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Developing Algebraic Notation Through Number Patterns [1]

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Abstract:
All the mathematics teachers in a Maltese secondary school were involved in setting and correcting a task involving the use of algebraic symbolization to describe number patterns in a number of their classes. A focus interview was carried out with the teachers some time after this experience. As a group, the teachers identified some very well documented difficulties that students have with the use of letters in Algebra. The work also shows that tasks of the type investigated provide teachers with contexts that they may utilize to help students make some entry points into using letters as generalized number.

Introduction

Number patterns provide a variety of rich contexts that may be utilized to help students develop algebraic symbolization to express relationships. Such experiences often constitute an important feature in introducing Algebra in many reformed Algebra curricula. Various researchers recognize the potential of these activities for engaging students in generalizing and for helping them develop algebraic notation (see for example Mason (1996), NCTM (1997), Yerushalmy (2000)). Other studies (e.g. Sasman,

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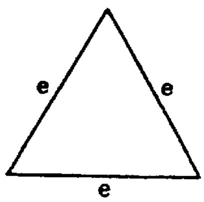
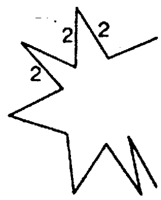
Linchevski & Olivier, (1999)) have focused on the cognitive difficulties evidenced by students working on generalizing patterns and describing the generalized pattern using algebra.

This paper tells the story of some teachers' experience in working with activities of this type in a Maltese secondary school. Given the specificity of algebraic thinking, it is clear that the subject matter constitutes a crucial aspect in the teaching and learning of the subject. For this reason it was considered relevant to start with a discussion of students' difficulties in beginning Algebra.

Theoretical Considerations

Traditionally, the algebra curriculum focuses on teacher taught procedures that students are encouraged to reproduce. The difficulties faced by secondary school students learning algebra in such set-ups have been well documented (e.g. Herscovics (1989), Kieran (1989), Kieran (1992), Küchemann (1981)). Many students face extensive difficulties in coming to terms with the different meanings attached to letters in algebra. On the other hand, those initiated into algebraic methods use letters with an ambiguity that allows for flexibility of use. For example the equation $3x + 5 = 17$, prompts the idea of x as a placeholder or unknown value for x. The relations $x + 3x = 4x$ and $x + y = y + x$ are hardly useful if x and y are viewed as placeholders standing for specific values of x and y. These statements are in fact more useful if the letters x and y are viewed as generalised numbers. The relation $y = 3x$ prompts notions of a function and also ideas of covariation, like that a change of 1 unit in the x-value corresponds to a change of 3 units in the y-value. Doing algebra involves choosing suitable interpretations for the letters being used and at times also shifting between different interpretations. This in itself is already difficult. To make matters worse, Algebra students are often hindered by incorrect meanings they associate with given algebraic statements.

Table 1: Responses to Question 9 in CSMS Algebra Test (14 year olds) (Adapted from Küchemann, 1981, p. 102)			
9. Find the perimeter of the shapes below:			
(i) 		(iv)  Part of this figure is not drawn. There are *n* sides altogether all of length 2.	
Correct Response :	3e	*Correct Response*:	2*n* or *n*2
Facility	94%	**Facility**	38%

As part of the Concepts in Secondary Science and Mathematics project, Küchemann's work involved the analysis of students' performance on the CSMS Algebra test (Küchemann 1981). The test was administered to 3000 British students in their second, third and fourth year of secondary schooling. British pupils start learning algebra in their first or second year of their secondary education, at age 12 or 13. In his analysis to students' responses, Küchemann identified some of the many possible interpretations that secondary students give to letters. Consider for example, the responses of the fourteen year-old sample (just under 1000 students) on question 9 of the CSMS test as shown in Table 1. While the answers to questions 9(i) and 9(iv) are similar, the facilities are in stark contrast. Although in question 9(iv), *n* is clearly defined as the number of sides of the shape; many seemed oblivious to this. They were unwilling to multiply the number of sides *n*, by 2, the length of each side. Instead, many counted the number of sides already drawn or else closed in the figure by adding one or more lines and gave an answer of 32 or 34. On the other hand, in question 9(i), the great majority of students were willing to multiply the length of the side, *e* , by 3, the number of sides. Küchemann argues that many students would have answered question 9(i) correctly by interpreting the letter *e* as just a *name* or *label* for each of the sides rather than an *unknown number*, the unknown length of each side. The letter *e* is regarded as an object, in this case the side, and the three sides are interpreted as 3*e*. Whilst an interpretation of the letter as object may have yielded a correct answer for question 9(i), such an interpretation often yields incorrect algebraic statements.

An often-quoted example illustrating the interpretation of letters as objects is the students and professors problem. The following problem was given to a group of 150 freshmen studying Engineering at the University of Massachusetts.

Write an equation, using the variables ***S*** and ***P*** to represent the following statement: "At this university there are six times as many students as professors."
Use ***S*** for the number of students and ***P*** for the number of professors.

More than a third of the students tested were unable to unable to write the correct equation, ***S*** = ***6P,*** in any form. The main error was to write the equation 6***S*** = ***P***, described by the authors as the reversed equation (Clement, Lochhead, and Monk 1981).

Basing on the response of students on interviews, the authors claim that students writing the reversed equation 6***S*** = ***P***, were interpreting the letters ***S*** and ***P*** as labels standing for *students* and *professors* rather than variables standing for the *number of students* and *number of professors* respectively. The equation 6***S*** = ***P*** was sometimes read as "there are six students for every one professor," while pointing to ***S*** and ***P*** as they uttered the terms ***students*** and ***professors*** respectively. Alternatively ***S*** = ***6P*** was read as "one student for every six professors. In both cases, the letters ***S*** and ***P*** were being conceived as ***students*** and ***professors***; in other words the letters were viewed objects. While working on these problems these students, who were not novices to algebraic symbolization, seem to have missed the abstraction that the letters stand for the ***number*** of students and professors.

Students sometimes associate letters with their position in the alphabet taking for example the letters *f*, *g* to represent the numbers 6 and 7 respectively (see for example Küchemann (1981); MacGregor and Stacey, 1997).

A theoretical interpretation that has been used to explain students' difficulties in interpreting algebraic symbolism is Sfard's theory of reification. Sfard & Linchevski (1994) argue that algebraic expressions can be viewed both operationally and structurally. Consider the expression x^2 - 9. One possible interpretation of this expression involves treating it as a prompt to substitute one or more values for *x*. This interpretation involves an operational view of the expression. Alternatively, the expression can be viewed as a number in its own right, depending on an unknown value of *x*. It may also be viewed as a function of *x* taking on different values with different values of *x*. This acceptance of the expression x^2 - 9 as a useful entity in its own right with an awareness that it may be manipulated to reveal other insights involves a structural view of this same expression. Sfard & Linchevski argue that an operational conception precedes a structural perception. They associate the move from the operational to the structural conception with the process of reification.

"Mathematical objects are an outcome of *reification* - of our mind's ability to envision the results of processes as permanent objects in their own right." (Sfard & Linchevski, 1994, p. 194)

Conceptual development in Algebra involves a lengthy process, involving various reifications. For example, it is one thing to view the expression $x^2 - 9$ as an expression involving a fixed unknown value of *x* and yet another to view it as a function. Both views are constructs of reification, both ideas are initially conceived from an operational viewpoint. Once conceived as objects in their own right, the properties of the reified constructs, in this case expression and function can be articulated and applied.

The authors maintain that student difficulties in algebra often result from an inability to make the necessary operational-structural connections, the structural viewpoint often resulting very difficult to achieve. For example, students are often found to be unwilling to give answers in the form of open algebraic expression. This phenomenon is described by Collis (1974) as an inability to accept lack of closure. In such cases, it may very well be the case that for the student, the letters are only useful in an operational sense, namely as numbers to be computed to arrive at a final numerical answer. On the other hand, some student errors, where students consistently assign the same incorrect meanings to letters, can be attributed to pseudo-structural conceptions. The student mistakes, discussed earlier, where letters are treated as object and where the position of letters in the alphabet is associated with their value qualify as pseudo-structural conceptions.

Other authors have also recognized the dual nature of mathematical symbols. Taking examples mainly from Arithmetic, Gray & Tall (1994) distinguish between viewing the same mathematical symbol as a process to viewing it as a concept. Using their terminology, a mathematical symbol is viewed as a *procept* when this can be interpreted both as a process and as a manipulable concept as the need arises. In analyzing some young children's use of algebraic symbols, Tall (2001) distinguishes between *evaluation algebra* and *manipulable algebra.* In evaluation algebra, symbolic expressions are viewed as processes of evaluation. Manipulable algebra is a more advanced stage where the symbols are viewed proceptually and are interpreted as processes or manipulable concepts as the need arises.

Although named differently, Sfard's operational and structural interpretations correspond to Gray& Tall's (1994) distinction between viewing a mathematical symbol as a process and viewing it as an object. The basic idea is the claim that actions, operations or processes become in turn conceived as mental objects in their own right that are also acted upon. This gives the learner more flexibility in that symbols can be viewed as process or else acted upon as opportune. As Mason (1996) puts it,

> Algebraic awareness requires, perhaps even consists of, necessary *shifts of attention*, which make it possible to be flexible in seeing written symbols
>
> - as expression and as value
> - as object and as process. (p.74)

In Malta, as in many traditional Algebra curricula, Algebra is taught starting from ideas connected with simplification of algebraic expressions, linear equations in one unknown and linear equations in two unknowns. The emphasis is on teacher taught procedures that students reproduce in exercises similar to those already worked out by the teacher. Considering this situation through the lenses of Sfard's model, one problem is immediately evident. Given that reification is a slow and difficult process, many students never get the chance to make the necessary accommodations to view Algebra as a tool for generalization. As Sfard and Linchevski (1994) point out, the curriculum literally reverses the order in which algebraic notions are related with the more difficult structural approach being assumed at the outset. With the missing foundation work linking the operational and structural approach, it is only natural that many never grasp flexibility in algebra making it possible for them to be able to shift between operational and structural representations as necessary in working with Algebra.

The search for suitable contexts to help students develop the necessary algebraic meanings is an essential part in reforming the algebra curriculum. (see for example Bednarz, Kieran and Lee (1996) and NCTM (1997)).The present study was undertaken with the conviction that number patterns offer a rich context for helping students come to terms with the use of letters to represent generalized numbers.

The Study: Preliminaries

The work being reported here started with a series of weekly meetings held between myself, Peter Vassallo, the Educational Officer responsible for the teaching of Mathematics in the school and the Mathematics teachers of a local Junior Lyceum for girls. On the other hand, I am employed with the Faculty of Education and am involved in the Mathematics Education component of the initial teacher-training course offered by the Faculty. Peter and myself set out with the idea of exploring ways of working with teachers in context. Peter was always present during our meetings and he proved to be a source of invaluable support throughout the work.

The preliminary meetings were held in school time between February and May 2000. As with other Junior Lyceums in Malta, the school is a government secondary school and caters for a particular catchment area. Students, in this case girls, are admitted into Junior Lyceum at ten to eleven years of age if they pass the entry examinations in Mathematics and another four subjects. Secondary education entails five years of schooling; in this particular school, there were about five classes in each Form. The classes were also streamed. The Form I classes were streamed on the basis of the students' results in their Junior Lyceum examinations. The other Forms were streamed according to the results obtained by the students in their annual examinations.

A lesson period had been allotted by the school administration making it possible for the five Mathematics teachers to be able to meet during this time. These meetings were very necessary to get to know the teachers and to come to grips with the context under which the teachers were working so as to ensure that the demands being made on the teachers could be met. One of the concerns that came out during the penultimate of these preliminary meetings was that many students fail to see the relevance of Algebra. I suggested that tasks involving patterns that can also be represented algebraically could help in this respect. During the next and final meeting during this academic year, I circulated a number of tasks of varying levels of difficulty that could be tried out with students. It was agreed that next year each teacher would be involved in choosing one or more of the activities. The teachers were to trial the chosen activities with their classes.

Three further meetings were held next year, during the month of November 2000. This new academic year there were four new teachers and one of the original five teachers had transferred to another school. During these meetings, the new teachers had the possibility to familiarize with the tasks. The teachers also suggested some minor revisions to improve the readability of some of the tasks and these were taken into account.

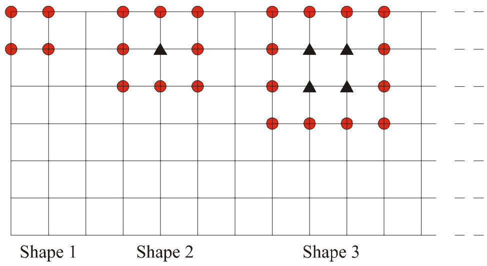
The teachers confirmed that none but the Form V students had ever been set similar problems at school. They were to set the tasks by giving the students individual worksheets and to let them work out the tasks individually or in small groups. Students were not to be given any prior hints as to how to work on the tasks. Also teachers were not to help the students on the tasks before the individual worksheets were collected. They were left free to decide which of the activities to try. However, they were encouraged to try at least one activity with each of their classes. They were also to collect and mark the activity sheets. The teachers agreed to try the tasks between December 2000 and March 2001. The tasks that were eventually tried out are shown in Figure 1. A focus group meeting was held later on to discuss the outcomes of the trials in the different classes.

The Focus Group Interview

An audio-taped focus group interview, lasting just about one and a half hours was held in May 2001. One of the teachers, here denoted by Carmen, was ill and could not attend the meeting. The other seven teachers then teaching Mathematics in the school, Peter and myself participated in this interview. In my role as discussion facilitator, I decided to elicit from the teachers their views about the students' performance on the tasks they had carried out with their classes. On my part, it was important that I would elicit their views and results, rather than dictate my own beliefs. The teacher comments that will be used for the purposes of this paper were prompted by questions relating to these issues:

(a) The tasks that were finally tried and with whom
(b) Help given during the tasks
(c) The instructional method used (individual/ pair work/ use of larger groups)
(d) Students' difficulties on the tasks

Circles and Triangles: Two Tasks



Task 1: Circles

The shapes above follow a pattern. Continue the pattern to draw shapes 4 and shape5.

(a) Continue the table below:

Shape Number	1	2	3	4	5
Number of circles	4	8			

(b) How many circles are there in shape 20? Explain your reasoning.

(c) How many circles are there in shape N?

Task 2: Triangles

Use the same pattern as in previous task. In this task, instead of finding the number of circles, the number of triangles used for different shapes are found.

(a) Continue the table below :

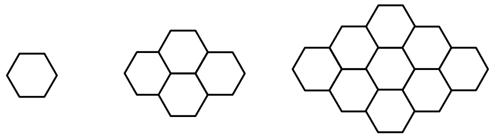
Shape Number	1	2	3	4	5
Number of triangles	0	1			

(b) How many triangles are there in shape 20? Explain your reasoning.

(c) How many triangles are there in shape N?

Hexagons

The three shapes in Figure below form a sequence.



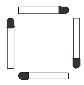
1st shape 2nd shape 3rd shape

a. How many hexagons are there in the first, second and third shape?

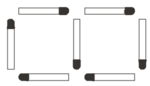
b. If the sequence is continued, how many hexagons are there in the:
- i) fourth shape
- ii) eighth shape
- iii) n th shape ?

Matches

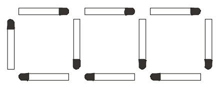
Arrangement 1



Arrangement 2



Arrangement 3



Arrangement	1	2	3	4	5	6	7	8
Number of matches	4	7	10					

a) Find the number of matches needed for arrangement 4. You may use matches or draw a diagram if you find this useful.

b) Complete the table.

c) How many matches are needed for arrangement 20?

d) How many matches are needed for arrangement 30?

e) Describe a rule that helps you to find out the number of matches according to the arrangement number.

f) How many matches are needed for arrangement N?

Figure 1: The tasks tried out by the teachers

During the focus group interview, the teachers collaborated fully. In their responses, they amplified on the reasons why they worked the way they did. For example, I asked all the teachers in turn about the help given while students were working on the task separately from the students' difficulties on the task. While in my questions I treated the two issues separately, some of the teachers' responses to my first question amplified on the students' difficulties. After transcribing the focus interview, it was clear that the teachers had a lot to say regarding the performance of the students on the tasks.

Anna had tried out Hexagons with her Form II classes. Most teachers had tried out the tasks Circles and Triangles. Stefan and Maria tried them out with their Form III classes, whilst Frank and Tanya tried them out with their Form IV and Form V classes. Vera who was teaching Mathematics to two Form V classes in the lowest achieving streams tried out the task Matches in both her classes.

All the teachers had tried out one or two tasks with their Mathematics classes, except for the Form I classes. In Junior Lyceums, Algebraic notation is introduced in Form I but the two teachers teaching this Form opted not to try any of the tasks with this Form. Not all the circulated tasks had been tried out and most teachers tried out the same tasks. Although Carmen was not present for the interview, on her invitation, I had visited one of her Form II classes, when she was trying out the task Hexagons with this class.

Copies of the students' handouts were also collected at the end of the interview. This made it possible for me to gain greater understanding into the teachers' comments about the students' performance on the tasks.

Students' performance on the tasks

The Form II classes

The major difficulty that was apparent in all Form II classes related to the students' difficulties with representing the observed patterns in algebraic symbolization. Even when they had found a global pattern that could be described verbally and used to find any specific term, many found it difficult to write this down as an expression for the *n* th term.

Both teachers working with Form II's tried out the task Hexagons. In this case the *n* th term of the sequence was n^2. In The following excerpt Anna explains how she found it necessary to help her students while working on the task. (All excerpts are translated from Maltese.)

Anna: I used Hexagons with Form II's. Now while they were finding out the 9th shape or something similar, there o.k. So they are working. But when they came to the *n* th term, there was panic. For one thing they did not understand what the *n* th term is. They had no idea. Because some drew the letter N. And they counted the corners in the letter N. Hopeless case! And I had to explain. And I explained that if I ask you "How many hexagons are there in the 100th shape, I don't think that you have to draw them. You need to find out something. And I insisted on this something. And many were arriving at the right answer. But there were still others who did not get to the right answer.

A similar situation had occurred in the class I had observed with Carmen. The class was a high achieving stream in Form II and the girls were set to work in pairs. They immediately started out working on the task. Initially some were using iteration to find the next term, but within a short time all the pairs were using the global pattern-squaring. Like in the excerpt above, the students got stuck when they came to describe the *n* th term. The students were visibly frustrated. After a short discussion with the teacher, we decided to tell them that *n* stands for any number. And we waited. After what seemed a very long wait, one of the students got the answer. Slowly, the other pairs were also coming to the answer. It seemed that everybody was accepting the pattern as n^2. While we were collecting the students' handouts, one of the students remarked; "But n^2 is not an answer!" While we were asking her about the meaning of n and n^2, she herself connected this problem with some work they had done a couple of weeks earlier in connection with constructing formulae.

Form III

Two teachers Stefan and Maria were teaching Form III's. Both tried Circles and Triangles, two patterns built on the same sequence of shapes with the *n* th shape being made up of $4n$ circles and $(n\text{-}1)^2$ triangles. In the following excerpt, Stefan and Maria are discussing the performance of different classes on these tasks.

Stefan: Yes, I saw a difference between the classes, the higher achieving got more involved. They completed the first task easily. The second task, the same as with the other classes. I gave them the tasks when we got back to school in January. They took it in their stride, you know, they weren't very worried.

Cettina: They managed to get it, did they?

Stefan: Yes, yes. But nobody managed to get $(n\text{-}1)^2$, not even the better classes. Verbally, yes they got it and many managed to find the number of triangles in shape 20.

Maria: In my case, there were some correct responses.

Stefan: $(n\text{-}1)^2$, nobody. Verbally yes.

Maria: But I did explain a bit.

Stefan: I didn't.

Maria: I was telling them, what is the meaning of shape *n*? They were saying the previous number squared. They were saying the previous number squared. I was saying, if I have 5, how do I write the previous number 4? Five minus 1.

Stefan: They managed as I said earlier. They used a different number for *n* and they worked as for part (b). So they said $n = 50$ and they used $n = 50$. This means that at least they noticed that *n* stands for any number.

In the of case Maria, who offered some help, some students did manage to finish the tasks, complete with the algebraic representation of the more difficult pattern, $(n\text{-}1)^2$ for the triangles task. In the case of Stefan, who gave no help whatsoever, nobody completed the algebraic representation required in the Triangles task. Although some described the pattern verbally for the triangles appropriately, none of the girls in Stefan's classes managed the description of this pattern in terms of *n*. This occurred in spite of the fact that Stefan's classes included the two highest achieving classes in Form III.

Form IV

Three teachers; Karen, Frank and Tanya tried out the same tasks, Circles and Triangles with their Form IV classes. One of the teachers, Karen offered some help with her classes. The other two did not. Even without help from their teachers, in this Form, there were some girls who managed the algebraic representation of the triangles pattern $(n\text{-}1)^2$. Still there were a lot of difficulties with the algebraic representations, as is evident from the following two excerpts, the first involving Frank.

Cettina: Any surprises in your case?

Frank: As such there were no surprises. The best class worked at the task and they
got the answer. The others, some did it, some did not. Look at this. (pointing to a student's hand-out). Because the children understand it could be how many circles are there in shape *n*, she told me (meaning she wrote down) 'it could be any number'. She understands that shape *n* could stand for any number, because *n* can represent any shape number. But she was not able to relate this, to get the formula, she could not do it.

Cettina: This was a fourth former. The 4*n*, it was the simplest.

Frank: She managed the first one. 'There are 80 circles. It is a multiplication of 4'(*reading from script*). But when she came to the second one she told me (wrote) '*n* can stand for any number and was not able to realize that all she had to do was 4*n*.

At another stage Tanya pointed out another peculiar response:

Tanya: … The fourths. (pause). The same, I have three classes. The lowest ability stream found it difficult. In fact the *n*, they counted as *a*, *b*, *c*, *d*, *e*, *f*, *g*… *m* was 13 and they ended with a formula m^2 instead of $(n\text{-}1)^2$. They put the letter *m*.

Maria: The letter before *n*.

Tanya: ***So they don't, (pause), At first I, (pause), and then I realized.***

Inaudible… sighs of surprise from the other teachers.

Anna: The *m* comes before *n*.

Form V

All the Form V classes had already done similar activities that year. Patterns involving the *n* th term are specifically mentioned in the Form V syllabus and problems of this type have repeatedly appeared in recent years in the Secondary Education Certificate (SEC) examination in Mathematics usually taken at the end of Form V. Vera was only teaching the two classes at the lower end of the achievement spectrum and she tried out a different task with these classes, Matches. With their Form V classes, Frank and Tanya tried out the same tasks they had presented to their Form IV classes, namely Circles and Triangles.

Both Frank and Tanya set the tasks individually in their Form V classes. Tanya had the best achieving class and she was very satisfied with the performance of these students. Some of the best students continued the triangles pattern in a different way than that intended. As they drew the next terms of the pattern, they were leaving an empty space at the center of their shapes. They ended up with a pattern that could not be easily represented in terms on *n*. As Tanya remarked during the interview, this could have been avoided had the question included a further term in the sequence.

Frank also set the tasks individually with his fifth formers. A number gave fully correct responses but many did not manage to complete the tasks.

Vera who was teaching the two lowest achieving classes set the task Matches where the *n*th term corresponding to the number of matches in the *n*th shape resulted to be $3n+1$. In this task, the pattern was not obvious from the diagrams. In the previous tasks described, the patterns involved were n^2 for the task Hexagons, $4n$ and $(n\text{-}1)^2$ for the Circles and Triangles task. The global patterns were quite transparent. This was not the case for this task, Matches. According to the teacher, the girls, working in groups of 3 to 4 students found plenty of difficulty with finding the *n*th term, with some giving answers like $3n$ and $n+3$. Still a few managed to complete it successfully.

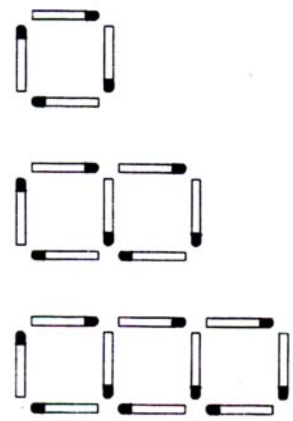
Evidence from students' handouts confirmed the claims made by Vera. Although the students could easily find successive terms by adding on three matches for the next term, many failed to find a correct expression for the *n*th term. When asked to describe a rule that gives the number of matches according to the arrangement number, these students generally wrote 'add 3'. There were a few exceptions. There were cases where the students used the general form of an arithmetic sequence to derive correctly the *n*th term, a method that had not have been covered in class but that the students would have met during private lessons. There were also cases where the students derived the *n*th term. In some cases, it is not quite clear how the students came to the correct answer. Since they were working in groups, some may have copied the answer. One particular student's work is shown in Figure 2. Clearly, the involvement of this particular student on the task is substantial. She was also able to split up the pattern in a way that was amenable to finding the *n*th term of the sequence. A possible interpretation of how she might have determined this is shown in Figure 3. As is clear from the right bottom corner of her work, where she expressed her result as $n+3 + 2(n-1)$, the student was successfully engaged in algebraic reasoning. In the process of describing the patterns she had discovered in terms of n and in the subsequent simplification of her answer, she was operating with and on the unknown n.

Matches

Arrangement 1

Arrangement 2

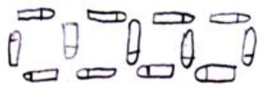
Arrangement 3



Arrangement	1	2	3	4	5	6	7	8
Number of matches	4	7	10	13	16	19	22	25

a) Find the number of matches needed for arrangement 4. You may use matches or draw a diagram if you find this useful.

13 matches



b) Complete the table.

c) How many matches are needed for arrangement 20?

61

9 10 11 12 13 14 15 16 17 18 19 20
28 31 34 37 40 43 46 49 52 55 58 61

d) What number of matches are needed for arrangement 30?

91

21 22 23 24 25 26 27 28 29 30
64 67 70 73 76 79 82 85 88 91

e) Describe a rule that helps you to find out the number of matches according to the arrangement number.

I done +3 after each numbers.

f) How many matches are needed for arrangement N?

3n+1

n=1 4 1+3
n=2 7 2+3+2
n=3 10 3+3+2+2
n=4 13 4+3+2+2+2 n+3+2(n-1)
n=5 16 5+3+2+2+2+2 n+3+2n-2
n=6 19 6+3+2+2+2+2+2 3n+1

Figure 2: The work on Matches of a Form V student

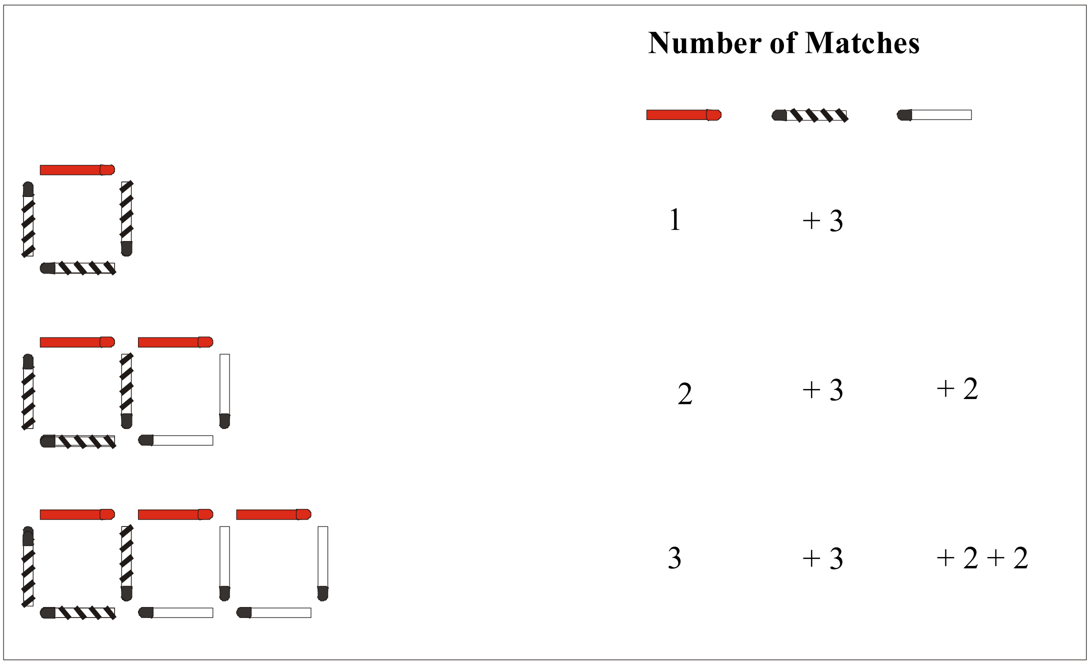


Figure 3: An interpretation of how the student may have sorted out the pattern to find the *n* th term

Discussion

All the tasks required the students to find distant terms in a sequence, the 8th term for Hexagons, the 20th term in Circles and Triangles, the 20th and 30th term in the task Matches. In all the tasks used, except for Hexagons, the students were also required to explain their reasoning. So the classroom observation of the teachers together with the evidence from students' scripts made it possible to distinguish between situations where the students failed at sorting out the global pattern and situations where students failed because of an inability to express the observed pattern algebraically. Apart from Vera, the tasks that the teachers chose for their classes involved finding global patterns that were very transparent in the sense that they were evident for many of their students. Consequently, students' difficulties with expressing the observed patterns algebraically became evident. This step, the writing of the recognized pattern in algebraic notation proved to be the major source of difficulty for many students.

Just a few weeks before the task Hexagons had been tried out, the Form II classes had covered a chapter on equations and formulae. Some of the questions in this chapter are actually devoted to students' construction of formulae. Still, while working on Hexagons, even when the students had recognized that the number of hexagons in the 8^{th} shape was 8^2, and so on, when unaided, they proved unable to write this down in terms of *n*. The prompting that was done in the class I visited and that described by Anna was certainly very minimal and was directed in helping the students use the algebraic convention, that *n* stands for any number. It is significant that even in the highest achieving classes, none of the girls came up with the answer before any help was given. This experience is consistent with the research findings described in the first section. Using Sfard's terminology, these students would have had no trouble with using the expression n^2 operationally; they would surely have coped for example with substituting the value for *n* in expressions. On the other hand, when no help was offered they could not express their simple pattern algebraically. The mere reminder by the teacher that *n* stands for any number helped many of the students come up with the answer n^2 themselves. With the slight help provided, it seems as if the students were making some steps in the direction of being able to make structural interpretations.

In the Form III classes, most students finished the task Circles complete with the algebraic generalization of $4n$ without any prompting from their teachers. On the other hand, none of the Form III students, including the higher achievers, managed the algebraic representation of the Triangles task, $(n\text{-}1)^2$, when no help was given. This in spite of the fact that many could describe the pattern verbally and could derive the 20^{th} term in the sequence using the pattern they had discovered. In the case of the teacher who did prompt the students on the meaning of *n* when they got stuck, some of the students did manage to come up with the correct algebraic representation of $(n\text{-}1)^2$. It is interesting to note that many of the Form III students were able, without prompting, to come up with the algebraic representation of $4n$. On the other hand, $(n\text{-}1)^2$ is structurally more complex, and although many had recognized the pattern, they could not articulate the pattern in an algebraic form unless some help was forthcoming.

The same tasks, Circles and Triangles were tried out in Form IV, and some of the Form V classes. A number of students in these forms including many of the students in the highest achieving class completed both tasks successfully, without any prompting. It is clear that at the beginning of the fourth form, a good number of students are gaining control over the structural complexities of algebraic representation. This is not to say that this is true of the majority of the students. Even in the fifth form, where students had already been introduced to these types of problems, many of the students who were not in the best achieving classes failed to arrive at the algebraic representation $(n\text{-}1)^2$. One of the excerpts refers to a Fourth former who deduced that the *n* th term for the Triangles task was m^2 because *m* precedes *n*. In this case the student was clearly assigning an incorrect meaning to the letters in algebra, a good exemplar of a pseudo-structural construction. This is not the first instance where students assign meanings to letters according to their position in the alphabet. I have seen it occurring in classes I have personally observed and it is also evident from literature. (Küchemann (1981); MacGregor and Stacey, 1997).

Matches, the task tried out with the two lowest achieving classes involved a global pattern of $3n$ +1. The iteration pattern, adding three matches for each successive arrangement of matches was easy for the girls to see, but the global pattern was not at all transparent. The fifth formers were already exposed to problems of this type and certainly some of their responses would have been influenced by prior instruction. Probably, some of the girls got the right answer using some teacher taught methods, in which case it is impossible to interpret the meanings the students were attaching to the letters. Even though the students had already done similar problems, a good number failed to find the *n* th term appropriately. A notable exception is the student whose work is shown in Fig 2. In writing the pattern as $n + 3 + (2n - 1)$ and simplifying, the student was using the letters structurally. She was using the expressions she derived as objects in their own right. In this case, the act of simplifying is not simply a rote procedure but involves a purposeful manipulation of the algebraic expression in order to obtain a neat concise symbolic description of the pattern involved.

The students' performance on the tasks is consistent with the findings from literature that were discussed in the first section. Even the first steps towards a structural representation of algebraic expressions involving generalised number is indeed a hurdle, that many of our students do not manage to overcome over their five years of secondary schooling. Except in the case of the task Matches, the global patterns were very transparent. In all these tasks (Hexagons, Circles and Triangles: Tasks 1 & 2) a number of students could describe the global pattern accurately but were found to have difficulty with representing the pattern using algebraic symbolization. For example in Hexagons, while working on their own, most students recognized the global pattern and worked out the number of hexagons in the eight shape by squaring the number 8. Yet these Form II students were unable to express this pattern in terms of *n*, the generalized shape number. Since the global pattern was so transparent, the students' difficulties in using algebraic representation became more explicit. At the same time, the very act of some of the teachers to get the students to think about the meaning of *n* was sufficient for many to overcome their difficulties; at least for the specific example in question. Using the distinction made by Tall between evaluative and manipulative algebra (Tall 2001), such transparent patterning tasks can be viewed as very rich learning experiences with the teachers having the possibility of helping their students with building some bridges from evaluative to manipulative algebra. Tasks like Matches where the global pattern is less transparent are more difficult and this is possibly the reason why they were not a popular choice with the teachers. In such tasks, students will not be able to express the pattern algebraically without recourse to some work at the level of manipulatitive algebra. Both types of patterning problems provide teachers with contexts where the students can make some entry points into using letters as generalized numbers and tasks of this nature could certainly be utilized more fully in the curriculum starting from the earliest Forms with the more transparent patterns.

Clearly the secondary algebra curriculum is wider in scope than the concepts tackled in this work. Rather than focusing on rote algebraic procedures, students need a wider variety of contextual problems if we want more students to make meaning out of their school algebra. Evidence from abroad affirms that this is by no means a simple issue and points to the need for an upheaval of the Algebra curriculum as is stated all too clearly in one of the NCTM documents.

> The school algebra curriculum must be reconsidered from the ground up, rather than just tinkering with the present curriculum. The challenge is to build a connected and coherent algebra strand by introducing important algebraic ideas at the appropriate

> grade level and point in the curriculum sequence and to build on this foundation throughout the rest of the K-12 curriculum. (NCTM, 1997, p.14)

By all accounts, curricular reform is a complex business and a serious attempt at reforming the Algebra curriculum is indeed a great challenge. As Gravemeijer (1977) argues, reform efforts in Mathematics Education are seriously hampered unless ample attention is given to develop instructional sequences that fit the reform. Teachers not only have to find effective activities for the various topics; they also have to sort out how to organise them within the current academic year. At the same time they may have little or no control over the manner in which prerequisite material had been covered in previous years. Teachers have little chance of overcoming all the hurdles unless they are given support through the design of sequences of instructional tasks that are tailored to fit the reform efforts.

Other factors impinging seriously on attempts at curricular reform relate to assessment issues. On the one hand, a serious evaluation of the proposed curricular changes is necessary to inform decision-makers on the effectiveness of the proposed curriculum changes. Teachers are an essential feature of such a scenario and they are indeed important partners if a serious attempt is to be made at reforming the Algebra curriculum. Another crucial issue is the need for assessment to be aligned to the reform. Students and teachers cannot be expected to give due credit to the desired reform outcomes unless the assessment methods used in schools and other examinations that may seriously affect students' lives do likewise.

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