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Optimal Execution Strategies with Generalized Price Impact Models

同志社大学, 商学部　久納　誠矢
Seiya KUNO
Faculty of Commerce, Doshisha University

大阪大学, 大学院・経済学研究科 & 数理・データ科学教育研究センター　大西　匡光
Masamitsu OHNISHI
Graduate School of Economics & Center for Mathematical Modeling and Data Science
Osaka University

大阪大学, 大学院・経済学研究科　下清水　慎
Makoto SHIMOSHIMIZU
Graduate School of Economics, Osaka University

1 Introduction

In the security market analysis, we usually assume that each trade of buying or selling does not affect the stock price. However, there are some kind of institutional traders called "large traders" who affect the market price through their own large trades, and the price change incurred by the large traders is referred to as "price impact."

Large traders must recognize these price impacts as "liquidity risk" and construct the execution strategy considering the liquidity risk. The field of "optimal execution problem considering liquidity risk," is a hot topic in recent years among many academic researchers or practitioners, in particular after the financial crisis of 2007–2008.

The pioneering work in this field was done by Bertsimas and Lo (1998) [2]. They discussed the optimization problem of minimizing the expected execution cost in a discrete–time framework via a dynamic programming approach, and showed that it is optimal to equal the execution volume throughout the trading epochs. However, their model does not take into account any risks. Therefore, Almgren and Chriss (2000) [1] derive the optimal execution strategy considering both the execution cost and the volatility risk. They also incorporate the idea of efficient frontier into their analysis. As for Kuno and Ohnishi (2015) [5] and Kuno, Ohnishi, and Shimizu (2017) [6], they construct models with the residual effect of the price impact, solve optimization problem of maximizing the expected utility payoff from the final wealth, and lead to optimal execution strategies.

As a trend of the previous papers including those mentioned above, many researchers focus on the behavior of the institutional traders. Moreover, they suppose that there is no price impact of the traders other than the large traders. We call such traders as trading crowd as Huberman and Stanzl (2004) [3]. In addition, they did not consider any model with a number of large traders in spite of the fact that there are many large traders in the real market.

The purpose of this article is to obtain the optimal execution strategy of maximizing the expected utility payoff from the large trader's final wealth, which is the same approach with those introduced in [5] and [6].

2 Price Impact Model with Non-Large Traders' Effects

In this model, we assume there is one large trader (for example, a life insurance company or a trust company) in a discrete–time framework $t = 1, \ldots, T, T+1$. Then, the large trader is going to purchase Q volume of one risky asset by the time $T+1$ ($\in \mathbb{Z}_+ := \{1, 2, \ldots\}$). We also suppose that the large trader has the CARA utility function with the absolute risk aversion parameter R.

2.1 Market Model

We assume that q_t represents the large amount of orders submitted by the large trader at time t $(= 1, \dots, T)$. Then, $\overline{Q}_t$ is the number of shares remained to purchase by the large trader at time t $(= 1, \dots, T, T+1)$. From this assumption, $\overline{Q}_1 = Q$ and

$$\overline{Q}_{t+1} = \overline{Q}_t - q_t, \quad t = 1, \dots, T \tag{1}$$

is satisfied.

The market price (or quotation price) of the risky asset at time t is represented by p_t. Because the large trader submit large amount of orders, the execution price become not p_t, but $\widehat{p}_t$ with the additive execution cost. In the following, we denote λ_t as the price impact per share occurred by the large trader and κ_t as the one by the trading crowd. Then, a sequence of independent random variables v_t follows the normal distribution with the mean μ_{v_t} and the variance $\sigma_{v_t}^2$, that is, $v_t \sim N(\mu_{v_t}, \sigma_{v_t}^2)$, which represents the execution volume of trading crowd at time t.

From the assumption above, we set the execution price in the form of the linear price impact model as follow:

$$\widehat{p}_t = p_t + (\lambda_t q_t + \kappa_t v_t), \quad t = 1, \dots, T. \tag{2}$$

With the deterministic price reversion rate α_t $(\in [0, 1])$ and the resilience speed ρ $(\in [0, \infty))$, we define the residual effect of past price impact as follows:

$$r_{t+1} = \sum_{k=1}^{t} (\lambda_k q_k + \kappa_k v_k)\alpha_k \mathrm{e}^{-\rho((t+1)-k)} = [r_t + (\lambda_t q_t + \kappa_t v_t)\alpha_t]\mathrm{e}^{-\rho}, \quad t = 1, \dots, T. \tag{3}$$

Some public news or information of the economic situation have an impact on the price. Therefore, we define independent random variable ε_t $(t = 1, \dots, T)$ as the effect of the public news/information about economic situation between t and $t+1$, and assume ε_t follows the normal distribution with the mean μ_{ε_t} and the variance $\sigma_{\varepsilon_t}^2$, that is, $\varepsilon_t \sim N(\mu_{\varepsilon_t}, \sigma_{\varepsilon_t}^2)$. We suppose that the two stochastic process, v_t $(t = 1, \dots, T)$ and ε_t $(t = 1, \dots, T)$, are mutually independent. However, we can derive the similar results without this assumption (that is, if they follow a bivariate normal distribution).

The definition of ε_t gives rise to the fundamental price $p_t^f := p_t - r_t$ as follows:

$$p_{t+1}^f = p_{t+1} - r_{t+1} = p_t^f + (\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t) + \varepsilon_t, \quad t = 1, \dots, T. \tag{4}$$

From (2) , (3) and (4), the execution price is calculated as

$$p_{t+1} = p_t - (1 - \mathrm{e}^{-\rho})r_t + (\lambda_t q_t + \kappa_t v_t)\{\alpha_t \mathrm{e}^{-\rho} + (1 - \mathrm{e}^{-\rho})\} + \varepsilon_t, \quad t = 1, \dots, T. \tag{5}$$

In this context, $(\lambda_t q_t + \kappa_t v_t)(1 - \alpha_t)$, $(\lambda_t q_t + \kappa_t v_t)\alpha_t$ and $(\lambda_t q_t + \kappa_t v_t)\alpha_t \mathrm{e}^{-\rho}$ represents the parmanent impact, the temporary impact, and the transient impact respectively. Moreover, if $\rho \to \infty$, the model is attributed to the parmanent impact model, and if $\alpha_t = 1$, the model is attributed to the transient impact model. Also, if $\kappa_t = 0$ or $\sigma_{v_t} = 0$, the model is attributed to [6].

Finally, the wealth process w_t is

$$w_{t+1} = w_t - \hat{p}_t q_t = w_t - \{p_t + (\lambda_t q_t + \kappa_t v_t)\}q_t, \quad t = 1, \dots, T. \tag{6}$$

2.2 Formulation as a Markov Decision Process

In this subsection, we formulate the large trader's problem as a discrete–time Markov decision process. The time horizon is finite, $1, \dots, T, T+1$. The state of the process at time $t \in \{1, \dots, T, T+1\}$ is a 4–tuple, and is denoted as $s_t = (w_t, \overline{Q}_t, p_t, r_t) \in \mathbb{R}^4 =: S$. Also, for $t \in \{1, \dots, T\}$, an allowable action chosen at state s_t is an execution volume $q_t \in \mathbb{R} =: A$ so that the set A of admissible actions is independent of the current sate s_t.

When an action q_t is chosen in a state s_t at time $t \in \{1, \dots, T\}$, a transition to a next state $s_{t+1} = (w_{t+1}, \overline{Q}_{t+1}, p_{t+1}, r_{t+1}) \in S$ occurs according to the law of motion precisely described in the previous subsection which is symbolically denoted by a system dynamics function $h_t : \mathrm{S} \times \mathrm{A} \times (\mathbb{R} \times \mathbb{R}) \longrightarrow \mathrm{S}$:

$$s_{t+1} = h_t(s_t, q_t, (v_t, \varepsilon_t)), \quad t = 1, \dots, T. \tag{7}$$

A utility payoff (or reward) arises only in a terminal state s_{T+1} at the end of horizon $T + 1$ as

$$r_{T+1}(s_{T+1}) := \begin{cases} -\exp\{-Rw_{T+1}\} & \text{if } \overline{Q}_{T+1} = 0; \\ -\infty & \text{if } \overline{Q}_{T+1} \neq 0, \end{cases} \tag{8}$$

which means a hard constraint enforcing the large trader to execute all of the remaining volume $\overline{Q}_T$ at the maturity T, that is, $q_T = \overline{Q}_T$.

If we define a (history–independent) one–stage decision rule f_t at time $t \in \{1, \dots, T\}$ by a map from a state $s_t \in S = \mathbb{R}^4$ to an action $q_t = f_t(s_t) \in A = \mathbb{R}$, then a Markov execution strategy π is defined as a sequence of one–stage decision rules $\pi := (f_1, \dots, f_t, \dots, f_T)$. We denote the set of all Markov execution strategies as Π_M. Further, for $t \in \{1, \dots, T\}$, we define the sub execution strategy after time t of a Markov execution strategy $\pi = (f_1, \dots, f_t, \dots, f_T) \in \Pi$ as $\pi_t := (f_t, \dots, f_T)$, and the entire set of π_t as $\Pi_{M,t}$.

By definition (8), the value function under an execution strategy π becomes an expected utility payoff arising from the terminal wealth w_{T+1} of the large trader with the absolute risk aversion R:

$$V_1^\pi[s_1] = \mathbb{E}_1^\pi[r_{T+1}(s_{T+1})|s_1] = \mathbb{E}_1^\pi[-\exp\{-Rw_{T+1}\} \cdot 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}|s_1], \tag{9}$$

where, for $t \in \{1, \dots, T\}$, $\mathbb{E}_t^\pi$ is a conditional expectation given s_t at time t under π.

Then, for $t \in \{1, \dots, T\}$ and $s_t \in \mathrm{S}$, we further let

$$V_t^\pi[s_t] = \mathbb{E}_t^\pi[r_{T+1}(s_{T+1})|s_t] = \mathbb{E}_t^\pi[-\exp\{-Rw_{T+1}\} \cdot 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}|s_t] \tag{10}$$

be the expected utility payoff at time t under the strategy π. It is noted that the expected utility payoff $V_t^\pi[s_t]$ depends on the Markov execution policy $\pi = (f_1, \dots, f_t, \dots, f_T)$ only through the sub execution policy $\pi_t := (f_t, \dots, f_T)$ after time t.

Now, we define the optimal value function as follows:

$$V_t[s_t] = \sup_{\pi \in \Pi_M} V_t^\pi[s_t], \quad s_t \in \mathrm{S},\ t = 1, \dots, T. \tag{11}$$

From the principle of optimality, the optimality equation (Bellman Equation, or dynamic programming equation) becomes

$$V_t[s_t] = \sup_{q_t \in \mathrm{R}} \mathbb{E}[V_{t+1}[h_t(s_t, q_t, v_t, \varepsilon_t)]s_t], \quad s_t \in \mathrm{S},\ t = 1, \dots, T. \tag{12}$$

3 Optimal Execution Strategy

The optimal dynamic execution strategy π is acquired by solving the above equation (12) backwardly on time t from maturity.

Theorem 3.1 (Optimal Execution Strategy and Optimal Value Function) *The optimal execution volume at time t, denoted by q_t^*, becomes the affine function of the remianing execution volume $\overline{Q}_t$ and the cumulative residual effect r_t at time t. That is,*

$$q_t^* = f_t(s_t) = a_t + b_t\overline{Q}_t + c_t r_t, \quad t = 1, \dots, T. \tag{13}$$

Moreover, the optimal value function $V_t(s_t)$ at time t $(= 1, \ldots, T)$ is represented as follow:

$$V_t[s_t] = -\exp\{-R[w_t - p_t\overline{Q}_t - G_t\overline{Q}_t^2 - H_t\overline{Q}_t + I_t\overline{Q}_t r_t + J_t r_t^2 + L_t r_t + Z_t]\}. \tag{14}$$

Where a_t, b_t, c_t; $G_t, H_t, I_t, J_t, L_t, Z_t$, $t = 1, \ldots, T$ are deterministic functions dependent on the problem parameters, and can be computed backwardly in time t.

From the theorem above, the optimal execution volume q_t^* depends on the state $s_t = (w_t, p_t, \overline{Q}_t, r_t)$ of the decision process only through the remianing execution volume $\overline{Q}_t$ and the cumulative residual effect r_t, not through the wealth w_t or market price p_t. Furthermore , if the orders of the trading crowds are deterministic, then optimal dynamic execution strategy is in a class of the static (and non-randomized) execution strategy.

4 Numerical Examples

In this section, we illustrate some numerical examples and show some properties of the optimal execution strategies defned above. The maturity is $T = 10$, and the large trader plans to execute the volume $Q = 100000$ within a time period T at the beginning. We assume the time homogeneity of the time-dependent parameter $\mu_{v_t}, \sigma_{v_t}, \mu_{\varepsilon_t}, \sigma_{\varepsilon_t}, \alpha_t, \lambda_t, \kappa_t$ for simplicity. To obtain the explicit form of the optimal execution volume, we assume that the price impact of the trading crowd is deterministic, i.e. $\sigma_{v_t} = 0$, $t = 1, \ldots, T$. We set the benchmark as $\mu_{v_t} \equiv 2.0, \mu_{\varepsilon_t} \equiv 1.0, \sigma_{\varepsilon_t} \equiv 0.02, \alpha_t \equiv 0.5, \lambda_t \equiv 0.001, \kappa_t \equiv 0.005, \rho = 0.1, R = 0.001$.

4.1 Possibility of Weak Arbitrage

In this subsection, we consider about a possibility of a gain from a round trip trading. For a sequence $\boldsymbol{q} := (q_1, \ldots, q_T) \in \mathbb{R}^T$, a static (and non-randomized) execution strategy $\pi = (f_1, \ldots, f_T) \in \Pi_M$ defined by $f_t(s_t) = q_t$ for any $s_t \in S = \mathbb{R}$ is called a round trip trading schedule if it satisfies the condition $\sum_{t=1}^T q_t = 0$. In particular, zero–trade schedule is the (trivial) round trip trading schedule defined by $\mathbf{0} = (0, \ldots, 0)$.

According to [3], an opportunity of an arbitrage in a weak sense is a round trip trading schedule $(\pi =)$ $\boldsymbol{q}$ if

$$\mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}|s_1] - w_1 = \mathbb{E}_1^{\boldsymbol{q}}[w_{T+1} - w_1|s_1] = \mathbb{E}_1^{\boldsymbol{q}}\left[-\sum_{t=1}^{T}\widehat{p}_t q_t \middle| s_1\right] > 0 \tag{15}$$

where $\widehat{p}_t$, $t = 1, \ldots, T$ is the execution price defined in Section 2.

If the initial execution volume $\overline{Q}_1 = Q = 0$, the zero-trade schedule $(\pi =)$ $\mathbf{0}$ obviously satisfies the hard terminal constraint $\overline{Q}_{T+1} = 0$ and results in a final wealth $w_{T+1} = w_1$ with certainty. Therefore, an opportunity $\boldsymbol{q}$ of an arbitrage in a weak sense does strictly better than the zero–trade schedule $\mathbf{0}$ with respect to the risk-neutral criterion. Furthermore, with respect to our expected utility criterion, if we have $V_1^{\mathbf{0}}[s_1] < V_1^{\boldsymbol{q}}[s_1]$ for some round trip trading schedule $\boldsymbol{q}$, then, by Jensen's inequality, we obtain

$$-\exp\{-Rw_1\} = V_1^{\mathbf{0}}[s_1] < V_1^{\boldsymbol{q}}[s_1] = \mathbb{E}_1^{\boldsymbol{q}}[-\exp\{-Rw_{T+1}\}|\, s_1] \le -\exp\left\{-R\,\mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}|s_1]\right\}, \tag{16}$$

which implies $w_1 < \mathbb{E}_1^{\boldsymbol{q}}[w_{T+1}|s_1]$, that is, $\boldsymbol{q}$ is also an arbitrage in a weak sense.

In the following, we show three cases of the trades where the large trader execute a round trip trade, i.e., $\overline{Q}_1 = \overline{Q}_{T+1} = 0$: $\mu_{\varepsilon_t} = 1$, $\mu_{\varepsilon_t} = 0$, $\mu_{\varepsilon_t} = -1$

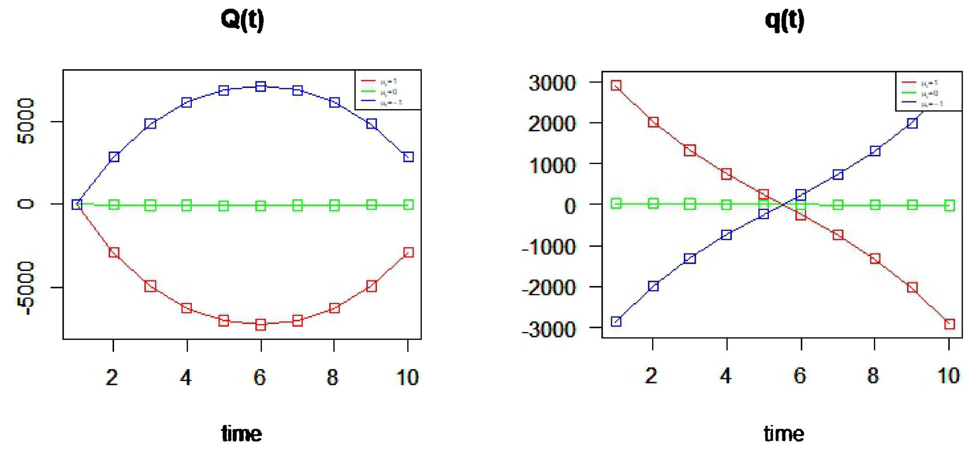

Figure 1. Remaining Execution Volume $\overline{Q}_t$ and Execution Volume q_t $(t = 1, \ldots, T)$

From these graphs, we find that, in the cases $\mu_{\varepsilon_t} \neq 0$, the large trader is able to increase the expected utility of the final wealth by starting from $\overline{Q}_1 = 0$. Therefore, if $\mu_{\varepsilon_t} \neq 0$, then there exist round trip trades which satisfy an arbitrage in a weak sense.

4.2 Comparative Statics

4.2.1 The Effect of Risk Aversion

First, we show the following three cases: $R = 0.001$, $R = 0.5$, and $R = 1$.

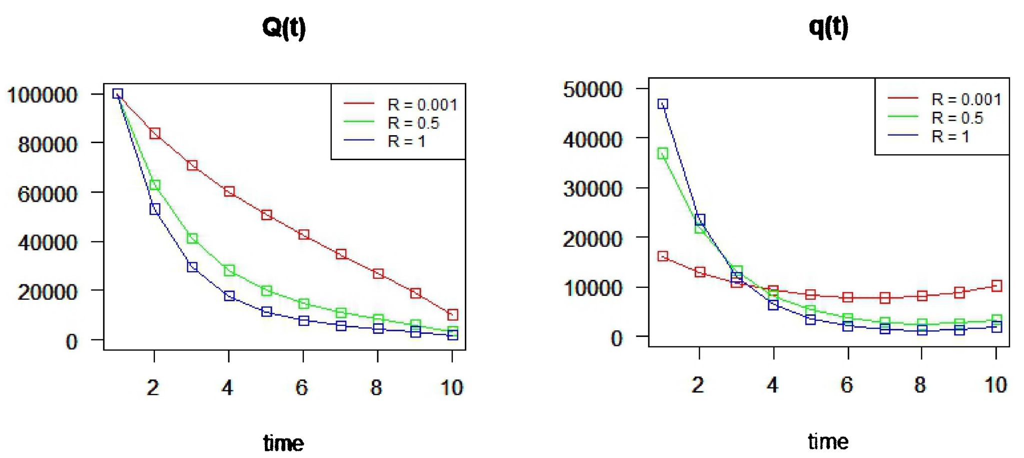

Figure 2. Remaining Execution Volume $\overline{Q}_t$ and Execution Volume q_t $(t = 1, \ldots, T)$

As these graphs show, the more risk averse the large trader is, the faster he or she executes. That is because the more risk-averse trader tends to avoid the price risk as possible.

4.2.2 The Effect of α_t

Next, we see the execution volume of the three cases: $\alpha_t = 0.01$, $\alpha_t = 0.5$, and $\alpha_t = 1$.

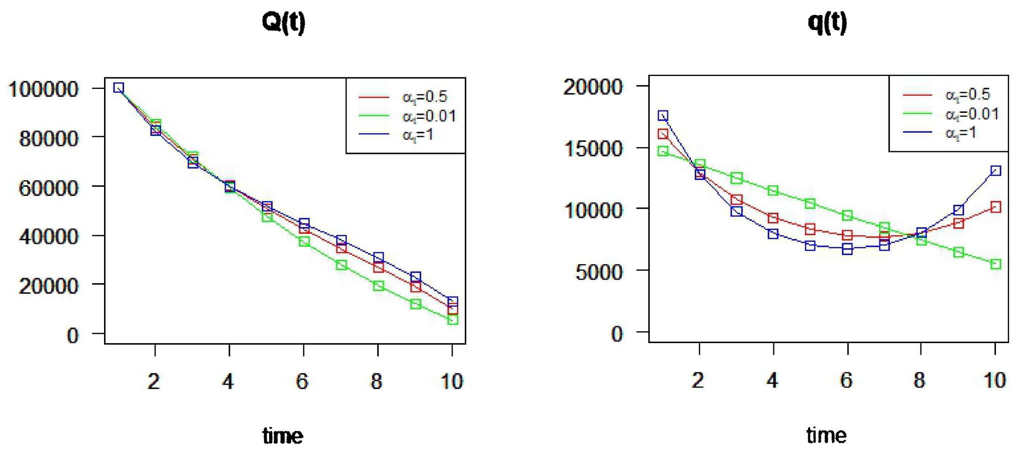

Figure 3. Remaining Execution Volume $\overline{Q}_t$ and Execution Volume q_t $(t = 1, \ldots, T)$

These graphs illustrate that the faster the price reverses, the more the large trader tends to execute at the beginning and at the end of the trading.

4.2.3 The Effect of σ_{ε_t}

We show the three cases: $\sigma_{\varepsilon_t} = 0.02$, $\sigma_{\varepsilon_t} = 0.5$, and $\sigma_{\varepsilon_t} = 1$.

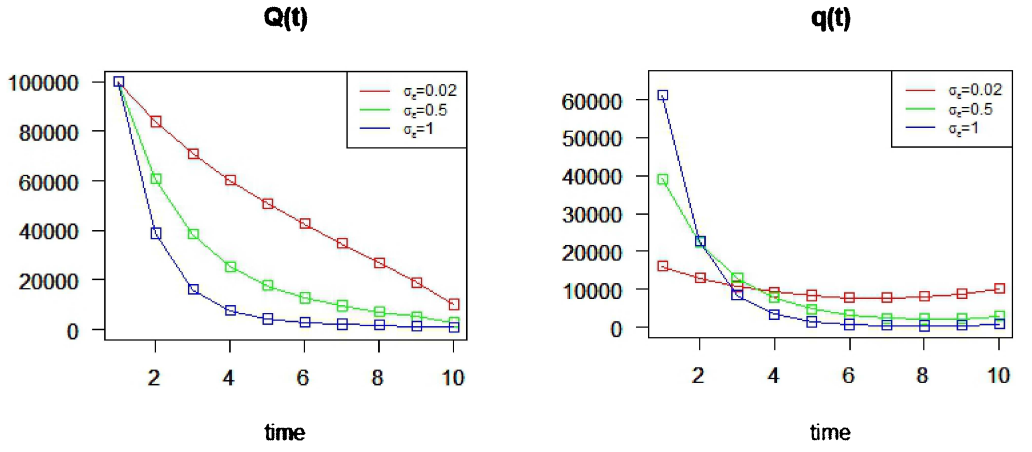

Figure 4. Remaining Execution Volume $\overline{Q}_t$ and Execution Volume q_t $(t = 1, \ldots, T)$

These graphs illustrate that if the variance of the effect of the public news is large, the large trader executes more faster for fear that the execution price will be pushed up by the public information.

4.2.4 The Effect of the resilience speed

Finally, we demonstrate the three cases: $\rho = 0.1$, $\rho = 0.5$, and $\rho = 1$.

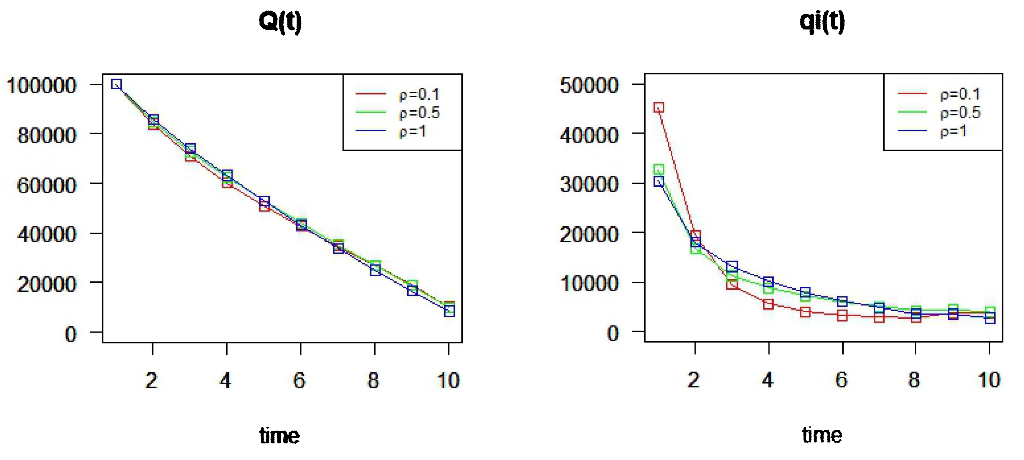

Figure 5. Remaining Execution Volume $\overline{Q}_t$ and Execution Volume q_t $(t = 1, \ldots, T)$

We can interpret from these graphs that as the resilience speed increase, the large trader executes their order submit slowly.

5 Conclusion and Future Research

In this article, we derived the optimal execution strategy in the case of single–large trader, and showed some features of that strategy. However, we did not consider the situation where there are other large traders. Hence, the study on the case of non–single large trader model would be remained for our future research.

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