

# Temporal-Multi-Agent Logics with Multi-valuations

Vladimir V. Rybakov

*Institute of Mathematics and Informatics, Siberian Federal University,*  
Ave. Svobodnui 79, Krasnoyarsk, 660 041, and (part time)  
Institute of Informatics Systems of the Siberian Branch of the RAS, Novosibirsk,  
Russian Federation, e-mail: Vladimir\_Rybakov@mail.ru

## Abstract

The paper studies a new approach to multi-agency using temporal relational models with multi-valuations. A kernel distinction from the standard relational models is introduction of separate valuation for each agent and then computation the global valuation using the all individual agent's valuation. We discuss this approach, illustrate it with examples and demonstrate that this is not a mechanical combination of standard models, but much more thin and sophisticated modeling knowledge and computation truth values in multi-agent environment. Based at these models we define logical language with temporal formulas and introduce logics based at classes of such models. The main mathematical problem we are dealing with is the satisfiability problem. We solve it and find deciding algorithms. In the final part of our paper we discuss interesting open problems for possible further investigations.

**Keywords:** *multi-agent logics, information, knowledge, temporal logic, multi-valuations at relational models, satisfiability problem, solving algorithms*

## 1 Introduction

Logical foundations of Information Sciences, and Computer Sciences have been widely studied for reasoning about correctness, consistency and reliability of information. In particular, multi-agent logics, e.g. with modalities interpreted as agent's operations, or oriented to model checking, were used for study interaction and autonomy, effects of cooperation (cf. e.g. Woldridge and Lomuscio [1], Woldridge [2,3], Lomuscio et al [4]), Babenyshev and Rybakov [5,6,7,8], Rybakov [9]). For example, representation of agent's interaction (as a dual of common knowledge) was suggested in Rybakov [9]. A concept of common knowledge for agent's was formalized and profoundly analyzed in Fagin et al [10] using as a base agent's knowledge (S5-like) modalities. Knowledge, as a concept itself, came from multi-agency, since individual knowledge may be obtained only from interaction of agents, learning.

The conception of knowledge was in a focus of AI and Logic in Computer Science for long ago. As a general field, knowledge-representation is a part of AI which is devoted to designing computer representations for capture information about the world that can be used to solve complex problems. The approach to model knowledge in terms of symbolic logic, probably, may be dated to the end of 1950. At 1962 Hintikka wrote the book: *Knowledge and Belief*, the first book-length work to suggest using modalities to capture the semantics of knowledge. This book laid much of the groundwork for the subject, but a great deal of research has taken place since that time.

Some important feature of multi-agent environment is the observation that receiving of knowledge, interaction of agents, cooperation, – occur during some intervals of time,

and the length of this interval might be very importance. To capture this observation CS often use symbolic (mathematical) temporal logic. Historically, investigations of temporal logic in framework of mathematical/philosophical logic being based at modal systems was originated by Arthur Prior in the late 1950s.

Since then temporal logic has been (and is) very active area in mathematical logic, information sciences, AI and CS overall (cf. e.g. – Gabbay and Hodkinson [11,12,13]). One of important cases of such logics is the linear temporal logic  $\mathcal{LTL}$ , which was used for analyzing protocols of computations, verification of consistency. Automaton technique to solve satisfiability in this logic was developed by Vardi [14,15]). Temporal ontology and temporal discourse was investigated and discussed in van Benthem [16]. Further, to evolve mathematical tools of  $\mathcal{LTL}$ , the solution for admissibility problem for  $\mathcal{LTL}$  was found in Rybakov [17], the basis for admissible rules of  $\mathcal{LTL}$  was obtained in Babenyshev and Rybakov [18]. Recently modeling multi-agency in assumption of non-transitive time was studied in Rybakov [19,20].

This paper is devoted to study a new approach to multi-agency using temporal relational models. These models have many valuations, a separate one for each agent, and the global one which to be computed from individual once by special rules. Main distinction from the standard approach is new rules for computation truth values of formulas (which will use switches of valuations). We will illustrate with examples why this is not merely a mechanical combination of usual rules. Using such models we define logics for classes of models and study their properties. The main mathematical problem we are dealing with is the satisfiability problem. We solve it and find deciding algorithms. In the final part of our paper we discuss interesting open problems and possible further investigations.

## 2 Motivation, Definitions, Notation

Before to define the language of logical systems describing multi-agent's environment, we preliminarily motivate the background for its introduction. First, just to recall, we outline the notion of relational models in an informal way. These models are often used for analysis and representation information (cf. e.g. relational databases). Relational models usually may be viewed as tuples  $\langle W, \{R_i \mid i \in I\}, V \rangle$  which have a base set  $W$  - the set of worlds (or states) of these models, a set of binary relations  $\{R_i \mid i \in I\}$  on these worlds (i.e. any  $R_i$  is a subset of  $W \times W$ ) and  $V$  is a valuation of a set  $Prop$  of propositional variables (letters) in these models. That is, for any  $p \in Prop$ ,  $V(p) \subseteq W$ . Then if  $w \in W$  and  $w \in V(p)$ , we say  $p$  is true at the world  $w$ .

The relations  $R_i$  are usually refereed as particular accessibility relations between the worlds (or alternatively states). Then usually a logical language, typically based on Boolean logic and using special logical operations modeling properties of these relations, is introduced. Formulas of these languages are terms constructed out of letters by means of logical operations, the formulas describe properties of the models. Special rules defining computation of the truth values of the formulas are introduced; and the logic usually is defined as the set of all formulas which are true at any world of such specified models.

Looking at these general framework we first discuss the way to embed multi-agent approach. And a first idea is to consider many valuations -  $V_1, \dots, V_k$  in such models instead of only the one unique fixed one. Then any  $V_i$  represents view-point of the agent  $i$  on truth of the atomic statement - propositional letters, and  $w \in V_i(p)$  would mean that the agent  $i$  think that the statement  $p$  is true at the world  $w$ .

Now we introduce relational models with which we will work. Let  $Prop$  be a set of propositional letters.

**Definition 1.** *A temporal Linear  $k$ -model with agent's multi-valuations is the structure*

$$\mathcal{M} := \langle \mathcal{N}, \leq, \text{Next}, V_1, \dots, V_k, V_0 \rangle,$$

where  $\mathcal{N}$  is the set of all natural numbers,  $\leq$  is the standard order on  $\mathcal{N}$ , Next is the binary relation, where  $a \text{ Next } b$  if  $b = a + 1$ , any  $V_j$  is a valuation of Prop (that is for any  $p \in \text{Prop}$ ,  $V_j(p) \subseteq \mathcal{N}$ ).

We will use the convenient notation  $\text{Next}(n) = m$  to represent  $n \text{ Next } m$  (i.e. to consider Next as the relation and as the function).

These models have a wide range of applications: they may represent (i) computational runs (in particular – threads, as often for usual linear temporal logic), (ii) surfing via networks, Internet, databases collections ( $\mathcal{N}$  then will represent sequence of steps in the search), (iii) sequences of queries for relational databases, (iv) evolutions of social objects in time, etc. Any  $a \in \mathcal{N}$  is called a state (or alternatively, as in Kripke semantics - a world),  $V_j(p)$  represents the set of all states where the atomic statement (proposition)  $p$  is true from viewpoint of the agent  $j$ . For all  $a \in \mathcal{N}$  and any  $p \in \text{Prop}$  we write

$$(\mathcal{M}, a) \models_{V_j} p \Leftrightarrow a \in V_j(p),$$

and say  $p$  is true at  $a$  w.r.t.  $V_j$ . But  $V_0$  here is a special valuation - global one - chosen by these models to fix objective truth relation, this valuation, in a sense, summarizes opinions of all agents. Ways to construct  $V_0$  out of all  $V_j$  may be different. E.g. we may consider

$$\begin{aligned} \text{(I)} \quad & (\mathcal{M}, a) \models_{V_0} p \Leftrightarrow \\ & ||\{j \mid (\mathcal{M}, a) \models_{V_j} p, j \neq 0\}|| > ||\{j \mid (\mathcal{M}, a) \not\models_{V_j} p, j \neq 0\}||, \end{aligned}$$

This means the majority of agents believe that  $p$  is true.

$$\begin{aligned} \text{(II)} \quad & (\mathcal{M}, a) \models_{V_0} p \Leftrightarrow \\ & ||\{j \mid (\mathcal{M}, a) \models_{V_j} p, j \neq 0\}|| \geq ||\{j \mid (\mathcal{M}, a) \not\models_{V_j} p, j \neq 0\}||, \end{aligned}$$

This would mean that  $p$  is plausible.

$$\begin{aligned} \text{(III)} \quad & (\mathcal{M}, a) \models_{V_0} p \Leftrightarrow \\ & (||\{j \mid (\mathcal{M}, a) \models_{V_j} p, j \neq 0\}||) / (||\{j \mid (\mathcal{M}, a) \not\models_{V_j} p, j \neq 0\}||) > 3, \end{aligned}$$

(for  $||\{j \mid (\mathcal{M}, a) \not\models_{V_j} p, j \neq 0\}|| \neq 0$ ). This would mean  $p$  is true from viewpoint of dominating majority of agents..

There are very many ways to express what means global valuation and what indeed means the dominant part of agents. Maybe the agent's opinion may be considered with an appropriate prescribed weights; maybe depending on different states, the rules to compute global valuation may be different, etc. In the very limit point we may assume  $V_0$  to be arbitrary, which does not depend on all  $V_j$  - it is the opinion of a total dominant - the only true what  $V_0$  thinks to be true.

Now we discuss how to express truth values of statements describing properties of models. For this we fix a logical language, which use formulas built up from a (potentially infinite) set  $\text{Prop}$  of atomic propositions (synonymously - propositional letters, variables).

**Definition 2.** The set  $\text{Form}$  of all formulas for our multi-agent logic contains  $\text{Prop}$  and is closed w.r.t. applications of Boolean logical operations  $\wedge, \vee, \neg, \rightarrow$ , the unary operations  $N_i$  (next) ( $i \in [0, k]$ ) and the binary operations  $U_i$ ,  $i \in [0, k]$  (until, each one for the agent  $i$ ).

The formula  $N_i\varphi$  has meaning:  $\varphi$  holds in the next time point (state) for the agent  $i$ ;  $\varphi U_i \psi$  can be read:  $\varphi$  holds until  $\psi$  will be true in the opinion of the agent  $i$ .

Thus, we defined our semantics - models, and defined formulas - logical language. Now we need rules for computation truth values at our models for compound, long formulas. Let a temporal linear  $k$ -model with agent's multi-valuations

$$\mathcal{M} := \langle \mathcal{N}, \leq, \text{Next}, V_1, \dots, V_k, V_0 \rangle,$$

be given. That is, for any letter  $p \in \text{Prop}$   $V_i(p) \subseteq \mathcal{N}$ . If  $a \in \mathcal{N}$  and  $a \in V_i(p)$  we write  $(\mathcal{M}, a) \models_{V_i} p$  and say that  $p$  is true at  $a$  w.r.t. the valuation  $V_i$ . The truth values may be expanded from letters to all formulas as follows.

**Definition 3.**

$$\forall p \in \text{Prop}, (\mathcal{M}, a) \models_{V_j} p \Leftrightarrow a \in \mathcal{N} \wedge a \in V_j(p);$$

$$(\mathcal{M}, a) \models_{V_j} (\varphi \wedge \psi) \Leftrightarrow (\mathcal{M}, a) \models_{V_j} \varphi \wedge (\mathcal{M}, a) \models_{V_j} \psi;$$

$$(\mathcal{M}, a) \models_{V_j} \neg\varphi \Leftrightarrow \text{not}[(\mathcal{M}, a) \models_{V_j} \varphi];$$

$$(\mathcal{M}, a) \models_{V_j} N_i\varphi \Leftrightarrow \forall b[(a \text{ Next } b) \Rightarrow (\mathcal{M}, b) \models_{V_i} \varphi];$$

$$(\mathcal{M}, a) \models_{V_j} (\varphi U_i \psi) \Leftrightarrow \exists b[(a \leq b) \wedge ((\mathcal{M}, b) \models_{V_i} \psi) \wedge$$

$$\forall c[(a \leq c < b) \Rightarrow (\mathcal{M}, c) \models_{V_i} \varphi]].$$

We may define other logical operations using the postulated ones. The modal operations  $\Box_i$  (necessary for agent  $i$ ) and  $\Diamond_i$  (possible for agent  $i$ ) might be defined via temporal operations as follows:  $\Diamond_i p := \top U_i p$ ,  $\Box_i p := \neg \Diamond_i \neg p$ . It might be easily verified that then

$$(\mathcal{M}, a) \models_{V_j} \Diamond_i \varphi \Leftrightarrow \exists b \in \mathcal{N}[(a \leq b) \wedge (\mathcal{M}, b) \models_{V_i} \varphi];$$

$$(\mathcal{M}, a) \models_{V_j} \Box_i \varphi \Leftrightarrow \forall b \in \mathcal{N}[(a \leq b) \Rightarrow (\mathcal{M}, b) \models_{V_i} \varphi];$$

Now we would like to illustrate with examples that the chosen language is flexible to describe correctness of information in multi-agent environment.

*Example 1. Agents (1) and (2) are in opposition for tomorrow:*

$$(\mathcal{M}, a) \models_{V_j} [N_1 p \rightarrow N_2 \neg p] \wedge [N_2 p \rightarrow N_1 \neg p].$$

This formula says that if one agent think tomorrow  $p$  will be true, the another one think the opposite.

*Example 2. Agents (1) and (2) are in opposition for truth of incontestable facts:*

$$(\mathcal{M}, a) \models_{V_j} [\Box_1 p \rightarrow \Box_2 \neg p] \wedge [\Box_2 p \rightarrow \Box_1 \neg p].$$

This formula says that now and always in future agents have opposite opinion, if one think that a fact is always true, then another one think it always must be false.

*Example 3. Agent (1) eventually out-argues agent (2)*

$$(\mathcal{M}, a) \models_{V_1} p \wedge N_1(p \wedge (p U_1 \Box_2 \neg p)).$$

The formula says that  $p$  is true now in opinion of agent (1) and will be true some interval of time in future, but then  $p$  will be false in opinion of (2).

*Example 4. The fact is always possible in the opinion at least one agent but never possible to be true in opinion of all agents*

$$(\mathcal{M}, a) \Vdash_{V_0} \Box_0 \left[ \bigvee_{i \in [1, k]} \Diamond_i p \wedge \Box_0 (p \rightarrow \neg (\bigwedge_{i \in [1, k]} \Diamond_i p)) \right].$$

*Example 5. A fact  $p$  is always possible but suspicious:*

$$(\mathcal{M}, a) \Vdash_{V_0} \Box_0 \left[ \bigvee_{i \in [1, k]} \Diamond_i p \right] \wedge \neg \Box_0 \Diamond_0 p.$$

This says that the fact  $p$  is always (from viewpoint of the global agent 0) possible by the opinion of at least one agent. But  $p$  is not always possible in opinion of the global agent (0).

Now we pause briefly to discuss why the approach we offer is indeed innovative, why we can not look at it as simply a mechanical combination of  $k$  - examples of the standard linear temporal logic. Why it is really new and interesting, why standard technique cannot directly work here.

That all is a consequence of the fact that in our definition of rules for computation truth values of formulas, cf. above, recall:

$$(\mathcal{M}, a) \Vdash_{V_j} N_i \varphi \Leftrightarrow \forall b [(a \text{ Next } b) \Rightarrow (\mathcal{M}, b) \Vdash_{V_i} \varphi];$$

$$(\mathcal{M}, a) \Vdash_{V_j} (\varphi U_i \psi) \Leftrightarrow \exists b [(a \leq b) \wedge ((\mathcal{M}, b) \Vdash_{V_i} \psi) \wedge$$

$$\forall c [(a \leq c < b) \Rightarrow (\mathcal{M}, c) \Vdash_{V_i} \varphi]].$$

So, we switch here the valuations for temporal operations: if the valuation is some  $V_j$  and we compute truth value for a temporal operation with index  $i$  we switch  $j$  to  $i$  and use further the valuation  $V_i$ . That seems correct and well justified: if a temporal statement refers to an agent  $i$ , the opinion about truth for future to be its one. We give below one illustrating examples. Consider the following:

*Examples:* Here (2) and (3) are agent's indexes,  $N_2, N_3$  are logical operations with rules for computation their truth values defined above.

$$(\mathcal{M}, a) \Vdash_{V_1} p \wedge N_2(\neg p \wedge \Box_3 p);$$

$$(\mathcal{M}, a) \Vdash_{V_1} p \wedge N_2(\neg p \wedge \Box_3(\neg p N_3 p \rightarrow (N_2(N_2(p U_2 q))))).$$

As you may see the computation of truth values in these formulas switches the valuations. Therefore the standard technique cannot be directly applied here. That is in particular because the standard rule of exchanging equivalents does not work.

Indeed if for a model  $\mathcal{M}$ ,

$$\forall a, (\mathcal{M}, a) \Vdash_{V_0} \Box_0((p \rightarrow q) \wedge (q \rightarrow p)),$$

it, generally speaking, does not imply

$$\forall a, (\mathcal{M}, a) \Vdash_{V_0} \Box_1((p \rightarrow q) \wedge (q \rightarrow p)).$$

Assume a class  $\mathcal{K}$  of described models is given. We may assume that the rules of definition of the global valuation  $V_0$  via agent's valuations  $V_i$ ,  $1 \leq i \leq k$  are fixed and are the same for all models and all states of these models. Though the agent's valuations themselves may be various (that looks as most general case) but the rules imposed on the agent's valuations are the same for all states. For example, rules for agent's valuations may be with the limitation: for all states  $a$ ,

$$[\|\{i \mid (\mathcal{M}, a) \Vdash_{V_i} p\}\| > k/2 + 1] \Rightarrow [\forall i(1 \leq i \leq k \Rightarrow (\mathcal{M}, a) \Vdash_{V_i} p)]. \quad (1)$$

This means a uniform opinion, – if a majority of agent’s believe that a fact is true then all of them think it is true.

**Definition 4.** A formula  $\varphi$  is said to be satisfiable in  $\mathcal{K}$  if there is a model  $\mathcal{M} \in \mathcal{K}$  and a state  $a \in \mathcal{M}$  such that  $(\mathcal{M}, a) \Vdash_{V_j} \varphi$  for some  $j$ .

Satisfiability problem for  $\mathcal{K}$  is to resolve for any given formula if it is satisfiable in some model from  $\mathcal{K}$ . Assuming that  $\mathcal{K}$  is chosen we may define the logic  $\mathcal{L}(\mathcal{K})$  of this class, e.g. as follows:

$$\mathcal{L}(\mathcal{K}) := \{\varphi \mid \varphi \in \text{Form}, \forall \mathcal{M} \in \mathcal{K}, \forall a \in \mathcal{M}, \forall V_j [(\mathcal{M}, a) \Vdash_{V_j} p]\}.$$

Assuming that all  $V_j$  are equal and  $V_0$  is the same as any  $V_j$  and all of them to be arbitrary, we obtain that  $\mathcal{L}(\mathcal{K})$  is just the standard linear temporal logic  $\mathcal{LTL}$ . Bigger than this, any  $j$ -fragment of any logic  $\mathcal{L}(\mathcal{K})$  for the valuation  $V_j$  will be  $\mathcal{LTL}$ . But, if combinations of different temporal and modal operations for distinct agents are allowed, the possibility to describe properties of multi-agent reasoning are much wider. For example, if (1) holds we have

$$[\bigvee_{X, X \subseteq \{1, \dots, k\}, \|X\| > k/2 + 1} [\bigwedge_{i \in X} N_i p] \Rightarrow \bigwedge_{i \in \{1, \dots, k\}} N_i p] \in \mathcal{L}(\mathcal{K}). \quad (2)$$

Satisfiability problem for the logic  $\mathcal{L}(\mathcal{K})$  generated by some  $\mathcal{K}$ , is the satisfiability problem for the class  $\mathcal{K}$  itself. For brevity in the sequel we write  $\mathcal{L}$  instead  $\mathcal{L}(\mathcal{K})$  assuming  $\mathcal{K}$  to be fixed. By a model  $\mathcal{M}$  (if not specified otherwise) we understand a model from  $\mathcal{K}$ .

### 3 Satisfiability Problem

We will need the special following modification of the  $k$ -models, – models  $\mathcal{M}_{+Circle}$ . Recall that for  $n, m \in \mathcal{N}$  with  $n < m$   $[n, m]$  denotes the closed interval of all numbers situated between  $n$  and  $m$  and these numbers  $n, m$  themselves.

**Definition 5.** Any  $\mathcal{M}_{+Circle}$  model has the following structure. For  $n, c(m), m \in \mathcal{N}$ , where  $0 < n < c(m) \leq m$ ,  $\mathcal{M}_{+Circle} = \langle [n, m], \leq, \text{Next}, V_1, \dots, V_k, V_0 \rangle$  where  $\text{Next}(m) := c(m)$ .

The rules for computation the truth values of formulas in such models w.r.t. any  $V_j$  are defined exactly as earlier in the models, simply for states bigger than  $c(m)$  the order  $\leq$  to be replaced by possible runs via sequences by  $\text{Next}$ . More precisely we define  $(\mathcal{M}_{+Circle}, a) \Vdash_{V_j} (\varphi U_i \psi)$  as follows. If  $a \in [0, c(m)]$  the definition is as earlier, if  $a > c(m)$ ,

$$(\mathcal{M}_{+Circle}, a) \Vdash_{V_j} (\varphi U_i \psi) \Leftrightarrow \exists b[(a \leq b \leq m) \wedge ((\mathcal{M}_{+Circle}, b) \Vdash_{V_i} \psi) \wedge$$

$$\forall c[(a \leq c < b) \Rightarrow (\mathcal{M}_{+Circle}, c) \Vdash_{V_i} \varphi] \bigvee$$

$$\exists d[(d > c(m)) \wedge ((\mathcal{M}_{+Circle}, d) \Vdash_{V_i} \psi) \wedge$$

$$\forall c[(a \leq c \leq m) \Rightarrow (\mathcal{M}_{+Circle}, c) \Vdash_{V_i} \varphi] \wedge$$

$$\forall c[(c(m) \leq c < d) \Rightarrow (\mathcal{M}_{+Circle}, c) \Vdash_{V_i} \varphi].$$

So, these rules act in accordance with the intuition of what is circled bypath by  $\text{Next}$ . For any formula  $\varphi$ ,  $\text{Sub}(\varphi)$  is the set of all its subformulas.

Let  $\text{Tm}(\varphi)$  be the temporal degree of  $\varphi$ . Recall that the temporal degree of formulas is defined inductively: temporal degree of letters is 0, (ii) temporal degree of any formula with

a temporal operation as the main one is the maximal temporal degree of the components plus 1; (iii) temporal degree of any formula with a Boolean logic operation, as the main one, is the maximal temporal degree of the components. Recall that  $k$  is the number of agents in our models. Denote  $f(\varphi) := 2 \times 2^{|Sub(\varphi)|} \times k + 3$ . By the size of a model we mean the number of states in this model.

**Theorem 6.** *If a formula  $\varphi$  is satisfiable in a model  $\mathcal{M}$  at a state by a valuation  $V_j$ , then there exists a finite model of kind  $\mathcal{M}_{+Circle}$  with size at most  $f(\varphi)$  satisfying  $\varphi$  at the world 0 by its own  $V_j$ .*

Proof. Let  $\mathcal{M} := \langle \mathcal{N}, \leq, \text{Next}, V_1, \dots, V_k, V_0 \rangle$  be given and  $(\mathcal{M}, a) \Vdash_{V_j} \varphi$ . We evidently may assume that  $a = 0$ . For any  $b \in \mathcal{M}$ , let

$$\forall j \in [0, k], \text{Sub}_j(b) := \{\alpha \in \text{Sub}(\varphi) \mid (\mathcal{M}, b) \Vdash_{V_j} \alpha\}.$$

Let for all  $b \in \mathcal{M}$ ,

$$\text{Desc}(b) := \{\text{Sub}_j(b) \mid j \in [0, k]\}.$$

Let for all  $b \in \mathcal{M}$ ,

$$\text{Ftr}(b) := \{\text{Desc}(c) \mid c \geq b\}.$$

A simple observation is that there is some  $c_m \in \mathcal{M}, c_m > 3$  such that  $\forall d, g \geq c_m, \text{Ftr}(d) = \text{Ftr}(g)$ . This is the case because the sets  $\text{Ftr}(d)$  may only decrease while increasing  $d$ . Take such minimal  $c_m$ .

For any  $x \geq 1$ , track of realizers from  $x$  is the minimal interval  $[x, y]$  (denoted in the sequel as  $[x, \text{Rls}(x)]$ ) starting from  $x$  such that

$$\begin{aligned} & \forall (\varphi_1 U_j \varphi_2) \in \text{Sub}(\varphi) \\ & [\exists i(\mathcal{M}, x) \Vdash_{V_i} (\varphi_1 U_j \varphi_2) \wedge (\mathcal{M}, x) \Vdash_{V_i} \neg \varphi_2 \Rightarrow \\ & \exists y \in \text{Rls}(x) ((\mathcal{M}, y) \Vdash_{V_i} \varphi_2) \wedge \forall z (x \leq z < y) (\mathcal{M}, z) \Vdash_{V_i} \varphi_1)] \bigwedge \\ & \forall (N_j \varphi_1) \in \text{Sub}(\varphi) [\exists i(\mathcal{M}, x) \Vdash_{V_i} N_j \varphi_1 \Rightarrow (x+1) \in \text{Rls}(x)]. \end{aligned}$$

That minimal interval might be large but nonetheless it exists, we denote it by  $[x, \text{Rls}(x)]$ . Now we consider  $c_m$  and  $[c_m, \text{Rls}(c_m)]$ .

By our definition of  $c_m$  there is  $d_m > \text{Rls}(c_m) + 2$  such that  $\text{Desc}(d_m) = \text{Desc}(c_m)$ . Take such smallest  $d_m$  and define now that  $\text{Next}(d_m) := c_m + 1$  and delete all states from  $\mathcal{M}$  which are strictly bigger than  $d_m$ . Denote the obtained model by  $\mathcal{M}_{+Circle}$ . As we noted before formulation of our theorem, the rules for truth values of formulas in such model w.r.t. any  $V_i$  are defined exactly as earlier in the original  $k$ -models, simply for states bigger than  $c_m$  the order  $\leq$  to be replaced by all possible sequences of states by  $\text{Next}$ .

**Lemma 7.** *For any subformula  $\psi$  from  $\text{Sub}(\varphi)$  and any  $a \in \mathcal{M}_{+Circle}$  where  $a \geq c_m$  and any  $V_j$ ,*

$$(\mathcal{M}, a) \Vdash_{V_i} \psi \Leftrightarrow (\mathcal{M}_{+Circle}, a) \Vdash_{V_i} \psi,$$

Proof. We will prove it by induction on the length of  $\psi$ . For letters, it is evidently true. Inductive steps for Boolean logical operations are evident as well. Let  $\psi = N_j \varphi_1$ . If  $a \geq c_m$  and  $a < d_m$  the conclusion

$$(\mathcal{M}, a) \Vdash_{V_i} N_j \varphi_1 \Leftrightarrow (\mathcal{M}_{+Circle}, a) \Vdash_{V_i} N_j \varphi_1,$$

follows immediately from the inductive assumption.

If  $a = d_m$  then  $Next(d_m) := c_m + 1$  and by indicative hypothesis

$$(\mathcal{M}, c_m + 1) \Vdash_{V_i} \varphi_1 \Leftrightarrow (\mathcal{M}_{+Circle}, c_m + 1) \Vdash_{V_i} \varphi_1.$$

Therefore by  $Desc(d_m) = Desc(c_m)$  we obtain

$$(\mathcal{M}, d_m) \Vdash_{V_i} N_j \varphi_1 \Leftrightarrow (\mathcal{M}_{+Circle}, d_m) \Vdash_{V_i} N_j \varphi_1.$$

Thus, the inductive proof for the operations  $N_i$  is completed. Consider now that case when  $\psi = \varphi_1 U_j \varphi_2 \in Sub(\varphi)$ .

Assume first that

$$(\mathcal{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2.$$

Then there exists smallest  $b \in \mathcal{M}$  such that  $b \geq a$  and  $(\mathcal{M}, b) \Vdash_{V_i} \varphi_2$ , and else either

(i): for all  $c$ , where  $a \leq c < b$ ,  $(\mathcal{M}, b) \Vdash_{V_j} \varphi_1$ , or otherwise

(ii):  $b = a$ .

Assume first  $b \leq d_m$ ; then by inductive assumption we obtain  $(\mathcal{M}_{+Circle}, b) \Vdash_{V_i} \varphi_2$ .

If (ii) holds (that is  $b = a$ ) then  $(\mathcal{M}_{+Circle}, a) \Vdash_{V_i} \varphi_2$  and hence we immediately obtain  $(\mathcal{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ . If (ii) is not a case but (i) is, then we have that, for all  $c$ , where  $a \leq c < b$ ,  $(\mathcal{M}_{+Circle}, b) \Vdash_{V_i} \varphi_1$  holds by inductive hypothesis, and consequently,

$$\mathcal{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_i \varphi_2.$$

Assume now that  $b > d_m$ . Then the following holds  $(\mathcal{M}, d_m) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ . Applying  $Desc(d_m) = Desc(c_m)$  we obtain  $(\mathcal{M}, c_m) \Vdash_{V_i} \varphi_1 U_i \varphi_2$ . Using  $d_m > Rls(c_m)$ ,  $a \leq d_m$  and the inductive assumption, from the fact that  $(\mathcal{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$  we have that there is the path in  $\mathcal{M}_{+Circle}$  by  $Next$  leading from  $a$  into a state in  $[c_m, Rls(c_m)]$  where  $\varphi_2$  is true w.r.t.  $V_i$  in  $\mathcal{M}_{+Circle}$  and that along this path always  $\varphi_1$  is true w.r.t.  $V_i$  in  $\mathcal{M}_{+Circle}$ . That is we obtain  $(\mathcal{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ , which is what we need.

In opposite direction, assume now that

$$(\mathcal{M}_{+Circle}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2.$$

By definition of the model  $\mathcal{M}_{+Circle}$  we have that  $a \leq d_m$  and there is a path by  $Next$  in the  $\mathcal{M}_{+Circle}$  from  $a$  into some closest  $b \leq d_m$  satisfying  $\varphi_2$  w.r.t.  $V_i$  where, along this path, always  $\varphi_1$  is true w.r.t.  $V_i$  in the model  $\mathcal{M}_{+Circle}$ . If

the path does not go via  $c_m + 1$  (3)

then for all  $c$  where  $a \leq c < b \leq d_m$ ,  $(\mathcal{M}_{+Circle}, c) \Vdash_{V_i} \varphi_1$ . Then using the inductive assumption we conclude that for all such  $c$ ,  $(\mathcal{M}, c) \Vdash_{V_j} \varphi_1$  holds, and consequently we obtain  $(\mathcal{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ .

Assume now that

the path goes via  $c_m + 1$ . (4)

Then  $(\mathcal{M}_{+Circle}, c_m + 1) \Vdash_{V_i} \varphi_1 U_j \varphi_2$  and using (3) we obtain  $(\mathcal{M}, c_m + 1) \Vdash_{V_i} \varphi_1 U_j \varphi_2$  and applying the inductive assumption we conclude  $(\mathcal{M}, a) \Vdash_{V_i} \varphi_1 U_j \varphi_2$ .  $\square$

**Lemma 8.** For any subformula  $\psi$  from  $Sub(\varphi)$  and any  $a \in \mathcal{M}_{+Circle}$  and any  $V_j$ ,

$$(\mathcal{M}, a) \Vdash_{V_j} \psi \Leftrightarrow (\mathcal{M}_{+Circle}, a) \Vdash_{V_j} \psi,$$

*Proof.* The proof immediately follows from our previous lemma and the fact that the initial part of the model  $\mathcal{M}$  before  $c_m$  while its transformation into the model  $\mathcal{M}_{+Circle}$

let intact. So the verification is a routine standard computation by induction on length the formulas.  $\square$

Thus now by Lemma 8 we have that the model  $\mathcal{M}_{+Circle}$  also satisfies the formula  $\varphi$  and this model is finite. We only need now to reduce the size of this model to a bound computable from the size of  $\varphi$ .

**Lemma 9.** *There is a model  $\mathcal{M}_{+Circle}$  satisfying  $\varphi$  and having the size at most  $f(\varphi)$ .*

*Proof.* We will use previous notation and proved above facts. Thus,  $\mathcal{M}_{+Circle}$  satisfies  $\varphi$ . Take the smallest state 0 from  $\mathcal{M}_{+Circle}$ . First, recall that  $2 \leq c_m - 1$ . Choose the biggest  $b \in [1, c_m - 1]$  such that

$$Desc(1) = Desc(b),$$

if one exists. In particular, it may happen that  $b = 1$ , then we do nothing at this stage. Otherwise we delete all states from  $[1, b]$  in  $\mathcal{M}_{+Circle}$  and denote the resulting model by  $\mathcal{M}_{+Circle}(1, b)$ . We will show that for all  $s \in \mathcal{M}_{+Circle}(1, b)$ , any  $\psi \in Sub(\varphi)$  and any  $i$ ,

$$(\mathcal{M}_{+Circle}, s) \Vdash_{V_i} \psi \Leftrightarrow (\mathcal{M}_{+Circle}(1, b), s) \Vdash_{V_i} \psi. \quad (5)$$

For  $s \geq b$  this statement is evident. It remains only to consider the case when  $s = 0$ . For  $\psi$  to be a letter it is evident, and inductive steps of the proof by the length of  $\psi$  for Boolean operations are again evident. Now assume that for a  $\psi$  (5) is proven and  $N_j\psi \in Sub(\varphi)$ . We claim that

$$(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} N_j\psi \Leftrightarrow (\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} N_j\psi. \quad (6)$$

This follows from (5) and our choice of  $b$  above with  $Desc(1) = Desc(b)$ .

Let for  $\psi_1$  and  $\psi_2$  the statement (5) is proven and  $\psi_1 U_j \psi_2 \in Sub(\varphi)$ .

Then if  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1 U_j \psi_2$  we have either  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_2$  and by inductive assumption we receive  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_2$  and we obtain  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_1 U_j \psi_2$ .

Or otherwise  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \neg\psi_2$  and then  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1$ , and besides we have that  $(\mathcal{M}_{+Circle}, 1) \Vdash_{V_i} \psi_1 U_j \psi_2$ . Then by our choice of  $b$  above with  $Desc(1) = Desc(b)$  we obtain  $(\mathcal{M}_{+Circle}, b) \Vdash_{V_i} \psi_1 U_j \psi_2$ . Therefore  $(\mathcal{M}_{+Circle}(1, b), b) \Vdash_{V_i} \psi_1 U_j \psi_2$ . By inductive assumption it follows that  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1$  implies  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_1$ . That overall gives to us that

$$(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_1 U_j \psi_2.$$

In opposite direction, assume that  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_1 U_j \psi_2$ . Then, if we assume that  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_2$ , this by inductive assumption yields the statement  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_2$  and consequently  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1 U_j \psi_2$ .

Assume now that  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \neg\psi_2$ . Then  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_1$  and we have  $(\mathcal{M}_{+Circle}(1, b), b) \Vdash_{V_i} \psi_1 U_j \psi_2$ . Therefore  $(\mathcal{M}_{+Circle}, b) \Vdash_{V_j} \psi_1 U_j \psi_2$  since  $b \geq b$  and (5).  $(\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_i} \psi_1$  yields  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1$ . So,  $(\mathcal{M}_{+Circle}, 0) \Vdash_{V_i} \psi_1 U_j \psi_2$ . Thus we proved that

$$(\mathcal{M}_{+Circle}, 0) \Vdash_{V_j} \psi_1 U_j \psi_2 \Leftrightarrow (\mathcal{M}_{+Circle}(1, b), 0) \Vdash_{V_j} \psi_1 U_j \psi_2. \quad (7)$$

This statement concludes the proof of (5).

Now we continue the proof of our Lemma 9. Considering  $b$  as we initially did above with 0 in  $\mathcal{M}_{+Circle}$ , and subsequently making similar reformations moving to  $c_m$ , we do as much steps as much various  $Desc(s)$  may happen, so finite, effectively bounded amount of steps. So, we then receive a model similar to  $\mathcal{M}_{+Circle}$ , but which has at most  $2^{|Sub(\varphi)|} \times k + 3$  states before  $c_m$ . Since this stage, we make similar rarefication in the loop path in  $\mathcal{M}_{+Circle}$  from  $c_m + 1$  to itself  $c_{m+1}$ . This concludes the proof of our Theorem 6.

**Theorem 10.** *If a formula  $\varphi$  is satisfiable in a finite model  $\mathcal{M}_{+Circle}$  then it is satisfiable in some  $k$ -model  $\mathcal{M}$ .*

Proof. Let  $\mathcal{M}_{+Circle} = \langle [n, c(m)] \cup [c(m), m], \leq, \text{Next}, V_1, \dots, V_k, V_0 \rangle$  where  $\text{Next}(m) := c(m)$ , and for some  $i$ ,  $(\mathcal{M}_{+Circle}) \models_{V_i} \varphi$ . Consider the infinite  $k$ -model  $\mathcal{M}$  with the following structure: the base set  $\mathcal{N}$  of this model is the sequence of all states  $[n, c(m)] \cup [c(m) + 1, m]$  and the infinite amount of states combined from the intervals of states situated in the interval  $[c(m) + 1, m]$  repeated one by one, where  $\text{Next}(m) = c(m) + 1$ . The valuations  $V_j$  on this model to be just transferred from the model  $\mathcal{M}_{+Circle}$ . This is immediate to show (simple computation by induction of length the formulas) that, for any (absolutely any) formula  $\psi$  constructed out of letters from  $\varphi$ ,

$$\forall a \in \mathcal{M}_{+Circle}, \forall V_i [( \mathcal{M}_{+Circle}, a ) \models_{V_i} \psi \Leftrightarrow ( \mathcal{M}, a ) \models_{V_i} \psi.]$$

So, this model  $\mathcal{M}$  will satisfy the formula  $\varphi$ .  $\square$

Recall that a logic  $\mathcal{L}(\mathcal{K})$  is decidable if for any formula  $\varphi$  we may compute if  $\varphi \in \mathcal{L}(\mathcal{K})$ . Observe that  $\varphi \in \mathcal{L}(\mathcal{K})$  iff  $\neg\varphi$  is not satisfiable in  $\mathcal{L}(\mathcal{K})$ . From Theorems 6 and 10 we immediately obtain

**Theorem 11.** *The satisfiability problem for  $\mathcal{L}(\mathcal{K})$  is decidable (so the logic  $\mathcal{L}(\mathcal{K})$  is decidable). For a formula  $\varphi$  to be satisfiable it is sufficient to check its satisfiability at models  $\mathcal{M}_{+Circle}$  of size at most  $f(\varphi)$ .*

## 4 Linear Interval Multi-Agent Temporal Logic

In this section we will consider the case when the models are not linear and even non-transitive, but are compound from some fragments of our temporal models from the previous section. We think that the case is indeed interesting and useful for applications. The matter is that the assumption that all computational runs are linear and potentially infinite is too strong. In fact, always the all resources are limited, they may be sufficiently big, but with some assumed upper bound. We aim to represent such limitation as follows.

Let we chop the set of all natural numbers  $N$  into an infinite sequence of closed intervals:  $[s_i, s_i + k_i], i \in N$ ; that is we assume that  $N = \bigcup_{i \in N} [s_i, s_i + k_i]$ ,  $s_i < s_{i+1}$ , and that  $s_i + k_i \geq s_{i+1}$ ,  $s_i + k_i < s_{i+1} + k_{i+1}$ .

So, we admit that the intervals may have possible nonempty not one state overlap, that is, it could be that  $[s_i, s_i + k_i] \cap (s_{i+1}, s_{i+1} + k_{i+1}] \neq \emptyset$ , so these intervals may have a non-empty and not one state common part.

Temporal Interval Linear  $k$ -model with agent's multi-valuations is the structure

$$\mathcal{M} := \langle \bigcup_{i \in N} [s_i, s_i + k_i], \leq, \text{Next}, V_0, V_1, \dots, V_k \rangle,$$

where  $\leq$  is the standard linear order on  $N$ ,  $\text{Next}$  is the standard next binary relation, and any  $V_i$  is a valuation. But for  $V_0$  we as earlier assume that  $V_0$  is a global valuation computed via valuations of the agents by some common rules (as we assumed before).

That models are intended to describe real computation in bounded time. In the case when  $s_{i+1} = s_i + 1$  it is just somewhat like immediate passing of information. In case when  $[s_i, s_i + k_i] \cap (s_{i+1}, s_{i+1} + k_{i+1}] \neq \emptyset$ , this models the situation when it might be that the next computational run started before the previous one was completed and they work sharing the resources and the information.

For such models we may define truth values of formulas exactly the same way as in the previous section - for pure linear time, with only a distinction on definition truth

values of the formulas containing operations  $U_m$  - until ones. The definition is as follows:  
 $\forall a \in [s_{i+1}, s_{i+1} + k_{i+1}] \setminus [s_i, s_i + k_i]$ ,

$$(\mathcal{M}, a) \Vdash_{V_j} (\varphi \ U_m \psi) \Leftrightarrow \exists b \in \mathcal{M}[(a \leq b \leq s_{i+1} + k_{i+1}) \wedge ((\mathcal{M}, b) \Vdash_{V_m} \psi) \wedge \forall c[(a \leq c < b) \Rightarrow ((\mathcal{M}, c) \Vdash_{V_m} \varphi)]]].$$

Thus, any  $U_m$  works as usual but is bounded by the upper boundary of the local run - by  $s_i + k_i$ . This corresponds very well with usual intuition concerning computational procedures and the computational runs - the solution (a state satisfying the formula) should (if exists) be reached before the end of computation for the current local computational possess. We think that structures of such models and their properties are clear. But nonetheless we give below some illustrative examples.

*Examples.*

$$(I) \quad \mathcal{M} := \langle \bigcup_{i \in N} [s_i, s_i + k_i], \leq, \text{Next}, V_0, V_1, \dots, V_k \rangle,$$

where  $s_i := i^2, k_i := (i+1)^2 - i^2$  so bounds are squares of numbers. Here the intersection of the time intervals are only bounds - numbers  $i^2$ .

$$(II) \quad \mathcal{M} := \langle \bigcup_{i \in N} [s_i, s_i + k_i], \leq, \text{Next}, V_0, V_1, \dots, V_k \rangle,$$

where  $s_i := (10 \times i), k_i := (10 \times (i+1)) - (10 \times i) + 5$ . Now the intersection of the time intervals are not only bounds, e.g.  $[s_0, s_0 + k_0] \cup [s_1, s_1 + k_1] = [11, 15]$ .

As in Section 2 earlier, we denote an arbitrary class of any such models by  $\mathcal{K}$  and denote the logic generated by this class as  $\mathcal{L}(\mathcal{K})$ ; the satisfiability of formulas and decidability of the logic defined as earlier.

For any model  $\mathcal{M}$  described above, the model  $\mathcal{M}^{(-)}$  is the one obtained from  $\mathcal{M}$  by deleting all states of all intervals  $[s_i, s_{i+1}]$  situated strictly bigger than certain fixed number  $n$  in  $(s_m, s_m + k_m)$ , and by defining  $\text{Next}(n) = s_m$ .

**Lemma 12.** *If a formula  $\varphi$  is satisfiable in a model  $\mathcal{M}$  at 0 by a valuation  $V_j$ , then there exists a finite model of kind  $\mathcal{M}^{(-)}$  satisfying  $\varphi$  at the world 0 by its own  $V_j$  where the size of  $\mathcal{M}^{(-)}$  is at most  $2^{\|\text{Sub}(\varphi)\|} \times k + 3$  and the number of the states  $s_i$  in this model is at most the temporal degree of the formula  $\varphi$  plus 2.*

*Proof.* Let a model

$$\mathcal{M} := \langle \bigcup_{i \in N} [s_i, s_i + k_i], \leq, \text{Next}, V_0, V_1, \dots, V_k \rangle,$$

satisfies a formula  $\varphi$ :  $(\mathcal{M}, 0) \Vdash_{V_j} \varphi$ . Let the temporal degree of  $\varphi$  is  $m$ . Using the standard argument on temporal degree of the formulas, we may assume that the model is shortened now by deleting all states strictly bigger than  $s_{m+2} + k_{m+2}$  and defining  $\text{Next}(s_{m+2} + k_{m+2}) = s_{m+2} + k_{m+2}$ , where  $m$  is the temporal degree of  $\varphi$ . And this model will satisfy  $\varphi$  at 0 as well.

So, now we assume that our model  $\mathcal{M}$  has this structure. The number of intervals  $[s_i, s_i + k_i]$  in this model is at most  $m+2$ . Now we will rarefy this model starting from the bottom interval  $[s_0, s_0 + k_0]$ . For  $[s_0, s_0 + k_0]$  we carry out the proof exactly as in Lemma 9 starting from considering the interval  $[s_0, s_0 + k_0]$  as  $[1, c_m - 1]$  in Lemma 9 making rarefication as it is shown there. This transformation will not change the truth values of subformulas of  $\varphi$ , and, in particular,  $s_0$  and  $s_0 + k_0$  will remain intact. Since this point we continue this rarefication procedure for resulting  $[s_1, s_1 + k_1]$  and so forth. In at most  $m+2$  steps this procedure will be completed. And the resulting model will satisfy  $\varphi$  at 0.  $\square$

**Lemma 13.** *If a formula  $\varphi$  is satisfiable in a finite model  $\mathcal{M}^{(-)}$  described in Lemma 12 at the state 0 by a valuation  $V_j$ , then  $\varphi$  may be satisfied in an infinite model  $\mathcal{M}$  of our class.*

Proof. Proof is a standard argument using the temporal degree of formulas.  $\square$

From Lemmas 12 and 13 we immediately infer

**Theorem 14.** *The satisfiability problem for the logic  $\mathcal{L}(\mathcal{K})$  is decidable. For verification that a formula  $\varphi$  is satisfiable it is sufficient to check its satisfiability at models  $\mathcal{M}^{(-)}$  of size at most  $2^{\|Sub(\varphi)\|} \times k + 3$ .*

## 5 Conclusion

We think that research from this paper may be essentially extended, many interesting problems remain open. The case when the global valuation would be computed via valuation of agent's at states not uniformly, – but by rules specific for any state, is not considered yet. Another venue not explored yet is the computation of truth at states when we consider many valued values (e.g. from some intervals of possible truth values, as e.g. in Łukasiewicz logics, or as in Fuzzy Logic). The open yet question is to consider models with lacunae in computational runs. That is the one, – when the agents may see not all future, – but have some lacunae of not visible intervals, which they do not see, and when rules for computation truth values for temporal and modal operations are accordingly enrolled. The extension of our results to branching time logic is very interesting task. Investigation of admissibility for rules and validity of rules in such logics is very interesting. The admissibility of rules was an area of most attraction for the author for a long time (cf. [17,21,22]) and also many strong results about admissibility were obtained by other researchers (cf. e.g. [23,24,25,26]). That area is very closer to unification problematic (cf. [27,28,23,29]) and it is very interesting to extend the unification theory to logics within framework of this our paper.

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