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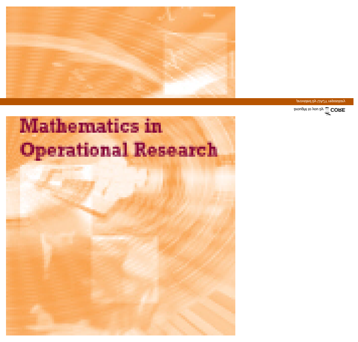

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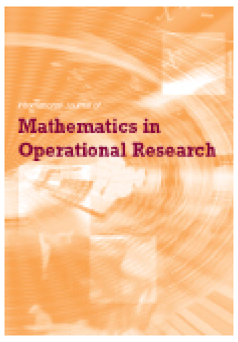

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A forward with backward inventory policy algorithm for nonlinear increasing demand and shortage backorders

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Abstract: The traditional inventory policies have been developed for constant demand processes. In reality, demand is not always stable; it might have an increasing pattern. In this paper, a forward with backward inventory policy algorithm is developed to determine the operational parameters of an inventory system with a nonlinear increasing demand rate, shortage backorders and a finite planning horizon. Numerical experiments are also conducted to compare the results with the existing techniques and to illustrate the applicability of the proposed technique.

Keywords: inventory; nonlinear increasing demand pattern; shortage backorders; forward with backward inventory policy algorithm.

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1 Introduction

Zhou et al. (2004) mentioned that inventory models considering shortage backorders can be classified into two categories. They are:

1. Inventory followed by shortage (IFS) where each cycle starts with replenishment and end with shortage.
2. Shortage followed by inventory (SFI) where each cycle starts with shortage before replenishment arrives.

Before, Teng et al. (1997) investigated two categories of inventory model above by taking into consideration whether shortages are allowed or not allowed in the last cycle of the planning horizon.

Complexity of the development of inventory model arises when demand is not stable. To the best of author's knowledge, initial research on IFS policy for shortage backorders case with positive linear trend in demand was done by Deb and Chaudhuri (1987) who developed a heuristic method by assuming that the shortage period in each cycle is a constant fraction of the length of the cycle. Later, Dave (1989a) corrected and tested the work of Deb and Chauduri (1987) by using Donaldson's (1977) example and the result from the corrected method was better than that of Silver (1979) and even close to the optimal solution provided by Dave (1989b). Other optimal solution was provided by Murdeshwar (1988) and Hariga (1994). It is noted that while Deb and Chauduri (1987), Dave (1989b) and Murdeshwar (1988) developed IFS policies for the case where shortages are not allowed in the last cycle, Hariga (1994) developed IFS policy for the case where shortages are allowed in the last cycle.

Other research on inventory policy for linear demand considering shortage backorder was conducted by Goyal et al. (1996) who concluded through empirical experiments that SFI policy often perform better than IFS policy. Further work was done by Teng et al. (1997) who compared among four inventory models, i.e.

1 IFS policy without shortage allowance in the last cycle

2 IFS policy with shortage allowance in the last cycle

3 SFI policy without shortage allowance in the last cycle

4 SFI policy with shortage allowance in the last cycle.

From their results Teng et al. (1997) concluded that model 4 is the best among the four investigated models, which provides the lowest total cost. Other research was done by Goyal and Giri (2000) who stated that the comparisons conducted by Teng et al. (1997) were invalid. They improved the method to make valid comparisons among the four models and came up with the conclusion that when inventory starts with zero demand rates, model 3 will provide the lowest cost, while if inventory starts with positive demand rate, model 4 is the best. It should be noted that, when model 4 is employed, there exist shortages in the last replenishment cycle, and this means that total demands of the whole planning horizon will not be met.

In reality, demand is not always linear increasing; it may have a nonlinear increasing pattern. Development of IFS policy for nonlinear increasing demand with shortage backorders case was done by Hariga (1994), who developed an exact solution procedure and also Astanti and Luong (2009) who developed the heuristic technique based on consecutive method.

For SFI policy with nonlinear increasing demand pattern and inventory starts with positive demand rate, Yang et al. (2002) proposed a forward recursive algorithm for the case when shortages are allowed in the last replenishment cycle. For this case, if shortages are assumed to be completely backlogged, an additional replenishment at the end of planning horizon should be taken into consideration to ensure that total demands is fulfilled. The additional replenishment will affect the total cost function and the optimal solution should be revised. However, this fact has not been discussed in the research of Yang et al. (2002).

A recent research for nonlinear increasing demand considering shortage backorders where inventory starts with zero demand rate, was also conducted by Yang (2006) who developed a backward recursive algorithm, compared among the four models investigated by Teng et al. (1997), and came up with the conclusion that model 4 always provides the lowest total cost. It is noted that this conclusion is contrary to the conclusion of Goyal and Giri (2000) for linear increasing demand case, which stated that, when inventory starts with zero demand rate, model 3 will provide a better cost performance.

Other researches on inventory policy problem for nonlinear increasing demand rate were conducted by considering more characteristics of inventory policy model in to the model such as deterioration rate, partial backlog and time value of money. However, the solution methodology are developed under the assumption of the very specific nonlinear increasing demand pattern such as quadratic demand pattern (Panda et al., 2009a, 2009b; Sarkar et al. 2010; Sanni and Chukwu, 2016; Vandana and Sharma, 2016), polynomial demand pattern (Bai and Kendall, 2008; Lukas and Hofman, 2016), exponential demand pattern (Wu, 2002), and ramp-type demand pattern (Kawakatsu, 2011; Manna and Chiang 2010; Roy and Chaudhuri 2011; Valliathal and Uthayakumar, 2016). Astanti and Luong (2014) developed a repetitive forward rolling technique for determining inventory policy for nonlinear increasing demand pattern and considering shortage.

From above literature reviews, it can be seen that past researches conducted for finding exact solution for the case of nonlinear increasing demand considering shortage backorders have not ensured yet that total demand over a pre-established planning horizon will always be fulfilled (Hariga, 1994). In addition, the exact solution methodologies were developed for very specific nonlinear demand pattern. The research presented in this paper therefore focuses on the development of an inventory policy for more general nonlinear increasing demand pattern (i.e., any log-concave function) and shortage backorders case in such a way that the total demand over a pre-established planning horizon can always be met. By working on more general demand pattern, it is expected that the result of this research can be applied to solve problems with various nonlinear demand pattern that may appear in practical situation including some patterns that have been individually discussed in past research, i.e., quadratic, polynomial, exponential, and ramp-type demand patterns.

In this research, a forward with backward inventory algorithm is proposed where the proposed algorithm consist of two steps. The first step is a procedure to help determine the replenishment times and intermediate shortage starting point simultaneously for two consecutive cycles in the planning horizon is proposed. Then, the proposed technique from the first step will be incorporated into a forward with backward rolling procedure for every two consecutive cycles in the planning horizon to help determine all replenishment times and shortage starting points in such a way that the total inventory cost will be gradually reduced until no improvement can be realized.

The remaining parts of this paper are organized as follows. Section 2 presents the mathematical model in which the expression of total inventory cost will be derived. In Section 3, the proposed technique to find the two replenishment times and the intermediate shortage starting point for any two consecutive cycles in the planning horizon will be derived, followed by the development of the forward with backward rolling procedure in Section 4. Numerical experiments to illustrate the applicability of the proposed forward technique are then presented in Section 5. Sensitivity analysis on the effect of the predefined number of cycles will be conducted in Section 6. And then, some concluding remarks will be discussed in Section 7.

2 Mathematical model

The following notation will be used throughout the paper:

- H is the length of planning horizon under consideration
- $f(t)$ is the demand rate at time t, which is assumed to be an increasing log-concave function
- c_1 is the ordering cost per order
- c_2 is the holding cost per unit per unit time
- c_3 is the shortage cost per unit per unit time
- n is the number of replenishment cycles in the planning horizon
- t_i is the i^{th} replenishment time ($i = 1, 2, \ldots, n$)
- s_i is the i^{th} shortage starting point ($i = 1, 2, \ldots, n, n + 1$), which is also the starting point of the i^{th} cycle $[s_i, s_{i+1}]$, except that $s_{n+1} = H$
- $I(t)$ is the inventory level at time t, which should be evaluated after the replenishment arrives at time $t = t_i$ in the i^{th} cycle $[s_i, s_{i+1}]$.

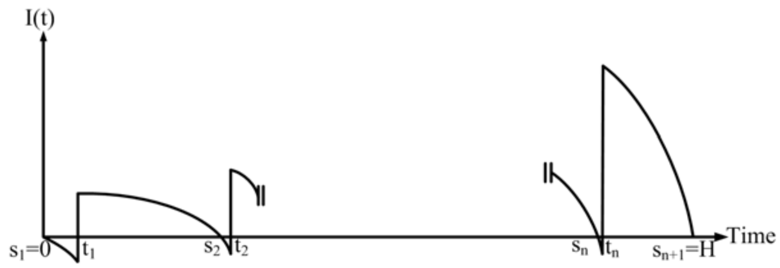
The behavior of the inventory level function is illustrated in Figure 1. For the development of the mathematical model, the following assumptions are also used:

a Replenishment orders are made only at time t_i ($i = 1, 2, \ldots, n$).

b Lead time is negligible, i.e., replenishment is instantaneous

c Shortages are permitted at the beginning of each cycle but no shortages are permitted at the end of planning horizon (i.e., $s_{n+1} = H$)

Figure 1 Inventory level over the whole planning horizon



From the above assumptions, the expression of total inventory cost function, which includes ordering cost, holding cost, and shortage cost; of the inventory system during a planning horizon H when n orders are placed is expressed as follows:

$$TC(n, \{s_i\}, \{t_i\}) = nc_1 + c_2 \sum_{i=1}^{n} I_i + c_3 \sum_{i=1}^{n} S_i \tag{1}$$

in which

- I_i is the cumulative holding inventory during cycle i
- S_i is the cumulative shortage during cycle i.

The expressions of cumulative holding inventory I_i and cumulative shortage S_i for each cycle i from s_i to s_{i+1} will be derived in Sections 2.1 and 2.2 below:

2.1 Cumulative holding inventory I_i

If $F(t)$ denotes the cumulative demand from time 0 to time t then

$$F(t) = \int_0^t f(t)dt$$

The inventory level at time t $\in [t_i, s_{i+1}]$ in cycle i can be expressed as:

$$I(t) = \int_t^{s_{i+1}} f(\tau)d\tau \quad t_i \le t \le s_{i+1}$$

Hence, the cumulative holding inventory I_i in cycle i can be determined as:

$$I_i = \int_{t_i}^{s_{i+1}} I(t)dt = \int_{t_i}^{s_{i+1}} \int_t^{s_{i+1}} f(\tau)d\tau dt$$

$$I_i = \left(s_{i+1} - t_i\right) F\left(s_{i+1}\right) - \int_{t_i}^{s_{i+1}} F(t)dt \tag{2}$$

2.2 Cumulative shortage S_i

The shortage level at time $t \in [s_i, t_i]$ in cycle i can be expressed as:

$$S(t) = \int_{s_i}^{t} f(\tau)d\tau \quad s_i \le t \le t_i$$

Hence, the cumulative shortage S_i in cycle i (i = 1, 2, …, n) can be determined as:

$$S_i = \int_{s_i}^{t_i} S(t)dt = \int_{s_i}^{t_i} \int_{s_i}^{t} f(\tau)d\tau dt$$

$$S_i = \left(s_i - t_i\right) F\left(s_i\right) + \int_{s_i}^{t_i} F(t)dt \tag{3}$$

From equations (2) and (3), the expression of the total inventory cost can be defined as follows:

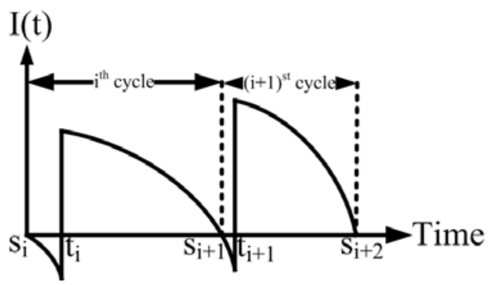
$$TC\left(n,\{s_i\},\{t_i\}\right)=nc_1+c_2\sum_{i=1}^{n}\left\{\left(s_{i+1}-t_i\right)F\left(s_{i+1}\right)-\int_{t_i}^{s_{i+1}}F(t)dt\right\}$$
$$+c_3\sum_{i=1}^{n}\left\{\left(s_i-t_i\right)F\left(s_i\right)+\int_{s_i}^{t_i}F(t)dt)\right\} \tag{4}$$

3 Proposed technique to determine replenishment times and shortage starting point for two consecutive cycles

Consider two consecutive cycles i and $(i + 1)$ of the planning horizon in which s_i and s_{i+2} are fixed. The technique proposed in this section is developed to help determine the two replenishment times t_i, t_{i+1} and the intermediate shortage starting point s_{i+1} so as to minimize total inventory cost of the two cycles (see Figure 2 for the illustration). The total inventory cost of the two consecutive cycles started with cycle i, denoted by $TC2_i$, is determined as follows.

$$TC2_i=2c_1+c_2\left\{\left(s_{i+1}-t_i\right)F\left(s_{i+1}\right)-\int_{t_i}^{s_{i+1}}F(t)dt\right\}$$
$$+c_2\left\{\left(s_{i+2}-t_{i+1}\right)F\left(s_{i+2}\right)-\int_{t_{i+1}}^{s_{i+2}}F(t)dt\right\}$$
$$+c_3\left\{\left(s_i-t_i\right)F\left(s_i\right)+\int_{s_i}^{t_i}F(t)dt\right\}$$
$$+c_3\left\{\left(s_{i+1}-t_{i+1}\right)F\left(s_{i+1}\right)+\int_{s_{i+1}}^{t_{i+1}}F(t)dt\right\} \tag{5}$$

Figure 2 Inventory levels of the two consecutive cycles



The optimal values of t_i, t_{i+1} and s_{i+1} (if exist) are the solutions of the following set of equations:

$$\frac{\partial TC2_i}{\partial t_i} = 0, \frac{\partial TC2_i}{\partial t_{i+1}} = 0, \text{ and } \frac{\partial TC2_i}{\partial s_{i+1}} = 0$$

or equivalently,

$$(c_2 + c_3)F(t_i) - c_2F(s_{i+1}) - c_3F(s_i) = 0 \tag{6}$$

$$(c_2 + c_3)F(t_{i+1}) - c_2F(s_{i+2}) - c_3F(s_{i+1}) = 0 \tag{7}$$

$$(c_2 + c_3)s_{i+1} - c_3t_{i+1} + c_2t_i = 0 \tag{8}$$

or

$$F(t_i) = \frac{c_2F(s_{i+1}) + c_3F(s_i)}{c_2 + c_3} \tag{9}$$

$$F(t_{i+1}) = \frac{c_2F(s_{i+2}) + c_3F(s_{i+1})}{c_2 + c_3} \tag{10}$$

$$s_{i+1} = \frac{c_2t_i + c_3t_{i+1}}{c_2 + c_3} \tag{11}$$

The unique existence of the solution $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$ of the set of equations (9), (10), and (11) can be confirmed through the following iterative procedure:

a In the first iteration, assign s_i to be the starting value of s_{i+1}: $s_{i+1}^{(0)} \leftarrow s_i$; then

- From (9), $t_i^{(0)}$ can be found
- From (10), $t_{i+1}^{(0)}$ can be determined which is a value that satisfy the condition:

$$s_i = s_{i+1}^{(0)} < t_{i+1}^{(0)} < s_{i+2}$$

b In the next iteration, $s_{i+1}^{(1)}$ will be determined from equation (11) based on $t_i^{(0)}$ and $t_{i+1}^{(0)}$; then $t_i^{(1)}$ and $t_{i+1}^{(1)}$ will be determined from equations (9) and (10) based on $s_{i+1}^{(1)}$. It is noted that $s_{i+1}^{(1)} > s_{i+1}^{(0)}$ and hence, $t_i^{(1)} > t_i^{(0)}$ and $t_{i+1}^{(1)} > t_{i+1}^{(0)}$.

The above procedure will be performed until the series $\{s_{i+1}^{(k)}\}$ converges. The convergence of the series $\{s_{i+1}^{(k)}\}$ can be ensured due to the fact that $\{s_{i+1}^{(k)}\}$ is increasing and has an upper bound of s_{i+2}. From the procedure, it can be seen that the set of equations (9), (10), (11) has unique solution t_i^*, t_{i+1}^* and s_{i+1}^*.

In brief, the following step-by-step procedure can be employed to determine $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$;

Step 0 k = 0; assign $s_{i+1}^{(0)} \leftarrow s_i$

Step 1

a Determine $t_i^{(k+1)}$ and $t_{i+1}^{(k+1)}$ from equations (9) and (10) by using bisection method.

b Determine $s_{i+1}^{(k+1)}$ from $t_i^{(k+1)}$ and $t_{i+1}^{(k+1)}$ using equation (11).

c Check if $s_{i+1}^{(k+1)} - s_{i+1}^{(k)} > \varepsilon$ then update $k = k+1,$ go back to step 1a.

Otherwise, stop. The current values $t_i^{(k+1)}, t_{i+1}^{(k+1)}, s_{i+1}^{(k+1)}$ will be recorded as the solution for t_i^*, t_{i+1}^*, and s_{i+1}^*.

In the next paragraphs, the unique solution $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$ of (9), (10), (11) determined by the above procedure is optimal will be proven by investigating the Hessian matrix of the total cost function *TC*2*i* and proving that the Hessian matrix is positive definite at $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$.

It is noted that the Hessian matrix of *TC*2*i* can be expressed as:

$$J = \begin{bmatrix} \dfrac{\partial^2 TC2_i}{\partial t_i^2} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial t_{i+1}} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial s_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial t_{i+1} \partial t_i} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1}^2} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1} \partial s_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial s_{i+1} \partial t_i} & \dfrac{\partial^2 TC2_i}{\partial s_{i+1} \partial t_{i+1}} & \dfrac{\partial^2 TC2_i}{\partial s_{i+1}^2} \end{bmatrix}$$

Consider the following determinants:

$$J_1 = \left| \dfrac{\partial^2 TC2_i}{\partial t_i^2} \right| \qquad J_2 = \begin{vmatrix} \dfrac{\partial^2 TC2_i}{\partial t_i^2} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial t_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial t_{i+1} \partial t_i} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1}^2} \end{vmatrix} \qquad J_3 = \begin{vmatrix} \dfrac{\partial^2 TC2_i}{\partial t_i^2} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial t_{i+1}} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial s_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial t_{i+1} \partial t_i} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1}^2} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1} \partial s_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial s_{i+1} \partial t_i} & \dfrac{\partial^2 TC2_i}{\partial s_{i+1} \partial t_{i+1}} & \dfrac{\partial^2 TC2_i}{\partial s_{i+1}^2} \end{vmatrix}$$

In order to prove that the Hessian matrix *J* is positive definite at $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$ it is sufficient to prove that J_1, J_2, J_3 are positive at $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$ [see Rao (2009)]. In fact,

$$J_1 \Big|_{\{t_i^*, t_{i+1}^*, s_{i+1}^*\}} = \left| \dfrac{\partial^2 TC2_i}{\partial t_i^2} \right| \Big|_{\{t_i^*, t_{i+1}^*, s_{i+1}^*\}} = (c_2 + c_3) f(t_i^*) > 0$$

$$J_2\Big|_{\{t_i^*,t_{i+1}^*,s_{i+1}^*\}} = \left|\begin{matrix} \dfrac{\partial^2 TC2_i}{\partial t_i^2} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial t_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial t_{i+1}\partial t_i} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1}^2} \end{matrix}\right|\Bigg|_{\{t_i^*,t_{i+1}^*,s_{i+1}^*\}}$$

$$= \left|\begin{matrix} (c_2+c_3)f(t_i^*) & 0 \\ 0 & \begin{matrix}\left[(c_2+c_3)s_{i+1}^* - c_2 t_i^* - c_3 t_{i+1}^*\right] f'(s_{i+1}^*) \\ +\left[c_2+c_3\right] f(s_{i+1}^*)\end{matrix} \end{matrix}\right|$$

It is noted that at $t_i^*, t_{i+1}^*, s_{i+1}^*$, we have $(c_2+c_3)s_{i+1}^* - c_2 t_i^* - c_3 t_{i+1}^* = 0$ [see equation(8)]. Hence,

$$J_2\Big|_{\{t_i^*,t_{i+1}^*,s_{i+1}^*\}} = \left|\begin{matrix} (c_2+c_3)f(t_i^*) & 0 \\ 0 & (c_2+c_3)f(s_{i+1}^*) \end{matrix}\right| = (c_2+c_3)^2 f(t_i^*) f(s_{i+1}^*) > 0$$

$$\begin{aligned} J_3\Big|_{\{t_i^*,t_{i+1}^*,s_{i+1}^*\}} &= \left|\begin{bmatrix} \dfrac{\partial^2 TC2_i}{\partial t_i^2} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial t_{i+1}} & \dfrac{\partial^2 TC2_i}{\partial t_i \partial s_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial t_{i+1}\partial t_i} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1}^2} & \dfrac{\partial^2 TC2_i}{\partial t_{i+1}\partial s_{i+1}} \\ \dfrac{\partial^2 TC2_i}{\partial s_{i+1}\partial t_i} & \dfrac{\partial^2 TC2_i}{\partial s_{i+1}\partial t_{i+1}} & \dfrac{\partial^2 TC2_i}{\partial s_{i+1}^2} \end{bmatrix}\right|\{t_i^*, t_{i+1}^*, s_{i+1}^*\} \\ &= \left|\begin{matrix} (c_2+c_3)f(t_i^*) & 0 & -c_2 f(s_{i+1}^*) \\ 0 & (c_2+c_3)f(t_{i+1}^*) & -c_3 f(s_{i+1}^*) \\ -c_2 f(s_{i+1}^*) & -c_3 f(s_{i+1}^*) & (c_2+c_3)f(s_{i+1}^*) \end{matrix}\right| \\ &= (c_2+c_3)f(t_i^*)\left[(c_2+c_3)^2 f(t_{i+1}^*) f(s_{i+1}^*) - c_3{}^2\left(f(s_{i+1}^*)\right)^2\right] \\ &\quad -c_2 f(s_{i+1}^*)\left[c_2(c_2+c_3) f(t_{i+1}^*) f(s_{i+1}^*)\right] \end{aligned} \tag{12}$$

In order to prove that $J_3\Big|_{\{t_i^*,t_{i+1}^*,s_{i+1}^*\}} > 0$ it will be derived that

$$(c_2+c_3)^2 f(t_{i+1}^*) f(s_{i+1}^*) - c_3{}^2\left(f(s_{i+1}^*)\right)^2 > c_2(c_2+c_3) f(t_{i+1}^*) f(s_{i+1}^*) \tag{13}$$

and

$$(c_2+c_3) f(t_i^*) > c_2 f(s_{i+1}^*) \tag{14}$$

First, it is noted that the inequality (13) is equivalent to

$$c_2 c_3 f(t_{i+1}^*) f(s_{i+1}^*) + c_3{}^2 f(s_{i+1}^*)\left[f(t_{i+1}^*) - f(s_{i+1}^*)\right] > 0$$

which holds true due to the facts that $t_{i+1}^* > s_{i+1}^*$ and $f(.)$ is an increasing function.

Second, inequality (14) can be derived based on the assumption that the demand rate function $f(.)$ is an increasing log-concave function (see Appendix).

From inequalities (13) and (14), it can be easily seen that $J_3\Big|_{\{t_i^*, t_{i+1}^*, s_{i+1}^*\}}$ is also positive. This completes the proof that the unique solution $\{t_i^*, t_{i+1}^*, s_{i+1}^*\}$ of (9), (10), (11) is optimal.

4 Forward with backward rolling technique

In this section, a forward with backward rolling technique will be proposed to help adjust the predefined replenishment times t_i's (i = 1, 2, …, n) and shortage starting points s_i's (i = 2, …, n) for the planning horizon of length H so that the total inventory cost can be gradually reduced. The proposed procedure is as follows:

Step 1 Divide the planning horizon of length H into n equal cycles in which cycle i goes from s_i to s_{i+1} ($i = 1,2,\ldots,n$; $s_1 = 0$; $s_{n+1} = H$). Assign a large value for the total cost function TC = Inf.

Step 2

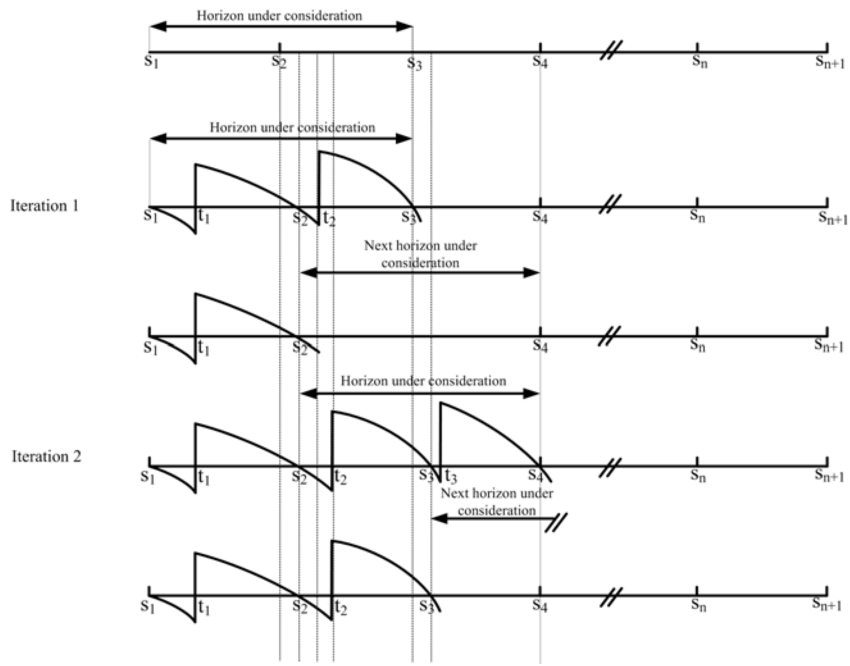
- a Forward move
  1. Set i = 1; consider two consecutive cycle i and i + 1; apply the procedure discussed in Section 3 to determine the optimal solution $\{t_i, t_{i+1}, S_{i+1}\}$ $\{t_i, t_{i+1}, s_{i+1}\}$ of the set of equations (9), (10), and (11).

     Record the value of t_i and update the value of s_{i+1} by the newly found value. Go to step 2.a.2.
  2. Update $i = i + 1$. If $i < n$, go back to step 2.a.1. Otherwise, record also the value of t_{i+1} found in the last iteration and go to step 2.a.3.
  3. Determine the total cost TC and check if the total cost function has been improved (i.e., reduced). If yes, go to step 2b. If no, stop the iterative procedure.
- b Backward move
  1. Set $I = n - 1$; consider two consecutive cycle i and i + 1; apply the procedure discussed in section 3 to determine the optimal solution $\{t_i, t_{i+1}, s_{i+1}\}$ of the set of equations (9), (10), and (11).

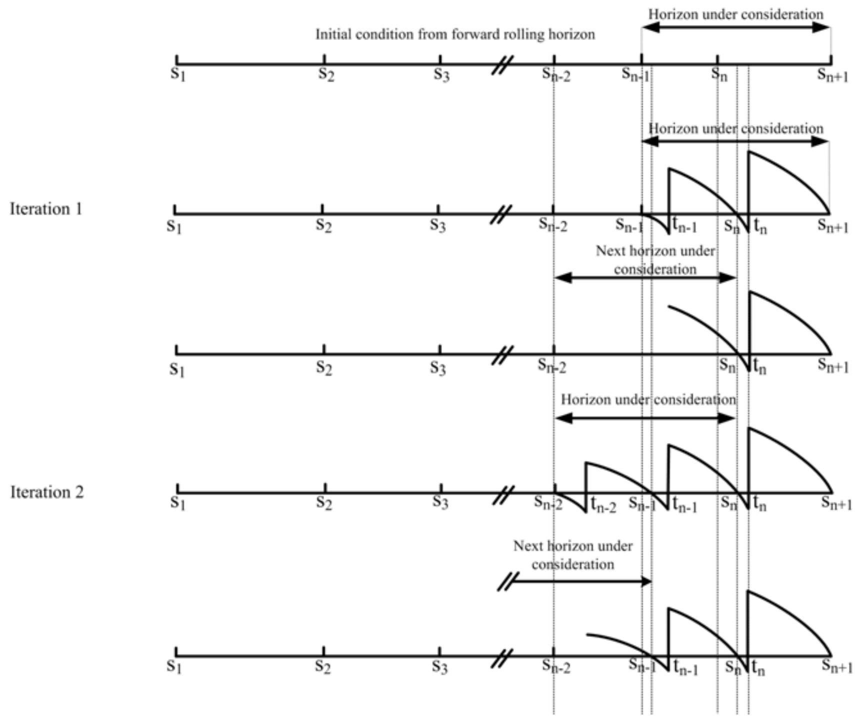
     Record the value of t_{i+1} and update the value of a_{i+1} by the newly found value. Go to step 2.b.2.
  2. Update $i = i - 1$. If $i > 0$, go back to step 2.b.1. Otherwise, record also the value of t_i found in the last iteration and go to step 2.b.3.
  3. Determine the total cost TC and check if the total cost function has been improved (i.e., reduced). If yes, go to step 2a. If no, stop the iterative procedure.

The forward with backward rolling procedures discussed above are illustrated in Figures 3(a) and 3(b).

Figure 3 (a) Forward rolling procedure (b) Backward rolling procedure



(a)



(b)

5 Numerical experiments

In this section, numerical experiments are conducted to illustrate the applicability of the proposed method. Three examples will be considered here.

5.1 Example 1 (Yang, 2006)

Consider the demand function of the form: $f(t) = bt^u$ with $u = 2$, $b = 900$. The other parameters are set as follows: $H = 1$, $c_1 = 4.5$, $c_2 = 1$, $c_3 = 3.5$.

The number of cycles n used in this sample problem is determined based on the formula developed by Teng (1996) for the case of linear increasing demand, where

$$n = \text{rounded integer of } \left\{\left[c_2 c_3 HF(H)\right] / \left[2c_1\left(c_2 + c_3\right)\right]\right\}^{1/2} = 5$$

Sensitivity analysis can be conducted later to find the appropriate value of n. The step-by-step procedure to determine t_i ($i = 1, 2, \ldots, n$) and s_i ($i = 2, \ldots, n$) is presented below:

Step 0 Determine the initial values of s_i ($i = 1, 2, \ldots, n$): For $s_1 = 0$, $s_2 = 0.2$, $s_3 = 0.4$, $s_4 = 0.6$, $s_5 = 0.8$, $s_6 = \text{H} = 1.0$, set $TC = \text{Inf}$.

Step 1 Forward move

- Iteration 1: consider cycles 1 and 2 which goes from $s_1 = 0$ to $s_3 = 0.4$. Solve the set of equations (9), (10), and (11), then the following values are found: $t_1 = 0.1740$, $t_2 = 0.3197$ and $s_2 = 0.2873$.

  Record the value of t_1 and update s_2 from 0.2 to 0.2873
- Iteration 2: consider cycles 2 and 3 which goes from $s_2 = 0.2873$ to $s_4 = 0.6$. Solve the set of equations (9), (10) and (11), then the following values are found: $t_2 = 0.3459$, $t_3 = 0.5044$ and $s_3 = 0.4692$.

  Record the value of t_2 and update s_3 from 0.4 to 0.4692
- Iteration 3: consider cycles 3 and 4 which goes from $s_3 = 0.4692$ to $s_5 = 0.8$. Solve the set of equations (9), (10) and (11), then the following values are found: $t_3 = 0.5226$, $t_4 = 0.6925$ and $s_4 = 0.6547$.

  Record the value of t_3 and update s_4 from 0.6 to 0.6547.
- Iteration 4: consider cycles 4 and 5 which goes from $s_4 = 0.6547$ to $s_6 = 1.0$. Solve the set of equations (9), (10) and (11), then the following values are found: $t_4 = 0.7061$, $t_5 = 0.8838$ and $s_5 = 0.8443$.

  Record the values of t_4, t_5 and update s_5 from 0.8 to 0.8443.

  Determine total cost TC from expression (1): $TC = 43.67$. Continue to step 2.

Step 2 Backward move

- Iteration 1: consider cycles 4 and 5 which goes from $s_4 = 0.6547$ to $s_6 = 1.0$. Solve the set of equations (9), (10) and (11), then the following values are found: $t_4 = 0.7061$,
  $t_5 = 0.8838$ and $s_5 = 0.8443$.

  Record the value of t_5 and update s_5.

It is noted that this iteration can be ignored because the solution should be exactly the same as in iteration 4 of step 1.

- Iteration 2: consider cycles 3 and 4 which goes from $s_3 = 0.4692$ to $s_5 = 0.8443$. Solve the set of equations (9),(10), and (11), then the following values are found: $t_3 = 0.5332$, $t_4 = 0.7244$ and $s_4 = 0.6817$

  Record the value of t_4 and update s_4 from 0.6547 to 0.6817
- Iteration 3: consider cycles 2 and 3 which goes from $s_2 = 0.2873$ to $s_4 = 0.6817$. Solve the set of equations (9), (10) and (11), then the following values are found: $t_2 = 0.3684$, $t_3 = 0.5655$ and $s_3 = 0.5217$.

  Record the value of t_3 and update s_3 from 0.4692 to 0.5217.
- Iteration 4: consider cycles 1 and 2 which goes from $s_1 = 0$ to $s_3 = 0.5217$. Solve the set of equations (9), (10) and (11), then the following values are found: $t_1 = 0.2270$, $t_2 = 0.4619$ and $s_2 = 0.3747$.

  Record the values of t_1, t_2 and update s_2 from 0.2873 to 0.3747.

Determine total cost TC from expression (1): $TC = 42.24$. Since there is improvement in total cost, another forward move step will be conducted. The forward with backward rolling procedure will be repeated until no improvement in total cost can be realized.

For the current example, the intermediate replenishment schedules after the first forward step and the first backward step, as well as the final replenishment schedule are shown in Table 1. It is noted that the total cost $TC = 40.51$ resulted from the proposed technique in this example is exactly the same as the one reported by Yang (2006).

Table 1 Replenishment schedule of example 1

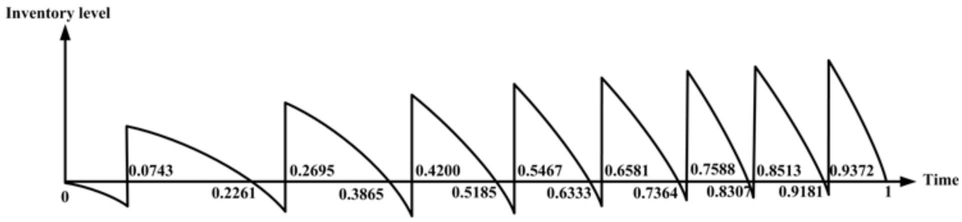
			After the first forward step				
i	*1*	*2*	*3*	*4*	*5*	*6*	*Total cost*
t_i^*	0.1740	0.3459	0.5226	0.7061	0.8838	-	43.67
S_i^*	0	0.2873	0.4692	0.6547	0.8443	1.0000	
			After the first backward step				
t_i^*	0.2270	0.4169	0.5655	0.7244	0.8838	-	42.24
S_i^*	0	0.3747	0.5217	0.6817	0.8443	1.0000	
			Final replenishment schedule				
t_i^*	0.2760	0.5070	0.6708	0.8043	0.9198	-	40.51
S_i^*	0	0.4556	0.6343	0.7746	0.8941	1.0000	

5.2 Example 2 (Yang et al., 2002)

Consider the demand function of the form: $f(t) = (a + bt)^u$ with $u = 2$, $a = 10$, $b = 30$. The other parameters are set as follows: $H = 1$, $c_1 = 4.5$, $c_2 = 1$, $c_3 = 3.5$.

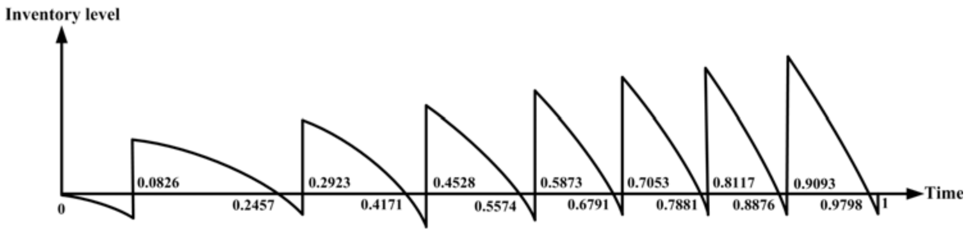
For n = 8, which is also determined from the formula of Teng (1996), the replenishment schedule in this example can be determined based on the proposed method in a similar way as in example 1. The detailed results are shown in Table 2 and illustrated in Figure 4.

Figure 4 Replenishment schedule of example 2



The corresponding total inventory cost for this example is TC = 67.21. It is noted that the total inventory cost reported by Yang et al. (2002) for this example is 66.13 with n = 7 However, in the resulting replenishment schedule reported by Yang et al. (2002), although shortages were assumed to be completely backlogged, there still exist a shortage period at the end of the planning horizon, and the author did not mention how to deal with this shortage (see Figure 5 for illustration).

Figure 5 Replenishment schedule of example 2



Source: Yang et al. (2002)

In this example, there are two practical ways to fulfill the demand at the end of the planning horizon. They are:

1. adding one more replenishment at the end of the planning horizon with the replenishment quantity exactly equals to the shortage amount
2. increasing the time coverage of the last replenishment cycle.

The associated costs of these two adjustments are shown also in Table 2 for comparison purpose. From the results in Table 2, it can be seen that, if the total demand in the planning horizon is completely fulfilled, the total inventory cost resulted from the proposed technique is smaller when it is compared with the result from the adjusted Yang's models.

Table 2 Comparisons between the proposed technique and the adjusted Yang's models

i	*Proposed method*		*Adjusted Yang's model (1)*		*Adjusted Yang's model (2)*	
	s_i	t_i	s_i	t_i	s_i	t_i
1	0	0.0743	0	0.0826	0	0.0826
2	0.2261	0.2695	0.2457	0.2923	0.2457	0.2923
3	0.3865	0.4200	0.4171	0.4528	0.4171	0.4528
4	0.5185	0.5467	0.5574	0.5873	0.5574	0.5873
5	0.6333	0.6581	0.6791	0.7053	0.6791	0.7053
6	0.7364	0.7588	0.7881	0.8117	0.7881	0.8117
7	0.8307	0.8513	0.8876	0.9093	0.8876	0.9093
8	0.9181	0.9372	0.9798	1.0000	1.0000	-
9	1.0000		1.0000		-	
TC	67.21 (n = 8)		70.63 (n = 8)		67.58 (n = 7)	

5.3 Example 3

Consider the demand function of Example 1, which is in the form $f(t) = bt^u$ with $u = 2$, $b = 900$. The other parameters are set as the combination of following values: $H = 1$, 1.5, and 2, $c_1 = 3.5$ and 4.5, $c_2 = 1$, $c_3 = 3.5$ and 4.5. In order to demonstrate the capability of the proposed algorithm, the same problems are also solved using Yang's (2006) method and a Nelder-Mead algorithm. The comparison of the results is presented in Table 3.

Table 3 Comparisons among the proposed technique, Yang's method, and Nelder-Mead algorithm

No.	*Problem parameters*						*Total cost of the solution*		
	b	u	H	c_1	c_2	c_3	*Yang's method (2006)*	*Proposed algorithm*	*Nelder-Mead algorithm*
A	900	2	1	4.5	1	3.5	40.51	40.51	42.17
B	900	2	1.5	4.5	1	3.5	90.91	90.56	96.53
C	900	2	2	4.5	1	3.5	169.41	160.95	178.69
D	900	2	1	3.5	1	4.5	36.89	36.94	38.13
E	900	2	1.5	3.5	1	4.5	83.72	82.22	83.72
F	900	2	2	3.5	1	4.5	159.71	146.33	163.48

From Table 3, it can be seen that the proposed algorithm is consistently able to provide a good solution, in which it is able to obtain five out of six problems with the smallest total cost. It is noted that the result of proposed algorithm of problem D is slightly worse than Yang's method, however, the deviation is very small, i.e., about 0.14%.

6 Sensitivity analysis

From Section 5, a good initial value of the replenishment cycle n derived by Teng (1996), for linear increasing demand pattern is used. In order to find the best value of n, sensitivity analysis is conducted to investigate the effect of n on the total cost function. For the two examples discussed, the summarized sensitivity analysis results are presented in Tables 4 and 5, respectively. From the results in Tables 4 and 5, it can be seen that the values of n for example 1 and 2 are respectively 5 and 8.

Table 4 Sensitivity analysis on the effect of n for example 1

i	$n = 4$		$n = 5$		$n = 6$	
	s_i	t_i	s_i	t_i	s_i	t_i
1	0.0000	0.3087	0.0000	0.2760	0.0000	0.2518
2	0.5097	0.5671	0.4556	0.5070	0.4157	0.4625
3	0.7096	0.7504	0.6343	0.6708	0.5787	0.6120
4	0.8665	0.8997	0.7746	0.8043	0.7067	0.7338
5	1.0000		0.8941	0.9198	0.8157	0.8392
6			1.0000		0.9124	0.9333
7					1.0000	
TC	40.56		*40.51*		41.97	

Table 5 Sensitivity analysis on the effect of n for example 2

i	$n = 4$		$n = 5$		$n = 6$	
	s_i	t_i	s_i	t_i	s_i	t_i
1	0.0000	0.0852	0.0000	0.0743	0.0000	0.0658
2	0.2518	0.2994	0.2261	0.2695	0.2055	0.2454
3	0.4265	0.4629	0.3865	0.4200	0.3540	0.3851
4	0.5694	0.5999	0.5185	0.5467	0.4769	0.5032
5	0.6934	0.7202	0.6333	0.6581	0.5841	0.6073
6	0.8045	0.8286	0.7364	0.7588	0.6805	0.7015
7	0.9060	0.9281	0.8307	0.8513	0.7688	0.7881
8	1.0000		0.9181	0.9372	0.8507	0.8687
9			1.0000		0.9275	0.9443
10					1.0000	
TC	67.28		67.21		68.17	

7 Conclusions

We have developed and solved the inventory replenishment problem for nonlinear increasing demand pattern considering shortage backorders. A forward with backward inventory policy algorithm has been developed to determine the replenishment times and the shortage points so as to minimize the total inventory cost. Comparing with the other techniques developed in the past, the proposed technique results in either the same or better cost performance. In addition, unlike the past-developed techniques that require specific functional forms of the demand pattern, the proposed technique can be employed for a more general demand pattern (i.e., any log-concave function).

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Appendix

By the assumption that the demand function *f*(*t*) is a log-concave function, the following inequality holds true:

$$\frac{f'(t)}{f(t)} \geq \frac{f'(s)}{f(s)} \text{ with } t \leq s. \tag{A1}$$

Fix $t = t_i^*$, multiply both sides of (A1) by *f*(*s*), and integrate with respect to *s* from t_i^* to s_{i+1}^*, we have:

$$\int_{t_i^*}^{s_{i+1}^*} f'(s)ds \leq \int_{t_i^*}^{s_{i+1}^*} \frac{f'(t_i^*)}{f(t_i^*)} f(s)ds$$

or equivalently,

$$f(s_{i+1}^*) - f(t_i^*) \leq \frac{f'(t_i^*)}{f(t_i^*)} \int_{t_i^*}^{s_{i+1}^*} f(s)ds \tag{A2}$$

Multiplying both sides of (A2) by c_2, we have:

$$c_2\left[f(s_{i+1}^*) - f(t_i^*)\right] \leq c_2 \frac{f'(t_i^*)}{f(t_i^*)} \int_{t_i^*}^{s_{i+1}^*} f(s)ds \tag{A3}$$

Noted from equation (6) in the main text that

$$(c_2 + c_3)F(t_i) - c_2 F(s_{i+1}) - c_3 F(s_i) = 0 \Leftrightarrow c_2 \int_{t_i}^{s_{i+1}} f(t)dt = c_3 \int_{s_i}^{t_i} f(t)dt$$

We also have: $c_2 \int_{t_i^*}^{s_{i+1}^*} f(t)dt = c_3 \int_{s_i^*}^{t_i^*} f(t)dt$

Hence, inequality (A3) can be rewritten as:

$$c_2\left[f(s_{i+1}^*) - f(t_i^*)\right] \leq c_3 \frac{f'(t_i^*)}{f(t_i^*)} \int_{s_i^*}^{t_i^*} f(s)ds \tag{A4}$$

Applying the Cauchy's mean value theorem for the two functions *f*(.) and *F*(.), i.e., the demand rate and the cumulative demand functions, in the interval $\left[s_i^*, t_i^*\right]$ there should exist a value $x \in \left[s_i^*, t_i^*\right]$ such that:

$$f'(x)\left[F(t_i^*) - F(s_i^*)\right] = F'(x)\left[f(t_i^*) - f(s_i^*)\right]$$

or equivalently,

$$\int_{s_i^*}^{t_i^*} f(s)ds = \frac{f(x)}{f'(x)}\left[f\left(t_i^*\right) - f\left(s_i^*\right)\right] \tag{A5}$$

Replacing (A5) into the right-hand side of (A4) we have

$$c_2\left[f\left(s_{i+1}^*\right) - f\left(t_i^*\right)\right] \le c_3 \frac{f'\left(t_i^*\right)}{f\left(t_i^*\right)} \frac{f(x)}{f'(x)}\left[f\left(t_i^*\right) - f\left(s_i^*\right)\right] \tag{A6}$$

Due to the fact that $\frac{f'(x)}{f(x)} \ge \frac{f'\left(t_i^*\right)}{f\left(t_i^*\right)}$, inequality (A6) implies that:

$$c_2\left[f\left(s_{i+1}^*\right) - f\left(t_i^*\right)\right] \le c_3\left[f\left(t_i^*\right) - f\left(s_i^*\right)\right]$$

or,

$$\left(c_2 + c_3\right) f\left(t_i^*\right) \ge c_2 f\left(s_{i+1}^*\right) + c_3 f\left(s_i^*\right) \tag{A7}$$

From (A7), we can derive:

$$\left(c_2 + c_3\right) f\left(t_i^*\right) \ge c_2 f\left(s_{i+1}^*\right) \tag{QED}$$