_4

Godsil and Gutman [5] showed that the matching polynomial of the graph G is the average signed characteristic polynomial $\mathbb{E}_s[\det(xI - A_s)]$, where $\mathbb{E}_s$ is the expected value over all signings s of G. By [9], the roots of the matching polynomial $\mu_G(x)$ of a d-regular graph G are bounded in absolute value by $2\sqrt{d-1}$. This means that the roots of the average signed characteristic polynomial are within the Ramanujan bounds, but it does not mean that one of these polynomials has roots that achieve these bounds. The key insight that Marcus, Spielman, and Srivastava used to make this leap was to show that the signed characteristic polynomials form an interlacing family (The definition of this term can be found at the beginning of section 2.2.3 of this dissertation). This implies that one of the signed characteristic polynomials has roots that are bounded by the largest root of the average of all signed characteristic polynomials. Since this average is bounded by $2\sqrt{d-1}$, this means that one of these signings gives rise to a Ramanujan 2-lift of G. One can then prove that there is a Ramanujan 2- lift of this Ramanujan 2-lift using the same method, and conclude that there exists an infinite tower of bipartite Ramanujan 2-lifts of d-regular graphs. We remark that this is an existence-only proof, in contrast with the explicit examples of Lubotzky, Phillips, Sarnak, and Morgenstern. This means that we do not have an effective way to find examples of large Ramanujan graphs of arbitrary degree because the number of 2-lifts of a graph grows exponentially with the number of edges, so the resulting search space becomes enormous.

2.2 Cyclic Ramanujan 3-lifts

2.2.1 Eigenvalues for Cyclic n-lifts

To generalize the work of Marcus, Spielman, and Srivastava, we will first define a generalization of 2-lifts of a graph. In this section we will only work with graphs that have at most one edge between any pair of vertices.

Definition 2.2.1. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *n-lift* of G is a graph H with vertex set $\{v_i | v \in V(G), i \in \{1, 2, \ldots, n-1, n\}\}$ and edge set $\{e_i | e \in E(G), i \in \{1, 2, \ldots, n-1, n\}\}$ with the following properties:

(1) For every edge e connecting vertices v and u in G, each e_i connects some v_j to some u_k.

(2) For every edge e connecting vertices v and u in G and each v_i in the fibre of v, exactly one of the e_i has v_i as an endpoint.

Each edge e_i in the fibre of the edge $e = (v, u)$ bijectively connects each v_i to some u_j, so each possible fibre for e can be thought of as corrseponding to a permutation in the symmetric group S_n. Note that, in contrast with the case of 2-lifts, there is a difference between choosing $\{(v_1, u_2), (v_2, u_3), \ldots, (v_n, u_1), \}$ as the fibre of an edge and choosing $\{(v_1, u_n), (v_2, u_1), \ldots, (v_n, u_{n-1})\}$, so when we define a generalization of signings of a graph, we must specify orientations for the edges of the graph. Let G be an oriented d-regular graph with adjacency matrix A. Let H be an n-lift of G. For each edge in G, there are $n!$ possible collections of n edges that could lie above that edge in H, with each possibility corresponding to a permutation in the symmetric group S_n. For a permutation $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, we choose the edges lying above the oriented edge $v \to u$ by adding an edge $v_i \to u_{\sigma(i)}$ for all i. As we are not able to work directly with matrices whose entries are permutations in S_n, we will only consider a smaller class of n-lifts called *cyclic n-lifts*.

Definition 2.2.2. An n-lift of the graph G is called a *cyclic n-lift* if the collection of edges lying above each edge in the base graph corresponds to an element of the cyclic subgroup of S_n generated by the permutation $(12 \cdots n)$.

Next we generalize the definition of a signing of the edges of a graph.

Definition 2.2.3. Let μ_n be the group of nth roots of unity under multiplication. A *labeling* is a map $s\colon E(G) \to \mu_n$ assigning each oriented edge of the graph G to an element of μ_n. We will also use the notation s^m to refer to the labeling defined by $s^m(e) = s(e)^m$.

The different labelings of G are in bijective correspondence with the cyclic n-lifts of G because we can label an edge that is lifted according to the permutation $(123\cdots n)^m$ with the number $e^{2mi\pi/n} = \varphi^m$.

Definition 2.2.4. We define the *labeled adjacency matrix* A_s to be the $|V(G)|\times|V(G)|$ matrix whose ij entry is

$$\begin{cases} s(e_{ij}), & \text{if } G \text{ has an oriented edge } v_i \to v_j \\ s(e_{ji})^{-1}, & \text{if } G \text{ has an oriented edge } v_j \to v_i \\ 0, & \text{if there is no edge connecting the two vertices.} \end{cases}$$

Note that by this definition, labeled adjacency matrices are Hermitian.

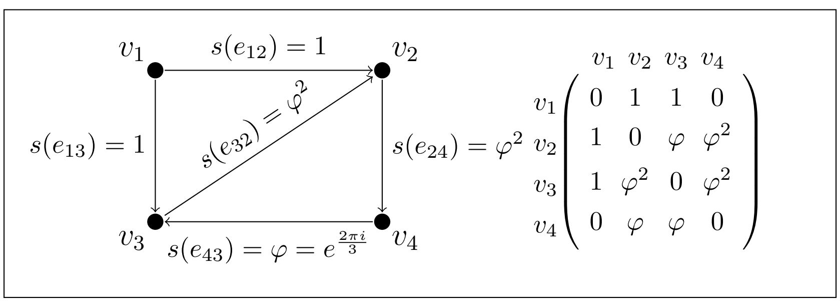


Figure 2.3. An example of a labeled adjacency matrix for a labeling of a graph by μ_3

We can now give our description of the spectra of cyclic n-lifts.

Theorem 2.2.5. *Let G be an oriented graph with adjacency matrix A. Let H be the cyclic n-lift of G corresponding to the labeling s, and let B be the adjacency matrix of H. Then the eigenvalues of B are precisely those of $A_{s^0}, A_{s^1}, \ldots,$ and $A_{s^{n-1}}$ combined.*

Proof. Let A_m be the $|V(G)| \times |V(G)|$ matrix where the ij entry is 1 if G has an oriented edge $v_i \to v_j$ with $s(e_{ij}) = \varphi^m$ or G has an oriented edge $v_j \to v_i$ with

$s(e_{ji}) = \varphi^{-m}$, and the ij entry is 0 otherwise. Note that the ij entry of A_m is also the ji entry of A_{n-m}, so $A_m = A_{n-m}^T$. Under this notation, the adjacency matrix B of the n-lift of G corresponding to this signing can be arranged as

$$\begin{pmatrix} A_0 & A_1 & \dots & A_{n-1} \\ A_{n-1} & A_0 & \dots & A_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_0 \end{pmatrix}.$$

We can decompose A_s as

$$A_s = A_0 + \varphi A_1 + \varphi^2 A_2 + \dots + \varphi^{n-1} A_{n-1}.$$

We can similarly decompose A_{s^m} as

$$A_{s^m} = A_0 + \varphi^m A_1 + \varphi^{2m} A_2 + \dots + \varphi^{m(n-1)} A_{n-1}.$$

Therefore, if v is an eigenvector of A_{s^m}, then

$$\begin{pmatrix} v \\ \varphi^m v \\ \varphi^{2m} v \\ \vdots \\ \varphi^{-m} v \end{pmatrix}$$

is an eigenvector of B with the same eigenvalue. We now show that these vectors are independent. Let $\{v_{i,m}\}$ be the eigenvectors of A_{s^m}. Suppose that for some constants $\{a_{i,m}\}$

$$\sum_{i,m} a_{i,m} \begin{pmatrix} v_{i,m} \\ \varphi^m v_{i,m} \\ \varphi^{2m} v_{i,m} \\ \vdots \\ \varphi^{-m} v_{i,m} \end{pmatrix} = 0.$$

Then for any m, we get the equation

$$\sum_i a_{i,0} v_{i,0} + \varphi^m \sum_i a_{i,1} v_{i,1} + \cdots + \varphi^{m(n-1)} \sum_i a_{i,n-1} v_{i,n-1}.$$

Since the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varphi & \varphi^2 & \dots & \varphi^{n-1} \\ 1 & \varphi^2 & \varphi^4 & \dots & \varphi^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \varphi^{n-1} & \varphi^{2(n-2)} & \dots & \varphi^{(n-1)^2} \end{pmatrix}$$

is invertible (its 4th power is $n^2 I$), this can only happen if each $\sum\limits_i a_{i,m} v_{i,m}$ is 0. However, since for each m, the $v_{i,m}$ are the eigenvectors of A_{s^m}, they are linearly independent. This means that each $a_{i,m}$ must be 0, so the eigenvectors of B we have found are independent. Therefore, the eigenvalues of B are precisely those of $A_{s^0}, A_{s^1}, A_{s^2}, \dots,$ and $A_{s^{n-1}}$ combined. □

In the case of 3-lifts, this means that the only new eigenvalues are those of A_s because $A_{s^2} = \overline{A_s}$, so the two matrices have the same eigenvalues, and $A_{s^0} = A$, so its eigenvalues are the old eigenvalues.

Corollary 2.2.6. *Let B be the adjacency matrix of the 3-lift of the graph G corresponding to the labeling s of G. Then every eigenvalue of B is an eigenvalue of either the adjacency matrix A of G or the labeled adjacency matrix A_s.*

2.2.2 Average Labeled Characteristic Polynomial

Let $f_s(x) = \det(xI - A_s)$. A generalization of the proof of Theorem 3.6 in Marcus, Spielman, and Srivastava shows that the expected characteristic polynomial of A_s is the matching polynomial.

Theorem 2.2.7. *Let G be a graph with m vertices. Let $\mathbb{E}_{s\in\mu_n^m}[f_s(x)]$ denote the average of the characteristic polynomials $f_s(x)$ over all possible labelings of the graph $s \in \mu_n^m$. Then we have*

$$\mathbb{E}_{s\in\mu_n^m}[f_s(x)] = \mu_G(x).$$

Proof. For any set S, let $\mathrm{Sym}(S)$ be the set of permutations of the set S and let $\mathrm{Der}(S)$ be the subset of Sym(S) consisting of the permutations with no fixed points (also known as *derangements*). Let $|\pi|$ be the number of inversions in the permutation π (i.e $|\pi|$ is even if and only if π is an even permutation). For any m, let $[\![m]\!]$ be the finite set $\{1, \ldots, m\}$. Let s_{ij} denote the ij entry of A_s.

$\mathbb{E}_s[\det(xI - A_s)]$
$= \mathbb{E}_s[\sum_{\sigma\in\mathrm{Sym}([\![m]\!])} (-1)^{|\sigma|} \prod_{i=1}^{m} (xI - A_s)_{i,\sigma(i)}]$
$= \sum_{k=0}^{m} x^{m-k} \sum_{S\subset[\![m]\!],|S|=k} \sum_{\pi\in\mathrm{Der}(S)} \mathbb{E}_s[(-1)^{|\pi|} \prod_{i\in S} s_{i,\pi(i)}](-1)^{|S|}.$

Note that each s_{ij} is independent from all other entries except for s_{ji} and $\mathbb{E}[s_{ij}] = 0$, so only the products given by permutations π consisting entirely of 2-cycles survive. Furthermore, the product will be 0 unless each of these 2-cycles in the permutation consists of vertices in G that are connected by an edge. These permutations represent the perfect matchings on $G(S)$, where $G(S)$ denotes the subgraph of G consisting of the vertices indexed by S and all edges in G connecting these vertices. These perfect matchings consist of $|S|/2$ inversions when $|S|$ is even, and they do not exist when $|S|$ is odd. Since s_{ij} and s_{ji} are inverse roots of unity when G has an $i \to j$ edge, this gives us

$\mathbb{E}_s[\det(xI - A_s)]$

$= \sum_{k=0}^{m} x^{m-k} \sum_{|S|=k} \sum_{\text{perfect matchings } \pi \text{ on } G(S)} (-1)^{|S|/2} \cdot 1$
$= \mu_G(x)$

This completes the proof of the theorem. □

2.2.3 Interlacing Families

Definition 2.2.8. We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces* a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n$$

We say that polynomials $f_1, \ldots, f_k$ have a common interlacing if there exists a g that interlaces each f_i.

Let $S_1, \ldots, S_m$ be finite sets and for each assignment $s_1, \ldots, s_m \in S_1 \times \cdots \times S_m$, let $f_{s_1,\ldots,s_m}(x)$ be a real-rooted polynomial of degree n with a positive leading coefficient. For a partial assignment $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$ with $k < m$, we define

$$f_{s_1,\ldots,s_k} = \sum_{s_{k+1} \in S_{k+1}, \ldots, s_m \in S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m}$$

and

$$f_{\emptyset} = \sum_{s_1 \in S_1, \ldots, s_m \in S_m} f_{s_1,\ldots,s_m}.$$

We call $\{f_{s_1,\ldots,s_k}\}_{s_1,\ldots,s_k}$ an *interlacing family* if for all $k < m$ and all $s_1, \ldots, s_k \in S_1 \times \cdots \times S_k$, the polynomials in $\{f_{s_1,\ldots,s_k,t}\}_{t \in S_{k+1}}$ have a common interlacing.

We can generalize the argument of Marcus, Spielman, and Srivastava to show that our more general labeled characteristic polynomials also form an interlacing family. To do this, we will prove a statement analogous to Theorem 5.1 in their paper.

Theorem 2.2.9. *Let $p_{1,0}, p_{1,1}, \ldots, p_{1,n-1}, p_{2,0}, \ldots, p_{m,n-1}$ be numbers in [0,1] where $p_{i,0} + \cdots + p_{i,n-1} = 1$ for any fixed i. Then the following polynomial is real-rooted:*

$$\sum_{s\in\mu_n^m}(\prod_{i:s_i=1} p_{i,0})(\prod_{i:s_i=\varphi} p_{i,1})\cdots(\prod_{i:s_i=\varphi^{n-1}} p_{i,n-1})f_s(x)$$

To prove this, we will use a generalized version of the argument of Marcus, Spielman, and Srivastava using real stable polynomials.

Definition 2.2.10. The polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is *real stable* if it is either the zero polynomial or $f(z_1, \ldots, z_n) \neq 0$ whenever each z_i has a strictly positive imaginary part.

We will use the following facts about real stable polynomials:

Lemma 2.2.11. *Let $A_1, \ldots, A_m$ be positive semidefinite matrices. Then the polynomial $\det(z_1A_1 + \cdots + z_mA_m)$ is real stable.*

Lemma 2.2.12. *If $f(z_1, \ldots, z_n)$ is real stable, then $f(z_1, \ldots, z_{n-1}, c)$ is real stable for any real number c.*

Lemma 2.2.13. *Let $p_0, \ldots, p_{n-1}$ be non negative real numbers and let $u_0, \ldots, u_{n-1}$ be variables. Let ∂_x denote the operation of partial differentiation with respect to x. The operator $T = 1 + p_0\partial_{u_0} + \cdots + p_{n-1}\partial_{u_{n-1}}$ preserves real stability.*

The first two of these three statements appear directly in Marcus, Spielman, and Srivastava. The third is a slight generalization of Corollary 6.4, to which the same proof applies [11].

Lemma 2.2.14. *Let A be an $m \times m$ invertible matrix with complex entries, let $a_0, \ldots, a_{n-1}$ be vectors in $\mathbb{C}^m$, and let $p_0, \ldots, p_{n-1}$ be non-negative real numbers that add up to 1. Let Z_{u_i} be the operator defined by setting the variable u_i to 0. Then we have*

$$Z_{u_0}\cdots Z_{u_{n-1}}(1 + p_0\partial_{u_0} + \cdots + p_{n-1}\partial_{u_{n-1}})\det(A + u_0a_0\overline{a_0^T} + \cdots + u_{n-1}a_{n-1}\overline{a_{n-1}^T})$$
$$= p_0\det(A + a_0\overline{a_0^T}) + \cdots + p_{n-1}\det(A + a_{n-1}\overline{a_{n-1}^T}).$$

Proof. By the matrix determinant lemma, for any invertible matrix A and real number t,

$$\det(A + ta\overline{a^T}) = \det(A)(1 + t\overline{a^T}A^{-1}a)$$

Applying ∂_t to this gives us

$$\partial_t \det(A + ta\overline{a^T}) = \det(A)(\overline{a^T}A^{-1}a)$$

This implies:

$$Z_{u_0} \ldots Z_{u_{n-1}}(1 + p_0\partial_{u_0} + \ldots + p_{n-1}\partial_{u_{n-1}}) \det(A + u_0 a_0\overline{a_0^T} + \ldots + u_{n-1}a_{n-1}\overline{a_{n-1}^T}) = \det(A)(1 + p_0(\overline{a_0^T}A^{-1}a_0) + \ldots + p_{n-1}(\overline{a_{n-1}^T}A^{-1}a_{n-1}))$$

Decomposing 1 into a sum of the p_i and applying the matrix determinant lemma, we see that this is equal to:

$$p_0 \det(A + a_0\overline{a_0^T}) + \ldots + p_{n-1} \det(A + a_{n-1}\overline{a_{n-1}^T})$$

□

Theorem 2.2.15. *Let $a_{1,0}, \ldots, a_{1,n-1}, a_{2,0}, \ldots, a_{m,n-1}$ be vectors in $\mathbb{C}^m$, and let $p_{1,0}$, $\ldots, p_{1,n-1}$, $p_{2,0}, \ldots, p_{m,n-1}$ be non-negative real numbers where for any fixed i, $p_{i,0} + \ldots + p_{i,n-1} = 1$. Let D be a positive semidefinite $m \times m$ matrix. Then the following polynomial is real-rooted:*

$$P(x) = \sum_{partitions\ \{S_0,\ldots,S_{n-1}\}\ of\ [\![m]\!]} (\prod_{i\in S_0} p_{i,0}) \ldots (\prod_{i\in S_{n-1}} p_{i,n-1}) det(xI + D +$$

$$\sum_{i\in S_0} a_{i,0}\overline{a_{i,0}^T} + \cdots + \sum_{i\in S_{n-1}} a_{i,n-1}\overline{a_{i,n-1}^T})$$

Proof. Let $u_{1,0}, \ldots, u_{1,n-1}, u_{2,0}, \ldots, u_{m,n-1}$ be variables and define

$$Q(x, u_{1,0}, \ldots, u_{1,n-1}, u_{2,0}, \ldots, u_{m,n-1}) =$$
$$det(xI + D + \sum_i u_{i,0}a_{i,0}\overline{a_{i,0}^T} + \cdots + \sum_i u_{i,n-1}a_{i,n-1}\overline{a_{i,n-1}^T})$$

By lemma 2.2.11, Q is real stable.

Let $T_i = 1 + p_{i,0}\partial_{u_{i,0}} + \cdots + p_{i,n-1}\partial_{u_{i,n-1}}$. We claim that

$$P(x) = (\prod_{i=1}^{m} Z_{u_{i,0}} \ldots Z_{u_{i,n-1}} T_i) Q(x, u_{1,0}, \ldots, u_{1,n-1}, u_{2,0}, \ldots, u_{m,n-1})$$

We show by induction that

$$(\prod_{i=1}^{k} Z_{u_{i,0}} \ldots Z_{u_{i,n-1}} T_i) Q(x, u_{1,0}, \ldots, u_{1,n-1}, u_{2,0}, \ldots, u_{m,n-1})$$
$$= \sum_{\text{partitions } \{S_0,\ldots,S_{n-1}\} \text{ of } [\![k]\!]} (\prod_{i \in S_0} p_{i,0}) \ldots (\prod_{i \in S_{n-1}} p_{i,n-1}) det(xI + D + \sum_{i \in S_0} a_{i,0}\overline{a_{i,0}^T} + \cdots +$$
$$\sum_{i \in S_{n-1}} a_{i,n-1}\overline{a_{i,n-1}^T} + \sum_{i>k} (u_{i,0}a_{i,0}\overline{a_{i,0}^T} + \cdots + u_{i,n-1}a_{i,n-1}\overline{a_{i,n-1}^T}))$$

The $k = 0$ case is the definition of Q, the induction step follows from Lemma 2.2.14, and the $k = m$ case is our claim. Since each Z and T operator preserves real stability, this means that P is real stable, and a real stable polynomial in one variable must be real-rooted.

□

Let d be the regularity of G. To prove that

$$\sum_{s \in \mu_n^m} (\prod_{i:s_i=1} p_{i,0})(\prod_{i:s_i=\varphi} p_{i,1}) \ldots (\prod_{i:s_i=\varphi^{n-1}} p_{i,n-1}) det(xI - A_s) \tag{2.1}$$

is real-rooted, we can prove that

$$\sum_{s \in \mu_n^m} (\prod_{i:s_i=1} p_{i,0})(\prod_{i:s_i=\varphi} p_{i,1}) \ldots (\prod_{i:s_i=\varphi^{n-1}} p_{i,n-1}) det(xI + dI - A_s) \tag{2.2}$$

is real-rooted because their roots differ by d.

Let e_u be the elementary unit vector in the u direction. For each edge (u, v) we define n different rank 1 matrices:

$$L_{u,v}^{\varphi^j} = (e_u - \varphi^j e_v)(e_u - \varphi^{-j} e_v)^T$$

This allows us to write $dI - A_s$ as

$$\sum_{(u,v)\in E} L_{u,v}^{s(u,v)}$$

so if we set $D = 0$ and $(e_u - \varphi^j e_v) = a_{u,v,j}$, we can rewrite the polynomial (2) as

$$\sum_{s\in\mu_n^m} (\prod_{i:s_i=1} p_{i,0})(\prod_{i:s_i=\varphi} p_{i,1}) \dots (\prod_{i:s_i=\varphi^{n-1}} p_{i,n-1}) det(xI + \sum_{s(u,v)=1} a_{u,v,1}\overline{a_{u,v,1}^T} + \dots + \sum_{s(u,v)=\varphi^{n-1}} a_{u,v,\varphi^{n-1}}\overline{a_{u,v,\varphi^{n-1}}^T})$$

Since this is of the form appearing in Theorem 2.2.15, it is real-rooted.

2.2.4 Cyclic Ramanujan 3-lifts

It now follows from Theorem 2.2.9 that the labeled characteristic polynomials form an interlacing family:

Theorem 2.2.16. *The $\{f_s\}_{s\in\mu_n^m}$ are an interlacing family.*

Proof. Lemma 4.5 in [11] states that a set of polynomials of the same degree with positive leading coefficients has a common interlacing if and only if all convex combinations of those functions are real-rooted. Therefore, it suffices to show that for any $k \leq m-1$, any partial assignment $s_1 \in \mu_n^m, \dots, s_k \in \mu_n^m$, and every set of non-negative real numbers $p_0, \dots, p_{n-1}$ with $p_0 + \dots + p_{n-1} = 1$, the following polynomial is real-rooted:

$$p_0 f_{s_1,\dots,s_k,1}(x) + \dots + p_{n-1} f_{s_1,\dots,s_k,\varphi^{n-1}}(x)$$

This follows from Theorem 2.2.9 when we set $p_{k+1,j} = p_j$, $p_{k+2,j}, \dots, p_m = 1/n$ for all j, and $p_{i,j} = 1$ if $s_i = \varphi^j$, and 0 otherwise. $\square$

By Theorem 4.4 in [11], this means that there exists some labeling $s = (s_1, \ldots, s_m)$ for which the largest root of $f_{s_1,\ldots,s_m}$ is less than the largest root of $f_\emptyset$. By Theorem 2.2.7, $f_\emptyset$ is a scalar multiple of the matching polynomial. Lemma 3.5 in [11] states that the roots of this polynomial are bounded in absolute value by $2\sqrt{d-1}$. This means that there is some labeling s for which highest root of f_s is within the Ramanujan bounds. If G is bipartite, this implies that all roots of f_s are within the Ramanujan bounds because the spectrum of a bipartite graph is symmetric. Therefore, if $n = 3$ and the base graph is Ramanujan, this labeling corresponds to a Ramanujan graph because all of this graph's new eigenvalues are roots of $f_{s_1,\ldots,s_m}$. It is always possible to choose a bipartite Ramanujan base graph for any regularity d. For, example we could start with the d-regular complete bipartite graph on $2d$ vertices, which is a bipartite Ramanujan graph. This allows us to conclude that infinite towers of cyclic Ramanujan 3-lifts of graphs on any regularity exist.

Theorem 2.2.17. *For any integer $d \geq 2$, there exists an infinite tower of bipartite Ramanujan d-regular graphs $\{G_1, G_2, G_3, \ldots\}$ in which each G_{i+1} is a cyclic 3-lift of G_i.*

2.3 Resonance and Higher Degrees

If we try to adapt our argument to lifts of higher degrees, we will need to consider polynomials of higher degrees that have all of the new eigenvalues of a lift as roots. We can do this by working with products of characteristic polynomials of different powers of the same labeling $f_{s^{i_1}}(x)f_{s^{i_2}}(x)\ldots f_{s^{i_k}}(x)$. For example, if $n = 4$ or $n = 5$, the new eigenvalues of the n-lift associated with the labeling s are the roots of $f_s(x)f_{s^2}(x)$. We will, therefore, examine the averages of these products over all labelings.

Definition 2.3.1. Suppose that $\{a_1, \ldots, a_k\}$ is a solution to the equation $a_1i_1 + \cdots + a_ki_k \equiv 0 \bmod n$ with each $a_i \in \{-1, 0, 1\}$. Then we call this solution a *resonance* *condition* on the exponents $i_1, \dots, i_k$ mod n. We call the solution where each $a_i = 0$ the *trivial resonance condition.*

Theorem 2.3.2. *Let s be a labeling of the graph G by nth roots of unity. Then there is no nontrivial resonance condition on the exponents $i_1, \dots, i_k$ mod n if and only if we have*

$$\mathbb{E}_s[f_{s^{i_1}}(x) f_{s^{i_2}}(x) \dots f_{s^{i_k}}(x)] = \mu_G(x)^k.$$

Proof. By expanding each determinant, we find that the expression $\mathbb{E}_s[\det(xI - A_{s^{i_1}}) \dots \det(xI - A_{s^{i_k}})]$ is equal to

$$\mathbb{E}_s[(\sum_{\sigma_1 \in \mathrm{Sym}([\![m]\!])} (-1)^{|\sigma_1|} \prod_{i=1}^{m} (xI - A_{s^{i_1}})_{i,\sigma_1(i)}) \cdot \dots \cdot (\sum_{\sigma_k \in \mathrm{Sym}([\![m]\!])} (-1)^{|\sigma_k|} \prod_{i=1}^{m} (xI - A_{s^{i_k}})_{i,\sigma_k(i)})].$$

The coefficient of x^{kn-K} in this expression is

$$\sum_{\substack{S_1,\dots,S_k \subset [\![n]\!] \\ |S_1|+\dots+|S_k|=K}} \sum_{\pi_1 \in \mathrm{Der}(S_1),\dots,\pi_k \in \mathrm{Der}(S_k)} \mathbb{E}_s[(-1)^{|\pi_1|} \prod_{i \in S_1} s_{i,\pi_1(i)}^{i_1} \dots (-1)^{|\pi_k|} \prod_{i \in S_k} s_{i,\pi_k(i)}^{i_k}](-1)^K.$$

We can reduce a product of products of the form $\prod_{i \in S_1} s_{i,\pi_1(i)}^{i_1} \cdots \prod_{i \in S_k} s_{i,\pi_k(i)}^{i_k}$ into a product of independent random variables by combining powers of $s_{i,j}$'s and writing $s_{j,i}$ as $s_{i,j}^{-1}$. When we do so, the exponent of each $s_{i,j}$ will be of the form $a_1 i_1 + \dots + a_k i_k$ with each $a_i \in \{-1, 0, 1\}$. If this exponent is not divisible by n, then $\mathbb{E}[s_{i,j}^{a_1 i_1 + \dots + a_k i_k}] = 0$. Therefore, if the only solution to the equation $a_1 i_1 + \dots + a_k i_k \equiv 0$ mod n with each $a_i \in \{-1, 0, 1\}$ is the trivial solution, the only surviving terms are those in which each π_i represents a perfect matching on $G(S_i)$ (There are other surviving terms if we do not have this condition). If we have this condition, then the coefficient of x^{kn-K} then becomes 0 if K is odd and

$$(-1)^{\frac{K}{2}} \sum_{\{(C_1,\dots,C_k) \in \mathbb{Z}_{\geq 0}^k | C_1 + \dots + C_k = K/2\}} m(G, C_1) \dots m(G, C_k)$$

if K is even, where $m(G,N)$ denotes the number of matchings on the graph G consisting of N edges. Since these are the coefficients of $\mu_G(x)^k$, this means that if there is no nontrivial resonance condition, the two polynomials are equal.

To prove the converse, let α be the minimal number of nonzero coefficients in any nontrivial resonance condition on the exponents $i_1, \ldots, i_k$ mod n, and suppose that there are N resonance conditions with this minimal number of nonzero coefficients. Let β be the minimal number of edges in any cycle in the graph G, and suppose that there are M β-cycles in the graph. We observe from our formula for the coefficient of x^{kn-K} that the coefficients of $x^{kn-\alpha\beta}$ in $\mathbb{E}_s[f_{s^{i_1}}(x)f_{s^{i_2}}(x)\ldots f_{s^{i_k}}(x)]$ and $\mu_G(x)^k$ differ by $2NM$.

□

Corollary 2.3.3. *Suppose that for all resonance conditions on the exponents* $i_1, \ldots, i_k$ *mod* n*,* $a_1 = \cdots = a_w = 0$*. Then* $\mathbb{E}_s[f_{s^{i_1}}(x)f_{s^{i_2}}(x)\ldots f_{s^{i_k}}(x)]$ *is divisible by* $\mu_G(x)^w$*.*

Proof. For each $i \leq w$, the only surviving terms will be those in which π_i represents a perfect matching on S_i. For such a permutation, we have $\prod_{i\in S_1} s_{i,\pi_1(i)}^{i_1} = 1$. Therefore, we can rewrite the coefficient of x^{kn-K} in $\mathbb{E}_s[f_{s^{i_1}}(x)f_{s^{i_2}}(x)\ldots f_{s^{i_k}}(x)]$ as

$$\sum(-1)^{\frac{|S_1|+\cdots+|S_w|}{2}}\mathbb{E}_s[(-1)^{|\pi_{w+1}|}\prod_{i\in S_{w+1}} s_{i,\pi_{w+1}(i)}^{i_{w+1}}\ldots(-1)^{|\pi_k|}\prod_{i\in S_k} s_{i,\pi_k(i)}^{i_k}](-1)^K,$$

where the sum is taken over the set $\{(S_1,\ldots,S_k,\pi_1,\ldots,\pi_w,\pi_{w+1},\ldots,\pi_k) \mid S_1,\ldots,S_k \subset [\![n]\!], |S_1|+\cdots+|S_k| = K, \pi_i$ is a perfect matching on $G(S_i)$ if $i \leq w, \pi_i \in \mathrm{Der}(S_i)$ if $i > w\}$, so $\mathbb{E}_s[f_{s^{i_1}}(x)f_{s^{i_2}}(x)\ldots f_{s^{i_k}}(x)]$ is divisible by $\mu_G(x)^w$ □

We can use this result to see which parts of our argument from the previous sections can and cannot be applied to n-lifts with $n > 3$. Note that, for $n > 3$, there is no nontrivial resonance condition on the exponents 1 and 2 mod n, so the preceding theorem is applicable to the polynomials $f_s(x)f_{s^2}(x)$, and we are able to conclude that $\mathbb{E}_s[f_s(x)f_{s^2}(x)]$ is real-rooted and its roots are within the Ramanujan bound. However, this does not allow us to conclude that one of these polynomials has roots within the Ramanujan bound because they do not form an interlacing family in the same way that the polynomials $f_s(x)$ do. For $n > 5$, the eigenvalues of the lift corresponding to the labeling s will include the roots of $f_s(x)$, $f_{s^2}(x)$, and $f_{s^3}(x)$, and we have a nontrivial resonance condition given by $1 + 2 - 3 = 0$. Therefore, we do not have a result placing the roots of the average of the desired polynomials within the Ramanujan bound.

2.4 Fixed Subgraphs

We now fix a subgraph H of the graph G. We will consider the average characteristic polynomial $\mathbb{E}_s[\det(xI - A_s)]$ over all labelings s of G by the multiplicative group of nth roots of unity μ_n in which every edge in H is mapped to the identity permutation. The resulting polynomial can be interpreted combinatorially as an extended matching polynomial.

Definition 2.4.1. We define an *extended matching* m on the graph G with respect to H to be a disjoint union of edges in G and cycles in H. The *extended matching polynomial* of G with respect to H is then defined as

$$\mu_{G,H}(x) = \textstyle\sum_m (-1)^{c(m)} 2^{z(m)} x^{N-v(m)}.$$

Here, the sum is taken over all extended matchings m on G with respect to H, $c(m)$ is the number of edges and cycles in the extended matching, $z(m)$ is the number of cycles in the extended matching, N is the number of vertices in G, and $v(m)$ is the number of vertices that are included in an edge or cycle of m.

This is a generalization of an interpretation of the characteristic polynomial given in [8]. Once extended matchings are defined, we can prove the following:

Theorem 2.4.2. *Let L_H be the set of labelings s of G by a cyclic group for which $s(e) = 1$ for all edges e in H. Then we have $\mathbb{E}_{s\in L_H}[\det(xI - A_s)] = \mu_{G,H}(x)$.*

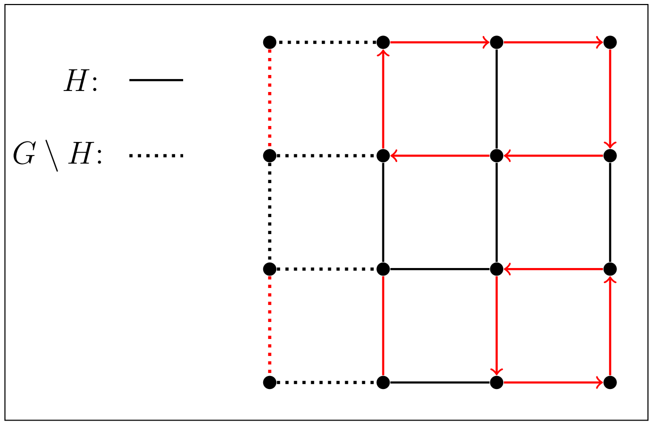


Figure 2.4. An extended matching on G with respect to H (The edges and cycles in the extended matching appear in red)

Proof. Let s_{ij} denote the ij entry of A_s. As in the proof of the analogous statement for labeled adjacency matrices, we have

$$\mathbb{E}_s[\det(xI - A_s)] = \sum_{k=0}^{N} x^{N-k} \sum_{S \subset [\![n]\!], |S|=k} \sum_{\pi \in \mathrm{Der}(S)} \mathbb{E}_s[(-1)^{|\pi|} \prod_{i \in S} s_{i,\pi(i)}](-1)^{|S|}.$$

For the edges $e_{ij} \in G \setminus H$, $\mathbb{E}[s_{ij}] = 0$ and these s_{ij} are independent from all entries other than s_{ji}. Therefore, the surviving products will be those consisting of 2-cycles and cycles whose edges all come from H, so these permutations correspond to perfect extended matchings on $G(S)$ with respect to $G(S) \cap H$. Note that for an extended matching m with $z(m)$ cycles, there are $2^{z(m)}$ permutations corresponding to m because each cycle has two inverse permutations corresponding to it. We also observe that the number of cycles of odd size (i.e. cycles with an even number of involutions) in a perfect extended matching m is even if and only if $|S|$ is even. Therefore, the sign of term contributed by m is $(-1)^{c(m)}$, and the entire contribution of m is $(-1)^{c(m)} 2^{z(m)} x^{N-v(m)}$. $\square$

Note that we could have alternatively defined extended matchings as disjoint unions of edges in G and oriented cycles in H, in which case we would no longer include the $2^{z(m)}$ in the definition of $\mu_{G,H}$, and we would be able to reach the same conclusion as in Theorem 2.4.2. We will refer to extended matchings by this definition as *partially oriented* extended matchings.

Observe that when we take $H = \emptyset$, we simply get the definition of the matching polynomial (since H contains no cycles), so $\mu_{G,\emptyset}(x) = \mu_G(x)$. On the other hand, if we take $H = G$, we find that the coefficient of $x^{N-v(m)}$ in $\mu_{G,G}$ is $\sum_m (-1)^{c(m)} 2^{z(m)}$ where the sum is taken over all subgraphs of G with $v(m)$ vertices whose components are all edges or cycles. Sachs showed that this polynomial is the characteristic polynomial of G [15], so the extended characteristic polynomial interpolates between the characteristic polynomial and matching polynomial of G. For any choice of fixed subgraph H, it is clear that the signed characteristic polynomials form an interlacing family because we already know that this is the case when $H = \emptyset$, which is a stronger condition. Therefore, the argument used by Marcus, Spielman, and Srivastava can be used to show the following:

Theorem 2.4.3. *For any bipartite graph G and subgraph H, there is a 2-lift of G that is trivial over the subgraph H whose largest new eigenvalue is no larger than the largest root of $\mu_{G,H}(x)$.*

This argument also works if we choose a non-trivial signing to fix the edges in H with. In particular, if G is d-regular and Ramanujan and the largest root of $\mu_{G,H}(x)$ is less than $2\sqrt{d-1}$, then one of these 2-lifts is also Ramanujan. If we can find conditions on H for which for which this condition holds, then we can narrow down the search space that [11] entails because for every edge that we include in H, we reduce the search space by a factor of two. We see a very simple example of this when we take H to be a spanning tree of G; since H contains no cycles, we have $\mu_{G,H}(x) = \mu_G(x)$, and we know that the roots of $\mu_G(x)$ are within the Ramanujan bound, so we are able to conclude that there is a Ramanujan 2-lift of G corresponding to a signing for which $s(e) = 1$ for all $e \in H$. If G has N vertices, this means that we only need to check one in every 2^{N-1} 2-lifts of G instead of all of them.

As an example where H is not contractible, we take G to be the complete bipartite graph $K_{3,3}$ and H to be a hexagonal subgraph. If we require the lifts to be trivial over H, the average signed characteristic polynomial that we get is $x^6 - 9x^4 + 18x^2 - 8$. Since the largest root of this polynomial is approximately 2.5243, which is less than $2\sqrt{2}$, we can conclude that there is a Ramanujan 2-lift of G in which the fibre of H is a pair of disjoint hexagons. On the other hand, we could also only consider signings in which the sign of the first edge of H is -1, and the rest of the edges in H have a sign of 1. In this case, the average signed characteristic polynomial that we get is $x^6 - 9x^4 + 18x^2 - 4$. The largest root of this polynomial is approximately 2.4903, which is also less that $2\sqrt{2}$, so there is also a Ramanujan 2-lift of G in which the fibre of H is a 12-gon.

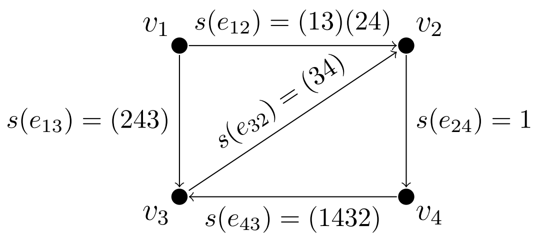
2.5 Characters

To generalize our results, we would like to analyze labelings of graphs by non-cyclic groups. To do so, we need a way of making an adjacency matrix out of a labeling, but in order for this to make sense, the entries need to be made into complex numbers rather than elements of a finite group. One way of doing this is by applying characters to the group elements. Let χ be the character of an irreducible non-trivial representation ρ of the finite group Γ. Let G be a d-regular directed graph. Let $s : E^+(G) \to \Gamma$ be a map assigning an element of S to each oriented edge of G. We call s a Γ-*labeling* of G. Note that by this definition labelings as we have previously defined them are S_n-labelings. Let $A_{s,\chi}$ be the matrix whose u, v entry is

$$\begin{cases} \chi(s(u,v)) & \text{if } (u,v) \text{ is an oriented edge,} \\ \chi(s(v,u)^{-1}) & \text{if } (v,u) \text{ is an oriented edge,} \\ 0 & \text{if there is no oriented edge } (u,v) \text{ or } (v,u). \end{cases}$$

.

See Figure 2.5 for an example of a matrix $A_{s,\chi}$ where χ is the unique irreducible 2-dimensional character of S_4 whose table can be found in Figure 2.6. Like with labeled adjacency matrices, we now consider the average characteristic polynomial of $A_{s,\chi}$ over all Γ-labelings of G.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 2.5. An example of an S_4 labeling s of a graph and the matrix $A_{s,\chi}$ where χ is the unique irreducible 2-dimensional character of S_4.

class:	1	(12)	(123)	(1234)	(12)(34)
χ	2	0	-1	0	2

Figure 2.6. The character table of χ.

Theorem 2.5.1. $\mathbb{E}_s[\det(xI - A_{s,\chi})] = \mu_G(x)$

Proof. Let s_{ij} denote the ij entry of $A_{s,\chi}$. As in the proof of the analogous statement for labeled adjacency matrix, we have

$$\mathbb{E}_s[\det(xI - A_{s,\chi})] = \textstyle\sum_{k=0}^{m} x^{m-k} \displaystyle\sum_{S \subset [\![n]\!], |S|=k} \sum_{\pi \in \mathrm{Der}(S)} \mathbb{E}_s[(-1)^{|\pi|} \prod_{i \in S} s_{i,\pi(i)}](-1)^{|S|}.$$

Again, we have that each s_{ij} is independent from all other entries except for s_{ji} and $\mathbb{E}[s_{ij}] = 0$ (due to orthogonality with the trivial character), so only the products given by permutations π consisting entirely of 2-cycles survive. By the Schur orthogonality relation, $\mathbb{E}_s[s_{ij}s_{ji}] = 1$ for any pair of vertices $\{i, j\}$ that are connected by an edge of the graph. These permutations represent the perfect matchings on $G(S)$. Therefore, as before, the only contributing permutations represent perfect matchings on $G(S)$ and we find

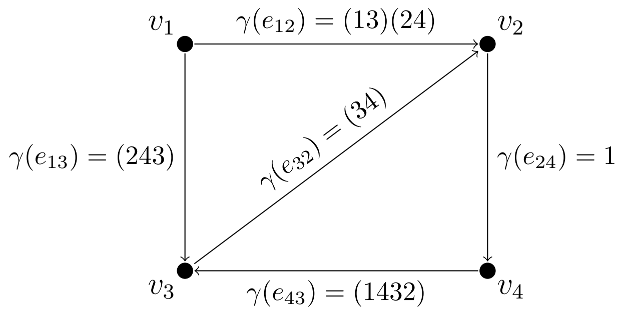
$$\begin{aligned}
&E_s[\det(xI - A_{s,\chi})] \\
&= \textstyle\sum_{k=0}^{m} x^{m-k} \sum\limits_{|S|=k} \sum\limits_{\text{perfect matchings } \pi \text{ on } S} (-1)^{|S|/2} \cdot 1 \\
&= \mu_G(x)
\end{aligned}$$

□

2.6 Representations and Extended Multi-Matchings

A recent paper by Hall, Puder, and Sawin generalizes the work of Marcus, Spielman, and Srivastava by labeling graphs by a group and using matrices formed by replacing ones in the graph's adjacency matrix with block matrices. These blocks are formed by applying a finite dimensional representation to the group that the edges of the graph have been labeled with [7]. We fix a graph G containing n vertices and a d-dimensional representation π of the group Γ. For a labeling $\gamma : E(G) \to \Gamma$, we define the matrix $A_{\gamma,\pi}$ to be the $dn \times dn$ matrix obtained by replacing the uv entry of the adjacency matrix of G with the $d \times d$ block $\sum\limits_{e:u \to v} \pi(\gamma(e)) + \sum\limits_{e:v \to u} \pi(\gamma(e))^{-1}$ (We use a block of zeros if there is no edge from u to v or v to u).

As an example, we label the edges the graph seen in Figure 2.7 with elements of S_4 and then construct the matrix $A_{\gamma,\pi}$ where π is the 2-dimensional representation defined on a generating set by $\pi((12)) = \pi((34)) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$ and $\pi((23)) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.



$$\begin{array}{cc} & \begin{array}{cccccccc} \multicolumn{2}{c}{v_1} & \multicolumn{2}{c}{v_2} & \multicolumn{2}{c}{v_3} & \multicolumn{2}{c}{v_4} \end{array} \\ \begin{array}{c} v_1 \\ \\ v_2 \\ \\ v_3 \\ \\ v_4 \\ \\ \end{array} & \left(\begin{array}{cc:cc:cc:cc} 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 1 & -1 & -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

$\pi((12)) = \pi((34)) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$ $\qquad$ $\pi((23)) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

Figure 2.7. An S_4-labeling γ of a graph and the matrix $A_{\gamma,\pi}$ where π is a 2-dimensional representation of S_4.

Hall, Puder, and Sawin define two conditions on the representation π which they refer to as $(\mathcal{P}1)$ and $(\mathcal{P}2)$ that are required to prove certain results. The representation (Γ, π) is said to satisfy $(\mathcal{P}1)$ if all exterior powers $\wedge^m \pi$ with $0 \leq m \leq d$ are irreducible and pairwise-nonisomorphic. The representation (Γ, π) is said to satisfy $(\mathcal{P}2)$ if $\pi(\Gamma)$ is generated by matrices that are conjugate to a diagonal matrix of the form

$$\begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

with $\lambda \neq 1$ a root of unity. Such matrices are called *pseudo-reflections.*

These conditions play complementary roles in the work of Hall, Puder, and Sawin. The condition $(\mathcal{P}2)$ is needed to show that the average characteristic polynomial of $A_{\gamma,\pi}$ is real-rooted and that one of these polynomials has a largest root that is smaller than that of this average.

The condition ($\mathcal{P}1$) is used to prove the following generalization of 2.2.7 (Theorem 1.8 in [7]):

Theorem 2.6.1. *Let G be connected. Let $\mathcal{C}_{d,G}$ be the set of all d-coverings of the graph G. For every pair (Γ, π) satisfying ($\mathcal{P}1$) with $\dim(\pi) = d$, the following holds:*

$$\mathbb{E}_{\gamma}[\phi_{\gamma,\pi}(x)] = \mathbb{E}_{F \in \mathcal{C}_{d,G}}[\mu_F(x)]$$

where $\phi_{\gamma,\pi}(x)$ is the characteristic polynomial of the matrix $A_{\gamma,\pi}$

In this section, we will prove a generalization of this theorem in which we restrict to the set of labelings that are trivial over a subgraph H of G. It can also be viewed as a generalization of Theorem 2.4.2.

Theorem 2.6.2. *Let H be a subgraph of the connected graph G and let (Γ, π) be a pair satisfying ($\mathcal{P}1$). Let L_H be the set of labelings γ of G by Γ for which $\gamma(e) = 1$ for all edges e in H. Let $\mathcal{C}_{d,G,H}$ be the set of d-coverings of G that are trivial over H, i.e., for every edge (u, v) in H, the fibre of (v, u) is $\{(v_0, u_0), \ldots, (v_{d-1}, u_{d-1})\}$. Then we have*

$$\mathbb{E}_{\gamma \in L_H}[\phi_{\gamma,\pi}(x)] = \mathbb{E}_{F \in \mathcal{C}_{d,G,H}}[\mu_{F,\tilde{H}}(x)]$$

where $\tilde{H}$ is the fibre of H in F.

Note that since for every $F \in \mathcal{C}_{d,G,H}$, F is trivial over H, the subgraph $\tilde{H}$ will be a disjoint union of d copies of H in each of these graphs.

In this section, we will adopt some of the notation used by Hall, Puder, and Sawin: We will regard the edge set $E(G)$ of the graph G as containing both e and $-e$ for every edge in G and $E^+(G)$ will contain only one orientation of each edge. We denote the head and tail vertices of the edge e as $h(e)$ and $t(e)$, respectively.

In order to prove Theorem 2.6.2, we must define a generalization of d-multi-matchings, a concept defined in [7] to prove Theorem 2.6.1. In Definition 2.8 of

[7], a *d-multi-matching* on the graph G is defined to be a function $m\colon E(G) \to \mathbb{Z}_{\geq 0}$ with $m(e) = m(-e)$ of all $e \in E(G)$ and for every vertex $v \in V(G)$, we have $\sum_{i \in V(G)} m(e_{v,i}) \leq d$. For the purposes of this section, it is also useful to think of a d-multi-matching on G as a formal sum of non-oriented edges in G where no vertex is contained by more than d of the edges in the sum.

We now define our generalization.

Definition 2.6.3. An *extended d-multi-matching* m of the graph G with respect to the subgraph H is a formal sum of non-oriented edges in G and oriented cycles in H where for each vertex v in G, the total number of edges and oriented cycles in m containing v is less than or equal to d. We will write the set of all extended d-multi-matchings of G with respect to H as $\mathrm{Mult}_d(G, H)$.

We say that a partially oriented extended matching M of a d-covering $F \in \mathcal{C}_{d,G,H}$ *projects* to the extended multi-matching m of G with respect to H if for every edge e in G, the number of edges in M belonging to the fibre of e is the number of times e appears in m and for each oriented cycle c in H, the number of cycles in M belonging to the fibre of c is the number of times c appears in m.

Note that when H is empty, the Definition 2.6.3 becomes equivalent to the definition of d-multi-matchings. We also observe that each partially oriented extended matching on a d-covering of G projects to a unique extended multi-matching. Therefore, we can write the expected extended matching polynomial as

$$\mathbb{E}_{F \in \mathcal{C}_{d,G,H}}[\mu_{F,\tilde{H}}(x)] = \sum_{m \in \mathrm{Mult}_d(G,H)} (-1)^{c(m)} W_d(m) x^{nd-|m|}$$

where $|m|$ is the total number of vertices included in the edges and cycles of m (counted with multiplicity), $c(m)$ is the number of edges and cycles in m, and $W_d(m)$ is the average number of partially oriented extended matchings projecting to m in a random element of $\mathcal{C}_{d,G,H}$.

To give an example of an extended multi-matching, we consider the graph G and subgraph H appearing in Figure 2.8. We can use the notation e_{vw} for a non-oriented edge between vertices v and w and the notation $c_{v_1v_2\ldots v_k}$ for the oriented cycle that is comprised of the vertices $v_1, \ldots, v_k$ and contains an oriented edge from with v_2 as its head and v_2 as its tail. Under this notation, $2e_{im} + e_{bf} + c_{fjnoplkg} + 2c_{fgkj} + c_{cdhg} + c_{cghd}$ is an example of an extended 4-multi-matching on G with respect to H. Note that it can also be thought of as a 5-multi-matching or a 6-multi-matching, but not a 3-multi-matching because the vertex f appears in 4 of the terms in this sum (counted with multiplicity), and no other vertex appears as often.

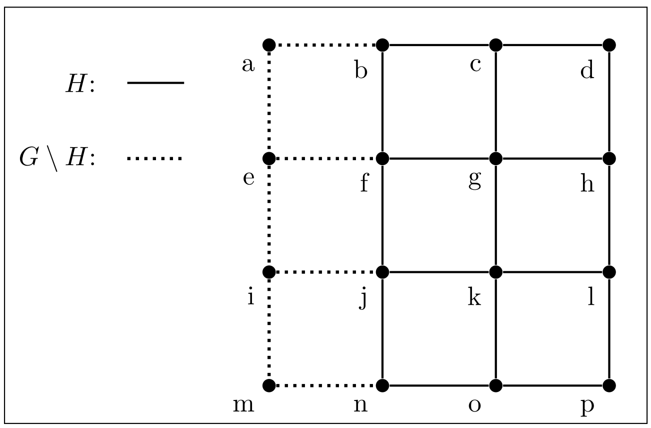


Figure 2.8. A graph G with a subgraph H.

Proof. To prove Theorem 2.6.2 we will follow the argument in section 3 of [7]. For an edge $e \in E(G)$, let $A_{\gamma,\pi}(e)$ be the $dn \times dn$ matrix consisting of n^2 blocks of size $d \times d$ where each block is a the zero matrix except for the block corresponding to $(h(e), t(e))$, where we have $\pi(\gamma(e))$. By this notation, we can write $A_{\gamma,\pi} = \sum_{e \in E(G)} A_{\gamma,\pi}(e)$. As in [7], we consider pairs of partitions $(\dot{R}, \dot{C})$ of the rows and columns of a $nd \times nd$ matrix and let

$T = \{(\dot{R}, \dot{C}) \mid \dot{R} = (R_x, R_{e_1}, R_{e_2}, \dots)$ and $\dot{C} = (C_x, C_{e_1}, C_{e_2}, \dots)$ are partitions of $[\![nd]\!]$ into $1 + |E(G)|$ parts indexed by $\{x\} \cup E(G)$ with $R_x = C_x$ and $|R_e| = |C_e|$ for all $e \in E(G)\}$.

Let $\text{sgn}(\dot{R}, \dot{C})$ be the sign of the permutation matrix obtained by assigning, for each l, the $|R_l| \times |R_l|$ identity matrix to the (R_l, C_l) minor. Assume without loss of generality that π maps the elements of Γ to unitary matrices, so we have $A_{\gamma,\pi}(e) = A_{\gamma,\pi}(-e)^*$ for all $e \in E(G)$.

For the matrix A and subsets R and C of the rows and columns of A with $|R| = |C|$, let $|A|_{R,C}$ be the determinant of the (R, C)-minor of A (Let it be 1 if R and C are empty). Equation 3.4 in [7] tells us that the expected characteristic polynomial $\mathbb{E}_{\gamma \in L_H}[\phi_{\gamma,\pi}(x)]$ is equal to

$$\sum_{(\dot{R},\dot{C}) \in T} \text{sgn}(\dot{R}, \dot{C}) x^{|R_x|} (-1)^{nd-|R_x|} \prod_{e \in E^+(G)} \mathbb{E}_{\gamma \in L_H}[|A_{\gamma,\pi}(e)|_{R_e,C_e} |A_{\gamma,\pi}(-e)|_{R_{-e},C_{-e}}].$$

Since $A_{\gamma,\pi}(e) = A_{\gamma,\pi}(-e)^*$, we have

$$\mathbb{E}_{\gamma \in L_H}[|A_{\gamma,\pi}(e)|_{R_e,C_e} |A_{\gamma,\pi}(-e)|_{R_{-e},C_{-e}}] = \mathbb{E}_{\gamma \in L_H}[|A_{\gamma,\pi}(e)|_{R_e,C_e} \overline{|A_{\gamma,\pi}(e)|_{C_{-e},R_{-e}}}].$$

The value of the expectation on the right hand side of this equation depends on whether or not e belongs to the subgraph H. Let B_v be the set of d indices of rows and columns corresponding to the vertex v. In both cases, the expectation is zero unless $R_e, C_{-e} \subseteq B_{h(e)}$ and $C_e, R_{-e} \subseteq B_{t(e)}$ because otherwise, one of the minors will have a zero row or zero column. Assume that we do have these inclusions. We first consider $e \in G \setminus H$. Since the signings in L_H can take any value in Γ on e, the expectation is equivalent to one taken over all elements of the group Γ. This means that we can apply the same argument as in [7]: Under the assumption that our representation satisfies ($\mathcal{P}1$), it follows from the Peter-Weyl Theorem (Theorem

3.3 in [7]) that the expectation is $\binom{d}{|R_e|}^{-1}$ if $|R_e| = |R_{-e}|$, $R_e = C_{-e}$, and $C_e = R_{-e}$, and the expectation is zero otherwise. Now suppose $e \in H$. Since $\gamma \in L_H$, the matrix $A_{\gamma,\pi}(e)$ has no dependence on γ. Let $r_{v,i}$ refer to the index of the ith row and column corresponding to the vertex v. Since the $(h(e), t(e))$ block of $A_{\gamma,\pi}(e)$ is the identity matrix, the expectation is 1 if there exist two subsets $S_1, S_2 \subseteq [d]$ for which $R_e = \{r_{h(e),i}\}_{i\in S_1}$, $C_e = \{r_{t(e),i}\}_{i\in S_1}$, $R_{-e} = \{r_{t(e),i}\}_{i\in S_2}$, and $C_{-e} = \{r_{h(e),i}\}_{i\in S_2}$, and the expectation is zero otherwise. We will refer to the subset of T for which $\prod_{e\in E^+(G)} \mathbb{E}_{\gamma\in L_H}[|A_{\gamma,\pi}(e)|_{R_e,C_e}|A_{\gamma,\pi}(-e)|_{R_{-e},C_{-e}}] \neq 0$ as T_H^{sym}. We now have

$$\mathbb{E}_{\gamma\in L_H}[\phi_{\gamma,\pi}(x)] = \sum_{(\dot{R},\dot{C})\in T_H^{\mathrm{sym}}} \mathrm{sgn}(\dot{R},\dot{C}) x^{|R_x|}(-1)^{nd-|R_x|} \prod_{e\in E^+(G\setminus H)} \binom{d}{|R_e|}^{-1}.$$

Now suppose that $(\dot{R},\dot{C}) \in T_H^{\mathrm{sym}}$. Let $e_1 \in E(H)$ and $r_{h(e_1),i} \in R_{e_1}$. Since $(\dot{R},\dot{C}) \in T_H^{\mathrm{sym}}$, this means that $r_{t(e_1),i} \in C_{e_1}$. Since $R_x = C_x$, there is some $e_2 \in E(G)$ with $h(e_2) = t(e_1)$ such that $r_{t(e_1),i} \in R_{e_2}$. Furthermore, e_2 must belong to H because we do not have $R_{e_2} = C_{-e_2}$ unless $e_2 = -e_1$. By making this argument repeatedly, we can conclude that there is a cycle of edges $e_1, \ldots, e_w \in H$ such that $r_{h(e_j),i} \in R_{e_j}$ and $r_{t(e_j),i} \in C_{e_j,i}$. We can think of the set $\{r_{e_j,i}\}_{j\in[\![w]\!]}$ as an oriented cycle in the ith copy of H in a disjoint union of d copies of H if $w > 2$, or as a non-directed edge in the ith copy of H if $w = 2$.

Since $(\dot{R},\dot{C}) \in T_H^{\mathrm{sym}}$, our partitions assign the same set of rows and columns to a subset of $[\![nd]\!]$ (i.e $\bigcup_{e\in E(H)} R_e = \bigcup_{e\in E(H)} C_e$), so we can define $(\dot{R},\dot{C})|_H$ as the pairs of partition of a subset of $[\![nd]\!]$ given by the sets R_e and C_e for $e \in E(H)$. By our discussion, $(\dot{R},\dot{C})|_H$ corresponds to a partially oriented extended matching on a disjoint union of d copies of H with respect to the entire disjoint union. Since $\bigcup_{e\in E(G\setminus H)} R_e = \bigcup_{e\in E(G\setminus H)} C_e$, we can similarly define $(\dot{R},\dot{C})|_{G\setminus H}$. As discussed in [7], $(\dot{R},\dot{C})|_{G\setminus H}$ projects to the d-multi-matching on $G \setminus H$ given by $\sum_{e\in E^+(G\setminus H)} |R_e|e$.

Therefore, we can define the projection map $\eta : T_H^{\mathrm{sym}} \to \mathrm{Mult}_d(G,H)$ mapping $(\dot{R},\dot{C})$ to the sum of the projection of $(\dot{R},\dot{C})|_{G\setminus H}$ and the projection of the partially oriented extended matching corresponding to $(\dot{R},\dot{C})|_H$. With this notation, we can write:

$$\mathbb{E}_{\gamma\in L_H}[\phi_{\gamma,\pi}(x)] = \sum_{m\in\mathrm{Mult}_d(G,H)} \sum_{\substack{(\dot{R},\dot{C})\in T_H^{\mathrm{sym}} \\ \eta(\dot{R},\dot{C})=m}} \mathrm{sgn}(\dot{R},\dot{C}) x^{|R_x|}(-1)^{nd-|R_x|} \prod_{e\in E^+(G\setminus H)} \binom{d}{|R_e|}^{-1}.$$

Observe that if $\eta(\dot{R},\dot{C}) = m$, then $|R_x| = nd - |m|$. As in our discussion of extended matchings, the number of cycles with an odd number of edges in m is even if and only if $|m|$ is even, so $\mathrm{sgn}(\dot{R},\dot{C})(-1)^{|m|} = (-1)^{c(m)}$, and we have

$$\mathbb{E}_{\gamma\in L_H}[\phi_{\gamma,\pi}(x)] = \sum_{m\in\mathrm{Mult}_d(G,H)} \sum_{\substack{(\dot{R},\dot{C})\in T_H^{\mathrm{sym}} \\ \eta(\dot{R},\dot{C})=m}} x^{nd-|R_x|}(-1)^{c(m)} \prod_{e\in E^+(G\setminus H)} \binom{d}{|R_e|}^{-1}.$$

The number of $(\dot{R},\dot{C})|_{G\setminus H}$ projecting to m is $\prod_{v\in V(G\setminus H)} \binom{d}{m(e_{v,1}),\ldots,m(e_{v,\deg(v)})}$, so if we let $C_H(m)$ be the number of $(\dot{R},\dot{C})$ such that $(\dot{R},\dot{C})|_{G\setminus H}$ projects to the multi-matching $m|_{G\setminus H}$ on $G \setminus H$ that agrees with m on $G \setminus H$, we can write $\mathbb{E}_{\gamma\in L_H}[\phi_{\gamma,\pi}(x)]$ as

$$\sum_{m\in\mathrm{Mult}_d(G,H)} x^{nd-|R_x|}(-1)^{c(m)} \prod_{e\in E^+(G\setminus H)} \binom{d}{|R_e|}^{-1} \prod_{v\in V(G\setminus H)} \binom{d}{m(e_{v,1}),\ldots,m(e_{v,\deg(v)})} C_H(m).$$

It is shown in section 2 of [7] that $\prod_{e\in E^+(G\setminus H)} \binom{d}{|R_e|}^{-1} \prod_{v\in V(G\setminus H)} \binom{d}{m(e_{v,1}),\ldots,m(e_{v,\deg(v)})}$ is the average number of matchings projecting to $m|_{G\setminus H}$ in a random d-covering of $G\setminus H$. Therefore, $\prod_{e\in E^+(G\setminus H)} \binom{d}{|R_e|}^{-1} \prod_{v\in V(G\setminus H)} \binom{d}{m(e_{v,1}),\ldots,m(e_{v,\deg(v)})} C_H(m) = W_d(m)$ and we have proven Theorem 2.6.2.

□

CHAPTER 3

TIERED TREES AND WEIGHT

3.1 Background on Alternating Trees

Let $[\![n]\!]$ refer to the set $\{1, \ldots, n\}$. An *alternating tree* (sometimes called a *maxmin tree*) T is a tree on n vertices where each vertex is given a label from $[\![n]\!]$ (each label is used once) and each vertex represents either a local maximum or local minimum relative to its neighbors. The *weight* $w(T) \in \mathbb{Z}_{\geq 0}$ of an alternating tree can be found either by a recursive definition or by computing the external activity of T within a certain larger graph. We will first explain the latter method of computing $w(T)$. Every alternating tree T is a spanning tree sitting inside a larger *alternating graph* G in which each local maximum is connected to every local minimum that has a smaller label. We order the edges of G *lexicographically*: the edge connecting the vertices labeled by n_1 and n_2 is said to come before the edge with vertices labeled by m_1 and m_2 if either $\min\{n_1, n_2\} < \min\{m_1, m_2\}$ or both $\min\{n_1, n_2\} = \min\{m_1, m_2\}$ and $\max\{n_1, n_2\} < \max\{m_1, m_2\}$ (i.e. the lexicographic order for the edges of K_4 labeled by $\{1, 2, 3, 4\}$ would be $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$). We say that an edge $e \in G \setminus T$ is *externally active* if, of all of the edges in the unique cycle that $\{e\} \cup T$ contains, e is the one that comes first lexicographically. The weight $w(T)$ of the alternating tree is the number of externally active edges in $G \setminus T$.

The recursive definition of $w(T)$ is as follows:

(i) If T has only one vertex, $w(T) = 0$

(ii) If T has multiple vertices, let v be the vertex with the lowest label and let $\{T_i\}$ be the set of components of the forest we obtain by deleting v from T. We then define

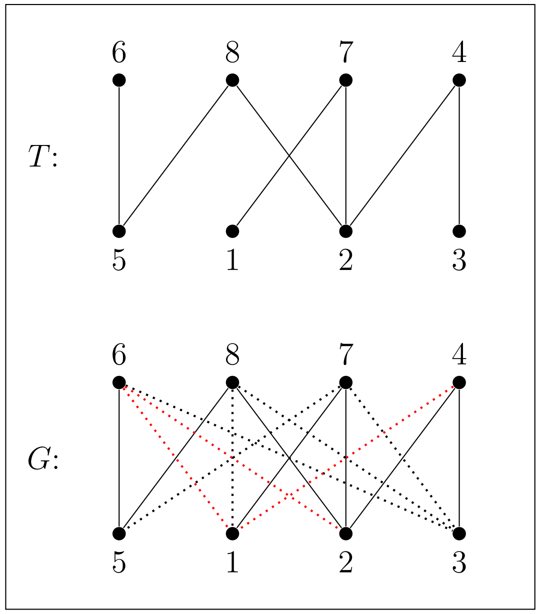


Figure 3.1. An alternating tree of weight 3. The 3 externally active edges of $G \setminus T$ appear in red.

$w(T) = \sum_i w(T_i) + w_i$ where w_i is the number of local maxima in T_i whose label is less than that of the vertex in T_i that was connected to v.

An example of a weight computation based on this recursive definition can be found in Figure 3.3.

3.2 Tiered Trees

We now define a generalization of alternating trees called *tiered trees*.

Definition 3.2.1. Let G be a graph with n vertices labeled by $[\![n]\!]$. A *tiering function* $t\colon V \to [\![m]\!]$ is a function assigning each vertex v of G a tier $t(v)$ such that for any edge (v, v') with $v > v'$, $t(v) > t(v')$. A pair (G, t) of a labeled graph G and a tiering function on that graph is called a *tiered graph*. If G is a tree, we call the pair a *tiered tree*.

Observe that if $m = 2$, this becomes a definition of alternating trees, with the tiering function indicating whether we have a local minimum or maximum at each vertex.

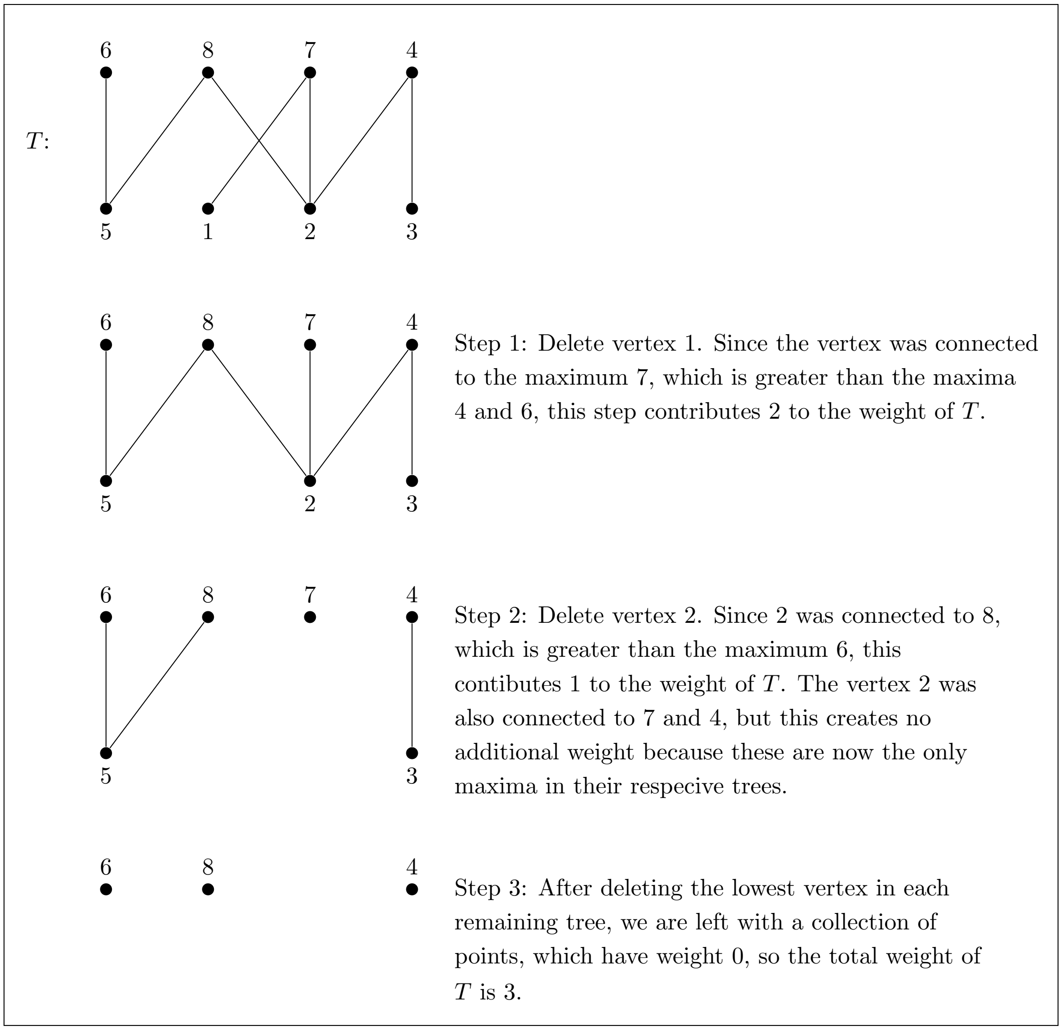


Figure 3.2. Finding the weight of the same tree using the recursive definition.

We can extend the definitions of the weight of an alternating tree to apply to tiered trees. We will define this weight recursively, then show that it is equivalent to an external activity-based definition.

Definition 3.2.2. The weight $w(T)$ of a tiered tree T is defined recursively as follows:
(i) If T has only one vertex, $w(T) = 0$.
(ii) If T has multiple vertices, let v be the vertex with the lowest label and let $\{T_i\}$ be the set of components of the forest we obtain by deleting v from T. We then define $w(T) = \sum_i w(T_i) + w_i$ where w_i is defined as follows: Let u_i be the vertex in T_i to

which v was connected in T. The integer w_i is the number of vertices u in T_i with $u < u_i$ and $t(u) > t(v)$.

Theorem 3.2.3. *Let T be a tiered tree, and let G be the tiered graph obtained from adding every possible edge to T, i.e., G has the same vertex set and tiering function, but it contains the edge (v, v') with $v > v'$ if and only $t(v) > t(v')$. The weight $w(T)$ of the tiered tree is the number of externally active edges in $G \setminus T$.*

Proof. The theorem follows from the claim that w_i is the number of externally active edges with a lower vertex of v and a higher vertex in T_i. To prove this, we note that for any u in T_i, the edge (u, v) is in G if and only if $t(u) > t(v)$ because v is the lowest remaining label. If $t(u) > t(v)$, the first two edges in the unique cycle of $\{(u, v)\} \cup T$ lexicographically are (v, u) and (v, u_i) because these are the only two edges in the cycle that include the lowest vertex v. Therefore, (u, v) is externally active if and only if u is in a higher tier than v and $u < u_i$. Since w_i is the number of vertices in T_i with this property, this completes the proof. □

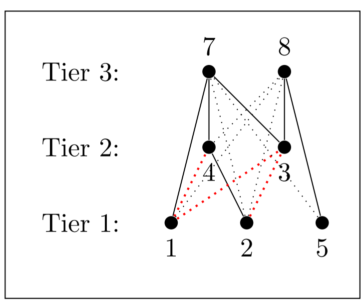


Figure 3.3. A tiered tree of weight 3. The edges of $G \setminus T$ appear as dotted lines, and the externally active edges are red.

3.3 Counting 3-tiered Trees

In this section, we will describe a way of enumerating 3-tiered trees by relating them to alternating trees. We note that the conjectures leading to Corollaries 3.3.4 and 3.3.5 were formulated as part of joint work with William Dugan.

Let $T^w_{a_1,a_2,a_3,\ldots,a_m}$ be the number of m-tiered trees of weight w with $|t^{-1}(i)| = a_i$. As a consequence of our bijection between permutations and weight zero alternating trees (to be shown in Section 3.4), $T^0_{a,0,b}$ is equal to the Eulerian number $A(a+b-1, a-1)$, which is the number of permutations on $a+b-1$ letters with $a-1$ descents. The following figure displays all values of $T^0_{a,b,c}$ with $a+b+c = 6$:

$$
\begin{array}{c}
T^0_{6,0,0} \\
T^0_{5,1,0}\ T^0_{5,0,1} \\
T^0_{4,2,0}\ T^0_{4,1,1}\ T^0_{4,0,2} \\
T^0_{3,3,0}\ T^0_{3,2,1}\ T^0_{3,1,2}\ T^0_{3,0,3} \\
T^0_{2,4,0}\ T^0_{2,3,1}\ T^0_{2,2,2}\ T^0_{2,1,3}\ T^0_{2,0,4} \\
T^0_{1,5,0}\ T^0_{1,4,1}\ T^0_{1,3,2}\ T^0_{1,2,3}\ T^0_{1,1,4}\ T^0_{1,0,5} \\
T^0_{0,6,0}\ T^0_{0,5,1}\ T^0_{0,4,2}\ T^0_{0,3,3}\ T^0_{0,2,4}\ T^0_{0,1,5}\ T^0_{0,0,6}
\end{array}
=
\begin{array}{c}
0 \\
1 \quad 1 \\
26 \quad 57 \quad 26 \\
66 \quad 302 \quad 302 \quad 66 \\
26 \quad 302 \quad 627 \quad 302 \quad 26 \\
1 \quad 57 \quad 302 \quad 302 \quad 57 \quad 1 \\
0 \quad 1 \quad 26 \quad 66 \quad 26 \quad 1 \quad 0
\end{array}
$$

Figure 3.4. The number of weight zero tiered trees on 6 vertices for each tier type

We can observe from Figure 3.4 that, for 3-tiered trees of weight 0 with 6 vertices, the number of weight 0 trees of a given tier type will be unchanged after applying any permutation to the populations of the tiers. In fact, it is more generally true that for any weight w, any tier populations $a_1, \ldots, a_n$, and any permutation σ on n letters that we have $T^w_{a_1,\ldots,a_n} = T^w_{\sigma(a_1),\ldots,\sigma(a_n)}$. As stated in [16], this can be shown using geometric results in [6], but we do not have an elementary proof.

We can view all weight 0 trees of type $\{a, c, b\}$ as coming from trees of type $\{a + d, 0, b + c - d\}$ for some $0 \leq d \leq c$ after moving d vertices from tier 1 and $c - d$ vertices from tier 3 into tier 2.

Theorem 3.3.1. *Fix four positive integers $a, b, n,$ and m. There is a bijection between the following two sets:*

$\mathcal{T}_1$: The set of weight 0 trees of type $\{a, n + m, b\}$ in which exactly m of the tier 2 vertices connect directly to a vertex in tier 3.

$\mathcal{T}_2$: The set of trees of type $\{a + n, 0, b + m\}$ (of any weight) together with subsets A of their sets of vertices with the following properties: A contains n vertices in tier 1 and m vertices in tier 3, any endpoint of any externally active edge belongs to A, and no two vertices in A are connected by an edge. We will call the elements of A distinguished vertices*.*

Proof. Given an element of set $\mathcal{T}_1$, we will denote the set of tier 2 vertices that are connected to tier 3 as $\{v'_i\}_{i \in [\![m]\!]}$, and we will refer to the set of tier 2 vertices not connected to any tier 3 vertex as $\{s_i\}_{i \in [\![n]\!]}$.

We define a map g on $\mathcal{T}_1$ whose image consists of tiered trees of type $\{a+n, 0, b+m\}$ (We will later show that the image is $\mathcal{T}_2$.) as follows: First, we move every s_i from tier 2 to tier 3. Note that this may create externally active edges between elements of $\{v'_i\}$ and $\{s_i\}$, but no other externally active edges will be created. These vertices s_i that we moved into tier 3 now become the distinguished tier 3 vertices of our tree. Consider the set of edges between vertices in tier 1 and vertices in tier 2, and order them lexicographically. Let e_j be the element of this set that comes jth lexicographically. Let w_j be the tier 1 endpoint of e_j and let v_j be the tier 2 endpoint. Note that the any v_j is one of the vertices v'_i, but a single vertex in $\{v'_i\}$ can have multiple indices as an element of $\{v_j\}$ (It could also not appear at all). We now delete each of the edges e_j connecting a tier 1 vertex to a tier 2 vertex.

Starting with w_1, we will now construct a new edge from each v_j. For a given j, this edge will terminate in the connected component of v_j. Let C_1 be the set of vertices in the component of v_1 whose path to v_1 contains no tier 2 vertex with a label less than that of v_1. We add an edge between w_1 and the lowest-labeled tier 3 vertex with a label greater than that of v_1 in C_1. We will call this vertex u_1. We repeat this process with each w_j in order of increasing j to obtain a connected tree, analogously defining C_j and u_j. Finally, we move all of the v_j into tier 1 and they become the tier 1 distinguished vertices of our tree.

We now define a map f with domain $\mathcal{T}_2$ which we will show to be the inverse of g. When starting with an element of $\mathcal{T}_2$, we will refer to the set of tier 1 distinguished vertices as $\{v'_i\}_{i\in[\![m]\!]}$, and we will refer to the set of tier 3 distinguished vertices as $\{s_i\}_{i\in[\![n]\!]}$. When applying f, we begin by moving all of the s_i into tier 2. Note that this cannot create any new externally active edges because all tier 3 vertices have higher labels than all of their neighbors. Next, we move all of the v'_i into tier 2 (We may do this because each of these vertices is not connected to any s_i). This may create externally active edges between the v'_i and certain tier 1 vertices. We identify the externally active edge between a v'_i and a tier 1 vertex that comes first lexicographically and add it to our tree. We then delete the edge in the cycle that has been created that comes second lexicographically. We then repeat this process until we no longer have any externally active edges. See Example 3.3.6 for an example of how to apply the maps f and g.

To show that our two maps are inverses, we will need to prove the following lemma about the map g:

Lemma 3.3.2. *Before connecting w_j to u_j, there is no vertex in C_j with a label less than that of w_j, and no tier 3 vertex with a label less than that of v_j.*

Proof. We prove the lemma by induction. Let S_1 be the set of vertices in C_1 whose labels are less than that of w_1, and let S_2 be the set of tier 3 vertices in C_1 whose labels are less than that of v_1. If $S_1 \cup S_2$ is nonempty, then there exists some element $w \in S_1 \cup S_2$ such that the path from w to v_1 contains no other elements of $S_1 \cup S_2$. If $w \in S_1$, then (w, v_1) is an externally active edge, contradicting the fact that the component of v_1 only has externally active edges between elements of $\{v_i'\}$ and $\{s_i\}$. On the other hand, if $w \in S_2$, then (w_1, w) was an externally active edge in the tree we had before removing the edge (w_1, v_1), contradicting the fact that our original tree had weight 0.

Before proceeding with the induction step, we will prove some consequences that hold if the result is true for all $i \leq j$. For a fixed j if the lemma holds for all $i \leq j$, then the tree we obtain by connecting w_j to u_j has an externally active edge (w_i, v_i) for each $i \leq j$. Furthermore, the only externally active edges of this tree (other than those between the v_i''s and s_i's) are between tier 1 vertices and tier 2 vertices, and these edges do not come before (w_j, v_j) lexicographically. To see that $e_j = (w_j, v_j)$ becomes externally active after connecting w_j to u_j, note that e_j can only fail to be externally active if either u_j has a label less than that of v_j, or the path from u_j to v_j contains a vertex with a label less than that of w_j. We will now show that connecting w_j to u_j creates no externally active edges that would contradict our claim.

Suppose that after connecting w_j to u_j, we have an externally active edge (w, u), where w has a lower label than u. Since the cycle that (w, u) creates includes the edge (w_j, u_j), w must have a label less than or equal to that of w_j. Note that the only edges in the original path between u and w that do not appear in our new path between u and v have both endpoints belonging to some set $C_i \cup \{w_i\}$ with $i \leq j$ consisting of vertices whose labels are not smaller than that of w_j. If $w \neq w_j$, this means that (w, u) must have been an externally active edge before we removed any edges because the second vertex in the path from w to u has not changed either.

Now suppose that (w_j, u) becomes externally active. If $u \in C_j$, then u must be tier 2 vertex because all tier 3 vertices in C_j have a label less than or equal to that of

u_j. This means that the label of u must be greater than that of v_j, which implies that (w_j, u) comes after (w_j, v_j) lexicographically. On the other hand, suppose $u \notin C_j$. This implies that there is some tier 2 vertex v_i' whose label is less than that of v_j in the path from v_j to u. Since (w_j, u) is externally active, and since the path between w_j and u contains v_i', the label of v_i' must be greater than that of w_j. This implies that (w_j, v_i') is externally active. But since the path from w_j to v_i' has only been altered by vertices whose labels are no less than that of w_j, and the label of v_j is greater than that of v_i', the edge (w_j, v_i') must have been externally active in the original tree. This gives us a contradiction because it would imply that the original tree had nonzero weight.

We will now continue the proof of Lemma 3.3.2 by induction. Suppose that the lemma is true for all $i < j$. Let S_1 be the set of vertices in C_j whose labels are less than that of w_j, and let S_2 be the set of tier 3 vertices in C_j whose labels are less than that of v_j. If $S_1 \cup S_2$ is nonempty, then there exists some element $w \in S_1 \cup S_2$ such that the path from w to v_j contains no other elements of $S_1 \cup S_2$. If $w \in S_1$, then (w, v_j) is an externally active edge. However, the path beween w and v_j has not been affected by connecting w_j and u_j, so (w, v_j) was externally active after connecting w_{j-1} and u_{j-1}. Since the label of w is less than that of w_j and that of w_{j-1}, the edge (w, v_j) comes before (w_{j-1}, v_{j-1}) lexicographically. This contradicts the fact we proved earlier that connecting w_{j-1} and u_{j-1} cannot create externally active edges between tiers 1 and 2 that come before (w_{j-1}, v_{j-1}) lexicographically. Suppose that $w \in S_2$. This means that (w_j, w) is externally active, but the only edges in our original path from w to w_j that do not also appear in our new path from w_j to w are either of the form $(w_i, v_i)_{\bar{j}}$ for $i \leq j$ or have both vertices belonging to some C_i with $i < j$. As none of these vertices has a label below that of w_j, and the label of v_j is greater than that of w, this means that (w_j, w) was externally active in our original graph, giving us a contradiction. Therefore, the vertex w cannot exist and the lemma is proved. $\square$

Resuming our proof of Theorem 3.3.1, we can see that this lemma implies that the image of g is contained in $\mathcal{T}_2$ because after moving the vertices v_i' into tier 1, the only remaining externally active edges will be between the distinguished vertices in $\{v_i'\}$ and $\{s_i\}$. As another consequence of our lemma, we can see that if we were to connect w_j to any tier 3 vertex in C_j other than u_j or v_j, the edge (w_j, u_j) would become an externally active edge. Therefore, connecting v_j to u_j is the only way to connect w_j to C_j that results in a tree with externally active edges between tier 1 and tier t vertices and no others except those between the s_i's and v_i''s. We now consider a class G of functions that turn elements of $\mathcal{T}_1$ into trees of type $\{a+n, 0, b+m\}$ with n distinguished vertices in tier 1 and m distinguished vertices in tier 3. We define G to consist of all functions whose formula is exactly the same as g except we are not required to choose u_j as the vertex to which we connect v_j, and we can instead choose any tier 3 vertex in C_j whose label is greater than that of v_j. As we have shown g is the only element of G whose image is contained in the domain of f.

Now, suppose that we apply the following procedure to an element T in $\mathcal{T}_2$: We first apply f to T. Secondly, we move the tier 2 vertices in $f(T)$ with no edge between them and a tier 3 vertex into tier 3 and begin treating them as distinguished vertices. Third, we delete the edges that we added when applying f to T. Fourth, we add the edges that we deleted when applying f to T in the reverse of the order that we added them when applying f. Lastly, we move the remaining tier 2 vertices into tier 1 and treat them as distinguished vertices. Obviously, the end result of this process will be T. We now claim that applying the last applying the last four steps of this process to $f(T)$ is equivalent to applying an element of G to it. To see this, first note that the only edges between tiers 1 and 2 in $f(T)$ are those that we added when applying f to T. We can call these edges $\{w_j, v_j\}$ and index them in decreasing lexicographic order. In the fourth step of our process, we will connect each w_j to some other vertex in order of increasing j. Defining C_j in the same way as in the definition of g, we see that this vertex must be a tier 3 vertex in C_j because if, when applying f, we had an externally active edge (x, y) with x in tier 1 and y in tier 2, and there was some tier 2 vertex z with a lower label than y in the path from x to y, then (x, z) would be an externally active edge that comes before (x, y) lexicographically, so it would need to be added first. This shows the last four steps of our process are equivalent to applying an element of G to $f(T)$, and since the result belongs to the domain of f, this element of G must be g, so $g(f(T)) = T$ for any $T \in \mathcal{T}_2$.

We can make a similar argument to show that $f(g(T)) = T$ for any $T \in \mathcal{T}_1$. Suppose that we apply the following procedure to an element T in $\mathcal{T}_1$: We first apply g to T. Secondly, we move the tier 3 distinguished vertices in $f(T)$ into tier 2. Third, we move the tier 1 distinguished vertices into tier 2. Next, we add the edge that comes first among those we deleted in the first step and delete the last edge we added when applying g to T. We then add the deleted edge that comes second lexicographically and remove the second-to-last edge that we added, and so on until we have rebuilt T. Since adding the edge (w_j, u_j) when applying g turns the deleted edge e_j into an externally active edge, and since the resulting tree has no externally active edges between tiers 1 and 2 that come before e_j lexicographically (including the e_i with $i > j$ does not affect this because it does not change anything in the proof of Lemma 3.3.2), each edge that we add to $g(T)$ using this process will be the externally active edge that comes first lexicographically among the externally active edges between tiers 1 and 2. Also, the edges that we delete will be the ones that come second in the cycles created because they will be the only other edge in the cycle that has the lowest-labeled vertex of the cycle as an endpoint. Therefore, our process is equivalent to first applying g to T, then applying f to $g(T)$, so $f(g(T)) = T$, and the theorem is proved.

$\square$

Our first corollary to this theorem will allow us to compute the number of type $\{a, 1, b\}$ weight 0 trees in terms of Eulerian numbers.

Corollary 3.3.3. $T^0_{a,1,b} = (b+1)T^0_{a,0,b+1} + (a+1)T^0_{a+1,0,b}$.

Since we have the formula $T^0_{a,0,b} = A(a+b-1, a-1)$ and Eulerian numbers have the recursive relation $A(n,m) = (n-m)A(n-1, m-1) + (m+1)A(n-1, m)$, the preceding corollary gives us another corollary:

Corollary 3.3.4. $T^0_{a,1,b} = T^0_{a+1,0,b+1}$.

Since every tree of type $\{a+1, 0, b+1\}$ has $a+b+1$ edges, there are $(a+1)(b+1) - (a+b+1) = ab$ ways to distinguish the vertices of a weight 0 tree of type $\{a+1, 0, b+1\}$ that correspond to a weight 0 tree of type $\{a, 2, b\}$. Also, a weight 1 tree of type tree of type $\{a+1, 0, b+1\}$ can be turned into a weight 0 tree of type $\{a, 2, b\}$ if we distinguish the endpoints of its externally active edge, so the preceding theorem also gives us a formula for the number of weight 0 trees of type $\{a, 2, b\}$:

Corollary 3.3.5. $T^0_{a,2,b} = \binom{b+2}{2}T^0_{a,0,b+2} + \binom{a+2}{2}T^0_{a+2,0,b} + abT^0_{a+1,0,b+1} + T^1_{a+1,0,b+1}$.

The previous theorem does not enable us to write similar formulas for $T^0_{a,c,b}$ with $c > 2$ because we do not have useful expressions for the sizes of some of the sets involved.

Example 3.3.6. We will now present an example of a weight 0 tree of type $\{3, 2, 3\}$ being turned into a tree of type $\{4, 0, 4\}$ with distinguished vertices and back using the maps g and f.

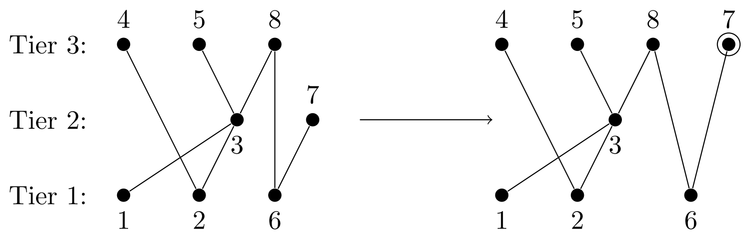


Figure 3.5. Step 1 of applying the map g.

When applying the map g, we first move each tier 2 vertex that does not share an edge with a tier 3 vertex into tier 3. We also distinguish these vertices (indicated in this figure by circling them). Note that in this case, this creates an externally active edge between vertices 3 and 7.

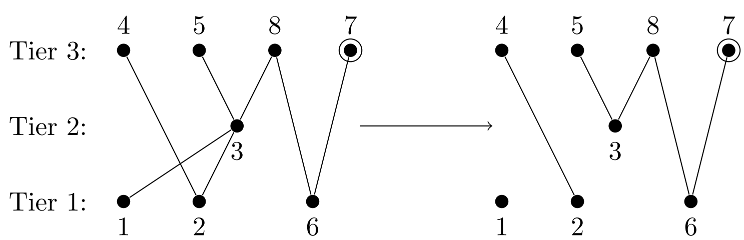


Figure 3.6. Step 2 of applying the map g.

Next, we delete each edge connecting a vertex in tier 2 to a vertex in tier 1.

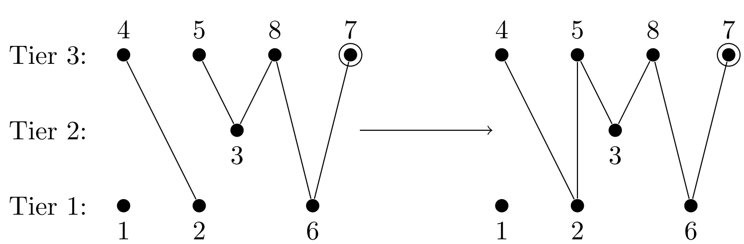


Figure 3.7. Step 3 of applying the map g.

Of the edges we deleted, the one between vertices 2 and 3 comes last lexicographically, so we replace this edge by connecting vertex 2 to a tier 3 vertex in the component of vertex 3 (Note: When we start with only one tier 1 vertex not connected to a tier 3 vertex, the definition of C_i reduces to the component of that unique vertex v_1). Since 5 is the smallest number we see in tier 3 of the component of vertex 3, we connect vertices 2 and 5 with an edge.

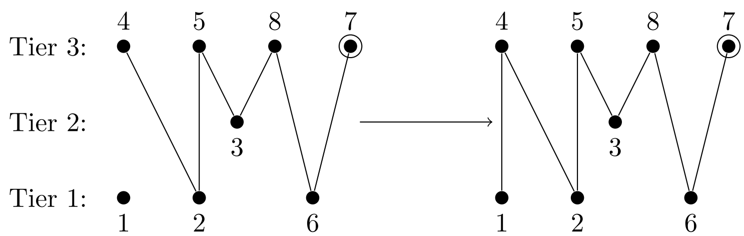


Figure 3.8. Step 4 of applying the map g.

Next, we replace the edge between vertices 1 and 3. Note that the component of vertex 3 now also contains vertex 4, so we add an edge between vertex 1 and vertex 4.

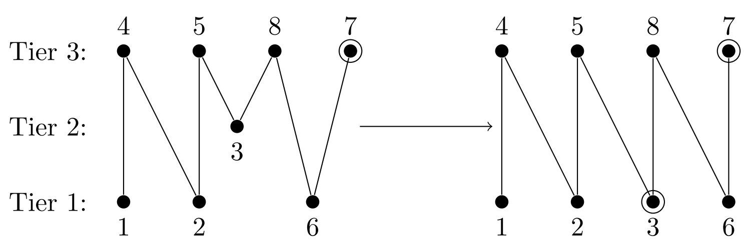


Figure 3.9. Step 5 of applying the map g.

Lastly, we move vertex 3 (in general, all remaining tier 2 vertices) into tier 1 and distinguish it. Observe that we do have one externally active edge between vertices 3 and 7, but this is allowed because these two vertices are distinguished.

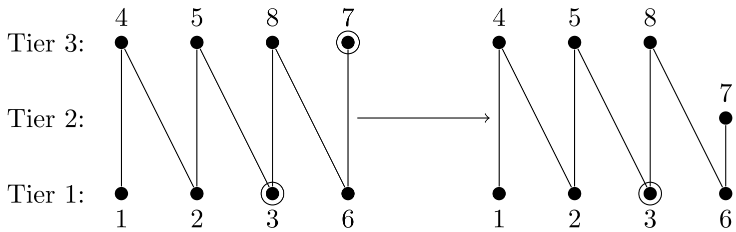


Figure 3.10. Step 1 of applying the map f.

When applying f, we first move tier 3 distinguished vertices into tier 2.

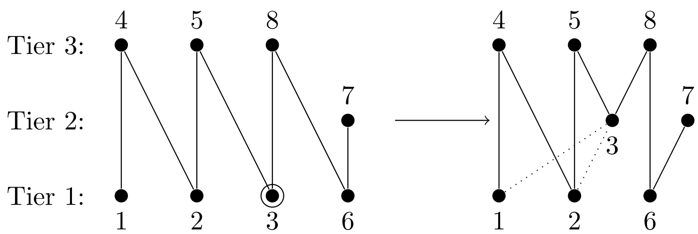


Figure 3.11. Step 2 of applying the map f.

Next, we move the tier 1 distinguished vertices into tier 2. We see that in this case, this creates two externally active edges (indicated by dotted lines).

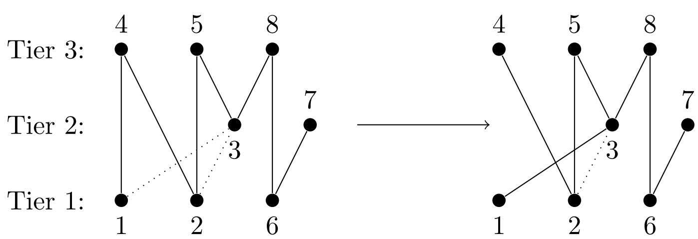


Figure 3.12. Step 3 of applying the map f.

Of the two externally active edges, the one between vertices 1 and 3 comes first lexicographically, so we add this edge to the tree. We delete the edge connecting vertex 1 to vertex 4 because our new edge would complete a cycle of five edges in which the edge connecting vertex 1 to vertex 4 comes second lexicographically.

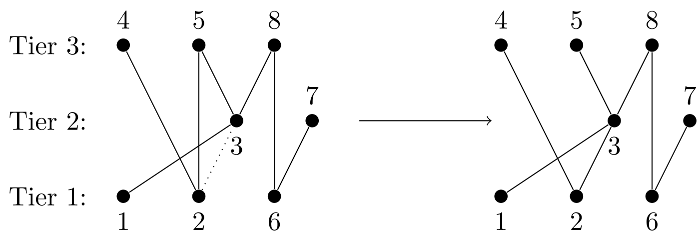


Figure 3.13. Step 4 of applying the map f.

Lastly, we add the one remaining externally active edge. Since this creates a triangular cycle in which the edge between vertices 2 and 5 comes second lexicographically, we delete that edge. We can see that by applying g to our original tree, and then applying f to the result, we have recovered the original tree.

3.4 Alternating Trees and Eulerian Numbers

3.4.1 Alternating Trees and Permutations

In this section, we will focus on alternating trees. All results appearing in this section come from joint work with Dugan, Gunnells, and Steingrimsson that can be found in [16]. We will begin by showing that there are exactly $(n-1)!$ alternating trees of weight 0 on n vertices and that the numbers of maxima that they have are determined by the Eulerian numbers.

Theorem 3.4.1. *There is a bijection between the set of permutations in the symmetric group S_n with k descents and the set of weight 0 alternating trees with $n+1$ vertices and $k+1$ local maxima.*

Proof. We will define the bijection h recursively. We map the identity permutation $1, 2, 3, \ldots, n-1, n$ to the unique alternating tree on $n+1$ vertices with one maximum, and we map the permutation $n, n-1, \ldots 3, 2, 1$ to the unique alternating tree on $n+1$ vertices with one minimum (see Figure 3.14). We will think of the group S_n as the subgroup of S_{n+1} consisting of permutations ending in the symbol $n+1$, and our bijection will turn elements of this subgroup into alternating trees on $n+1$ vertices. We will represent elements of this subgroup as ordered sequences of $1, \ldots, n+1$ ending in $n+1$. Given a permutation π in this subgroup, we break it up into sequences on the left and right of 1: $\pi = \pi_L \cdot 1 \cdot \pi_R$ (where the operation $\cdot$ is concatenation of subsequences). We then further break up π_L by letting π_1 be the subsequnce starting at the beginning of π_L and ending in the largest symbol m_1 in π_L, then letting π_2 be the subsequence beginning at the symbol following m_1 and ending at the largest remaining symbol m_2 in π_L. We continue in this fashion until every symbol in π_L belongs to some π_i for some positive integer i. This results in the expression $\pi = \pi_1 \cdot \pi_2 \cdot \cdots \cdot \pi_l \cdot 1 \cdot \pi_R$. Let $d(\pi_j)$ denote the number of descents in the subsequence π_j. Note that by definition, the subsequence π_R and each of the subsequences π_i ends in the largest symbol appearing in the subsequence, so by induction on the number of vertices, we can construct from each of these subsequences π_j a weight 0 alternating tree on the symbols appearing in π_j with $d(\pi_j) + 1$ maxima (We can do this by temporarily relabeling the symbols with consecutive integers starting with one in the unique way that preserves all inequalities between the symbols in π_j). We do this with each subsequence and then add an edge between the vertex 1 and the lowest-labeled maximum in each of the trees corresponding to π_R and the π_i. Since we have chosen the lowest-labeled maximum in each tree, it is clear from the recursive definition of weight that the resulting tree will have weight 0. Since the resulting tree has $(d(\pi_1) + 1) + \cdots + (d(\pi_l) + 1) + (d(\pi_R) + 1)$ maxima, and π has $d(\pi_1) + \cdots + d(\pi_l) + d(\pi_R) + l$ descents, we can see that this tree has the correct number of maxima.

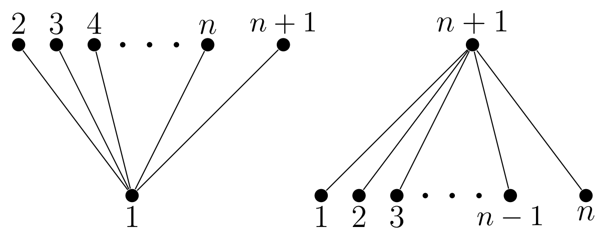


Figure 3.14. The unique alternating trees on $n+1$ vertices with one minimum and one maximum, respectively.

We can see that this map is bijective because we can recursively define an inverse. When applying this inverse map to an alternating tree, we first delete all edges that have the vertex 1 as an endpoint. We let the block B_R consist of the labels of vertices in the component of the vertex $n+1$. We call the other trees in our resulting forest $T_1, \ldots, T_l$ where T_1 is the tree containing the highest labeled vertex not in, T_2 is the one containing the highest-labeled vertex not in T_1, and so on. We let the block B_i consist of the labls of vertices in T_i. Our resulting permutation is obtained by writing $B_1 \cdot \cdots \cdot B_l \cdot 1 \cdot B_R$ where the order of the symbols within each block is determined by induction on the number of vertices. Finally, we remove the $n+1$ at the end to give us a permutation belonging to S_n. It is easy to check that our two maps are inverses of one another. $\square$

As an example, we will construct the weight 0 tree associated with the permutation 42687153 in S_8. To do this, we first begin thinking of it as the permutation $\pi =$ 427861539 in S_9. Since 8 is the largest symbol appearing to the left of 1 in π, and 6 is the largest symbol between 8 and 1, the resulting decomposition into subsequences is $\pi = 4278 \cdot 6 \cdot 1 \cdot 539$. Therefore, we must find the weight 0 alternating trees associated with the subsequence 4278, 6, and 539, so that we can connect them to the vertex 1. Since the lowest symbol in 4278 is 2, this subsequence decomposes into $4 \cdot 2 \cdot 78$. This means that we should connect the vertex 2 to the vertex 4 and the only maximum in the tree associated with 78 (see Figure 3.15). Likewise, since 3 is the lowest symbol in 539, it decomposes as $5 \cdot 3 \cdot 9$, so both vertices 5 and 9 become maxima that we connect to vertex 5. Since the symbol 6 is alone in its subsequence, vertex 6 will become a maximum that will only be connected to vertex 1. We now add an edge between vertex 1 and the lowest maximum in each of the trees associated with the subsequences 4278, 6, and 538 (vertices 4, 6, and 5, respectively). This gives us the weight 0 tree seen in Figure 3.16.

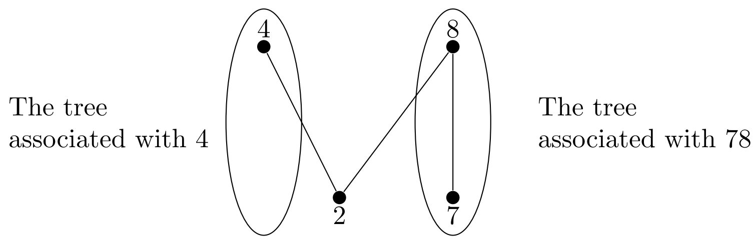


Figure 3.15. The weight 0 alternating tree associated with the subsequence 4278.

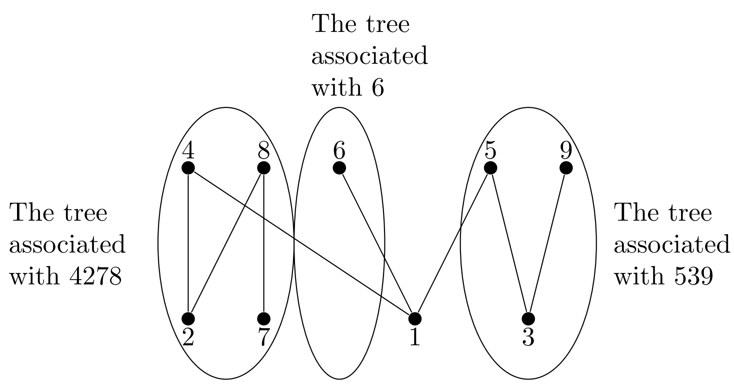


Figure 3.16. The weight 0 alternating tree on 9 vertices associated with the permutation 42687153 in S_8.

3.4.2 Weights of Permutations

Note that our instructions for applying the map h^{-1} to an alternating tree T do not require that T have weight 0, so it makes sense to expand the domain of h^{-1} to the set of all alternating trees on $n+1$ vertices. After expanding the domain of h^{-1}, we will rename this function h' to reflect the fact that it is no longer injective. This allows us to define the *weight* of a permutation.

Definition 3.4.2. Let π be a permutation in S_n. The *weight* $w(\pi)$ of the permutation π is the highest weight of all trees T for which $h'(T) = \pi$.

Observe that we can find the weight of the permutation π by applying a map similar to h to it, but instead choosing to connect vertices to the highest available maximum in a tree instead of the lowest, and then computing the weight of the resulting tree. In our last example, this would mean connecting vertex 1 to vertices 8 and 9 instead of vertices 4 and 5, resulting in a tree of weight 2 (see Figure 3.17).

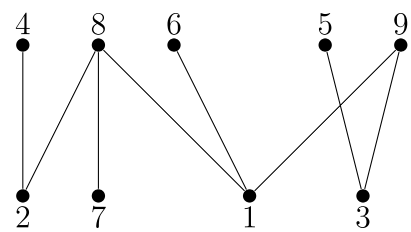


Figure 3.17. The tree on 9 vertices of maximal weight associated with the permutation 42687153 in S_8.

We are now able to define a q-Eulerian polynomial based on weights of permutations.

Definition 3.4.3. For any $\pi \in S_n$, let $d(\pi)$ be the number of descents that the permutation π has. We define the polynomial $E_n(x, q)$ as follows:

$$E_n(x, q) = \sum_{\pi \in S_n} x^{d(\pi)} q^{w(\pi)}$$

For example, we have $E_4(x, q) = 1 + x(q^2 + 3q + 7) + x^2(q^2 + 4q + 6) + x^3$. As discussed in [16], this is different from other q-Eulerian polynomials such as those defined by Carlitz and Stanley.

Theorem 3.4.4. *The number of weight 0 permutations in S_n with l descents is equal to the number of partitions of $[\![n]\!]$ into $l+1$ nonempty subsets (This number is the Stirling number of the second kind $\left\{ {n \atop d} \right\}$).*

Proof. Observe that the permutation $\pi \in S_n$ has weight 0 if and only if, when applying the map h to it, we obtain a decomposition $\pi = \pi_1 \cdot \pi_2 \cdot \cdots \cdot \pi_l \cdot 1 \cdot \pi_R$ where each subsequence corresponds to an alternating tree with only one maximum. Since the number of maxima in each tree is determined by the number of ascents in each subsequence, this is equivalent to having all of the symbols in each subsequence being listed in ascending order. Therefore, π has weight 0 if and only if it is of the form $a_{1,1}a_{1,2}\ldots m_1 \cdot a_{2,1}a_{2,2}\ldots m_2 \cdot \cdots \cdot a_{l,1}a_{l,2}\ldots m_l \cdot 1 \cdot a_{R,1}a_{R,2}\ldots m_R$ with the following properties:

(1) For any $i \in \{1,\ldots,l,R\}$, and any $j_2 > j_1$, we have $m_i > a_{i,j_2} > a_{i,j_1}$.

(2) For any $i_1, i_2 \in \{1,\ldots,l\}$, with $i_2 > i_1$, we have $m_{i_2} > m_{i_1}$.

Therefore, we can map the permutation $a_{1,1}a_{1,2}\ldots m_1 \cdot a_{2,1}a_{2,2}\ldots m_2 \cdot \cdots \cdot a_{l,1}a_{l,2}\ldots m_l \cdot 1 \cdot a_{R,1}a_{R,2}\ldots m_R$ to the partition of $[\![n]\!]$ consisting of the subsets $\{a_{1,1}a_{1,2}\ldots m_1\}$, $\{a_{2,1}a_{2,2}\ldots m_2\},\ldots,\{a_{l,1}a_{l,2}\ldots m_l\}$, and $\{1, a_{R,1},\ldots,a_{R,m}\}$. Note that π has l descents (one beginning at each m_i), and the partition consists of $l+1$ nonempty subsets. This map has a clear inverse that we can make by writing the elements of each subset in a partition of $[\![n]\!]$ in ascending order, placing the sets that do not include 1 in order of decreasing maximal elements, and finally adding the set containing 1 to the end, so the map is bijective, and the theorem is proved. □

BIBLIOGRAPHY

[1] N.; Milman V.D. Alon, *λ1, isoperimetric inequalities for graphs, and superconcentrators*, J. Combin. Theory (1985). MR 0782626

[2] Yonatan Bilu and Nathan Linial, *Constructing expander graphs by 2-lifts and discrepancy vs. spectral gap*, 2003.

[3] Jozef Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284** (1984). MR 0743744

[4] David Forge, *Linial arrangements and local binary search trees*, 2014.

[5] C.D. Godsil and I. Gutman, *On the matching polynomial of a graph*, Algebraic Methods in Graph Theory, Vol. I, volume I of Colloquia Mathematica Societatis Janos Bolyai, 25 (1981). MR 0782626

[6] Paul E. Gunnells, Emmanuel Letellier, and Fernando Rodriguez Villegas, *Torus orbits on homogeneous varieties and Kac polynomials of quivers*, submitted, 2013.

[7] Chris Hall, Doron Puder, and William F. Sawin, *Ramanujan coverings of graphs*, 2015.

[8] F. Harary, *The determinant of the adjacency matrix of a graph*, SIAM Rev., 4 (1962).

[9] O. J. Heilman and E. H. Lieb, *Theory of monomer-dimer systems*, Communications in Mathematical Physics **25(3)** (1972).

[10] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277. MR 963118

[11] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Ann. of Math. (2) **182** (2015), no. 1, 307–325. MR 3374962

[12] Moshe Morgenstern, *Existence and explicit constructions of $q+1$ regular Ramanujan graphs for every prime power q*, J. Combin. Theory Ser. B **62** (1994), no. 1, 44–62. MR 1290630

[13] A. Postnikov, *Intransitive trees*, J. Combin. Theory Ser. A **79** (1997).

[14] Alexander Postnikov and Richard P. Stanley, *Deformations of Coxeter hyperplane arrangements*, J. Combin. Theory Ser. A **91** (2000), no. 1-2, 544–597, In memory of Gian-Carlo Rota. MR 1780038

[15] H. Sachs, *Uber teiler, faktoren und charakteristische polynome von graphen. ii*, Wiss. Z. Techn. Hochsch Ilmenau, 13 (1967).

[16] William Dugan, Sam Glennon, Paul Gunnells, and Einar Steingrimsson, *Tiered Trees, Weights, and q-Eulerian Polynomials.*