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Hankel operators and measures

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Abstract. In this paper we consider Hankel operators of Schatten p-classes. If μ is a (finite) positive Borel measure on $\mathbb{D}$, it induces an infinite Hankel matrix H_μ. There are several results for conditions that H_μ belong to Schatten p-classes. We loot at these results and its generalizations.

Keywords. Hankel operators, Carleson measures, Hamburger moment problem, Schatten p-class.

1. Introduction

In 1920, H. Hamburger showed that the classical Hamburger moment problem with data $(\mu_n)_{n\geq 0}$ has a solution if and only if the infinite Hankel matrix $\{\mu_{j+k}\}_{j,k\geq 0}$ is positive semi-definite. This paper deals with the results in this direction.

This paper consists of two parts. In Section 2, we introduce basic notions and preliminary results such as Hankel operators and matrices, the Hamburger moment problem and the Carleson measures. In Section 3, we consider Hankel matrices H_μ induced by measures.

2. Preliminaries

In this section we introduce some basic notions and known results.

Let

$$\mathbb{D} := \{z : |z| < 1\}$$

be the open unit disk and let

$$\mathbb{T} := \{z : |z| = 1\}$$

be the unit circle. Let m be the normalized Lebesgue measure on $\mathbb{T}$, that is $m(\mathbb{T}) = 1$.

For $1 \leq p \leq \infty$, we let

$$L^p(\mathbb{T}) := L^p(\mathbb{T}, m).$$

Let $H(\mathbb{D})$ be the class of all analytic functions in $\mathbb{D}$.

For a function f on $\mathbb{D}$ and $0 < r < 1$, let f_r be the function on $\mathbb{T}$ defined by

$$f_r(e^{i\theta}) = f(re^{i\theta}).$$

For $1 \leq p \leq \infty$, let H^p be the set of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} \|f_r\|_{L^p} < \infty.$$

Thanks.

If $f \in H^p$, then for almost all point $e^{i\theta} \in \mathbb{T}$, f has a nontangential limit $f^*(e^{i\theta})$. In this case f^* is an $L^p(\mathbb{T})$-function. If we identify f with f^*, then H^p is a closed subspace of $L^p(\mathbb{T})$. Note that

$$\sum_{n\geq 0} a_n z^n \in H^2 \quad \text{if and only if} \quad \sum_{n\geq 0} |a_n|^2 < \infty.$$

Let

$$\mathcal{A} := \{f \in C(\overline{\mathbb{D}}) : f \in H(\mathbb{D})\}$$

be the disk algebra. Since the disc algebra $\mathcal{A}$ contains every analytic trigonometric polynomials, it is dense in the Hardy space H^p for each $1 \leq p < \infty$.

Now we introduce the definition of Hankel operators. Let P be the orthogonal projection of $L^2(\mathbb{T})$ onto H^2, that is, is $f = \sum_{n=-\infty}^{\infty} a_n z^n$, then

$$Pf := \sum_{n\geq 0} a_n z^n.$$

Define the unitary operator $J : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$(Jf)(z) := \bar{z} f(\bar{z}).$$

Definition. For $\varphi \in L^\infty(\mathbb{T})$, define $H_\varphi : H^2 \to H^2$ by

$$H_\varphi f := J(I - P)(\varphi f)$$

for $f \in H^2$. The operator H_φ is called the Hankel operator with simbol φ.

The matrix of H_φ with respect to the orthonormal basis $\{z^n : n \geq 0\}$ is

$$\begin{pmatrix} \hat{\varphi}(-1) & \hat{\varphi}(-2) & \hat{\varphi}(-3) & \cdots \\ \hat{\varphi}(-2) & \hat{\varphi}(-3) & \hat{\varphi}(-4) & \cdots \\ \hat{\varphi}(-3) & \hat{\varphi}(-4) & \hat{\varphi}(-5) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where

$$\hat{\varphi}(n) := \int_{\mathbb{T}} \varphi(z) z^{-n}\, dm(z)$$

is the n-th Fourier coefficient of φ. Such a matrix is called a Hankel matrix.

Let $(\mu_n)_{n\geq 0}$ be a sequence of complex numbers. The classical *Hamburger moment problem with data* $(\mu_n)_{n\geq 0}$ is to find a positive measure μ on the real line $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |t|^n\, d\mu(t) < \infty$$

and

$$\mu_n = \int_{\mathbb{R}} t^n\, d\mu(t)$$

for every $n = 0, 1, 2, \ldots$. The following theorem gives a solution of the Hamburger moment problem with data $(\mu_n)_{n\geq 0}$. Recall that the infinite matrix $(\mu_{j,k})_{j,k\geq 0}$ is said to be *positive semi-deifnite* if

$$\sum_{j,k\geq 0} \mu_{j,k} x_k \bar{x}_j \geq 0$$

for every finitely supported sequence $(x_j)_{j\geq 0}$.

Theorem. [Ham] Let $(\mu_n)_{n\geq 0}$ be a sequence of complex numbers. The Hamburger moment problem with data $(\mu_n)_{n\geq 0}$ is solvable if and only if the Hankel matrix (μ_{j+k}) is positive semi-definite.

A positive measure μ on $\mathbb{D}$ is called a *Carleson measure* if there is a constant C such that

$$\int_{\mathbb{D}} |f(z)|^2 \mu(z) \leq C\|f\|_2^2$$

for all $f \in H^2$, that is, the identical embedding operator $I_\mu : H^2 \to L^2(\mu)$, $I_\mu f = f$, is bounded. By the Carleson embedding theorem [Car], a positive measure μ on $\mathbb{D}$ is a Carelson measure if and only if

$$\sup_I \frac{|\mu|(R_I)}{m(I)} < \infty,$$

where the supremum is taken over all subarcs $I \subset \mathbb{T}$, and

$$R_I := \{z \in \mathbb{D} : \frac{z}{|z|} \in I \text{ and } 1 - |z| \leq m(I)\}.$$

Observe that if μ is supported on $(-1, 1)$, then μ is a Carleson measure if and only if there is a constant C such that

$$\mu((1-t, 1)) \leq Ct \quad \text{and} \quad \mu((-1, -1+t)) \leq Ct$$

for every $0 < t < 1$.

A positive measure μ on $\mathbb{D}$ is called a *vanishing Carleson measure* if the identical embedding operator $I_\mu : H^2 \to L^2(\mu)$, $I_\mu f = f$ is compact.

3. Hankel operators and measures

Let μ be a (finite) positive Borel measure on $\mathbb{D}$. Define

$$\mu_n := \int_{\mathbb{D}} t^n \, d\mu(t)$$

and define

$$H_\mu := \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \\ \mu_1 & \mu_2 & & \\ \mu_2 & & & \\ & & & \ddots \end{pmatrix}.$$

Note that H_μ is an infinite Hankel matrix. The following theorems tell us when H_μ is bounded or compact operator.

Theorem. [Pe] Let $(\alpha_n)_{n\geq 0}$ be a complex sequence. Let $\Gamma = \{\alpha_{j+k}\}_{j,k\geq 0}$ be a nonnegative Hankel matrix. The following are equivalent:

(i) Γ determines a bounded operator on ℓ^2.
(ii) There is a positive Carleson measure μ supported on $(-1, 1)$ such that $\Gamma = H_\mu$.
(iii) $|\alpha_n| \leq \text{const}(1+n)^{-1}$.

Theorem. [Pe] Let $(\alpha_n)_{n\geq 0}$ be a complex sequence. Let $\Gamma = \{\alpha_{j+k}\}_{j,k\geq 0}$ be a nonnegative Hankel matrix. The following are equivalent:

(i) Γ determines a compact operator on ℓ^2.
(ii) There is a positive vanishing Carleson measure μ supported on $(-1, 1)$ such that $\Gamma = H_\mu$.
(iii) $\lim_{n\to\infty} \alpha_n(1+n) = 0$.

Let H be a Hilbert space with $\langle \cdot, \cdot \rangle_H$ and let $(e_n)_{n\geq 0}$ be an orthonormal basis. A positive (bounded) operator A on H is of *trace class* if

$$\sum_{n\geq 0} \langle Ae_n, e_n \rangle_H < \infty.$$

Definition. Let $p > 0$. An operator A on H is of *Schatten p-class* if $|A|^p$ is of trace class. Let $S_p(H)$ denote the class of all Schatten p-class operators on H. An operator $A \in S_2(H)$ is called a Hilbert-Schmidt class operator.

In 2014, C. Chatzifountas, D. Girela and J. A. Peláez have given a characterization of infinite Hankel matrices H_μ which are of Schatten p-class:

Theorem. [CGP] Assume $1 < p < \infty$. Let μ be a (finite) positive Borel measure on $[0, 1)$. Then $H_\mu \in S_p(H^2)$ if and only if

$$\sum_{n\geq 0} (n+1)^{p-1} \mu_n^p < \infty.$$

Kiwon Lee generalized this theorem as follows (not published): Assume $1 < p < \infty$. Let μ be a (finite) positive Borel measure supported on $(-1, 1)$. Then $H_\mu \in S_p(H^2)$ if and only if

$$\sum_{n\geq 0} (n+1)^{p-1} |\mu_n|^p < \infty.$$

In 2010, P. Galanopoulos and J. A. Peláez.

Theorem. [GP] Let μ be a (finite) positive Borel measure on $[0, 1)$ and suppose that H_μ is bounded on H^2. Then $H_\mu \in S_2(H^2)$ if and only if

$$\int_{[0,1)} \frac{\mu([t,1))}{(1-t)^2}\, d\mu(t) < \infty.$$

To generalize this theorem to a measure on $\mathbb{D}$ we introduce some notations:
For $0 < t < 1$,

$$\mathbb{D}_t := \{z : |z| < t\},$$
$$A_t := \{z : |z| > t\},$$

and

$$\mathbb{T}_t := \{z : |z| = t\}.$$

Note that $\overline{\mathbb{D}_t} = \mathbb{D}_t \cup \mathbb{T}_t$, $\overline{A_t} = A_t \cup \mathbb{T}_t$, and $\mathbb{D} = \mathbb{D}_t \cup A_t \cup \mathbb{T}_t$. The following theorem gives a sufficient condition that H_μ belongs to $S_2(H^2)$.

Theorem. Let μ be a (finite) positive Borel measure on $\mathbb{D}$. If

$$\int_{\mathbb{D}} \frac{\mu(\overline{A}_{|z|})}{(1-|z|^2)}\, d\mu(z) < \infty,$$

then $H_\mu \in S_2$.

Remark. If μ is Lebesgue measure on $\mathbb{D}$, then

$$H_\mu = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 0 & & \\ 0 & & & \\ & & & \ddots \end{pmatrix}.$$

Hence H_μ is of finite rank. On the other hand,

$$\int_{\mathbb{D}} \frac{\mu(\overline{A}_{|z|})}{(1-|z|^2)}\, d\mu(z) = \int_0^{2\pi}\int_0^1 \frac{\pi(1-r^2)}{(1-r)^2} r\, dr d\theta = \infty.$$

Hence the converse of the above theorem is not true.

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