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ON REFLEXIVITY OF C -SYMMETRIC OR SKEW- C -SYMMETRIC OPERATORS

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ABSTRACT. We present results concerning reflexivity and hyperreflexivity of a subspace of all C -symmetric operators from [6] and a subspace of all skew- C -symmetric operators from [2] with a given conjugation C . We also give a description of their preannihilators.

1. INTRODUCTION

Let \mathcal{H} denote a complex separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and let $B(\mathcal{H})$ be the Banach algebra of all bounded linear operators on \mathcal{H} .

Recall that C is a *conjugation* on \mathcal{H} if $C : \mathcal{H} \rightarrow \mathcal{H}$ is an antilinear, isometric involution, i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator T in $B(\mathcal{H})$ is said to be *C -symmetric* if $CTC = T^*$. Denote by $\mathcal{C} = \{T \in B(\mathcal{H}) : CTC = T^*\}$ the subspace of all C -symmetric operators. Operators which are C -symmetric have been lately studied by many authors (see [3], [4], [5]). In this class there are for example Jordan blocs, truncated Toeplitz operators and Hankel operators. An operator $T \in B(\mathcal{H})$ is called to be *skew- C -symmetric* if $CTC = -T^*$. Denote by $\mathcal{C}^s = \{T \in B(\mathcal{H}) : CTC = -T^*\}$ the subspace of all skew- C -symmetric operators. It follows directly from the definition that \mathcal{C} and \mathcal{C}^s are weak* closed. It is worth to note that any operator $T \in B(\mathcal{H})$ can be written as a sum of a C -symmetric operator and a skew- C -symmetric operator. Indeed, $T = A + B$, where $A = \frac{1}{2}(T + CTC)$ and $B = \frac{1}{2}(T - CTC)$.

Recall that the predual to $B(\mathcal{H})$ is the space of trace class operators denoted by τc with the dual action $\langle T, f \rangle = \text{tr}(Tf)$, where $T \in B(\mathcal{H})$ and $f \in \tau c$. The norm in τc is denoted by $\|\cdot\|_1$ and called the trace norm. Denote by F_k the set of all operators which have rank at most k . Rank one operators are usually written as $x \otimes y$, where $x, y \in \mathcal{H}$,

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and $(x \otimes y)z = \langle z, y \rangle x$ for $z \in \mathcal{H}$. Moreover, $\text{tr}(T(x \otimes y)) = \langle Tx, y \rangle$. For a closed subspace $\mathcal{S} \subset B(\mathcal{H})$ denote by \mathcal{S}_\perp the *preannihilator* of \mathcal{S} defined by $\mathcal{S}_\perp = \{t \in \tau\mathcal{C} : \text{tr}(St) = 0 \text{ for all } S \in \mathcal{S}\}$.

Recall that the *reflexive closure* of a subspace $\mathcal{S} \subset B(\mathcal{H})$ is given by

$$\text{Ref } \mathcal{S} = \{T \in B(\mathcal{H}) : Tx \in [\mathcal{S}x] \text{ for all } x \in \mathcal{H}\},$$

here $[\cdot]$ is the norm-closure. A subspace \mathcal{S} is called *reflexive*, if $\mathcal{S} = \text{Ref } \mathcal{S}$ and \mathcal{S} is called *transitive*, if $\text{Ref } \mathcal{S} = B(\mathcal{H})$. A subspace $\mathcal{S} \subset B(\mathcal{H})$ is called *k-reflexive* if $\mathcal{S}^{(k)} = \{S^{(k)} : S \in \mathcal{S}\}$ is reflexive in $B(\mathcal{H}^{(k)})$, where $S^{(k)} = S \oplus \cdots \oplus S$ and $\mathcal{H}^{(k)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Recall after [10, 8] that a weak* closed subspace \mathcal{S} is *k-reflexive* if and only if operators of rank at most k are linearly dense in \mathcal{S}_\perp , i.e., $\mathcal{S}_\perp = [\mathcal{S}_\perp \cap F_k]$. On the other hand, transitivity means that there are no rank-1 operators in the preannihilator of \mathcal{S} , i.e., $\mathcal{S}_\perp \cap F_1 = \{0\}$.

The definition of *k-hyperreflexivity* was introduced in [1, 7] and is a stronger property than *k-reflexivity*, which means that each *k-hyperreflexive* subspace is *k-reflexive*. A subspace \mathcal{S} is called *k-hyperreflexive* if there is a constant $c > 0$ such that

$$(1) \quad \text{dist}(T, \mathcal{S}) \leq c \cdot \sup\{|\text{tr}(Tt)| : t \in F_k \cap \mathcal{S}_\perp, \|t\|_1 \leq 1\},$$

for all $T \in B(\mathcal{H})$. Note that $\text{dist}(T, \mathcal{S}) = \inf\{\|T - S\| : S \in \mathcal{S}\}$ and the supremum on the right hand side of (1) we denote by $\alpha_k(T, \mathcal{S})$. The smallest constant for which the inequality (1) is satisfied is called the *k-hyperreflexivity constant* and is denoted $\kappa_k(\mathcal{S})$. If $k = 1$, the letter k will be omitted.

In this paper we present results concerning reflexivity and hyperreflexivity of subspaces \mathcal{C} and \mathcal{C}^s proved in [6] and [2]. It is shown that the subspace of all \mathcal{C} -symmetric operators is transitive (hence far from being reflexive) and 2-reflexive or even 2-hyperreflexive. It means that the preannihilator of \mathcal{C} does not contain any rank-one operators and rank-two operators are linearly dense in the preannihilator. Moreover, we describe all rank-two operators in this preannihilator. However, the subspace of all skew- \mathcal{C} -symmetric operators have much better properties: it is reflexive (so very far from being transitive) and hyperreflexive.

2. PREANNIHILATOR

Let \mathcal{H} be a complex separable Hilbert space with an antilinear involution \mathcal{C} . Now we will present results describing the structure of preannihilator of the subspace \mathcal{C} . First theorem says that there are no rank-1 operators in the preannihilator.

Theorem 2.1 (Theorem 2.1 [6]). *Let \mathcal{C} be the set of \mathcal{C} -symmetric operators. The subspace \mathcal{C} is transitive.*

The next theorem gives a full description of rank-2 operators in \mathcal{C}_\perp .

Theorem 2.2 (Theorem 3.1 [6]). *Let \mathcal{C} be the set of all \mathcal{C} -symmetric operators. Then*

$$F_2 \cap \mathcal{C}_\perp = \{h \otimes g - Cg \otimes Ch : h, g \in \mathcal{H}\}.$$

Let now consider some examples of conjugations given in [3] in the context of Theorem 2.2.

Example 2.3. A natural example of a conjugation in $l^2(\mathbb{N})$ is given by

$$C(z_0, z_1, z_2, \dots) = (\bar{z}_0, \bar{z}_1, \bar{z}_2, \dots).$$

In this case

$$\mathcal{C}_\perp \cap F_2 = \{h \otimes g - \bar{g} \otimes \bar{h} : h, g \in l^2(\mathbb{N})\}.$$

Example 2.4. Consider the classical Hardy space H^2 and take a non-constant inner function α . Denote by $K_\alpha^2 = H^2 \ominus \alpha H^2$. For $f \in K_\alpha^2$ and $h \in H^2$ the formula

$$C_\alpha f = \alpha \bar{z} f$$

defines a conjugation $C = C_\alpha$ on K_α^2 . Then

$$\mathcal{C}_\perp \cap F_2 = \{h \otimes g - \alpha \bar{z} \bar{g} \otimes \alpha \bar{z} \bar{h} : h, g \in K_\alpha^2\}.$$

Example 2.5. Let ϱ be a bounded, positive continuous weight on the interval $[-1, 1]$, symmetric with respect to the midpoint of the interval: $\varrho(t) = \varrho(-t)$ for $t \in [0, 1]$. Then the formula

$$Cf(t) = \overline{f(-t)}$$

defines a conjugation on $L^2([-1, 1], \varrho dt)$. In this case

$$\mathcal{C}_\perp \cap F_2 = \{h(\cdot) \otimes g(\cdot) - \overline{g(-(\cdot))} \otimes \overline{h(-(\cdot))} : h, g \in L^2([-1, 1], \varrho dt)\}.$$

Example 2.6. Consider the isometric antilinear operator

$$C(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$$

on \mathbb{C}^2 . Then

$$\mathcal{C}_\perp \cap F_2 = \{(h_1, h_2) \otimes (g_1, g_2) - (\bar{g}_2, \bar{g}_1) \otimes (\bar{h}_2, \bar{h}_1) : (h_1, h_2), (g_1, g_2) \in \mathbb{C}^2\}.$$

Now we will consider the preannihilator of the subspace of all skew- \mathcal{C} -symmetric operators.

Lemma 2.7. *Let C be a conjugation in a complex Hilbert space \mathcal{H} and $h, g \in \mathcal{H}$. Then*

- (1) $C(h \otimes g)C = Ch \otimes Cg$,
- (2) $h \otimes g - Cg \otimes Ch \in \mathcal{C}^s$.

In [3, Lemma 2] it was shown that

$$\mathcal{C} \cap F_1 = \{\alpha \cdot h \otimes Ch : h \in \mathcal{H}, \alpha \in \mathbb{C}\}.$$

The next proposition shows that it is also a description of the rank-one operators in the preannihilator of \mathcal{C}^s .

Proposition 2.8 (Proposition 2.2 [2]). *Let \mathcal{C} be a conjugation in a complex Hilbert space \mathcal{H} . Then*

$$\mathcal{C}_\perp^s \cap F_1 = \mathcal{C} \cap F_1 = \{\alpha \cdot h \otimes Ch : h \in \mathcal{H}, \alpha \in \mathbb{C}\}.$$

Lemma 2.9. *Let \mathcal{C} be a conjugation in a complex Hilbert space \mathcal{H} . Then*

$$\mathcal{C}_\perp^s \cap F_2 \supset \{h \otimes g + Cg \otimes Ch : h, g \in \mathcal{H}\}.$$

The following examples illustrate the result presented in Proposition 2.8.

Example 2.10. Note that for different conjugations we obtain different subspaces. Let $\mathcal{C}_1(x_1, x_2, x_3) = (\bar{x}_3, \bar{x}_2, \bar{x}_1)$ be a conjugation on \mathbb{C}^3 . Then

$$\mathcal{C}_1^s = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

and

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} a & b & * \\ c & * & b \\ * & c & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Rank-one operators in \mathcal{C}_1 and in $(\mathcal{C}_1^s)_\perp$ are of the form $\alpha(x_1, x_2, x_3) \otimes (\bar{x}_3, \bar{x}_2, \bar{x}_1)$ for $\alpha \in \mathbb{C}$.

If we now consider another conjugation $\mathcal{C}_2(x_1, x_2, x_3) = (\bar{x}_2, \bar{x}_1, \bar{x}_3)$ on \mathbb{C}^3 , then

$$\mathcal{C}_2^s = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & -a & c \\ -c & -b & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\},$$

and

$$\mathcal{C}_2 = \left\{ \begin{pmatrix} a & * & b \\ * & a & c \\ c & b & * \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Similarly, rank-one operators in \mathcal{C}_2 and also in $(\mathcal{C}_2^s)_\perp$ are of the form $\alpha(x_1, x_2, x_3) \otimes (\bar{x}_2, \bar{x}_1, \bar{x}_3)$.

Example 2.11. Let C be a conjugation in \mathcal{H} . Consider the conjugation $\tilde{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$ in $\mathcal{H} \oplus \mathcal{H}$ (see [9]). An operator $T \in B(\mathcal{H} \oplus \mathcal{H})$ is skew- \tilde{C} -symmetric, if and only if $T = \begin{pmatrix} A & B \\ D & -CA^*C \end{pmatrix}$, where $A, B, D \in B(\mathcal{H})$ and B, D are skew- C -symmetric. Moreover, rank-one operators in \tilde{C}_\perp^s are of the form $\alpha(f \oplus g) \otimes (Cg \oplus Cf)$ for $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

The following example gives a description of skew- C -symmetric operators in a case of model space K_α^2 equipped with the conjugation C_α defined in Example 2.4.

Example 2.12. Let H^2 be the Hardy space, and let α be a non-constant inner function. As in Example 2.4 consider the conjugation $C_\alpha h = \overline{\alpha z h}$ on the space $K_\alpha^2 = H^2 \ominus \alpha H^2$. By S_α and S_α^* denote the compressions of the unilateral shift S and the backward shift S^* to K_α^2 , respectively. Recall after [11] that the kernel functions in K_α^2 for $\lambda \in \mathbb{C}$ are projections of appropriate kernel functions k_λ onto K_α^2 , namely $k_\lambda^\alpha = k_\lambda - \overline{\alpha(\lambda)}\alpha k_\lambda$. Denote by $\tilde{k}_\lambda^\alpha = C_\alpha k_\lambda^\alpha$. Since S_α and S_α^* are C_α -symmetric (see [3]), for a skew- C_α -symmetric operator $A \in B(K_\alpha^2)$ we have

$$(2) \quad \langle AS_\alpha^n k_\lambda^\alpha, (S_\alpha^*)^m \tilde{k}_\lambda^\alpha \rangle = \langle C_\alpha (S_\alpha^*)^m \tilde{k}_\lambda^\alpha, C_\alpha AS_\alpha^n k_\lambda^\alpha \rangle = \\ - \langle S_\alpha^m C_\alpha \tilde{k}_\lambda^\alpha, A^* C_\alpha S_\alpha^n k_\lambda^\alpha \rangle = - \langle AS_\alpha^m k_\lambda^\alpha, (S_\alpha^*)^n \tilde{k}_\lambda^\alpha \rangle,$$

for all $n, m \in \mathbb{N}$. Note that if $n = m$, then

$$(3) \quad \langle AS_\alpha^n k_\lambda^\alpha, (S_\alpha^*)^n \tilde{k}_\lambda^\alpha \rangle = 0.$$

In particular, we may consider the special case $\alpha = z^k$, $k > 1$. Then the equality (3) implies that a skew- C_α -symmetric operator $A \in B(K_{z^k}^2)$ has the matrix representation in the canonical basis with 0 on the diagonal orthogonal to the main diagonal. Indeed, let $A \in B(K_{z^k}^2)$ have the matrix $(a_{ij})_{i,j=0,\dots,k-1}$ with respect to the canonical basis. Note that $C_{z^k} f = z^{k-1} \bar{f}$, $k_0^{z^k} = 1$, $\tilde{k}_0^{z^k} = z^{k-1}$. Hence for $0 \leq n \leq k-1$ we have

$$0 = \langle AS_\alpha^n 1, (S_\alpha^*)^n z^{k-1} \rangle = \langle Az^n, z^{k-n-1} \rangle = a_{n,k-n-1}.$$

Moreover, from the equality (2) we can obtain that

$$\langle Az^n, z^{k-m-1} \rangle = - \langle Az^m, z^{k-n-1} \rangle,$$

which implies that $a_{n,k-m-1} = -a_{m,k-n-1}$ for $0 \leq m, n \leq k-1$.

3. REFLEXIVITY

In this section we present results concerning reflexivity of the space of all C -symmetric operators and the subspace of all skew- C -symmetric operators.

Theorem 3.1 (Theorem 4.1 [6]). *Let \mathcal{H} be a complex separable Hilbert space with an antilinear involution C . The subspace $\mathcal{C} \subset B(\mathcal{H})$ of all C -symmetric operators is 2-reflexive.*

In the case of the space of all skew- C -symmetric operators we can obtain a stronger result.

Theorem 3.2 (Theorem 3.1 [2]). *Let C be a conjugation in a complex Hilbert space \mathcal{H} . The subspace \mathcal{C}^s of all skew- C -symmetric operators on \mathcal{H} is reflexive.*

Recall that a single operator $T \in B(\mathcal{H})$ is called reflexive if the weakly closed algebra generated by T and the identity is reflexive. In [9] authors characterized normal skew symmetric operators and by [12] we know that every normal operator is reflexive. Hence one may wonder, if all skew- C -symmetric operators are reflexive. The following simple example shows that it is not true.

Example 3.3. Consider the space \mathbb{C}^2 and a conjugation $C(x, y) = (\bar{x}, \bar{y})$. Note that operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is skew- C -symmetric. The weakly closed algebra $\mathcal{A}(T)$ generated by T consists of operators of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Hence $\mathcal{A}(T)_\perp = \left\{ \begin{pmatrix} t & s \\ s & -t \end{pmatrix} : t, s \in \mathbb{C} \right\}$. It is easy to see, that $\mathcal{A}(T)_\perp \cap F_1 = \{0\}$, which implies that T is not reflexive.

4. HYPERREFLEXIVITY

Hyperreflexivity is a stronger property than reflexivity. Here we present results concerning hyperreflexivity of the subspaces \mathcal{C} and \mathcal{C}^s . Since \mathcal{C} is transitive, it cannot be hyperreflexive. However, we can prove the following:

Theorem 4.1 (Theorem 4.2 [6]). *Let \mathcal{H} be a complex separable Hilbert space and let C be a conjugation on \mathcal{H} . The subspace $\mathcal{C} \subset B(\mathcal{H})$ of all C -symmetric operators is 2-hyperreflexive with constant 1.*

The subspace \mathcal{C}^s is reflexive. It can be proved that it also has the stronger property – hyperreflexivity.

Theorem 4.2 (Theorem 4.1 [2]). *Let C be a conjugation in a complex Hilbert space \mathcal{H} . Then the subspace $C^s \subset B(\mathcal{H})$ of all skew- C -symmetric operators is hyperreflexive with the constant $\kappa(C^s) \leq 3$ and 2-hyperreflexive with $\kappa_2(C^s) = 1$.*

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