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Stirling Numbers of Sunflower Graphs

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1 Abstract

A Stirling number of the second kind, $S(n, k)$, is the number of ways to take all of the elements from an n element set and put them into k subsets, so that the subsets are non-empty and pairwise disjoint. To get the graphical Stirling number for a graph G, we add the restriction that any two vertices that are adjacent in G cannot be in the same subset. The traditional Stirling numbers are the graphical Stirling number where the graph is empty. We find graphical Stirling numbers for sunflower graphs, which are powers of paths joined at a single vertex. We approach the problem in two different ways, (1) by finding the chromatic polynomial and (2) recursively. Our results include the Stirling number for what we refer to as a complete sunflower graph, as well as a few other cases for sunflower graphs. We then form a general conjecture for the chromatic polynomial of a sunflower graph, which would then provide us with the graphical Stirling number for a sunflower graph using the Principle of Inclusion Exclusion. We also find several recursive formulas for finding graphical Stirling numbers, such as the graphical Stirling number for graph G with vertex v with a complete neighborhood, $S(G, k) = S(G - v, k - 1) + (k - \deg(v)) \cdot S(G - v, k)$. We end with a discussion of possible future work.

Contents

2 Introduction

2.1 Key Definitions and Ideas

Stirling numbers, named for James Stirling, deal with counting the number of partitions of a set into a predetermined number of parts. They fall into the field of mathematics called combinatorics. While they have been studied thoroughly in the past, we felt inclined to explore the relationship between Stirling numbers and graphs. To discuss Stirling numbers and their properties in relation to graphs, we must first familiarize ourselves with the definition of a Stirling number.

Definition 1. A *partition* of a set A is a set of non-empty subsets of A such that for all $a \in A$, a is in one and only one of the subsets. We call each subset a *part* of the partition.

Definition 2. Let $n, k \in \mathbb{N}$. The *Stirling numbers of the second kind*, denoted $S(n,k)$, are the number of ways to partition n distinct objects into k nonempty parts. We let $S(n,0) = 0$ for $n \geq 1$, $S(0,0) = 1$, and $S(n,k) = 0$ where $k > n$.

Building on the well-known Stirling numbers of the second kind, we can explore other types of Stirling numbers in which we keep the requirements of $S(n,k)$, but add restrictions onto which of the n objects may be placed into the same part in a partition. For instance, what if objects labeled 1 through n must be a predetermined distance apart in order to be in the same part in a partition? This particular restriction leads us to the distance Stirling numbers.

Definition 3. Let n, k, and d be natural numbers. The *distance Stirling numbers of the second kind*, denoted $S^d(n,k)$, is the number of ways to partition a set of n distinct elements into k nonempty disjoint sets $U_1, U_2, ..., U_k$ such that for all $1 \leq i \leq k$, if $n_0, n_1 \in U_i$, then $|n_1 - n_0| > d$.

We note here that we define this slightly differently from [1]. This is because we will relate these distance Stirling numbers to powers of paths.

The traditional Stirling numbers of the second kind are a special case where $d = 0$. That is, when we say $S(n,k)$, in the context of distance Stirling numbers, we are referring to $S^0(n,k)$. This is because the requirement is that $|n_1 - n_0| > 0$ and the distance between any two distinct natural numbers is always at least 1.

We also take a result from [1], specifically Theorem 7. In terms of our notation, we have

$$S^d(n,k) = S(n-d, k-d), \text{ where } n \geq k \geq d.$$

Since distance Stirling numbers (with $d \geq 1$) have objects that cannot be placed in the same part, we know that when $k = d$ and $d \geq 1$, $S^d(n,k) = S(n-d, 0) = 0$. We can make this result more general and say that $S^d(n,k) = 0$ with $k \leq d$ and $d \geq 1$ since we would get either a negative number of parts or 0 parts when using the Mohr and Porter theorem above. Another result of this definition of distance Stirling numbers is that $S^d(n, d+1) = 1$. The proof of this result is below.

Theorem 1. Let $d, n \in \mathbb{N}$ and $n \geq d$. Then $S^d(n, d+1) = 1$.

Proof. In [1] the authors prove $S^d(n,k) = S(n-d, k-d)$ where $n, k, d \in \mathbb{N}$ and $n \geq k \geq d$. If we let $k = d+1$, then

$$\begin{aligned} S^d(n, d+1) &= S(n-d, (d+1)-d) \\ &= S(n-d, 1) \\ &= 1. \end{aligned}$$

Thus, $S^d(n, d+1) = 1$. □

Intuitively, this proof makes sense because we know right away that we will need one more part than our distance in order to partition the objects.

For example, if we have 5 objects and our distance is 2, we can partition it into 3 parts. We know that 1 cannot be with 2 or 3, so each of those objects must be in separate parts. So we can start by placing 1, 2, and 3 in their own parts. Now we know that 4 can only be in the same part as 1, so we place 4 with 1. Now 5 only has one possible part into which it can be placed, the part with object 2. Had we placed object 5 with 1 before we had placed object 4, we would not have had any possible part into which 4 could have gone. This gives us the partition below.

$$14 \qquad 25 \qquad 3$$

Therefore, this is the only possible partition for this example, $S^2(5,3) = 1$.

Since our goal is to eventually find Stirling numbers for graphs, it would be ideal to examine what the graph corresponding to a distance Stirling number would look like. We can represent the restrictions of objects in the same part being greater than d apart as a graph. We place a labeled vertex corresponding to each object and make any two vertices that are d or fewer apart (and therefore cannot be in the same part) adjacent by placing an edge between them. This process yields a power of a path.

Definition 4. Given a connected graph G and two vertices v and w we say that the *distance* between v and w is the number of edges in a shortest path between v and w. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ for some natural number n. The k^{th} *power* of G has the same vertex set and we say that two vertices v_i and v_j in the k^{th} power graph are adjacent if and only if they have distance $d(v_i, v_j) \leq k$ in G. We denote the k^{th} power of G as $(G)^k$.

Powers of graphs can be more easily comprehended through examples, which we provide in Figures 1 and 2, where we look at P_4 and P_4^2:

From here we can construct the 'petals' of our sunflower graphs.

Definition 5. Let P_ℓ be a path of length ℓ. A *petal graph*, is a d^{th} power of a path P_ℓ with an end vertex identified as a root vertex. We denote it as $(P_\ell)^d$. We say that ℓ is the length of the petal, and d is the distance.

We will soon be looking at partitioning the vertex set of these graphs, so will first examine what the vertex set is:

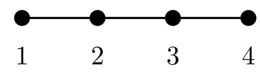


Figure 1: A path of four vertices (raised to the first power), P_4.

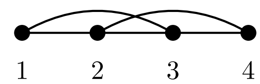


Figure 2: A path of four vertices raised to the second power, P_4^2.
In P_4^2, we add the edges $\{1,3\}$ and $\{2,4\}$ since the distance between these vertices in P_4 is 2.

Definition 6. Let $d, p, \ell \in \mathbb{N}$ such that $d \leq \ell$. A *petal graph*, denoted as $(P_\ell)^d$, has the vertex set,

$$V((P_\ell)^d) = \{v_t \mid 0 \leq t \leq \ell\}$$

where we say v_0 is the *root* of the graph. The edge set for this graph is,

$$E((P_\ell)^d) = \{\{v_t, v_{t'}\} \in V\left((P_\ell)^d\right) \times V\left((P_\ell)^d\right) : |t - t'| \leq d\}.$$

Now to construct the sunflower graph, we can identify multiple petal graphs at their root vertices, as the definition below shows. An example of a sunflower graph is given in Figure 3.

Definition 7. Let $\mathcal{P} = \left\{(P_{\ell_1})^{d_1}, (P_{\ell_2})^{d_2}, ..., (P_{\ell_i})^{d_i}\right\}$ be a set of petal graphs. We identify the root vertices of each petal graph as a single vertex and call it the *center vertex* of our new graph. We call this graph a *sunflower graph*, and denote it as $F_\mathcal{P}$.

In a lot of our theorems, the distance and length of our petals are consistent throughout the graph. We denoted these sunflower graphs as $F_{\ell^p}^d$ to mean the sunflower graph with $\mathcal{P} = \{(\mathcal{P}_\ell)^d, \ldots, (\mathcal{P}_\ell)^d\}$ and $|\mathcal{P}| = p$.

Now that we have defined our primary graphs of interest, we can define what a Stirling Number of such graphs would look like. To do so, we must first define stable partitions.

Definition 8. A *stable partition* of a graph G is a set partition of the vertex set of G with the property that no two elements in the same part are adjacent.

The term stable partition is motivated by the concept of stable sets in graph theory. In a graph a *stable set*, also known as an *independent set* is a set of vertices that have no edges between them. Each part of our stable partition can have no edges between them, hence the term.

Now, as we can make the logical leap from partitions of a set to our traditional Stirling numbers, we can do the same by making a similar leap from stable partitions of a graph to the Stirling number of a graph.

Definition 9. For a natural number k, the *graphical Stirling number* of graph G, $S(G,k)$, is the number of stable partitions with exactly k parts.

Note that if $k \geq |V(G)|$ then $S(G,k) = 0$. Also, if $k \leq 0$ then $S(G,k) = 0$. We'll later see that if $k > \chi(G)$ where $\chi(G)$ is the chromatic number, then $S(G,k) = 0$.

2.2 Previous Work

The graphical Stirling number has been studied before. Galvin and Thanh [2] suggest that they were first studied in a PhD thesis by Tomescu in 1971 [3]. In this section we provide a summary of some of the previous work that has been done.

The main paper that inspired us to look at this problem is that of Mohr and Porter [1]. In this paper they define the distance Stirling numbers to be the Stirling numbers of the second kind where numbers in the

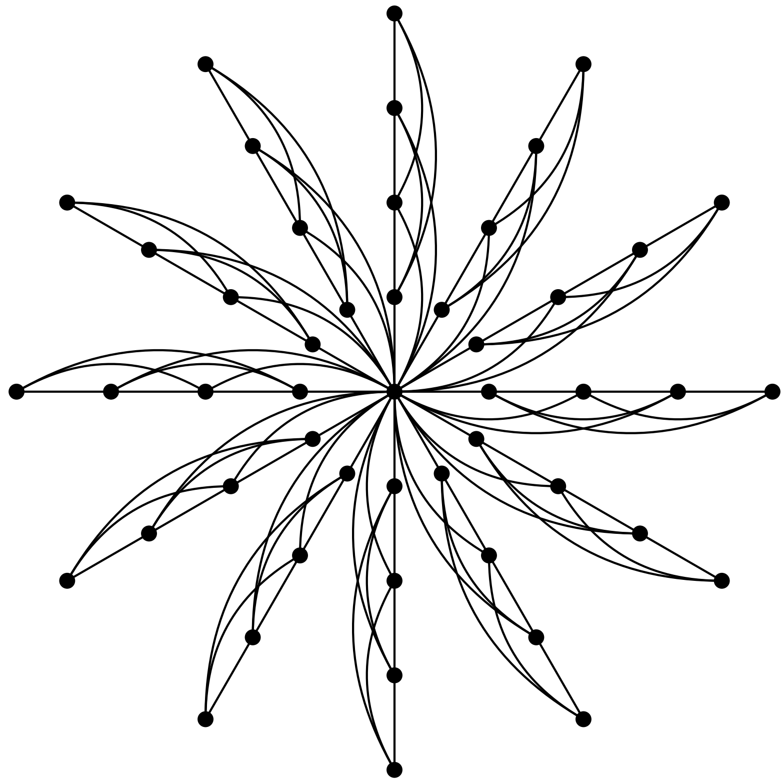


Figure 3: The Sunflower Graph, $F^3_{4^{12}}$.

same part must have difference at least d. Here they use the chromatic polynomial, inclusion-exclusion, and induction. Their main result is to relate the distance Stirling numbers to the traditional Stirling numbers.

At a similar time, Duncan and Peele published [4]. In this paper they, like Mohr and Porter, find the same results on powers of paths. The paper also discusses Bell numbers which are the total number of ways to place n elements in boxes so that each box is nonempty for any number of boxes. The Bell numbers and the Stirling numbers are related in that we can sum up the Stirling numbers for each number of boxes and get the Bell number. Letting $S(G, k)$ be the graphical Stirling number for G into k parts and $B(G)$ be the Bell number of the graph G then $\sum_{n\geq 1} S(G, k) = B(G)$. This paper also introduced us to the technique of addition and contraction which we use in Conjecture 21

There are two additional papers of interest that also discuss the "distance Stirling numbers" or $S(G, k)$ for G being the power of a path graph. Chu and Wei provide algebraic arguments [5] while Shattuck provides a combinatorial proof [6].

We found [4] by Duncan and Peele to be of particular interest. In this paper they found Bell and Stirling Numbers for graphs.

Definition 10. The *Bell number*, B_n, is the number of partitions of n objects (into any number of parts).

Definition 11. The *Bell number of a graph G*, $B(G)$, is the number of stable partitions of G.

So, the key difference between $B(G)$ and $S(G, k)$ is that with graphical Stirling numbers we are partitioning the vertices of the graph into a predetermined number of parts. For an example, we will find $B(G)$ for $G = P_4$, the path with 4 vertices (Figure 4). According to [4], $B(P_n) = B_{n-1}$, so $B(P_4) = B_3 = 5$. The stable partitions of this graph are:

$$1 - 2 - 3 - 4$$

$$13 - 2 - 4 \qquad 14 - 2 - 3 \qquad 1 - 24 - 3$$

$$13 - 24$$

where each $-$ represents a separation between parts.

So we have verified that $B(P_4) = 5$. We can also compute the graphical Stirling numbers of this graph. One of the above partitions is with 4 parts, so $S(P_4, 4) = 1$; three are with 3 parts, so $S(P_4, 3) = 3$; one is with 2 parts, so $S(P_4, 2) = 1$. Since there is at least one pair of vertices that cannot be placed in the same part, we know $S(P_4, 1) = 0$. We notice something here that holds true generally as well, that we can find Bell numbers in terms of Stirling numbers: $B(G) = \sum_{i=1}^{n} S(G, i)$.

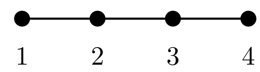


Figure 4: A path of four vertices, P_4.

While this paper dealt mostly with Bell numbers, it introduced us to the ideas of deletion-contraction and addition-contraction, which we found to be useful tools. These tools are explained in Section 3.

Another paper that touched on the same basic topics and paved the way for our research was [1]. In it, they proved the following theorem about distance Stirling numbers, which we also prove here (although our definition of distance Stirling numbers varies slightly, it is the same general proof).

Theorem 2. [1] The distance Stirling numbers of the second kind $S^1(n, k)$ satisfy the recurrence relation $S^1(n, k) = S^1(n-1, k-1) + (k-1) \cdot S^1(n-1, k)$ for $k \leq n$.

Proof. By the definition of a distance Stirling number, we know that we have n distinct elements partitioned into k distinct sets where any pair of elements in the same part are greater than one apart. We have two disjoint cases when it comes to just one element. Let the objects be labeled $1, \ldots, n$ and consider the element 1.

The first case is when 1 is in a set by itself. If 1 is in a set by itself we can just take the Stirling number of $S^1(n-1, k-1)$ since we are taking one element out and one set out for 1.

The second case is when 1 is in a set not by itself. Since 1 is in a set with other elements we have $S^1(n-1, k)$ ways to partition the other elements into k sets. Note 1 can go into every one of the k sets except for the single set with 2 in it. Thus we multiply $S^1(n-1, k)$ by $k-1$. These cases are disjoint since 1 can't be in a set by itself and also be in a set with other numbers. Therefore we add both cases and we can get $S^1(n, k)$. Therefore, $S^1(n, k) = S^1(n-1, k-1) + (k-1) \cdot S^1(n-1, k)$. $\square$

2.3 Properties of Stirling Numbers

In this section we will be talking about some of the properties that are known about Stirling numbers and some properties that we have found. A well known property of Stirling numbers is the following recurrence relation.

Theorem 3. Let $n, k \in \mathbb{N}$. The traditional Stirling numbers of the second kind $S(n, k)$ satisfy the recurrence relation

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

for $1 \leq k \leq n$.

Proof. Let n and k be natural numbers such that $1 \leq k \leq n$. Also let $S(n, k)$ denote the Stirling numbers of the second kind. By definition, $S(n, k)$ counts the number of ways to partition the elements from a set B where $|B| = n$ into k disjoint nonempty subsets. First let B be a set with n distinct elements and identify one distinct element $p \in B$. We will consider what happens when p is placed into its own part and what will happen when p is placed into a part with other elements of B.

In the first case, we take the element p and place it into its own subset, we know there is only one way to do this. We then consider how to partition the set $B - \{p\}$ into $k-1$ disjoint nonempty subsets. By the definition of a Stirling number of the second kind, this is exactly $S(n-1, k-1)$ since $|B - \{p\}| = n-1$. We can do this because p took up one of the k subsets, and only $k-1$ subsets remain to partition the remaining set $B - \{p\}$. So in this case we have $S(n-1, k-1)$

In the second case, we temporarily disregard the element p and look at how to partition the set $B - \{p\}$ into k stable parts. Again, we know that $|B - \{p\}| = n - 1$ so by the definition of a Stirling number of the second kind, this is $S(n-1, k)$. Then we consider what happens to the element $p \in B$ that we originally disregarded. Since we already have k stable parts, we now count how many ways there are to place the element p into one of these k stable parts. There are k ways to do this, so we have $k \cdot S(n-1, k)$ ways to partition our set B.

Note that in both cases we took a set B and partition its elements into k stable parts. We know these two cases are disjoint since one case counts the number of ways to partition a distinct element p into its own stable part. The second case on the other hand counts the number of ways to partition a distinct element p into an already partitioned set $B - \{p\}$.

So using both cases, we have $S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$ as needed. □

Stirling numbers can be used to count many different objects. One of the things that Stirling numbers can be used for is trying to find the number of surjections bewtween a finite domain and codomain.

Theorem 4. Let N represent the number of surjections from a domain of n elements, to a codomain of k elements. Then

$$N = k!S(n, k).$$

Proof. Stirling numbers of the second kind count the number of ways of placing n objects into k nonempty boxes. In other words, we have a function that maps the n elements of a domain D to the k elements of a codomain C. Since we require that no box is empty, each element in the codomain is "mapped-to"; so, we have a surjective function. When we count the number of surjections, we must take into account which elements in D are being mapped to which elements in C. For example, choose $x_1, x_2 \in D$ and $y_1, y_2 \in C$. A surjection in which $f(x_1) = y_1$ and $f(x_2) = y_2$ is *different* than a surjection in which $f(x_1) = y_2$ and $f(x_2) = y_1$. However, for the Stirling numbers if the placement has y_1 in a box by itself and y_2 in another box by itself, then it does not matter if you switch the order of the boxes. Therefore, we must multiply $S(n, k)$ by $k!$ to get the number of surjections. □

Theorem 4 relates the number of surjections to the Stirling numbers. In Section 3 we discuss how we can use this theorem and the Principle of Inclusion Exclusion to find a formula for the Stirling numbers.

Next we explored some properties of Stirling numbers. One fact that has been useful is the size of the largest part of any partition of n elements into k non empty subsets.

Theorem 5. In any set partition of n elements into k nonempty parts, the size of the largest part is at most $n - k + 1$.

Proof. Since we need *nonempty* parts and our goal is to attain a partition in which we have as many objects as possible in one part, we first place one object into each of the first $k - 1$ of our k parts. This leaves us with $n - (k - 1)$ objects remaining, and one empty part. We place these $n - (k - 1) = n - k + 1$ objects into the remaining part, and therefore have no empty parts, and one "largest" part with $n - k + 1$ objects. □

Having the knowledge from Theorem 5 we can prove the following theorem.

Theorem 6. For $n \in \mathbb{N}$, $S(n, n-1) = \binom{n}{2}$.

Proof. Using Theorem 5, the maximum number of elements in one box so that none are empty is $n - k + 1$. Since our k here is $n - 1$, we have

$$\begin{aligned} n - (n-1) + 1 &= n - n + 1 + 1 \\ &= 2. \end{aligned}$$

Thus there is only one case to consider, where one part has two elements which means all the other parts only have one element. It follows that there are $\binom{n}{2}$ ways to choose which two elements go into the same part since the order of the parts do not matter. □

Using Theorem 5 again we can find an explicit formula for $S(n, n-2)$, the number of ways to partition a set of size n into $n-2$ parts.

Theorem 7. For $n \in \mathbb{N}$ such that $n \geq 4$, $S(n, n-2) = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$.

Proof. Using Theorem 5, the maximum number of elements in one part so that none are empty is $n-k+1$. Since our k here is $n-2$, we have

$$\begin{aligned} n-(n-2)+1 &= n-n+2+1 \\ &= 3. \end{aligned}$$

Therefore, we have two cases to consider. The first case is when we have three elements in one box which then means that the other boxes only have one element. The second case is when there are two boxes with two elements and then the rest of the boxes have one element.

For our first case, the number of ways to choose 3 elements for one part is $\binom{n}{3}$. Since the order of the parts do not matter and no box is empty, there is only 1 way to replace the remaining elements.

For the second case, we have two parts with two elements and the rest with one. We have $\binom{n}{2}$ options for the first part and it follows that we have $\binom{n-2}{2}$ options for the second part. We are not done because we are over counting since the first box we can have $\{1,2\}$ then the second box could be $\{3,4\}$ but then in a different example we can have the first box be $\{3,4\}$ and the second box be $\{1,2\}$ which means each partition is getting counted twice. So we need to exclude some of the combinations by multiplying by $\frac{1}{2}$. For each of the options, we have to choose the elements that go into the two boxes so we are counting one of the Stirling arrangements twice.

Thus, the second term by the multiplication axiom is $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$. Therefore $S(n, n-2) = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$. $\square$

While we began our exploration with the traditional Stirling numbers our goal was to look at graphical Stirling numbers. In the next section we discuss properties of our main graph of interest - sunflower graphs.

2.4 Properites of Sunflower Graphs

One property of sunflower graphs we quickly found was the number of edges. In the next theorem we find the number of edges in one petal.

Theorem 8. The number of edges in a petal with length ℓ and distance d with $\ell \geq d$ is

$$|E((P_\ell)^d)| = d(\ell - d + 1) + \sum_{i=1}^{d}(d-i).$$

Proof. Let there be a petal with length ℓ and distance d. We will think of how many edges a certain vertex has by considering how many other vertices it reaches to in front of it with the given distance. This helps us not to count edges twice since if we count that vertex 1 is connected to vertex 2, we do not need to then count that vertex 2 is connected to vertex 1. As soon as we get to the vertex in the graph in which it has less than d other vertices in front of it, it cannot reach the full distance. The first vertex to have this problem of not having enough vertices in front of it will have $(d-1)$ edges, the next $(d-2)$ edges. This goes all the way to the end until you have the last vertex as $(d-d) = 0$ edges, which follows from our reasoning since there are no more vertices in front of it. Hence we get $\sum_{i=0}^{d}(d-i)$. Consider the first vertex in the graph and how many other vertices it reaches to. It will always reach d vertices in front of it since $\ell \geq d$. Since there is $\ell+1$ total vertices and d vertices that do not reach to their full distance, there will be $(\ell+1)-d = \ell-d+1$ vertices that reach their full d other vertices, hence $d(\ell-d+1)$ edges.

For illustration, see Figure 5, a petal with length 6 and distance 3 in which the vertices are labeled with how many other vertices it reaches to.

$\square$

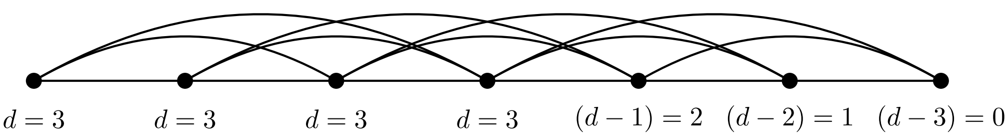


Figure 5: Length 6 distance 3

By knowing how many edges are on just one petal we can use that to find the number of edges on the whole sunflower graph.

Corollary 9. The number of edges in a sunflower graph, $F^d_{\ell^p}$, is

$$|E(F^d_{l^p})| = p\left(d(\ell - d) + \sum_{i=0}^{d}(d-i)\right).$$

Proof. From what we know from Theorem 8 we can use that to prove this corollary. Since we know that the number of edges on a petal is $d(\ell - d + 1) + \sum_{i=1}^{d}(d-i)$ we can take that and multiply by the number of petals. That would give us $p\big(d(\ell - d) + \sum_{i=0}^{d}(d-i)\big)$. □

Our overall goal is to find $S(G,k)$ for each k. It turns out that the smallest k for which $S(G,k) \neq 0$ is $\chi(G)$, which we will define next.

Definition 12. A *proper coloring* on a graph, G, is a coloring of the vertices such that no pair of adjacent vertices are the same color.

Definition 13. [7] The *chromatic number* of a graph, G, denoted $\chi(G)$, is the fewest colors needed for G to have a proper coloring.

We can consider the set of vertices with a given color to be a part in our stable partition since vertices of the same color can't be adjacent. Since the chromatic number gives the fewest colors that can be used, it is also the fewest number of parts we can have in a partition. Thus $S(G,k) = 0$ when $k < \chi(G)$. In the next theorem we find the chromatic number of sunflower graphs that have all petals the same length.

Theorem 10. The chromatic number of any sunflower graph with petals that have the same distance is

$$\chi(F^d_{\ell^p}) = d + 1.$$

Proof. We will show that $\chi(F^d_{\ell^p}) \leq d+1$ by showing that $F^d_{\ell^p}$ can be colored with $d+1$ colors. Since our definition of a sunflower graph prohibits any vertices on different petals from being connected, we will talk about coloring one petal, all other petals can be colored in the same way. We can start by coloring the first vertex on the petal (the root vertex of our sunflower graph). Then we color each of the d neighbors of that vertex using the remaining d colors. The next vertex on the petal may then be the same color as the root vertex, the next may be the same as the vertex nearest the root, and so on. This means that after coloring the first $d+1$ vertices on each petal, the remaining vertices can be colored without adding any additional colors. Therefore, the minimum number of colors needed to properly color a petal (and therefore an entire sunflower graph) is $d+1$. So,

$$\chi(F^d_{\ell^p}) \leq d + 1.$$

Consider the center vertex v of F. Let $N_P[v]$ be the closed neighborhood of v on one petal. By definition of a sunflower graph $N_P[v]$ that is complete. There are $d+1$ vertices in this closed neighborhood since v has degree d in the petal. Since the vertices are all adjacent we know that we must use a different color on each vertex. Thus we need $d+1$ colors to properly color $N_P[v]$ and thus need at least $d+1$ colors to properly color the whole graph. So,

$$\chi(F^d_{\ell^p}) \geq d + 1.$$

Thus we have shown that $\chi(F^d_{\ell^p}) = d + 1$. □

There are many proper colorings of a sunflower graph, the one described in the proof of Theorem 10 is only one. For another example of a proper coloring using $d+1$ colors see Figure 6.

3 Tools

In this section we discuss tools we found in different literature and also tools that we use to address our problem. In particular, we provide background for the Principle of Inclusion Exclusion, the chromatic polynomial, deletion-contraction, and addition-contraction.

3.1 The Principle of Inclusion Exclusion

The Principle of Inclusion Exclusion allows us to find the number of elements in a union of sets where the intersections between sets is nonempty. This technique can be found in many combinatorics books.

Theorem 11. (Principle of Inclusion Exclusion [8]) Let U be some universal set and let $A_1, A_2, ..., A_k$ be subsets of U with $k \in \mathbb{N}$. Then

$$\begin{aligned}\left|\overline{\bigcup_{i=1}^{k} A_i}\right| &= |U| - |A_1| - |A_2| - ... - |A_k| + |A_1 \cap A_2| + |A_1 \cap A_3| + ... \\ &+ |A_{k-1} \cap A_k| + ... + (-1)^k |A_1 \cap A_2 \cap ... \cap A_k| \\ &= |U| - \sum_{k=1}^{n} (-1)^k \left(\sum_{1 \le i_1 \le i_2 \le ... \le i_k \le n} |A_{i_1} \cap ... A_{i_k}| \right).\end{aligned}$$

We'll first consider an application of the Principle of Inclusion-Exclusion with surjections. Recall from Theorem 4, we can relate the number of surjections directly to the Stirling numbers. Instead, one can find the number of surjections between a domain and codomain using the Principle of Inclusion-Exclusion.

Theorem 12. Let N represent the number of surjections from a domain of n elements, to a codomain of k elements. Then

$$N = \sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i.$$

Proof. Let $D = \{1, 2, \ldots, n\}$ be the domain and let $C = \{1, 2, \ldots, k\}$ be the codomain. Let U be all functions that map D to C. Choose $i \in C$ and let A_i be the functions from D to C in which i is not "mapped-to"; that is, for all $x \in D$, $f(x) \neq i$. So, $|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k|$ represents all the functions from D to C that "miss" at least one element in C (and are therefore not surjections).

Therefore, $N = |U| - |A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k| = |(A_1 \cup A_2 \cup \cdots \cup A_k)^c|$ represents the number of surjections, N, from D to C. Therefore, we want to use Theorem 11 to find

$$|U| - (|A_1| + |A_2| + \ldots) + (|A_1 \cap A_2| + \ldots) - \ldots$$

Since we defined U as the set of functions mapping D of n elements to C of k elements, we can represent it as k^n. Since we defined $|A_i|$ to be the number of functions "missing" element $i \in C$, we can represent all these with $(k-1)^n$. We know there is $\binom{k}{1}$ of them since there are $\binom{k}{1}$ ways to choose which element we are going to miss. So, we have $\binom{k}{1}(k-1)^n$ ways to miss a single element in our codomain C. Similarly, all of the ways to miss two different elements in C, $|A_i \cap A_j|$, can be represented by $(k-2)^n$, and we know there are $\binom{k}{2}$ ways to do so. Continuing in this manner, we have

$$\begin{aligned}&\binom{k}{0}k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \ldots \\ &= \sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i.\end{aligned}$$

The top limit of the sum is k, because there are k elements in C to miss. Thus, we have:

$$N = \sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i.$$

□

This next corollary combines two earlier theorems to find another way to express Stirling numbers.

Corollary 13. Let $S(n,k)$ be the Stirling number of the second kind. Then

$$S(n,k) = \frac{\sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i}{k!}.$$

Proof. From Theorem 12, we know that the number of surjections, N, from a domain of n elements to a codomain of k elements is $\sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i$. From Theorem 4, we know that N can also be represented as $k!S(n,k)$. Setting these two quantities equal to each other, we have

$$k!S(n,k) = \sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i.$$

Dividing each side by $k!$ yields

$$S(n,k) = \frac{\sum_{i=0}^{k} \binom{k}{i} \cdot (k-i)^n \cdot (-1)^i}{k!}$$

□

3.2 The Chromatic Polynomial

One way of finding graphical Stirling numbers relies on first finding the chromatic polynomial of a graph. Recall the definition of a proper coloring seen in Definition 12. We define the chromatic polynomial in Definition 14.

Definition 14. For a graph G and a natural number k, the *chromatic polynomial,* $\chi(G,k)$*,* is the number of proper colorings of a graph G with at most k colors.

As an example of a chromatic polynomial consider the graph in Figure 7. Since this is a complete graph, we know that each vertex must have its own distinct color. So to color this graph with at most k colors, we know that there are $k(k-1)(k-2)(k-3)(k-4)$ ways. So

$$\chi(K_5,k) = k(k-1)(k-2)(k-3)(k-4).$$

Another common tool that we use is the inclusive chromatic polynomial, which we define below.

Definition 15. The *inclusive chromatic polynomial*, $\tilde{\chi}(G,k)$, represents the number of proper colorings of a graph G with exactly k colors.

To find the inclusive chromatic polynomial we apply The Principle of Inclusion Exclusion to the chromatic polynomial.

Theorem 14. Given the chromatic polynomial of the graph, the inclusive chromatic polynomial is given by

$$\tilde{\chi}(G,k) = \sum_{j=0}^{k-\chi(G)} (-1)^j \binom{k}{j} \chi(G,k-j)$$

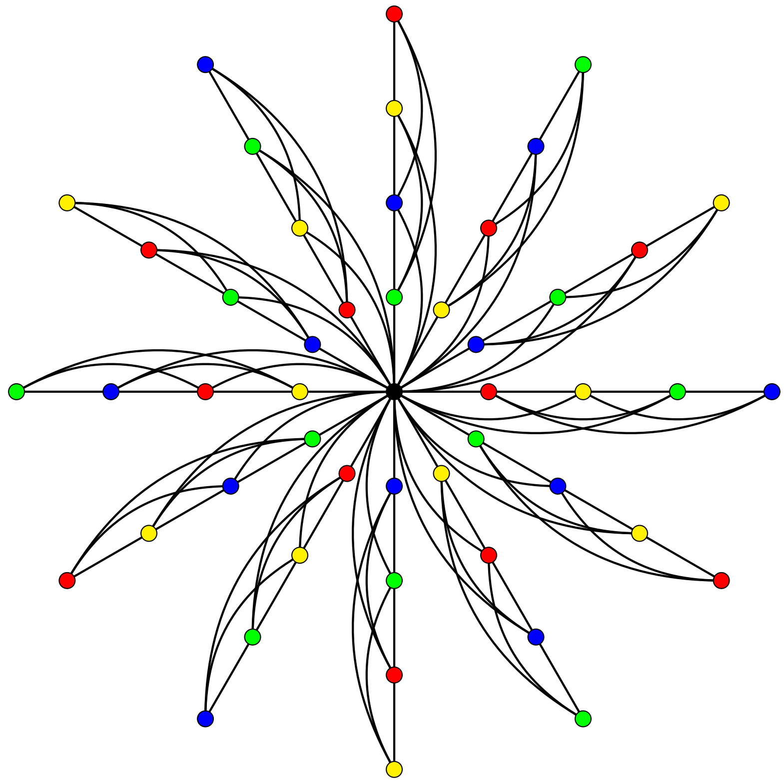


Figure 6: This is a proper coloring of the sunflower graph, $F^3_{4^{12}}$.

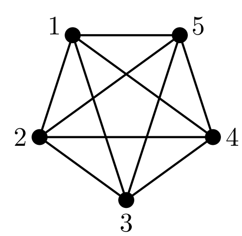


Figure 7: The complete graph of 5 vertices, K_5.

Proof. The chromatic polynomial with k colors gives us the way to properly color a graph with at most k colors. To get from at most k colors to exactly k colors, we use inclusion-exclusion (Theorem 11).

We let U be the universal set of all the proper colorings of our graph G with *at most* k colors. We label our colors $1, 2, \dots, k$. Let set $A_i \subseteq U$ contain the proper colorings missing color i for $1 \leq i \leq k$. Then $|A_i|$ is the number of proper colorings that are missing color i. Then $|A_i \cap A_j|$ is the number of proper colorings that are missing colors i and j and so forth. To get *exactly* k colors, we must eliminate the cases in which we are missing one or more of the colors. Thus in this representation, $|(A_1 \cup A_2 ... \cup A_k)^c|$ is the number of ways to color G with *exactly* k colors as to not include any colorings that use any fewer than k colors. Thus we must use the Principle of Inclusion Exclusion. We want,

$$|U| - (|A_1| + |A_2| + \dots) + (|A_1 \cap A_2| + \dots) - \dots$$

Since we defined U as the set of all proper colorings with at most k colors, we can represent it as $\chi(G, k)$. Since we defined $|A_i|$ to be the number of proper colorings missing a single color i, we can represent all these with $\chi(G, k-1)$. We know there is $\binom{k}{1}$ sets A_i since there are $\binom{k}{1}$ ways to choose which color we are going to miss. Similarly, all of the $|A_i \cap A_j|$ can be represented by $\chi(G, k-2)$ and we know there are $\binom{k}{2}$ pairs, A_i, A_j, to consider. Continuing in this manner, we have

$$\begin{aligned}
\binom{k}{0}\chi(G,k) - \binom{k}{1}\chi(G,k-1) + \binom{k}{2}\chi(G,k-2) - \cdots + (-1)^k\binom{k}{k}\chi(G,k-k) \\
= \sum_{j=0}^{k}(-1)^j\binom{k}{j}\chi(G,k-j) \\
= \sum_{j=0}^{k-\chi(G)}(-1)^j\binom{k}{j}\chi(G,k-j).
\end{aligned}$$

We can make the last term in the sum $k - \chi(G)$ since when $j = k - \chi(G)$ we get the number of colors is $k - (k - \chi(G)) = \chi(G)$ and so $\chi(G, k-j)$ will be 0 for $j > k - \chi(G)$.

□

Theorem 15. Given the inclusive chromatic polynomial of a graph G, $\tilde{\chi}(G,k)$, the Stirling number of that graph is,

$$S(G,k) = \frac{1}{k!}\tilde{\chi}(G,k).$$

Proof. The inclusive chromatic polynomial gives us the number of colorings with exactly k colors. The step to get from colorings to partitions is to divide by $k!$. Thus,

$$S(G,k) = \frac{1}{k!}\tilde{\chi}(G,k).$$

□

To recap how to find the Stirling number of a graph G we can

- find the chromatic polynomial of G, $\chi(G,k)$
- find the inclusive chromatic polynomial of G, $\tilde{\chi}(G,k)$, using Theorem 14
- find the graphical Stirling number, $S(G,k)$, using Theorem 15

We use this idea in Section 4. For now, we note that when $k = \chi(G)$, the fewest colors required for a proper coloring, there is no inclusion-exclusion to be done since cases with fewer than the chromatic number will result in no proper colorings. We can also see this when we substitute $k = \chi(G)$ into Theorem 14.

Corollary 16. The Stirling number of a graph, G, with $k = \chi(G)$ parts is,

$$S(G,\chi(G)) = \frac{1}{k!}\chi(G,\chi(G)).$$

Proof. Let there be a graph, G, with chromatic number $\chi(G)$. Using Theorem 14 and Theorem 15,

$$\begin{aligned}
S(G,\chi(G)) &= \frac{1}{k!}\sum_{j=0}^{\chi(G)-\chi(G)}(-1)^j\binom{\chi(G)}{j}\chi(G,\chi(G)-j) \\
&= \frac{1}{k!}\sum_{j=0}^{0}(-1)^j\binom{\chi(G)}{j}\chi(G,\chi(G)-j) \\
&= \frac{1}{k!}(-1)^0\binom{\chi(G)}{0}\chi(G,\chi(G)-0) \\
&= \frac{1}{k!}\chi(G,\chi(G)).
\end{aligned}$$

□

3.3 Addition-Contraction and Deletion-Contraction

We found the idea of Addition-Contraction in [4]. Addition-contraction and deletion-contraction help us with the process of finding the chromatic polynomial and Stirling number of a graph. To start, we must define a few more terms.

Definition 16. An *edge deletion* e in G, denoted as $G - e$ is the graph obtained by removing an edge e in the graph G. An *edge addition* e of a graph G is the graph obtained by setting the edge e that is not already an edge of G incident to two vertices in G.

Definition 17. The *open neighborhood* of a vertex v in a graph G is a set of vertices in G with the condition that the vertices in G are adjacent to v not including v. Formally, we have $N_G(v) = \{w \in G : (v, w) \in E(G)\}$. We denote this as $N_G(v)$ or $N(v)$ when the graph is unambiguous.

Definition 18. Let $e = (u, v)$ be an edge from an edge set $E(G)$. The graph G/e, the *contraction of edge* e is the graph obtained when the vertices u and v become one vertex w not already in the vertex set of G. The vertex w gets the neighborhood $N_G(w) = N_G(u) \cup N_G(v)$

Deletion-contraction leads us to a recursive formula where we find the graphical Stirling number of a larger graph based on the graphical Stirling number of smaller graphs.

Theorem 17. Let G be a simple graph and let $e \in E(G)$ be an edge. Then

$$S(G, k) = S(G - e, k) - S(G/e, k).$$

Proof. Let G be a simple graph, and let e be an edge in the edge set of G, in other words $e \in E(G)$. From this we know that the edge $e = \{u, v\}$ exists. We want $S(G, k)$ which is the number of ways to partition the vertex set $V(G)$ into k disjoint nonempty sets such that no pair of vertices are adjacent. To do this we will consider what happens when we delete the edge e from the graph G, and then what happens when we contract the edge e in graph G.

In the first case, we delete the edge e from the graph G. This means that the edge set of G no longer contains the edge e in its set. After deleting the edge e we get the graph $G - e$. The graphical Stirling number of the second kind for $G - e$ counts the number of ways that the vertices u and v can be placed into different parts, and also the number of ways they can be placed into the same part.

In the second case, we contract the edge e from the graph G. This means that the edge set of G no longer contains the edge e in its set. After contracting the edge e, we can think of vertices u and v as becoming the saem vertex. From the contraction, we get the graph G/e. The graphical Stirling number of the second kind $S(G/e, k)$ counts the number of ways that the vertices u and v can be partitioned into the same set.

So in case one we overcount because we remove an edge e and that means that the two vertices that define the edge can now also go into the same part when placed into k disjoint nonempty subsets. In case two we count the overcount, that is we count the ways when the vertices that define the edge e are exactly in the same part and then subtract this from the overcount. Putting case one and two together gives us what we are looking for, that is

$$S(G, k) = S(G - e, k) - S(G/e, k).$$

□

It is important to note that the method of deletion-contraction also works works for the chromatic polynomial. We state this theorem here without proof, but the proof is essentially the same as the one above by replacing the word "part" with "color".

Theorem 18. Let G be a simple graph and let $e \in E(G)$. Then

$$\chi(G, k) = \chi(G - e, k) - \chi(G/e, k).$$

Using a similar argument one can also use addition-contraction, where we add an edge not already in the graph in one case, and contract it in the other. We make this argument in terms of the chromatic polynomial.

Theorem 19. Let G be a simple graph and let $a, b \in V(G)$ be two vertices that are not adjacent in G. Then

$$\chi(G, k) = \chi(G + ab, k) + \chi(G/ab, k),$$

where $ab \in E(G)$ is an edge added to G that makes a and b adjacent.

Proof. Let G be a simple graph, and let $a, b \in V(G)$ be any two vertices in G that are not adjacent. We want $\chi(G, k)$, which is the number of ways to properly color the vertex set $V(G)$ with at most k colors. Since vertices a and b are not adjacent in G, we know that in any proper coloring of G, they may or may not be the same color. We will consider what happens when we add the edge ab to the graph G, and then what happens when we contract the two vertices a and b in the graph G.

In the first case, we add edge ab to G so that it connects vertices a and b and is now in the edge set of G. Now that a and b are adjacent, we know that in any proper coloring of this graph, which we call $G + ab$, a and b are *different* colors. So $\chi(G + ab, k)$ counts the number of colorings of our original graph G in which a and b are different colors.

In the second case, we contract the two vertices a and b in the graph G. After contracting, we can think of vertices a and b as becoming a pair. From the contraction, we get the graph G/ab. We know that since a and b are now the same vertex in G/ab, any proper coloring of G will have a and b as the same color. Therefore, the chromatic polynomial $\chi(G/ab, k)$ counts the number of colorings of our original graph G in which a and b are the same color.

So in case one we count the number of colorings of G in which a and b are different colors because a and b are adjacent in the graph we create when we add edge ab. In case two we count count the number of colorings of G where a and b are the same color because a and b will always get the same color when coloring $\chi(G/ab, k)$. Putting case one and two together gives us what we are looking for (all colorings of G in which a and b are either the same color or different colors); that is,

$$\chi(G, k) = \chi(G + e, k) + \chi(G/e, k).$$

□

As before, this argument also applies to the Stirling number, so we get the following theorem, for which we omit the proof.

Theorem 20. Let G be a simple graph and let $a, b \in V(G)$ be two vertices that are not adjacent in G. Then

$$S(G, k) = S(G + ab, k) + S(G/ab, k),$$

where $ab \in E(G)$ is an edge added to G that makes a and b adjacent.

With the tools of inclusion-exclusion, the chromatic polynomial, and addition-contraction and deletion-contraction we can go on to find the graphical Stirling numbers for sunflower graphs. In Section 4 we expand on these ideas.

4 Using Chromatic Polynomials to Find Graphical Stirling Numbers

Theorems 14 and 15 works once we have the chromatic polynomial of the graph. So naturally, the next step is to be able to find the chromatic polynomial so that we may use Theorems 14 and 15.

We were able to use addition-contraction discussed in Section 3 (Theorem 19) repeatedly on an individual petal of our sunflower graphs to find a general formula for the chromatic polynomial of a petal. This meant we were one step closer to finding the chromatic polynomial for a sunflower graph. We, unfortunately, did not have time to find an elegant proof for this result but it is of significance. Of course, we will provide our conjecture and some reasoning.

1

1 1

1 3 1

1 7 6 1

1 15 25 10 1

Figure 8: Stirling Number Recurrence Triangle

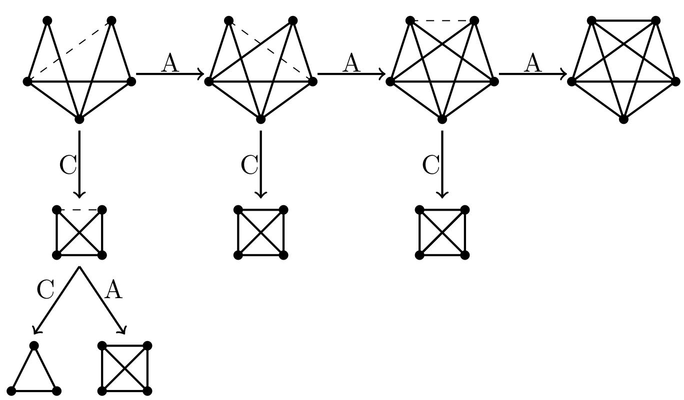
Conjecture 21. Let $d, \ell, k \in \mathbb{N}$. The chromatic polynomial of a petal with length ℓ and distance d with k colors is,

$$\sum_{i=0}^{\ell-d} S(\ell - d + 1, \ell - d + 1 - i)\chi(K_{\ell+1-i}, k).$$

The reasoning for this conjecture depends on using addition-contraction to get our petals to be complete graphs.

Since our petal has length ℓ, there are $\ell + 1$ vertices in the petal. Thus the biggest complete graph that we will make by addition-contraction is $K_{\ell+1}$ since we never add any vertices, just edges. Following addition-contraction, will ever only make one of these $K_{\ell+1}$ graphs.

Here is an example of a petal that we did addition contraction on repeatedly. This petal in particular is P_4^2. The dashed lines represent which edge we are adding and contracting in that step.



As you can see, we result in $1\chi(K_5, k) + 3\chi(K_4, k) + 1\chi(K_3, k)$, $1, 3, 1$ being a row of numbers in the Stirling number recurrence triangle (provided below).

This may look complicated at first because we are using so many different chromatic polynomials but the chromatic polynomial of a complete graph is extremely easy, demonstrated after Definition 14. The big idea to note in the conjecture is that while we are doing addition contraction repeatedly, the regular Stirling numbers are popping up as our coefficients! Of course you may distribute out and simplify and they will disappear but there must be some connection there.

Once we have the chromatic polynomial of a single petal, we need a way to join the chromatic polynomial of many petals together to make a sunflower graph.

Lemma 22. Let $G_1, G_2, \ldots, G_m$ be graphs, each with a designated root vertex. Let G be the graph where we identify $G_1, \dots, G_m$ at their root vertices. Then

$$\chi(G,k) = k \prod_{i=1}^{m} \frac{\chi(G_i,k)}{k}.$$

Proof. Consider $\chi(G,k)$, the number of proper colorings of G with k colors. Partition these colorings into sets C_j, for $1 \leq j \leq k$, such that a coloring is in C_j if the color of the center vertex is j. Let $(C_j)_i$ be the number of colorings of G_i such that the root vertex is color j. Given any coloring of G_i with the root vertex is color j, we can get a coloring of G_i with root vertex color $m \neq j$ by switching the labels j and m. In this way we generate all colorings exactly once. Thus,

$$k \cdot (C_j)_i = \chi(G_i,k).$$

Since each of these colorings of G_i can be paired with a coloring of G_j and still be a proper coloring of G we get that

$$C_j = \prod_{i=1}^{m} \frac{\chi(G_i,k)}{k}.$$

Since there are k options for the root vertex, there are

$$k \prod_{i=1}^{m} \frac{\chi(G_i,k)}{k}$$

colorings in total. □

Now that we have a general formula for the chromatic polynomial of a sunflower graph, that means we are able to use Theorem 14 and 15.

Before we had Conjecture 21, we often looked at specific examples and searched for patterns and found reasoning in that. One particular type we looked at was the complete sunflower graph:

Definition 19. A *complete sunflower graph* is a special case of a sunflower graph $F_{\mathcal{P}}$ where all the elements in the set $\mathcal{P}$ are defined as $(P_d)^d$ for some $d \in \mathbb{N}$.

Theorem 23. The chromatic polynomial for a complete sunflower graph (where length and distance are the same) can be written as

$$\chi(F^d_{d^p},k) = k \prod_{i=1}^{d} (k-i)^p$$

Proof. First, we note that since the distance and the length are equal, we have a complete sunflower graph. As a result, we know that all vertices on the same petal are adjacent and therefore cannot be the same color. We will label the vertices of the graph as $V_{i,j}$ where i refers to the i^{th} petal and j refers to the j^{th} vertex on the petal (as counted from the center). We begin by coloring the root vertex of the graph, $V_{i,0}$. There are k possible colors to use. Then concentrate on one petal, to $V_{1,1}$, and since it is each connected to the root vertex, there are $k-1$ options, which when considering the whole graph, gives us $(k-1)^p$ ways to color the first vertex on each petal. So thus far, we have $k(k-1)^p$ ways to color the root vertex and all of the $V_{i,1}$. We now move to the next vertex on each petal, $V_{i,2}$. Since each of these are connected to the two vertices preceding it, there are $(k-2)^p$ options. We continue sequentially down each petal in this manner, and because each vertex will be connected to all vertices preceding it on its petal, we see that the chromatic polynomial will take the form

$$k(k-1)^p(k-2)^p(k-3)^p \cdots.$$

The last term in this product will be the chromatic polynomial for the last vertex on each of the petals in the graph. Since our length is d, we know that these vertices are connected to the d vertices preceding them on their respective petals, so we can color them in $(k-d)^p$ ways. Therefore, our chromatic polynomial for this graph is

$$k(k-1)^p(k-2)^p(k-3)^p \cdots (k-d)^p,$$

which can be represented as the product

$$k\prod_{i=1}^{d}(k-i)^p.$$

□

After looking at a complete sunflower graph, we took one step down and looked at the case where instead of $\ell = d$, $\ell = d+1$.

Theorem 24. Let ℓ, d, p, and $k \in \mathbb{N}$. The chromatic polynomial for a sunflower graph with length, ℓ, one more than distance, d,

$$\chi(F^d_{(d+1)^p}, k) = k(k-d)^p\prod_{i=1}^{d}(k-i)^p.$$

Proof. Using the same method we employed in Theorem 23, we label the vertices of the graph as $V_{i,j}$ where i refers to the i^{th} petal and j is the distance from the center. We note that with this case, the center vertex is connected to every vertex on each petal, except the last one; therefore it is connected to the first d vertices on each petal. This allows us to color the first d vertices on each petal the same way we did with the complete sunflower in Theorem 23. This gives us $k(k-1)^p(k-2)^p\cdots(k-d)^p$ ways to color root vertex and the first d vertices on each petal. This leaves us to color the last vertex on each petal. Since these vertices are not adjacent to the root vertex of the graph, and are only adjacent to the previous d vertices on their respective petals, there are $k-d$ possible colors to use for each of them. Thus, we have a chromatic polynomial for the entire graph:

$$\begin{aligned}\chi(F^d_{(d+1)^p}, k) &= k(k-1)^p(k-2)^p\cdots(k-d)^p(k-d)^p\\ &= k(k-d)^p\prod_{i=1}^{d}(k-i)^p.\end{aligned}$$

□

We were also able to develop an algorithm to find the chromatic polynomial, although we did not have time to officially prove why it works. We will of course give our conjecture with some reasoning and an example. It is important to note that several examples using the following algorithm yielded the same results as Conjecture 21, which furthered our belief that they are both true.

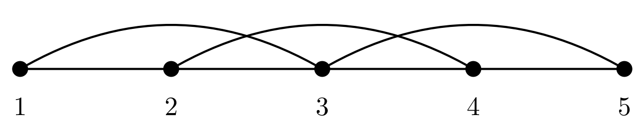
Conjecture 25. Let $\ell, d, p, k, i \in \mathbb{N}$. The chromatic polynomial of a sunflower graph with identical petals, $F^d_{\ell^p}$, with k colors can be defined as follows,

$$\chi(F^d_{\ell^p}, k) = k(k-d)^{(\ell-d+1)p}\prod_{i=1}^{d-1}(k-i)^p.$$

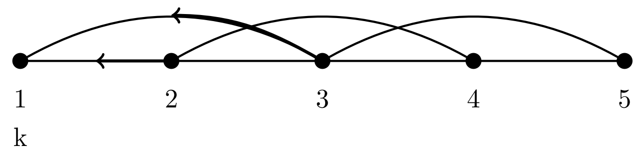
Our algorithm relies on us giving the graph an orientation.

Definition 20. An *orientation* of a simple graph is an assignment of a direction to each edge in the graph. This results in a *directed graph* or *digraph*. In a digraph each directed edge is called an *arc*. The *outdegree* of a vertex v, denoted $d^+(v)$ is the number of arcs that are directed away from v.

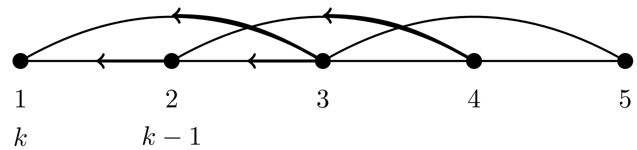
For the algorithm, like most other of our processes, we start with the root vertex of a petal. Observe P^2_4.



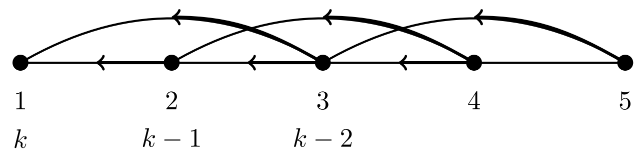
We first consider how many color options we have for vertex 1, which is k. Then whenever we do this, we orient the edges incident to that vertex towards the vertex, which we will represent by using arrows. Thus after the first step in our algorithm, our graph is



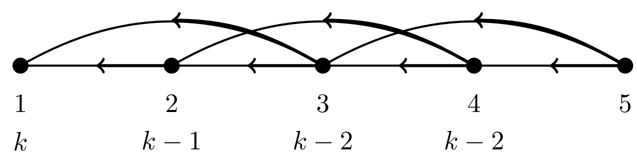
Then for vertex 2, we subtract one from k for each outdegree it now has. That is, we know we can color this vertex in $k - d^+(2)$ ways. Thus vertex 2 has $k-1$ options (which makes sense since we have already considered coloring a vertex adjacent to it). Resulting in,



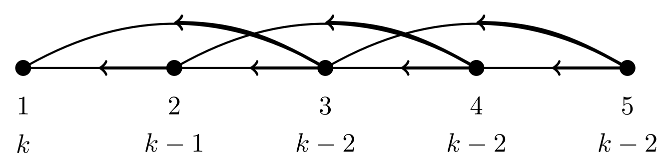
The next step is then to look at vertex 3 and its outdegree and simultaneously adding the arrows to the edges not already marked with an arrow.



Then again for vertex 4,



Then lastly, vertex 5,



Thus we conclude $\chi(P_4^2, k) = k(k-1)(k-2)^3$. This result matches with our other results.

Most of the results with the chromatic polynomial that we had time to find pertain to sunflower graphs where all of the petals are identical. Of course, this may not always be the case. We have a lot of the parts to get towards a more general case such as Theorem 14, 15, and Lemma 22, but we ran out of time to work out the fine details.

This next section focuses on finding recursions for graphical Stirling numbers in a general setting.

5 Using Recursions to Find Graphical Stirling Numbers

We've learned about Stirling numbers, previous work, and relating those ideas to graphs. We saw that we can first find the chromatic polynomial of a graph to find a graphical Stirling number. One of the main ideas in previous work is that the traditional Stirling numbers can be computed via a recursion, as seen in Theorem 3. Likewise, we have can do similar things in order to find the graphical Stirling numbers of a given sunflower graph.

As mentioned in the initial sections, there are many things known about the traditional Stirling numbers, and so any progress we can make on relating graphical Stirling numbers to the traditional Stirling numbers gives us a big advantage on raw calculations.

This leads to the following theorem where we relate the graphical Stirling number of a star graph to the traditional Stirling numbers. Note that a star graph is a special case of a sunflower graph, we'll consider the star graph in the following theorem.

Theorem 26. Let $F_{\mathcal{P}}$ be a sunflower graph with petal graph set such that $|\mathcal{P}| = p$ for some natural number p, and

$$\mathcal{P} = \{(P_1)^1, ..., (P_1)^1\}.$$

We relate the graphical Stirling number of the second kind for $F_{\mathcal{P}}$ to the traditional Stirling numbers of the second kind as follows,

$$S(F_{\mathcal{P}}, k) = S(p, k-1).$$

Proof. Let $F_{\mathcal{P}}$ be a sunflower graph with petal graph set such that $|\mathcal{P}| = p$ for some natural number p, and

$$\mathcal{P} = \{(P_1)^1, ..., (P_1)^1\}.$$

By the definition of a graphical Stirling number, we want to partition the vertex set of $F_{\mathcal{P}}$ into k disjoint nonempty subsets. First we note that we identified the root of every petal graph and identified it as one vertex. Next we note that each petal only has length and distance 1 and so, every other vertex in the vertex set of $F_{\mathcal{P}}$ is adjacent to the root vertex by definition. This means that the root vertex has to be in a set by itself. We are left with $k-1$ sets where we can place the remaining vertices. Next we know that each petal only has two vertices since they are of length 1. Since one vertex is already identified as the root and we have already placed it in a part by itself, we only have one vertex per petal graph left to place into $k-1$ disjoint nonempty subsets. This means we are looking for the number of ways to place $p \cdot 1$ vertices into $k-1$ disjoint nonempty subsets. Since none of the p vertices are adjacent to each other, there are no restrictions on where they can go. This is exactly the definition of a traditional Stirling number of the second kind. So we have $S(p, k-1)$ ways to place the remaining elements into $k-1$ parts.

Putting all this information together we get

$$S(F_{\mathcal{P}}, k) = S(p, k-1).$$

$\square$

Example. As an example, consider a special case of a sunflower graph $\mathcal{S} = F^1_{1^4}$ also known as a star graph with 4 leaves. We can easily use this theorem to find that, $S(\mathcal{S}, k) = S(4, k)$ since this sunflower graph has 4 petals. $\square$

We can do the same for any star graph, but even then these are only special cases of our sunflower graphs. However, whenever we have a sunflower whose petals are the same length and are complete graphs, we have another theorem. It allows us to use recurrence to define the graphical Stirling number.

Theorem 27. The number of stable partitions of a complete sunflower graph where $k = \chi$ by relation is

$$S(F^{(k-1)}_{(k-1)^p}, k) = S(F^{k-2}_{(k-2)^p}, k-1) \cdot (k-1)^{(p-1)}$$

Proof. By Theorem 23, we have $S(F^{(k-1)}_{(k-1)^p}, k) = \displaystyle\prod_{i=1}^{k-1} (k-i)^{p-1}$, and $S(F^{(k-2)}_{(k-2)^p}, k-1) = \displaystyle\prod_{i=1}^{k-2} (k-1-i)^{p-1}$.

We will start by dividing $S(F^{(k-1)}_{(k-1)^p}, k)$ by $S(F^{k-2}_{(k-2)^p}, k-1)$.

$$\frac{S(F^{(k-1)}_{(k-1)^p},k)}{S(F^{(k-2)}_{(k-2)^p},k-1)} = \frac{\prod_{i=1}^{k-1}(k-i)^{p-1}}{\prod_{i=1}^{k-2}(k-1-i)^{p-1}}$$
$$= \frac{(k-1)^{p-1}(k-2)^{p-1}...(k-(k-1)^{p-1}}{(k-2)^{p-1}(k-3)^{p-1}...(k-1-(k-2))^{p-1}}$$
$$= (k-1)^{p-1}.$$

We see that when we divide the two Stirling numbers we receive $(k-1)^{p-1}$. Therefore we know that we can multiply $S(F^{(k-2)}_{(k-2)^p},k-1)$ by $(k-1)^{p-1}$ to get $S(F^{(k-1)}_{(k-1)^p},k)$.

□

Example. For this example we will find $S(F^4_{4^4},5)$ from our relation in Theorem 27. We will use $S(F^3_{3^4},4)$ to find the Stirling number we are looking for. Using that relation we receive,

$$\begin{aligned} S(F^3_{3^4},4)(5-1)^{4-1} &= S(F^3_{3^4},4)(4)^3 \\ &= S(F^3_{3^4},4)(64) \\ &= 216(64) \\ &= 13824. \end{aligned}$$

That means that $S(F^4_{4^4},5) = 13824$ which agrees with all of our other ways of finding this number. □

The recursion for a special case of a sunflower graph called a star graph, Theorem 26, has a quick interpretation of its graphical Stirling numbers in terms of the traditional Stirling numbers. The thing that makes Theorem 26 special is what we will use it to prove another recursion for a complete sunflower graph.

We noted that the center vertex of the star graph is adjacent to all of the other vertices in the graph. We can define this idea as follows.

Definition 21. [9] A set $S \subseteq V$ of vertices in a graph G with vertex set V is a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

With this definition, we can prove the following.

Theorem 28. Let G be a graph, and let v be a vertex in G. If $\{v\}$ is a dominating set of G, then the graphical Stirling number of the graph G satisfies the following recursion,

$$S(G,k) = S(G-v,k-1).$$

Proof. Let G be a graph, and let v be a vertex in G such that $\{v\}$ is a dominating set of G. To find the graphical Stirling number of G we consider two disjoint cases.

In Case I, we put the vertex v into its own part, this leaves us with the vertex set of $G-v$ to partition into $k-1$ parts.

In Case II, we separate the vertex set of $G-v$ into k parts and consider how many different ways to can place the vertex v into one of these k parts. Since v is a dominating vertex of G, it means that it is adjacent to each of the vertices in the vertex set of $G-v$. This tells us that v cannot be in any one of the parts created from $G-v$, and thus there are zero ways to first partition $G-v$ into k parts and then placing v in one of these k parts.

Adding the two disjoint cases we get $S(G,k) = S(G-v,k-1) + 0 \cdot S(G-v,k)$, or just,

$$S(G,k) = S(G-v,k-1),$$

as needed. □

Theorem 28 leads us to a new complete sunflower graph recursion.

Corollary 29. Let F be a complete sunflower graph and let v be an end vertex of one of the petals in F. The graphical Stirling number of F satisfies the following recursion,

$$S(F,k) = S(F-v,k).$$

This last theorem requires us to have a Sunflower graph whose petals are themselves complete graphs. We can do some modifications and build the following more relaxed theorem. This theorem applies to any graph G that has a complete open neighborhood. We will show later how it relates to sunflower graphs.

Recall the definition of open neighborhood from Definition 17. In the next definition we expand on this definition.

Definition 22. Given a graph G and a vertex v we say the induced open neighborhood graph is the graph consisting of $N(v)$ together with all the edges between those vertices. We say an open neighborhood is *complete* if all the vertices in $N_G(v)$ are adjacent to each other in G, that is, its induced open neighborhood graph is a complete graph.

Theorem 30. Let G be a graph, and let v be a vertex in G. If the open neighborhood of a vertex v, denoted $N(v)$, is complete, then the graphical Stirling number of G satisfies the recurrence relation,

$$S(G,k) = S(G-v,k-1) + (k-\deg(v)) \cdot S(G-v,k).$$

Proof. Let G be a connected graph, and let v be a vertex in G. Furthermore, assume that the neighborhood $N(v)$ is complete, and let $|V(G)| = n$ for some natural number n. By definition of a graphical Stirling number, we are looking for the number of ways to take the vertex set of G and place them into a stable partition of k parts. In notation, we are looking for $S(G,k)$.

We can consider two disjoint cases. Case I is when the vertex v is placed in its own stable part. Case II is when v is not in a part by itself.

In case one, we place the vertex v in its own part, there is only one way to do this. This leaves us with the rest of the graph $G-v$ to place into $k-1$ parts. So this gives us $S(G-v,k-1)$.

In case two, we consider when vertex v is not in a part by itself. We first consider how to place the graph $G-v$ into k parts. Then we must consider what part the remaining vertex v will get placed in. By the definition of a graphical Stirling number we know that v cannot be in the same part as any of the vertices of the induced neighborhood graph. Also since the induced neighborhood subgraph is complete, that means that the vertices in this graph cannot be in the same part as each other. So we can say there exist $\deg(v)$ parts so that the vertices in $N(G)$ are in their own part. Thus we have $k-\deg(v)$ ways to place vertex v into the k parts previously created. This gives us $(k-\deg(v)) \cdot S(G-v,k)$.

Note that in both cases we took a graph G and placed its vertices into k parts. We know these two cases are disjoint since one case counts the number of ways to partition a distinct vertex v into its own part. The second case on the other hand counts the number of ways to partition a distinct element v into an already partitioned set $G-v$. Thus putting this all together, we get

$$S(G,k) = S(G-v,k-1) + (k-\deg(v)) \cdot S(G-v,k)$$

as needed. □

From this recursion notice that even though the neighborhood of a vertex in a graph has to be complete, it doesn't require a whole petal on a sunflower graph to be complete as the last recurrence did. However, using Theorem 30, we have the following general recurrence for sunflower graphs.

Corollary 31. Let $F_{\mathcal{P}}$ be a sunflower graph with

$$\mathcal{P} = \{(P_{\ell_1})^{d_1}, ..., (P_{\ell_i})^{d_i}\}$$

for some natural numbers ℓ_i and d_i, such that $d_i \le \ell_i$ for all natural numbers i. Also let $v \in (P_{\ell_j})^{d_j}$ be an end vertex for some natural number $j \le i$. The graphical Stirling number of the second kind for F_P satisfies the following recursion,

$$S\left(F_{\mathcal{P}},k\right) = S\left(F_{\mathcal{P}}-v,k-1\right) + (k-deg(v)) \cdot S\left(F_{\mathcal{P}}-v,k\right).$$

Proof. Let $F_{\mathcal{P}}$ be a sunflower graph with

$$\mathcal{P} = \{(P_{\ell_1})^{d_1}, ..., (P_{\ell_i})^{d_i}\}$$

for some natural numbers ℓ_i and d_i, such that $d_i \leq \ell_i$ for all natural numbers i. Also let $v \in (P_{\ell_j})^{d_j}$ be an end vertex for some natural number $j \leq i$.

Now consider the vertex v as defined. We know that the neighborhood of v is going to depend on what it's adjacent to. We know that v is adjacent to distance d vertices from itself. However, since each petal is a power of a path, we also know that the induced neighborhood graph $N(v)$ is also complete. This is because each vertex in the induced subgraph $N(v)$ they are adjacent to vertices that are less than or equal to d_i by definition of a petal graph.

So by Theorem 30, we know that

$$S(F_{\mathcal{P}}, k) = S(F_{\mathcal{P}} - v, k-1) + (k - deg(v)) \cdot S(F_{\mathcal{P}} - v, k)$$

as needed. □

6 Conclusion and Future Work

In conclusion, our research consisted of looking at restrictions on Stirling numbers and how they can be represented by graphs. In contrast, we then looked at graphs and then tried to find their Stirling numbers. To start, we looked at proper colorings of a graph and were able to find a way to get the Stirling number once we had the chromatic polynomial of said graph. We also looked into recursions on these sunflower graphs, just as there are recursions on the regular Stirling numbers.

Coming to the end of our research we still have some questions that we would love to be answered in the future. One of the main problems is going off of Conjecture 21. We would like to find an elegant proof for that conjecture.

We would also like to add more to our recursion work. Most of our recursion work centered around identifying a vertex and then constructing two cases that represent where that vertex would end up when considering a set partition of the vertex set of a graph under very specific circumstances. However we would like to extend this to other graph properties. For example, what happens when we identify two vertices a and b of a graph G such that their neighborhoods are complete and nothing more. Or even more generally, can we create a relation where we start with say any dominating set and see what that tells us.

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