DTU Library

A periodic H2 state feedback controller for a rotor-blade system

Camino, J. F.; Santos, I. F.

Published in:
Proceedings of the International Conference on Noise and Vibration Engineering

Publication date:
2018

Document Version
Peer reviewed version

Link back to DTU Orbit

A periodic $\mathcal{H}_2$ state feedback controller for a rotor-blade system

J. F. Camino[1] **and I. F. Santos**[2]

[1] University of Campinas
School of Mechanical Engineering
13083-860, Campinas, SP, Brazil
e-mail: **camino@fem.unicamp.br**

[2] Technical University of Denmark
Department of Mechanical Engineering
DK-2800, Kgs. Lyngby, Denmark

Abstract

Many engineering applications have as their main component rotor-blade systems whose dynamics can be represented by a linear time-varying model. Since rotor-blade systems exhibit periodic dynamics, standard linear time-invariant analysis and synthesis techniques cannot be directly used and are not able to guarantee closed-loop stability and performance. Although there exist many results for periodic systems, the design of controllers for such systems is, in general, a difficult task. Its practical application is challenging, from the computational and experimental aspects. This paper presents the application of a periodic $\mathcal{H}_2$ control problem in a rotor-blade system in order to attenuate the tip vibration. The proposed control design is based on a periodic Riccati differential equation (PRDE). The Floquet-Lyapunov theory is used to represent the dynamics in an adequate coordinate system, so that the PRDE can be efficiently solved. A robustness analysis is also perfomed. Numerical experiments show the effectiveness of the proposed approach.

1 Introduction

Linear time-varying (LTV) differential equations are widely used to model natural phenomena, engineering systems, and many other situations. For instance, time-varying models have been used to describe the dynamic behavior of an idealized human operator in [35, 48] and to estimate the stiffness of human elbow joint during cyclic voluntary movement in [6]. In the field of economics, the dynamics of the yield curve and key macroeconomic variables can be modeled through linear time-varying models [7]. In the rendezvous problem for satellites in elliptic orbits [40], the relative motion of a deputy satellite with respect to the chief satellite can be described by a time-varying linearized equation of motion.

In a trajectory tracking problem for mechanical systems [36], it is common to use a generator to provide an off-line open-loop optimal trajectory, basically composed of an ideal system state and control input, which satisfies the nonlinear system dynamics and ensures that the system output follows a given reference. Then, considering that the design objective is to keep the actual plant state near its ideal value, under small perturbations, a linear time-varying model can be used for control design [4]. In [44], an LTV model has been used in the steps to obtain a flatness-based open-loop control law. A sequence of linear time-varying approximations is used in [2, 16] to solve a nonlinear optimal control problem.

Periodic systems are a particular class of LTV systems that appear frequently in engineering applications, since machinery with rotating parts is found in many mechanical systems, such as vehicles, airplanes, ships

and wind-turbines, among others [20, 37]. Active control of rotor-blade systems has been an attractive approach to improve performance, suppress vibrations, and prolong machinery lifetime [23, 32, 46]. However, due to the periodic time-varying nature of the problem, applications of classical LTI analysis and synthesis techniques are not suitable.

One of the most important results available to analyze periodic systems is known as Floquet-Lyapunov decomposition. It provides a coordinate change that reduces a homogeneous periodic system to a linear time-invariant system. This decomposition was used in [24] to analyze helicopter rotor dynamics, in [47] to design a periodic state feedback controller for a mistuned bladed disk, and in [41] to design a periodic observer-based controller for mechanical systems. It was shown in [15] that the vibration of rotor-blade systems can be attenuated using control forces acting on the hub. The presence of parametric vibration effects due to vibration coupling between rotor and blades was analyzed in [14]. A periodic LQR controller was proposed in [28] with guaranteed performance and closed-loop stability, and a discrete-time periodic $\mathcal{H}_2$ controller was proposed in [10] for vibration reduction using actively twisted blades.

The $\mathcal{H}_2$ norm has been extensively used to quantify system performance as an objective function for a large class of optimal control problems [39, 45]. The $\mathcal{H}_2$ control problem consists of stabilizing the closed-loop system while minimizing its $\mathcal{H}_2$ norm, from the disturbance input to the performance output [13, 39, 45]. The notion of the $\mathcal{H}_2$ norm and synthesis conditions for the $\mathcal{H}_2$ state feedback control problem for continuous-time periodic systems are found in [17, 18]. Algorithms to solve the underlying set of periodic time-varying Lyapunov and Riccati equations [1, 8, 9], which frequently appear in control and systems theory, have been extensively investigated in the last decades [19, 26, 30, 31].

In this paper, an $\mathcal{H}_2$ state feedback controller is designed to attenuate the tip vibration of a rotor-blade system. The periodic time-varying control design is formulated in terms of a matrix Riccati differential equation. Although the design guarantees stability and performance, an analysis of the robustness of the $\mathcal{H}_2$ controller with respect to stability is also provided.

2   Control design

This section presents the proposed periodic $\mathcal{H}_2$ optimal control design, used to improve the performance and robustness of the rotor-blade system.

To access the best achievable performance, it is assumed henceforth that all the states are available for feedback and, thus, the controller is a full state feedback gain. In case not all states are available for feedback, a periodic observer can be designed, since the separation principle still holds [4, 50], with possible decrease in performance and robustness [5, 21].

Consider the following linear time-varying system

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) \\
z(t) &= C_z(t)x(t) + D_{zu}(t)u(t) + D_{zw}(t)w(t)
\end{aligned}
\tag{1}
$$

where all the matrices have compatible dimensions, are continuous functions of time, and bounded. Then, the closed-loop system using the state feedback law

$$
u(t) = K(t)x(t)
$$

is given by

$$
\begin{aligned}
\dot{x}(t) &= A_{cl}(t)x(t) + B_{cl}(t)w(t) \\
z(t) &= C_{cl}(t)x(t) + D_{cl}(t)w(t)
\end{aligned}
\tag{2}
$$

with

$$
A_{cl}(t) = A(t) + B_u(t)K(t), \quad B_{cl}(t) = B_w(t), \quad C_{cl}(t) = C_z(t) + D_{zu}(t)K(t), \quad D_{cl}(t) = D_{zw}(t)
$$

Before presenting the control design, some well-known results are introduced in the next section. For a detailed exposition, see the comprehensive sources [8, 11, 52] and references therein.

2.1 Preliminary facts

Consider the following linear time-varying system H given by

$$H := \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)v(t) \\ y(t) = C(t)x(t) + D(t)v(t) \end{cases} \qquad (3)$$

where all the matrices have compatible dimensions, are continuous functions of time, and bounded. The vectors $y(t)$ and $v(t)$ are the output and the input signals, respectively.

A convenient notion of stabilizability and detectability is given in the next definition [38].

Definition 1. *System* (3) *is said to be stabilizable if there exists a bounded matrix function $K(t)$ such that $\dot{x}(t) = (A(t) - B(t)K(t))x(t)$ is exponentially stable. System* (3) *is said to be detectable, if there exists a bounded matrix function $L(t)$ such that $\dot{x}(t) = (A(t) - L(t)C(t))x(t)$ is exponentially stable.*

The next Lemma 2 provides a classical version of the Lyapunov stability theorem for continuous-time linear time-varying systems [38], which was first proved for the discrete-time case in [3]. A version for periodic systems can be found in [8].

Lemma 2. *Assume the pair $(A(t), B(t))$ is stabilizable and the pair $(A(t), C(t))$ is detectable. Then, either of the following statements is a sufficient condition for system* (3) *to be exponentially stable:*

1. *There exists a bounded nonnegative definite matrix solution $L_c(t)$ of the equation*

$$\dot{L}_c(t) = L_c(t)A'(t) + A(t)L_c(t) + B(t)B'(t)$$

2. *There exists a bounded nonnegative definite matrix solution $L_o(t)$ of the equation*

$$-\dot{L}_o(t) = A'(t)L_o(t) + L_o(t)A(t) + C'(t)C(t)$$

In system theory, mainly in optimal control problems [29, 49], the Riccati equation plays an important role [1, 9, 33]. The next Lemma 3 states the conditions for the solution of the Riccati differential equation to be unique and stabilizing [38].

Lemma 3. *Assume the pair $(A(t), B(t))$ is stabilizable and the pair $(A(t), C(t))$ is detectable. Then, there exists a unique bounded solution $X(t) \geq 0$ of the following Riccati equation*

$$0 = \dot{X}(t) + A'(t)X(t) + X(t)A(t) - X(t)B(t)B'(t)X(t) + C'(t)C(t)$$

Furthermore, the system $\dot{x}(t) = (A(t) - B(t)B'(t)X(t))x(t)$ is exponentially stable.

The next Lemma 4 provides another important result, called Floquet-Lyapunov decomposition [11, 22, 52], which can be used to transform the homogeneous T-periodic system

$$\dot{x}(t) = A(t)x(t), \qquad A(t + T) = A(t) \qquad (4)$$

into a linear time-invariant system.

Lemma 4. *Consider the T-periodic linear system* (4). *Let the associated transition matrix be denoted by $\Phi_A(t, t_0)$. Define a matrix function $P(t)$ and a constant matrix W via the equations*

$$P(t) = e^{Wt}\Phi_A(0, t) \qquad and \qquad e^{WT} = \Phi_A(T, 0) \qquad (5)$$

Then, the transition matrix $\Phi_A(t, t_0)$ can be written as

$$\Phi_A(t, t_0) = P^{-1}(t)e^{W(t-t_0)}P(t_0)$$

Moreover, system (4) *is reducible in the sense of Lyapunov and its origin $x = 0$ is an exponentially stable equilibrium if, and only if, W is Hurwitz.*

From this previous result, it becomes clear that the class of linear periodic systems is reducible in the sense of Lyapunov [22]. Furthermore, the change of variable $\eta(t) = P(t)x(t)$ transforms (4) into the linear time-invariant system

$$\dot{\eta}(t) = W\eta(t) \tag{6}$$

To see this fact, observe that the system dynamics in the η-coordinate is given by

$$\dot{\eta} = \hat{A}(t)\eta(t)$$

with

$$\hat{A}(t) = P(t)A(t)P^{-1}(t) + \dot{P}(t)P^{-1}(t)$$

whose transition matrix can be shown to be

$$\Phi_{\hat{A}}(t,t_0) = P(t)\Phi_A(t,t_0)P^{-1}(t_0) = e^{W(t-t_0)}$$

which is exactly the transition matrix of system (6).

The monodromy matrix Ψ_A, which is usually defined as the transition matrix computed over one period, i.e. $\Psi_A = \Phi_A(T,0)$, is a crucial element for the stability analysis of periodic systems. The characteristic roots of $e^{WT} = \Psi_A$ are called the *multipliers associated with* $A(t)$ and the characteristic roots of W are called *characteristic exponents*. As a direct consequence of Lemma 4, the stability of (4) is characterized by the location of these roots.

Using the Floquet-Lyapunov transformation above, the periodic system (3) becomes

$$\begin{aligned} \dot{\eta}(t) &= \hat{A}\eta(t) + \hat{B}(t)v(t) \\ y(t) &= \hat{C}(t)\eta(t) + \hat{D}(t)v(t) \end{aligned} \tag{7}$$

with

$$\hat{A} = W, \qquad \hat{B}(t) = P(t)B(t), \qquad \hat{C}(t) = C(t)P^{-1}(t), \qquad \hat{D}(t) = D(t)$$

Notice that the system matrix $A(t)$ is transformed into a constant matrix W. However, all the other matrices still remain time-varying. In this new coordinate, it is possible to apply an extra similarity transformation to convert W, for instance, to a real block diagonal form.

A procedure for computing the Floquet-Lyapunov transformation using Chebyshev polynomials is presented in [42]. Another procedure that uses a Fourier series expansion to transform a periodic system into an LTI system in the modal form is provided in [51] and applied in [15]. Modal analysis techniques for linear time-varying systems can be found in [12, 27, 34].

2.2 Periodic state feedback $\mathcal{H}_2$ synthesis

The $\mathcal{H}_2$ norm for linear time-invariant (LTI) systems can be computed from the controllability or the observability gramians [43, 53]. The computation of these norms is readily performed by solving the corresponding algebraic Lyapunov equation. For linear time-varying (LTV) systems, there exist distinct characterizations of the $\mathcal{H}_2$ norm, which are straightforward extensions of the LTI case, but, in general, they are not equivalent [45]. As shown in the next Lemma 5, the $\mathcal{H}_2$ norm for a T-periodic system can be defined in a similar way as the LTI case, using the controllability and the observability gramians [17].

Lemma 5. *Consider system H given by (3) with $D(t) = 0$. Assume the T-periodic system matrix $A(t)$ is exponentially stable. Let the controllability gramian $L_c(t)$ and the observability gramian $L_o(t)$ be bounded solutions of the following equations*

$$\begin{aligned} \dot{L}_c(t) &= L_c(t)A'(t) + A(t)L_c(t) + B(t)B'(t) \\ -\dot{L}_o(t) &= A'(t)L_o(t) + L_o(t)A(t) + C'(t)C(t) \end{aligned}$$

Then, the $\mathcal{H}_2$ norm of (3) *is given by*

$$\|H\|_2^2 = \frac{1}{T} \int_0^T \text{Tr}\left[C(\tau)L_c(\tau)C'(\tau)\right] d\tau = \frac{1}{T} \int_0^T \text{Tr}\left[B'(\tau)L_o(\tau)B(\tau)\right] d\tau$$

The proposed $\mathcal{H}_2$ optimal control problem consists in finding a state feedback gain $K(t)$ such that the $\mathcal{H}_2$ norm of the closed-loop system is minimized. The next Lemma 6 provides, as shown in [18], a Riccati differential equation that characterizes the solution of the $\mathcal{H}_2$ state feedback control problem for system (1), with $D_{zw}(t) = 0$.

Lemma 6. *Consider system* (1) *with $D_{zw}(t) = 0$. Assume that $(A(t), B_u(t))$ is stabilizable, $D_{zu}(t)$ is injective for each t, and that the periodic system $(A(t), B_u(t), C_z(t), D_{zu}(t))$ does not have invariant zeros in the unit circle. Then, the optimal solution of the $\mathcal{H}_2$ problem is*

$$K(t) = -R^{-1}(t)\left(B_u'(t)X(t) + D_{zu}'(t)C_z(t)\right)$$

where $R(t) = D_{zu}'(t)D_{zu}(t)$, and $X(t)$ is the stabilizing positive semidefinite solution of the periodic Riccati differential equation

$$\dot{X}(t) + A'(t)X(t) + X(t)A(t) + C_z'(t)C_z(t)$$
$$- \left(X(t)B_u(t) + C_z'(t)D_{zu}(t)\right)R^{-1}(t)\left(B_u'(t)X(t) + D_{zu}'(t)C_z(t)\right) = 0$$

There exist appropriate techniques that can be used to solve the periodic Riccati differential equation above [19, 26, 30, 31].

2.3   Robustness analysis

It is interesting to verify how robust the stability of the closed-loop system is, using the proposed periodic $\mathcal{H}_2$ state feedback controller, when the system model is uncertain. For this analysis, different approaches can be used.

The necessary and sufficient conditions for asymptotic stability of continuous-time linear time-varying systems and, in particular, for periodic systems presented in [25], are explored. These conditions provide a function $\bar{\rho}(t)$ that bounds all the system trajectories, for a prescribed set of initial conditions, as stated in the next Lemma 7 from [25].

Lemma 7. *For a given $\rho_0 > 0$, let $\bar{\rho}(t) = \rho_0 \lambda_{\max}^{1/2}(X(t, t_0))$, where $X(t, t_0)$ is the solution of the following Lyapunov differential equation*

$$\frac{\partial}{\partial t}X(t, t_0) = A(t)X(t, t_0) + X(t, t_0)A'(t), \quad X(t_0, t_0) = I$$

and let

$$\rho_T = \max_{t \in [t_1, t_1+T]} \bar{\rho}(t), \quad t_1 \geq t_0$$

Then, the T-periodic system (4) *is asymptotically stable if, and only if,*

$$\rho_T < \infty \quad \text{and} \quad \bar{\rho}(t_1 + T) < \bar{\rho}(t_1), \quad \forall t_1 \geq t_0$$

Moreover, there always exists an initial condition $x_0 \in \Omega_0$, with $\Omega_0 = \{x \mid x'x \leq \rho_0^2, \rho_0 > 0\}$, such that the associated trajectory reaches the bound $\bar{\rho}(t)$, in the sense that $x'(t)x(t) = \bar{\rho}^2(t)$.

It is also possible to analyze the region of stability of the closed-loop system, using the computed $\mathcal{H}_2$ controller, as a function of the system's uncertainties. For this test, the stability is verified by computing the multipliers associated with the closed-loop systems, obtained by the feedback connection of each uncertain open-loop model with the $\mathcal{H}_2$ controller.

3 Rotor-blade dynamics

The two-dimensional rotor-blade system shown in Figure 1 consists of a rotor with four flexible blades, rotating in a suspended hub, with angular velocity Ω. The hub motion is described in the inertial (x, y)-coordinate system. The angular position of blade 1 is given by $\theta(t)$. A detailed derivation of the model is found in [14, 15].
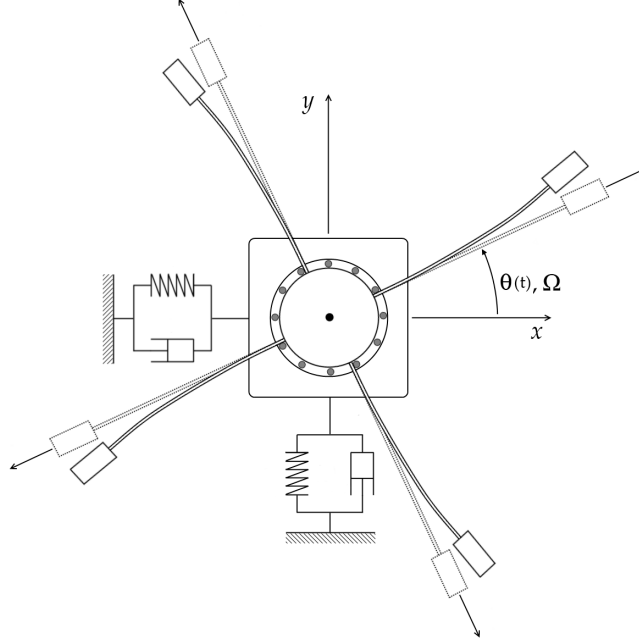

Figure 1: Two-dimensional model of the rotor-blade system.

Considering that the system rotates at the constant angular velocity $\Omega = 300$ rpm, the equation of motion is given by

$$M(t)\ddot{q}(t) + D(t)\dot{q}(t) + S(t)q(t) = Q_u u(t) + p(t) + w(t) \tag{8}$$

where $M(t)$ is the mass matrix, $D(t)$ is the damping matrix, $S(t)$ is the stiffness matrix, Q_u is the input matrix, $u(t)$ is the control input force, $p(t)$ is the periodic force due to the unbalance, and $w(t)$ is the exogenous disturbance acting on the system.

The system matrices, with period $T = 2\pi/\Omega = 0.2$ seconds, are given by

$$M(t) = \begin{bmatrix} 10.99 & 0 & -0.135S_0 & -0.135S_1 & -0.135S_2 & -0.135S_3 \\ 0 & 9.09 & 0.135C_0 & 0.135C_1 & 0.135C_2 & 0.135C_3 \\ -0.135S_0 & 0.135C_0 & 0.161 & 0 & 0 & 0 \\ -0.135S_1 & 0.135C_1 & 0 & 0.161 & 0 & 0 \\ -0.135S_2 & 0.135C_2 & 0 & 0 & 0.161 & 0 \\ -0.135S_3 & 0.135C_3 & 0 & 0 & 0 & 0.161 \end{bmatrix}$$

$$D(t) = \begin{bmatrix} 1.200 & 0 & -0.270\omega C_0 & -0.270\omega C_1 & -0.270\omega C_2 & -0.270\omega C_3 \\ 0 & 1.500 & -0.270\omega S_0 & -0.270\omega S_1 & -0.270\omega S_2 & -0.270\omega S_3 \\ 0 & 0 & 0.800 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.800 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.800 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.800 \end{bmatrix}$$

$$S(t) = \begin{bmatrix} 6.6 \times 10^4 & 0 & 0.135\omega^2 S_0 & 0.135\omega^2 S_1 & 0.135\omega^2 S_2 & 0.135\omega^2 S_3 \\ 0 & 7.7 \times 10^4 & -0.135\omega^2 C_0 & -0.135\omega^2 C_1 & -0.135\omega^2 C_2 & -0.135\omega^2 C_3 \\ 0 & 0 & \hat{S}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{S}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{S}_3 \end{bmatrix}$$

$$Q_u = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.299 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.299 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.299 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.299 \end{bmatrix}$$

where $S_k = \sin(\omega t + \pi k/2)$, $C_k = \cos(\omega t + \pi k/2)$ and $\hat{S}_k = 1961 + 0.102\omega^2 - 16.85 S_k$, with $\omega = 2\pi/T = 10\pi$ rad/s, and $k = 0, 1, 2, 3$.

The matrices $M(t)$, $D(t)$, $S(t)$, and Q_u above are obtained from the system matrices given in the appendix of [14] after removing the second bending mode of the blades, i.e., after removing the 4th, 6th, 8th, 10th lines and columns.

By defining the state space vector $x(t) = \begin{bmatrix} q'(t) & \dot{q}'(t) \end{bmatrix}'$, equation (8) can be written in following the state space form

$$\dot{x}(t) = A(t)x(t) + B_u(t)u(t) + B_w(t)w(t) + f(t) \tag{9}$$

where

$$A(t) = \begin{bmatrix} 0 & I \\ -M^{-1}(t)S(t) & -M^{-1}(t)D(t) \end{bmatrix},$$

$$B_u(t) = \begin{bmatrix} 0 \\ M^{-1}(t)Q_u \end{bmatrix}, \quad B_w(t) = \begin{bmatrix} 0 \\ M^{-1}(t) \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ M^{-1}(t) \end{bmatrix} p(t)$$

The displacement vector q has six generalized coordinates that account for the hub motion, in the (x, y)-directions, and the tip deflection of each one of the four blades. Thus, the system has 6 degrees of freedom and the state x has 12 entries.

The mechanical system has six actuators that can independently control all six degrees of freedom. Two actuators are located in the hub, providing forces in the (x, y)-directions, and the other four are located in each one of the blades.

As observed in [14], the force $p(t)$, due to unbalances, is usually known or well estimated when the system rotates at constant speed. Thus, it is possible to cancel out this effect, as performed in [28], using the following feedforward control law

$$u_f(t) = -\left(B_u'(t)B_u(t)\right)^{-1} B_u'(t)f(t)$$

The next section presents the numerical results in which the unbalance is neglected, since it is assumed that a preliminary feedforward control law can always be applied to cancel its effect.

4 Numerical results

For the numerical simulations, the matrices $A(t)$, $B_u(t)$, and $B_w(t)$ are provided in Section 3 and the matrices $C_z(t)$, $D_{zu}(t)$, and $D_{zw}(t)$ are given by

$$C_z(t) = 10 \begin{bmatrix} N \\ 0_{6,12} \end{bmatrix}, \quad N = \text{blkdiag}(0_2, I_4, 0_2, I_4), \quad D_{zu}(t) = \sqrt{0.1} \begin{bmatrix} 0_{12,6} \\ I_6 \end{bmatrix}, \quad D_{zw}(t) = 0$$

where $0_{n,m}$ and $I_{n,m}$ are the zero and the identity matrices of size $n \times m$, respectively, and the operation $\mathrm{blkdiag}(X_1, \ldots, X_n)$ denotes the block diagonal matrix concatenation of its arguments.

Using Lemma 4, a Floquet-Lyapunov decomposition can be computed to represent system (1) in a new coordinate $\eta(t)$, as shown in (7), so that the system dynamic matrix $A(t)$ is transformed into a constant matrix W. In the first step of this process, the monodromy matrix $\Phi_A(T, 0)$, generated by the open-loop matrix $A(t)$, is computed and, subsequently, matrix W and the matrix function $P(t)$ are determined using (5). Afterwards, a similarity transformation is used to represent W in a block diagonal form. The obtained matrix W is given by $W = \mathrm{blkdiag}(W_1, \ldots, W_6)$, with

$$W_1 = \begin{bmatrix} -0.5253 & 0.6891 \\ -0.6891 & -0.5253 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.5605 & 12.8302 \\ -12.8302 & -0.5605 \end{bmatrix}, \quad W_3 = \begin{bmatrix} -2.5467 & 9.2207 \\ -9.2207 & -2.5467 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} -1.6028 & 11.8287 \\ -11.8287 & -1.6028 \end{bmatrix}, \quad W_5 = \begin{bmatrix} -2.4885 & 12.5150 \\ -12.5150 & -2.4885 \end{bmatrix}, \quad W_6 = \begin{bmatrix} -2.4885 & 12.5148 \\ -12.5148 & -2.4885 \end{bmatrix}$$

The eigenvalues of W are given by: $-0.5253 \pm \mathrm{j}0.6891$, $-0.5605 \pm \mathrm{j}12.8302$, $-1.6028 \pm \mathrm{j}11.8287$, $-2.4885 \pm \mathrm{j}12.5150$, $-2.4885 \pm \mathrm{j}12.5148$, and $-2.5467 \pm \mathrm{j}9.2207$. Thus, the open-loop system is asymptotically stable, although lightly damped mainly due to the eigenvalues near the imaginary axis.

Using Lemma 6, the periodic $\mathcal{H}_2$ state feedback controller is designed by solving the associated periodic Riccati differential equation. The $\mathcal{H}_2$ norm of the open-loop system, computed using Lemma 5, is 39.5693. On the other hand, the $\mathcal{H}_2$ norm of the closed-loop system is 15.5883, which corresponds to a decrease of approximately 60%.

A time domain simulation is performed using a Gaussian white noise input $w(t)$ with zero mean and co-variance matrix equals to $100I$. The system is assumed to be at rest with initial condition $x_0 = x(0) = 0$. Figure 2 and Figure 3 show, respectively, the hub position in the x-direction and the blade 1 tip deflection, for the open-loop (dashed black line) and the closed-loop (solid red line) systems. The other blades have similar behavior as blade 1.
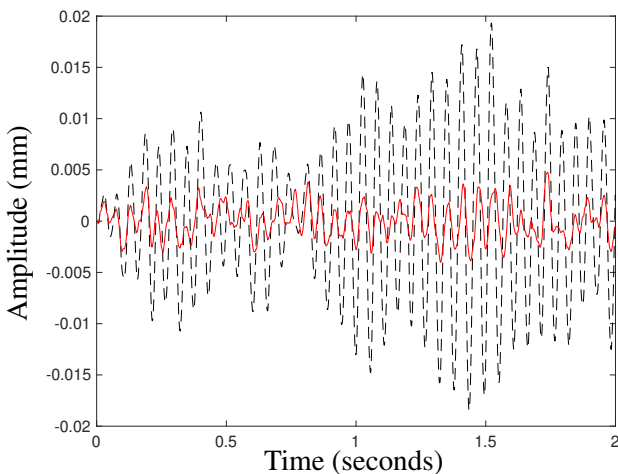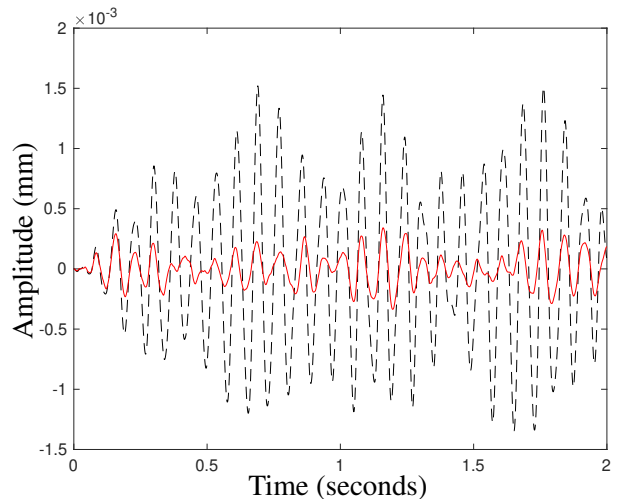

Figure 2: Blade 1 tip deflection.

Figure 3: Hub position in the x-direction.

The next Figure 4 shows the curve given by function $\bar{\rho}(t)$ (solid black line) from Lemma 7, proposed in [25], and the 2-norm of the solution $x(t)$ (dotted red line) of the closed-loop system, for a set of 100 randomly generated initial conditions satisfying $x_0' x_0 = \rho_0^2 = 1$. As it can be observed from this figure, the function $\bar{\rho}(t)$ bounds $\|x(t)\|_2 = (x'(t)x(t))^{1/2}$ for all system trajectories and, clearly, there must exist an initial condition x_0 such that the boundary $x'(t)x(t) = \bar{\rho}^2(t)$ is reached.

To verify how robust the closed-loop stability is under parameter variation, it is assumed that the mass matrix $M(t)$ varies in the range $\alpha M(t)$, for $\alpha \in [10^{-3}, 70]$, and the damping matrix $D(t)$ varies in the range $\beta D(s)$,
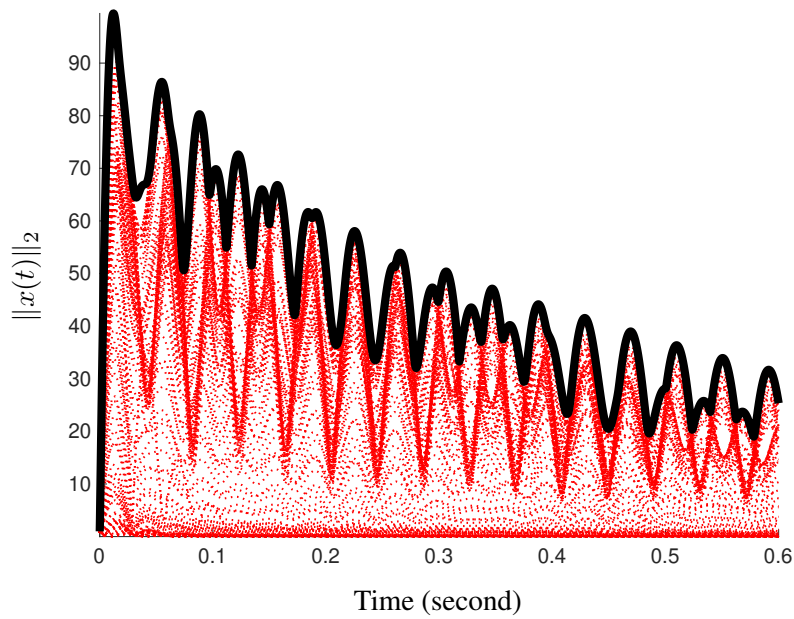
Figure 4: Function $\bar{\rho}(t)$ (solid black line) and $\|x(t)\|_2$ (dotted red line) for a set of 100 randomly generated initial conditions such that $x_0' x_0 = 1$.

for $\beta \in [0, 15]$. Since $M(t)$ needs to be invertible, the lower bound for α was chosen as 10^{-3}. The upper bounds for α and β were arbitrarily chosen. For a mesh of points in the $\alpha\beta$-region, the maximum absolute value γ of the 12 multipliers associated with each closed-loop system is computed.
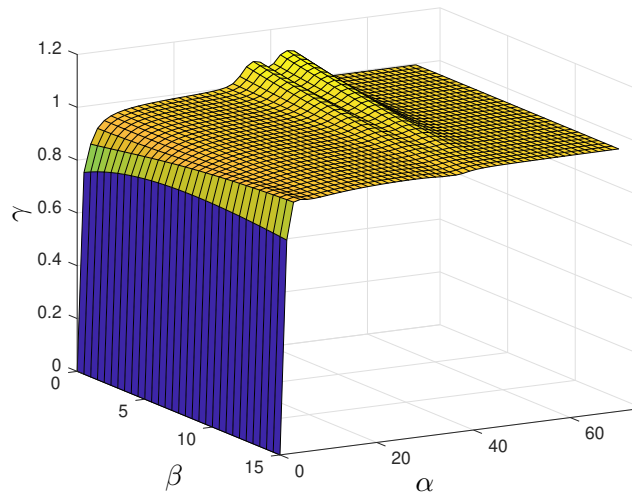

Figure 5: Stability surface defined by α, β and γ.

Figure 5 shows the plot of the parametric surface defined by α, β and γ. The closed-loop system is asymptotically stable if, and only if, $\gamma < 1$. Notice that for small values of α, which correspond to a mass matrix whose periodic entries have small amplitude, stability is guaranteed for any β. However, the system becomes unstable in the region where γ exceeds unity, which occurs in the vicinity of $\alpha = 40$ for a set of values of β. This fact is illustrated in Figure 6, which is a zoom of Figure 5, and in Figure 7, which shows the view of the $\alpha\beta$-plane for $\gamma \leq 1$. In Figure 7, the stable region is represented by the yellow mesh, and the unstable region otherwise.
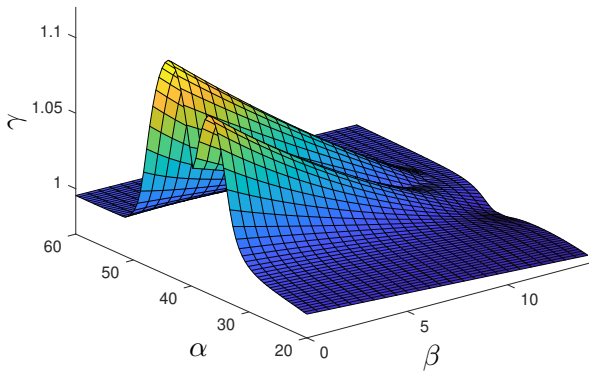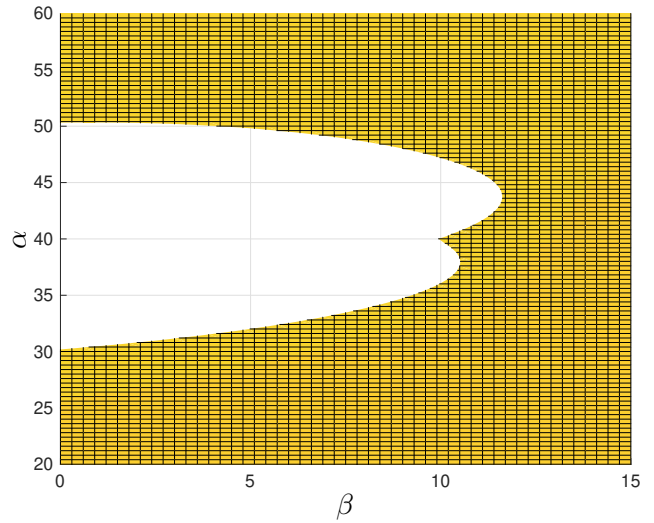
Figure 6: Zoom of the stability surface.

Figure 7: View of the $\alpha\beta$-plane for $\gamma \leq 1$.

5 Conclusion

In this paper, a periodic $\mathcal{H}_2$ state feedback controller was designed to attenuate the tip vibration of a two-dimensional rotor-blade system. The periodic dynamics of this system has 6 degrees of freedom: two related to the hub motion and one for each blade transverse deflection. The system is fully actuated. The unbalanced was neglected, since its effect can be canceled by a feedforward control law.

A Floquet-Lyapunov decomposition was applied to the system such that the system matrix $A(t)$ is reduced to a constant matrix W. A similarity transformation is also used to represent W in a block diagonal form with the characteristic exponent on the diagonal. Afterwards, the $\mathcal{H}_2$ state feedback controller is computed from the solution of a periodic Riccati differential equation.

The $\mathcal{H}_2$ norm of the open- and closed-loop systems are computed from the solution of a periodic Lyapunov differencial equation. It was readily verified that the periodic controller was capable of reducing significantly the vibration of the tip deflection, as was expected. The reduction measured in terms of the two-norm of closed-loop system was approximately 60%. However, different choice for the output matrices C_z and D_{zu} would lead to different performance and attenuation level.

An analysis of the robustness of the $\mathcal{H}_2$ controller was performed a posteriori. An envelop that bounds all the system trajectories, for a prescribed set of initial conditions, was computed. It was assumed that the mass matrix and the damping matrices were uncertain matrices inside a given region of uncertainty, and the stability of the closed-loop system for each uncertain model was analyzed. It was readily verified that the $\mathcal{H}_2$ controller is robust for most of the uncertain models, but there are some combinations of mass and stiffness matrices for which the closed-loop system will be unstable.

Acknowledgments

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank. *Matrix Riccati Equations in Control and Systems Theory*. Birkhäuser, Basel, Switzerland, 2003.

[2] Z. Aganovic and Z. Gajic. The successive approximation procedure for finite-time optimal control of bilinear systems. *IEEE Transactions on Automatic Control*, 39(9):1932–1935, Sept. 1994.

[3] B. D. O. Anderson and J. B. Moore. Detectability and stabilizability of time-varying discrete-time linear systems. *SIAM Journal on Control and Optimization*, 19(1):20–32, Jan. 1981.

[4] M. Athans. The role and use of the stochastic linear-quadratic-Gaussian problem in control system design. *IEEE Transactions on Automatic Control*, 16(6):529–552, 1971.

[5] M. Athans. A tutorial on the LQG/LTR method. In *American Control Conference*, pages 1289–1296, Athens, Greece, June 1986.

[6] D. J. Bennett, J. M. Hollerbach, Y. Xu, and I. W. Hunter. Time-varying stiffness of human elbow joint during cyclic voluntary movement. *Experimental Brain Research*, 88:433–442, 1992.

[7] F. Bianchi, H. Mumtaz, and P. Surico. The great moderation of the term structure of UK interest rates. *Journal of Monetary Economics*, 56(6):856–871, 2009.

[8] S. Bittanti, P. Colaneri, and G. D. Nicolao. The periodic Riccati equation. In Bittanti et al. [9], pages 127–162.

[9] S. Bittanti, A. J. Laub, and J. C. Willems, editors. *The Riccati Equation*. Springer-Verlag, New York, USA, 1991.

[10] C. Brillante, M. Morandini, and P. Mantegazza. Periodic controllers for vibration reduction using actively twisted blades. *The Aeronautical Journal*, 120(1233):1763–1784, Nov. 2016.

[11] R. W. Brockett. *Finite Dimensional Linear Systems*. John Wiley & Sons, New York, USA, 1970.

[12] I. Bucher and D. J. Ewins. Modal analysis and testing of rotating structures. *Philosophical Transactions of The Royal Society of London*, 359:61–96, Jan. 2001.

[13] B. M. Chen and A. Saberi. Necessary and sufficient conditions under which an H_2 optimal control problem has a unique solution. *International Journal of Control*, 58(2):337–348, 1993.

[14] R. H. Christensen. *Active Vibration Control of Rotor-Blade Systems, Theory and Experiment*. PhD thesis, Dept. of Mechanical Engineering, Technical University of Denmark, Kgs. Lyngby, Denmark, 2004.

[15] R. H. Christensen and I. F. Santos. Design of active controlled rotor-blade systems based on time-variant modal analysis. *Journal of Sound and Vibration*, 280(3):863–882, 2005.

[16] T. Çimen and S. P. Banks. Nonlinear optimal tracking control with application to super-tankers for autopilot design. *Automatica*, 40:1845–1863, 2004.

[17] P. Colaneri. Continuous-time periodic systems in H_2 and H_∞. Part I: Theoretical aspects. *Kybernetika*, 36(2): 211–242, 2000.

[18] P. Colaneri. Continuous-time periodic systems in H_2 and H_∞. Part II: State feedback problems. *Kybernetika*, 36 (3):329–350, 2000.

[19] L. Dieci. Numerical integration of the differential Riccati equation and some related issues. *SIAM J. Numer. Anal.*, 29:781–815, 1992.

[20] A. D. Dimarogonas, S. A. Paipetis, and T. G. Chondros. *Analytical Methods in Rotor Dynamics*. Elsevier, New York, USA, 1983.

[21] J. C. Doyle. Guaranteed margins for LQG regulators. *IEEE Transactions on Automatic Control*, 23(4):756–757, 1987.

[22] N. P. Erugin, editor. *Linear Systems of Ordinary Differential Equations, with Periodic and Quasi-Periodic Coefficients*, volume 28 of *Mathematics in Science and Engineering*. Academic Press, 1966.

[23] R. Firoozian and R. Stanway. Active vibration control of turbomachinery: A numerical investigation of modal controllers. *Mechanical Systems and Signal Processing*, 2(3):243–264, 1988.

[24] P. P. Friedmann. Numerical methods for determining the stability and response of periodic systems with applications to helicopter rotor dynamics and aeroelasticity. *Computers & Mathematics with Applications*, 12(1): 131–148, Jan. 1986.

[25] G. Garcia, P. L. D. Peres, and S. Tarbouriech. Necessary and sufficient numerical conditions for asymptotic stability of linear time-varying systems. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 5146–5151, Cancun, Mexico, Dec. 2008.

[26] J. J. Hench, C. S. Kenney, and A. J. Laub. Methods for the numerical integration of Hamiltonian systems. *Circuits, Systems, and Signal Processing*, 13(6):695–732, 1994.

[27] H. Irretier. Mathematical foundations of experimental modal analysis in rotor dynamics. *Mechanical Systems and Signal Processing*, 13(2):183–191, 1999.

[28] C. S. Jakobsen, J. F. Camino, and I. F. Santos. Rotor-blade vibration control using a periodic LQR controller. In *Proceedings of the 2013 SBAI/DINCON*, pages 1–07, Fortaleza, CE, Brazil, Oct. 2013.

[29] R. E. Kalman. Contribution to the theory of optimal control. In *Bol. Soc. Mat. Mexicana*, volume 5, pages 102–119, 1960.

[30] H. Kano and T. Nishimura. Periodic solutions of matrix Riccati equations with detectability and stabilizability. *International Journal of Control*, 29(3):471–487, 1979.

[31] C. S. Kenney and R. B. Lepnik. Numerical integration of the differential matrix Riccati equation. *IEEE Transactions on Automatic Control*, 30(10):962–970, Oct. 1985.

[32] Y. A. Khulief. Vibration suppression in rotating beams using active modal control. *Journal of Sound and Vibration*, 242(4):681–699, 2001.

[33] P. Lancaster and L. Rodman. *Algebraic Riccati Equations*. Oxford University Press, New York, USA, 1995.

[34] K. Liu. Extension of modal analysis to linear time-varying systems. *Schock and Vibration*, 226(1):149–167, 1999.

[35] D. T. McRuer and E. S. Krendel. Dynamic response of human operators. Technical report, WADC-TR-56-524, Oct. 1957.

[36] M. J. V. Nieuwstadt and R. M. Murray. Real-time trajectory generation for differentially flat systems. *International Journal of Robust and Nonlinear Control*, 8:995–1020, 1998.

[37] J. S. Rao. *Rotor Dynamics*. John Wiley & Sons, New York, USA, 1983.

[38] R. Ravi, A. M. Pascoal, and P. P. Khargonekar. Normalized coprime factorizations and the graph metric for linear time-varying systems. In *Proceedings of the 29th IEEE Conference on Decision and Control*, pages 1241–1246, Honolulu. Hawaii, Dec. 1990.

[39] M. Rotea. The generalized H_2 control problem. *Automatica*, 29(2):373–385, Mar. 1993.

[40] R. E. Sherrill, A. J. Sinclair, S. C. Sinha, and T. A. Lovell. Lyapunov-Floquet control of satellite relative motion in elliptic orbits. *IEEE Transactions on Aerospace and Electronic Systems*, 51(4):2800–2810, Oct. 2015.

[41] S. C. Sinha and P. Joseph. Control of general dynamic systems with periodically varying parameters via Liapunov-Floquet transformation. *ASME Journal of Dynamic Systems, Measurement, and Control*, 116(4):650–658, 1994.

[42] S. C. Sinha and D. H. Wu. An efficient computational scheme for the analysis of periodic systems. *Schock and Vibration*, 151(1):91–117, 1991.

[43] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, Inc., Bristol, USA, 1998.

[44] M. Staudecker, K. Schlacher, and R. Hansl. Passivity based control and time optimal trajectory planning of a single mast stacker crane. In *Proceedings of the 17th IFAC World Congress*, pages 875–880, Seoul, korea, July 2008.

[45] A. A. Stoorvogel. The robust $\mathcal{H}_2$ control problem: A worst-case design. *IEEE Transactions on Automatic Control*, 38(9):1358–1370, 1993.

[46] G. Szász and G. T. Flowers. Vibration suppression in bladed-disk assemblies with deliberate mistuning via magnetic bearings. *Journal of Vibration Control*, 6(6):903–921, 2000.

[47] G. Szász and G. T. Flowers. Time periodic control of a bladed disk assembly using shaft based actuation. *ASME Journal of Vibration and Acoustics*, 123(3):395–398, 2001.

[48] W. W. Wierwille. A theory for optimal deterministic characterization of time-varying human operator dynamics. *IEEE Transactions on Human Factors in Electronics*, 6(1):53–61, Sept. 1965.

[49] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 16(6):621–634, Dec. 1971.

[50] J. C. Willems and S. K. Mitter. Controllability, observability, pole allocation, and state reconstruction. *IEEE Transactions on Automatic Control*, 16(6):582–595, 1971.

[51] J. Xu and R. Gasch. Modale behandlung linearer periodisch zeitvarianter bewegungsgleichungen. *Archive of Applied Mechanics*, 65(3):178–193, 1995.

[52] V. A. Yakubovich and V. M. Starzhinskii. *Linear differential equations with periodic coefficients*, volume 1–2. John Wiley & Sons, New York, USA, 1975.

[53] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, USA, 1996.