On right-angled polygons in hyperbolic space

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Abstract We study oriented right-angled polygons in hyperbolic spaces of arbitrary dimensions, that is, finite sequences $(S_0, S_1, \ldots, S_{p-1})$ of oriented geodesics in the hyperbolic space H^{n+2} such that consecutive sides are orthogonal. It was previously shown by Delgove and Retailleau (Ann Fac Sci Toulouse Math 23(5):1049–1061, 2014. https://doi.org/10.5802/afst.1435) that three quaternionic parameters define a right-angled hexagon in the 5-dimensional hyperbolic space. We generalise this method to right-angled polygons with an arbitrary number of sides $p \geq 5$ in a hyperbolic space of arbitrary dimension.

Keywords Hyperbolic space · Clifford matrix · Cross ratio · Right-angled polygon · Golden ratio

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1 Introduction

For $n \geq 0$, let H^{n+2} denote the real hyperbolic $(n + 2)$-space. The boundary of this space can be described with Clifford vectors. These are special elements of the Clifford algebra $\mathcal{C}_n$, which is the unitary associative algebra generated by n elements $i_1, \ldots, i_n$ such that $i_j i_l = -i_l i_j$, $i_l^2 = -1$ for $l \neq j$. The group of orientation preserving isometries $\mathrm{Isom}^+(H^{n+2})$ of the hyperbolic space H^{n+2} can be expressed with Clifford matrices. These are 2×2 matrices

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with coefficients in the multiplicative group generated by Clifford vectors and with Ahlfors determinant 1.

In this context, we describe hyperbolic right-angled polygons for which we mean right-angled closed edge paths in $n + 2$ dimensions. We show how to construct a hyperbolic right-angled polygon Π_p of p sides, $p > 4$, by prescribing a parameter set consisting of $p - 3$ Clifford vectors in ∂H^{n+2}. Such a construction is achieved in an arbitrary dimension. No connection between the dimension of the space and the number of sides of the polygon is required.

Similar objects have already been studied in dimension 2 and 3 by Thurston [8] and by Fenchel [9], who studied right-angled hexagons. Costa and Martínez [5] studied right-angled polygons with an arbitrary number of sides in the hyperbolic plane. More recently Delgove and Retailleau [7] classified right-angled hexagons in H^5. In their work, 2×2 quaternionic matrices having Dieudonné determinant 1 are used in order to describe the direct isometries of H^5. While this approach based on quaternions is very convenient, it can not be extended to arbitrary dimensions. By using Clifford matrices instead, we are able to generalise the construction to any dimension. Particularly, 2×2 quaternionic Clifford matrices are used to describe direct isometries of H^4.

In the first section we develop more precisely the connection between hyperbolic space and the Clifford algebra. Then we discuss the role of the cross ratio for Clifford vectors and its geometrical interpretation. Our main result, the algorithmic construction of Π_p, is presented in the second section. In the last part we treat the case when the convex hull of the p vertices of the polygon Π_p give rise to a hyperbolic $(p - 1)$-simplex. A necessary condition for its realisation is stated. As a conclusion we discuss in more details a special case in 4 dimensions, supposing that all the edges of the edge path have equal length. By exploiting the work of Dekster and Wilker [6] we explicitly state a necessary and sufficient condition for realisability depending on such a side length. Surprisingly, it turns out that the side length must be related to the golden ratio $\gamma = \frac{1+\sqrt{5}}{2}$.

2 The real Clifford algebra and hyperbolic space

In this section we present the notion of Clifford algebra and its relation to isometries of hyperbolic space. For a more complete description we refer to the works of Ahlfors [1,2], Vahlen [13] and Waterman [14] (see also [11, Section 7]).

2.1 The real Clifford algebra $\mathcal{C}_n$

Consider the real Clifford algebra $\mathcal{C}_n$ generated by $i_1, \ldots, i_n$, that is

$$\mathcal{C}_n = \langle i_1, \ldots, i_n \mid i_j \, i_l = -i_l \, i_j, \, i_l^2 = -1 \text{ for } l \neq j \rangle,$$

which is a unitary associative real algebra. Every element x of the algebra $\mathcal{C}_n$ can be uniquely written as $x = \sum x_I I$, where $x_I \in \mathbb{R}$ and the sum is taken over all the products $I = i_{k_1} \cdots i_{k_m}$, with $1 \leq k_1 < \cdots < k_m \leq n$ and $1 \leq m \leq n$. Here the empty product I_0 is included and identified with $i_0 := 1$. Hence $\mathcal{C}_n$ is a 2^n-dimensional real vector space. In particular we can identify $\mathcal{C}_0$ with $\mathbb{R}$, $\mathcal{C}_1$ with $\mathbb{C}$ and $\mathcal{C}_2$ with $\mathbb{H}$, the Hamiltonian quaternions. To each element $x = \sum x_I I$ we associate a norm as given by $|x|^2 = \sum x_I^2$, inducing a Euclidean structure on $\mathcal{C}_n$. Denote with $\Re(x)$ the coefficient x_0, called the *real part* of x, while $\Im(x) = x - \Re(x)$ is called the *non-real part* of x. If $\Re(x) = 0$ we will refer to x as a *pure element* of $\mathcal{C}_n$.

On $\mathcal{C}_n$ there are three well-known involutions. Let $x \in \mathcal{C}_n$, $x = \sum x_I I$. Then:

(i) $x^* = \sum x_I I^*$, where I^* is obtained from $I = i_{k_1} \cdots i_{k_m}$ by reversing the order of the factors, that is $I^* = i_{k_m} \cdots i_{k_1}$;

(ii) $x' = \sum x_I I'$, where I' is obtained from $I = i_{k_1} \cdots i_{k_m}$ by replacing each factor i_k with $-i_k$, that is $I' = (-i_{k_1}) \cdots (-i_{k_m}) = (-1)^m I$;

(iii) $\overline{x} = (x^*)' = (x')^*$.

The involutions (i) and (iii) are anti-automorphisms, while the involution (ii) is an automorphism.

Of particular interest are Clifford elements of the form $x = x_0 + x_1 i_1 + \cdots + x_n i_n$, called *Clifford vectors*. The set

$$\mathbb{V}^{n+1} = \{x_0 + x_1 i_1 + \cdots + x_n i_n \mid x_0, \ldots, x_n \in \mathbb{R}\}$$

of all Clifford vectors is an $(n+1)$-dimensional real vector space, naturally isomorphic to the Euclidean space $\mathbb{R}^{n+1}$. Notice that for an element $x \in \mathbb{V}^{n+1}$ we have $x^* = x$ and hence $\overline{x} = x'$ as well as $x + \overline{x} = 2\Re(x)$ and $x\overline{x} = \overline{x}x = |x|^2$. Moreover every non-zero vector x has an inverse given by $x^{-1} = \frac{\overline{x}}{|x|^2}$. Hence finite products of non-zero vectors are invertible and they form the so-called *Clifford group* Γ_n. Observe that we have $\Gamma_n = \mathcal{C}_n \setminus \{0\}$ only for $n \in \{0, 1, 2\}$.

2.2 Square root of a Clifford vector

Next we introduce the notion of the square root of a Clifford vector. It will be a generalisation of the square root of quaternions (see [10] for example) in the following way:

Proposition 1 *Let $y \in \mathbb{V}^{n+1} \setminus \{0\}$ be a Clifford vector. If $y \notin \mathbb{R}_{<0}$, then there exist exactly two elements $x_1, x_2 \in \mathbb{V}^{n+1}$ such that $x_1^2 = x_2^2 = y$; x_1 and x_2 are both called a* square root *of y. If $y \in \mathbb{R}_{<0}$, we have the three following situations depending on n:*

- *If $n = 0$, then there is no element $x \in \mathbb{V}^1$ such that $x^2 = y$,*
- *If $n = 1$, then there are exactly two elements $x_1, x_2 \in \mathbb{V}^2$ such that $x_1^2 = x_2^2 = y$,*
- *If $n \geq 2$, then there are uncountably many square roots of y.*

Proof Suppose that $x^2 = y$, with $x, y \in \mathbb{V}^{n+1} \setminus \{0\}$. Then $\overline{x}^2 = \overline{y}$ and $|x|^2 = |y|$. We have the following two equations:

$$x(\overline{x} + x) = x\overline{x} + x^2 = |y| + y, \tag{1}$$

$$(x + \overline{x})^2 = x^2 + 2x\overline{x} + \overline{x}^2 = y + 2|y| + \overline{y} = 2(\Re(y) + |y|). \tag{2}$$

Observe that the term $2(\Re(y) + |y|) \geq 0$.

Now let $y \notin \mathbb{R}_{<0}$, then we have $\Re(y) + |y| > 0$, and the element

$$x := \frac{|y| + y}{\sqrt{2(\Re(y) + |y|)}} \in \mathbb{V}^{n+1} \tag{3}$$

satisfies $x^2 = y$. Indeed,

$$x^2 = \frac{|y|^2 + 2|y|y + y^2}{2\Re(y) + 2|y|} = \frac{(\overline{y} + 2|y| + y)y}{2\Re(y) + 2|y|} = y.$$

Notice that in the special case if $y \in \mathbb{R}_{>0}$, the identity (3) yields $x = \pm\sqrt{y}$ as desired. For $y \notin \mathbb{R}$ the square roots of y have to lie in the plane spanned by 1 and y which is isomorphic

to $\mathbb{C}$, ensuring the non-existence of more than two roots. By abuse of notation the square root x of y is denoted by $\sqrt{y} := x$.

Let $y \in \mathbb{R}_{<0}$. For $n = 0$ or 1 the assertion is trivial. Let $n \geq 2$. We can write $y = -z^2$ for some $z \in \mathbb{R}_{>0}$. In this case consider $x := z \cdot u$ where u is a pure Clifford vector with norm 1. In general for any pure Clifford vector we have

$$0 = (u + \overline{u})u = u\overline{u} + u^2 = |u|^2 + u^2,$$

which implies $u^2 = -|u|^2$. Hence $x^2 = z^2 u^2 = -z^2 |u|^2 = -z^2$. $\qquad\square$

Remark 1 Notice that Proposition 1 remains true for $y \in \Gamma_2 = \mathbb{H}\backslash\{0\}$ since $y + \overline{y} = 2\Re(y)$ still holds. However, it does not hold for a general element of $\mathcal{C}_n$ or even Γ_n, $n \geq 3$. Indeed, for an arbitrary $y \in \Gamma_n$ one has $y + \overline{y} \neq 2\Re(y)$. For example let $y = i_1 i_2 i_3 \in \Gamma_n$, $n \geq 3$. Then $y + \overline{y} = 2i_1 i_2 i_3$. Hence Eq. (2) does not hold.

Remark 2 For the square root $\sqrt{y}$ of a Clifford vector $y \in \mathbb{V}^{n+1}\backslash\mathbb{R}_{\leq 0}$ we have:

- For all positive $\mu \in \mathbb{R}_{>0}$, $\sqrt{\mu y} = \sqrt{\mu}\sqrt{y}$,
- For the inverse $\sqrt{y^{-1}} = \sqrt{y}^{-1} = \frac{1}{|y|}\sqrt{\overline{y}}$.
- The square root of $-y$ can be found by a rotation of $90°$: $\sqrt{-y} = i\sqrt{y}$ for some pure Clifford vector i with $i^2 = -1$. This also holds for negative $y \in \mathbb{R}_{<0}$.

2.3 Clifford matrices and hyperbolic isometries

We now take a look at matrices having entries in the extended Clifford group $\Gamma_n \cup \{0\}$. These matrices will be used to explicitly represent direct isometries of the hyperbolic space $\boldsymbol{H}^{n+2}$ (see for example [14] and [11, Section 7]).

A *Clifford matrix* is a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

$$a, b, c, d \in \Gamma_n \cup \{0\}, \quad ab^*, cd^*, c^*a, d^*b \in \mathbb{V}^{n+1}, \quad ad^* - bc^* \in \mathbb{R}\backslash\{0\},$$

where $ad^* - bc^*$ is the *Ahlfors determinant* of A. Denote the set of such matrices by $\mathrm{GL}(2, \mathcal{C}_n)$. By a result of Vahlen and Maass [2, p. 221] the set

$$\mathrm{SL}(2, \mathcal{C}_n) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathcal{C}_n) \mid ad^* - bc^* = 1 \right\} \tag{4}$$

of Clifford matrices with Ahlfors determinant 1 is a multiplicative group.

Each element $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_n)$ has the inverse matrix $T^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$.

Furthermore $\mathrm{SL}(2, \mathcal{C}_n)$ is generated by the matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix},$$

where $t \in \mathbb{V}^{n+1}$ and $a \in \Gamma_n$ (see for example [11, Section 7]).

The group $\mathrm{SL}(2, \mathcal{C}_n)$ plays an important role in our investigation since it is closely related to the group of orientation preserving isometries of the hyperbolic $(n + 2)$-space realised in the upper half-space according to

$$\boldsymbol{H}^{n+2} = \left\{ x = (x_0, x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} \mid x_{n+1} > 0 \right\}$$
$$\cong \mathbb{V}^{n+1} \times \mathbb{R}_{>0}.$$

The compactification $\overline{H^{n+2}}$ is given by the union of H^{n+2} with the boundary set $\partial H^{n+2} = \mathbb{V}^{n+1} \cup \{\infty\}$ of points at infinity of H^{n+2}.

Consider the projective group

$$\mathrm{PSL}(2, \mathcal{C}_n) = \mathrm{SL}(2, \mathcal{C}_n)/\{\pm I\}.$$

It is known that this group acts bijectively on $\mathbb{V}^{n+1} \cup \{\infty\}$ by

$$T(x) = (ax + b)(cx + d)^{-1} \tag{5}$$

with $T(-c^{-1}d) = \infty$, $T(\infty) = ac^{-1}$ if $c \neq 0$, and $T(\infty) = \infty$ otherwise. By Poincaré extension, the action (5) can be extended to the upper half-space H^{n+2}. In this way we obtain an isomorphism between $\mathrm{PSL}(2, \mathcal{C}_n)$ and the group $\mathrm{Möb}^+(n + 1)$ of orientation preserving Möbius transformations of $\mathbb{V}^{n+1} \cup \{\infty\}$ (see [4,14]). Since the group $\mathrm{Isom}^+\left(H^{n+2}\right)$ of orientation preserving isometries of H^{n+2} is isomorphic to $\mathrm{Möb}^+(n+1)$, we get the following identification:

$$\mathrm{Isom}^+\left(H^{n+2}\right) \cong \mathrm{Möb}^+(n + 1) \cong \mathrm{PSL}(2, \mathcal{C}_n). \tag{6}$$

Therefore any direct isometry of H^{n+2} can be represented by a Clifford matrix in $\mathrm{PSL}(2, \mathcal{C}_n)$.

Finally, we remark that Möbius transformations act triply transitively on $\mathbb{V}^{n+1} \cup \{\infty\}$ (see [15, Section 6], for example). That is, given two triplets $\{x_1, x_2, x_3\}$ and $\{x_1', x_2', x_3'\}$ of distinct points in the boundary, there always exists a transformation $T \in \mathrm{Möb}(n + 1)$ with $T(x_i) = x_i'$. For $n = 0$ this map is unique and for $n = 1$ it is unique if one demands that it preserves the orientation. In higher dimensions this map is not unique anymore.

2.4 The cross ratio

As in the classical case, we shall use the cross ratio to study configurations of points in $\mathbb{V}^{n+1} \cup \{\infty\}$.

Definition 1 Let x, y, z, w be four pairwise different Clifford vectors in $\mathbb{V}^{n+1}$. Then

$$[x, y, z, w] := (x - z)(x - w)^{-1}(y - w)(y - z)^{-1} \in \Gamma_n \backslash \{0\} \tag{7}$$

is called the *cross ratio* of x, y, z and w.

We extend the definition (7) by continuity to $\mathbb{V}^{n+1} \cup \{\infty\}$, allowing x, y or w to be ∞, by

$$[\infty, y, z, w] = (y - w)(y - z)^{-1} \quad \text{for } x = \infty, \tag{8}$$

and similarly for $y = \infty$ and $w = \infty$. Moreover in an analogous way we put

$$[x, y, \infty, w] = (x - w)^{-1}(y - w).$$

The cross ratio satisfies the following transformation behaviour (see [4, Lemma 6.2]):

$$[T(x), T(y), T(z), T(w)] = (cz + d)^{*-1}[x, y, z, w](cz + d)^*, \tag{9}$$

for all $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_n)$.

Hence, the real part and the norm of the cross ratio $[x, y, z, w]$ of four vectors are invariant under the action of T. However, the cross ratio itself is not an invariant.

We specialise the cross ratio in the following way: consider two oriented geodesics s, t in H^{n+2} whose endpoints s^-, s^+ and t^-, t^+ are four distinct points in $\mathbb{V}^{n+1} \cup \{\infty\}$.

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Definition 2 The *cross ratio* $\Delta(s, t)$ *of s and t* is defined by

$$\Delta(s, t) := \left[s^-, s^+, t^-, t^+\right]. \tag{10}$$

Lemma 1 *Let s and t be two geodesics as above. If s and t intersect then* $\Delta(s, t) = \Delta(t, s)$. *If s and t are disjoint, then* $\Delta(s, t) = \Delta(t, s)$ *if one of the endpoints is* ∞ *or if the cross ratios are real, otherwise the two cross ratios are conjugate.*

Proof Assuming one of the endpoints to be infinity, let $s = (x, \infty)$ with $x \in \mathbb{V}^{n+1}$. We can apply a translation $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ such that s is mapped to $(0, \infty)$. By (9), any translation leaves the cross ratio unchanged. Using (8) it is easy to see that $\Delta(s, t) = \Delta(t, s)$.

Let now s and t be two arbitrary geodesics with no endpoint at infinity. We know that we can always find an isometry $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathcal{C}_n)$ mapping the two endpoints of one of the geodesics to 0 and ∞. Using (9) and what we have just discussed above we get

$$
\begin{aligned}
(ct^- + d)^{*-1}[s^-, s^+, t^-, t^+](ct^- + d)^* &= \left[T(s^-), T(s^+), T(t^-), T(t^+)\right] \\
&= \left[T(t^-), T(t^+), T(s^-), T(s^+)\right] \\
&= (cs^- + d)^{*-1}[t^-, t^+, s^-, s^+](cs^- + d)^*.
\end{aligned}
$$

Hence the two cross ratios $\Delta(s, t)$ and $\Delta(t, s)$ are conjugate. This implies that if the cross ratios are real, then the equality $\Delta(s, t) = \Delta(t, s)$ holds. In particular, if two geodesics intersect, then $\Delta(s, t) = \Delta(t, s)$ by Proposition 2 below. $\square$

Now consider three geodesics r, s and t in $\boldsymbol{H}^{n+2}$ with pairwise different endpoints r^-, r^+, s^-, s^+ and t^-, t^+ in $\mathbb{V}^{n-1} \cup \{\infty\}$.

Definition 3 The quantity

$$\Delta(r, s, t) := \left[s^+, s^-, r^+, t^+\right] \tag{11}$$

is called the *double bridge cross ratio* of (r, s, t).

Definition 4 The ordered triple (r, s, t) is called a *double bridge* if s is orthogonal to r and t such that $r \neq t$. If $|\Delta(r, s, t)| > 1$, then the intersections $r \cap s$ and $s \cap t$ do not coincide and we call the double bridge *properly oriented*.

Consider a properly oriented double bridge (r, s, t). The norm of $\Delta(r, s, t)$ encodes the hyperbolic length of the geodesic segment $[r, t]$ on s between r and t. Indeed, assume w.l.o.g. that the endpoints of s in the double bridge (r, s, t) are $s^- = 0$ and $s^+ = \infty$ (see Fig. 1). The hyperbolic distance δ of two points $p, q \in s$ in $\boldsymbol{H}^{n+2}$ with $p_{n+2} > q_{n+2}$ is equal to (see [3, p. 131])

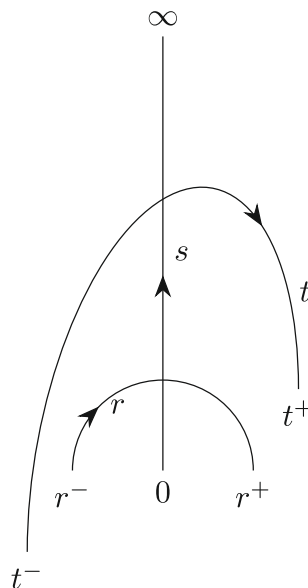$$\delta = \log\left(\frac{p_{n+2}}{q_{n+2}}\right).$$

On the other hand, by (7) we get

$$|\Delta(r, s, t)| = |[\infty, 0, r^+, t^+]| = \frac{|t^+|}{|r^+|}.$$

If we take $p = s \cap t$ and $q = r \cap s$, we conclude that $\delta = \log(|\Delta(r, s, t)|)$.

The following results will be of importance:

Fig. 1 Double bridge

Proposition 2 *Two hyperbolic geodesics s and t intersect if and only if their cross ratio $\Delta(s, t) \in \mathbb{R}_{<0}$. Furthermore s and t are perpendicular if and only if $\Delta(s, t) = -1$.*

Proof Since hyperbolic isometries act triply transitively, there is an isometry represented by $A \in \mathrm{SL}(2, \mathcal{C}_n)$ mapping s and t into $(0, \infty)$ and $(1, x)$, $x \in \mathbb{V}^{n+1}$. Then, by (7) and (8), the cross ratio of $A(s)$ and $A(t)$ equals $\Delta(A(s), A(t)) = [0, \infty, 1, x] = x^{-1}$, and the assertions follow for $A(s)$ and $A(t)$. Moreover, by (9), a real cross ratio stays invariant under isometry. $\square$

Proposition 3 *Let $s = (0, \infty)$ and $t = (1, y)$ with $y \neq 0, \infty$ be two disjoint geodesics in H^{n+2}. Then the common perpendicular l is $(-\sqrt{y}, \sqrt{y})$. This perpendicular is unique up to orientation.*

Proof Let $l = (z, w)$ denote the common perpendicular between s and t. By Proposition 2 and by (8), we get

$$\Delta(s, l) = [0, \infty, z, w] = -1. \tag{12}$$

and

$$\Delta(t, l) = [1, y, z, w] = -1. \tag{13}$$

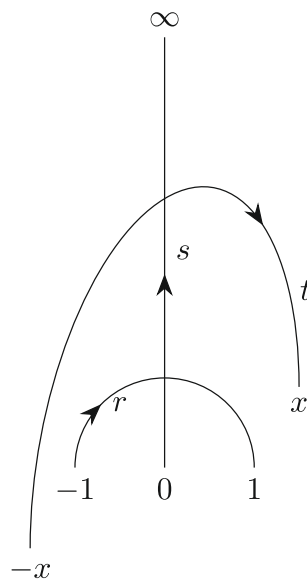Equation (12) yields $z = -w$. The Eq. (13) states that

$$(1 - z)(1 + z)^{-1} = -(y - z)(y + z)^{-1}. \tag{14}$$

It is easy to see that $(1 - z)(1 + z)^{-1} = (1 + z)^{-1}(1 - z)$, so that

$$(1 - z)(y + z) = -(1 + z)(y - z).$$

By expanding the above equation we obtain $y = z^2$. Notice that by construction, since s and t are disjoint, we have $y \notin \mathbb{R}_{<0}$. Hence, by applying Proposition 1, the result follows for $l = (\pm \sqrt{y}, \mp\sqrt{y})$. $\square$

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Fig. 2 Standard configuration double bridge

$$\infty$$

$$s$$

$$t$$

$$r$$

$$x$$

$$-1 \qquad 0 \qquad 1$$

$$-x$$

3 The main theorem

3.1 Preliminaries

Our aim is to construct oriented right-angled polygons in hyperbolic space from a minimal number of prescribed parameters.

Definition 5 An oriented right-angled polygon with p sides in $\boldsymbol{H}^{n+2}$ (or p-gon for short), $n \geq 0$, is a p-tuple of oriented geodesics $(S_0, S_1, \ldots, S_{p-1})$ with $S_{i-1} \neq S_{i+1}$ for i (mod p) and such that S_i is orthogonal to S_{i+1} for $0 \leq i \leq p-2$ and S_{p-1} is orthogonal to S_0.

We usually denote it by Π_p.

We call such a p-gon Π_p *non-degenerate* if consecutive intersections do not coincide (that is $S_{i-1} \cap S_i \neq S_i \cap S_{i+1}$ for i (mod p)) and the double bridges (S_{i-1}, S_i, S_{i+1}), i (mod p), are properly oriented.

We can take $p \geq 5$ since the simplest case of a right-angled polygon is the pentagon. There cannot be a hyperbolic rectangle since the common perpendicular of two geodesics S_0 and S_2 is unique. Hence if there was a hyperbolic rectangle (S_0, S_1, S_2, S_3), two geodesics would have to be identical.

Note that it is no restriction to only consider p-gons in $\boldsymbol{H}^{p-1}$ since the convex hull of p geodesics can at most have dimension $p - 1$. Hence, we will always refer to this case.

Recall that the one-point compactified vector space $\mathbb{V}^{p-2} \cup \{\infty\}$ forms the boundary of hyperbolic $(p-1)$-space

$$\boldsymbol{H}^{p-1} = \left\{ (x, y) \in \mathbb{V}^{p-2} \times \mathbb{R}_{>0} \right\}.$$

Consider the standard configuration double bridge (r, s, t) similar to Sect. 2.4 with $r = (-1, 1)$, $s = (0, \infty)$ and $t = (-x, x)$ for $x \in \mathbb{V}^{p-2} \setminus \{-1, 0, 1\}$ (see Fig. 2).

A small computation shows that the double bridge cross ratio is given by

$$\Delta((-1, 1), (0, \infty), (-x, x)) = [\infty, 0, 1, x] = x. \tag{15}$$

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If conversely the first two geodesics of this double bridge and a desired double bridge cross ratio q are given, one can construct the third geodesic as $(-q, q)$. In the general case this is not easy since the Clifford vectors do not commute. In view of (15) we shall start with the configuration given by the geodesics $(-1, 1)$, $(0, \infty)$ and $(-x, x)$. If the double bridges are supposed to be properly oriented, this poses the immediate restriction $|x| > 1$.

To construct more geodesics we will have to apply certain isometries to achieve this configuration from a general double bridge. These isometries depend on the double bridge cross ratios in the right-angled polygon Π_p they are part of.

Definition 6 For a set of given Clifford vectors $\{q_1, \ldots, q_{p-3}\} \subset \mathbb{V}^{p-2} \setminus \{0\}$ define the isometries ϕ_i of upper half-space by the following Möbius transformations:

$$\phi_i : x \mapsto \sqrt{-2\,q_i}^{-1} (x + q_i)(x - q_i)^{-1} \sqrt{-2\,q_i}, \quad 1 \le i \le p - 3. \tag{16}$$

If $q_i \in \mathbb{R}_{>0}$, choose $\sqrt{-2\,q_i} := \sqrt{2\,q_i}\, i_1$.

Let Φ_i be the concatenation $\Phi_i := \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_1$.

Note that the isometries ϕ_i carry the two geodesics $(0, \infty)$ and $(-q_i, q_i)$ into the geodesics $(-1, 1)$ and $(0, \infty)$ of a double bridge in the aforementioned setting. However, these isometries are not uniquely defined by this property. We will always apply these ϕ_i if we need an isometry which maps given geodesics to specific other geodesics in a polygon Π_p.

The Clifford matrix corresponding to ϕ_i is

$$\begin{pmatrix} \sqrt{-2\,q_i}^{-1} & q_i \sqrt{-2\,q_i}^{-1} \\ \sqrt{-2\,q_i}^{-1} & -q_i \sqrt{-2\,q_i}^{-1} \end{pmatrix}. \tag{17}$$

The inverse $\phi_i^{-1}(x) = \sqrt{-q_i}\,(1 + x)(1 - x)^{-1} \sqrt{-q_i}$ is represented by the matrix

$$\begin{pmatrix} q_i \sqrt{-2\,q_i}^{-1} & q_i \sqrt{-2\,q_i}^{-1} \\ \sqrt{-2\,q_i}^{-1} & -\sqrt{-2\,q_i}^{-1} \end{pmatrix}. \tag{18}$$

Repeatedly applying these isometries to geodesics in a Π_p enables us to standardise the cross ratio of a double bridge in a p-gon and eliminate the problem of the cross ratio not being invariant under isometries (Fig. 3).

3.2 The theorem

Definition 7 Let $(S_0, \ldots, S_{p-1})$ be a right-angled p-gon. Define the *gauged double bridge cross ratios* $\tilde{\Delta}_i$ for $i = 1, \ldots, p - 3$ by the following recursive definition:

$$\tilde{\Delta}_1 := \Delta(S_0, S_1, S_2), \tag{19}$$

$$\tilde{\Delta}_{i+1} := \Delta(\Phi_i(S_i), \Phi_i(S_{i+1}), \Phi_i(S_{i+2})). \tag{20}$$

The Clifford vectors q_i which are needed to define the maps Φ_i are calculated along the way as

$$q_i = \tilde{\Delta}_i. \tag{21}$$

These gauged double bridge cross ratios will be the parameters describing the non-degenerate right-angled p-gons in $\boldsymbol{H}^{p-1}$ in the Theorem 1 below. Hence consider the set

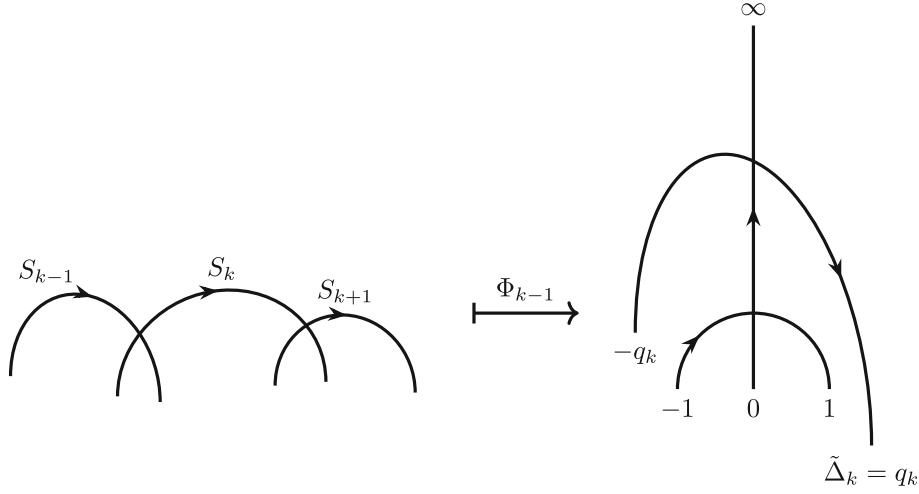$$\mathcal{P}_p := \{(q_1, \ldots, q_{p-3}) \mid q_i \in \mathbb{V}^{p-2}, |q_i| > 1, 1 \le i \le p - 3\} \tag{22}$$

Fig. 3 Gauging by an isometry

of $(p-3)$-tuples of non-zero Clifford vectors. Denote by

$$\text{RAP}_p := \Big\{ \left(S_0, \ldots, S_{p-1}\right) \text{ non-degenerate right-angled polygon in } \boldsymbol{H}^{p-1}$$

$$\text{with } S_0 = (-1, 1)\,, \ S_1 = (0, \infty) \Big\} \tag{23}$$

the set of non-degenerate right-angled polygons with p sides. The calculation of the gauged double bridge cross ratios gives a map $\tilde{\Delta} : \text{RAP}_p \to \mathcal{P}_p$. Denote the image of this map by $\mathcal{P}_p^* := \tilde{\Delta}\left(\text{RAP}_p\right) \subset \mathcal{P}_p$. This is the set of parameters which will yield a non-degenerate Π_p in the construction below.

Theorem 1 *The map $\tilde{\Delta} : \text{RAP}_p \to \mathcal{P}_p^*$ is a bijection. The inverse map can be given as an explicit construction of a right-angled p-gon Π_p from a tuple of $p - 3$ parameters in $\mathcal{P}_p^*$.*

3.3 Proof of Theorem 1

Bijectivity It is sufficient to prove the injectivity of $\tilde{\Delta}$ since it is surjective by definition. Note that in the standard configuration double bridge of Fig. 2, there is a one-to-one correspondence of Clifford vectors x and geodesics $t = (-x, x)$ as given by Eq. (15). Now assume there are two p-gons $\Pi_p = (S_0, \ldots, S_{p-1})$, $\Pi_p' = (S_0', \ldots, S_{p-1}') \in \text{RAP}_p$ such that $\tilde{\Delta}\left(\Pi_p\right) = \tilde{\Delta}\left(\Pi_p'\right) = \left(q_1, \ldots, q_{p-3}\right)$. By definition $S_0 = S_0'$ and $S_1 = S_1'$. By the above correspondence we also have $S_2 = S_2'$. Furthermore the maps $\phi_1, \ldots, \phi_{p-3}$ are the same for both Π_p and Π_p' since these maps are defined by $q_1, \ldots, q_{p-3}$ as given in Eq. (16). Therefore the map Φ_i yields the same one-to-one correspondence between geodesics and Clifford vectors in both p-gons.

Construction of the polygon Π_p The inverse map $\tilde{\Delta}^{-1}$ is given by the construction of a Π_p from $p - 3$ parameters $q_1, \ldots, q_{p-3} \in \mathbb{V}^{p-2}$.

Assume we are given $p - 3$ parameters $(q_1, \ldots, q_{p-3}) \in \mathcal{P}_p^*$.

Start The first two geodesics are fixed as $S_0 = (-1, 1)$ and $S_1 = (0, \infty)$. Since this is the standard configuration double bridge considered above, we find $S_2 = (-q_1, q_1)$ if we demand $\Delta(S_0, S_1, S_2) = q_1$.

The geodesic S_3 To find the endpoints of S_3, we benefit from the isometry ϕ_1^{-1} above which maps $(-1, 1)$ to $(0, \infty)$ and $(0, \infty)$ to S_2. If q_2 was the cross ratio of a double bridge involving $(-1, 1)$ and $(0, \infty)$, the third geodesic would be $(-q_2, q_2)$. Since S_3 is part of the double bridge starting with $(0, \infty)$ and S_2, S_3 can be found by applying ϕ_1^{-1} to $(-q_2, q_2)$, that is $S_3 = (\phi_1^{-1}(-q_2), \phi_1^{-1}(q_2))$.

The next geodesic in the general case The further procedure expands the previous idea. First we note that the next geodesic is given by the parameter q_3. The geodesic S_4 would then be the image of $(-q_3, q_3)$ under the isometry Φ_2^{-1} mapping $(-1, 1)$ and $(0, \infty)$ to S_2 and S_3, respectively.

In general, assuming we have calculated the geodesics $S_0, \ldots, S_k$ for some k with $2 \leq k \leq p - 3$, we can use Φ_{k-1}^{-1} in order to obtain $S_{k+1} = (\Phi_{k-1}^{-1}(-q_k), \Phi_{k-1}^{-1}(q_k))$.

Existence of the last geodesic After using all the parameters $q_1, \ldots, q_{p-3}$, we have determined the geodesics $S_0, \ldots, S_{p-2}$. As a consequence of Proposition 2 the last common perpendicular between S_0 and S_{p-2} exists and is unique as long as

$$\Delta\left(S_0, S_{p-2}\right) \notin \mathbb{R}_-. \tag{24}$$

This is ensured by the set $\mathcal{P}^* \subset \mathcal{P}$. □

Remark 3 Since the Clifford vectors do not commute, one cannot directly compute the common perpendicular S_{p-1} using the equations

$$\Delta\left(S_{p-1}, S_0\right) = -1, \quad \Delta\left(S_{p-1}, S_{p-2}\right) = -1. \tag{25}$$

However, one can use an isometry to obtain a nice configuration where the terms in the equations above commute. Writing $S_{p-2} = (a, b)$, consider the isometry

$$\psi : x \mapsto \alpha^{-1}(1 + x)(1 - x)^{-1}\alpha^{-1} \tag{26}$$

where $\alpha := \sqrt{-(1 + a)(1 - a)^{-1}}$. This isometry maps S_0 to $(0, \infty)$ and S_{p-2} to $(1, c)$ where $c := \alpha^{-1}(1 + b)(1 - b)^{-1}\alpha^{-1}$.

Hence, by Proposition 3

$$S_{p-1} = \left(\psi^{-1}\left(-\sqrt{c}\right), \psi^{-1}\left(\sqrt{c}\right)\right) \tag{27}$$

modulo orientation where ψ^{-1} is given by

$$\psi^{-1}(x) = (\alpha x \alpha - 1)(\alpha x \alpha + 1)^{-1}. \tag{28}$$

Remark 4 A major drawback is that we cannot explicitly describe $\mathcal{P}_p^*$. One can take a set of parameters $(q_1, \ldots, q_{p-3}) \in \mathcal{P}_p$, apply the above construction and afterwards check whether the created object actually is a non-degenerate right-angled p-gon.

If all the parameters q_i have norm $|q_i| > 1$ the proper orientation of the geodesics $S_1, \ldots, S_{p-3}$ is automatically guaranteed. So one needs to check the orientation of S_0, S_{p-2} and S_{p-1}. This can be done by calculating the norm of the double bridge cross ratios with the respective geodesic as the central one. Since the norm of the cross ratio is invariant under isometry we do not have to use the gauged double bridge cross ratios at this point. If the

orientation of S_{p-1} is wrong, one can just invert it. If the orientation of S_{p-2} is wrong, one needs to replace the parameter q_{p-3} by $-q_{p-3}$ and the construction yields the same Π_p just with the inverted orientation of S_{p-2}. If the orientation of S_0 is wrong, one can replace the parameter q_1 by $-q_1$. This introduces a factor i to the left and to the right of the map ϕ_1^{-1}, where i is a root of -1 in the plane spanned by 1 and q_1; respectively $i = i_1$ if q_1 is real. Such a map is a rotation of $180°$ in the plane spanned by 1 and i.

After some exemplary calculations, we conjecture that for $p = 5$ the set

$$\{(q_1, q_2) \in \mathcal{P}_5 \mid \Re(q_1) \neq 0, q_1 \not\perp q_2\} \tag{29}$$

yields non-degenerate right-angled 5-gons up to orientation.

4 Right-angled polygons with full span

One natural question which arises when studying right-angled polygons is the question of the dimension of the resulting object. In this section we consider right-angled p-gons which have the highest possible dimension. This is the case if the p intersection points are the vertices of a $(p-1)$-simplex. Thus the parameters will be taken from a $(p-2)$-dimensional Clifford vector space $\mathbb{V}^{p-2} \subset \mathcal{C}_{p-3}$.

4.1 A necessary condition for the realisation of $(p-1)$-simplices

If we want some set of parameters to yield a simplex, we need to pass to a new dimension with every new geodesic in the construction. This basic idea results in the following theorem:

Theorem 2 *If the parameters $q_1, \ldots, q_{p-3} \in \mathcal{C}_{p-3}$ give rise to a right-angled polygon Π_p whose intersection points are the vertices of a simplex, then the parameters together with 1 have to form a basis of the Clifford vectors according to $\langle 1, q_1, \ldots, q_{p-3} \rangle = \mathbb{V}^{p-2}$.*

This theorem is a consequence of the following lemma:

Lemma 2 *Let $(S_0, S_1, \ldots, S_k)$, $k \geq 2$ be a finite sequence of geodesics in upper-half space $\boldsymbol{H}^{p-1}$ such that $S_0 = (-1, 1)$, $S_1 = (0, \infty)$ and $S_{i-1} \perp S_i$ for $i = 1, \ldots, k$. Furthermore denote by $q_i := \tilde{\Delta}\,(S_{i-1}, S_i, S_{i+1})$ the gauged double bridge cross ratios of the respective double bridges for $i = 1, \ldots, k-1$ and write $S_i = (S_i^-, S_i^+)$ for all geodesics.*
Then the linear subspace of $\mathbb{V}^{p-2}$ spanned by the endpoints of the geodesics is the same as the subspace spanned by $\{1, q_0, q_1, \ldots, q_{k-1}\}$. In symbols this means

$$\langle S_0^\pm, S_2^\pm, S_3^\pm, \ldots, S_k^\pm \rangle = \langle 1, q_1, q_2, \ldots, q_{k-1} \rangle. \tag{30}$$

The geodesic S_1 is left out since $\infty \notin \mathbb{V}^{p-2}$.

Proof We prove this by induction over k. For $k = 2$ the lemma is plain, since $S_2 = (-q_1, q_1)$.

Hence, we have to prove $\langle 1, q_1, q_2, \ldots, q_{k-1}, S_{k+1}^\pm \rangle = \langle 1, q_1, q_2, \ldots, q_k \rangle$. We know that S_{k+1} is given as the image of $(-q_k, q_k)$ under the isometry Φ_{k-1}^{-1}. This isometry is given as a concatenation of the maps $\phi_i^{-1} : x \mapsto \sqrt{-q_i}\,(1 + x)(1 - x)^{-1}\sqrt{-q_i}$, $1 \leq i \leq k - 1$. If $q_i \notin \mathbb{R}$, ϕ_i^{-1} restricts to an isometry on $\boldsymbol{H}^3$ where the boundary is given as $\partial \boldsymbol{H}^3 = \langle 1, q_i \rangle \cup \{\infty\}$. Likewise, ϕ_i restricts to an isometry on $\boldsymbol{H}^4$ where the boundary is given as $\partial \boldsymbol{H}^4 = \langle 1, q_i, q_k \rangle \cup \{\infty\}$. The case $q_i \in \mathbb{R}$ follows in the same manner, by yielding isometries leaving corresponding subspaces $\boldsymbol{H}^2$ and $\boldsymbol{H}^3$ invariant. Thus follows the statement. □

Fig. 4 Hyperbolic pentagon with right-angled cyclic edge path
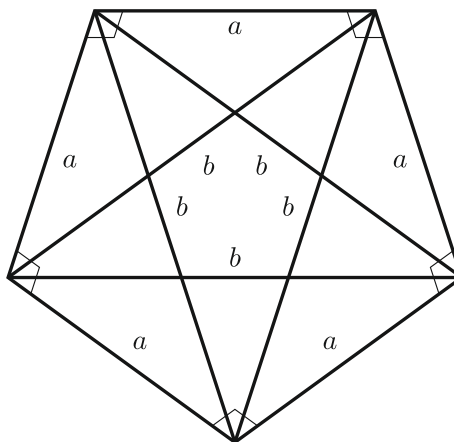
Notice that the theorem above does not give a sufficient condition. If the parameters q_i are pairwise orthogonal to each other and pure Clifford vectors then the geodesics S_0 and S_{p-2} will contribute sides of length 0.

4.2 Hyperbolic 4-simplices with an orthogonal cyclic edge path

In the end, it would be nice to have an a priori condition on the parameters of at least some family of pentagons. Dekster and Wilker [6] proved a criterion for the existence of n-simplices with vertices $p_1, \ldots, p_{n+1}$ with given side and diagonal lengths $l_{ij} = \mathrm{d}\left(p_i, p_j\right)$, $1 \leq i < j \leq n + 1$ in a Euclidean, spherical or hyperbolic space $X \in \left\{ \boldsymbol{E}^n, \boldsymbol{S}^n, \boldsymbol{H}^n \right\}$. They call a symmetric $(n + 1) \times (n + 1)$-matrix $L = (l_{ij})$ *allowable* if $l_{ii} = 0$ and $l_{ij} > 0$ for $i \neq j$. The matrix L is called *realisable* in the space X if there are $n + 1$ points $p_1, \ldots, p_{n+1}$ in X with the given distances $\mathrm{d}\left(p_i, p_j\right) = l_{ij}$. They gave a criterion for realisability in each of the three above cases. We are especially interested in the hyperbolic case.

Theorem 3 [6, Theorem 1 (hyperbolic case)] *Let $L = (l_{ij})$ be an allowable $(n+1) \times (n+1)$-matrix and let its entries be used to form the $(n \times n)$-matrix $S = (s_{ij})$ where*

$$s_{ij} = \cosh l_{i,n+1} \, \cosh l_{j,n+1} - \cosh l_{ij}.$$

Then L is realisable if and only if the eigenvalues of S are non-negative. If L is realisable then the dimension of each realisation is equal to the rank of S.

Now we can easily treat the case of a hyperbolic pentagon having a cyclic edge path along which all sides have the same length (Fig. 4). With [6] we can get a criterion on the side lengths and due to symmetry it might be possible to find the corresponding orientations of the sides.

Lemma 3 *A right-angled hyperbolic pentagon $\Pi_5 = (S_0, \ldots, S_4)$ with all side lengths equal to $a > 0$ is realisable as a 4-simplex if and only if $\cosh(a) < \gamma$, where $\gamma = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio.*

Proof By using hyperbolic trigonometry (see for example [12, Section 3.5]) we obtain the relation $\cosh(b) = \cosh^2(a)$. We can now construct the two matrices L and S as in [6, Theorem 1]. We get

$$L = \begin{pmatrix} 0 & a & b & b & a \\ a & 0 & a & b & b \\ b & a & 0 & a & b \\ b & b & a & 0 & a \\ a & b & b & a & 0 \end{pmatrix}.$$

Let us define $x := \cosh(a)$. We then have

$$S = \begin{pmatrix} x^2 - 1 & x^3 - x & x^3 - x^2 & 0 \\ x^3 - x & x^4 - 1 & x^4 - x & x^3 - x^2 \\ x^3 - x^2 & x^4 - x & x^4 - 1 & x^3 - x \\ 0 & x^3 - x^2 & x^3 - x & x^2 - 1 \end{pmatrix}.$$

By Dekster's and Wilker's Theorem, the matrix L is realisable as a 4-simplex if and only if all the eigenvalues of S are positive. This is true if and only if S is positive definite. By Sylvester's criterion, it is enough to check that all the top left minors of S have positive determinant:

$$\det_1 = x^2 - 1,$$
$$\det_2 = x^4 - 2x^2 + 1 = (x^2 - 1)^2 = (x + 1)^2(x - 1)^2,$$
$$\det_3 = -x^8 + 2x^7 + x^6 - 2x^5 - 2x^4 + 3x^2 - 1,$$
$$\det_4 = \det(S) = 2x^{10} - 10x^9 + 15x^8 - 15x^6 + 2x^5 + 10x^4 - 5x^2 + 1.$$

Notice that $x > 1$ since a must be greater than 0. Hence $\det_1$ and $\det_2$ are always greater than 0. Furthermore, $\det_3$ is positive whenever $-1 < x < \frac{1-\sqrt{5}}{2}$ or $1 < x < \frac{1+\sqrt{5}}{2}$, hence only the latter has to be considered. The determinant of S is positive everywhere except in $\frac{1-\sqrt{5}}{2}$, 1, $\frac{1+\sqrt{5}}{2}$, where it vanishes. Combining everything we obtain that S is positive definite whenever $1 < x < \gamma$, giving us the desired result. □

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