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NOTES ON PRODUCTS OF LINDELÖF SPACES WITH POINTS G_δ

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ABSTRACT. In this note, under some extra assumptions, we study some constructions of regular T_1 Lindelöf spaces with points G_δ whose product have a large extent.

1. INTRODUCTION

For a topological space X, the *Lindelöf degree* of X, $L(X)$, is the minimal cardinal κ such that every open cover of X has a subcover of size $\leq \kappa$. A space X is *Lindelöf* if $L(X) = \omega$, that is, every open cover of X has a countable subcover. The *extent* of X, $e(X)$, is $\sup\{|C| \mid C \subseteq X \text{ is closed and discrete}\}$. It is clear that $|X| \geq L(X) \geq e(X)$.

It is well-known that the product of compact spaces is compact. In contrast with compact spaces, it is also known that the product of Lindelöf spaces needs not to be Lindelöf; If S is the Sorgenfrey line, S is Lindelöf but $e(S \times S) = 2^\omega \leq L(S \times S)$. This fact suggests the following natural question:

Question 1.1. Are there Lindelöf spaces X and Y with $e(X \times Y) > 2^\omega$?

For this question, Shelah [3] and Gorelic [1] proved the following consistency results:

Fact 1.2. *(1) Under $V = L$, there are regular T_1 Lindelöf spaces X and Y with points G_δ such that $e(X \times Y) = (2^\omega)^+$.*

(2) Suppose CH. Then there is a σ-closed, ω_2-c.c. forcing notion which forces the following statement: There are regular T_1 Lindelöf spaces X and Y with points G_δ such that $e(X \times Y) = 2^{\omega_1}$ and 2^{ω_1} is arbitrary large.

However it is still open whether the existence of such spaces is provable from ZFC. In this note, we will give relatively simple proofs of Shelah and Gorelic's results. First, we show that the Cohen forcing creates such spaces:

Theorem 1.3. *The Cohen forcing forces the following: There are regular T_1 Lindelöf spaces X and Y with points G_δ such that $e(X \times Y) = (2^{\omega_1})^V$.*

We also prove the following theorem. A space X is a *P-space* if every G_δ-subset of X is open in X.

Theorem 1.4. *Suppose there is a regular T_1 Lindelöf P-space X with character $\leq \omega_1$. Then there are regular Lindelöf spaces X_0 and X_1 with points G_δ such that $e(X_0 \times X_1) = |X|$.*

It is known that under $V = L$, there is a regular T_1 Lindelöf P-space of weight ω_1 and size $(2^\omega)^+ = \omega_2$ (Juhász-Weiss [2]).

2. In the Cohen forcing extension

For Theorem 1.3, we prove the following which would be interesting in its own right.

Proposition 2.1. *Let X be a zero-dimensional T_1 Lindelöf space X with points G_δ. Then the Cohen forcing forces the following statement: There are zero-dimensional T_1 Lindelöf spaces X_0 and X_1 with points G_δ such that $e(X_0 \times X_1) = |X|$.*

We can obtain Theorem 1.3 by the combination of the proposition with the following fact:

Fact 2.2 (Usuba [4]). *The Cohen forcing forces the following: There exists a zero-dimensional T_1 Lindelöf space X with points G_δ such that $|X| = (2^{\omega_1})^V$.*

We start the proof of the proposition. Fix a space X as in the assumption of the proposition. For each $x \in X$, take open sets G^x_n $(n < \omega)$ such that $\bigcap_{n<\omega} G^x_n = \{x\}$. By the assumption for X, we may assume that each G^x_n is clopen and $G^x_0 \supseteq G^x_1 \supseteq \cdots$. Let $H^x_n = G^x_n \setminus G^x_{n+1}$. Note that the following:

1. H^x_n is clopen.
2. $H^x_n \cap H^x_m = \emptyset$ for every $n < m < \omega$.
3. $x \notin H^x_n$.
4. $G^x_m = \{x\} \cup \bigcup_{m \leq n < \omega} H^x_n$.

Let $\mathbb{C}$ be the Cohen forcing notion $2^{<\omega}$. Take a $(V, \mathbb{C})$-generic G, and we work in $V[G]$. Let $a = \{n < \omega \mid \bigcup G(n) = 0\}$ and $b = \{n < \omega \mid \bigcup G(n) = 1\}$. We define the space X_a as the following manner. For $x \in X$, let $W^x_a = \bigcup\{H^x_n \mid n \in a\} \cup \{x\}$. W^x_a is a closed subset of X. Then the topology of X_a is generated by the family $\{O \subseteq X \mid O \text{ is open in } X\} \cup \{W^x_a \mid x \in X\}$ as a subbase. One can check that X_a is a zero-dimensional T_1-space with points G_δ. We define X_b by the same way but replacing a by b. X_a and X_b are finer spaces than X. We shall show that X_a and X_b are required spaces.

Lemma 2.3. $e(X_a \times X_b) = |X|$. *Namely, the diagonal $\Delta = \{\langle x, x\rangle \mid x \in X\}$ is a closed discrete subset of $X_a \times X_b$.*

Proof. Since X_a and X_b are Hausdorff, it is clear that Δ is closed. To see that Δ is discrete, take $\langle x, x\rangle \in \Delta$. Consider $W^x_a \times W^x_b$. It is obvious that $W^x_a \times W^x_b$ is an open neighborhood of $\langle x, x\rangle$ in $X_a \times X_b$. We check that $\Delta \cap (W^x_a \times W^x_b) = \{\langle x, x\rangle\}$. Take

$\langle y,y\rangle \in \Delta \cap (W_a^x \times W_b^x)$ and suppose $y \neq x$. Since $W_a^x = \bigcup\{H_n^x \mid n \in a\} \cup \{x\}$ and $W_b^x = \bigcup\{H_n^x \mid n \in b\} \cup \{x\}$, there are $n_a \in a$ and $n_b \in b$ such that $y \in H_{n_a}^x \cap H_{n_b}^x$. $n_a \neq n_b$ because $a \cap b = \emptyset$. However then $H_{n_a}^x$ is disjoint from $H_{n_b}^x$, this is a contradiction. □

Lemma 2.4. *X_a and X_b are Lindelöf.*

Proof. We prove it only for X_a. X_b can be checked by the same argument. Our argument which will be used in this proof came from Usuba [5].

Let $\mathcal{U}$ be an open cover of X_a. We may assume that every element of $\mathcal{U}$ is of the form $O \cap W_a^{x_0} \cap \cdots \cap W_a^{x_n}$ for some open set O in X and $x_0, \ldots, x_n \in X$. Let $W_a^{x_0,\ldots,x_n} = W_a^{x_0} \cap \ldots \cap W_a^{x_n}$. Take a $\mathbb{C}$-name $\dot{\mathcal{U}}$ for $\mathcal{U}$, and let $\dot{a}$ be a name for a.

Return to V. Let $p \in \mathbb{C}$ be such that $p \Vdash_{\mathbb{C}}$"$\dot{\mathcal{U}}$ is an open cover of $X_{\dot{a}}$". Take a sufficiently large regular cardinal θ, and a countable $M \prec H_\theta$ containing all relevant objects. We see that $p \Vdash_{\mathbb{C}}$"$\{O \cap W_{\dot{a}}^{x_0,\ldots,x_n} \in \dot{\mathcal{U}} \mid O, x_0, \ldots, x_n \in M\}$ is a cover of $X_{\dot{a}}$". Since M is countable, we have that $p \Vdash_{\mathbb{C}}$"$\dot{\mathcal{U}}$ has a countable subcover" as required.

In order to show it, fix $x^* \in X$ and $p' \leq p$. We will find $r \leq p'$ and $O, x_0, \ldots, x_n \in M$ with $r \Vdash_{\mathbb{C}}$"$x^* \in O \cap W_{\dot{a}}^{x_0,\ldots,x_n} \in \dot{\mathcal{U}}$". For a condition $q \leq p'$ and $x \in X$, let $a_q = \{n \in \mathrm{dom}(q) \mid q(n) = 0\}$ and $W_q^x = \bigcup\{H_n^x \mid n \in a_q\} \cup G_{\mathrm{dom}(q)}^x$. Then for $x_0, \ldots, x_n \in X$, let $W_q^{x_0,\ldots,x_n} = W_q^{x_0} \cap \cdots \cap W_q^{x_n}$. Note that $W_q^{x_0,\ldots,x_n}$ is open in X.

Now let $\mathcal{V}$ be the set of all $O \cap W_q^{x_0,\ldots,x_n}$ such that $q \leq p'$ and $q \Vdash_{\mathbb{C}}$"$O \cap W_{\dot{a}}^{x_0,\ldots,x_n} \in \dot{\mathcal{U}}$". We claim that $\mathcal{V}$ is an open cover of X. Take $y \in X$. Then there are $q \leq p$, open $O \subseteq X$, and $x_0, \ldots, x_n \in X$ such that $q \Vdash_{\mathbb{C}}$"$y \in O \cap W_{\dot{a}}^{x_0,\ldots,x_n} \in \dot{\mathcal{U}}$". Clearly $y \in O$, and $y \in G_0^{x_i}$ for every $i \leq n$. We see $y \in W_q^{x_i}$. Fix $i \leq n$.

Case 1: $y = x_i$. Then trivially $y \in W_q^{x_i}$.

Case 2: $y \neq x_i$. Then there is a unique $m < \omega$ with $y \in H_m^{x_i}$. If $m \geq \mathrm{dom}(q)$ or $m \in \mathrm{dom}(q)$ but $q(m) = 1$, we can take a condition $q' \leq q$ with $m \in \mathrm{dom}(q')$ and $q'(m) = 1$. $q' \Vdash_{\mathbb{C}}$"$m \notin \dot{a}$", hence $q' \Vdash_{\mathbb{C}}$"$y \notin W_{\dot{a}}^{x_i} \supseteq W_{\dot{a}}^{x_0,\ldots,x_n}$", this is a contradiction. Thus $m \in \mathrm{dom}(q)$ and $q(m) = 0$, hence $y \in W_q^{x_i}$.

In either cases, we have $y \in W_q^{x_i}$, hence $y \in \bigcap_{i \leq n} W_q^{x_i} = W_q^{x_0,\ldots,x_n}$.

By the elementarity of M, we have $\mathcal{V} \in M$. Since X is Lindelöf, $\mathcal{V}$ has a countable subcover $\mathcal{V}'$. We may assume that $\mathcal{V}' \in M$. $\mathcal{V}' \subseteq M$ because $\mathcal{V}'$ is countable. Now, we can take $O \cap W_q^{x_0,\ldots,x_n} \in \mathcal{V}'$ with $x^* \in O \cap W_q^{x_0,\ldots,x_n}$. It is clear that $O, x_0, \ldots, x_n \in M$, and $q \Vdash_{\mathbb{C}}$"$O \cap W_{\dot{a}}^{x_0,\ldots,x_n} \in \dot{\mathcal{U}}$". Finally we have to find $r \leq q$ with $r \Vdash_{\mathbb{C}}$"$x^* \in O \cap W_{\dot{a}}^{x_0,\ldots,x_n}$", this completes our proof.

Since $x^* \in W_q^{x_0,\ldots,x_n}$, for each $i \leq n$, if $x^* \neq x_i$ then $x^* \in H_k^{x_i}$ for some $k < \omega$. Hence there is a large $m < \omega$ such that $m > \mathrm{dom}(q)$ and if $x^* \neq x_i$ then $x^* \in H_k^{x_i}$ for some $k < m$. Define $r \leq q$ by $\mathrm{dom}(r) = m$, and $r(k) = 0$ for every $\mathrm{dom}(q) \leq k < m$. We check that $r \Vdash_{\mathbb{C}}$"$x^* \in W_{\dot{a}}^{x_0,\ldots,x_n}$". It is clear $r \Vdash_{\mathbb{C}}$"$x^* \in W_{\dot{a}}^{x_i}$" if $x^* = x_i$. Suppose $x^* \neq x_i$. We can find $k < m$ such that $x^* \in H_k^{x_i}$. By the choice of r, we have $r \Vdash_{\mathbb{C}}$"$k \in \dot{a}$", hence $r \Vdash_{\mathbb{C}}$"$x^* \in H_k^{x_i} \subseteq W_{\dot{a}}^{x_i}$". □

Remark 2.5. As in [5], for each positive $n < \omega$ we can prove the following: In the Cohen forcing extension, there is a regular Lindelöf space X with points G_δ such that X^n is Lindelöf but $e(X^{n+1}) = (2^{\omega_1})^V$.

3. Using P-spaces

In this section, we prove Theorem 1.3. Fix a regular Lindelöf T_1 P-space X with character $\leq \omega_1$. Let $Y = \{x \in X \mid \chi(x, X) = \omega_1\}$. Note that every $x \in X \setminus Y$ is an isolated point.

Let S be the Sorgenfrey line, namely, the underlying set of S is the real line $\mathbb{R}$, and the topology of S is generated by the family $\{[r, s) \mid r, s \in \mathbb{R}\}$ as an open base. S is a first countable regular T_1 Lindelöf space.

For a subset $A \subseteq X$, let $[\![A]\!] = \bigcup\{\{x\} \times \mathbb{R} \mid x \in A \cap Y\} \cup (A \setminus Y)$. By the assumption, for each $x \in Y$, there is a sequence $\langle G^x_\alpha \mid \alpha < \omega_1\rangle$ such that:

(1) G^x_α is clopen in X.
(2) $\langle G^x_\alpha \mid \alpha < \omega_1\rangle$ is $\subseteq$-decreasing, and $G^x_\alpha = \bigcap_{\beta<\alpha} G^x_\beta$ if α is limit.
(3) $\bigcap_{\alpha<\omega_1} G^x_\alpha = \{x\}$.

Fix an injection $\sigma : \omega_1 \to \mathbb{R}$. For $x \in Y$, $\alpha < \omega_1$, and open $O \subseteq S$, let $W(x, \alpha, O) = \bigcup\{[\![G^x_\beta \setminus G^x_{\beta+1}]\!] \mid \alpha \leq \beta, \sigma(\beta) \in O\} \cup (\{x\} \times O)$.

We define the space X_0 as in [4] using X and S. The underlying set of X_0 is $[\![X]\!]$. The topology of X_0 is generated by the family $\{[\![W]\!] \mid W \subseteq X \text{ is open}\} \cup \{W(x, \alpha, O) \mid x \in Y, \alpha < \omega_1, O \subseteq S \text{ is open}\}$ as an open base. We know that X_0 is a regular T_1 Lindelöf space with points G_δ (see [4]).

For X_1, let S' be the space $\mathbb{R}$ equipped with the reverse Sorgenfrey topology, namely, it is generated by the family $\{(r, s] \mid r, s \in \mathbb{R}\}$ as an open base. S' is also a first countable regular T_1 Lindelöf space. We define X_1 by the same way to X_0 but replacing S with S'. Again, X_1 is a regular Lindelöf space with points G_δ.

We show that $e(X_0 \times X_1) = |X|$. Let $\Delta = \{\langle x, x\rangle \mid x \in X \setminus Y\} \cup \{\langle\langle x, r\rangle, \langle x, r\rangle\rangle \mid x \in Y, r \in \mathbb{R}\}$. One can check that Δ is closed in $X_0 \times X_1$. We see that Δ is discrete. If $x \in X \setminus Y$, then x is isolated in X, hence $\langle x, x\rangle$ is also isolated in $X_0 \times X_1$. Let $x \in Y$ and $r \in \mathbb{R}$. Let $O_0 = [r, r+1)$ and $O_1 = (r-1, r]$. O_0 is open in S with $r \in O_0$, and O_1 is open in S' with $r \in O_1$. Moreover we have $O_0 \cap O_1 = \{r\}$. Consider open sets $W(x, 0, O_0) \subseteq X_0$ and $W(x, 0, O_1) \subseteq X_1$. By the choice of O_0 and O_1, we have $W(x, 0, O_0) \cap W(x, 0, O_1) = \{\langle x, r\rangle\}$. Then $\Delta \cap (W(x, 0, O_0) \times W(x, 0, O_1)) = \langle\langle x, r\rangle, \langle x, r\rangle\rangle$, as required.

Remark 3.1. The existence of a regular T_1 Lindelöf P-space with character $\leq \omega_1$ and size $> \omega_1$ is independent from ZFC.

References

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