Bunched Logics
A Uniform Approach

Simon Robert Docherty

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
of
University College London.

Department of Computer Science
University College London

May 2, 2019

I, Simon Robert Docherty, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Bunched logics have found themselves to be key tools in modern computer science, in particular through the industrial-level program verification formalism Separation Logic. Despite this—and in contrast to adjacent families of logics like modal and substructural logic—there is a lack of uniform methodology in their study, leaving many evident variants uninvestigated and many open problems unresolved.

In this thesis we investigate the family of bunched logics—including previously unexplored intuitionistic variants—through two uniform frameworks. The first is a system of duality theorems that relate the algebraic and Kripke-style interpretations of the logics; the second, a modular framework of tableaux calculi that are sound and complete for both the core logics themselves, as well as many classes of bunched logic model important for applications in program verification and systems modelling. In doing so we are able to resolve a number of open problems in the literature, including soundness and completeness theorems for intuitionistic variants of bunched logics, classes of Separation Logic models and layered graph models; decidability of layered graph logics; a characterisation theorem for the classes of bunched logic model definable by bunched logic formulae; and the failure of Craig interpolation for principal bunched logics. We also extend our duality theorems to the categorical structures suitable for interpreting predicate versions of the logics, in particular hyperdoctrinal structures used frequently in Separation Logic.

Impact Statement

As a work of formal logic this thesis' impact is primarily academic. We identify a number of different logic communities to which we believe this research will be of benefit.

- For the mathematical logic community, this work resolves a number of open problems in the bunched logic literature and situates this branch of logic in a mathematically substantial framework.

- For the proof theory community, this work yields new insights about the construction of tableaux calculi—in particular, the identification of tableaux systems with theories of coherent logic—that will undoubtably prove useful for automated reasoning, the construction of new systems and our mathematical and philosophical understanding of the tableau method.

- For the computational logic community, this work significantly broadens the understanding of 'resource semantics', a central notion in modern program verification. The new logics and the general mathematical techniques used may find use in more bespoke verification formalisms and other application areas suggested in the thesis (e.g., quantum information theory, rewriting systems and process algebra).

- For the program verification community, this work resolves an issue regarding the incompleteness of bunched logic for the classes of 'memory models' typically used in separation logic-style program verification formalisms by producing a modular framework of tableaux proof systems that are sound and complete for any choice of class of memory model.

This impact is witnessed by publications in top conference venues [78, 79, 80, 81], with two further journal papers [82, 83] (both invited submissions following the top ranking of [78] and [80] at their respective conferences). In particular, [79] was an invited submission to the top ranking AI conference IJCAI as part of their

Sister Conferences programme showcasing the best papers at more specialised conferences to a wider audience.

Potential impact outside of the academic community lies with the modular proof systems for memory models constructed in Part III of the thesis. These are proved correct but not implemented; however, their implementation could form the basis for a program verification tool that is parametric in choice of memory model.

Acknowledgements

First and foremost I would like to thank my supervisor David Pym for his support and guidance over the past four years. David has been a great scientific inspiration, and not just for co-creating the branch of logic that this thesis is dedicated to investigating. In the course of my studies he has imparted on me a philosophical understanding of logic that has substantially deepened my appreciation of the field, as well as broadened my perception of its applicative possibilities.

He has also been a great help in navigating the world of academia as a fledging researcher. This has taken many forms, from teaching me about the precision eradication of whitespace for punishing conference paper page restrictions, to guiding me through my first grant proposal. More than anything, I thank him for the many times he set me straight when the amorphous task of completing a PhD began to overwhelm or caused me to lose confidence.

I extend these thanks to the whole PPLV group for creating a stimulating and supportive environment to do research in. Being so close to the research of academics like Alexandra Silva and James Brotherston has been a great source of inspiration, while the support, solidarity and friendship of my fellow PhD students has made many a long day shorter. I also thank UCL and the EPSRC for funding the studentship that made my work possible.

I would also like to thank my examiners Samson Abramsky and Peter O'Hearn for their dilligent and thoughtful examination of this thesis. Their comments have greatly improved the presentation of this work.

My final thanks go to my partner Maya. Doing a PhD can be a lonely enterprise, but her love has made it substantially less so. Thanks to her I remained (somewhat) human through the arduous task of writing up, and I hope to be able to repay that to her long into the future.

I dedicate this thesis to my parents Caroline and Robert. I have them to thank for the curiosity and determination that has led me through my entire education up to this point. Without their material and emotional support this work wouldn't be possible.

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Chapter 1

Introduction

This thesis is about a species of non-classical logic called *bunched logic*. Bunched logics are curious systems, with their strong semantic motivation and full set of standard propositional connectives aligning them with modal logic, despite their proof theory being strongly rooted in considerations particular to substructural logics, which typically diverge radically from classical logic in their interpretation of the standard connectives.

In this introductory chapter we outline some of the concepts that led to the formulation of the principal bunched logic, BI, as well as its most characteristic features. We then highlight some of the gaps in the theory of bunched logics more generally, and explicate the purpose of the thesis: the specification of a uniform methodology for bunched logics, capable of resolving a number of open problems in the literature. We close the chapter with a road map for the rest of the thesis.

1.1 From Classical to Non-Classical Logic

1.1.1 Substructural Logics

While, *classically*, mathematical logic concerned itself with the formal analysis of the principles of sound reasoning and truth in an absolute sense, the study of modern logic is dominated by an application-driven approach that looks to alternative logical systems designed for specialised reasoning tasks. For example, modal logic [27] provides an analysis of qualified truths—for example, what is *possibly* true and what is *necessarily* true—while fuzzy logic [222] generalises the binary notions of true and false to a continuum of truth values in order to capture vagueness and approximate reasoning—for example, given fuzzy information φ, ψ is *more likely* to be true than false. These systems, distinct as they are from the propositional and predicate logic advanced as the formal language of mathematics, are designated *non-classical logics*.

A particularly interesting class of such systems are those that have come to

be known as *substructural logics* [193]. These logics arise through an analysis of Gentzen's [103, 104] formulation of classical and intuitionistic logic as sequent calculi. Gentzen designed the sequent calculus to analyse logical consequence, represented in his proof system by sequents $\Gamma \vdash \varphi$, where Γ is a finite list of formulae $\varphi_0, \ldots, \varphi_n$, called here a *context*, to be read as: φ is a *consequence* of the *assumptions* φ_i in Γ. The calculus is specified by rules for the valid derivation of new sequents and these rules can broadly be categorised two ways. First, are the *logical rules*, specifying how to introduce each of the connectives on the left and the right side of sequents. For example, for conjunction $\wedge$, there are the rules

$$\langle \mathrm{L}\wedge \rangle \quad \frac{\Gamma, \varphi_i \vdash \psi}{\Gamma, \varphi_0 \wedge \varphi_1 \vdash \psi} \quad \langle \mathrm{R}\wedge \rangle \quad \frac{\Gamma \vdash \varphi \quad \Sigma \vdash \psi}{\Gamma, \Sigma \vdash \varphi \wedge \psi},$$

where $i = 0$ or 1 in $\langle \mathrm{L}\wedge \rangle$. In contrast, the *structural rules* specify the manner in which the contexts Γ can be manipulated. Explicitly, these rules are

$$\langle \mathrm{W} \rangle \quad \frac{\Gamma \vdash \psi}{\Gamma, \varphi \vdash \psi} \quad \langle \mathrm{E} \rangle \quad \frac{\Gamma, \varphi, \psi, \Sigma \vdash \chi}{\Gamma, \psi, \varphi, \Sigma \vdash \chi} \quad \langle \mathrm{C} \rangle \quad \frac{\Gamma, \varphi, \varphi \vdash \psi}{\Gamma, \varphi \vdash \psi},$$

where W stands for *Weakening*, E for *Exchange*, and C for *Contraction*. Each of these rules has an intuitive operational reading: weakening states that the addition of extra assumptions preserves consequence, exchange states that the order of assumptions does not affect consequence, and contraction states that the number of instances of a particular assumption do not affect consequence.

While these are suitable assumptions to make for consequence in mathematics, it began to be understood that these rules were responsible for some of the quirks that make these logics ill-suited for other reasoning domains like natural language. Of the many paradoxes of material implication, fallacies of *relevance* are often the most perplexing for the novice student of logic. Our pre-theoretic understanding of implication strongly inclines us to believe the antecedent of a *if ...then* statement should have some connection to the consequent, which leads us to regard propositions like "if it is raining then I will get wet" as sensical, and propositions like "if green is purple then pigs can't fly" as nonsensical. From the perspective of classical logic, however, these are both true propositions.

Fallacies of relevance occur at the level of validity: for example, $\varphi \to (\psi \to \varphi)$ is a valid formula of classical and intuitionistic logic, where ψ can be an arbitrary formula that has nothing to do with φ. From the perspective of the sequent calculus, the mechanism facilitating the derivation of formulae like this is the structural rule of weakening, which allows one to add *irrelevant* assumptions. Such considerations naturally motivate the investigation of *relevant logics* [9] in which $\Gamma \vdash \varphi$ is interpreted as a valid logical consequence iff the assumptions in Γ are just those

necessary to entail φ. In such systems, it is of course necessary to exclude the structural rule of weakening.

Similar considerations arise in the use of formal logic in linguistics. Here, propositional atoms are *types* of grammatical atoms—for example, determiners (e.g., the), adjectives (e.g., good) and nouns (e.g., dog)—which can be combined with the connectives to form types of sentences. Thus a sequent $\Gamma \vdash \varphi$ is now read as a typing judgement: a string is of type φ iff it is obtained as the concatenation of strings of the types in Γ. Simple considerations of how sentences of natural language are formed immediately brings such an enterprise into conflict with the structural rules of Gentzen's calculus: in particular, the *order* of words in a sentence is of vital importance to its meaning and well-formedness. Categorial grammar [145, 147] (known in its proof-theoretic form as the Lambek calculus) reflects this fact by dropping the structural rule of exchange, as well as weakening and contraction.

Substructural logics can broadly be thought of as logics obtained through the excision of one or more of the structural rules. Such systems include the aforementioned examples of the family of relevant logics and the Lambek calculus. While the structural rules appear to be quite simple, their removal defines systems quite unlike classical logic.

1.1.2 Linear Logic

One of the most influential analyses of the effect of removing structural rules is due to Girard [105], in his landmark work defining linear logic. Girard's contention was that the constructive character of intuitionistic logic lay in its control over the application of contraction and weakening: while in the classical sequent calculus, contexts occur on either side of the turnstile, $\vdash$, and the structural rules can be applied on both the left and the right, in the intuitionistic calculus, the structural rules are restricted to the left, and it is this that allows the derivation of such characteristic constructive features as the disjunction property. Girard sought to generalise this further, by additionally disallowing weakening and contraction on the left.

In the presence of the contraction and weakening rules contexts are fairly malleable entities, and it isn't difficult to use them to show that the following alternative right rule for $\wedge$ is equivalent to the one previously given:

$$\langle \mathrm{R}\wedge' \rangle \quad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}.$$

Essentially, when the premises of $\langle \mathrm{R}\wedge \rangle$ hold, the contexts Γ and Σ can always be massaged into an identical context through repeated applications of weakening and contraction to give premises allowing the application of $\langle \mathrm{R}\wedge' \rangle$. As such, the

sequent calculus could just as well have been defined with $\langle R\wedge'\rangle$ as a primitive rule. *Without* contraction and weakening, however, these two rules become distinct, and this begs the question: which should have priority as a basic rule in a substructural system? Girard's innovative answer was *both*. This results in a splitting of conjunction into an *additive* component, &, governed by a rule of the same form as $\langle \mathrm{R}\wedge'\rangle$

$$\langle \mathrm{R}\& \rangle \quad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \,\&\, \psi},$$

and a *multiplicative* component, $\otimes$, governed by a rule of the same form as $\langle \mathrm{R}\wedge\rangle$

$$\langle \mathrm{R}\otimes \rangle \quad \frac{\Gamma \vdash \varphi \quad \Sigma \vdash \psi}{\Gamma, \Sigma \vdash \varphi \otimes \psi}.$$

A similar analysis sees disjunction split into an additive and a multiplicative connective.

In relevant logics the lack of weakening demands that the formulae in the antecedent are all relevant to the entailment of the consequent; in a system without *both* contraction *and* weakening, this is strengthened further to enable a *resource-sensitive* reading of sequents. Now, not only are the formulae that occur in the context precisely the formulae required for the consequent, the number of occurences of each formula also matters.

Perhaps the most innovative feature of linear logic is the use of the exponentials ! and ? to allow contraction and weakening to be applied locally. Informally, if the occurence of φ in a context is read as the availability of one φ to be used to derive a consequence, $!\varphi$ is read as the availability of as many φ's as we want. This is enforced by 'local' weakening and contraction rules that only apply to formulae marked with !:

$$\langle !W \rangle \quad \frac{\Gamma \vdash \psi}{\Gamma, !\varphi \vdash \psi} \qquad \langle !C \rangle \quad \frac{\Gamma, !\varphi, !\varphi \vdash \psi}{\Gamma, !\varphi \vdash \psi}.$$

The presence of the exponentials recovers a lot of expressivity lost from the removal of the structural rules. In particular, intuitionistic logic can be faithfully encoded in linear logic using the exponentials, with intuitionistic implication $\varphi \to \psi$ given by $!\varphi \multimap \psi$, where $\multimap$ is the multiplicative implication of linear logic. This sets up a principled balance between expressivity and control: the use of weakening and contraction is completely tracked by the exponentials.

Two major applications of linear logic utilise this operational reading of sequents. The first is in the use of logic as a goal-oriented programming language, most famously implemented non-linearly in the language Prolog. In such languages

a goal (e.g., a query to be answered) is specified by a sequent and computation is given by "the process of building a cut-free sequent proof bottom-up" [161] for it. The additional control over the dynamics of the sequent calculus that linear logic provides offers a significant refinement on languages based on classical or intutionistic logic. An overview of such linear logic programming languages is given by Miller [161]. Miller also highlights the second major application of linear logic in computer science: the use of linear logic as a type theory for programs, in line with the Curry-Howard correspondence between intuitionistic proofs and programs. Abramsky [2] outlines the necessary ingredients for this: linear logic formulae give a language of types, allowing programs to be interpreted by linear logic proofs and computation by a proof theoretic procedure known as proof normalisation. Resource-sensitivity and the additional control over the dynamics of proof are related, for example, to the controlled use of arguments during computation and the implementation of concurrent algorithms.

1.1.3 The Descriptive View vs the Intrinsic View

Abramsky [4] neatly characterises two views on the relation of logic to structure:

1. The *descriptive view*: in which logical formulae are understood as assertions describing properties of structures. This might fruitfully be considered a semantics-first view on logic, where the syntactic machinery of logic is delegated to reason about a semantic structure of interest;

2. The *intrinsic view*: in which the syntactic machinery of logic *embodies* structure. Here Abramsky uses the example of intuitionistic logic in the Curry-Howard correspondence: logical formulae are not assertions *about* some semantic model of functional programming, instead logical formulae together with proof theory provides an instance *of* functional programming.

The applications of linear logic in computer science that we have just discussed fall quite cleanly into the second of these characterisations. In modern logic, the example *par excellence* of the descriptive view is the many ways in which the semantics of modal logic have been put to use modelling a range of philosophical and computational phenomena.

In essence, the Kripke semantics for modal logic is based on a set of *possible worlds*, where accessibility between worlds is represented by relations. In the most basic example, we have a set of worlds X with a binary accessibility relation R. The truth of formulae formed with the possibility operator $\Diamond$ is defined as follows:

$$x \vDash \Diamond\varphi \text{ iff there exists } y \text{ such that } Rxy \text{ and } y \vDash \varphi.$$

That is, $\Diamond\varphi$ is true at the world x iff there is a world y accessible from x where φ holds. The terminology "possible worlds" is an artefact of modal logic's philosophical origins, as its original intended purpose was to analyse the notion of possible and necessary truth. However we might usefully think of the possible worlds as states of a system (e.g., a labelled transition system), with the accessibility relation determined by computation steps. This idea has been massively influential in computer science in the past forty years, with particular varieties of modal logic like CTL [58] and dynamic logic [114] providing the means to verify that systems meet their specification and do not fault through essentially semantic means.

Resource, be it memory or CPU time, is a central notion in computer science, and for this reason it may seem that linear logic would be well-suited to play a similar modelling role to these modal logics. Unfortunately, despite the resource-sensitive operational reading of linear logic's proof theory, none of the semantic approaches to linear logic really reflect the notion of resource.

The standard treatment of linear logic's semantics is known as *phase semantics* [106], and involves the interpretation of linear logic formulae in structures called *phase spaces*: pairs $(M,\bot)$ in which M is a commutative monoid and $\bot$ an arbitrary set. An operation on subsets $A \subseteq M$ is defined by $A^\bot := \{m \in M \mid \forall n \in A : mn \in \bot\}$ and all formulae can be interpreted by *closed* subsets $A \subseteq M$, satisfying $A = A^{\bot\bot}$, called *facts*. For example, the multiplicative implication $\varphi \multimap \psi$ is interpreted, given facts $[\![\varphi]\!]$ and $[\![\psi]\!]$ interpreting φ and ψ, by the fact $[\![\varphi \multimap \psi]\!] = \{m \in M \mid \forall n \in [\![\varphi]\!] : mn \in [\![\psi]\!]\}$. While this restriction to closed subsets of a phase space allows one to give a sound and complete interpretation of the logic (in particular, neatly handling the non-distributive additive connectives and the notoriously tricky exponentials), it has the unfortunate side effect of making meaningful models extremely difficult to find. Moreover, it resists a natural interpretation in terms of resource.

For a long time it remained open if there even was a Kripke-style semantics for the full logic, although Kripke models for the fragment without exponentials were given by Allwein & Dunn [8] early on. It was only recently that Coumans et al. [67] gave a Kripke semantics for which the exponentials could also be interpreted, but for soundness to hold a number of artificial conditions must be imposed on the relational structure, and, once again, it remains open if any real-world phenomenon of independent interest can be found that satisfies their definition.

What, then, would a logic with a resource-sensitive *semantics* look like? One answer is the *logic of bunched implications*, the principal member of a family of logics that this thesis is dedicated to investigating. This logic is formulated in a similar fashion to linear logic, but also has a sound and complete Kripke-style semantics arising out of a simple analysis of the abstract notion of resource—that is,

objects that can be *composed* and *compared*.

1.2 Bunched Logics

1.2.1 The Logic of Bunched Implications

Formulated by O'Hearn & Pym [177], the logic of bunched implications (BI) is a logic that captures many of the salient features of linear logic's proof theory while enjoying a set-theoretic semantics that allows it to be used to make declarative assertions about resource in much the same way modal logic can be used vis-à-vis computation systems. It also arises through considerations relating to the semantics of ALGOL-like programming languages [194, 195] and category theoretic constructions [74], but is perhaps most clearly understood through a continuation of our proof theoretic analysis. Directly, it is obtained as the free combination of intuitionistic propositional logic and multiplicative intuitionistic linear logic, and this requires some interesting adjustments to the sequent calculus.

As discussed when introducing substructural logics, the structural rules of the sequent calculus specify the legal ways in which contexts can be restructured. In effect, these rules can be seen as axioms governing the behaviour of the *context former* , when seen as an operation on formulae. For example, the Exchange rule $\langle E \rangle$ specifies that the comma is commutative, while the Contraction rule $\langle C \rangle$ specifies that it is idempotent. In Linear Logic, certain properties of the context former do not universally hold (because of the absence of the corresponding structural rules) but the exponentials allow the specification of instances where the properties *do* hold.

BI departs from this set-up by enforcing this distinction at the level of structural rules rather than at the level of formulae. To facilitate this, a natural generalisation is required: from one context former to *two*. The original context former , lacks Weakening and Contraction, just as with Linear Logic, but it is now joined by a new context former ; for which all of the structural rules hold. Of course, this increases the structural complexity of contexts. While before contexts were lists of formulae separated by commas, now contexts have a tree structure, with internal nodes given by the context formers and formulae as leaves. Such tree-structured contexts originate in relevant logic [18] and are called *bunches* (hence *bunched* implications). As an example, Figure 1.1 shows the bunch $((\varphi\,;\psi)\,,\chi)\,;\eta$ represented in tree form.

Using bunched contexts to facilitate it, BI arises as the combination of the sequent calculus for intuitionistic logic (giving the additives of BI) and the sequent calculus for the multiplicative fragment of intuitionistic linear logic. Letting $\Gamma(\Delta)$ denote a bunch in which Δ occurs as a subbunch, the rules governing additive conjunction $\wedge$ and multiplicative conjunction $*$ are given as follows.

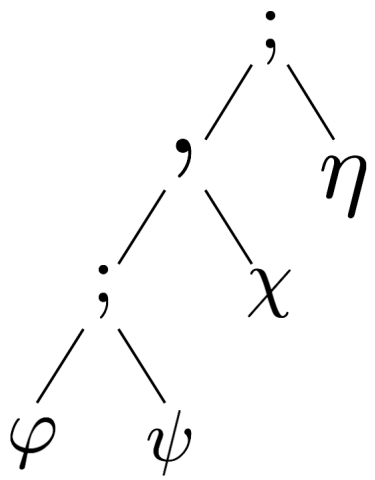


Figure 1.1: The bunch $((\varphi\,;\psi)\,,\chi)\,;\eta$.

$$\langle \mathrm{L}\wedge\rangle \quad \frac{\Gamma(\varphi\,;\psi)\vdash\chi}{\Gamma(\varphi\wedge\psi)\vdash\chi} \qquad \langle \mathrm{R}\wedge\rangle \quad \frac{\Gamma\vdash\varphi \quad \Gamma\vdash\psi}{\Gamma\vdash\varphi\wedge\psi}$$

$$\langle \mathrm{L}*\rangle \quad \frac{\Gamma(\varphi\,,\psi)\vdash\chi}{\Gamma(\varphi*\psi)\vdash\chi} \qquad \langle \mathrm{R}*\rangle \quad \frac{\Gamma\vdash\varphi \quad \Sigma\vdash\psi}{\Gamma\,,\Sigma\vdash\varphi*\psi}$$

The move to two context formers also indicates a further splitting of connectives that was not considered by Girard. Consider the rules for implication in the standard sequent calculus,

$$\langle \mathrm{L}\to\rangle \quad \frac{\Gamma\vdash\varphi \quad \Sigma,\psi\vdash\chi}{\Sigma,\Gamma,\varphi\to\psi\vdash\chi} \qquad \langle \mathrm{R}\to\rangle \quad \frac{\Gamma,\varphi\vdash\psi}{\Gamma\vdash\varphi\to\psi}.$$

These rules induce the multiplicative implication $\multimap$ of linear logic when the structural rules are missing, through which intuitionistic implication can only be recovered with the use of the exponential ! . The use of two context formers is suggestive, however, of two *coexisting* implications, given by analogues of the standard rules for each context former: the standard intuitionistic implication $\to$ and a multiplicative implication $-\!\!*$ (hence bunched *implications*).

$$\langle \mathrm{L}\to\rangle \quad \frac{\Gamma\vdash\varphi \quad \Sigma(\Sigma';\psi)\vdash\chi}{\Sigma(\Sigma';\Gamma;\varphi\to\psi)\vdash\chi} \qquad \langle \mathrm{R}\to\rangle \quad \frac{\Gamma;\varphi\vdash\psi}{\Gamma\vdash\varphi\to\psi}$$

$$\langle \mathrm{L}-\!\!*\rangle \quad \frac{\Gamma\vdash\varphi \quad \Sigma(\Sigma',\psi)\vdash\chi}{\Sigma(\Sigma',\Gamma,\varphi-\!\!*\psi)\vdash\chi} \qquad \langle \mathrm{R}-\!\!*\rangle \quad \frac{\Gamma,\varphi\vdash\psi}{\Gamma\vdash\varphi-\!\!*\psi}.$$

Similarly to linear logic, BI can be given a type theoretic presentation [174, 62] and used as a logic programming language [13]. There are major differences between the logics, however, and this can be seen at this proof theoretic level. For one,

the "additives" of BI specify precisely intuitionistic logic, whereas for the additives of linear logic many properties of intuitionistic logic (such as the distributive law for conjunction and disjunction) fail. This is arguably quite a superficial comparison, however, as the additives of linear logic are still linear. Perhaps more forceful is the fact that distinctions are maintained even with the embedding of intuitionistic logic into linear logic. For example, in linear logic intuitionistic implication arises as $!\varphi \multimap \psi$ and the sequent $!\varphi \multimap \psi \vdash \varphi \multimap \psi$ is provable; however $\varphi \to \psi \vdash \varphi \mathrel{-\!\!*} \psi$ is *not* provable in BI.

The starkest difference can be seen with BI's semantics however: unlike linear logic, the notion of resource is instantiated at the level of a simple Kripke-style set theoretic semantics. Models for BI are given by a set X of resources with a commutative, monoidal operation $\cdot$ —interpreted as the *composition* of resources— and an order $\succcurlyeq$—allowing the *comparison* of resources. The Kripke-style semantics on such structures can straightforwardly be seen as the extension of Kripke's [142] preorder semantics for intuitionistic logic with Urquhart's [213] semantics for the multiplicative fragment of intuitionistic linear logic. In particular, at a resource r we can evaluate the truth of formulae of the form $\varphi * \psi$ and $\varphi \mathrel{-\!\!*} \psi$ as follows.

$$r \vDash \varphi * \psi \text{ iff there exists } s,t \text{ such that } r \succcurlyeq s \cdot t, s \vDash \varphi \text{ and } t \vDash \psi$$

This states that $\varphi * \psi$ is true of a resource r if part of r can be decomposed into separate resources s and t such that φ is true of s and ψ is true of t. Note that this clause directly corresponds to the decomposition of the bunch Γ, Σ in the rule $\langle \mathrm{R}* \rangle$ when read bottom-up.

$$r \vDash \varphi \mathrel{-\!\!*} \psi \text{ iff for all } s, \text{ if } s \vDash \varphi \text{ then } r \cdot s \vDash \psi$$

This states that $\varphi \mathrel{-\!\!*} \psi$ is true of a resource r if every composition of r with a resource for which φ holds results in a resource for which ψ holds. Note that this is of precisely the same form as the semantics for $\multimap$: however, in the case of BI, we do not have the semantically ad hoc restriction to any special subsets of X. It should also be noted that these clauses diverge from even the operational reading of linear logic sequents, in which the principal idea is that of the *number of uses* of a resource. BI's semantics is instead based on a *sharing* interpretation, in which the truth of the multiplicatives is witnessed by the ability to split up or put together resources for a particular goal. This idea has been wildly impactful, with the logic (and in particular, these semantic ideas) forming the core of the industrial-level program verification formalism Separation Logic [129, 197, 221].

1.2.2 From BI to a Family of Bunched Logics

BI has inspired a number of logics that can broadly be grouped under the name *bunched logics*. Though for mathematical reasons most are not presented as bunched sequent calculi, they all are formed by a similar guiding principle: the extension of intuitionistic or classical propositional logic with fragments of substructural logics, in particular linear logics. A key motivation behind these logics is their resource interpretation, elucidating the somewhat mysterious meaning of the multiplicative connectives that appear in substructural and linear logics, and suggesting innovative uses of logic as a modelling technology.

While the formulation and widespread application in program verification of Boolean BI (the variant of BI in which the intuitionistic propositional logic fragment is replaced with classical propositional logic) indicated the usefulness of going beyond BI, the idea of a family of bunched logics was first proposed by Pym [187], who suggested that two further bunched logics, De Morgan BI and Classical BI, could be obtained by extending BI and BBI (respectively) with a multiplicative negation. In the case of CBI, this idea was executed by Brotherston & Calcagno [39], who motivated the logic with a resource semantics based on dualisable resource and a display calculus proof theory. A further group of bunched logics were given by Brotherston & Villard [45], who considered *subclassical* bunched logics obtained by extending BBI with multiplicative disjunction and falsum together with axioms governing their interactions. These logics also have a strong semantic motivation, obtained by considering sets of resources that both compose and intersect. Taking a different perspective, Collinson et al. [63] weakened the multiplicative structure of BBI to obtain layered graph logic (LGL), motivated by a semantics given on structured directed graphs of the sort utilised in complex systems modelling. Alongside these, a large number of extensions of BI and BBI with modal and hybrid operators have also been defined [44, 68, 98].

Much is missing from the theory of these logics, however. Perhaps most stark, given bunched logics' origin in BI, is the lack of metatheory for the evident intuitionistic variants of the logics just outlined. In the cases of Classical, Hybrid and Subclassical bunched logics, this has much to do with the methodology behind their soundness and completeness theorems, which rely on an indirect argument utilising translations into modal logic that only work for logics extending classical propositional logic. This has seemingly prevented the formulation of resource semantics for these logics. Beyond this, much typical logical theory—decidability results, duality theory, expressivity limits, Craig interpolation—is missing from the bunched logic literature.

This thesis is about addressing this gap. Our guiding philosophy is *uniformity*

and *modularity*: in short, to what extent can this family of logics be presented in a uniform/modular framework, and what new metatheory can be proved by doing so? To this end we rationally reconstruct the family of bunched logics (previously unconsidered intuitionistic variants included) as a family of extensions of a new basic bunched logic: the intuitionistic variant of layered graph logic. This presentation is systematised in two ways: first, by a uniform duality theory that relates algebraic and Kripke-style presentations of the logics, through which the soundness and completeness of resource semantics is obtained as a corollary; second, by a modular tableaux proof theory suitable for both consequence and validity in the base logics *and* restrictions to classes of models used in applications.

In doing so we're able to prove a number of other results: decidability for layered graph logics, a characterisation theorem for the classes of bunched logic model definable by bunched logic formulae, the failure of Craig interpolation for principal bunched logics and the extension of duality to the categorical structures that interpret predicate versions of the logics. The techniques used are sufficiently general and extendable to be easily applied to specify new bunched logics. As an example, we define a new bunched logic inspired by concurrent Kleene algebra [178] for which the duality theoretic and tableaux-based approach to soundness and completeness is applied.

1.3 Road Map for the Thesis

This thesis is split into four parts for the purpose of organisation. The first part, A Family of Bunched Logics introduces the bunched logics that the rest of the thesis is dedicated to investigating. While this part is introductory in nature, due to its substantial size we advise that the core of the thesis (Parts II and III) can effectively be understood after reading Chapter 2 on weak bunched logics, with Chapters 3 and 4 used as a reference if necessary. Throughout this part a number of examples of applications of bunched logics are given for the interested reader – it should be noted that, with the exception of Separation Logic in Chapter 3, these are inessential for the comprehension of the core of the thesis, and should simply be understood as demonstrative of the utility of bunched logics themselves.

In Chapter 2 we begin with the layered graph logics, the bunched logics with the 'weakest' multiplicative structure. In particular, we introduce Intuitionistic Layered Graph Logic, from which the existent Layered Graph Logic can be obtained as an extension. In doing so we give Hilbert-style proof systems and Kripke semantics for both logics, and these form the basis of the Hilbert systems and Kripke semantics for every other logic examined in the thesis. We pay particular attention to the layered graph models that give the logics their name, explaining how they

arise for both logics and indicating potential applications for them. This chapter is based on material from the publications *Intuitionistic Layered Graph Logic* [78], *Intuitionistic Layered Graph Logic (Abridged Version)* [79] and the journal paper *Intuitionistic Layered Graph Logic: Semantics and Proof Theory* [82].

In Chapter 3 we introduce BI and BBI as extensions of layered graph logics. In particular, we introduce our version of resource semantics (generalising that of O'Hearn & Pym [177] and inspired by the work of Coecke et al. [60] and Fritz [94] in quantum information theory) and explain the logical relationship between our models and those found in the literature. We also take an opportunity to introduce Separation Logic, a program verification formalism based on (B)BI. While this is *not* a thesis about Separation Logic, many concerns particular to Separation Logic inspire our work: for example, the logic CKBI introduced in Chapter 4, the duality theory for predicate (B)BI given in Chapter 8 and the tableaux calculi for Separation Logic-like models given in Chapter 10. We finish this chapter by highlighting a range of examples from computer science that yield models of the logics. Parts of this chapter are based on material from the journal paper *Stone-Type Dualities for Separation Logics* [83].

In Chapter 4 we introduce a range of logics that extend BI and BBI, finishing our summary of the bunched logics under investigation. We begin with DMBI and CBI, logics obtained by extending BI and BBI (respectively) with a multiplicative negation. Similarly to Chapter 2, we focus on the previously unexplored intuitionistic variant DMBI's resource semantics, obtaining CBI as a particular extension. Next we consider Subclassical Bunched Logics, extending BI and BBI with multiplicative disjunction and falsum. Once again, the variant extending BI is new, and is the focus of this section. We finish the chapter with logics extending BBI. First, a class of modal logics we call—after Courtault et al.'s [68] Logic of Separating Modalities—*separating modal logics*. We explain how the connectives of BBI can be used together with a single modal operator to obtain a range of normal modalities interpreted by a notion of accessibility offset by the composition of resources. We also discuss the problems with defining separating modal logics extending BI. Second, a new bunched logic we call Concurrent Kleene BI, or CKBI. This logic is inspired by work [178] relating Concurrent Separation Logic to concurrent Kleene algebra and acts as something of a test case for the methods of the thesis. For simplicity we do not consider the evident intuitionistic variant, though note that its specification is straightforward using the techniques of the thesis. Parts of this chapter are based on material from the journal paper *Stone-Type Dualities for Separation Logics* [83].

The second part, Algebra and Duality for Bunched Logics, is something of

a rejoinder to a comment of O'Hearn & Pym [177] in their paper introducing BI. There they state

> "We are not looking for an algebraic semantics here, where one takes (say) a Heyting algebra with enough structure to model the multiplicatives; this would just be a collapsed version of the [categorical] semantics and would not be very informative"

before defining BI's Kripke-style resource semantics. In contrast to this, we consider just such an algebraic semantics and show via duality theory that the resource semantics of bunched logics is in fact encoded within it, in much the same way that the Kripke semantics of modal logic were anticipated by Jónsson & Tarski's duality theory for Boolean algebras with operators. Moreover, we are able to use this perspective to resolve a number of open problems in bunched logic. This indicates that while, just as in the case of modal logic, having the Kripke-style semantics provides the catalyst for identifying potent applications, the algebraic perspective is still incredibly important.

In Chapter 5 we provide the necessary preliminaries for our duality theoretic approach: an overview of basic notions from algebra and topology, as well as a presentation of the Esakia and Stone duality theorems connecting the algebraic and topological semantics of intuitionistic and classical propositional logic. As bunched logics extend these logics, naturally our duality theorems must extend these theorems, and so we give them in full as a preliminary step. Of particular importance in this chapter is the notion of *prime predicate*, a technical device we introduce (generalising one of Galmiche & Larchey-Wendling [99]) that we will repeatedly use throughout the rest of this part of the thesis.

In Chapter 6 we get to work setting up duality theory for bunched logics. For each logic this comes in two steps. First, we relate an algebraic semantics that we introduce to the Kripke semantics given in Part I by producing functors that transform one into the other and give representation theorems for bunched logic algebras that are based on them. This is sufficient for many of the applications we then go on to consider, but in order to obtain true duality we must introduce topology to the Kripke semantics; and thus we do so as a second step, introducing bunched logic spaces. Reflecting the structure of Part I, we start with duality theory for layered graph logics, and then extend these results to each logic by introducing the additional required structure that corresponds to the manner in which each bunched logic arises as an extension of the basic bunched logics. This chapter is based on the publication *A Stone-Type Duality Theorem for Separation Logic* [80] and the journal paper *Stone-Type Dualities for Separation Logics* [83].

We apply the duality theory of Chapter 6 in Chapter 7, resolving a number of open problems in bunched logic. First, we obtain the soundness and completeness of the Kripke semantics of each logic simultaneously as a corollary of the previous chapter's work. In particular, this gives the first completeness theorem for the resource semantics of the range of intuitionistic variants of bunched logics that had previously not been investigated. It also highlights duality theory as a powerful technique for proving soundness and completeness, reobtaining results for the classical variants of bunched logics that had previously been given through lengthy translations into equivalent modal logics. Second, we prove the decidability of the layered graph logics by proving a finite model property for the algebraic semantics and discuss the non-extension of the results to other bunched logics. Next, we investigate the expressivity of bunched logics by proving a bunched logic variant of the Goldblatt-Thomason theorem for modal logic. That theorem outlines precisely the classes of modal logic model that can be captured by a set of modal logic formulae—that is, the classes C of modal logic model such that there exists a set of formulae Σ such that a model $\mathscr{X}$ is in C iff the formulae of Σ are valid in $\mathscr{X}$—and we prove an analogous classification for bunched logics. Finally, we consider Craig Interpolation for bunched logics, proving that it fails for BBI and CBI, and reducing the problem for BI and DMBI to a much simpler one. Parts of this chapter are based on material from the journal papers *Stone-Type Dualities for Separation Logics* [83] and *Intuitionistic Layered Graph Logic: Semantics and Proof Theory* [82].

In the final chapter of this part, Chapter 8, we extend the dualities of Chapter 6 from propositional bunched logics to predicate bunched logics. This is of particular interest because Separation Logic is actually based on predicate (B)BI, not propositional (B)BI. First we outline categorical structures suitable to interpret predicate versions of the logic: 'algebraic' structures called bunched logic hyperdoctrines and new 'Kripke-style' structures called indexed bunched logic frames. We show that the standard model of Separation Logic is an instance of an indexed frame, and use this as inspiration for an example of a predicate ILGL model. Finally we extend the propositional dualities to these structures. This chapter is based on the publication *A Stone-Type Duality Theorem for Separation Logic* [80] and the journal paper *Stone-Type Dualities for Separation Logics* [83].

Part III, Proof Theory for Bunched Logics, is entirely concerned with proof theory: in particular, setting up a modular tableaux calculus framework sufficient for all of the logics under investigation. Our work here is particularly motivated by the fact that many of the classes of bunched logic model used in applications are in fact incomplete for standard bunched logic proof systems. Our goal is to give a uniform proof theory that also specifies proof systems for which these classes of

model are complete.

In Chapter 9 we set up the basic tableaux calculi for the propositional logics introduced in Part I. These are given by specifying *logical rules* that relate to the way formulae decompose into subformulae and *frame rules* that encode the structure of Kripke models that makes resource semantics sound. The frame rules are given by a uniform translation schema that converts axioms defining Kripke models into proof rules, taking advantage of the fact that all such axioms belong to a special fragment of first-order logic called coherent logic. Tableaux calculi for each logic can then be build modularly by adding or removing the appropriate logical and frame rules. We show that the tableaux calculi themselves can be given as theories of coherent logic—something that in fact holds for tableaux systems more generally—and use metatheory particular to that fragment to prove them sound and complete. Material in this chapter is based on the publication *Modular Tableaux Calculi for Separation Theories* [81].

In Chapter 10 we specify tableau calculi for classes of bunched logic model of interest in applications. First, we consider the classes of bunched logic model specified by *separation theories*: first-order axioms that encode properties of the memory models used in Separation Logic. We show that these are all specified by coherent formulae, and so can be modularly added to the tableaux calculi framework defined in Chapter 9 to give sound and complete proof systems for classes of memory models. We finish by considering the class of layered graph models of ILGL. Since the states of these models are graphs and thus have internal structure that can't be specified by coherent axioms, the method used for separation theories does not work. Instead, we build a new tableaux calculus from scratch that controls the structure added in derivations in such a way that the failure to find a tableau proof of a formula generates a layered graph countermodel. This countermodel extraction is then used to prove the system is sound and complete for the class of layered graph models. This chapter is based on the publications *Modular Tableaux Calculi for Separation Theories* [81] and *Intuitionistic Layered Graph Logic* [78].

We finish with Part IV, Conclusions & Further Work. As its name suggests, this part summarises the contents of the thesis and suggests a range of further research inspired by its results. In particular, we suggest the use of the new bunched logics as modelling technologies, the application of duality theory to prove further metatheoretic results and the implementation of the proof systems given in Part III.

Throughout the thesis some basic category theoretic notions are used without definition. These can all be found in any introductory text on category theory (e.g., [15, 154]), but we collect them in the Appendix for ease of reference.

Part I

A Family of Bunched Logics

Introduction to Part I

This part is dedicated to setting up the logics that will be investigated in this thesis. While some of these systems are well investigated, some have only been postulated previously and others are new. We organise the introduction of the logics systematically, starting with the bunched logic with the weakest multiplicative structure in Chapter 2. In Chapter 3 we explain how the well-known logic of bunched implications can be obtained as an extension of that weak logic while giving an overview of its metatheory and applications. In Chapter 4 we survey a range of bunched logics, both investigated and new, that can be obtained as extensions of bunched implication logics, corresponding to the addition of multiplicative counterparts to disjunction, falsum, negation and modality.

While this part is introductory in nature, due to its substantial size we advise that the core of the thesis (Parts II and III) can effectively be understood after reading Chapter 2 on weak bunched logics, with Chapters 3 and 4 used as a reference if necessary. Throughout this part a number of examples of applications of bunched logics are given for the interested reader – it should be noted that, with the exception of Separation Logic in Chapter 3, these are inessential for the comprehension of the core of the thesis, and should simply be understood as demonstrative of the utility of bunched logics themselves.

Chapter 2

Layered Graph Logics

We begin our investigation of the family of bunched logics with the most basic systems: layered graph logics. These logics extend classical or intuitionistic additives with the weakest possible multiplicative conjunction and its associated implications.

Consider again the analysis of the sequent calculus outlined in Chapter 1 for BI. In order to obtain coexisting additives and multiplicatives we moved to a system with two types of context former, each axiomatised differently. This yields tree-structured contexts called *bunches*. For the additive context former ";" (henceforth *semi-colon*) the structural properties that determine intuitionistic logic hold: associativity of context formation, together with Contraction—in the bunch $\Gamma(\varphi\,;\varphi)$ the 'extraneous' instance of φ can be removed to give the bunch $\Gamma(\varphi)$)—Weakening—$\Gamma(\varphi)$ can be 'weakened' with an extra assumption ψ to obtain the bunch $\Gamma(\varphi\,;\psi)$—and Exchange—$\Gamma(\varphi\,;\psi)$ is identical to $\Gamma(\psi\,;\varphi)$. In addition there is a unit for semi-colon given by the empty additive bunch. In contrast, the multiplicative context former "," (henceforth *comma*) only satisfies associativity and exchange, and has its own unit given by the empty multiplicative bunch.

For layered graph logic, the context former comma fails to satisfy *all* of the structural properties—thus additionally dropping associativity and exchange—and has no unit. This is reflected in a multiplicative conjunction that fails to be associative, commutative and idempotent. Importantly, as in the case for Contraction, removing the structural rule of Exchange causes another splitting of connectives. The introduction rule for implication is given by

$$\langle \mathrm{R}{-\!\!*}\rangle \frac{\Gamma\,,\varphi \vdash \psi}{\Gamma \vdash \varphi \mathrel{-\!\!*} \psi}.$$

Of course, in the presence of Exchange this is equivalent to

$$\langle \mathrm{R}{-\!\!*}'\rangle \frac{\varphi\,,\Gamma \vdash \psi}{\Gamma \vdash \varphi \mathrel{-\!\!*} \psi},$$

but this equivalence collapses when Exchange is removed. Accordingly, the absence of Exchange naturally suggests two implications should be associated with the multiplicative conjunction, corresponding to the two possible rules: $-\!\!*$ and $*\!\!-$, respectively. Taken on its own, the multiplicative conjunction and it's two implications define the non-associative Lambek calculus [146].

The variant of layered graph logic with classical additives was introduced and investigated by Collinson et al. [63]. In that work the emphasis was on possible applications of the logic, with soundness and completeness of the logic with respect to sequent calculus, natural deduction, Hilbert and display calculus systems only proved for an algebraic semantics of which the intended semantics generates a particular case. Moreover, the evident variant with *intuitionistic additives* (à la BI) is new, and so this chapter is dedicated to its introduction. In Part II we will fill in some of the metatheoretic gaps in the literature for both of these systems.

Every other logic we consider in this thesis can be obtained as an extension of the layered graph logics that is obtained by adding axioms and/or connectives. In turn, we can develop metatheory for the layered graph logics, and then systematically extend it to this additional structure to obtain it for the rest of the bunched logic family. While this is a strong mathematical motivation for the formulation of the logics, there is also a principled semantic justification in the form of the layered graph models that give them their name. In these models the multiplicative conjunction is interpreted by the separating of directed graphs into layers; one layer above another. As we will see, this interpretation is naturally non-commutative (in fact, the composition of layers is *contra-commutative*), non-associative and cannot be given a unit, so the dropping of the additional structural properties of the sequent calculus is well motivated semantically.

This chapter is based on the papers *Intuitionistic Layered Graph Logic* [78], *Intuitionistic Layered Graph Logic (Abridged Version)* [79] and *Intuitionistic Layered Graph Logic: Semantics and Proof Theory* [82].

2.1 Syntax and Semantics

We now specify the logics LGL and ILGL. LGL stands for Layered Graph Logic and is the variant with classical additives introduced by Collinson et al. [63, 64]. Appropriately, ILGL stands for Intuitionistic Layered Graph Logic and is the variant with intuitionistic additives.

Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of layered graph logic $\mathrm{Form}_{\mathrm{LGL}}$ is generated by the grammar

$$\varphi ::= \mathrm{p} \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi -\!\!* \varphi \mid \varphi *\!\!- \varphi,$$

0. $\dfrac{}{\neg\neg\varphi \vdash \varphi}$ 1. $\dfrac{}{\varphi \vdash \varphi}$ 2. $\dfrac{}{\varphi \vdash \top}$

3. $\dfrac{}{\bot \vdash \varphi}$ 4. $\dfrac{\eta \vdash \varphi \quad \eta \vdash \psi}{\eta \vdash \varphi \wedge \psi}$ 5. $\dfrac{\varphi \vdash \psi_1 \wedge \psi_2}{\varphi \vdash \psi_i}$

6. $\dfrac{\varphi \vdash \psi}{\eta \wedge \varphi \vdash \psi}$ 7. $\dfrac{\eta \vdash \psi \quad \varphi \vdash \psi}{\eta \vee \varphi \vdash \psi}$ 8. $\dfrac{\varphi \vdash \psi_i}{\varphi \vdash \psi_1 \vee \psi_2}$

9. $\dfrac{\eta \vdash \varphi \to \psi \quad \eta \vdash \varphi}{\eta \vdash \psi}$ 10. $\dfrac{\eta \wedge \varphi \vdash \psi}{\eta \vdash \varphi \to \psi}$ 11. $\dfrac{\xi \vdash \varphi \quad \eta \vdash \psi}{\xi * \eta \vdash \varphi * \psi}$

12. $\dfrac{\eta * \varphi \vdash \psi}{\eta \vdash \varphi \mathrel{-\!\!*} \psi}$ 13. $\dfrac{\xi \vdash \varphi \mathrel{-\!\!*} \psi \quad \eta \vdash \varphi}{\xi * \eta \vdash \psi}$ 14. $\dfrac{\eta * \varphi \vdash \psi}{\varphi \vdash \eta \mathrel{*\!\!-} \psi}$

15. $\dfrac{\xi \vdash \varphi \mathrel{*\!\!-} \psi \quad \eta \vdash \varphi}{\eta * \xi \vdash \psi}$

Figure 2.1: Hilbert rules for layered graph logics. $i = 1$ or 2 for 5. and 8.

with additive negation defined by $\neg\varphi := \varphi \to \bot$.

We note that our notation differs here compared from that found in the work of Collinson et al. [63, 64] and our previous work [78, 79, 82]. There, the multiplicative conjunction is given by $\blacktriangleright$, with the associated implications written $\mathrel{-\!\!\blacktriangleright}$ and $\blacktriangleleft\!\!-$. While this notation has the benefit of directly presenting the non-commutativity of the conjunction, we use the $*$ notation associated with the logic of bunched implications uniformly across all logics we consider. This allows us to present rules, algebras, and constructions that instantiate the same structure independently of the logic under consideration uniformly across this thesis.

Hilbert-style rules for LGL and ILGL are given in Figure 2.1. Rules $0-10$ specify classical propositional logic, $1-10$ intuitionistic logic and $11-15$ the non-associative Lambek calculus. Accordingly, $0-15$ gives a proof system for LGL, while $1-15$ gives a proof system for ILGL.

Proofs in the system are constructed inductively: in the base case, an instance of a zero premiss rule $\dfrac{}{\varphi \vdash \psi}$ is a proof of $\varphi \vdash \psi$. Given an instance of a rule

$$\frac{\varphi_1 \vdash \psi_1 \cdots \varphi_n \vdash \psi_n}{\varphi \vdash \psi}$$

together with proofs of each premiss $\varphi_i \vdash \psi_i$, we obtain a proof of $\varphi \vdash \psi$ by concatenating the proofs of each premiss with the rule instance. If a proof exists of

$\varphi \vdash \psi$ we say $\varphi \vdash \psi$ is *provable*. If one exists of $\top \vdash \varphi$ we say φ is provable.

The following deduction theorem can be verified directly from the rules and holds for all of the bunched logics we investigate. In particular, it enables us to straightforwardly interpret sequents $\varphi \vdash \psi$ as encodings of implications.

Theorem 2.1 (Deduction Theorem for Bunched Logics)**.** *For any bunched logic formulae φ, ψ, $\varphi \vdash \psi$ is provable iff $\varphi \to \psi$ is provable.* □

The most general semantics of LGL and ILGL is given on Kripke structures we call *(I)LGL frames*.

Definition 2.2 ((I)LGL Frame)**.** *An* ILGL frame *is a triple $\mathscr{X} = (X, \succcurlyeq, \circ)$ where X is a set, $\succcurlyeq$ a preorder on X and $\circ : X^2 \to \mathscr{P}(X)$ a binary operation. An* LGL frame *is an ILGL frame for which the order $\succcurlyeq$ is equality $=$.*

ILGL frames are a simple combination of the structures appropriate for interpreting intuitionistic propositional logic [142] and the non-associative Lambek calculus [86]—a preorder $(X, \succcurlyeq)$ and a non-determinstic composition $\circ : X^2 \to \mathscr{P}(X)$ respectively. This formal construction suffices to define a sound and complete semantics for the logic, but we will see when presenting layered graph models of the logic in Section 2.2 that meaningful examples of ILGL frames do in fact exist.

A *valuation* is a mapping $\mathscr{V} : \mathrm{Prop} \to \mathscr{P}(X)$; if X carries an order $\succcurlyeq$ that valuation is *persistent* if $x \in \mathscr{V}(\mathrm{p})$ and $y \succcurlyeq x$ implies $y \in \mathscr{V}(\mathrm{p})$. To understand the necessity of persistence requires some unpicking of the philosophical interpretation of intuitionistic logic's preorder semantics. In the Kripke semantics of intuitionistic logic the order is interpreted temporally [142]. The judgement $x \Vdash \mathrm{p}$ is thus interpreted as stating "by state x, p has been verified"; similarly, $x \nVdash \mathrm{p}$ is interpreted "at state x, p has yet to be verified". One might consider p to range over mathematical propositions, with verification supplied by the existence of a proof. It is understood that once something has been verified (or proved) it remains so. Hence valuations are necessarily persistent.

For intuitionistic logic persistence extends to the satisfaction of all formulae: $x \Vdash \varphi$ and $y \succcurlyeq x$ implies $y \Vdash \varphi$. This follows from the distinctive intuitionistic interpretation of negation and implication: $x \Vdash \neg\varphi$ iff φ does not get verified at any future state after x, and $x \Vdash \varphi \to \psi$ iff verification of φ at a future state after x will allow verfication of ψ. Crucucially, the archetypal classical reasoning principle $\varphi \vee \neg\varphi$ is not valid for this semantics: $\mathrm{p} \vee \neg\mathrm{p}$ will not hold at a state x at which $x \nVdash \mathrm{p}$ but at some future state y, $y \Vdash \mathrm{p}$, as $x \Vdash \neg\mathrm{p}$ would contradict persistence.

This purely philosophical interpretation is thus mathematically sound: persistence is in fact necessary to give the soundness of additive implication when it

x	$\vDash_{\mathscr{Y}}$	p	iff $x \in \mathscr{V}(\mathrm{p})$
x	$\vDash_{\mathscr{Y}}$	$\top$	always
x	$\vDash_{\mathscr{Y}}$	$\bot$	never
x	$\vDash_{\mathscr{Y}}$	$\varphi \wedge \psi$	iff $x \vDash_{\mathscr{Y}} \varphi$ and $x \vDash_{\mathscr{Y}} \psi$
x	$\vDash_{\mathscr{Y}}$	$\varphi \vee \psi$	iff $x \vDash_{\mathscr{Y}} \varphi$ or $x \vDash_{\mathscr{Y}} \psi$
x	$\vDash_{\mathscr{Y}}$	$\varphi \to \psi$	iff for all $y \succcurlyeq x$, $y \vDash_{\mathscr{Y}} \varphi$ implies $y \vDash_{\mathscr{Y}} \psi$
x	$\vDash_{\mathscr{Y}}$	$\varphi * \psi$	iff there exists x', y, z s.t. $x \succcurlyeq x' \in y \circ z$, $y \vDash_{\mathscr{Y}} \varphi$ and $z \vDash_{\mathscr{Y}} \psi$
x	$\vDash_{\mathscr{Y}}$	$\varphi \mathrel{-\!\!*} \psi$	iff for all x', y, z s.t. $x' \succcurlyeq x$ and $z \in x' \circ y$: $y \vDash_{\mathscr{Y}} \varphi$ implies $z \vDash_{\mathscr{Y}} \psi$
x	$\vDash_{\mathscr{Y}}$	$\varphi \mathrel{*\!\!-} \psi$	iff for all x', y, z s.t. $x' \succcurlyeq x$ and $z \in y \circ x'$: $y \vDash_{\mathscr{Y}} \varphi$ implies $z \vDash_{\mathscr{Y}} \psi$

Figure 2.2: Satisfaction for (I)LGL. LGL is given by the case where $\succcurlyeq$ is $=$.

cannot be decomposed using Boolean negation and prevents the validation of intuitionistically unsound principles like the laws of double negation and excluded middle.

The requirement of persistence should not be seen as a restriction on bunched logics with intuitionistic additives. Instead, variants with intuitionistic additives should be seen as being the right tools to reason about properties that are naturally persistent. For example, properties of models that are naturally inherited from part to whole—for example, the property of a path existing in a subsystem *persists* to the whole system—or, in logics with resource interpretations, properties that only need to be verified up to 'having enough'—for example, having sufficient resource to perform an action *persists* to greater amounts of resource.

An (I)LGL frame $\mathscr{X}$ together with a persistent valuation $\mathscr{V} : \mathrm{Prop} \to \mathscr{P}(X)$ gives a(n) *(I)LGL model* $\mathscr{M} = (\mathscr{X}, \mathscr{V})$. Given a(n) (I)LGL model $\mathscr{M}$, the satisfaction relation $\vDash_{\mathscr{Y}} \subseteq X \times \mathrm{Form}_{\mathrm{LGL}}$ is inductively generated by the clauses in Figure 2.2. The clauses for the standard connectives are straightforwardly the Kripke semantics for intuitionistic propositional logic. The clauses for the multiplicatives are guarded by $\circ$-statements to ensure the structural properties discussed earlier do not hold for $*$ as well as the adjoint relationship between $*$, $\mathrel{-\!\!*}$ and $\mathrel{*\!\!-}$. These clauses are not quite the standard semantics for the non-associative Lambek calculus, however: our clauses are additionally guarded by $\succcurlyeq$-statements to ensure persistence holds. Nonetheless, as we will see in Section 2.2 when we introduce layered graph models, this can be given a clear spatial interpretation.

We now fix some notation. Given a frame $\mathscr{X}$, the judgement $\mathscr{X} \vDash \varphi$ asserts that for every possible valuation $\mathscr{V}$ on $\mathscr{X}$ and every state $x \in X$, $x \vDash_{\mathscr{Y}} \varphi$. The judgement $\varphi \vDash_{\mathscr{Y}} \psi$ asserts that for every state x of the model $\mathscr{M}$, whenever $x \vDash_{\mathscr{Y}} \varphi$, it follows $x \vDash_{\mathscr{Y}} \psi$. $\varphi \vDash \psi$ asserts that $\varphi \vDash_{\mathscr{Y}} \psi$ holds for all models $\mathscr{M}$. Finally, $\vDash \varphi$

— read, φ is *valid* — asserts that $\top \vDash \varphi$. It is straightforward to prove that for all φ, ψ, whenever $\varphi \vdash \psi$ is provable in the Hilbert system, $\varphi \vDash \psi$ holds in the semantics. First we must show that persistence extends to all formulae for any ILGL model.

Lemma 2.3. *For any ILGL model $\mathscr{M}$, the satisfaction relation is persistent. That is, if $x \vDash \varphi$ and $y \succcurlyeq x$ then $y \vDash \varphi$.*

Proof. We give the case for $\mathrel{*\!\!-}$: the others are similar. Suppose $x \vDash \varphi \mathrel{*\!\!-} \psi$ and $y \succcurlyeq x$. We have that for all $x' \succcurlyeq x$, if $w \in z \circ x'$ and $z \vDash \varphi$ then $w \vDash \psi$. Thus if $y' \succcurlyeq y$ and $z \in w \circ y'$ with $z \vDash \varphi$, by transitivity $y' \succcurlyeq x$ so $w \vDash \psi$ by the assumption. Hence $y \vDash \varphi \mathrel{*\!\!-} \psi$. □

This allows us to prove soundness by an inductive argument on (I)LGL proofs.

Theorem 2.4 (Soundness of (I)LGL)**.** *$\varphi \vdash \psi$ is provable implies $\varphi \vDash \psi$. Similarly, φ is provable implies φ is valid.*

Proof. Soundness follows by showing that validity of premisses leads to validity of the conclusion for each rule. We demonstrate with the rule

$$15.\quad \frac{\xi \vdash \varphi \mathrel{*\!\!-} \psi \quad \eta \vdash \varphi}{\eta * \xi \vdash \psi}$$

for ILGL. Suppose $\xi \vDash \varphi \mathrel{*\!\!-} \psi$ and $\eta \vDash \varphi$. Let x be an arbitrary state in an ILGL model such that $x \vDash_{\mathscr{V}} \eta * \xi$. Then $x \succcurlyeq x' \in y \circ z$ with $y \vDash \eta$ and $z \vDash \xi$. By assumption $y \vDash_{\mathscr{V}} \varphi$ and $z \vDash_{\mathscr{V}} \varphi \mathrel{*\!\!-} \psi$ so $x \succcurlyeq x' \in y \circ z$ entails $x \vDash \psi_{\mathscr{V}}$ by persistence, as required. □

In Chapter 7 we will return to the opposite direction of *completeness*: $\varphi \vDash \psi$ implies that $\varphi \vdash \psi$ is provable.

2.2 Layered Graphs

The key motivation behind the formulation of layered graph logic is the modelling of *complex systems*. Complex systems can be defined as the field of science that studies, on the one hand, how it is that the emergent behaviour of a system, be it natural or synthetic, derives from the behaviours of its constituent parts and, on the other, how said system interacts with its environment. A commonly employed and highly effective concept that helps to manage the difficulty in conceptualizing and reasoning about complex systems is that of *layering*: the system is considered to consist of a collection of interconnected layers each of which has a distinct, identifiable role in the system's operations. Layers can be informational or physical and both kinds may be present in a specific system.

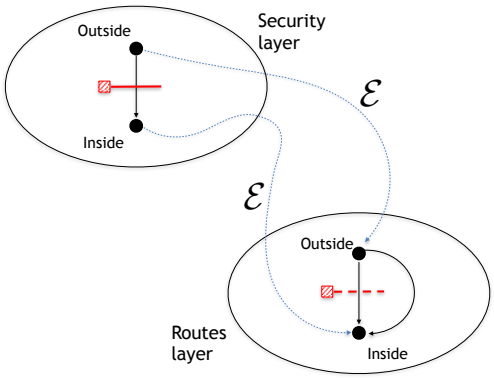


Figure 2.3: Layered graph representation of Schneier's gate.

In the work introducing LGL [63, 64] the systems under consideration were graph models of physical architecture layered with the security policies intended to apply to them. An illuminating example of the kind of mismatch that can arise when such layering isn't taken into account is highlighted by Schneier [205], recreated in Figure 2.3. To access a car park a token must be input at a barrier on the road. However, the space either side of the barrier is totally unsecured, with tiremark tracks indicating that many drivers choose to simply drive around the barrier instead. Here the intended security policy (a token-based barrier system) is undermined by the architecture it is applied to (a road with driveable paths beside it).

Graphs provide a suitably abstract setting for a wide variety of modelling purposes, and layered graphs already form a component of many existing systems modelling approaches. For example, both social networks [35] and transportation systems [143], have been modelled by a form of layered graph in which multiple layers are given by relations over a single set of nodes. A key feature of the TCP/IP conceptual model of communications on the Internet [57] is its separation into layers. This form of layering is not immediately represented in terms of graphs. However, the form of its information flows may be captured quite naturally using layered graphs [63]. Elsewhere layered graph models have been deployed to solve problems related to telecommunications networks [111] and to aid the design of P2P systems for businesses [219]. A bigraph [164] is a form of layered graph that superimposes a spatial *place graph* of locations and a *link graph* designating communication structure on a single set of nodes. Such graphs provide models of distributed systems and have been used to generalize process models like Petri nets and the π-calculus [163]. Similar ideas have also been used to give layered models of biological systems [153]. More generally, multilayer networks have become ubiquitous in a range of complex system modelling approaches (see [141] for a survey).

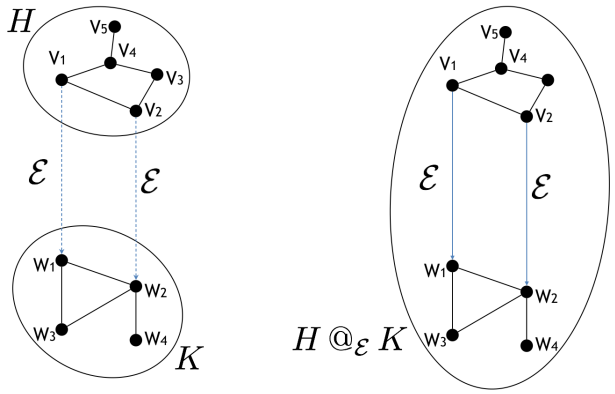


Figure 2.4: The graph composition $H @_{\mathscr{E}} K$.

2.2.1 The Layered Graph Construction

A notion of layered graph can be given that supplies a semantics to the layered graph logics. Informally, two layers in a directed graph are connected by a specified set of edges, each element of which starts in the upper layer and ends in the lower layer. This definition contrasts with prior accounts in which the layering structure is left implicit [92, 180], and generalises others which consider only a restricted class of layered graphs [182]. Requiring the layering structure to be explicit is necessary for it to be interpreted by multiplicative conjunction, while considering a more general class of layered graphs permits the modelling of a wider variety of examples.

Given a directed graph, $\mathscr{G}$, we refer to its *vertex set* by $V(\mathscr{G})$. Its edge set is given by a subset $E(\mathscr{G}) \subseteq V(\mathscr{G}) \times V(\mathscr{G})$, while its set of subgraphs is denoted $Sg(\mathscr{G})$. Here, H is a subgraph of $\mathscr{G}$ iff $V(H) \subseteq V(\mathscr{G})$ and $E(H) \subseteq V(\mathscr{G})$. We thus overload set theoretic inclusion to also refer to the subgraph relation: $H \subseteq \mathscr{G}$ iff $H \in Sg(\mathscr{G})$. For a distinguished edge set $\mathscr{E} \subseteq E(\mathscr{G})$, the reachability relation $\rightsquigarrow_{\mathscr{E}}$ on subgraphs of $\mathscr{G}$ is defined $H \rightsquigarrow_{\mathscr{E}} K$ iff there exist $u \in V(H)$ and $v \in V(K)$ such that $(u,v) \in \mathscr{E}$.

This generates a partial composition $@_{\mathscr{E}}$ on subgraphs. Let $\downarrow$ denote definedness and $\uparrow$ denote undefinedness. For subgraphs H and K, $H @_{\mathscr{E}} K \downarrow$ iff $V(H) \cap V(K) = \emptyset, H \rightsquigarrow_{\mathscr{E}} K$ and $K \not\rightsquigarrow_{\mathscr{E}} H$ with output given by the graph union of the two subgraphs and the $\mathscr{E}$-edges between them. Formally, if $H @_{\mathscr{E}} K \downarrow$, then $H @_{\mathscr{E}} K$ is defined by $V(H @_{\mathscr{E}} K) = V(H) \cup V(K)$ and $E(H @_{\mathscr{E}} K) = E(H) \cup E(K) \cup \{(u,v) \mid u \in V(H), v \in V(K)$ and $(u,v) \in \mathscr{E}\}$. Figure 2.4 shows a situation in which $H @_{\mathscr{E}} K$ is defined, as well as the composition itself.

A number of properties can be proved about $@_{\mathscr{E}}$ that indicate that (I)LGL is well suited for reasoning about it.

Proposition 2.5 (c.f. Proposition 2.3 [63])**.**

1. *No (left or right) unit can exist for* $@_{\mathscr{E}}$.

2. $@_{\mathscr{E}}$ *is not necessarily associative.*

3. $@_{\mathscr{E}}$ *is anti-commutative.*

Proof. For 1, suppose E is a right unit. Then for all subgraphs H, $H @_{\mathscr{E}} E = H$. However $H @_{\mathscr{E}} E \downarrow$ implies H and E are disjoint. Thus E can only possibly be the empty graph. However if this was the case, it would be impossible for $H \rightsquigarrow_{\mathscr{E}} E$ as E contains no vertex to be the target of an $\mathscr{E}$-arrow. Hence $H @_{\mathscr{E}} E \uparrow$, a contradiction. A similar argument also suffices for the existence of a left unit.

For 2, consider the graph G defined $V(G) = \{x,y,z\}$ and $E(G) = \{(x,y),(x,z)\} = \mathscr{E}$. Let $\{w\}$ designate the graph consisting of the single vertex w. Then $(\{x\} @_{\mathscr{E}} \{y\}) @_{\mathscr{E}} \{z\} \downarrow$, but—because $\{y\} @_{\mathscr{E}} \{z\} \uparrow$—the composition $\{x\} @_{\mathscr{E}} (\{y\} @_{\mathscr{E}} \{z\})$ is undefined. Hence $@_{\mathscr{E}}$ is not associative.

Finally, 3. follows immediately from the defintion of $@_{\mathscr{E}}$. □

We define a class of (I)LGL frames based on partial compositions $@_{\mathscr{E}}$ called (ordered) scaffolds. We first need the definition of *admissible subgraph set*: a subset $X \subseteq Sg(\mathscr{G})$ such that, for all $G,H \in Sg(\mathscr{G})$, if $G @_{\mathscr{E}} H \downarrow$, then $G,H \in X$ iff $G @_{\mathscr{E}} H \in X$. Admissible subgraph sets specify the layers of the model and act as the carrier of an (I)LGL frame. They are defined in order to exclude 'degenerate' cases of layering from a modelling perspective. For example, two disjoint subgraphs G and H may designate distinct, non-interacting regions in a systems model. However, their disjoint union would be interpreted as layered over the subgraph K if $G \rightsquigarrow_{\mathscr{E}} K$, even if $H \not\rightsquigarrow_{\mathscr{E}} K$. The solution is to specify that $G \cup H \notin X$.

Definition 2.6 ((Ordered) Scaffold)**.** *An* ordered scaffold *is a tuple* $(\mathscr{G},\mathscr{E},X,\succcurlyeq)$ *where* $\mathscr{G}$ *is a directed graph,* $\mathscr{E}$ *a distinguished edge set,* X *an admissible subgraph set and* $\succcurlyeq$ *a preorder on* X*. It is a* scaffold *when* $\succcurlyeq$ *is* $=$*.*

Clearly an ordered scaffold defines an ILGL frame $(X, @_{\mathscr{E}}, \preccurlyeq)$, thus ordered scaffolds are suitable for interpreting ILGL; similarly, a scaffold gives an LGL frame $(X, @_{\mathscr{E}})$. There are a number of candidates for $\succcurlyeq$ for any scaffold (for example, the subgraph relation or the preorder generated by layering); moreover, $\succcurlyeq$ can be used to model additional structure. For now, assume $\succcurlyeq$ is simply the subgraph ordering: $K \succcurlyeq H$ iff $K \supseteq H$. We can give an extremely intuitive understanding of all of the semantic clauses for ILGL from the perspective of a scaffold ordered by the subgraph relation.

As a simple example, suppose all propositional atoms p state the existence of particular paths and $G \vDash_{\mathscr{V}} \mathrm{p}$ is read: G contains the path p. This is, of course, persistent with respect to the order $\succcurlyeq$: if G contains a path and $H \supseteq G$ then H also contains that path. Thus $G \vDash_{\mathscr{V}} \mathrm{p} \to \mathrm{q}$ is read: for any extension H of G, if H

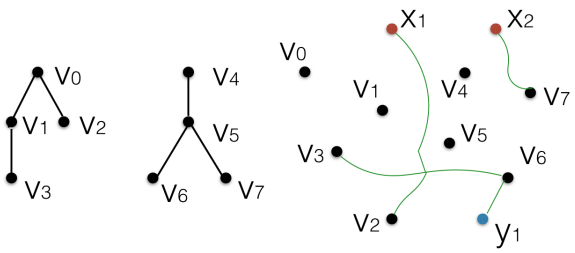


Figure 2.5: Place and link graphs.

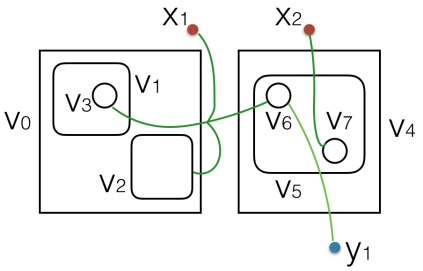


Figure 2.6: Bigraph.

contains the path p, it must also contain the path q. $G \vDash_{\mathscr{V}} \mathrm{p} * \mathrm{q}$ is read: G contains a subgraph that can be split into layers H and K such that H contains the path p and K contains the path q. The implications $-\!\!*$ and $*\!\!-$ express emergent properties of the system being modelled by the scaffold. If $G \vDash_{\mathscr{V}} \mathrm{p} -\!\!* \mathrm{q}$ then for any extension G' of G, if G' is layered over H and H contains the path p then the layered graph $G' @_{\mathscr{E}} H$ contains the path q. Here one might imagine the path q extends a path that necessarily exists in any extension of G. $*\!\!-$ is the same, except H is layered over G'.

While the completeness theorem we will give in Chapter 7 is given for a wider class of frames than scaffolds, in Chapter 10 we will show that a proof system that is sound and complete for the layered graph semantics of ILGL can be given.

2.2.2 A Bigraph Model of ILGL

We now give an example of a layered graph model in which the order is given by something distinct from a subgraph relation. At this juncture we emphasise that the logic and its layered graph semantics is only really capturing static, structural aspects of such systems. Realistic complex systems modelling requires a notion of evolution or dynamics at the very least, if not also resources and processes. Such an extension is outside of the scope of this thesis.

A bigraph [164] is comprised of a set of nodes on which a *place graph* and a *link graph* are defined. The place graph has the structure of a disjoint union of trees (a forest), while the link graph is a hypergraph on which one edge can connect many nodes. The place graph denotes spatial relationships, while the link graph denotes the communication structure of the system.

The link graph has additional structure: finite sets of labelled vertices $\{x_1,\ldots,x_n\}$, $\{y_1,\ldots,y_m\}$ denoting *inner names* and *outer names* respectively. These act as interfaces to enable the composition of bigraphs: if the outer names of a bigraph match the inner names of another, their link graphs may be connected at these vertices. Bigraphs are thus ideal for modelling distributed systems, which are similarly compositional. Bigraphical Reactive Systems (BRS) provide a dy-

namics for such models by defining transitions that reconfigure spatial relations and connectivity through graph rewriting. Such systems generalise a wealth of process calculi, including π-calculi and the CCS.

Figure 2.6 shows a bigraph and Figure 2.5 its consituent parts. The structure of the place graph is visually realised in the bigraph by the containment of its nodes. We now show how a system of composed bigraphs can be encoded as an ordered scaffold. Given we work with directed graphs, we model *directed* bigraphs [112].

We begin with a single bigraph. First, consider the link graph G. We can replace each hyperedge with a vertex attached to which we add an edge for each connection of the hyperedge. This obtains a directed graph with the same path information. For the place graph, note that a forest can straightforwardly be seen as a partial order on its vertices. This gives an order $\succcurlyeq$ on the set of subgraphs $\{\{v\} \mid v \text{ a vertex of the place graph}\}$. We also specify $G \succcurlyeq G$.

Now we consider a system of composed bigraphs. Given bigraphs $(G, \succcurlyeq)$, $(H, \succcurlyeq')$ where G has the same outer names as H's inner names, we can connect the outer name vertices of G to the inner name vertices of H with new edges. We collect all such edges as $\mathscr{E}$. Thus the composition $G \mathbin{@}_{\mathscr{E}} H$ denotes the composition of the link graphs $(G, \succcurlyeq)$ and $(H, \succcurlyeq')$, and we can take the disjoint union of the partial orders to obtain a bigraph $(G \mathbin{@}_{\mathscr{E}} H, \succcurlyeq \sqcup \succcurlyeq')$. In this way we obtain an ordered scaffold with the admissible subgraph set given by the closure under composition of the set $\{\{v\} \mid v \text{ a vertex of a place graph }\}$ together with each link graph G, and order generated by the union of the partial orders defined by the place graphs.

A propositional theory for such models can be given by decorating nodes of the bigraph with resources r. Intuitively, we interpret nodes of the place graph as locations which contain these resources. Then if a node x is decorated with a resource r, every place node that includes x is also interpreted as containing r. This generates a persistent valuation, where $G \vDash \mathrm{r}$ iff G contains the resource r. This can be extended further with propositional atoms $r \mapsto r'$ that are interpreted as stating the existence of a path from a location containing r to a location containing r'.

Chapter 3

Logics of Bunched Implications

We now consider O'Hearn and Pym's [177] logics of bunched implications, (B)BI. These variants are the most well investigated of all the bunched logics, owing in part to the incredible success of program verification techniques based on their semantics.

Much work has been done on the proof theory of these logics. BI enjoys a proof theory based on the bunched sequent calculus (sketched in Chapter 1) and a natural deduction system both formulated by O'Hearn & Pym [177, 187], as well as a labelled tableaux calculus (used to prove decidability of the logic) due to Galmiche et al. [101] and a display calculus due to Brotherston [38]. BI has been proved complete with respect to these systems for a number of different semantics: a topological semantics based on Grothendieck sheaves due to O'Hearn et al. [189], a Kripke style semantics based on composable resources due to O'Hearn & Pym [177] (generalised to coalgebra by Dahlqvist & Pym [70]) and an algebraic semantics given by extending Heyting algebras with the structure of a residuated commutative monoid, again due to O'Hearn & Pym [177]. This class of algebras has recently been investigated in more detail by Galatos & Jipsen [97].

The bunched sequent calculus is sufficiently well behaved that proofs in the calculus can be modelled by a categorical semantics in *doubly closed categories* (DCCs). DCCs are categories carrying two symmetric monoidal closed structures: one Cartesian (for the interpretation of the intuitionistic fragment) and one not (for the multiplicative connectives). This categorical interpretation motivates a corresponding bunched type system, the $\alpha\lambda$-calculus, introduced by Pym [187] and O'Hearn [174]. This system has been extended further to encorporate polymorphism by Collinson et al. [62]. A further computational application is provided by Armelìn & Pym's [13] formulation of *bunched logic programming*, a logic programming language based on a fragment of BI.

The proof theory of BBI is no less investigated than BI, although complications

arise when trying to formulate a bunched sequent calculus with good computational properties like cut elimination, something that Brotherston [38] speculates may not exist based on his analysis of the relation between BI's bunched sequent and display calculi. Brotherston [38] gives a display calculus and Park et al. [181] give a nested sequent calculus, both satisfying cut elimination, while other proof theory for BBI relies on the direct representation of the semantics via labels: examples include Larchey-Wendling's [148] labelled tableau calculus for BBI interpreted on models with a partial monoidal composition and Hóu et al.'s [127] labelled sequent calculus. All of these systems are proved complete with respect to the Kripke-style semantics and many of these proofs are obtained by encoding the Hilbert system for BBI in the respective calculus, a system proved complete by Galmiche & Larchey-Wendling [99]. It is not known if a proof system for BBI exists in which proofs can be interpreted categorically, however.

In contrast to BI, BBI is known to be undecidable. An algebraic proof of undecidability (in the context of residuated Boolean algebras) was given by Kurucz et al. [144], but remained unknown to the bunched logic community until the publication of Brotherston & Kanovich [42, 43] and Larchey-Wendling & Galmiche's [149] undecidability proofs. Undecidability appears to stem from the fact that BBI is highly expressive, something indicated by Galmiche & Larchey-Wendling's faithful embeddings of intuitionistic propositional logic, the modal logic S4 and BI itself into BBI. Its expressivity allows the encoding of models of computation like Minsky machines in its semantics. Brotherston & Villard [44] have also investigated the limits of BBI's expressivity, highlighting a number of properties common to memory models of the logic that cannot be defined by BBI formulae.

It is the *resource semantics* of the logic which have had the most impact, however. The key application of the logic is Separation Logic, a program verification formalism based on the theory of a particular model of (B)BI initially developed by Reynolds [196, 197], Ishtiaq & O'Hearn [129], and O'Hearn et al. [179]. We discuss Separation Logic in more detail in Section 3.2. Pym & Tofts [190], Collinson & Pym [61] and Anderson & Pym [10] have developed an approach to process algebra encorporating resource semantics for the use in systems and simulation modelling. In another direction, Abramsky & Väänänen [7] show that the semantics of BI naturally encapsulates the semantics of the dependence-sensitive logic IF: we discuss this example further in Section 3.3. Of course, the interpretation of multiplicative connectives as operations on resources has stimulated much research into the formulation of *other* bunched logics that we will discuss in Chapter 4.

For the most part, this chapter is concerned with understanding bunched implication logics as extensions of layered graph logics. In particular, the semantic

16. $\dfrac{}{(\varphi * \psi) * \xi \vdash \varphi * (\psi * \xi)}$ 17. $\dfrac{}{\varphi * \psi \vdash \psi * \varphi}$ 18. $\dfrac{}{\varphi * \top^* \dashv\vdash \varphi}$

Figure 3.1: Hilbert rules for logics of bunched implications.

structures we consider (directly extending the frames and semantics for layered graph logics) are defined slightly different to those typically found in the literature. We explain how these semantic approaches are related and show that our approach generates the same class of valid formulae as prior approaches. We also take an opportunity to introduce Separation Logic and highlight some other (optional, for the interested reader) models of bunched implication logic that can be found in computer science. Parts of this chapter are based on the journal paper *Stone-Type Dualities for Separation Logics* [83].

3.1 Syntax and Semantics

Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of the logics of bunched implications $\mathrm{Form}_{\mathrm{BI}}$ is generated by the grammar

$$\varphi ::= \mathrm{p} \mid \top \mid \bot \mid \top^* \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi,$$

with additive negation defined by $\neg\varphi ::= \varphi \to \bot$.

Figure 3.1 gives the rules that need to be added to the Hilbert systems of the layered graph logics to obtain systems for (B)BI. To obtain a system for BI—the system with intuitionistic additives—we add these rules to the system for ILGL, and for BBI—the system with classical additives—we add the rules to the system for LGL. These rules represent additional structural rules of the sequent calclulus: 16. is associativity of the multiplicative context former, 17. is the Exchange rule and 18.—with $\dashv\vdash$ indicating that the judgement can be read in either direction—represents the addition of a unit for the multiplicative context former. One simple consequence of adding these rules is that the connectives $\mathrel{-\!\!*}$ and $\mathrel{*\!\!-}$ become equivalent, exactly inverting the splitting of multiplicative implication into two connectives when Exchange does not hold. Hence we do not consider $\mathrel{*\!\!-}$ in the grammar of (B)BI.

The most general semantics of BI and BBI is given on Kripke structures we call *(B)BI frames*. Appropriately, these structures extend LGL and ILGL frames.

Definition 3.1 ((B)BI Frame)**.** *A* BI frame *is a triple* $\mathscr{X} = (X, \succcurlyeq, \circ, E)$ *where* $(X, \succcurlyeq, \circ)$ *is an ILGL frame,* $E \subseteq X$ *and the following conditions are satisfied (with*

$$x \models_{\mathscr{M}} \top^* \text{ iff } x \in E$$

Figure 3.2: Satisfaction for (B)BI.

outermost universal quantification omitted for readability):

$$\begin{array}{ll} \textit{(Commutativity)} \quad z \in x \circ y \to z \in y \circ x & \textit{(Closure)} \quad e \in E \wedge e' \succcurlyeq e \to e' \in E \\ \textit{(Unit Existence)} \quad \exists e \in E(x \in x \circ e) & \textit{(Coherence)} \quad e \in E \wedge x \in y \circ e \to x \succcurlyeq y \\ \multicolumn{2}{l}{\textit{(Associativity)}\ t' \succcurlyeq t \in x \circ y \wedge w \in t' \circ z \to \exists s, s', w'(s' \succcurlyeq s \in y \circ z \wedge w \succcurlyeq w' \in x \circ s')} \end{array}$$

A BBI frame *is a BI frame for which the order $\succcurlyeq$ is equality $=$.*

A (B)BI frame $\mathscr{X}$ together with a persistent valuation $\mathscr{V}$ gives a (B)BI model $\mathscr{M}$, and for such a model the satisfaction relation $\models_{\mathscr{M}} \subseteq X \times \mathrm{Form}_{\mathrm{BI}}$ is inductively generated by the satisfaction clauses for layered graph logic, extended with the condition for $\top^*$ given in Figure 3.2. By Closure this clause clearly satisfies persistence, so persistence of satisfaction is a corollary of the case for layered graph logic.

To expound the meaning of these structures we outline an interpretation that is strictly more general than the O'Hearn & Pym [177] analysis sketched in Chapter 1. Recent work by Coecke et al. [60] and Fritz [94] outlines an algebraic framework for *resource theories* inspired by notions from quantum information theory. Such theories are represented by ordered commutative monoids in which composition straightforwardly represents the composition of resources but the order $\succcurlyeq$ is generated by a background theory of the conversion of resources: for example, the conversion of chemical substances into others via chemical reactions.

Our version of resource semantics can be seen as a synthesis of the original resource semantics of O'Hearn & Pym [177] together with this later analysis. The benefits of this synthesis are twofold: on the one hand it expands the convertibility analysis to account for partiality, a property that has been crucial for both the metatheory and application of bunched logic; on the other, it broadens our view of the kind of phenomena bunched logic can reason about by considering a strictly more general interpretation of the necessary semantic structures for BI. Crucially, the mathematical and logical content for both is identical.

The carrier set X of a BI frame is considered to be a set of resources, ordered by a *conversion ordering* $\succcurlyeq$. We distinguish between two different interpretations of the order and name them for the kind of propositional atoms each viewpoint is suitable for reasoning about. The first is the *invariant interpretation*. In the invariant interpretation we read $x \preccurlyeq y$ as "x converts to y". Persistence of the valuation $\mathscr{V}$ has the consequence that propositional atoms must be *invariants of conversion*:

if p is true at x and x converts to y then p is true at y. In contrast, the *sufficiency interpretation* reads $y \succcurlyeq x$ as "y converts to x". Persistence here means that propositional atoms are *judgements of sufficiency*: if p is true at x then any resource y that converts into x is also sufficient to make p true.

Let's pause to consider some simple examples exemplifying these perspectives. We consider the chemical reaction model of Coecke et al. [60] and the money model of Pym et al. [189] in their respective analyses of resource. The first, corresponds to the invariant interpretation. The basic idea is that, given a particular experimental set up, it is possible to convert chemical substances into others. For example, the chemical equation describing the conversion of one methane and two oxygen molecules into one carbon dioxide and two water molecules is given by

$$CH_4 + 2O_2 \to CO_2 + 2H_2O.$$

We can consider the resource set X to be the set of all chemical substances and $\succcurlyeq$ to be the conversion ordering generated by chemical reactions using the invariant interpretation. That is, the above chemical equation is witnessed as $CH_4 + 2O_2 \preccurlyeq CO_2 + 2H_2O$. One of the simplest facts about chemical reactions is that the number and types of atoms are necessarily preserved by conversion. For example, $CH_4 + 2O_2$ is comprised of 9 atoms—1 carbon atom, 4 hydrogen atoms and 4 oxygen atoms—and so too is $CO_2 + 2H_2O$. We can thus give a persistent valuation witnessing this *invariant* by defining $\mathscr{V} : \mathbb{N} \times \text{Periodic} \to \mathscr{P}(X)$ where $x \in \mathscr{V}(n, y)$ if the substance x contains n y-atoms.

For an example of a sufficiency interpretation we can consider O'Hearn & Pym's [177] original motivating example for BI. Consider the natural numbers $\mathbb{N}$ ordered by the standard $\geq$. We interpret $n \in \mathbb{N}$ as designating the cash value £n, while the ordering can be seen as that generated by the notion of conversion "spending some money elsewhere" under the sufficiency interpretation. For example, £$10 \succcurlyeq$ £6 because £10 converts to £6 via spending £4 elsewhere. Now propositional atoms are given by items one might purchase with their money. Suppose an apple costs £1. Then any amount of money over £1 is sufficient to purchase an apple. The valuation $\mathscr{V}$ defined by £$x \in \mathscr{V}(\text{Item})$ iff £x is enough to buy Item is thus persistent and witnesses the *sufficiency* of the amount of resource to perform a given task.

Every frame comes equipped with a notion of resource composition $\circ$. We do not commit to some aspects of the composition at this stage (e.g. partial vs total, deterministic vs non-deterministic) for two reasons: first, it is known that the Hilbert system for (B)BI is incomplete when restricted to total and/or deterministic composition [150]; second, well-motivated BI models can be found for all com-

binations of these properties. Hence we state that the composition is an operation $\circ : X^2 \to \mathscr{P}(X)$, which is sufficient to subsume all possible variations. What *is* enforced is that this composition is commutative (given by the axiom Commutativity) and that it satisfies a generalised notion of associativity (given by the axiom Associativity) that is compatible with partiality and the ordering. As we progress further we will explain the conditions under which this axiom can be replaced with a more familiar looking one, For now we simply state that such conditions hold for our running example, and thus our interpretation of both the $+$ in chemical equations and the $+$ given by addition on $\mathbb{N}$ as $\circ$ follows from the standard associativity property $x+(y+z) = (x+y)+z$.

The readings of the semantic clauses for $*$ and $-\!\!*$ are subtly different depending on whether our BI frame is interpreted for invariance or sufficiency. Recall that the semantic clause for $\varphi * \psi$ is given by

$$x \vDash_{\mathscr{M}} \varphi * \psi \text{ iff there exists } x', y, z \text{ s.t. } x \succcurlyeq x' \in y \circ z, y \vDash_{\mathscr{M}} \varphi \text{ and } z \vDash_{\mathscr{M}} \psi.$$

Under the invariance interpretation, this is read as "there is a resource x' made up of a φ-resource y and a ψ-resource z such that x' converts to x. In the chemistry example, we have that $CO_2 + 2H_2O \vDash_{\mathscr{M}} (1C \wedge 4H) * 4O$ because $CH_4 \vDash_{\mathscr{M}} 1C \wedge 4H$ (is made up of 1 carbon atom and 4 hydrogen atoms), $2O_2 \vDash_{\mathscr{M}} 4O$ (is made up of 4 oxygen atoms) and the chemical equation $CH_4 + 2O_2 \to CO_2 + 2H_2O$ holds. Thus $*$ formulae witness the *conversion invariant* given by the distribution of properties across the components required to obtain the resource currently at hand through composition and conversion.

Under the sufficiency interpretation, this is read as "x can be converted into a resource x' that is sufficient to be split into separate resources, y and z such that y is a φ-resource and z is a ψ-resource. In the money example, if apples cost £1 and a bunch of bananas cost £2 then £5 $\vDash_{\mathscr{M}}$ apple $*$ banana because we can spend £2 elsewhere and still have sufficient cash to buy an apple for £1 and separately a bunch of bananas with our remaining £2. Thus $*$ formulae witness the *sufficiency* of the current resource at hand to be split to perform two parallel tasks, modulo a conversion freely available in the system.

The semantic clause for $\varphi -\!\!* \psi$ is given by

$$x \vDash_{\mathscr{M}} \varphi -\!\!* \psi \text{ iff for all } x', y, z \text{ s.t. } x' \succcurlyeq x \text{ and } z \in x' \circ y : y \vDash_{\mathscr{M}} \varphi \text{ implies } z \vDash_{\mathscr{M}} \psi.$$

Under the invariance interpretation, this is read as "for any x' that the resource x converts into, a successful composition with a φ-resource y gives a ψ-resource z. In

the chemistry example we have $CO_2 \vDash_{\mathscr{M}} 2H \mathrel{-\!\!*} (C \wedge 2O \wedge 2H)$: anything that CO_2 converts to still has 1 carbon atom and 2 oxygen atoms. Thus if it is combined with anything that contains 2 hydrogen atoms, the resulting composition of chemical substances will contain the atoms that originated in CO_2, as well as those hydrogen atoms.

Under the sufficiency interpretation, this is read as "for any resource x' that can be converted to x, successfully composing with a φ-resource y gives a ψ-resource z". In the money example, $£1 \vDash \text{apple} \mathrel{-\!\!*} \text{banana}$: for any amount of money greater than or equal to £1, adding sufficient money to buy an apple (£1) gives you sufficient money to buy a bunch of bananas.

We finally consider the E component of a BI frame. Under the invariance interpretation, E is interpreted as the set of resources convertible *from* unit resources: the *free resources* of the system. Free resources play an important role in the resource theories of quantum information theory [60]. For example, when considering entanglement as a resource that can be used for the quantum information processing task of teleportation, the free resources are those given by classical communication and local operations (e.g., those that do not increase entanglement). We thus interpret $x \vDash_{\mathscr{M}} \top^*$ as simply stating x is a free resource: a property that is conversion-invariant.

Under the invariance interpretation, the axiom Unit Existence states that all resources have some compatible unit resource in E and Closure ensures that everything convertible from a resource in E is also in E. The Coherence axiom has two important roles. First, it (together with the other axioms governing E) ensures that E is comprised of the free resources and *only* the free resources. Let $e \in E$. Then by unit existence there exists $e' \in E$ that is a unit for e: $e \in e \circ e'$. By Commutativity, $e \in e' \circ e$ and by Coherence $e \succcurlyeq e'$ so e is a free resource. That every free resource is in E is a trivial consequence of Unit Existence and Closure. Second, it enforces a minimal coherence property between $\circ, \succcurlyeq$ and E: if you can obtain x by composing y with a free resource, then from the perspective of the system you can obtain x by conversion from y alone. In the chemistry example, because of conservation of mass only the empty chemical substance $\emptyset$ is a free element. Thus $E = \{\emptyset\}$.

Under the sufficiency interpretation, E is interpreted as the set of resources that convert *into* unit resources. We call these the *unit-convertible* resources of the frame. Intuitively, unit convertibles are simply the resources that can get used up by conversion in the system, and we can interpret $x \vDash_{\mathscr{M}} \top^*$ as stating x is unit-convertible. Like in the invariance interpretation, it follows from the BI frame axioms that E is comprised *only* of the unit-convertible resources. Coherence now states that if x is obtained by composing y with a unit-convertible resource e, x con-

verts to y by the conversion of the e-component of x into a compatible unit for x. In the money model, sadly any amount of money can be converted to nothing by spending it all. Hence $E = \mathbb{N}$.

We've yet to talk about BBI frames, but they have a very simple interpretation as a specific case of the analysis just outlined. Essentially, BBI frames are an analysis of resource where conversion is either not possible or ignored, as demonstrated by the order of a BBI frame being $=$. A lot is simplified for the definition of BBI frame: Closure becomes redundant, the other E rules now simply specify that E is the set of unit resources and Associativity collapses to the Simple Associativity axiom

$$t \in x \circ y \wedge w \in t \circ z \to \exists s(s \in y \circ z \wedge w \in x \circ s). \tag{3.1}$$

The semantic clause for $*$ now witnesses the direct decomposition of a resource while that for $-\!\!*$ witnesses the direct composition of a resource with an arbitrary one satisfying the antecedent.

Some of this simplification also holds for BI frames that satisfy special properties. To aid the understanding of how and why this is possible it is instructive to examine the soundness proof for BI.

Theorem 3.2 (Soundness of (B)BI)**.** *$\varphi \vdash \psi$ is provable implies $\varphi \vDash \psi$. Similarly, φ is provable implies φ is valid.*

Proof. We focus on the case we will concentrate on for the rest of this section: $*$-associativity for BI. Assume $r \vDash (\varphi * \psi) * \xi$. Then $r \succcurlyeq w \in t' \circ z$ such that $t' \vDash \varphi * \psi$ and $z \vDash \xi$. It follows that $t' \succcurlyeq t \in x \circ y$ with $x \vDash \varphi$ and $y \vDash \psi$. From here we can apply the BI frame axiom Associativity to obtain s, s', w' such that $s' \succcurlyeq s \in y \circ z$ and $w \succcurlyeq w' \in x \circ s'$. It follows that $x \vDash \varphi$ and $s' \vDash (\psi * \xi)$ so $w \vDash \varphi * (\psi * \xi)$, and thus by persistence $r \vDash \varphi * (\psi * \xi)$. □

Now in most presentations of BI [177, 189, 101] the satisfaction clause for $-\!\!*$ is given by

$$x \vDash \varphi -\!\!* \psi \text{ iff for all } y, z : z \in x \circ y \text{ and } y \vDash \varphi \text{ implies } z \vDash \psi, \tag{3.2}$$

with $=$ instead of $\in$ in presentations of BI models as monoids. This is precisely what the semantic clause for $-\!\!*$ works out as in our semantics for the special case of BBI models. However this is not sound in general for our BI models (which are defined in such a way to directly extend ILGL) as this clause does not always satisfy persistence. In the original monoid models (where $\circ$ is total and deterministic) an additional condition called *bifunctorality* is assumed. Bifunctorality is a very natural condition relating the composition $\circ$ with the conversion ordering $\succcurlyeq$ and

dictates that given $x' \succcurlyeq x$ and $y' \succcurlyeq y$ it follows that $x' \circ y' \succcurlyeq x \circ y$. Our running examples of chemical reactions and money (the natural numbers) are bifunctorial: if two different chemical reactions are possible they can be run in parallel and $+$ is straightforwardly monotone with respect to the $\leq$ on $\mathbb{N}$.

For a bifunctorial (total deterministic) ordered monoid we have that the clause (3.2) satisfies persistence: if $x \vDash \varphi \mathrel{-\!\!*} \psi$ and $x' \succcurlyeq x$, then for any y, $x' \circ y \succcurlyeq x \circ y$ by bifunctorality, and so $y \vDash \varphi$ implies $x' \circ y \vDash \psi$ by our assumption together with persistence of satisfaction for ψ. Soundness of $*$-associativity also follows from the standard associativity of $\circ$ in such a model: if $x \vDash (\varphi * \psi) * \xi$ there exists y, z such that $x \succcurlyeq y \circ z$ with $y \vDash (\varphi * \psi)$ and $z \vDash \xi$. Thus there exist w, v such that $y \succcurlyeq w \circ v$ with $w \vDash \varphi$ and $v \vDash \psi$. By associativity of $\circ$ and bifunctorality, $w \circ (v \circ z) = (w \circ v) \circ z \preccurlyeq y \circ z \preccurlyeq x$ so $x \vDash \varphi * (\psi * \xi)$.

Generalising bifunctorality for partial $\circ$ in order to permit proofs of these properties is a delicate matter; the previous arguments can collapse without a guarantee that certain compositions exist. Cao et al. [51] identify one such generalisation, calling it the *Downwards Closed* property:

$$z \in x \circ y \wedge x \succcurlyeq x' \wedge y \succcurlyeq y' \rightarrow \exists z' (z \succcurlyeq z' \wedge z' \in x' \circ y'). \tag{3.3}$$

This is a stronger coherence property that the minimum required of a BI frame to soundly interpret the logic. On either conversion interpretation of $\succcurlyeq$ the property essentially says conversion of components lifts to conversion of compositions: a very natural idea. What is perhaps less natural is the enforcement of the existence of $z' \in x' \circ y'$ as x' and y' may be incompatible in general.

By a similar argument to that for bifunctoral monoids, a model being Downwards Closed is sufficient to prove that the simpler semantic clause (3.2) satisfies persistence, and in this case the two possible clauses for $\mathrel{-\!\!*}$ are equivalent. It also entails that satisfaction of the Simple Associativity axiom (3.1) is sufficient to prove soundness of $*$-associativity. Suppose $x \vDash (\varphi * \psi) * \xi$. Then there exists v, w, x', y, y', z such that $x \succcurlyeq x' \in y \circ z$ and $y \succcurlyeq y' \in v \circ w$ with $v \vDash \varphi$, $w \vDash \psi$ and $z \vDash \xi$. By Downwards Closed (3.3) there exists $x'' \in y' \circ z$ such that $x' \succcurlyeq x$ and by Simple Associativity $y' \in v \circ w$ and $x'' \in y' \circ z$ implies there exists t such that $x'' \in v \circ t$ and $t \in w \circ z$. Hence $x \vDash \varphi * (\psi * \xi)$ as required.

Similarly, the satisfaction clause for $*$ is occasionally (albeit rarely) given as

$$x \vDash \varphi * \psi \text{ iff there exists } y, z \text{ such that } x \in y \circ z, y \vDash \varphi \text{ and } z \vDash \psi \tag{3.4}$$

in presentations of BI [129, 119]. Once again, this is both what our semantic clause

reduces to in the special case of a BBI model and also not sound in general for BI models as persistence can fail. It is, however, satisfied on models that have what Cao et al. call the *Upwards Closed* property:

$$z \in x \circ y \wedge z' \succcurlyeq z \to \exists x', y' (z' \in x' \circ y' \wedge x' \succcurlyeq x \wedge y' \succcurlyeq y). \tag{3.5}$$

The conversion interpretation of this property is that conversion drops from composition down to components. This is perhaps less natural on our reading than Downwards Closed is, given that conversion may only be possible because of the combination of the resources x and y. Looking once again at the chemical reaction $CH_4 + 2O_2 \to CO_2 + 2H_2O$, we have $CH_4 \not\to CO_2$, $2O_2 \not\to 2H_2O$, $CH_4 \not\to 2H_2O$ and $2O_2 \not\to CO_2$: atoms from both components of $CH_4 + 2O_2$ are required for each component CO_2 and $2H_2O$. This property *is* satisfied by our money model, however.

Nonetheless, the property makes mathematical sense, and similarly to the case for Downwards Closed and $-\!\!*$, this condition makes the two possible satisfaction clauses of $*$ equivalent. It also renders the Simple Associativity axiom sufficient to prove soundness of $*$-associativity because of the direct decomposition in the simpler satisfaction clause for $*$. A structure for modelling BI which is both upwards and downwards closed can thus be defined with the Simple Associativity axiom and has its semantics presented identically to BBI. This is the case for the original BI model of Separation Logic presented by Ishtiaq and O'Hearn [129], a fact that is commented upon, albeit without formulating the sufficient conditions for it to always be the case.

Cao et al. show that any structure satisfying either Upwards or Downwards Closed, together with Simple Associativity, can be conservatively transformed into a sound model of BI satisfying all three. They note that applying each transformation in sequence from a structure satisfying only Simple Associativity does not result in a sound Upwards and Downwards Closed BI model though. However it *is* the case that any BI frame can be conservatively transformed into one satisfying all three properties: the key is starting with the more general BI frame axiom Associativity, a possible BI model that the authors do not consider.

Proposition 3.3. *Any BI frame $\mathscr{X}$ can be transformed into an Upwards and Downwards Closed, Simple Associative frame $\mathscr{X}^{\Uparrow\Downarrow}$ with the same carrier such that for any valuation $\mathscr{V} : \mathrm{Prop} \to \mathscr{P}(X)$ every element x satisfies the same formulae in $\mathscr{X}^{\Uparrow\Downarrow}$ as it does in $\mathscr{X}$.*

Proof. Given a BI frame $\mathscr{X} = (X, \succcurlyeq, \circ, E)$, define its upwards and downwards closure by $\mathscr{X}^{\Uparrow\Downarrow} = (X, \succcurlyeq, \circ^{\Uparrow\Downarrow}, E)$ where $x \in y \circ^{\Uparrow\Downarrow} z$ iff there exist x', y', z' such that

$x \succcurlyeq x'$, $y' \succcurlyeq y$, $z' \succcurlyeq z$ and $x' \in y' \circ z'$ (cf. [51]). That this is Upwards and Downwards Closed is straightforward. To see that Simple Associativity is satisfied, suppose $t \in x \circ^{\Uparrow\Downarrow} y$ and $w \in t \circ^{\Uparrow\Downarrow} z$. By definition this entails that there are t', x', y' such that $t \succcurlyeq t' \in x' \circ y'$ with $x' \succcurlyeq x$ and $y' \succcurlyeq y$. Similarly there are w', t'', z' such that $w \succcurlyeq w' \in t'' \circ z$ with $t'' \succcurlyeq t$ and $z' \succcurlyeq z$. Applying Associativity for $\circ$ we obtain s, s', w'' such that $s' \succcurlyeq s \in y' \circ z'$ and $w' \succcurlyeq w'' \in x' \circ s'$. We thus obtain $s \in y \circ^{\Uparrow\Downarrow} z$ immediately. To see that $w \in x \circ^{\Uparrow\Downarrow} s$ note that $w \succcurlyeq w' \succcurlyeq w'' \in x' \circ s'$ with $x' \succcurlyeq x$ and $s' \succcurlyeq s$. A straightforward inductive argument shows that the satisfaction relations generated by any valuation $\mathscr{V}$ are equivalent for these models. □

Not only this, but any structure satisfying all three properties also satisfies the BI frame axiom Associativity.

Proposition 3.4. *Let $(X, \preccurlyeq, \circ)$ satisfy Simple Associativity and Upwards and Downwards Closed. Then $(X, \preccurlyeq, \circ)$ also satisfies the BI frame axiom Associativity.*

Proof. Assume all three properties hold and suppose $t' \succcurlyeq t \in x \circ y$ and $w \in t' \circ z$. By Upwards Closed there exist x' and y' such that $t' \in x' \circ y'$, $x' \succcurlyeq x$ and $y' \succcurlyeq y$. By Simple Associativity we obtain s' such that $s' \in y' \circ z$ and $w \in x' \circ s'$. By Downwards Closed we obtain s such that $s' \succcurlyeq s \in y \circ z$ from $s' \in y' \circ z$ and $y' \succcurlyeq y$. By Downwards Closed again we obtain $w \succcurlyeq w' \in x \circ s'$ from $w \in x' \circ s', x' \succcurlyeq x$ and $s' \succcurlyeq s'$. Hence Associativity is satisfied. □

Thus BI frames and the semantics extending the layered graph logic semantics genuinely generalises the many different formulations of BI found in the literature. In Part II we'll see the mathematical reason for this in the form of topological duality. Most importantly, every sound choice one can make regarding the closure and associativity properties defines the same set of valid formulae. This allows us to prove theorems about BI and its intended models by straightforwardly extending those we prove for ILGL. One might suppose we could choose one of the more familiar presentations of BI just discussed and instead stipulate the ILGL frames satisfy the same closure properties for $\circ$ and $\preccurlyeq$, but unfortunately this would have the undesirable effect of excluding the intended models of ILGL.

3.2 Separation Logic

The principal application of (B)BI (and indeed, bunched logic more generally) is the program verification formalism Separation Logic. Introduced by Reynolds [196, 197], Ishtiaq & O'Hearn [129], and O' Hearn et al. [179], Separation Logic enabled a paradigm shift in the application of formal methods to the verification of programs that manipulate mutable data structures.

The problem of verifying such programs occurs at the level of *pointers*, which are programming language objects that store a memory address to be used in computations. Constructs that dynamically allocate memory (for example, lists and trees) are typically defined through the use of pointers. A program may utilise many pointers, and *aliasing* occurs when multiple pointers store the same memory address. If two pointers alias each other, an update of the value at the memory address they both reference can have serious ramifications in distinct parts of the program, and serious faults like buffer overflows and memory leaks can occur because of this happening. Traditional approaches to program verification (e.g., Hoare logic [118, 12]) are ill-equipped to identify these problems in a program because they are syntax-directed, and these issues may arise in syntactically distinct expressions (see Bornat [31] for an illustration of this issue).

This can be illustrated with a counterexample of Reynolds [198] to the validity of the *rule of constancy* for Hoare logic in the presence of pointers. Hoare logic is a proof system for deriving triples of the form $\{\varphi\}C\{\psi\}$, where φ, ψ are formulae of predicate logic (the *assertion language* of Hoare logic) and C is a program. In such a triple, φ is a *precondition* that holds of a state prior to the execution of C and ψ a *postcondition* that must hold after execution. The rule of constancy is given by

$$\frac{\{\varphi\}C\{\psi\}}{\{\chi \wedge \varphi\}C\{\chi \wedge \psi\}},$$

where C does not affect any of the free variables of χ. It can be thought of as a scalability rule, allowing the passage from a local specification of C to a global specification by considering χ to be an assertion satisfied by the state that isn't touched by C. However this rule is not sound for pointer programs. Let $[x] := n$ denote the assignment of the memory address specified by x to the value n, $x \mapsto n$ a predicate stating that the pointer named by x points to n and $x \mapsto -$ (defined $\exists m.x \mapsto m$) a predicate stating that the address x is active. Then

$$\frac{\{x \mapsto -\}[x] := 4\{x \mapsto 4\}}{\{y \mapsto 3 \wedge x \mapsto -\}[x] := 4\{y \mapsto 3 \wedge x \mapsto 4\}}$$

can fail if y is aliased by x.

The resource semantics of (B)BI solves this issue by allowing the definition of an assertion language that is able to express "φ, *and separately in memory* ψ" with the multiplicative conjunction $\varphi * \psi$ for a model in which memory is resource. A *heap* is a partial allocation of memory addresses to values: formally, a partial function $h : \mathbb{N} \to V$, where V is a value-set (generally $\mathbb{Z}$). Given heaps h and h',

$h\#h'$ denotes that $dom(h) \cap dom(h') = \emptyset$; $h \cdot h'$ denotes the union of functions with disjoint domains, which is defined iff $h\#h'$. The *empty heap*, $[]$, is defined nowhere.

Let H denote the set of all heaps. Then $\text{Heap}_{\text{BBI}} = (H, \cdot, \{[]\})$ is a BBI frame. Letting $h' \succcurlyeq h$ denote that h' extends h, $\text{Heap}_{\text{BI}} = (H, \succcurlyeq, \cdot, H)$ defines a BI frame. These frames generate the standard classical and intuitionistic propositional models of Separation Logic. In such models, atomic propositions are pointer assertions $x \mapsto y$ which are satisfied by heaps h for which $h(x) = y$ in the intuitionistic case and by the singleton heap $h = \{(x,y)\}$ in the classical case. In the case of Heap_{BI}, $\succcurlyeq$ may coarsely be thought of as the ordering generated by the notion of conversion given by a garbage-collector (a mechanism that automatically deallocates addresses when it determines them to be no longer in use) under the sufficiency interpretation. A more fine-grained analysis of this idea can be found in the work of Calcagno [48]. As everything can be converted into the empty heap in this fashion, the set of unit-convertibles is the entire set of possible heaps.

The assertion language of Separation Logic arises as an extension of the interpretation of (B)BI on this model to handle additional program constructs as well as additive quantification. The formulae of the assertion language are given by the following grammar, where the expressions E, E' are built using Booleans, variables, cons cells and atomic expressions.

$$\varphi ::= E = E' \mid E \mapsto E' \mid \top \mid \bot \mid \top^* \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi \mid \exists v.\varphi \mid \forall v.\varphi.$$

Denotations $\{\{E\}\} \in V$ of expressions E are determined by the *store* s, a partial function mapping variables to values $a \in V$. The store represents memory to which the values of variables is automatically allocated, while the heap represents dynamically allocated memory. Given a store s, $[s \mid v \mapsto a]$ is the store that is equal to s except that v maps to a;

Figure 3.3 gives all of the semantic clauses of Separation Logic's assertion language except those pertaining to the pointer predicate $\mapsto$. The classical interpretation of $\mapsto$ requires E to be the only active address in the current heap,

$$s,h \vDash E \mapsto F \text{ iff } \{\{E\}\}s = \text{dom}(h) \text{and } h(\{\{E\}\}s) = \{\{F\}\}s,$$

whereas the intuitionistic interpretation is the weaker judgement that E is *at least* one of the active addresses in the current heap

$$s,h \vDash E \mapsto F \text{ iff } \{\{E\}\}s \in dom(h) \text{and } h(\{\{E\}\}s) = \{\{F\}\}s.$$

The judgement, $s,h \vDash \varphi$ states that the assertion φ holds for a given store and heap,

$s,h \vDash E = E'$ iff $\{\{E\}\}s = \{\{E'\}\}s$

$s,h \vDash \top$

$s,h \nvDash \bot$

$s,h \vDash \varphi \wedge \psi$ iff $s,h \vDash \varphi$ and $s,h \vDash \psi$

$s,h \vDash \varphi \vee \psi$ iff $s,h \vDash \varphi$ or $s,h \vDash \psi$

$s,h \vDash \varphi \to \psi$ iff for all $h' \succcurlyeq h$, $h' \vDash \varphi$ implies $h' \vDash \psi$

$s,h \vDash \top^*$ iff $h \succcurlyeq []$

$s,h \vDash \varphi * \psi$ iff there exists h', h'' s.t. $h \# h'$, $h = h' \cdot h''$, $s,h' \vDash \varphi$ and $s,h'' \vDash \psi$

$s,h \vDash \varphi \mathrel{-\!\!*} \psi$ iff for all h' such that $h \# h', s,h' \vDash \varphi$ implies $s, h \cdot h' \vDash \psi$

$s,h \vDash \exists v.\varphi$ iff there exists $a \in \mathrm{V}$, $[s \mid v \mapsto a], h \vDash \varphi$

$s,h \vDash \forall v.\varphi$ iff for all $a \in \mathrm{V}$, $[s \mid v \mapsto a], h \vDash \varphi$

Figure 3.3: Satisfaction for Separation Logic. Classical Separation Logic is the case where $\succcurlyeq$ is $=$.

assuming that the free variables of φ are contained in the domain of s. Note that the clauses for $*$ and $\mathrel{-\!\!*}$ are identical in both the BI and BBI heap model: this is because the BI heap frame is both Upwards and Downwards Closed.

A sound version of the rule of constancy, called the *frame rule* can be given for this assertion language:

$$\frac{\{\,\phi\,\}\,C\,\{\,\psi\,\}}{\{\,\phi * \chi\,\}\,C\,\{\,\psi * \chi\,\}},$$

where χ does not include any free variables modified by the program C. The counterexample to the rule of constancy does not hold in this case, as if y and x are aliased the precondition $y \mapsto 3 * x \mapsto -$ does not hold. The frame rule enables the characteristic *local reasoning* of Separation Logic: it is possible to reason about just the bit of memory a program affects, and carry through that reasoning to the global memory state.

Further work by Calcagno et al. [49] develops abduction-based procedures to automatically infer suitable frame formulas χ to enable the scalability of this local reasoning to large code bases. As a result the Infer tool based on this formalism is implemented at industrial level, automatically fixing memory bugs for Facebook and Spotify. O'Hearn [175] and Brookes [36] famously extended Separation Logic to Concurrent Separation Logic, a formalism for the verification of parallel programs.

Since this landmark work, a veritable zoo of separation logics have been designed for bespoke reasoning tasks, all of which are given by considering variations of memory model and Hoare logic (for just a selection, see [11, 14, 32, 46, 53]). In

Chapter 10 we discuss further the variety of different models in the wild as well as proof theoretic techniques to organise and reason about them.

3.3 Examples of (B)BI Frames

We now describe a number of (B)BI frames beyond memory models that can be found throughout computer science. Some of these (like the examples generated by the notion of resource theory) are not investigated applications of (B)BI, and as such should be understood as potential connections to be made with bunched logic. We note that this Section is fairly substantial, and it should be emphasised, is not essential for the understanding of the core of the thesis. We include it for the interested reader in order to demonstrate the scope of (B)BI beyond Separation Logic, and to indicate possible new directions for research.

3.3.0.1 Traces

A very simple model of parallel computation can be given via traces. Straightforwardly, traces represent a sequence of computations, and a program can be associated to a set of traces, representing possible behaviour upon execution. Given two traces, there is a non-deterministic choice of *interleavings* of the sequences, which can be seen to correspond to possible behaviours of parallel execution. More complicated notions of interleaving can be given that correspond to more intricate notions of parallel computation—for example, *fair* interleaving that ensures that computations from one trace are not unduly privileged over the other—and a resource-sensitive version of the variant of the simple notion outlined here provides semantics for Concurrent Separation Logic [36].

Let Σ be an alphabet representing actions. The set of traces over Σ is given by the finite and infinite strings over Σ. Formally, $Tr(\Sigma) = \Sigma^* \cup \Sigma^\omega$. In particular there exists the empty trace ε given by the empty string. Let $\alpha, \beta \in Tr(\Sigma)$ be given by $\alpha = \alpha_0\alpha_1\alpha_2...$ and $\beta = \beta_0\beta_1\beta_2....$ An interleaving of α and β is a trace $\lambda = \lambda_0\lambda_1\lambda_2...$ such that α and β occur as subsequences—i.e. for some i_m and i_k, $\alpha = \lambda_{i_0}\lambda_{i_1}\lambda_{i_2}...$, $\beta = \lambda_{j_0}\lambda_{j_1}\lambda_{j_2}...$, where $i_0 < i_1 < i_2 < \cdots$ and $j_0 < j_1 < j_2 \cdots$—and for all n, $\lambda_n = \alpha_i$ or β_i for some i.

A BBI frame can be given by the set of traces, together with $\alpha \circ \beta$ defined to be the set of interleavings of α and β. It's immediate from the definition that $\circ$ satisfies Commutativity. Associativity is only slightly more complicated: suppose $\lambda \in \alpha \circ \beta$ and $\mu \in \lambda \circ \gamma$. Then we can obtain the required $\nu \in \beta \circ \gamma$ such that $\mu \in \alpha \circ \nu$ by taking the trace obtained by deleting the subsequence corresponding to α from μ. E is given by $\{\varepsilon\}$. Clearly $\alpha \circ \varepsilon = \{\alpha\}$ for all α, giving satisfaction of Unit Existence and Coherence.

To obtain a BI frame, we consider the subsequence ordering—$\alpha \succcurlyeq \beta$ iff β occurs as a subsequence of α—and take E to be the set of all traces. This ordering satisfies Downwards Closed with respect to interleaving, meaning the verification that this defines a BI frame is simpler: suppose $\lambda \in \alpha \circ \beta$ with α' a subsequence of α and β' a subsequence of β. Then the required trace $\lambda' \in \alpha' \circ \beta'$ with $\lambda \succcurlyeq \lambda'$ is found as the subsequence corresponding to the α' and β' subsequences of λ. Hence for soundnesss of $*$-associativity we need only check Simple Associativity, which follows from our consideration of the BBI frame. Closure is trivially satisfied while Coherence is straightforward: for any trace α, if γ is an interleaving of α and β then β occurs as a subsequence of γ by definition.

3.3.0.2 Rewriting Systems

A more sophisticated model of computation defined on strings is given by semi-Thue systems. Thue systems were invented by Thue in 1914 [212] in an attempt to positively resolve the word problem for finitely presented semigroups, a problem finally proved undecidable by Post [185] and Markov Jr. [158] in 1947. A Thue system is defined by an alphabet and a set of bi-directional rewriting rules for transforming words over that alphabet into other words. A *semi*-Thue system only permits rewriting rules in one direction, although any Thue system can be represented as a semi-Thue system which replaces each bi-directional rule with two one-directional rules. It can be shown that this simple idea is powerful enough to encode Turing machines [73] and as such many problems involving semi-Thue systems can be shown to be undecidable.

Formally (cf. [159]), a semi-Thue system over an alphabet Σ is a subset $S \subseteq \Sigma^* \times \Sigma^*$. An element $(u,v) \in S$ is called a rewriting rule, and S defines a one-step rewriting relation $\rightarrow_S$ as follows:

$$f \rightarrow_S g \text{ iff } \exists u,v,p,q \in \Sigma^*((u,v) \in S \wedge f = puq \wedge g = pvq)$$

Intuitively, the rewriting rule (u,v) states that the system can rewrite a word f in which u occurs as a subword by replacing u with v. The rewriting relation $\rightarrow_S^*$ is given as transitive and reflexive closure of the one-step rule $\rightarrow_S$ and relates words to those that can be obtained from a finite sequence of one-step rewritings.

A set of words over an alphabet Σ^* together with semi-Thue system defined upon it can easily be seen to generate a non-commutative BI frame suitable for modelling non-commutative BI. The composition $\circ$ is given by concatenation of words (which naturally fails Commutativity) while the order is given by the rewriting relation $\rightarrow_S^*$. Finally the set of unit-convertibles is given by $E = \{f \mid f \rightarrow_S^* \varepsilon\}$.

We can show that this structure satisfies Downwards Closure, making the ver-

ification that this is indeed a non-commutative BI frame simpler. Suppose $h = fg$, $f \to_S^* f'$ and $g \to_S^* g'$. Then clearly $h \to_S^* f'g'$ by performing the rewrites on the f component and then the rewrites on the g component. Hence we need only check Simple Associativity for the soundness of $*$-associativity, which follows from the associativity of concatenation. E is defined to satisfy Closure, $f\varepsilon = f = \varepsilon f$ for all f so the frame satisfies the non-commutative version of Unit Existence. Finally, if $h = fg$ and $g \to_S^* \varepsilon$ holds then $h \to_S^* f$ holds by rewriting the g component of h to the empty word, thus showing that the frame satisfies Coherence. One might also consider *term* rewriting systems where the composition of strings is partial (due to a grammar determining legitimate terms) which forms a non-commutative BI frame in a similar manner.

Consider Prop $= \Sigma$ and set $\mathscr{V}(a) = \{f \mid f \to_S^* a\}$. Then $f \vDash_{\mathscr{M}} a_0 * \cdots * a_n$ iff f can be rewritten to $a_0 \ldots a_n$ by the semi-Thue system S. The problem of determining whether $f \to_S^* g$ for an arbitrary f, g and S is called the accessibility problem for semi-Thue systems and is undecidable. Hence the satisfaction relation $\vDash_{\mathscr{M}}$ is in general undecidable and in some situations even the set of unit-convertibles E will be non-computable. While this undermines the ability to use BI to reason about term rewriting, we conjucture decision problems in bunched logic may be determined by considering models of rewriting systems.

3.3.0.3 Databases and Dependence

Our next example is given by Abramsky and Väänänen's [7] reformulation of Hodges' semantics [121, 122] for the independence-friendly logic IF [117] and Väänänen's Dependence Logic [216] as a model of BI. It should be emphasised that this subsection closely follows the material in the original paper. These logics are extensions of first-order logic with quantifiers and predicates suitable for reasoning about the (in)dependence of variables on (from) each other. This idea originates with branching quantifiers, often known as Henkin quantifiers in honour of their inventor Leon Henkin [115]. A simple example of a branching quantifier is

$$\begin{pmatrix} \forall x & \exists y \\ \forall z & \exists w \end{pmatrix} \varphi(x,y,z,w),$$

to be read: for all x there exists a y *dependent* on x and *independent* of z, and, *independently*, for all z, there exists w *dependent* on z and *independent* of x, such that $\varphi(x,y,z,w)$ holds. With branching quantifiers, formulae can be defined that characterise properties that are not definable in first-order logic; for example, infinite cardinality of domain. And indeed, the second-order character of these quantifiers is clearly exposed by their rendering with Skolem functions:

	x_0	x_1	x_2	x_3
s_1	$s_1(x_0)$	$s_1(x_1)$	$s_1(x_2)$	$s_1(x_3)$
s_2	$s_2(x_0)$	$s_2(x_1)$	$s_2(x_2)$	$s_2(x_3)$
s_3	$s_3(x_0)$	$s_3(x_1)$	$s_3(x_2)$	$s_3(x_3)$

Figure 3.4: A team as a database.

$\exists f \exists g \forall x \forall z \varphi(x, f(x), z, g(z))$.

Dependence logic permits the definition of branching quantifiers and allows the expression of (in)dependence at the level of atomic formulae. They can be given a compositional semantics by generalising the Tarskian semantics of first-order logic from a single variable assignment to a *set* of variable assignments called a *team*. A team can equivalently be seen as a database, in which the rows are given by the assignments (with the condition that identical rows of the database are identified) and columns are given by variables, with each entry in the database thus determined by the action of the assignment on the variable.

We begin with an overview of dependence logic and team semantics before presenting the model of BI. Let $\mathscr{A} = (A, \mathscr{I})$ be a model of a first-order language $\mathscr{L}$, where A is a carrier set and $\mathscr{I}$ assigns each n-ary predicate symbol in $\mathscr{L}$ to a subset $R \subseteq A^n$. Given a set of variables X, an assignment is given as a function $t : X \to A$. This enables the definition of the Tarskian truth definition on literals L (atomic $\mathscr{L}$-formulae or negations of atomic $\mathscr{L}$-formulae) whose free variables occur in X, with $\mathscr{A}, t \models_X L$ denoting that the assignment t of variables in L is sufficient to make L hold in $\mathscr{A}$. A *team* is a set $T \in \mathscr{P}(A^X)$ of assignments of the variables in X. The basic semantic clause for IF is given on literals L with respect to teams:

$$\mathscr{A}, T \models_X L \text{ iff } \forall t \in T : \mathscr{A}, t \models_X L.$$

That is, a literal L is satisfied in team semantics if it is true simultaneously of all the assignments in the team (rows in the database). This idea can then be used to express the (in)dependence between variables in calculating the truth value of formulae by considering a team where the assignment of a variable is varied while the others stay fixed. The dependence predicate $D(W, x)$ (where W is a subset of variables $W \subseteq X$ and x a variable) is defined as follows. The equivalence relation $\simeq_W$ on assignments is defined by $s \simeq_W t$ iff $\forall w \in W : s(w) = t(w)$. Then

$$\mathscr{A}, T \models_X D(W, x) \text{ iff } \forall s, t \in T : s \simeq_W t \text{ implies } s(x) = t(x)$$

Intuitively, this encodes the idea that the variable x is dependent on the variables in W by stating that equal assignments of W-variables leads to an equal assignment of

x for the assignments in the team. Next we give semantic clauses for $\wedge$ and $\vee$:

$$\begin{array}{lll} \mathscr{A},T \vDash_X \varphi \wedge \psi & \text{iff} & \mathscr{A},T \vDash_X \varphi \text{ and } \mathscr{A},T \vDash_X \psi \\ \mathscr{A},T \vDash_X \varphi \vee \psi & \text{iff} & \text{there exists } U,V \text{ s.t. } T = U \cup V, \mathscr{A},U \vDash_X \varphi \text{ and } \mathscr{A},V \vDash_X \psi. \end{array}$$

For the quantifiers, recall the notation $[t \mid x \mapsto a]$, which is the update of the assignment t that is equal to t everywhere except for at x, at which it has the value a. Then for a function $f : T \to A$ and a set $B \subseteq A$, we define $[T \mid x \mapsto f] = \{[t \mid x \mapsto f(t)] \mid t \in T\}$ and $[T \mid x \mapsto B] = \{[t \mid x \mapsto b] \mid b \in B\}$. Then

$$\begin{array}{lll} \mathscr{A},T \vDash_X \exists x.\varphi & \text{iff} & \text{there exists } f : T \to A \text{ s.t } \mathscr{A},[T \mid x \mapsto f] \vDash_{X \cup \{x\}} \varphi \\ \mathscr{A},T \vDash_X \forall x.\varphi & \text{iff} & \mathscr{A},[T \mid x \mapsto A] \vDash_{X \cup \{x\}} \varphi. \end{array}$$

Intuitively, the existential clause states that there is some update of the x-column of the database that makes φ true; the universal clause states that any possible update of the x-column makes φ true.

A model of BI subsuming the team semantics can easily be identified. Consider the frame $(\mathscr{P}(A^X), \succcurlyeq, \cup, \{\emptyset\})$ where $\mathscr{P}(A^X)$ is the set of A-valued teams on X—equivalently, the set of databases with attributes from X and entries from A—$\succcurlyeq$ is defined $T \succcurlyeq T'$ iff $T \subseteq T'$, $\cup$ is set union and $\emptyset$ is the empty set. $\succcurlyeq$ is Upwards and Downwards Closed with respect to $\cup$ so it is sufficient to check for Simple Associativity, which follows from the associativity of set union. Closure is trivial as only $\emptyset \succcurlyeq \emptyset$, and Unit Existence is easily shown. Finally, suppose $T = U \cup \emptyset$. Then clearly $T = U$ so $T \succcurlyeq U$, satisfying Coherence.

Let Prop be given by the literals of $\mathscr{L}$. Then the valuation defined $\mathscr{V}(\mathrm{L}) = \{T \mid \forall t \in T : \mathscr{A}, t \vDash_X L\}$ can be seen to be both persistent with respect to $\succcurlyeq$ and an encoding of the satisfaction clauses for literals in team semantics. The standard BI semantics generated from this model pick up the predicate- and quantifier-free team semantics clauses, with dependence logic and BI's $\wedge$ coinciding and dependence logic's $\vee$ given by $*$: in a sense, BI is the propositional logic of dependence.

Abramsky and Väänänen show that the presence of BI's additional connectives is very natural from the point-of-view of dependence logic. Intuitionistic implication can be used to define a new dependence-friendly universal quantifier $(\forall x \backslash x_1, \ldots, x_n)\varphi := \forall x(D(x_1, \ldots, x_n, x) \to \varphi)$, joining the already-definable dependence-friendly existential quantifier $(\exists x \backslash x_1, \ldots, x_n)\varphi := \exists x(D(x_1, \ldots, x_n, x) \wedge \varphi)$, both of which arise as adjoints. Further, they show that the dependence predicate can be defined using an unary predicate $C(x) := D(\emptyset, x)$, called *constancy* (straightforwardly read as *dependence on nothing*), together with intuitionistic conjunction and implication: $D(W, v) := (\bigwedge_{w \in W} C(w_i)) \to C(v)$.

3.3.0.4 Resource Theories

We finish our examples with a method for generating BI frames derivable from Coecke et al.'s [60] categorical definition of *resource theory*, a formalisation of a concept ubiquitous throughout quantum information theory. The basic idea is that quantum properties (e.g., entanglement) can be considered as resources that can be used to perform information processing tasks (e.g., teleportation of a quantum state over a distance). This idea can be formalised in a way that captures more than just quantum phenomena.

Mathematically, a resource theory is a *symmetric monoidal category* (SMC) (see Appendix) $(\mathscr{C}, \otimes, \mathbb{I})$, where $\mathscr{C}$ is a small category with resources as objects and resource conversions as morphisms, $\otimes$ is a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ satisfying appropriate monoidal properties and $\mathbb{I}$ an object (the unit resource) suitably compatible with $\otimes$. For our purposes, a resource theory provides the background information required to compute the conversion ordering $\succcurlyeq$ for a BI frame by explicitly representing the possible conversions as morphisms in the category. This ordering is defined on a commutative monoid obtained from the monoidal structure of the resource theory and it is easily seen that this structure generates a BI frame.

To obtain a BI frame, we take equivalence classes of the objects of $\mathscr{C}$ as the set of resources, setting $A \equiv B$ iff there exists morphisms $f : A \to B$ and $g : B \to A$. The order $\succcurlyeq$ is defined according to the invariance interpretation: $[x] \succcurlyeq [y]$ iff there exists an arrow f in $\mathscr{C}$ such that $f : y \to x$. We represent conversion this way in the frame to ensure E characterises the *free resources*, an important aspect of a resource theory. Due to the existence of identity arrows and closure under composition of arrows, this defines a preorder. Accordingly $E = \{x \mid \exists f : f : \mathbb{I} \to x\}$. $\circ$ is straightforwardly $\otimes$, and unlike the general case of BI frames, this is total and deterministic, due to its origin as a tensor on a category. Since the tensor $\otimes$ is a functor it is bifunctorial/downwards-closed with respect to $\succcurlyeq$: $f : x \to x'$ and $g : y \to y'$ give the existence of the arrow $f \otimes g : x \otimes y \to x' \otimes y'$. Simple Associativity is then immediate from the associativity of $\otimes$. Closure holds by definition and Coherence is shown by noting that given $f : \mathbb{I} \to y$ we obtain $id_x \otimes f : x = x \otimes \mathbb{I} \to x \otimes y$.

A *monotone* for a resource theory is a function $M : X \to \mathbb{R}$ satisfying $\exists f : a \to b$ implies $M(a) \geq M(b)$. This is interpreted as assigning a 'cost' to every resource in a way that reflects conversion: if a converts into b then a has a higher cost than b. A monotone on a resource theory generates a persistent valuation on its induced BI frame by considering upper bounds for the cost of each resource: for each $r \in \mathbb{R}$ set $x \in \mathscr{V}_M(r)$ iff $r \geq M(x)$.

Coecke et al. [60] and Fritz [94] describe a range of resource theories, including randomness, communication channels and a wide array of examples from

quantum information theory, and these all generate BI frames by this method. This provides interesting new potential application areas for BI outside of Separation Logic.

Chapter 4

Extensions of the Logics of Bunched Implications

In this chapter we give an overview of a number of bunched logics that are obtained by extending (B)BI. These extensions are obtained by considering multiplicative versions of more connectives than just $\wedge$ and $\top$. We begin with the *De Morgan bunched logics*, which are obtained with the addition of a multiplicative negation, which allows the definition of multiplicative disjunction and falsum. Of these, only Classical BI (the variant extending BBI) has been investigated in any depth. Brotherston & Calcagno [39] introduced CBI and proved it sound and complete with respect to a display calculus. In later work, Brotherston & Kanovich [43] proved CBI undecidable by the same methods used for BBI and propositional Separation Logic. De Morgan BI (the variant extending BI) has not been well studied. The possibility of a system like DMBI was postulated (and named) by Pym [187], but it has only been treated proof theoretically, as part of Brotherston's [38] uniform display calculus proof theory for bunched logics.

Lying intermediate between bunched implication logics and De Morgan bunched logics are Brotherston & Calcagno's [45] *subclassical bunched logics*, in which multiplicative disjunction and falsum are given as primitives in systems lacking a multiplicative negation. These logics arise as a framework of axiomatic extensions obtained by enforcing particular relationships between multiplicative conjunction, disjunction and falsum. Like with the De Morgan bunched logics, these have only been investigated for the variants extending BBI, with Brotherston & Calcagno giving a modular semantic and proof theoretic treatment. The evident subclassical extensions of BI are new.

Separating modal logics add a diamond modality to BBI, which can be used to define modalities that are off-set by resource composition. This idea has been implemented in resource-sensitive generalisations of the modal logics S4 and S5 by

Courtault et al. [68] and Galmiche et al. [98] respectively, and we give a general schema for produing separating modal logics extending BBI. It is currently not clear how to extend the schema to obtain similar logics extending BI, as Boolean negation is used in an essential way.

Finally, we consider a new bunched logic we call Concurrent Kleene BI, or CKBI. This logic is a formalisation of a system extending BBI postulated by O'Hearn [176] in the context of work connecting Concurrent Separation Logic to concurrent Kleene algebra. CKBI acts as a test case for the applicability of the frameworks we set up throughout this thesis: not only do we produce metatheory systematically for existing bunched logics, we are also able to easily output the same metatheory for novel extensions of existing logics. As CKBI is included as a simple test case, we defer the investigation of the evident intuitionistic variant to another occasion.

Section 4.4 is based on material from the journal paper *Stone-Type Dualities for Separation Logics* [83].

4.1 De Morgan Bunched Logics

In this section we define DMBI and CBI as extensions of BI and BBI respectively, and give a semantics on frames. Intuitively, the logics are designed to reason about the duality between positive and negative resource. In the context of the money model, we can reason about not just the size of our credit, but also the size of our *debt*. More generally, it can be thought of as a logic of dualisable resource, with the multiplicative negation interpreted by an involutive 'dualising' operation on resources.

4.1.1 Syntax and Semantics

Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of the De Morgan bunched logics $\mathrm{Form}_{\mathrm{DMBI}}$ is generated by the grammar

$$\varphi ::= \mathrm{p} \mid \top \mid \bot \mid \top^* \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \mathrel{-\!\!*}\varphi \mid \varphi * \varphi \mid \varphi \mathrel{*\!\!-} \varphi,$$

where additive negation is given by $\neg\varphi := \varphi \to \bot$, multiplicative falsum is given by $\bot^* := \mathrel{-\!\!*}\top^*$ and multiplicative disjunction is given by $\varphi \mathbin{⅋} \psi := \mathrel{-\!\!*}(\mathrel{-\!\!*}\varphi * \mathrel{-\!\!*}\psi)$.

Figure 4.1 gives Hilbert rules that need to be added to the proof systems of the bunched implication logics to obtain systems for the De Morgan bunched logics: to get DMBI, the Hilbert system for BI is extended with 19. and 20.; to get CBI, 19. and 20. are instead added to the Hilbert system for BBI. These rules enforce the two simple properties of the new multiplicative negation: 19. gives the multiplicative double negation law, dictating that $\mathrel{-\!\!*}$ is an involution on formulae, while 20. codifies

19. $\dfrac{}{\multimap\neg\multimap\neg\varphi \dashv\vdash \varphi}$ 20. $\dfrac{}{\multimap\neg\varphi \dashv\vdash \varphi \mathrel{-\!\!*} \bot^*}$

Figure 4.1: Hilbert rules for De Morgan bunched logics.

the relationship between $\multimap\neg$, $-\!\!*$ and $\bot^*$, mirroring the fact that for additive negation $\neg\varphi$ is syntactic sugar for $\varphi \to \bot$.

At first sight it seems that something contradictory is going on here: we have an 'intuitionistic' logic DMBI with a classical-looking negation satisfying an analogue of the double negation law for $\neg$, the presence of which is the difference between defining intuitionistic and classical additives. Is it possible that our $\multimap\neg$ somehow collapses DMBI logic into classical logic? For our Hilbert systems, the intuitionistically invalid principles are derived upon adding the double negation rule for $\neg$ because of the rule corresponding to the principal of explosion: $\dfrac{}{\bot \vdash \varphi}$. Since there is no such analogue for $\bot^*$, $\multimap\neg$ is strictly weaker than Boolean negation, and so the logic has two coexisting negations of differing strengths.

The structures for interpreting De Morgan bunched logics are called DMBI and CBI frames. In much the same way that BBI (BI) frames can be seen as the partial/non-deterministic analogues of (ordered) commutative monoids, CBI (DMBI) frames are the partial/non-deterministic analogues of (ordered) abelian groups.

Definition 4.1 (DMBI/CBI Frame)**.** *A DMBI frame is a tuple $\mathscr{X} = (X, \succcurlyeq, \circ, E, -)$ where $(X, \succcurlyeq, \circ, E)$ is a BI frame and $- : X \to X$ is an operation satisfying the following conditions (with outermost universal quantification omitted for readability):*

$$\textit{(Dual)}\quad x \succcurlyeq y \to -y \succcurlyeq -x \qquad \textit{(Involutive)}\quad --x = x$$
$$\textit{(Compatibility)}\quad z \in x \circ y \to -x \in -z \circ y.$$

A CBI frame is a DMBI frame for which the order $\succcurlyeq$ is equality $=$.

A DMBI/CBI frame $\mathscr{X}$ together with a persistent valuation $\mathscr{V}$ gives a DMBI/CBI model $\mathscr{M}$, and for such a model the satisfaction relation $\vDash_{\mathscr{M}} \subseteq X \times \mathrm{Form}_{\mathrm{DMBI}}$ is inductively generated by the satisfaction clauses for BI/BBI, extended with the condition for $\multimap\neg\varphi$ given in Figure 4.2. This clause is that for the *Routley star* negation operation of relevant logic [192]. In the case for DMBI, it is easy to see that the frame property Dual ensures persistence for this clause. If $x \vDash_{\mathscr{M}} \multimap\neg\varphi$ and $y \succcurlyeq x$, then by Dual $-x \succcurlyeq -y$. Since $-x \nvDash_{\mathscr{M}} \varphi$, necessarily $-y \nvDash_{\mathscr{M}} \varphi$ by persistence. Hence the persistence of satisfaction follows a corollary of the case for

BI.

Let's unpack this definition. The Compatibility axiom is best understood as enforcing that $-$ returns something like inverses for composition, in a sense that takes into account partiality and non-determinism. To see this, consider the case where $\circ$ is a total deterministic composition. In this setting Compatibility states that $-x = -(x \circ y) \circ y$. In an Abelian group (where $-$ returns inverses) this holds because $-(x \circ y) = -x \circ -y$, with associativity allowing us to cancel the y with $-y$ to be left with $-x$. Nothing in the definition of DMBI/CBI frame enforces either the distribution of $-$ over $\circ$ nor that $x \circ -x$ 'cancels', but Compatibility does allow us to make something like the same deduction from $-(x \circ y) \circ y$ to $-x$ in the nondeterministic setting. The analogy with inverses is further bolstered by the Involutive axiom stating that $-$ is an involution (a function that is its own inverse).

Dual is best understood with an example. Our general resource theoretic interpretation of BI frames from Chapter 3 does not extend straightforwardly to DMBI frames, since in many cases there is *prima facie* no involutive operation that reverses conversion. To return to the chemistry example, $-$ might be read as converting matter to antimatter, but it is extremely unlikely that antimatter chemical reactions are the perfect mirror of standard chemical reactions. The standard resource semantics of BI (generalised by our sufficiency interpretation) does make sense for DMBI, however. We can think of frames in which all resources occur as both credit x and debt and $-x$. Take the money model: this can be seen as a DMBI frame by extending $\mathbb{N}$ to $\mathbb{Z}$. Now positive integers n represent credit $£n$ and negative integers $-n$ represent debt $-£n$: if $£n \succcurlyeq £m$ then $-£m \succcurlyeq -£n$. The unit convertibles E are again given by $\mathbb{N}$: if you're in debt, no amount of spending can get you out of it. Suppose again that the price of a bunch of bananas is $£2$. Then $-£1 \vDash \nrightarrow$banana witnesses that the size of our debt is less than the price of a bunch of bananas. Another useful intuition is to think of the resources of a DMBI frame as processes in which the inputs and outputs can be switched to give a dual process.

We should also note that the definition of CBI frame here looks different to the notion given by Brotherston & Calcagno [39] but is in fact equivalent. There, a (multi-unit) CBI model is a tuple $(X, \circ, E, -, \infty)$ such that $(X, \circ, E)$ is a BBI frame, with $- : X \to X$ and $\infty \subseteq X$ satisfying, for all $x \in X$, $-x$ is the unique element such that $\infty \cap (-x \circ x) \neq \emptyset$. The frame properties Involutive and Compatibility are then proved as consequences of this definition in their Proposition 2.3 (1) and (3). As they discuss, the choice of ∞ is fixed by the choice of $-$, and it can easily be seen that defining $\infty = \{-e \mid e \in E\}$ on our CBI frames yields their CBI models. We choose this presentation as it simplifies the proofs of duality in Chapter 6 and allows a more straightforward construction of tableaux calculi in Chapter 9.

$$x \models_{\mathscr{M}} \dot{\neg}\varphi \text{ iff } -x \not\models_{\mathscr{M}} \varphi$$

Figure 4.2: Satisfaction for DMBI/CBI.

4.1.2 Examples of De Morgan bunched logic frames

We now state a number of examples of DMBI/CBI frames found that can be found in mathematics and computer science. Other CBI frames considered by Brotherston & Calcagno [39] include bit arithmetic, deny-guarantee permissions and generalised heaps.

4.1.2.1 Effect Algebras

A simple example which once again connects bunched logic to quantum mechanics is Foulis & Bennett's [93] *effect algebras*. Effect algebras formalise the algebraic structure of the effects of a quantum mechanical system, and have been suggested as a framework for reasoning about unsharp quantum measurements.

Formally, an effect algebra is a structure $(X, \oplus, (-)^{\perp}, 0, 1)$ such that $\oplus$ is a binary partial function that is commutative and associative (up to definedness), $0, 1 \in X$, 0 a unit for $\oplus$, and for every $x \in X$, $x^{\perp}$ the unique element such that $x \oplus x^{\perp} = 1$ and $x \oplus 1 \downarrow$ implies $x = 0$. An order can be defined on any effect algebra by setting $x \succcurlyeq y$ iff there exists z such that $x = y \oplus z$. This defines a CBI frame $(X, \oplus, \{0\}, (-)^{\perp})$ and a DMBI frame $(X, \succcurlyeq, \oplus, \{0\}, (-)^{\perp})$ because of the following properties of effect algebras (*cf.* [93, Lemma 2.3, Theorem 2.4]):

- $((x)^{\perp})^{\perp} = x$;
- $x \oplus y \downarrow$ implies $(x \oplus y)^{\perp} \oplus y \downarrow$ and $x^{\perp} = (x \oplus y)^{\perp} \oplus y$;
- $x \succcurlyeq y$ implies $y^{\perp} \succcurlyeq x^{\perp}$.

Other examples of effect algebras include regular languages with disjoint union and complement, and CCS-style [162] communicating actions [39].

4.1.2.2 Dualisable Resource Theories

In Chapter 3 we described how Coecke et al.'s [60] category theoretic formalisation of resource conversion could be used to generate BI frames. That construction was based on the fact that conversions were morphisms in a symmetric monoidal category: by taking an equivalence relation on objects based on mutual convertibility we obtain an ordered commutative monoid, a particular instance of BI frame. The equivalent algebraic structure in the case of DMBI would be an ordered abelian group, but what kind of category determines such a structure when we quotient with respect to its morphisms?

One answer is *compact closed categories*. Extensively studied by Kelly and Laplaza [139], such structures have more recently been used as the basis for a categorical treatment of quantum mechanics by Abramsky & Coecke [6]. We suggest that the notion of resource theory might be extended to a notion of dualisable resource theory by considering compact closed categories. The idea of representing resource debt in such categories can be traced to Martí-Oliet & Mesegeuer's work relating Petri nets and linear logic [152, 5], an idea that is further expanded upon in recent work by Genovese & Herold [102].

Formally, a compact closed category (cf. [3]) is a SMC $(\mathscr{C}, \otimes, \mathbb{I})$ such that for every object A of $\mathscr{C}$ there exists an object A^* (the *dual* of A), a 'unit' morphism $\eta_A : \mathbb{I} \to A^* \otimes A$ and a 'counit' morphism $\varepsilon_A : A \otimes A^* \to \mathbb{I}$ satisfying the triangle equalities $Id_A = (\varepsilon_A \otimes Id_A) \circ (Id_A \otimes \eta_A)$ and $Id_{A^*} = (Id_{A^*} \otimes \varepsilon_A) \circ (\eta_A \otimes Id_{A^*})$. Some consequences of these definitions are as follows:

- The assignment $A \mapsto A^*$ defines a contravariant endofunctor: given any $f : A \to B$ in a compact closed category, we may form the morphism $f^* : B \to A$ as $f^* = (1_{A^*} \otimes \varepsilon_A) \circ (1_{A^*} \otimes f \otimes 1_{B^*}) \circ (\eta_A \otimes 1_{B^*})$;
- $A \cong A^{**}$;
- $(A \otimes B)^* \cong A^* \otimes B^*$.

Consequently, when we perform the construction that obtains a BI frame from a SMC we are able to define $-[A] = [A^*]$. By 1., the DMBI frame axiom Dual is satisfied, by 2. the axiom Involutive, and by 3. the axiom Compatibility. Some characteristic examples of compact closed categories are the category of Sets with relations as morphisms, and the category of Conway games [134].

4.2 Sub-Classical Bunched Logics

Next we consider the sub-classical bunched logics introduced by Brotherston & Villard [45]. These logics give an alternative way to extend the logics of bunched implications with multiplicative disjunction $\mathbin{⅋}$ and falsum $\bot^*$: in De Morgan bunched logics, these connectives are defined using multiplicative negation, whereas in the subclassical bunched logics they occur as primitives of the language. The principal motivation for this is that heap-like (B)BI frames cannot be extended with an involution satisfying the DMBI/CBI frame properties; however, intersection operations on heaps are suggestive of an interpretation of $\mathbin{⅋}$. Formally, they can be seen as combinations of classical/intuitionistic propositional logic with fragments of Hyland & De Paiva's [128] *full intuitionistic linear logic*. While Brotherston & Villard con-

$$
21.\quad \frac{\eta \vdash \varphi \mathbin{\overset{*}{\vee}} \psi}{\eta \mathbin{\overset{*}{\setminus}} \varphi \vdash \psi}
\qquad
22.\quad \frac{\eta \mathbin{\overset{*}{\setminus}} \varphi \vdash \psi}{\eta \vdash \varphi \mathbin{\overset{*}{\vee}} \psi}
$$

$$
23.\quad \frac{\xi \vdash \varphi \quad \eta \vdash \psi}{\xi \mathbin{\overset{*}{\vee}} \eta \vdash \varphi \mathbin{\overset{*}{\vee}} \psi}
\qquad
24.\quad \frac{}{\varphi \mathbin{\overset{*}{\vee}} \psi \vdash \psi \mathbin{\overset{*}{\vee}} \varphi}
$$

Figure 4.3: Hilbert rules for basic Bi(B)BI.

sider only the subclassical bunched logics extending BBI, we additionally consider the evident variant extending BI.

Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of the subclassical bunched logics Form_{BiBI} is generated by the grammar

$$\varphi ::= \mathrm{p} \mid \top \mid \bot \mid \top^* \mid \bot^* \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathbin{\overset{*}{\vee}} \varphi \mid \varphi \mathrel{-\!\!*} \varphi \mid \varphi \mathbin{\overset{*}{\setminus}} \varphi,$$

where additive negation is defined by $\neg\varphi := \varphi \to \bot$.

The simplest subclassical bunched logics are called *basic bi-intuitionistic (B)BI*, or basic Bi(B)BI. Figure 4.3 gives Hilbert rules for basic Bi(B)BI: for basic Bi(B)BI these rules should be added to the system for (B)BI. In this basic case, very little is enforced for the new connectives: rules 21. and 22. enforce that the new disjunctive implication $\overset{*}{\setminus}$ (magic slash) is adjoint to $\overset{*}{\vee}$, in a similar manner to $*$ and $\mathrel{-\!\!*}$; 23. enforces montonicity of $\overset{*}{\vee}$; and 24. enforces that $\overset{*}{\vee}$ is commutative. We could of course consider an even more basic system in which 24. was dropped, but this increases the complexity of the semantics to such an extent that it does not seem to be well-motivated from a modelling point of view.

The reason for this simplicity is that the most basic subclassical logics drop all the De Morgan-like correspondences between $*$, $\overset{*}{\vee}$ and $\overset{*}{\neg}$ (which can be defined as $\overset{*}{\neg}\varphi := \varphi \mathrel{-\!\!*} \bot^*$). Including them all collapses the logics back to DMBI/CBI, excluding the heap models once more. Instead, a number of these correspondences can be added as axioms *without* collapsing the logic to a De Morgan bunched logic, but with the benefit of retaining heap-like models. In Figure 4.4 these axioms are given as Hilbert-style rules which can be added to the system for basic Bi(B)BI. The most interesting of these is Weak Distributivity, a way of enforcing a relationship between $*$ and $\overset{*}{\vee}$ that is strictly weaker than the analogous De Morgan law through which $\overset{*}{\vee}$ was defined in the De Morgan bunched logics.

Basic Bi(B)BI is interpreted on structures extending (B)BI frames called basic Bi(B)BI frames.

Definition 4.2 (Basic Bi(B)BI Frame)**.** *A* basic Bi(B)BI frame *is a structure* $\mathscr{X} = (X, \succcurlyeq, \circ, E, \triangledown, U)$ *such that* $(X, \succcurlyeq, \circ, E)$ *is a (B)BI frame,* $\triangledown : X^2 \to \mathscr{P}(X)$ *and* $U \subseteq$

Associativity	$\overline{\varphi \mathbin{⅋} (\psi \mathbin{⅋} \chi) \vdash (\varphi \mathbin{⅋} \psi) \mathbin{⅋} \chi}$	$\bot^*$ Weakening	$\overline{\varphi \vdash \varphi \mathbin{⅋} \bot^*}$
$\bot^*$ Contraction	$\overline{\varphi \mathbin{⅋} \bot^* \vdash \varphi}$	$⅋$ Contraction	$\overline{\varphi \mathbin{⅋} \varphi \vdash \varphi}$
Weak Distributivity	$\overline{\varphi * (\psi \mathbin{⅋} \chi) \vdash (\varphi * \psi) \mathbin{⅋} \chi}$		

Figure 4.4: Hilbert rules for subclassical bunched logics.

$x \Vdash_{\mathscr{M}} \quad \bot^* \quad$ iff $x \notin U$

$x \Vdash_{\mathscr{M}} \varphi \mathbin{⅋} \psi$ iff for all s,t,u, $x \preccurlyeq s \in t \bigtriangledown u$ implies $t \Vdash_{\mathscr{M}} \varphi$ or $u \Vdash_{\mathscr{M}} \psi$

$x \Vdash_{\mathscr{M}} \varphi \mathbin{⅋\!\!\!/} \psi$ iff there exist s,t,u such that $x \succcurlyeq s$, $u \in t \bigtriangledown s$, $u \Vdash_{\mathscr{M}} \varphi$ and $t \nVdash_{\mathscr{M}} \psi$

Figure 4.5: Satisfaction for Bi(B)BI. BiBBI is the case where $\succcurlyeq$ is $=$.

X, satisfying (with outermost universal quantification omitted for readability):

$$\textit{(Commutativity)} \quad z \in x \bigtriangledown y \to z \in y \bigtriangledown x; \quad \textit{(U-Closure)} \quad u \in U \wedge u \succcurlyeq u' \to u' \in U.$$

A basic Bi(B)BI frame $\mathscr{X}$ together with a persistent valuation $\mathscr{V}$ gives a basic Bi(B)BI model $\mathscr{M}$, and for such a model the satisfaction relation $\Vdash_{\mathscr{M}} \subseteq X \times \mathrm{Form}_{\mathrm{BiBI}}$ is inductively generated by the satisfaction clauses for (B)BI, extended with the clauses for $\bot^*$, $⅋$ and $⅋\!\!\!/$ given in Figure 4.5. Persistence for the $\bot^*$ clause is ensured by the frame property U-Closure, while persistence of the $⅋$ and $⅋\!\!\!/$ clauses can be shown in a similar manner to the cases for $*$ and $-\!\!*$.

Importantly, each of the optional subclassical axioms can be witnessed by a corresponding frame property on the frames interpreting basic Bi(B)BI. As usual, the case for BiBBI is obtained as the special case where $\succcurlyeq$ is $=$. We will prove these correspondences in Chapter 6.

Definition 4.3 (Subclassical Frame Properties)**.**

(Associativity)	$t' \preccurlyeq t \in x \bigtriangledown y \wedge w \in t' \bigtriangledown z \to \exists s, s', w'$
($\bot^*$ Weakening)	$u \in U \wedge x \in y \bigtriangledown u \to x \preccurlyeq y$
($\bot^*$ Contraction)	$\exists u \in U (w \in w \bigtriangledown u)$
($⅋$ Contraction)	$x \in x \bigtriangledown x$
(Weak Distributivity)	$t' \succcurlyeq t \in x_1 \circ x_2 \wedge t' \preccurlyeq t'' \in y_1 \bigtriangledown y_2 \to$ $\exists w (y_1 \in x_1 \circ w \wedge x_2 \in w \bigtriangledown y_2)$

Basic Bi(B)BI frames are somewhat artificial constructions that resist an in-

tuitive reading at the level of abstract resource, perhaps reflecting the somewhat specific motivation of obtaining some of the features of De Morgan bunched logics for heap models. One way to think of ∇ is as an intersection operation, with $z \in x \nabla y$ if z is a shared portion of the resources x and y. The clause for $\mathbin{\dot{\vee}}$ can then be read as stating that $\varphi \mathbin{\dot{\vee}} \psi$ holds of a resource x if every time x is obtained through the conversion of a shared part of resources t and u, one of t or u satisfies φ or ψ.

This idea is explicitly realised by the heap models of BiBBI given by Brotherston & Villard [45]. They give two possible heap intersection operations: one which outputs the compatible part of two heaps, and another which is defined only when the two heaps agree on the intersection of their domains. Explicitly, these are

$$h \cap_1 h'(l) = \begin{cases} h(l) & \text{if } l \in dom(h) \cap dom(h') \text{ and } h(l) = h'(l) \\ \uparrow & \text{otherwise} \end{cases}$$

$$h \cap_2 h' = \begin{cases} h \cap h' & \text{if } h(l) = h'(l) \text{ for all } l \in dom(h) \cap dom(h') \\ \uparrow & \text{otherwise} \end{cases}$$

When U is taken to be the set of all heaps, $\cap_1$ suffices to turn the standard heap model into a model of BiBBI satisfying associativity, $\mathbin{\dot{\vee}}$ contraction and weak distributivity, while $\cap_2$ suffices for a model satisfying $\mathbin{\dot{\vee}}$ contraction and weak distributivity. The standard heap ordering is sufficient to make both of these models of BiBI satisfying the same properties. A heap model of BiBBI satisfying all of the axioms is given by Brotherston & Villard by taking states to be pairs (h,x) where h and x are heaps for which there exists a heap h' such that $h = x \cdot h'$, interpreted as a piece of local memory together with an *environment* reflecting the wider machine state.

4.3 Separating Modal Logics

Next we consider separating modal logics. These logics extend BBI with resource modalities $\Diamond_r$ and include Courtault et al.'s [68] logic of separating modalities and Galmiche et al.'s [98] epistemic resource logic. In these papers the logics are introduced semantically and given a tableau proof theory with countermodel extraction. In models of these logics, for each $\Diamond_r$ in the signature, a resource $\lfloor r \rfloor$ is assigned to r. $\Diamond_r \varphi$ is then interpreted as stating that there exists a resource x that can be composed with the *local resource* $\lfloor r \rfloor$ to access a state satisfying φ. We generalise this to a schema for defining separating modal logics.

Let Prop be a set of atomic propositions, ranged over by p. The set of all

25. $\dfrac{}{\Diamond(\varphi \vee \psi) \vdash \Diamond\varphi \vee \Diamond\psi}$ 26. $\dfrac{}{\Diamond\bot \vdash \bot}$

Figure 4.6: Hilbert rules for separating modal logic.

formulae of separating modal logic $\mathrm{Form}_{\mathrm{SML}}$ is generated by the grammar

$$\varphi ::= \mathrm{p} \mid \top \mid \bot \mid \top^* \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi \mathrel{-\!\!*} \varphi \mid \Diamond\varphi$$

where additive negation is defined by $\neg\varphi := \varphi \to \bot$ and the necessity modality is defined by $\Box\varphi := \neg\Diamond\neg\varphi$. For each formula φ of SML, a *separating modality* $\Diamond_\varphi$ is defined by $\Diamond_\varphi\psi := \neg(\varphi \mathrel{-\!\!*} \neg\Diamond\psi)$. The idea of formula-labelled modalities can be traced back to Chellas' [54] relative necessity modalities for conditional logic (themselves interpreted as a form of implication, cohering somewhat with the use of $\mathrel{-\!\!*}$ in defining $\Diamond_\varphi$), and the separating modal logics can usefully be thought of as a substructural reconstruction of that idea.

Figure 4.6 gives Hilbert rules to be added to the system for BBI to obtain one for SML. These are simply axioms specifying that $\Diamond$ is interpreted as a *normal* modality. Further axioms may be added to specify that $\Diamond$ is a particular kind of modality: in the logic of separating modalities $\Diamond$ is an S4 modality, thus satisfying the axioms $\varphi \vdash \Diamond\varphi$ and $\Diamond\Diamond\varphi \vdash \Diamond\varphi$, whereas in epistemic resource logic $\Diamond$ is an S5 modality, thus satisfying the S4 axioms plus $\Diamond\Box\varphi \vdash \Box\varphi$.

SML is interpeted on structures that extend BBI frames with an accessibility relation that we call SML frames.

Definition 4.4 (SML Frame)**.** *An* SML frame *is a structure* $\mathscr{X} = (X, \circ, E, R)$ *such that* $(X, \circ, E)$ *is a BBI frame and* R *a binary relation on* X*.*

If $\Diamond$ is axiomatised by modal axioms with frame correspondents, the SML frame must also satisfy those frame correspondents. For example, for an S4 modality, R must be reflexive and transitive; for an S5 modality, R must additionally be symmetric. A SML frame together with a valuation $\mathscr{V}$ gives a SML model $\mathscr{M}$, and for such a model the satisfaction relation $\vDash_{\mathscr{M}} \subseteq X \times \mathrm{Form}_{\mathrm{SML}}$ is inductively generated by the clauses for BBI, together with the clause for $\Diamond$ given in Figure 4.7. This figure also includes the satisfaction clause for $\Diamond_\varphi$, obtained directly from the definition $\Diamond_\varphi\psi := \neg(\varphi \mathrel{-\!\!*} \neg\Diamond\psi)$. Intuitively, this clause states that $\Diamond_\varphi\psi$ is true at a resource x iff x can be composed with a resource satisfying φ, with that composition having access to a state at which ψ is true. If Prop contains atoms r that are interpreted by $\mathscr{V}$ to be true at a single state $\lfloor r \rfloor \in X$, the clause for $\Diamond_r$ is precisely that given in the primitive satisfaction clauses for the logic of separating modalities

$x \models_{\mathscr{M}} \Diamond\varphi$ iff there exists y such that Rxy and $y \models_{\mathscr{M}} \varphi$

$x \models_{\mathscr{M}} \Diamond_\varphi\psi$ iff there exists w,y,z such that $z \in x \circ y$, $y \models \varphi$, Rzw and $w \models \psi$.

Figure 4.7: Satisfaction for SML

and epistemic resource logic.

Typical examples of SML frames can be found by considering frames in which states are pairs (r,x), where r is a resource and x is a possible world, understood as being a state of a system. Broadly speaking, the BBI component of the logic is interpreted purely on the resource component, whereas the modal component is interpreted on resources together with states. Courtault et al. [68] give producer-consumer and timed petri nets models of the logic of separating modalities based on this notion of frame, while Galmiche et al. [98] give a range of such frames for epistemic resource logic suitable for security modelling.

The separating modalities $\Diamond_\varphi$ inherit the property of being normal from $\Diamond$, and are thus well behaved. This is most easily seen semantically. First note that $x \models \Diamond_\varphi\bot$ never holds, as there is no state at which $w \models \bot$. It is thus equivalent to $\bot$. For the distribution of $\Diamond_\varphi$ over $\vee$, we note that (*cf.* Chapter 6, Proposition 6.2), $\varphi \mathrel{-\!\!*} (\psi_1 \wedge \psi_2)$ is equivalent to $(\varphi \mathrel{-\!\!*} \psi_1) \wedge (\varphi \mathrel{-\!\!*} \psi_2)$. By applying De Morgan laws and the distribution of $\Diamond$ over $\vee$, it is easily seen that $\Diamond_\varphi(\psi_1 \vee \psi_2) := \neg(\varphi \mathrel{-\!\!*} \neg\Diamond(\psi_1 \vee \psi_2))$ is logically equivalent to $\neg(\varphi \mathrel{-\!\!*} \neg\Diamond\psi_1) \vee \neg(\varphi \mathrel{-\!\!*} \neg\Diamond\psi_2)$, or, $\Diamond_\varphi\psi_1 \vee \Diamond_\varphi\psi_2$. They do not, however, necessarily inherit any additional axioms from $\Diamond$: for example, $\Diamond$ being an S4 modality does not entail that $\Diamond_\varphi$ is an S4 modality.

These kinds of considerations help to understand why we do not consider the evident intuitionistic variant of SML. For one, our definition of $\Diamond_\varphi$ uses Boolean negation, something that we do not have access to if we begin from BI. When using intuitionistic negation, the semantic clause for $\Diamond_\varphi$ is not equivalent to anything capturing the idea of combining a local resource with a new resource to reach a new state. It follows that the modalities $\Diamond_\varphi$ must be defined primitively. However, intuitionistic modalities requires a certain amount of coherence between the order $\succcurlyeq$ and the accessibility relation R in order to be interpreted naturally (see Simpson [208] for a consideration of these issues). Unfortunately, none of the possible coherence conditions between $\succcurlyeq$ and R appear to be inherited by the induced accessibility relation that $\Diamond_\varphi$ would be required to be interpreted on to have its natural reading. We leave open the possibility of defining the analogous system extending BI.

Frame:	$\dfrac{\{p\}c\{q\}}{\{p*r\}c\{q*r\}}$	Concurrency:	$\dfrac{\{p_1\}c_1\{q_1\} \qquad \{p_2\}c_2\{q_2\}}{\{p_1*p_2\}c_1 \parallel c_2\{q_1*q_2\}}$
Skip:	$\dfrac{}{\{p\}\text{skip}\{p\}}$	Seq:	$\dfrac{\{p\}c_1\{q\} \qquad \{q\}c_2\{r\}}{\{p\}c_1;c_2\{r\}}$
NonDet:	$\dfrac{\{p\}c_1\{q\} \qquad \{p\}c_2\{q\}}{\{p\}c_1+c_2\{q\}}$	Iterate:	$\dfrac{\{p\}c\{p\}}{\{p\}\text{Iterate}(c)\{p\}}$
Disjunction:	$\dfrac{\{p_i\}c\{q\},\ \text{all } i\in I}{\{\bigvee_{i\in I} p\}c\{q\}}$	Consequence:	$\dfrac{p\le p' \qquad \{p\}c\{q\} \qquad q\le q'}{\{p'\}c\{q'\}}$

Figure 4.8: Rules for ASL^{--}.

4.4 Concurrent Kleene Bunched Logic

We finish this chapter with a new bunched logic, CKBI. It is motivated by work by O'Hearn et al. [178] connecting a basic version of Concurrent Separation Logic called ASL^{--} to concurrent Kleene algebra. We briefly recount the relevant definitions here. Figure 4.8 gives the proof rules for ASL^{--}.

Definition 4.5 (Concurrent Kleene Algebra (cf. [178]))**.**

1. *A* concurrent monoid $(M,\le,\parallel,;,\text{skip})$ *is a partial order* $(M,\le)$*, together with two monoids* $(M,\parallel,\text{skip})$ *(with* $\parallel$ *commutative) and* $(M,;,\text{skip})$ *satisfying the* exchange law
$$(p \parallel r);(q \parallel s) \le (p;q) \parallel (r;s).$$
It is complete *if* $(M,\le)$ *is a complete lattice.*

2. *A* concurrent Kleene algebra *(CKA) is a complete concurrent monoid where* $\parallel$ *and* ; *preserve joins in both arguments.*

3. *A* weak CKA *is a complete concurrent monoid together with a subset* $A\subseteq M$ *(the assertions of the algebra) such that i)* $\text{skip}\in A$*; ii)* A *is closed under* $\parallel$ *and all joins; iii)* $\parallel$ *restricted to* A *preserves all joins in both arguments; iv) for each* $a\in A$*,* $a;(-) : M\to M$ *preserves all joins; and v) for each* $m\in M$*,* $(-);m : A\to M$ *preserves all joins.*

4. *A CKA or weak CKA is* Boolean *if the underlying lattice is a Boolean algebra and intuitionistic if the underlying lattice is a Heyting algebra.* □

In concurrent Kleene algebra, $p \parallel q$ is interpreted as giving the parallel execution of programs p and q while $p;q$ is interpreted as giving the sequential execution p, then q. One of the key aspects of this definition is the exchange law, which enforces a "soften[ing of] true concurrency" [138]: it states that a program that runs p

27. $\dfrac{\xi \vdash \varphi \quad \eta \vdash \psi}{\xi\,;\eta \vdash \varphi\,;\psi}$ 28. $\dfrac{\eta\,;\varphi \vdash \psi}{\eta \vdash \varphi \multimap \psi}$ 29. $\dfrac{\xi \vdash \varphi \multimap \psi \quad \eta \vdash \varphi}{\xi\,;\eta \vdash \psi}$

30. $\dfrac{\eta\,;\varphi \vdash \psi}{\varphi \vdash \eta \mathrel{\triangleright\!\!-} \psi}$ 31. $\dfrac{\xi \vdash \varphi \mathrel{\triangleright\!\!-} \psi \quad \eta \vdash \varphi}{\eta\,;\xi \vdash \psi}$ 32. $\dfrac{}{\top^*\,;\varphi \dashv\vdash \varphi}$

33. $\dfrac{}{\varphi\,;\top^* \dashv\vdash \varphi}$ 34. $\dfrac{}{\varphi\,;(\psi\,;\chi) \dashv\vdash (\varphi\,;(\psi\,;\chi)}$ 35. $\dfrac{}{(\varphi * \psi)\,;(\chi * \xi) \vdash (\varphi\,;\chi) * (\psi\,;\xi)}$

Figure 4.9: Hilbert rules for concurrent Kleene bunched logic.

and r in parallel, followed by q and s in parallel can be implemented as a program that runs p then q in parallel to r then s.

O'Hearn et al. show that ASL^{--} is sound and complete for weak CKAs when Hoare triples $\{p\}c\{q\}$ are interpreted as inequalities $p;c \leq q$ (where $p,q \in A$ and $c \in M$) and $*$ is interpreted as $\|$ restricted to A. This is achieved via the construction of a predicate transformer model over ASL^{--} propositions. They also show that a trace model of ASL^{--} generates a Boolean CKA.

Elsewhere, O'Hearn [176] suggests that the structures involved could be used as inspiration for a bunched logic extending BBI. We define such a logic and call it *Concurrent Kleene BI* or CKBI. CKBI acts as a case study for the methods of this thesis, demonstrating the applicability of the duality and proof theoretic approaches to new extensions of bunched logics. We leave the evident intuitionistic variant extending BI to another occasion.

Let Prop be a set of atomic propositions, ranged over by p. The set of all formulae of the concurrent Kleene bunched logic Form_{CKBI} is generated by the grammar

$$\varphi ::= \mathrm{p} \mid \top \mid \bot \mid \top^* \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi * \varphi \mid \varphi\,;\varphi \mid \varphi \mathrel{-\!\!*} \varphi \mid \varphi \multimap \varphi \mid \varphi \mathrel{\triangleright\!\!-} \varphi$$

where additive negation is defined by $\neg\varphi := \varphi \to \bot$.

Figure 4.9 gives rules that can be added to the system for BBI to obtain a system for CKBI. These rules essentially dictate that the $(*, \mathrel{-\!\!*})$-free fragment of CKBI is non-commutative BBI, with $\top^*$ both the left and right unit for ;. The non-commutativity of ; ensures that it has two associated implications, just as in the case for the layered graph logics, and these are related to ; by proof rules enforcing adjointness. Finally, the exchange law relating the two multiplicative conjunctions $*$ and ; is axiomatised directly by 35.

CKBI is interpreted on structures extending BBI frames called CKBI frames.

Definition 4.6 (CKBI Frame)**.** *A* CKBI frame *is a structure* $\mathscr{X} = (X, \circ, E, \triangleright)$ *such that* $(X, \circ, E)$ *is a BBI frame and* $\triangleright : X^2 \to \mathscr{P}(X)$ *a binary operation satisfying (with*

$$
\begin{array}{lll}
x \models_{\mathscr{M}} \varphi ; \psi & \text{iff} & \text{there exists } y,z \text{ s.t. } x \in y \triangleright z,\ y \models_{\mathscr{M}} \varphi \text{ and } z \models_{\mathscr{M}} \psi \\
x \models_{\mathscr{M}} \varphi \multimap \psi & \text{iff} & \text{for all } y,z \text{ s.t. } z \in x \triangleright y\text{: } y \models_{\mathscr{M}} \varphi \text{ implies } z \models_{\mathscr{M}} \psi \\
x \models_{\mathscr{M}} \varphi \mathrel{\triangleright\!\!-} \psi & \text{iff} & \text{for all } y,z \text{ s.t. } z \in y \circ x\text{: } y \models_{\mathscr{M}} \varphi \text{ implies } z \models_{\mathscr{M}} \psi
\end{array}
$$

Figure 4.10: Satisfaction for CKBI.

outermost quantification omitted for readability):

$$
\begin{array}{ll}
\textit{(Unit Existence}_L\textit{)} & \exists e \in E(x \in e \triangleright x); \\
\textit{(Unit Existence}_R\textit{)} & \exists e \in E(x \in x \triangleright e); \\
\textit{(Coherence}_L\textit{)} & e \in E \wedge x \in e \triangleright y \to x = y; \\
\textit{(Coherence}_R\textit{)} & e \in E \wedge x \in y \triangleright e \to x = y; \\
\textit{(Associativity)} & \exists t(t \in x \triangleright y \wedge w \in t \triangleright z) \leftrightarrow \exists t'(t' \in y \triangleright z \wedge w \in x \triangleright t') \\
\textit{(Exchange)} & t \in w \circ y \wedge s \in x \circ z \wedge u \in t \triangleright s \to \\
 & \exists r, v(r \in w \triangleright x \wedge v \in y \triangleright z \wedge u \in r \circ u)
\end{array}
$$

A CKBI frame $\mathscr{X}$ together with a valuation $\mathscr{V}$ gives a CKBI model $\mathscr{M}$, and for such a model the satisfaction relation $\models_{\mathscr{M}} \subseteq X \times \mathrm{Form}_{\mathrm{CKBI}}$ is inductively generated by the satisfaction clauses for BBI, together with the clauses for $;, \multimap, \mathrel{\triangleright\!\!-}$ given in Figure 4.10.

A CKBI frame is straightforwardly interpreted as having a concurrent composition $\circ$ and a sequential composition $\triangleright$ that are related by the Exchange frame property. In Chapter 6 we will show this frame property corresponds to the exchange law: in particular, CKBI frames generate Boolean CKAs extended with residuals for $\|$ and ;. The traces model of ASL^{--} can be seen as an algebra generated by a CKBI frame, where $\circ$ is interleaving, $\triangleright$ is concatenation and E is the singleton set containing the empty trace (cf. the examples of Section 3.3). Another example is given by pomsets [107], with $\circ$ given by the parallel pomset composition, $\triangleright$ the series pomset composition, and E the singleton set containg the empty pomset.

Summary of Part I

In this part we introduced the family of bunched logics as extensions of the new basic bunched logic ILGL. This begins with the layered graph logics ILGL and LGL in Chapter 2, for which we specify Hilbert-type proof systems, Kripke semantics and a special class of models based on graphs carrying a structure of layers. In Chapter 3 those logics are extended to obtain the principal bunched logics BI and BBI. There we compared our Kripke semantics from those in the literature, introduced Separation Logic and gave a raft of examples of (B)BI models found in the computer science literature. Chapter 4 introduced the bunched logics obtained by extending BI and BBI. These include the previously explored CBI, as well as the previously uninvestigated DMBI, the subclassical bunched logics extending BI and BBI, a schema for defining a range of separating modal logics and the new bunched logic CKBI based on the interpretation of Concurrent Separation Logic in concurrent Kleene algebra.

Part II

Algebra and Duality for Bunched Logics

Introduction to Part II

This part of the thesis is dedicated to setting up a duality theoretic framework for understanding the semantics of bunched logics. In Chapter 5 we outline the necessary preliminaries to prove the duality theorems for bunched logics: basic algebraic and topological notions, together with the proofs of the duality theorems for classical and intutionistic propositional logic. In Chapter 6 we systematically give duality theorems for the structures associated with all of the bunched logics defined in Part I: A Family of Bunched Logics. This is applied in Chapter 7 to a range of metatheoretic problems: uniform soundness and completeness for the breadth of bunched logics, decidability results for layered graph logics, an analogue of the Goldblatt-Thomason theorem characterising classes of frames definable in bunched logic and the failure of Craig interpolation. In Chapter 8 we extend the duality theorems to the predicate case, capturing in particular (B)BI hyperdoctrines, which are widely invoked in the theory of higher-order separation logic.

Chapter 5

Algebraic and Topological Preliminaries

In this chapter we lay the necessary groundwork for the family of duality theorems that relate the syntax and semantics of bunched logics. To do so we first require some basic algebraic and topological material. Naturally, the dualities for bunched logics extend the dualities of Stone and Esakia for classical and intuitionistic propositional logics, so the remainder of the chapter is dedicated to the proofs of these results which we give in full detail in order to streamline the presentation of the bunched logic duality theorems.

5.1 Algebra

We begin with the basic lattice structure underlying all algebras of bunched logics.

Definition 5.1 (Distributive Lattice)**.** *A* distributive lattice *is an algebra* $\mathbb{A} = (A, \wedge, \vee)$ *where* $\wedge$ *and* $\vee$ *are associative and commutative binary operations—called* meet *and* join *respectively—on a set A satisfying, for all* $a, b, c \in A$*,*

$$\begin{array}{lll} \wedge \textit{ Absorption} & a \vee (a \wedge b) & = a \\ \vee \textit{ Absorption} & a \wedge (a \vee b) & = a \\ \textit{Distributive Law} & a \wedge (b \vee c) & = (a \wedge b) \vee (a \wedge c). \end{array}$$

A distributive lattice is bounded *if there exist elements* $\bot \neq \top$ *that are units for* $\vee$ *and* $\wedge$ *respectively: for all* $a \in \mathbb{A}$*,* $a \vee \bot = a$ *and* $a \wedge \top = a$*. This makes* $\bot$ *and* $\top$ *the minimum and maximum (respectively) of the partial order defined*

$$a \geq b \textit{ iff } a \vee b = a \textit{ iff } a \wedge b = b$$

We specify that $\top \neq \bot$ to exclude the degenerate single element algebra. We refer to elements of algebras by the variables a, b, c, d. We also abuse notation and

write $a \in \mathbb{A}$ to mean $a \in A$, where A is the carrier set of the algebra $\mathbb{A}$. The key lattice-theoretic concepts we require for our work is that of filters and ideals.

Definition 5.2 (Filters and Ideals)**.** *A* filter *on a distributive lattice $\mathbb{A}$ is a non-empty subset $F \subseteq A$ satisfying*

1. *Upwards Closure: $a \in F$ and $b \geq a$ implies $b \in F$;*

2. *Meet Closure: $a, b \in F$ implies $a \wedge b \in F$.*

A filter is proper *if it additionally satisfies $\bot \notin F$, and a proper filter is* prime *if it further satisfies $a \vee b \in F$ implies $a \in F$ or $b \in F$. An* ideal *is the order dual notion to a filter: a non-empty subset $I \subseteq A$ satisfying*

1. *Downwards Closure: $a \in I$ and $a \geq b$ implies $b \in I$;*

2. *Join Closure: $a, b \in I$ implies $a \vee b \in I$.*

It is proper *if it additionally satisfies $\top \notin I$, and a proper ideal is* prime *is it further satisfies $a \wedge b \in I$ implies $a \in I$ or $b \in I$.*

We use the variables F, G to refer to filters, I, J for ideals and H to refer to either, and we denote the set of all proper filters of an algebra by $\mathbb{F}_{\mathbb{A}}$; likewise the set of proper ideals by $\mathbb{I}_{\mathbb{A}}$. Filters and ideals can be understood intuitively from a variety of different directions, as they arise naturally in fields as diverse as topology and social choice theory. From a logician's perspective, a filter is a propositional theory, closed under a simple form of logical consequence. If $\bot$ is in the theory then that theory is 'improper' as it is inconsistent and thus contains every proposition.

Proposition 5.3. *The complement $\overline{F}$ of a prime filter F is a prime ideal; the complement $\bar{I}$ of a prime ideal I is a prime filter.* □

Given a non-empty subset X of an algebra $\mathbb{A}$, we define $[X) := \{a \in \mathbb{A} \mid \exists b_1, \ldots, b_n \in X : a \geq b_1 \wedge \cdots b_n\}$ to be the *filter generated by X*. It is easily seen that this is the least filter (with respect to set theoretic inclusion) containing the set X. Dually, we obtain the *ideal generated by X* as $(X] := \{a \in \mathbb{A} \mid \exists b_1, \ldots, b_n \in X : b_1 \vee \cdots \vee b_n \geq a\}$, and this is the least ideal containing X. For a singleton $\{a\}$ we write $[a)$ and $(a]$ for $[\{a\})$ and $(\{a\}]$. The next proposition uses the closure properties of filters and ideals to give useful identities for the generation of new filters and ideals from old ones.

Proposition 5.4 (cf. [89]). *Let F be a filter on $\mathbb{A}$, I an ideal on $\mathbb{A}$, $c \in \mathbb{A}$ and $f : \mathbb{A} \to \mathbb{A}'$ a homomorphism. Then*

$$[F \cup \{c\}) = \{a \in \mathbb{A} \mid \exists b \in F : a \geq b \wedge c\} \quad (I \cup \{c\}] = \{a \in \mathbb{A} \mid \exists b \in I : b \vee c \geq a\}$$
$$[f(F)) = \{a \in \mathbb{A}' \mid \exists b \in F : a \geq f(b)\} \qquad (f(I)] = \{a \in \mathbb{A}' \mid \exists b \in I : f(b) \geq a\}$$

□

We will frequently need to prove the existence of prime filters/ideals satisfying certain properties in order to prove our duality theoretic framework captures the structures associated with bunched logics. To this end we generalise a concept of Galmiche & Larchey-Wendling [99] that gives a systematic method for showing such prime filters/ideals exist. First some terminology: a $\subseteq$-*chain* is a sequence of sets $(X_\alpha)_{\alpha<\lambda}$ such that $\alpha \leq \alpha'$ implies $X_\alpha \subseteq X_{\alpha'}$. A basic fact about proper filters (ideals) is that the union of a $\subseteq$-chain of proper filters (ideals) is itself a proper filter (ideal). We lift the terminology to n-tuples of sets by determining $(X^1_\alpha, \ldots, X^n_\alpha)_{\alpha<\lambda}$ to be a $\subseteq$-chain if each $(X^i_\alpha)_{\alpha<\lambda}$ is a $\subseteq$-chain.

Definition 5.5 (Prime Predicate). *A* prime predicate *is a map $P : \mathbb{F}^n_{\mathbb{A}} \times \mathbb{I}^m_{\mathbb{A}} \to \{0,1\}$, where $n,m \geq 0$ and $n+m \geq 1$, such that*

a) Given a $\subseteq$-chain $(F^0_\alpha, \ldots, F^n_\alpha, I^0_\alpha, \ldots, I^m_\alpha)_{\alpha<\lambda}$ of proper filters/ideals, $min\{P(F^0_\alpha, \ldots, I^m_\alpha) \mid \alpha < \lambda\} \leq P(\bigcup_\alpha F^0_\alpha, \ldots, \bigcup_\alpha I^m_\alpha)$;

b) $P(\ldots, H_0 \cap H_1, \ldots) \leq max\{P(\ldots, H_0, \ldots), P(\ldots, H_1, \ldots)\}$.

A prime predicate is a property that can hold of an $(n+m)$-tuple of proper filters and ideals that is evaluated to true or false. The two simple conditions it must satisfy to be a prime predicate are that the truth of the property persists from a chain of tuples of proper filters and ideals to their component-wise union and that if an n-tuple of proper filters and ideals for which one component is an intersection $H_0 \cap H_1$ is evaluated true, at least one of the tuples of prime filters and ideals obtained by replacing that intersection with either H_0 or H_1 is also evaluated true. Our definition differs from that of Galmiche & Larchey-Wendling in two ways: first, our notion of prime predicate takes an argument of tuples of proper filters and ideals rather than a single filter, as we frequently need to prove the simultaneous existence of prime filters and ideals satisfying a particular condition; second, we build into the definition that we only consider proper filters and ideals, removing the need for their clause stating that the prime predicate evaluates to false for improper filters/ideals.

To justify the existence of prime filters we require throughout the next few chapters we give a Prime Extension Lemma, stating when the existence of appropriate proper filters and ideals allows us to determine the existence of prime filters

and ideals. It is a strict generalisation of Galmiche & Larchey-Wendling's Prime Extension Lemma ([99], Lemma 1) and also plays a similar role to the various 'Squeeze Lemmas' (traced to Routley & Meyer [200] by Dunn [88]) used to prove representation and duality theorems for relevant logics. The lemma requires the use of Zorn's lemma.

Lemma 5.6 (Zorn's Lemma)**.** *If $(S, \geq)$ is a partial order in which every totally-ordered subset has an upper bound, there exists a maximal element s: that is, for all $s' \in S$, $s \leq s'$ implies $s = s'$.* □

Zorn's lemma is equivalent to the axiom of choice, which itself entails the law of excluded middle for the set theoretic universe we're doing mathematics in. In short, our metatheory is classical. More philosophically-minded logicians might balk at the idea of giving 'classical' metatheory for 'intuitionistic' logics, but we reaffirm the message outlined in Chapter 2: we are interested in these logics for their use as a modelling technology, not necessarily because of their constructive interpretation. Recent work by Negri [172] investigates constructive reformulations of representation theorems through the use of analytic proof calculi, but the extension of this idea to bunched logics would represent substantial further work and would necessarily build on the present results.

Lemma 5.7 (Prime Extension Lemma)**.** *If P is an $(n+m)$-ary prime predicate and $F_0, \ldots, F_n, I_0, \ldots, I_m$ an $(n+m)$-tuple of proper filters and ideals such that $P(F_0, \ldots, F_n, I_0, \ldots, I_m) = 1$ then there exists a $(n+m)$-tuple of prime filters and ideals $F_0^{pr}, \ldots, F_n^{pr}, I_0^{pr}, \ldots I_m^{pr}$ such that $P(F_0^{pr}, \ldots, F_n^{pr}, I_0^{pr}, \ldots I_m^{pr}) = 1$.*

Proof. Consider the set $Z = \{(F_0, \ldots, F_n, I_0, \ldots, I_m) \mid P(F_0, \ldots, F_n, I_0, \ldots, I_m) = 1\}$. This is ordered by component-wise inclusion and by assumption is non-empty. By a) in the definition of prime predicate, any totally-ordered subset of this partial order has an upper bound given by taking the union in each component. Hence by Zorn's lemma there exists a maximal element $(F_0^{max}, \ldots, F_n^{max}, I_0^{max}, \ldots I_m^{max})$ of Z. We claim that each F_k^{max} is a prime filter and each I_k^{max} a prime ideal.

We concentrate on the case of F_k^{max}, as the argument for I_k^{max} is essentially dual. Suppose $a \vee b \in F_k^{max}$. We must show $a \in F_k^{max}$ or $b \in F_k^{max}$. Consider the two filters $[F^{max} \cup \{a\})$ and $[F^{max} \cup \{b\})$. There are two cases. First, suppose (*wolog*) that $[F_k^{max} \cup \{a\})$ is improper. Then there exists $x \in F_k^{max}$ such that $\bot = x \wedge a$ by Proposition 5.4. Thus by distributivity $x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) = x \wedge b \leq b$ so $b \in F_k^{max}$ and we're done. Next, suppose instead that both $[F_k^{max} \cup \{a\})$ and $[F_k^{max} \cup \{b\})$ are proper. We show that $F_k^{max} = [F_k^{max} \cup \{a\}) \cap [F_k^{max} \cup \{b\})$. The left-to-right

inclusion is trivial. For the right-to-left, suppose $x \in [F_k^{max} \cup \{a\}) \cap [F_k^{max} \cup \{b\})$. Then there exist $y, z \in F_k^{max}$ such that $x \geq y \wedge a, z \wedge b$. It follows that

$$x \geq (y \wedge a) \vee (z \wedge b) \geq (y \wedge z \wedge a) \vee (y \wedge z \wedge b) = (y \wedge z) \wedge (a \vee b),$$

and since $y \wedge z, a \vee b \in F_k^{max}$, we have that $x \in F_k^{max}$. Now by b) in the definition of prime predicate we have that either $P(\ldots, [F_k^{max} \cup \{a\}), \ldots) = 1$ or $P(\ldots, [F_k^{max} \cup \{b\}), \ldots) = 1$. Thus either $F_k^{max} = [F_k^{max} \cup \{a\})$ and $a \in F_k^{max}$ or $F_k^{max} = [F_k^{max} \cup \{b\})$ and $b \in F_k^{max}$ as otherwise the maximality of $(F_0^{max}, \ldots, F_n^{max}, I_0^{max}, \ldots I_m^{max})$ in Z is contradicted. □

One application of the prime extension lemma is what is often called the *prime filter theorem*, which states that a filter disjoint from an ideal I can be extended to a prime filter disjoint from I.

Theorem 5.8 (Prime Filter Theorem)**.** *If F a filter and I an ideal such that $F \cap I = \emptyset$ then there exists a prime filter $F' \subseteq F$ with $F' \cap I = \emptyset$.*

Proof. We show that the property $F' \cap I = \emptyset$ and $F \subseteq F'$ of proper filters F' defines a prime predicate. We first note that $F \cap I = \emptyset$ means $\bot \notin F$ and so F is proper and can thus be a subset of a proper filter. First it is clear that if we have a $\subseteq$-chain $(F'_\alpha)_{\alpha<\lambda}$ for which the property holds then it neccessarily holds of $\bigcup_{i \in I} F'_i$. Now suppose $F'_0 \cap F'_1 \cap I = \emptyset$ and $F'_0 \cap F'_1 \supseteq F$. Clearly $F'_0, F'_1 \supseteq F$ so suppose for contradiction that $x \in F'_0 \cap I$ and $y \in F'_1 \cap I$. We then have $x, y \leq x \vee y \in F'_0 \cap F'_1 \cap I$ by upwards closure for F'_i and closure under joins of I; a contradiction. Hence one of $F'_0 \cap I = \emptyset$ or $F'_1 \cap I = \emptyset$ must hold. □

5.2 Topology

We now briskly recall the topological notions that we will use for our duality theorems. These can all be found in any standard text on topology (e.g., [167]).

Definition 5.9. *A* topological space *is a pair $\mathscr{X} = (X, \mathscr{O})$ where X is a set and $\mathscr{O} \subseteq \mathscr{P}(X)$, satisfying:*

1. *$\emptyset, X \in \mathscr{O}$;*
2. *Given any $O_i \in \mathscr{O}$ indexed by $i \in I$, $\bigcup_{i \in I} O_i \in \mathscr{O}$;*
3. *$O_0, \ldots, O_n \in \mathscr{O}$ implies $\bigcap_{i=0}^{n} O_i \in \mathscr{O}$.*

The sets $O \in \mathscr{O}$ are called open*, the sets C with $\overline{C} \in \mathscr{O}$ are called* closed*, and if a set is both open and closed it is called* clopen*.*

The correct notion of morphism for a topological space is a continuous map, and topological spaces with continuous maps form a category Top.

Definition 5.10 (Continuous Map)**.** *A* continuous map $f : \mathscr{X} \to \mathscr{X}'$ *is a function* $f : X \to X'$ *such that for any open* O' *of* $\mathscr{X}'$*, the inverse image* $f^{-1}[O']$ *is open in* $\mathscr{X}$*.*

There are useful ways to characterise or generate a topology on a given set.

Definition 5.11 (Base)**.** *A* base $\mathscr{B}$ *for a topological space* $\mathscr{X} = (X, \mathscr{O})$ *is a subset* $\mathscr{B} \subseteq \mathscr{O}$ *such that every* $O \in \mathscr{O}$ *can be expressed as the union* $\bigcup_{i \in I} B_i$ *of base elements* $B_i \in \mathscr{B}$*. We say* $\mathscr{X}$ *is* zero-dimensional *if the topology has a base of clopen elements.*

Definition 5.12 (Subbase)**.** *A* subbase $\mathscr{S}$ *for a topological space* $\mathscr{X} = (X, \mathscr{O})$ *is a set* $\mathscr{S} \subset \mathscr{O}$ *such that the set of finite intersections of elements of* $\mathscr{S}$ *together with* X *forms a base for* $(X, \mathscr{O})$*.*

While one can find a number of suitable subbases for a given topological space $\mathscr{X}$, any collection of sets $\mathscr{S} \subseteq \mathscr{P}(X)$ defines a base for a unique topology on X by taking the finite intersections of elements of $\mathscr{S}$. An important topological property is compactness.

Definition 5.13 (Compact)**.** *A topological space* $\mathscr{X}$ *is* compact *if, for every open cover of* X *(a collection of open sets* O_i *such that* $X = \bigcup_{i \in I} O_i$ *there exists a finite subcover* $X = \bigcup_{j=0}^{i_m} O_{i_j}$*.*

A useful equivalent definition of compact space is as follows. The *finite intersection property* (FIP) holds for a family of sets $(X_i)_{i \in I}$ if every finite intersection of elements X_i is non-empty. A topological space is compact iff every family of closed sets with the FIP has non-empty intersection. Compact spaces have the nice property that any closed set C is also compact: that is, any open cover of a closed set C has a finite subcover. An extremely useful technique to verify a space is compact is Alexander's subbase theorem.

Theorem 5.14 (Alexander Subbase Theorem)**.** *Let* $\mathscr{S}$ *be a subbase for a topological space* $\mathscr{X}$*. If every cover of* X *by subbase elements has a finite subcover then* $\mathscr{X}$ *is compact.* □

We finish this section with a number of *separation properties* that the topological spaces we examine will satisfy.

Definition 5.15 (Hausdorff)**.** *A topological space $\mathscr{X}$ is* Hausdorff *if any distinct $x, y \in X$ can be separated by disjoint open sets: there exists $O, O' \in \mathscr{O}$ with $O \cap O' = \emptyset$, $x \in O$ and $y \in O'$.*

A useful property of maps between compact and Hausdorff spaces is as follows: if $f : \mathscr{C} \to \mathscr{H}$ is a continuous map between a compact space $\mathscr{C}$ and a Hausdorff space $\mathscr{H}$ then for any closed set C in $\mathscr{C}$, the image $f[C]$ will be closed in $\mathscr{H}$.

Definition 5.16 (Totally Disconnected)**.** *A topological space $\mathscr{X}$ is* totally disconnected *if any distinct $x, y \in X$ can be separated by disjoint open sets O, O' such that $X = O \cup O'$.*

An *ordered topological space* is a topological space equipped with a partial order $\succcurlyeq$ on its points. The following is a separation property for ordered spaces that expresses coherence between the topological structure and the order.

Definition 5.17 (Priestley Separation Axiom)**.** *An ordered topological space $(\mathscr{X}, \succcurlyeq)$ satisfies the* Priestley separation axiom *if, given any $x, y \in X$ such that $x \not\succcurlyeq y$, there exists an upwards-closed clopen set U such that $y \in U$ and $x \notin U$.*

5.3 Esakia Duality

In this section we recount the duality theorem for Heyting algebras; the algebraic structures that correspond to intuitionistic propositional logic in the same manner as Boolean algebras for classical propositional logic.

Definition 5.18 (Heyting Algebra)**.** *A* Heyting algebra *is an algebra $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot)$ such that $(A, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice and $\to$ is a binary operation satisfying, for all $a, b, c \in A$: $a \wedge b \leq c$ iff $a \leq b \to c$.*

It can in fact be shown that distributivity need not be assumed, as it can be proven from the residuation property of $\wedge$ and $\to$. We specify it for simplicity of presentation. Interpretations of intuitionistic logic on a Heyting algebra work as follows: let $\mathscr{V} : \mathrm{Prop} \to A$ be an assignment of elements of the algebra to propositional variables; this is uniquely extended to an interpretation $[\![-]\!]$ of every intutionistic logic formula by induction, with $[\![\mathrm{p}]\!] = \mathscr{V}(\mathrm{p})$, $[\![\top]\!] = \top$ and $[\![\bot]\!] = \bot$ as base cases:

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!] \quad [\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!] \quad [\![\phi \to \psi]\!] = [\![\phi]\!] \to [\![\psi]\!]$$

It is well known that intuitionistic logic is sound and complete with respect to this semantics.

Theorem 5.19 (Algebraic Soundness and Completeness)**.** *For all intutionistic logic formulae φ and ψ, $\varphi \vdash \psi$ is provable in the Hilbert system for intuitionistic logic iff $[\![\varphi]\!] \leq [\![\psi]\!]$ holds for all algebraic interpretations on Heyting algebras.* □

A *homomorphism* $f : \mathbb{A} \to \mathbb{A}'$ between algebras of the same type is a map $f : A \to A'$ that commutes with the algebraic operations in the signature: for each n-ary operation $\heartsuit$, and all $a_1, \ldots, a_n \in \mathbb{A}$, $f(\heartsuit(a_1, \ldots, a_n)) = \heartsuit'(f(a_1), \ldots, f(a_n))$. The category HA is comprised of Heyting algebras and homomorphisms between them.

We can also equip *intuitionistic frames* (preordered sets $(X, \succcurlyeq)$) with a notion of morphism to obtain a category. These can be found in the literature by the name bounded morphism or pseudo epi morphism, but we use the name *intuitionistic morphism* in-keeping with the convention we will set for bunched logic frame morphisms.

Definition 5.20 (Intuitionistic Morphism)**.** *An* intuitionistic morphism $g : (X, \succcurlyeq) \to (Y, \succcurlyeq')$ *is a map $g : X \to Y$ satisfying*

i) $x \succcurlyeq y$ implies $g(x) \succcurlyeq' g(y)$;

ii) $x' \succcurlyeq' g(y)$ implies there exists $x \in X$ such that $x \succcurlyeq y$ and $g(x) = x'$.

It is straightforward to see that the identity map $id : X \to X$ satisfies i) and ii). Further, the composition of intuitionistic frame morphisms is once again an intuitionistic frame morphism. Hence intuitionistic frames and intuitionistic morphisms form a category Int.

Definition 5.21 (Intuitionistic Complex Algebra)**.** *Given an intuitionistic frame $\mathscr{X}$, the* intuitionistic complex algebra *of $\mathscr{X}$ is given by $Com(\mathscr{X}) = (\mathscr{P}_{\succcurlyeq}(X), \cap, \cup, \Rightarrow_{\mathscr{X}}, X, \emptyset)$ where*

$$\begin{aligned} \mathscr{P}_{\succcurlyeq}(X) &= \{A \subseteq X \mid \text{ if } a \in A \text{ and } b \succcurlyeq a \text{ then } b \in A\} \\ A \Rightarrow_{\mathscr{X}} B &= \{x \mid x' \succcurlyeq x \text{ and } x' \in A \text{ implies } x' \in B\} \end{aligned}$$

Lemma 5.22. *The intuitionistic complex algebra of a intuitionistic frame is a Heyting algebra.*

Proof. First, note that $\mathscr{P}_{\succcurlyeq}(X)$ is closed under $\cap, \cup$ and $\Rightarrow_{\mathscr{X}}$, with $X, \emptyset \in \mathscr{P}_{\succcurlyeq}(X)$ trivially and the distributive laws holding for $\cap$ and $\cup$ straightforwardly. To prove the adjointness property of Heyting implication, assume $A \cap B \subseteq C$ and suppose $a \in A$ with $b \succcurlyeq a$ and $b \in B$. By upwards-closure of A, $b \in A \cap B$ so $b \in C$ as required. In the other direction, assume $A \subseteq B \Rightarrow_{\mathscr{X}} C$ and suppose $a \in A \cap B$. Then by assumption $a \in C$. □

Definition 5.23 (Prime Filter Intuitionistic Frame)**.** *Given a Heyting algebra* $\mathbb{A}$*, the* prime filter intuitionistic frame *of* $\mathbb{A}$*,* $Pr(\mathbb{A})$*, is given by the set of prime filters on* $\mathbb{A}$ *ordered by inclusion* $\supseteq$*.*

Henceforth we freely use $Pr(\mathbb{A})$ to refer to both the set of prime filters of the algebra $\mathbb{A}$ and the prime filter intuitionistic frame of $\mathbb{A}$. $\supseteq$ is a partial order, so $Pr(\mathbb{A})$ is an intuitionistic frame.

Lemma 5.24. *The prime filter intuitionistic frame of a Heyting algebra is an intuitionistic frame.* □

Theorem 5.25 (Representation Theorem for Heyting Algebras (cf. [89]))**.** *Every Heyting algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given a Heyting algebra* $\mathbb{A}$*, the map* $\theta_{\mathbb{A}} : \mathbb{A} \to Com(Pr(\mathbb{A}))$ *defined* $\theta_{\mathbb{A}}(a) = \{F \in Pr(\mathbb{A}) \mid a \in F\}$ *is an embedding.*

Proof. We first show that the map $\theta_{\mathbb{A}}$ is injective. Suppose $a \neq b$; *wolog* we may suppose $a \not\leq b$. Consider the filter $[a)$ and the ideal $(b]$. By assumption $[a) \cap (b] = \emptyset$, so by Theorem 5.8 there exists a prime $F \supseteq [a)$ with $F \cap (b] = \emptyset$. Then $b \notin F$ and $a \in F$ so $\theta_{\mathbb{A}}(a) \neq \theta_{\mathbb{A}}(b)$. It remains to prove the map is a homomorphism. By upwards-closure $\theta_{\mathbb{A}}(\top) = Pr(\mathbb{A})$ and by properness $\theta_{\mathbb{A}}(\bot) = \emptyset$. $\theta_{\mathbb{A}}(a \wedge b) = \theta_{\mathbb{A}}(a) \cap \theta_{\mathbb{A}}(b)$ since $a \wedge b \in F$ iff $a, b \in F$. Similarly, $\theta_{\mathbb{A}}(a \vee b) = \theta_{\mathbb{A}}(a) \cup \theta_{\mathbb{A}}(b)$ since—because of primeness—$a \vee b \in F$ iff $a \in F$ or $b \in F$.

To show $\theta_{\mathbb{A}}(a \to b) = \theta_{\mathbb{A}}(a) \Rightarrow_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$ is slightly more involved. First, two corner cases. If $a = \bot$ then both sides are $Pr(\mathbb{A})$ as $\top = \bot \to b$. The same holds if $b = \top$, as $\top = a \to \top$. We suppose $a \neq \bot$ and $b \neq \top$. Now, note that the left-to-right inclusion follows immediately from the fact that $a \wedge (a \to b) \leq b$. For the right-to-left, we assume $a \to b \notin F$ and prove there exists $G \supseteq F$ such that $a \in G$ and $b \notin G$. That this is possible follows from the assumption $a \neq \bot$ and $b \neq \top$ and the fact that $b \leq a \to b$ entails $b \notin F$. We can easily see that

$$P(G) = \begin{cases} 1 & \text{if } G \supseteq F, a \in G \text{ and } b \notin G \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. Given a $\subseteq$-chain $(G_\alpha)_{\alpha<\lambda}$, $P(G_\alpha) = 1$ for all α clearly implies $P(\bigcup_\alpha G_\alpha) = 1$. Further, if $P(G \cap G') = 1$ then $G, G' \supseteq F$, $a \in G, G'$ and $b \notin G$ or $b \notin G'$, so $P(G) = 1$ or $P(G') = 1$.

Consider the filter $G = [F \cup \{a\})$. Note that G is proper: if $\bot \in \alpha$ then there exists $x \in F$ such that $\bot = x \wedge a$. Consequently $a \to \bot \in F$. Since $\bot \to b \in F$ and $(a \to \bot) \wedge (\bot \to b) \leq a \to b$ we have a contradiction. We have $P(G) = 1$, hence by

the prime extension lemma there exists a *prime* filter G such that $P(G) = 1$, which is precisely what we need. □

θ is in fact a natural transformation that forms one part of a dual equivalence of categories between the category of Heyting algebras and a category of intuitionistic frames carrying topological structure. In the first instance, we can take advantage of the implicit topological structure on dual intuitionistic frames to lift the assignments *Pr* and *Com* to contravariant functors. It is straightforward to specify what the action on morphisms should be: $Pr(f) = f^{-1}$ and $Com(g) = g^{-1}$. Showing that this is well-defined requires more care. We must first investigate the structure of prime filter intuitionistic frames further. Our presentation of the remainder of this section is based on Morandi [166].

The first thing to note is that, as it is a collection of subsets of $Pr(\mathbb{A})$, $\mathscr{S} = \{\theta_{\mathbb{A}}(a) \mid a \in \mathbb{A}\} \cup \{\overline{\theta_{\mathbb{A}}(a)} \mid a \in \mathbb{A}\}$ is a subbase for a topology $\mathscr{O}_{\mathbb{A}}$ on $Pr(\mathbb{A})$. By Theorem 5.25, the base $\mathscr{B} = \{\bigcup_i^n S_i \mid S_i \in \mathscr{S}\} \cup \{Pr(\mathbb{A})\}$ that this generates for $\mathscr{O}_{\mathbb{A}}$ has a particularly straightforward form. For any $a_1, \ldots, a_n$, $\theta_{\mathbb{A}}(a_1) \cap \cdots \cap \theta_{\mathbb{A}} = \theta_{\mathbb{A}}(a_1 \wedge \cdots \wedge a_n)$. Similarly, for $b_1, \ldots, b_m$, $\overline{\theta_{\mathbb{A}}(b_1)} \cap \cdots \cap \overline{\theta_{\mathbb{A}}(b_m)} = \overline{\theta_{\mathbb{A}}(b_1 \vee \cdots \vee b_m)}$. Finally $\theta_{\mathbb{A}}(\top) \cap \theta_{\mathbb{A}}(\bot)) = Pr(\mathbb{A})$. Hence $\mathscr{B} = \{\theta_{\mathbb{A}}(a) \cap \overline{\theta_{\mathbb{A}}(b)} \mid a, b \in \mathbb{A}\}$.

Lemma 5.26. *The topology $\mathscr{O}_{\mathbb{A}}$ on $Pr(\mathbb{A})$ is 1. compact; 2. Hausdorff; and 3. satisfies the Priestley separation axiom with respect to $\subseteq$.*

Proof. 1. By the Alexander subbase theorem we need only show that a cover of $Pr(\mathbb{A})$ by base elements has a finite subcover. So suppose $Pr(\mathbb{A}) = \bigcup_{j \in J} \theta_{\mathbb{A}}(a_j) \cup \bigcup_{k \in K} \overline{\theta_{\mathbb{A}}(b_k)}$. Equivalently, $\bigcap_{k \in K} \theta_{\mathbb{A}}(b_k) \subseteq \bigcup_{j \in J} \theta_{\mathbb{A}}(a_j)$. Define $I = (\{a_j \mid j \in J\}]$ and $F = [\{b_k \mid k \in K\})$. If there exists $x \in F \cap I$ then there must be $a_{j_0}, \ldots, a_{j_n}, b_{k_0}, \ldots, b_{k_m}$ such that $b_{k_0} \wedge \cdots \wedge b_{k_m} \leq x \leq a_{j_0} \vee \cdots \vee a_{j_n}$. Then $\theta_{\mathbb{A}}(b_{k_0} \wedge \cdots \wedge b_{k_m}) \subseteq \theta_{\mathbb{A}}(a_{j_0} \vee \cdots \vee a_{j_n})$ giving us our finite subcover: $Pr(\mathbb{A}) = \bigcup_{i=0}^n \theta_{\mathbb{A}}(a_{j_i}) \cup \bigcup_{i=0}^m \overline{\theta_{\mathbb{A}}(b_{k_i})}$. Hence we assume for contradiction that $F \cap I = \emptyset$. By Theorem 5.8 this entails a prime $F' \supseteq F$ such that $F' \cap I = \emptyset$. Necessarily $F' \in \bigcap_{k \in K} \theta_{\mathbb{A}}(b_k) \subseteq \bigcup_{j \in J} \theta_{\mathbb{A}}(a_j)$ so for some a_j, $a_j \in F'$ and $F' \cap I \neq \emptyset$.

2. Let F and G be distinct prime filters. Then there exists (*wolog*) $a \in F \cap \overline{G}$. Thus $F \in \theta_A(a)$ and $G \in \overline{\theta_{\mathbb{A}}(a)}$ with $\theta_A(a) \cap \overline{\theta_{\mathbb{A}}(a)} = \emptyset$.

3. If $F \not\subseteq G$ then there exists $a \in F \cap \overline{G}$. Thus $F \in \theta_{\mathbb{A}}(a)$ (a clopen up-set) and $G \notin \theta_{\mathbb{A}}(a)$. □

Lemma 5.27. *For any homomorphism $f : \mathbb{A} \to \mathbb{A}'$, $f^{-1} : Pr(\mathbb{A}') \to Pr(\mathbb{A})$ is a continuous map.*

Proof. First we must verify that f^{-1} maps prime filters to prime filters. Let F be a prime filter. For upwards closure, let $a \in f^{-1}(F)$ with $b \geq a$. By definition $f(a) \in F$ and f is order-preserving so $f(a) \leq f(b) \in F$. Hence $b \in f^{-1}(F)$. For meet closure, if $a, b \in f^{-1}(F)$ then $f(a), f(b) \in F$, hence $f(a) \wedge f(b) = f(a \wedge b) \in F$. $f^{-1}(F)$ is proper because $f(\bot) = \bot \notin F$. Finally it is prime because $a \vee b \in f^{-1}(F)$ implies $f(a \vee b) = f(a) \vee f(b) \in F$ so $f(a) \in F$ or $f(b) \in F$. It is sufficient to verify continuity on the subbase elements and this is straightforward: $(f^{-1})^{-1}(\theta_{\mathbb{A}}(a)) = \theta_{\mathbb{A}'}(f(a))$ and $(f^{-1})^{-1}(\overline{\theta_{\mathbb{A}}(a)}) = \overline{\theta_{\mathbb{A}'}(f(a))}$. □

We now introduce the following notation. Given a subset $A \subseteq X$ of an ordered set $(X, \succcurlyeq)$, we call $\uparrow A := \{b \mid \exists a \in A(b \succcurlyeq a)\}$ the *upwards closure* of A. Similarly, $\downarrow A := \{b \mid \exists a \in A(a \succcurlyeq b)\}$ is the *downwards closure* of A.

Lemma 5.28. *For any clopen set C, and homomorphism f, $(f^{-1})^{-1}(\downarrow C) = \downarrow (f^{-1})^{-1}(C)$.*

Proof. Since $\mathscr{B}$ is a base for $\mathscr{O}_{\mathbb{A}}$, any open set can be written as $\bigcup_{i \in I} \theta_{\mathbb{A}}(a_i) \cap \overline{\theta_{\mathbb{A}}(b_i)}$. By compactness, this entails any clopen set can be written as $\bigcup_{i=0}^{m} \theta_{\mathbb{A}}(a_i) \cap \overline{\theta_{\mathbb{A}}(b_i)}$. $\downarrow$ commutes with union, so it suffices to verify the statement for $C = \theta_{\mathbb{A}}(a) \cap \overline{\theta_{\mathbb{A}}(b)}$. By the argument in the representation theorem, $\downarrow (\theta_{\mathbb{A}}(a) \cap \overline{\theta_{\mathbb{A}}(b)}) = \overline{\theta_{\mathbb{A}}(a \to b)}$. We thus have

$$\begin{aligned}
(f^{-1})^{-1}(\downarrow \theta_{\mathbb{A}}(a) \cap \overline{\theta_{\mathbb{A}}(b)}) &= (f^{-1})^{-1}(\theta_{\mathbb{A}}(a \to b)) \\
&= \overline{\theta_{\mathbb{A}'}(f(a) \to f(b))} \\
&= \downarrow (\theta_{\mathbb{A}'}(f(a)) \cap \overline{\theta_{\mathbb{A}'}(f(b))}) \\
&= \downarrow ((f^{-1})^{-1}(\theta_{\mathbb{A}}(a) \cap \overline{\theta_{\mathbb{A}}(b)})
\end{aligned}$$ □

Lemma 5.29. *For any closed set C of $\mathscr{O}_{\mathbb{A}}$, the set $\uparrow C$ is closed.*

Proof. We first show that the graph of $\subseteq$, $G(\subseteq)$, is a closed set of the product topology on $Pr(\mathbb{A}) \times Pr(\mathbb{A})$. For any $F \not\subseteq G$ we have a clopen up-set C with $F \in C$ and $G \in \overline{C}$, a clopen down-set. Thus $(F, G) \in C \times \overline{C} \subseteq \overline{G(\subseteq)}$, thus the complement of $G(\subseteq)$, $\overline{G(\subseteq)}$, is open and $G(\subseteq)$ is closed.

We now note that $\uparrow C = \pi_2((C \times Pr(\mathbb{A})) \cap G(\subseteq))$ where π_2 is the projection in the second component for the product space $Pr(\mathbb{A}) \times Pr(\mathbb{A})$. $(C \times Pr(\mathbb{A})) \cap G(\subseteq)$ is the intersection of closed sets, and thus closed. Since projections are continuous and the product space is compact, $\uparrow C$ is compact. Since $(Pr(\mathbb{A}), \mathscr{O}_{\mathbb{A}})$ is Hausdorff, $\uparrow C$ is thus closed. □

With these lemmas obtained we can show that the functors are well-defined.

Lemma 5.30. *The functors Pr and Com are well-defined.*

Proof. Given Lemmas 5.22 and 5.24 it just remains to show that, if $f : \mathbb{A} \to \mathbb{A}'$ is a homomorphism of Heyting algebras and $g : \mathscr{X} \to \mathscr{X}'$ an intuitionistic frame morphism then $f^{-1} : Pr(\mathbb{A}') \to Pr(\mathbb{A})$ is an intuitionistic frame morphism and $g^{-1} : Com(\mathscr{X}') \to Com(\mathscr{X})$ is a homomorphism of Heyting algebras.

We begin with f. By Lemma 5.27 we know f^{-1} maps prime filters to prime filters, and it is easy to see it satisfies the first condition for intuitionistic frame morphisms. For the second, suppose $f^{-1}(G) \subseteq F$. For any clopen set C containing F we therefore have $G \in (f^{-1})^{-1}(\downarrow C) = \downarrow (f^{-1})^{-1}(C)$ by Lemma 5.28. Hence $\uparrow \{G\} \cap (f^{-1})^{-1}(C) \neq \emptyset$ for any clopen C containing F. The family $\{(f^{-1})^{-1}(C) \mid C \text{ clopen and } F \in C\}$ has the FIP, hence so too does $\{\uparrow \{G\}\} \cup \{(f^{-1})^{-1}(C) \mid C \text{ clopen and } F \in C\}$. Since $\mathscr{O}_{\mathbb{A}}$ is Hausdorff, $\{G\}$ is closed, and by Lemma 5.29 so too is $\uparrow \{G\}$. Hence by compactness, the whole family has non-empty intersection. $\bigcap\{(f^{-1})^{-1}(C) \mid C \text{ clopen and } F \in C\} = (f^{-1})^{-1}(\bigcap\{C \mid F \in C\}) = (f^{-1})^{-1}(\{F\})$, so this entails there exists some F' such that $G \subseteq F'$ and $f^{-1}(F') = F$ as required.

For g, we just need to verify that g^{-1} respects the structure of the complex algebra. This is trivial in all cases but $\Rightarrow$. Suppose $x \in g^{-1}(A \Rightarrow_{\mathscr{X}'} B)$ and assume $y \succcurlyeq x$ with $g(y) \in A$. By monotonicity of g, $g(y) \succcurlyeq g(x)$, and so by our assumption $g(y) \in B$. Hence $x \in g^{-1}(A) \Rightarrow_{\mathscr{X}} g^{-1}(B)$. Suppose instead that $x \in g^{-1}(A) \Rightarrow_{\mathscr{X}} g^{-1}(B)$ and assume $y' \succcurlyeq' g(x)$ with $y' \in A$. By intuitionistic frame morphism condition ii) there exists y such that $y \succcurlyeq x$ and $g(y) = y'$. $g(y) = y' \in A$ so by assumption $y' \in B$ as required and $x \in g^{-1}(A \Rightarrow_{\mathscr{X}'} B)$. □

The existence of these functors has significant ramifications for intuitionistic logic. As we will investigate in detail in Chapter 7, the action on objects encodes the soundness and completeness of the Kripke semantics via the algebraic presentation of the proof system of intutionistic logic. Further, the action on morphisms allows us to examine many semantical properties algebraically (and vice versa).

Can we give an analogous representation theorem for intuitionistic frames? Unfortunately this is not possible: there exist intuitionistic frames $\mathscr{X}$ such that *no* intuitionistic morphism exists between $\mathscr{X}$ and $PrCom(\mathscr{X})$ (cf. Venema's comments [217, pg 352]). This means we cannot give a dual adjunction between these categories based on the representation theorem for Heyting algebras. However, by making the topological structure we've utilised explicit we *can* strengthen this relationship to a dual equivalence of categories. This is a particularly strong property: many metatheoretic properties of a logic can be shown to have an equivalent algebraic formulation; the dual equivalence then opens the door to the topological investigation of these properties. For example, Priestley [186] lists a number of algebraic properties of varieties of distributive lattices that have a dual formulation on

the topological side, while Sambin & Vaccaro [203] shows how to use the topological side of a duality to prove Sahlqvist's theorem on the correspondence between modal axioms and frame properties.

We begin by abstracting the ordered topological space $(Pr(\mathbb{A}), \mathscr{O}_{\mathbb{A}}, \supseteq)$ to a category of spaces. These spaces are named for Esakia, the mathematician who first discovered the duality [91], although Davey & Galati [72] note that Adams independently discovered the duality (reported as folklore in Priestley [186]).

Definition 5.31 (Esakia Space)**.** *An* Esakia Space *is a compact, partially ordered topological space $\mathscr{X} = (X, \mathscr{O}, \succcurlyeq)$ satisfying both the Priestley separation axiom and the property that for any clopen set C, the downwards closure $\downarrow C$ is also clopen.*

The clopen sets of an Esakia space being closed under downward closure is closely connected to Heyting implication, a fact that is explicitly drawn out in the proof that the prime filter space is an Esakia space. Removing this condition obtains Priesley spaces, the topological duals of distributive lattices.

Lemma 5.32. *For any Heyting algebra $\mathbb{A}$, the ordered topological space $Pr(\mathbb{A}) = (Pr(\mathbb{A}), \mathscr{O}_{\mathbb{A}}, \supseteq)$ is an Esakia space.*

Proof. By Lemma 5.26 we only need to show the clopen sets are closed under downward closure; however, in Lemma 5.28 we already saw that for any clopen C, $\downarrow C$ is the finite union of clopen sets $\overline{\theta_{\mathbb{A}}(a \to b)}$, and thus a clopen set. □

An *Esakia morphism* is a continuous intuitionistic morphism. Esakia spaces together with Esakia morphisms form a category Esa. Lemmas 5.27 and 5.30 show that for any Heyting homomorphism $f : \mathbb{A} \to \mathbb{A}'$, $f^{-1} : Pr(\mathbb{A}') \to Pr(\mathbb{A})$ is an Esakia morphism. Thus Pr is a functor from Heyting algebras to Esakia spaces.

In the other direction we need to construct a Heyting algebra from an Esakia space. We do something similar to the complex algebra construction: define $\mathscr{CL}_{\succcurlyeq}(\mathscr{X}) = \{C \in \mathscr{P}_{\succcurlyeq}(X) \mid \text{C clopen}\}$. We have already verified that upwards-closed sets are closed under the complex algebra operations; we can easily see that in addition the *clopen* upwards-closed sets of an Esakia space are closed under them. Straightforwardly, the upwards-closed clopen sets are closed under finite union and intersection, and $\emptyset$ and X are both clopen and upwards-closed. Finally, it is easily seen that $A \Rightarrow B = \overline{\downarrow A \cap \overline{B}}$. If A and B are clopen then $\downarrow A$ and $\overline{B}$ are clopen, hence so is their intersection, and so is the complement of that intersection.

We thus define the functor $Clop_{\succcurlyeq} : \text{Esa} \to \text{HeyAlg}$ by $Clop_{\succcurlyeq}(\mathscr{X}) = (\mathscr{CL}_{\succcurlyeq}(X), \cap, \cup, \Rightarrow, X, \emptyset)$ and $Clop_{\succcurlyeq}(g) = g^{-1}$. Continuity of g ensures g^{-1} maps clopen sets to clopen sets, and the fact it is order-preserving means it maps upwards-closed clopen sets to upwards-closed clopen sets. Finally Lemma 5.30 ensures g^{-1}

is a homomorphism of Heyting algebras. We now give the counterpart to θ on the Esakia space side: $\eta_{\mathscr{X}}(x) = \{C \in Clop_{\succcurlyeq}(\mathscr{X}) \mid x \in C\}$. This yields a prime filter: $x \in C$ and $C \subseteq C'$ implies $x \in C'$; $x \in C$ and $x \in C'$ implies $x \in C \cap C'$; $x \notin \emptyset$; $x \in C \cup C'$ implies $x \in C$ or $x \in C'$.

Theorem 5.33 (Esakia Duality)**.** *$\theta : Id_{\mathrm{HA}} \to Clop_{\succcurlyeq}Pr$ and $\eta : Id_{\mathrm{Esa}} \to PrClop^{Int}_{\succcurlyeq}$ form a dual equivalence of categories between* HeyAlg *and* Esa*.*

Proof. We show that the components of θ and η are isomorphisms as naturality is trivial. For θ we begin with injectivity. If $a \neq b$ then either $a \not\succcurlyeq b$ or $b \not\succcurlyeq a$: *wolog* assume $a \not\succcurlyeq b$. There exists a prime filter F containing a and not b by Theorem 5.8. Hence $\theta_{\mathbb{A}}(a) \neq \theta_{\mathbb{A}}(b)$.

For surjectivity, we know that for each $a \in \mathbb{A}$, $\theta(a)$ is an upwards-closed clopen set. We show for any upwards-closed clopen C, there exists $a \in \mathbb{A}$ such that $C = \theta_{\mathbb{A}}(a)$. Since C is upwards-closed, for any $F \in C, G \in \overline{C}$, $F \not\subseteq G$. There thus exists $a \in F \cap \overline{G}$. Then $\theta_{\mathbb{A}}(a)$ and $\overline{\theta_{\mathbb{A}}(a)}$ separate F and G. We thus have $\overline{C} = \bigcup_a \overline{\theta_{\mathbb{A}}(a)}$, where a ranges over all such a. By compactness there is a finite subcover $\overline{C} = \bigcup_{a_i}^{m} \overline{\theta_{\mathbb{A}}(a_i)} = \overline{\theta_{\mathbb{A}}(a_0 \wedge \cdots \wedge a_m)}$. Hence $C = \theta_{\mathbb{A}}(a_0 \wedge \cdots \wedge a_m)$.

We now attend to η. η_X is clearly order-preserving, and the Priestley separation axiom guarantees injectivity as well as $\eta_{\mathscr{X}}(x) \supseteq \eta_X(y)$ implies $x \succcurlyeq y$. Continuity follows from the fact that $\eta_{\mathscr{X}}^{-1}(\theta_{Clop_{\succcurlyeq}(\mathscr{X})}(C) = C$. As an injective and surjective order isomorphism is trivially a intuitionistic frame isomorphism, it only remains to show surjectivity. Suppose for contradiction $F \in \overline{\eta_{\mathscr{X}}(X)}$. We note that the image of $\mathscr{X}$ under $\eta_{\mathscr{X}}$ is closed because $\mathscr{X}$ is compact, $\eta_{\mathscr{X}}$ is continuous and $PrClop_{\succcurlyeq}(\mathscr{X})$ is Hausdorff. Hence the complement $\overline{\eta_{\mathscr{X}}(X)}$ is open and there exists a base element $\theta_{Clop_{\succcurlyeq}(\mathscr{X})}(U) \cap \overline{\theta_{Clop_{\succcurlyeq}(\mathscr{X})}(V)} \subseteq \overline{\eta_{\mathscr{X}}(X)}$ containing F. We thus have that the preimage of this base element with respect to $\eta_{\mathscr{X}}$ is $\emptyset$. $\eta_{\mathscr{X}}^{-1}(\theta_{Clop_{\succcurlyeq}(\mathscr{X})}(C)) = C$ means $\emptyset = U \cap \overline{V}$ so $U \subseteq V$. But then $\theta_{Clop_{\succcurlyeq}(\mathscr{X})}(U) \cap \overline{\theta_{Clop_{\succcurlyeq}(\mathscr{X})}(V)} = \emptyset$, a contradiction. □

5.4 Stone Duality

Conventionally (and chronologically) the Stone duality theorem [210, 132] relating Boolean algebras and compact Hausdorff spaces is presented prior to that for Heyting algebras and Esakia spaces. In keeping with our presentation of the bunched logics with classical additives as special cases of intuitionistic bunched logics we instead derive Stone duality as the special case of Esakia duality when $\succcurlyeq$ is $=$. The idea here is that by presenting results for the ordered frames interpreting bunched logics with intuitionistic additives we should automatically obtain the analogous results for bunched logics with classical additives when the order is equality.

Definition 5.34 (Boolean Algebra)**.** *A* Boolean algebra *is a Heyting algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot)$ *satisfying, where* $\neg a := a \to \bot$, $a \vee \neg a = \top$.

This is, of course, not how Boolean algebras are standardly presented. If $\neg$ is instead given as a primitive complementation operation then $a \to b := \neg a \vee b$ defines a Heyting implication and $a \to \bot = \neg a \vee \bot = \neg a$, meaning it satisfies our definition.

The structure of the prime filters on a Boolean algebra is greatly simplified by the property $a \vee \neg a = \top$. A *maximal filter* is a proper filter F such that for any other proper filter G, $F \subseteq G$ implies $F = G$. An *ultrafilter* is a proper filter F such that for all $a \in \mathbb{A}$, $a \in F$ or $\neg a \in F$. The following proposition is an easy consequence of the fact that $a \vee \neg a = \top$.

Proposition 5.35. *Let F be a filter on a Boolean algebra $\mathbb{A}$. F is a prime filter iff F is a maximal filter iff F is an ultrafilter.* □

Thus applying the prime filter frame construction for a Boolean algebra $\mathbb{A}$ results in a an intuitionistic frame in which the order is simply equality. The complex algebra of this frame is thus simply the power set of prime filters on $\mathbb{A}$. We obtain the Stone representation theorem immediately.

Theorem 5.36 (Stone Representation Theorem for Boolean algebras [210])**.** *Every Boolean algebra is isomorphic to a subalgebra of a power set algebra. Specifically, given a Boolean algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com(Pr(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr(\mathbb{A}) \mid a \in F\}$ is an embedding.* □

Com and *Pr* lift to functors in the same way as the intuitionistic case. To strengthen this to a dual equivalence of categories, topology enters the picture once again. Now, the topology generated by $\theta_{\mathbb{A}}$ is greatly simplified. Now, as $\overline{\theta_{\mathbb{A}}(a)} = \theta_{\mathbb{A}}(\neg a)$, we have that the base for $\mathscr{O}_{\mathbb{A}}$ is straightforwardly given by $\mathscr{B} = \{\theta_{\mathbb{A}}(a) \mid a \in \mathbb{A}\}$. This is a base of clopen sets, making $\mathscr{O}_{\mathbb{A}}$ *zero-dimensional*. We have that $\theta_{\mathbb{A}}(a) \cap \theta_{\mathbb{A}}(\neg a) = \emptyset$ and $\theta_{\mathbb{A}}(a) \cup \theta_{\mathbb{A}}(\neg a) = Pr(\mathbb{A})$ since prime filters are ultrafilters. Hence for any distinct F and G we can find $a \in F \cap \overline{G}$, meaning $F \in \theta_{\mathbb{A}}(a)$, $G \in \theta_{\mathbb{A}}(\neg a)$ and $\theta_{\mathbb{A}}(a) \cup \theta_{\mathbb{A}}(\neg a) = Pr(\mathbb{A})$, making the topology totally disconnected.

Definition 5.37 (Stone space)**.** *A* Stone space *is a compact, totally disconnected topological space $\mathscr{X} = (X, \mathscr{O})$. Equivalently, a Stone space is a compact, Hausdorff, zero dimensional topological space.*

Lemma 5.38. *For any Boolean algebra $\mathbb{A}$, $(Pr(\mathbb{A}), \mathscr{O}_{\mathbb{A}})$ is a Stone space.*

Proof. Immediate from Lemma 5.26 and the preceeding discussion. □

Lemma 5.39. *$(X,\mathscr{O})$ is a Stone space iff $(X,\mathscr{O},=)$ is an Esakia space.*

Proof. Assume $(X,\mathscr{O})$ is a Stone space. By definition this is compact. Further, for any clopen set C, trivially $\downarrow C = C$ and so is clopen. For the Priestley separation axiom, let $x \neq y$. Then we can use the Hausdorff property to find an open set O with $x \in O$ and $y \notin O$. Since the space is zero dimensional, there exists a clopen base element B such that $x \in B \subseteq O$. So $(X,\mathscr{O},=)$ is an Esakia space.

Now assume $(X,\mathscr{O},=)$ is an Esakia space. By assumption this is compact and, by Lemma 5.26, Hausdorff. To see that it is zero dimensional, let O be an open set. By Esakia duality, $O = \eta_{\mathscr{X}}^{-1}(O')$ for O' open in $Pr_{Clop_{\geqslant}(\mathscr{X})}Clop_{\geqslant}(\mathscr{X})$. $O' = \bigcup_i \theta_{Clop_{\geqslant}(\mathscr{X})}(A_i) \cup \overline{\theta_{Clop_{\geqslant}(\mathscr{X})}(B_i)}$ for some upwards-closed clopen sets A_i and B_i of $\mathscr{X}$. Since the order is $=$, these are simply clopen sets, and $\overline{\theta_{Clop_{\geqslant}(\mathscr{X})}(B_i)} = \theta_{Clop_{\geqslant}(\mathscr{X})}(\overline{B_i})$. We can thus rewrite O' as $O' = \bigcup_j \theta_{Clop_{\geqslant}(\mathscr{X})}(A_j)$. Thus $O = \bigcup_j \eta_{\mathscr{X}}^{-1}(\theta_{Clop_{\geqslant}(\mathscr{X})}(A_j)) = \bigcup_j A_j$, and so $\mathscr{O}$ has a clopen basis. □

For morphisms between Esakia spaces carrying the trivial order the intuitionistic frame conditions also become trivial and are simply continuous maps. We thus have a category Stone of Stone spaces and continuous maps, and by Esakia duality, it is dually equivalent to the category BA.

Theorem 5.40 (Stone Duality)**.** *$\theta : Id_{BA} \to ClopPr$ and $\eta : Id_{Stone} \to PrClop$ form a dual equivalence of categories between* BA *and* Stone*.* □

Chapter 6

Dualities for Propositional Bunched Logics

In this chapter we extend Esakia and Stone duality to the algebras and Kripke structures interpreting the bunched logics introduced in Part I. Duality theorems of this sort have a long history in non-classical logic, beginning with the landmark work of Jónsson & Tarski [133] extending Stone duality to Boolean algebras with operators, prefiguring Kripke's modal logic semantics over a decade before it was proposed. This was extended to distributive lattices with operators by Goldblatt [108], who made explicit the connection with modal logic and was able to use the duality theoretic framework to prove the Goldblatt Thomason theorem that characterises modal definability for logics interpreted on first-order definable frames.

One of the precursors to the present work is Urquhart's [215] duality theory for the structures interpreting relevant logics. As one might expect, given the similarity between bunched logics and relevant logics, similar arguments come into play for us. Urquhart's duality was generalised to residuated algebras called gaggles by Bímbo & Dunn [25], and it is their work that can be considered the foundation of ours: indeed, duality for the algebras associated to LGL can be found as a specific case of their result for Boolean gaggles. Our duality theoretic framework can helpfully be seen as a combination of Bímbo & Dunn's duality and modal logic-style correspondence theory: the duality theory of gaggles is used to give a duality theory for the most basic bunched logic algebras, and each extension of that logic is obtained by adding appropriate operations and showing that frame properties and the axioms defining the logics induce each other.

6.1 Layered Graph Logics

We begin our analysis with the weakest systems, LGL and ILGL. Each of the logics we consider can be obtained by extending the basic structures associated with

these logics and so we are able to systematically extend the theory in each case by accounting for just the extensions to the structure. First, we consider lattice-based algebras suitable for interpreting (I)LGL.

Definition 6.1 ((I)LGL Algebra)**.**

1. *An* ILGL algebra *is an algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \mathrel{*\!\!-})$ *such that* $(A, \wedge, \vee, \to, \top, \bot)$ *is a Heyting algebra and* $*, \mathrel{-\!\!*}, \mathrel{*\!\!-}$ *are binary operations on A satisfying, for all* $a, b, c \in A$*,*

$$a * b \leq c \text{ iff } a \leq b \mathrel{-\!\!*} c \text{ iff } b \leq a \mathrel{*\!\!-} c.$$

2. *A* LGL algebra *is an ILGL algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \mathrel{*\!\!-})$ *for which* $(A, \wedge, \vee, \to, \top, \bot)$ *is a Boolean algebra.*

Residuation of $*$, $\mathrel{-\!\!*}$ and $\mathrel{*\!\!-}$ with respect to the underlying lattice order entails a number of useful properties that are utilised in what follows.

Proposition 6.2 (cf. [131])**.** *Let* $\mathbb{A}$ *be an (I)LGL algebra. Then, for all* $a, b, a', b' \in A$ *and* $X, Y \subseteq A$*, we have the following:*

1. *If* $a \leq a'$ *and* $b \leq b'$ *then* $a * b \leq a' * b'$*;*
2. *If* $\bigvee X$ *and* $\bigvee Y$ *exist then* $\bigvee_{x \in X, y \in Y} x * y$ *exists and* $(\bigvee X) * (\bigvee Y) = \bigvee_{x \in X, y \in Y} x * y$*;*
3. *If* $a = \bot$ *or* $b = \bot$ *then* $a * b = \bot$*;*
4. *If* $\bigvee X$ *exists then for any* $z \in A$*,* $\bigwedge_{x \in X} (x \mathrel{-\!\!*} z)$ *and* $\bigwedge_{x \in X} (x \mathrel{*\!\!-} z)$ *exist with*

$$\bigwedge_{x \in X} (x \mathrel{-\!\!*} z) = (\bigvee X) \mathrel{-\!\!*} z \text{ and } \bigwedge_{x \in X} (x \mathrel{*\!\!-} z) = (\bigvee X) \mathrel{*\!\!-} z;$$

5. *If* $\bigwedge X$ *exists then for any* $z \in A$*,* $\bigwedge_{x \in X} (z \mathrel{-\!\!*} x)$ *and* $\bigwedge_{x \in X} (z \mathrel{*\!\!-} x)$ *exist with*

$$\bigwedge_{x \in X} (z \mathrel{-\!\!*} x) = z \mathrel{-\!\!*} (\bigwedge X) \text{ and } \bigwedge_{x \in X} (z \mathrel{*\!\!-} x) = z \mathrel{*\!\!-} (\bigwedge X); \text{ and}$$

6. $a \mathrel{-\!\!*} \top = a \mathrel{*\!\!-} \top = \bot \mathrel{-\!\!*} a = \bot \mathrel{*\!\!-} a = \top$. □

Interpretations of (I)LGL on an (I)LGL algebra work as follows: let $\mathscr{V} : \text{Prop} \to A$ be an assignment of elements of the algebra to propositional variables;

this is uniquely extended to an interpretation $[\![-]\!]$ of every (I)LGL formula by induction, with $[\![\mathrm{p}]\!] = \mathscr{V}(\mathrm{p}), [\![\top]\!] = \top$ and $[\![\bot]\!] = \bot$ as base cases:

$$\begin{array}{llllll} [\![\phi \wedge \psi]\!] & = [\![\phi]\!] \wedge [\![\psi]\!] & [\![\phi \vee \psi]\!] & = [\![\phi]\!] \vee [\![\psi]\!] & [\![\phi \to \psi]\!] & = [\![\phi]\!] \to [\![\psi]\!] \\ [\![\phi * \psi]\!] & = [\![\phi]\!] * [\![\psi]\!] & [\![\phi \mathrel{-\!\!*} \psi]\!] & = [\![\phi]\!] \mathrel{-\!\!*} [\![\psi]\!] & [\![\phi *\!\!- \psi]\!] & = [\![\phi]\!] *\!\!- [\![\psi]\!]. \end{array}$$

We prove that this is sound now, returning to completeness in Chapter 7.

Theorem 6.3 (Algebraic Soundness of (I)LGL)**.** *If $\varphi \vdash \psi$ is provable in the Hilbert system for (I)LGL, then for all interpretations $[\![-]\!]$ on (I)LGL algebras, $[\![\varphi]\!] \leq [\![\psi]\!]$.*

Proof. The proof proceeds by an inductive argument on (I)LGL proofs. As with the soundness theorem for frames, we demonstrate with the case for the rule

$$15. \quad \frac{\xi \vdash \varphi *\!\!- \psi \quad \eta \vdash \varphi}{\eta * \xi \vdash \psi}$$

Suppose $[\![\xi]\!] \leq [\![\varphi *\!\!- \psi]\!]$ and $[\![\eta]\!] \leq [\![\varphi]\!]$. By residuation $[\![\varphi]\!] * [\![\xi]\!] \leq [\![\psi]\!]$. Since $*$ is order preserving we have $[\![\eta * \xi]\!] = [\![\eta]\!] * [\![\xi]\!] \leq [\![\varphi]\!] * [\![\xi]\!] \leq [\![\psi]\!]$ as required. □

The structure of what follows mirrors that of Chapter 5, and will be replicated across each bunched logic. First, we equip the frames associated with the logic with an appropriate notion of morphism to obtain a category. Next we set up the dual functors *Com* and *Pr* for transforming algebras into frames (and vice versa) and homomorphisms into frame morphisms (and vice versa), and prove a representation theorem falls out of this relationship. Finally we add appropriate topological structure to the frames to obtain a category of topological spaces that is dually equivalent to the category of algebras.

We begin with ILGL morphisms. These necessarily extend the definition of intuitionistic morphism with back and forth conditions pertaining to $\circ$.

Definition 6.4 (ILGL Morphism)**.** *Given ILGL frames $\mathscr{X}$ and $\mathscr{X}'$, an* ILGL morphism *is a intuitionistic morphism $g : (X, \succcurlyeq) \to (X', \succcurlyeq')$ satisfying*

1. *$x \in y \circ z$ implies $g(x) \in g(y) \circ' g(z)$,*
2. *$g(x) \succcurlyeq' w'$ and $w' \in y' \circ' z'$ implies there exists $w, y, z \in X$ s.t. $x \succcurlyeq w$, $w \in y \circ z$, $g(y) \succcurlyeq' y'$ and $g(z) \succcurlyeq z'$,*
3. *$w' \succcurlyeq' g(x)$ and $z' \in w' \circ' y'$ implies there exists $w, y, z \in X$ s.t. $w \succcurlyeq x$, $z \in w \circ y$, $g(y) \succcurlyeq' y'$ and $z' \succcurlyeq' g(z)$, and*
4. *$w' \succcurlyeq g(x)$ and $z' \in y' \circ' w'$ implies there exists $w, y, z \in X$ s.t. $w \succcurlyeq x$, $z \in y \circ w$, $g(y) \succcurlyeq' y'$ and $z' \succcurlyeq' g(z)$.* □

LGL morphisms are the special case of ILGL morphisms where $\succcurlyeq$ is $=$, which collapses the above definition to this simpler one.

Definition 6.5 (LGL Morphism)**.** *Given LGL frames $\mathscr{X}$ and $\mathscr{X}'$, a* LGL morphism *is a map $g : \mathscr{X} \to \mathscr{X}'$ satisfying*

1. *$x \in y \circ z$ implies $g(x) \in g(y) \circ' g(z)$,*
2. *$g(x) \in y' \circ' z'$ implies there exists $y,z \in X$ s.t. $x \in y \circ z, g(y) = y'$ and $g(z) = z'$,*
3. *$z' \in g(x) \circ' y'$ implies there exists $y,z \in X$ s.t. $z \in x \circ y, g(y) = y'$ and $g(z) = z'$, and*
4. *$z' \in y' \circ' g(x)$ implies there exists $y,z \in X$ s.t. $z \in y \circ x, g(y) = y'$ and $g(z) = z'$.* □

A variant of this definition is used in the context of BBI by Brotherston & Villard [44] to demonstrate that the logic is not sufficiently expressive to axiomatise a number of properties common to models of separation logic. Urquhart [215] defines similar maps in order to define dualities for relevant logic and Bímbo & Dunn [25] generalise Urquhart's definition further to give morphisms that respect residuals on the dual algebras of frames for gaggles. Bunched logics can be seen to be extensions of gaggles with extra operators and/or axioms.

Proposition 6.6. *Given (I)LGL morphisms $f : \mathscr{X} \to \mathscr{X}'$ and $g : \mathscr{X}' \to \mathscr{X}''$, the composition $gf : \mathscr{X} \to \mathscr{X}''$ is an ILGL morphism.*

Proof. We verify property 4. in the definition of ILGL morphism: the others are similar. Suppose $gf(x) \succcurlyeq'' w''$ with $w'' \in y'' \circ z''$. Since g is an ILGL morphism we obtain $w',y',z' \in X'$ such that $f(x) \succcurlyeq' w'$, $w' \in y' \circ z'$, $g(y') \succcurlyeq'' y''$ and $g(z') \succcurlyeq'' z''$. Applying the property for f, we obtain $w,y,z \in X$ such that $x \succcurlyeq w$, $w \in y \circ z$, $f(y) \succcurlyeq' y'$ and $g(z) \succcurlyeq' z'$. Since g is order preserving, $gf(y) \succcurlyeq'' g(y') \succcurlyeq y''$ and $gf(z) \succcurlyeq'' g(z') \succcurlyeq'' z''$ and so the property holds. □

We thus obtain the categories (I)LGL of (I)LGL frames and (I)LGL morphisms.

Definition 6.7 (ILGL Complex Algebra)**.** *Given an (I)LGL frame $\mathscr{X}$, the* (I)LGL complex algebra *of $\mathscr{X}$ is given by $Com^{(I)LGL}(\mathscr{X}) = (\mathscr{P}_{\preccurlyeq}(X), \cap, \cup, \Rightarrow_{\mathscr{X}}, X, \emptyset, \bullet_{\mathscr{X}}, \mathrel{—\!\!\bullet}_{\mathscr{X}}, \mathrel{\bullet\!\!—}_{\mathscr{X}})$ where*

$$\begin{aligned} A \bullet_{\mathscr{X}} B &= \{x \mid \text{there exists } w,y,z \text{ s.t } x \succcurlyeq w, w \in y \circ z, y \in A \text{ and } z \in B\} \\ A \mathrel{—\!\!\bullet}_{\mathscr{X}} B &= \{x \mid \text{for all } w,y,z, \text{ if } w \succcurlyeq x, z \in w \circ y \text{ and } y \in A \text{ then } z \in B\} \\ A \mathrel{\bullet\!\!—}_{\mathscr{X}} B &= \{x \mid \text{for all } w,y,z, \text{ if } w \succcurlyeq x, z \in y \circ w \text{ and } y \in A \text{ then } z \in B\}. \end{aligned}$$

Lemma 6.8. *For an (I)LGL frame $\mathscr{X}$, the complex algebra $Com^{(I)LGL}(\mathscr{X})$ is a an ILGL algebra.*

Proof. The remaining verification is the residuation property of $\bullet_{\mathscr{X}}, \multimap_{\mathscr{X}}$ and $\bullet\!\!-_{\mathscr{X}}$. We show one of the bi-implications. Suppose $A \bullet_{\mathscr{X}} B \subseteq C$ and let $x \in A$ with w, y, z are such that $w \succcurlyeq x$, $z \in w \circ y$ and $y \in A$. Since A is upwards-closed, by assumption, $z \in A \bullet_{\mathscr{X}} B \subseteq C$. Hence $x \in B \multimap_{\mathscr{X}} C$. Assuming $A \subseteq B \multimap_{\mathscr{X}} C$ and $x \in A \bullet_{\mathscr{X}} B$, we obtain w, y, z such that $x \succcurlyeq w$, $w \in y \circ z$, $y \in A$ and $z \in B$. By assumption, it follows that $y \in B \multimap_{\mathscr{X}} C$, so $w \in C$. By upwards-closure of C, $x \in C$ as required. □

Definition 6.9 (Prime Filter (I)LGL Frame)**.** *Given an (I)LGL algebra $\mathbb{A}$, the* prime filter frame *of $\mathbb{A}$ is given by $Pr^{(I)LGL}(\mathbb{A}) = (Pr(A), \subseteq, \circ_{\mathbb{A}})$ where*

$$F \circ_{\mathbb{A}} F' = \{F'' \in Pr(\mathbb{A}) \mid \forall a \in F, \forall b \in F' : a * b \in F''\}.$$

Of course, because of the structure of prime filters on Boolean algebras, the order $\subseteq$ collapses to $=$ for a prime filter LGL frame, as we would expect.

Lemma 6.10. *The prime filter frame $Pr^{(I)LGL}(\mathbb{A})$ of an (I)LGL algebra $\mathbb{A}$ is an (I)LGL frame.* □

We can give a representation theorem for (I)LGL algebras using these constructions. For ILGL algebras this extends the representation theorem for Heyting algebras, whereas for LGL algebras this extends Stone's theorem. These results are closely related to various representation theorems for algebras with operators (e.g., [133], [108]). The key difference is the use of a single operation $\circ$ for the operator $*$ *and* its non-operator adjoints $-\!\!*$ and $*\!\!-$. The derived structure required to take care of these adjoints was not investigated in the frameworks of Jonsson-Tarski or Goldblatt but has been in the context of gaggle theory [25, 87]. There the result for LGL algebras can be found as a particular case of that for Boolean gaggles ([25], Theorem 1.4.16).

Theorem 6.11 (Representation Theorem for (I)LGL Algebras)**.** *Every (I)LGL algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given an (I)LGL algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{(I)LGL}(Pr^{(I)LGL}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr^{(I)LGL}(\mathbb{A}) \mid a \in F\}$ is an embedding.*

Proof. We prove the theorem for ILGL algebras; the case for LGL algebras can be obtained by substituting $\succcurlyeq$ for $=$ throughout. That $\theta_{\mathbb{A}}$ is an embedding and a homomorphism on the Heyting algebra operations is simply the representation

theorem for Heyting algebras. It thus remains to show that $\theta_{\mathbb{A}}$ respects $*$, $\mathrel{-\!\!*}$ and $\mathrel{*\!\!-}$. We focus on the case for $\mathrel{-\!\!*}$; the others are similar.

We first note the corner cases: for all $a,b \in A$ we trivially have that $\theta_{\mathbb{A}}(a \mathrel{-\!\!*} \top) = \theta_{\mathbb{A}}(a) \multimap_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(\top)$ and $\theta_{\mathbb{A}}(\bot \mathrel{-\!\!*} b) = \theta_{\mathbb{A}}(\bot) \multimap_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$ by Proposition 6.2 property 6. Hence it is sufficient to consider $a \mathrel{-\!\!*} b$ where $a \neq \bot$ and $b \neq \top$.For the inclusion $\theta_{\mathbb{A}}(a \mathrel{-\!\!*} b) \subseteq \theta_{\mathbb{A}}(a) \multimap_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$, assume $a \mathrel{-\!\!*} b \in F$ with F_0, F_1, F_2 such that $F \subseteq F_0$, $F_2 \in F_0 \circ_{\mathbb{A}} F_1$ and $a \in F_1$. Then $(a \mathrel{-\!\!*} b) * a \in F_2$ and so $b \in F_2$, since residuation entails $(a \mathrel{-\!\!*} b) * a \leq b$ and F_2 is upwards closed. Hence $F \in \theta_{\mathbb{A}}(a) \multimap_{Pr(\mathbb{A})} \theta_{\mathbb{A}}(b)$.

For the reverse inclusion, consider F such that $a \mathrel{-\!\!*} b \notin F$. We show, for proper filter G and proper ideal I, that (abusing notation for $\circ_{\mathbb{A}}$)

$$P(G,I) = \begin{cases} 1 & \text{if } \bar{I} \in F \circ_{\mathbb{A}} G, a \in G \text{ and } b \in I \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. We concentrate on the non-trivial verifications: suppose $P(G \cap G', I) = 1$. Clearly, $a \in G, G'$ so suppose for contradiction that there exists $c, c' \in F$, $d \in G$ and $d' \in G'$ such that $c * d, c' * d' \in I$. We have that $c'' := c \wedge c' \in F$ and $c'' * d, c'' * d' \in I$. This entails $c'' * (d \vee d') = (c'' * d) \vee (c'' * d') \in I$. Since $d \vee d' \in G \cap G'$ we have $c'' * (d \vee d') \notin I$ by assumption: a contradiction. Hence either $\bar{I} \in F \circ_{\mathbb{A}} G$ or $\bar{I} \in F \circ_{\mathbb{A}} G'$. If $P(G, I \cap I') = 1$ we clearly have $b \in I, I'$, so suppose for contradiction there exist $c, c' \in F$, $d, d' \in G$ such that $c * d \in I$ and $c' * d' \in I'$. $c'' = c \wedge c' \in F$ and $d'' = d \wedge d' \in G$ so we have $c'' * d'' \in \bar{I} \cup \overline{I'}$. This means $c'' * d'' \in \bar{I}$ or $c'' * d'' \in \overline{I'}$, but $c * d, c' * d' \geq c'' * d'' \in I \cap I'$, a contradiction. Thus either $\bar{I} \in F \circ_{\mathbb{A}} G$ or $\overline{I'} \in F \circ_{\mathbb{A}} G$.

Hence P is a prime predicate. By our assumption on a and b, $[a)$ and $(b]$ are a proper filter and a proper ideal respectively, and $P([a), (b]) = 1$: if $x \in F$ and $y \geq a$ then $x * y \not\leq b$, otherwise by residuation and monotonicity of $*$ we would have $x \leq a \mathrel{-\!\!*} b \in F$, a contradiction. Hence by Lemma 5.7 there exist prime G, and I with $P(F, I) = 1$. Letting $G' = \bar{I}$, we have the prime filters we require. □

We can also show that the assignment $Pr^{\text{(I)LGL}}(f) = f^{-1}$ and $Com^{\text{(I)LGL}}(g) = g^{-1}$ makes $Pr^{\text{(I)LGL}}$ and $Com^{\text{(I)LGL}}$ functorial.

Lemma 6.12. *The functors $Pr^{(I)LGL}$ and $Com^{(I)LGL}$ are well defined.*

Proof. We extend Lemma 5.30 to the additional properties of (I)LGL algebra homomorphisms and (I)LGL morphisms. For $Com^{\text{(I)LGL}}$ we need to show g^{-1} respects $*$, $\mathrel{-\!\!*}$ and $\mathrel{*\!\!-}$. We handle the case for $\mathrel{*\!\!-}$. Assume $x \in g^{-1}(A \bullet\!\!- _{\mathscr{Z}'} B)$ and suppose

w,y,z are such that $w \succcurlyeq x$, $z \in y \circ w$ and $g(y) \in A$. By order preservation and property 1. of ILGL morphism, $g(w) \succcurlyeq' g(x)$ and $g(z) \in g(y) \circ' g(w)$. By our assumption this entails $g(z) \in B$ as required. Now assume $x \in g^{-1}(A) \bullet\!\!- _{\mathscr{X}} g^{-1}(B)$ and suppose there exist w',y',z' with $w' \succcurlyeq g(x)$, $z' \in y' \circ' w'$ and $y' \in A$. By ILGL morphism property 4., there exist w,y,z such that $w \succcurlyeq x, z \in y \circ w, g(y) \succcurlyeq' y'$ and $z' \succcurlyeq' g(z)$. Upwards closure of A implies $g(y) \in A$, so $y \in g^{-1}(A)$. By assumption this entails $z \in g^{-1}(B)$ so $g(z) \in B$. By upwards closure of B, $z' \in B$, as required.

For $Pr^{\text{(I)LGL}}$ we need to show f^{-1} satisfies the (I)LGL morphism properties 1. to 4. We give the characteristic case of property 3. Suppose $F_{w'} \supseteq f^{-1}(F_x)$ and $F_{z'} \in F_{w'} \circ_{\mathbb{A}'} F_{y'}$. We show that the following map on proper filter/ideal pairs is a prime predicate

$$P(F,I) = \begin{cases} 1 & \text{if } \bar{I} \in F_x \circ_{\mathbb{A}} F, f^{-1}(F) \supseteq F_{y'} \text{ and } F_{z'} \supseteq f^{-1}(\bar{I}) \\ 0 & \text{otherwise} \end{cases}$$

First, assume for all $\alpha < \lambda$ $P(F_\alpha, I_\alpha) = 1$ for a $\subseteq$-chain $(F_\alpha, I_\alpha)_{\alpha<\lambda}$. Suppose for contradiction $a \in F_x$ and $b \in \bigcup_\alpha F_\alpha$ with $a * b \in \bigcup_\alpha I_\alpha$. Then for some β, $b \in F_\beta$ and for some β', $a * b \in I_{\beta'}$. There thus exists $\gamma \geq \beta, \beta'$ such that $b \in F_\gamma$ and $a * b \in I_\gamma$, but then $P(F_\gamma, I_\gamma) = 0$, a contradiction. If for all α, $f^{-1}(F_\alpha) \supseteq F_{y'}$ then clearly $f^{-1}(\bigcup_\alpha F_\alpha) \supseteq F_{y'}$. Similarly, if for all α, $F_{z'} \supseteq f^{-1}(\overline{I_\alpha})$, for any $a \in f^{-1}(\overline{\bigcup_\alpha I_\alpha})$ we have $f(a) \in \overline{I_\alpha}$ for all α, so $a \in F_{z'}$.

Now consider $P(F \cap F', I) = 1$. Clearly $f^{-1}(F), f^{-1}(F') \supseteq F_{y'}$. Suppose for contradiction that there exists $a, a' \in F_x$, $b \in F$ and $b' \in F'$ such that $a * b, a' * b' \in I$. We have $a'' = a \wedge a' \in F_x$ and by downwards closure and montonicity of $*$, $a'' * b, a' * b' \in I$. Hence $a'' * (b \vee b') = (a'' * b) \vee (a'' * b') \in I$. However $b \vee b' \in F \cap F'$, so $a'' * (b \vee b') \notin I$, a contradiction. Similarly, if $P(F, I \cap I') = 1$, we have $F_{z'} \supseteq f^{-1}(\bar{I} \cup \overline{I'})$, so $F_{z'} \supseteq f^{-1}(\bar{I}), f^{-1}(\overline{I'})$. Assume for contradiction that there exists $a, a' \in F_x$ and $b, b' \in F$ such that $a * b \in I$ and $a' * b' \in I'$. Then $a'' = a \wedge a' \in F_x$ and by downwards closure and monotonicity of $*$, $a'' * b \in I$ and $a'' * b' \in I'$. We have $b'' = b \wedge b' \in F$, so $a'' * b'' \in \overline{I \cap I'}$. However $a'' * b'' \leq a'' * b, a'' * b'$ so $a'' * b'' \in I \cap I'$, a contradiction. P is a prime predicate.

We show that the filter $F = [f[F_{y'}]]$ and ideal $I = (f[\overline{F_{z'}}])$ are proper, with $P(F,I) = 1$. First note that an equivalent characterisation of $\circ_{\mathbb{A}'}$ is

$$F \circ_{\mathbb{A}'} F' = \{F'' \mid \forall a,b : a \in F' \text{ and } b \notin F'' \text{ implies } a \twoheadrightarrow b \notin F\}.$$

Suppose $\bot \in F$. Then there exists $a \in F_{y'}$ such that $f(a) = \bot$. Let $b \notin F_{z'}$ be arbitrary. Then $a \twoheadrightarrow b \notin F_{w'}$. However $f(a \twoheadrightarrow b) = f(a) \twoheadrightarrow f(b) = \bot \twoheadrightarrow b = \top \in F_x$, so

$a \twoheadrightarrow b \in f^{-1}(F_x) \subseteq F_{w'}$, a contradiction. The case for I is similar. That $f^{-1}(F) \supseteq F_{y'}$ follows by the definition of F. For the other inclusion, assume $a \in f^{-1}(\bar{I})$ and $a \notin F_{z'}$. Then $f(a) \in f[\overline{F_{z'}}] \subseteq I$ so $a \notin f^{-1}(\bar{I})$, a contradiction. There hence exist prime F and I with $P(F,I) = 1$ by the Prime Extension Lemma. Setting $F_w = F_x$, $F_y = F$ and $F_z = \bar{I}$ gives the prime filters required for property 3. to hold.

Finally, for LGL morphisms, the conditions hold because maximality of prime filters makes all inclusions of prime filters into identities. □

To obtain a dual equivalence of categories we introduce topology to (I)LGL frames. Throughout this chapter we will define topological spaces corresponding to bunched logic algebras by instantiating structures which are simultaneously Stone/Esakia spaces and bunched logic frames, with the additional operations that correspond to the multiplicative structure of the logic satisfying certain coherence conditions with the underlying topology.

Definition 6.13 (ILGL Space)**.** *An* ILGL space *is a structure* $\mathscr{X} = (X, \mathscr{O}, \succcurlyeq, \circ)$ *such that:*

1. $(X, \mathscr{O}, \succcurlyeq)$ *is an Esakia space;*
2. $(X, \succcurlyeq, \circ)$ *is an ILGL frame;*
3. *The upwards-closed clopen sets of* $(X, \mathscr{O}, \preccurlyeq)$ *are closed under* $\bullet_{\mathscr{X}}, \multimap_{\mathscr{X}}, \bullet\!\!-_{\mathscr{X}}$*;*
4. *If* $x \notin y \circ z$ *then there exist upwards-closed clopen sets* C_1, C_2 *such that* $y \in C_1, z \in C_2$ *and* $x \notin C_1 \bullet_{\mathscr{X}} C_2$*.*

The coherence conditions on the composition $\circ$ are inspired by those found on the topological duals of gaggles [25]. An LGL space is obtained as the special case where $\succcurlyeq$ is $=$. This yields the following definition.

Definition 6.14 (LGL Space)**.** *An* LGL space *is a structure* $\mathscr{X} = (X, \mathscr{O}, \circ)$ *such that*

1. $(X, \mathscr{O})$ *is an Stone space;*
2. $(X, \circ)$ *is an LGL frame;*
3. *The clopen sets of* $(X, \mathscr{O})$ *are closed under* $\bullet_{\mathscr{X}}, \multimap_{\mathscr{X}}, \bullet\!\!-_{\mathscr{X}}$*;*
4. *If* $x \notin y \circ z$ *then there exist clopen sets* C_1, C_2 *such that* $y \in C_1, z \in C_2$ *and* $x \notin C_1 \bullet_{\mathscr{X}} C_2$*.*

The morphisms of (I)LGL spaces are the continuous (I)LGL morphisms, and this yields categories (I)LGLSp. We adapt the functor $Clop_{\succcurlyeq}$ of the Esakia duality to obtain a functor $Clop^{\text{ILGL}}_{\succcurlyeq} : \text{ILGLSp} \to \text{ILGLAlg}$ by setting $Clop^{ILGL}_{\succcurlyeq}(\mathscr{X}) = (\mathscr{CL}_{\succcurlyeq}(\mathscr{X}), \cap, \cup, \Rightarrow_{\mathscr{X}}, \bullet_{\mathscr{X}}, \multimap_{\mathscr{X}}, \bullet\!\!-_{\mathscr{X}})$ and $Clop^{\text{ILGL}}_{\succcurlyeq}(g) = g^{-1}$. Abusing notation, $Pr^{\text{ILGL}} : \text{ILGLAlg} \to \text{ILGLSp}$ is defined $Pr^{\text{ILGL}}(\mathbb{A}) = (Pr(\mathbb{A}), \mathscr{O}_{\mathbb{A}}, \supseteq, \circ_{\mathbb{A}})$ and $Pr^{\text{ILGL}}(f) = f^{-1}$.

Lemma 6.15. *The functors $Clop^{ILGL}_{\succcurlyeq}$ and Pr^{ILGL} are well-defined.*

Proof. To see $Clop^{\text{ILGL}}(\mathscr{X})$ is an ILGL algebra, note that by Esakia duality it is a Heyting algebra. By property 3. in the definition of ILGL space, the upwards-closed clopen sets of $\mathscr{X}$ are closed under $\bullet_{\mathscr{X}}, \multimap_{\mathscr{X}}$ and $\bullet\!\!-_{\mathscr{X}}$, and the fact that they satisfy the residuation property is proved in the same way as for complex algebras. That $Clop^{\text{ILGL}}(g)$ is a homomorphism of ILGL algebras follows from Esakia duality, together with Lemma 6.12.

For $Pr^{\text{ILGL}}(\mathbb{A})$, considering Esakia duality we only need to verify conditions 3. and 4. By Esakia duality, every upwards-closed clopen set is of the form $\theta_{\mathbb{A}}(a)$ for $a \in \mathbb{A}$. The representation theorem for ILGL algebras gives that $\theta_{\mathbb{A}}$ is a homomorphism, so the upwards-closed clopen sets are closed under $\bullet_{Pr^{\text{ILGL}}(\mathbb{A})}$, $\multimap_{Pr^{\text{ILGL}}(\mathbb{A})}$ and $\bullet\!\!-_{Pr^{\text{ILGL}}(\mathbb{A})}$. For the final condition, suppose $F \notin F' \circ_{\mathbb{A}} F''$. Then there exists $a \in F'$ and $b \in F''$ such that $a * b \notin F$. We have $F' \in \theta_{\mathbb{A}}(a)$ and $F'' \in \theta_{\mathbb{A}}(b)$ with $F \notin \theta_{\mathbb{A}}(a) \bullet_{Pr^{\text{ILGL}}(\mathbb{A})} \theta_{\mathbb{A}}(b)$. That $Pr^{\text{ILGL}}(f)$ is a morphism of ILGL spaces follows from Esakia duality and Lemma 6.12. □

The natural transformations we give are necessarily the same as that for Esakia duality. θ is as already defined and $\eta_{\mathscr{X}}(x) = \{C \in \mathscr{CL}_{\succcurlyeq}(\mathscr{X}) \mid x \in C\}$. This gives the duality theorem for ILGL algebras.

Theorem 6.16 (ILGL Duality)**.** *$\theta : Id_{\text{ILGLAlg}} \to Clop^{ILGL}_{\succcurlyeq}$ and $\eta : Id_{\text{ILGLSp}} \to Pr^{ILGL}Clop^{ILGL}_{\succcurlyeq}$ form a dual equivalence of categories between* ILGLAlg *and* ILGLSp.

Proof. The final verification required is that $x \in y \circ z$ iff $\eta_{\mathscr{X}}(x) \in \eta_{\mathscr{X}}(y) \circ \eta_{\mathscr{X}}(z)$. The left-to-right direction is trivial. For the right-to-left, suppose $x \notin y \circ z$. Then by condition 4. of ILGL space there exist upwards-closed clopens C_1 and C_2 such that $y \in C_1$, $z \in C_2$ and $x \notin C_1 \bullet_{Clop^{\text{ILGL}}_{\succcurlyeq}(\mathscr{X})} C_2$. The result then obtains immediately. □

Just as in the case for Stone duality, the duality for LGL algebras then obtains as an immediate corollary of ILGL duality. This is also obtainable as an instance of Bimbó & Dunn's duality theorem for Boolean gaggles ([25], Theorem 9.2.22).

Theorem 6.17 (LGL Duality)**.** *$\theta : Id_{\text{LGLAlg}} \to Clop_{LGL}$ and $\eta : Id_{\text{LGLSp}} \to Pr^{LGL}Clop^{LGL}$ form a dual equivalence of categories between* LGLAlg *and* LGLSp.

6.2 Logics of Bunched Implications

We next look to the logics of bunched implications (B)BI introduced in Chapter 3. We begin with (B)BI algebras.

Definition 6.18 ((B)BI Algebra)**.**

1. *A* BI algebra *is an algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*)$ *such that* $(A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \mathrel{-\!\!*})$ *is an ILGL algebra and* $(A, *, \top^*)$ *a commutative monoid: that is,* $*$ *is commutative and associative with* $\top^*$ *a unit for* $*$*.*

2. *A* BBI algebra *is a BI algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*)$ *for which* $(A, \wedge, \vee, \to, \top, \bot)$ *is a Boolean algebra.*

Interpretations of (B)BI on a (B)BI algebra are given in the same way as for (I)LGL algebras, with $[\![\top^*]\!] = \top^*$. Noting that the additional axioms for (B)BI correspond precisely to the equations corresponding to $(A, *, \top^*)$ being a commutative monoid we obtain a soundness theorem for algebraic interpretations.

Theorem 6.19 (Algebraic Soundness)**.** *If* $\varphi \vdash \psi$ *is provable in the Hilbert system for (B)BI, then for all interpretations* $[\![-]\!]$ *on (B)BI algebras* $[\![\varphi]\!] \leq [\![\psi]\!]$*.* □

Moving on to morphisms for (B)BI frames, we simply extend the definitions of (I)LGL morphisms. First we recall the definition of (B)BI frame (Definition 3.1): a *BI frame* is a tuple $\mathscr{X} = (X, \succcurlyeq, \circ, E)$ where $(X, \succcurlyeq, \circ)$ is an ILGL frame, $E \subseteq X$ and the following conditions are satisfied (with outermost universal quantification omitted for readability):

(Commutativity) $z \in x \circ y \to z \in y \circ x$ (Closure) $e \in E \wedge e' \succcurlyeq e \to e' \in E$
(Unit Existence) $\exists e \in E(x \in x \circ e)$ (Coherence) $e \in E \wedge x \in y \circ e \to x \succcurlyeq y$
(Associativity) $t' \succcurlyeq t \in x \circ y \wedge w \in t' \circ z \to \exists s, s', w'(s' \succcurlyeq s \in y \circ z \wedge w \succcurlyeq w' \in x \circ s')$

A *BBI frame* is a BI frame for which the order $\succcurlyeq$ is equality $=$.

Definition 6.20 ((B)BI Morphism)**.** *A* (B)BI morphism $f : \mathscr{X} \to \mathscr{X}'$ *is an ILGL (LGL) morphism satisfying the additional property*

5. $e \in E$ *iff* $f(e) \in E'$*.*

(B)BI frames together with (B)BI morphisms form a category (B)BI. In the case for BBI, this gives precisely the BBI morphisms of Brotherston & Villard [44].

Definition 6.21 (Complex BBI Algebra)**.** *Given a (B)BI frame $\mathscr{X}$, the complex algebra of $\mathscr{X}$, $Com^{(B)BI}(\mathscr{X})$ is given by extending $Com^{ILGL}(\mathscr{X})$ ($Com^{LGL}(\mathscr{X})$) with E.*

Lemma 6.22. *Given a (B)BI frame $\mathscr{X}$, $Com^{(B)BI}(\mathscr{X})$ is a (B)BI algebra.*

Proof. We give the argument for BI frames; the case for BBI frames is obtained from the specific case where $\succcurlyeq$ is $=$. Given Lemma 6.8, we just need to verify that $(\mathscr{P}_{\succcurlyeq}(X), \bullet_{\mathscr{X}}, E)$ is a commutative monoid. First note that $E \in \mathscr{P}_{\succcurlyeq}(X)$ by virtue of the axiom Closure. Commutativity of $\bullet_{\mathscr{X}}$ is easily derived from the frame axiom Commutativity. Further, the inclusion $A \subseteq A \bullet_{\mathscr{X}} E$ follows immediately from the axiom Unit Existence. Slightly more involved is $A \bullet_{\mathscr{X}} E \subseteq A$: let $x \in A \bullet_{\mathscr{X}} E$. Then there exists x', y, e such that $x \succcurlyeq x'$ with $x' \in y \circ e$, $y \in A$ and $e \in E$. By the frame axiom Coherence it follows that $x' \succcurlyeq y$, so by transitivity $x \succcurlyeq y$. Since A is upwards-closed, $x \in A$.

We finally come to associativity. We only need to verify the inclusion $(A \bullet_{\mathscr{X}} B) \bullet_{\mathscr{X}} C \subseteq A \bullet_{\mathscr{X}} (B \bullet_{\mathscr{X}} C)$ because of commutativity of $\bullet_{\mathscr{X}}$. Let $a \in (A \bullet_{\mathscr{X}} B) \bullet_{\mathscr{X}} C$. Then there exists w, t', z such that $a \succcurlyeq w \in t' \circ z$ with $t' \in A \bullet_{\mathscr{X}} B$ and $z \in C$. This entails the existence of x, y, t such that $t' \succcurlyeq t \in x \circ y$ with $x \in A$ and $y \in B$. By the frame axiom Associativity we thus have s, s', w' with $s' \succcurlyeq s \in y \circ z$ and $w \succcurlyeq w' \in x \circ s'$. Hence $s' \in B \bullet_{\mathscr{X}} C$ and, because $w' \preccurlyeq w \preccurlyeq a$, $a \in A \bullet_{\mathscr{X}} (B \bullet_{\mathscr{X}} C)$ as required. □

Definition 6.23 (Prime Filter (B)BI Frame)**.** *Given a (B)BI algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$, $Pr^{(B)BI}(\mathbb{A})$, is given by extending $Pr^{ILGL}(\mathbb{A})$ ($Pr^{LGL}(\mathbb{A})$) with $E_{\mathbb{A}} = \{F \in Pr(A) \mid \top^* \in F\}$.*

In the BBI case, this definition is essentially contained in Galmiche & Larchey-Wendling's [99] completeness theorem for the relational semantics of BBI, although they do further work to give a frame in which the set of units E is a singleton. The next lemma follows a similar argument to theirs, although we generalise it to our BI frames.

Lemma 6.24. *Given a (B)BI algebra $\mathbb{A}$, the prime filter frame $Pr^{(B)BI}(\mathbb{A})$ is a (B)BI frame.*

Proof. Commutativity of $\circ_{\mathbb{A}}$ can be read off the definition, given that $*$ is commutative for (B)BI. We also have that E satisfies Closure trivially. We are left to verify Associativity, Unit Existence and Coherence. We note that in the case for BBI, maximality of prime filters collapses all of the inclusions to equalities in what follows, so we just give the argument for BI.

First, Associativity. Assume $F_{t'} \supseteq F_t \in F_x \circ F_y$ and $F_w \in F_{t'} \circ F_z$. We show that

$$P(F) = \begin{cases} 1 & \text{if } F \in F_y \circ_{\mathbb{A}} F_z \text{ and } F_w \in F_x \circ_{\mathbb{A}} F \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. For a $\subseteq$-chain $(F_\alpha)_{\alpha<\lambda}$ such that $P(F_\alpha) = 1$ for all α, we straightforwardly have $P(\bigcup_\alpha F_\alpha) = 1$. If $P(F \cap F') = 1$, we have that $F, F' \in F_y \circ F_z$ immediately, so suppose for contradiction that there exists $a, a' \in F_x$, $b \in F$, $b' \in F'$ such that $a * b, a' * b' \notin F_w$. We have that $a'' = a \wedge a' \in F_x$ and $b \vee b' \in F \cap F'$ so $a'' * (b \vee b') = (a'' * b) \vee (a'' * b') \in F_w$. Since F_w is prime, either $a'' * b \in F_w$ or $a'' * b' \in F_w$. Thus $a * b \in F_w$ or $a' * b' \in F_w$ because $*$ is monotone, a contradiction.

Now consider the set $F = \{a \in \mathbb{A} \mid \exists b \in F_y, c \in F_z : a \geq b * c\}$. We show this is a proper filter satisfying $P(F) = 1$. First, suppose for contradiction that $\bot \in F$. Then there exists $b \in F_y$ and $c \in F_z$ such that $b * c = \bot$. Letting $a \in F_x$ be arbitrary, we have that $a * b \in F_t \subseteq F_{t'}$, so $(a * b) * c = a * (b * c) = a * \bot = \bot \in F_w$, contradicting that F_w is proper. F is clearly upwards-closed; to see it is closed under meets, consider $a, a' \in F$. Then there exists $b, b' \in F_y$ and $c, c' \in F_z$ such that $a \geq b * c$ and $a' \geq b' * c'$. We have that $b \wedge b' \in F_y$ and $c \wedge c' \in F_z$, and by monotonicity of $*$, $(b \wedge b') * (c \wedge c') \leq a * b, a' * b'$. Hence $(b \wedge b') * (c \wedge c') \leq (a * b) \wedge (a' * b') \leq c \wedge c'$ as required.

We now verify that $P(F) = 1$. Let $b \in F_y$ and $c \in F_z$. Clearly $b * c \in F$, so $F \in F_y \circ_{\mathbb{A}} F_z$. If $a \in F_x$ and $a' \geq b * c$ for $b \in F_y$ and $c \in F_z$, we have that $a * a' \geq a * (b * c) = (a * b) * c \in F_w$, since $a * b \in F_t \subseteq F_{t'}$ and $c \in F_z$. Thus $a * a' \in F_w$ and $F_w \in F_x \circ_{\mathbb{A}} F$ as required. We thus obtain a prime F with $P(F) = 1$ by the Prime Extension Lemma, which is precisely what is required to satisfy Associativity.

For Unit Existence, let F be an arbitrary prime filter. We show that

$$P(G) = \begin{cases} 1 & \text{if } F \in F \circ_{\mathbb{A}} G \text{ and } \top^* \in G \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. If $P(G_\alpha) = 1$ for all G_α in a $\subseteq$-chain $(G_\alpha)_{\alpha<\lambda}$ then clearly $F \in F \circ_{\mathbb{A}} \bigcup_\alpha G_\alpha$. Next, let $P(G \cap G') = 1$ and assume for contradiction that there exists $a, a' \in F$, $b \in G$ and $b' \in G'$ such that $a * b \notin F$ and $a' * b' \notin F$. $b \vee b' \in G \cap G'$ so for $a'' = a \wedge a' \in F$ we have $a'' * (b \vee b') = (a'' * b) \vee (a'' * b') \in F$. Since F is prime, either $a'' * b \in F$ or $a'' * b' \in F$. Hence, by monotonicity of $*$, either $a * b \in F$ or $a' * b' \in F$, a contradiction. Now consider the filter $[\top^*)$. We note that this can only fail to be proper when $\top^* = \bot$, but in that case it can be shown that for all $a \in \mathbb{A}$, $a = \bot$, and thus $\mathbb{A}$ is degenerate and not a BI algebra. Given any $a \in F$ and

$b \geq \top^*$, we have $a * b \geq a * \top^* = a \in F$, so $a * b \in F$. Since $P(\lceil \top^* \rceil) = 1$, there exists a prime filter F with $P(F) = 1$, and so Unit Existence is satisfied.

Finally, for Coherence, assume $F_x \in F_y \circ F_e$ where $\top^* \in F_e$. Then for all $a \in F_y$, $a * \top^* = a \in F_x$, so $F_y \subseteq F_x$ as required. □

We now obtain the representation theorem for (B)BI algebras immediately by noting that $\theta_{\mathbb{A}}(\top^*) = E_{\mathbb{A}}$.

Theorem 6.25 (Representation Theorem for (B)BI Algebras)**.** *Every (B)BI algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given a (B)BI algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{(B)BI}(Pr^{(B)BI}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr^{(B)BI}(\mathbb{A}) \mid a \in F\}$ is an embedding.* □

Lifting $Com^{(\mathrm{B})\mathrm{BI}}$ and $Pr^{(\mathrm{B})\mathrm{BI}}$ to functors is done in the standard way, and given our results for (I)LGL we just need to check that the assignment of morphisms interacts appropriately with $\top^*$ and the (B)BI morphism condition on E. This is straightforward though: property 5. of (B)BI morphism ensures that $g^{-1}(E') = E$, and $f(\top^*) = \top^*$ ensures that $F \in E_{\mathbb{A}'}$ iff $f^{-1}(F) \in E_{\mathbb{A}}$.

Lemma 6.26. *The functors $Com^{(B)BI}$ and $Pr^{(B)BI}$ are well-defined.* □

Once again, we topologise (B)BI frames to obtain a dual equivalence of categories.

Definition 6.27 (BI Space)**.** *A* BI space *is a structure $\mathscr{X} = (X, \mathscr{O}, \succcurlyeq, \circ, E)$ such that*

1. *$(X, \mathscr{O}, \succcurlyeq, \circ)$ is an ILGL space,*
2. *$(X, \succcurlyeq, \circ, E)$ is a BI frame, and*
3. *E is clopen in $(X, \mathscr{O})$.*

A morphism of BI frames is a continuous BI morphism, giving a category BISp. In the particular case of BI spaces with a trivial ordering we obtain BBI spaces.

Definition 6.28 (BBI Space)**.** *A* BBI space *is a structure $\mathscr{X} = (X, \mathscr{O}, \circ, E)$ such that*

1. *$(X, \mathscr{O}, \circ)$ is an LGL space,*
2. *$(X, \circ, E)$ is a BBI frame, and*
3. *E is clopen in $(X, \mathscr{O})$.*

The duality theorems for (B)BI algebras then follows immediately from duality for (I)LGL algebras, together with Lemmas 6.22 and 6.24. The only additional structure that needs to be taken care of is the constant $\top^*$ and E.

For BI we have $Pr^{\mathrm{BI}} : \mathrm{BIAlg} \to \mathrm{BISp}$ defined by $PrSp^{\mathrm{BI}}(\mathbb{A}) = (Pr(A), \mathscr{O}_{\mathbb{A}}, \supseteq, \circ_{\mathbb{A}}, E_{\mathbb{A}})$ and $Pr^{\mathrm{BI}}(f) = f^{-1}$; correspondingly, $Clop^{\mathrm{BI}}_{\succcurlyeq} : \mathrm{BISp} \to \mathrm{BIAlg}$ is given by $Clop^{\mathrm{BI}}_{\succcurlyeq}(\mathscr{X}) = (\mathscr{CL}_{\succcurlyeq}(\mathscr{X}), \cap, \cup, \Rightarrow_{\mathscr{X}}, X, \emptyset, \bullet_{\mathscr{X}}, \mathrel{-\!\!\bullet}_{\mathscr{X}}, E)$ and $Clop^{\mathrm{BI}}_{\succcurlyeq}(g) = g^{-1}$. η is as given previously. Since E is clopen in a BI space by definition, and upwards closed by the BI axiom Closure, $E \in \mathscr{CL}_{\succcurlyeq}(\mathscr{X})$. By the BI morphism property 5. the components $\eta_{\mathscr{X}}$ are isomorphic on E. The duality theorem then obtains.

Theorem 6.29 (BI Duality)**.** *θ and η form a dual equivalence of categories between* BIAlg *and* BISp. □

We note that Jipsen & Litak [130] independently proved the duality theorem for BI concurrently with the writing of this thesis. The case for BBI algebras obtains as a special case of BI duality.

Theorem 6.30 (BBI Duality)**.** *θ and η form a dual equivalence of categories between* BBIAlg *and* BBISp. □

6.3 De Morgan Bunched Logics

Next we consider the De Morgan bunched logics introduced in Chapter 4 Section 4.1, obtained by extending (B)BI with a multiplicative negation $\mathrel{-\!\!*}\neg$.

Definition 6.31 (DMBI/CBI Algebra)**.** *1. A* DMBI algebra *is an algebra $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*, \bot^*)$ such that $(A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*)$ is a BI algebra and, defining $\mathrel{-\!\!*}\neg a := a \mathrel{-\!\!*} \bot^*$, $\mathrel{-\!\!*}\neg\mathrel{-\!\!*}\neg a = a$ and $\mathrel{-\!\!*}\neg\top^* = \bot^*$.*

2. A CBI algebra *is a DMBI algebra $\mathbb{A}=(A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*, \bot^*)$ in which $(A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*)$ is a BBI algebra.*

We collect a number of useful properties of these algebras in the following proposition.

Proposition 6.32. *Let $\mathbb{A}$ be a DMBI or CBI algebra with $a, b, c \in \mathbb{A}$ and $X \subseteq X$. Then the following hold.*

1. *If $\bigvee X$ exists, then $\bigwedge \mathrel{-\!\!*}\neg X$ exists and $\mathrel{-\!\!*}\neg \bigvee X = \bigwedge \mathrel{-\!\!*}\neg X$;*
2. *If $a \leq b$ then $\mathrel{-\!\!*}\neg b \leq \mathrel{-\!\!*}\neg a$;*
3. *If $\bigwedge X$ and $\bigvee \mathrel{-\!\!*}\neg X$ exist then $\mathrel{-\!\!*}\neg \bigwedge X = \bigvee \mathrel{-\!\!*}\neg X$;*

4. $a * b \leq c$ *iff* $b * \mathrel{-\!\!*} \neg c \leq \mathrel{-\!\!*} \neg a$.

Proof. 1. If $\bigvee X$ exists we have $\mathrel{-\!\!*} \neg \bigvee X = \bigvee X \mathrel{-\!\!*} \bot^* = \bigwedge_{x \in X} \mathrel{-\!\!*} \neg x$ by Proposition 6.2.

2. If $a \leq b$ then $a \vee b = b$. Hence $\mathrel{-\!\!*} \neg b = \mathrel{-\!\!*} \neg (a \vee b) = \mathrel{-\!\!*} \neg a \wedge \mathrel{-\!\!*} \neg b$ so $\mathrel{-\!\!*} \neg b \leq \mathrel{-\!\!*} \neg a$.

3. Suppose $\bigwedge X$ and $\bigvee \mathrel{-\!\!*} \neg X$ exist. First note that by 2. $\mathrel{-\!\!*} \neg \bigwedge X \leq \bigvee \mathrel{-\!\!*} \neg X$ iff $\bigwedge X \geq \mathrel{-\!\!*} \neg \bigvee \mathrel{-\!\!*} \neg X$. However by 1. we have $\mathrel{-\!\!*} \neg \bigvee \mathrel{-\!\!*} \neg X = \bigwedge \mathrel{-\!\!*} \neg \mathrel{-\!\!*} \neg X = \bigwedge X$, so the inequality holds. In the other direction, $\bigvee \mathrel{-\!\!*} \neg X \leq \mathrel{-\!\!*} \neg \bigwedge X$ iff $\mathrel{-\!\!*} \neg \bigwedge X \geq \mathrel{-\!\!*} \neg a$ for all $a \in a \in X$. The right hand side of this biconditional clearly holds by $\bigwedge X \leq a$ for all $a \in X$.

4. First note that $b * (b \mathrel{-\!\!*} c) * (c \mathrel{-\!\!*} \bot^*) \leq c * (c \mathrel{-\!\!*} \bot^*) \leq \bot^*$. By residuation we have $a * b \leq c$ iff $a \leq b \mathrel{-\!\!*} c$. By 2. $\mathrel{-\!\!*} \neg (b \mathrel{-\!\!*} c) = (b \mathrel{-\!\!*} c) \mathrel{-\!\!*} \bot^* \leq \mathrel{-\!\!*} \neg a$. Now $b * (c \mathrel{-\!\!*} \bot^*) \leq (b \mathrel{-\!\!*} c) \mathrel{-\!\!*} \bot^*$ iff $b * (b \mathrel{-\!\!*} c) * (c \mathrel{-\!\!*} \bot^*) \leq \bot^*$, with the righthand statement true. Hence $b * \mathrel{-\!\!*} \neg c \leq \mathrel{-\!\!*} \neg a$.

□

As a result of this proposition, we have that for any DMBI or CBI algebra $\mathbb{A}$, $(A, \wedge, \vee, \mathrel{-\!\!*} \neg, \top, \bot)$ is a De Morgan algebra [165]. Thus $\mathrel{-\!\!*} \neg$ is a dual automorphism on the underlying bounded distributive lattice of $\mathbb{A}$. Now an algebraic interpretation of DMBI or CBI on a DMBI or CBI algebra extends one on the underlying BI or BBI algebra by additionally setting $[\![\mathrel{-\!\!*} \neg a]\!] = \mathrel{-\!\!*} \neg [\![a]\!]$. That this is sound follows straightforwardly from the additional De Morgan bunched logic Hilbert rules matching the defining properties of $\mathrel{-\!\!*} \neg$.

Theorem 6.33 (Algebraic Soundness of DMBI/CBI)**.** *If $\varphi \vdash \psi$ is provable in the Hilbert system for DMBI (CBI), then for all interpretations $[\![-]\!]$ on DMBI (CBI) algebras $[\![\varphi]\!] \leq [\![\psi]\!]$.* □

We recall the definition of DMBI/CBI frame (Definition 4.1): a DMBI frame is a tuple $\mathscr{X} = (X, \succcurlyeq, \circ, E, -)$ where $(X, \succcurlyeq, \circ, E)$ is a BI frame and $- : X \to X$ is an operation satisfying the following conditions (with outermost universal quantification omitted for readability):

$$\text{(Dual)} \quad x \succcurlyeq y \to -y \succcurlyeq -x \qquad \text{(Involutive)} \quad --x = x$$
$$\text{(Compatibility)} \quad z \in x \circ y \to -x \in -z \circ y.$$

A CBI frame is a DMBI frame for which the order $\succcurlyeq$ is equality $=$.

Definition 6.34 (DMBI/CBI Morphism)**.** *A DMBI (CBI) morphism $g : \mathscr{X} \to \mathscr{X}'$ is a BI (BBI) morphism satisfying the additional property*

6. *g(-x) = -g(x).*

DMBI (CBI) frames together with DMBI (CBI) morphisms form a category DMBI (CBI).

Definition 6.35 (Complex DMBI/CBI Algebra)**.** *Given a DMBI (CBI) frame $\mathscr{X}$, the complex algebra of $\mathscr{X}$, $Com^{DMBI}(\mathscr{X})$ ($Com^{CBI}(\mathscr{X})$) is given by extending $Com^{BI}(\mathscr{X})$ ($Com^{BBI}(\mathscr{X})$) with the set $U = \{x \in \mathscr{X} \mid -x \notin E\}$.*

Lemma 6.36. *Given a DMBI (CBI) frame $\mathscr{X}$, $Com^{DMBI}(\mathscr{X})$ ($Com^{CBI}(\mathscr{X})$) is a DMBI (CBI) algebra.*

Proof. First we note that U is an upwards-closed set. Suppose $u \in U$ and $u' \succcurlyeq u$. Since $-u \notin E$, and E is upwards-closed, we must have that $-u' \notin E$ as $-u \succcurlyeq -u'$. On the complex algebra we define the multiplicative negation by $\sim_{\mathscr{X}} A := A \multimap_{\mathscr{X}} U$, as guided by the definition of DMBI/CBI algebra. We must show that $\sim_{\mathscr{X}} \sim_{\mathscr{X}} A = A$ and $\sim_{\mathscr{X}} E = U$, and this follows immediately if $\sim_{\mathscr{X}} A = \{a \mid -a \notin A\}$; we verify this identity.

First assume $a \in \sim_{\mathscr{X}} A$. Let $e \in E$ be such that $a \in a \circ e$ by the frame axiom Unit Existence. Then by Compatibility, $-e \in a \circ -a$ and if $-a \in A$, we would have $-e \in U$, a contradiction as $--e = e \in E$. Now assume a is such that $-a \notin A$. Let $a' \succcurlyeq a$ with $b \in A$ and $c \in a' \circ b$. We assume for contradiction that $c \notin U$. Then $-c \in E$ and by Compatibility we have $-a' \in b \circ -c$. By the frame axiom Coherence $-a' \succcurlyeq b$, and by upwards-closure of A, $-a \in A$; a contradiction. Hence $c \in U$. □

Definition 6.37 (Prime Filter DMBI/CBI Frame)**.** *Given a DMBI (CBI) algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$, $Pr^{DMBI}(\mathbb{A})$ ($Pr^{CBI}(\mathbb{A})$ is given by extending $Pr^{BI}(\mathbb{A})$ ($Pr^{BBI}(\mathbb{A})$) with the operation $-_{\mathbb{A}} F := \overline{\dashv F}$.*

Lemma 6.38. *Given a DMBI (CBI) algebra $\mathbb{A}$, the prime filter frame $Pr^{DMBI}(\mathbb{A})$ ($Pr^{CBI}(\mathbb{A})$) is a DMBI (CBI) frame.*

Proof. We first note that $-_{\mathbb{A}}$ is a well-defined operation. Since $\dashv$ is a dual automorphism, if F is a prime filter, it follows that $\dashv F$ is a prime ideal. Thus $\overline{\dashv F}$ is a prime filter. We must check that the three DMBI frame axioms hold. If $F' \supseteq F$ then $\dashv F' \supseteq \dashv F$ and so $\overline{\dashv F} \supseteq \overline{\dashv F'}$, as required for Dual. $\dashv \dashv a = a$ straightforwardly entails that Involutive is satisfied.

Finally we verify Compatibility. Assume $F_z \in F_x \circ_{\mathbb{A}} F_y$ and let $c \in -_{\mathbb{A}} F_z$ and $d \in F_y$. For contradiction, suppose $c * d \notin -_{\mathbb{A}} F_x$. Then there necessarily exists $a \in F_x$ such that $c * d \leq \dashv a$. By Proposition 6.32 this entails $a * d \leq \dashv c$. Since $a \in F_x$ and $d \in F_y$ we have $a * d \in F_z$, and thus $\dashv c \in F_z$. However, $c \in -_{\mathbb{A}} F_z$ entails $c \notin \dashv F_z$, so $\dashv c \notin F_z$, a contradiction. Thus $c * d \in -_{\mathbb{A}} F_x$ as required. □

Theorem 6.39 (Representation Theorem for DMBI/CBI Algebras)**.** *Every DMBI (CBI) algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given an DMBI (CBI) algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{DMBI}(Pr^{DMBI}(\mathbb{A}))$ ($\theta_{\mathbb{A}} : \mathbb{A} \to Com^{CBI}(Pr^{CBI}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr(\mathbb{A}) \mid a \in F\}$ is an embedding.*

Proof. To obtain the theorem we simply have to show that $\theta_{\mathbb{A}}(\bot^*) = \{F \mid \top^* \notin -_{\mathbb{A}}F\}$. This follows immediately though: in one direction, we have that $\rightharpoonup\top^* = \bot^*$ by definition, so $\bot^* \in F$ implies $\top^* \in \rightharpoonup F$, so $\top^* \notin \overline{\rightharpoonup F}$ as required; in the other, if $\top^* \notin -_{\mathbb{A}}F$ then $\top^* \in \rightharpoonup F$ and so $\rightharpoonup\top^* = \bot^* \in F$. □

Once again we lift the complex algebra and prime filter frame assignments to functors by assigning the morphisms to their inverse image. We verify that this assignment is functorial.

Lemma 6.40. *The functors Com^{DMBI} (Com^{CBI}) and Pr^{DMBI} (Pr^{CBI}) are well-defined.*

Proof. We first verify the inverse image of a DMBI (CBI) homomorphism f respects the operation $-_{\mathbb{A}}$. We have that $a \in f^{-1}(-_{\mathbb{A}'}F)$ iff $f(a) \in -_{\mathbb{A}}F$ iff $f(a) \notin \rightharpoonup F$ iff $\rightharpoonup f(a) \notin F$ iff $f(\rightharpoonup a) \notin F$ iff $a \in -_{\mathbb{A}}f^{-1}(F)$.

Next, let g be a DMBI (CBI) morphism. By (B)BI morphism property 5. and DMBI/CBI morphism property 6. we have $x \in g^{-1}(U')$ iff $g(x) \in U'$ iff $-g(x) \notin E'$ iff $g(-x) \notin E'$ iff $-x \notin E$ iff $x \in U$. □

Topology now enters to obtain a dual equivalence of categories.

Definition 6.41 (DMBI Space)**.** *A* DMBI space *is a structure $\mathscr{X} = (X, \mathscr{O}, \succcurlyeq, \circ, E, -)$ such that*

1. *$(X, \mathscr{O}, \succcurlyeq, \circ, E)$ is a BI space,*
2. *$(X, \succcurlyeq, \circ, E, -)$ is a DMBI frame, and*
3. *$-$ is a continuous map.*

Definition 6.42 (CBI Space)**.** *A* CBI space *is a structure $\mathscr{X} = (X, \mathscr{O}, \circ, E, -)$ such that*

1. *$(X, \mathscr{O}, \circ, E)$ is a BBI space,*
2. *$(X, \circ, E, -)$ is a CBI frame, and*
3. *$-$ is a continuous map.*

As in previous cases, morphisms for these kinds of spaces are given by continuous DMBI (CBI) morphisms. This gives us the category of DMBI spaces DMBISp and the category of CBI spaces, CBISp. Just as the move from ILGL duality to BI duality simply required us to verify everything works as it should with regards to E and $\top^*$, in light of Lemmas 6.36, 6.38 and 6.40, moving from BI duality to DMBI duality just needs verifications on U and $\bot^*$.

We have the functor $Pr^{\text{DMBI}} : \text{DMBIAlg} \to \text{DMBISp}$ defined by $Pr^{\text{DMBI}}(\mathbb{A}) = (Pr(\mathbb{A}), \mathscr{O}_{\mathbb{A}}, \supseteq, \circ_{\mathbb{A}}, E_{\mathbb{A}}, -_{\mathbb{A}})$ and $Pr^{\text{DMBI}}(f) = f^{-1}$. Continuity of $-_{\mathbb{A}}$ can be verified on the subbase elements of $\mathscr{O}_{\mathscr{A}}$, and this holds because $(-_{\mathbb{A}})^{-1}[\theta_{\mathbb{A}}(a)] = \overline{\theta_{\mathbb{A}}(\sim a)}$ and $(-_{\mathbb{A}})^{-1}[\overline{\theta_{\mathbb{A}}(a)}] = \theta_{\mathbb{A}}(\sim a)$. In the other direction, we have the functor $Clop^{\text{DMBI}}_{\succcurlyeq} : \text{DMBISp} \to \text{DMBIAlg}$ defined $Clop^{\text{DMBI}}_{\succcurlyeq}(\mathscr{X}) = (\mathscr{CL}_{\succcurlyeq}(\mathscr{X}), \cap, \cup, \Rightarrow_{\mathscr{X}}, X, \emptyset, \bullet_{\mathscr{X}}, \mathrel{-\!\!\bullet}_{\mathscr{X}}, E, U)$ and $Clop^{\text{DMBI}}_{\succcurlyeq}(g) = g^{-1}$. That $U \in \mathscr{CL}_{\succcurlyeq}(\mathscr{X})$ follows from the fact that $U = \overline{-E}$. E is clopen, so $-E$ is clopen by continuity and so too is $\overline{-E}$. Further, E is upwards-closed, so $-E$ is downwards-closed, meaning $\overline{-E}$ is upwards-closed. We once again consider the collection of maps $\eta_{\mathscr{X}}(x) = \{C \in \mathscr{CL}_{\succcurlyeq}(\mathscr{X}) \mid x \in C\}$ to complete the duality.

Theorem 6.43 (DMBI Duality)**.** *θ and η form a dual equivalence of categories between* DMBIAlg *and* DMBISp*.*

Proof. The last remaining steps are to show that the components $\eta_{\mathscr{X}}$ are isomorphisms in DMBISp. The key step is to verify that $-_{Clop^{\text{DMBI}}_{\succcurlyeq}(\mathscr{X})}\eta_{\mathscr{X}}(x) = \eta_{\mathscr{X}}(-x)$, as the rest obtains from BI duality. Unpacking the definition, we must check $\overline{\{C' \mathrel{-\!\!\bullet}_{\mathscr{X}} U \mid C' \in \eta_{\mathscr{X}}(x)\}} = \eta_{\mathscr{X}}(-x)$. For the right-to-left inclusion, suppose $-x \in C$ and for contradiction $C = C' \mathrel{-\!\!\bullet}_{\mathscr{X}} U$ for some upwards-closed clopen C' such that $x \in C'$. Then by Unit Existence there exists $e \in E$ such that $-x \in -x \circ e$, and by Compatibility $-e \in -x \circ x$. By assumption this entails $-e \in U$, but $--e = e \in E$, a contradiction. Hence $C \in \overline{\{C' \mathrel{-\!\!\bullet}_{\mathscr{X}} U \mid C' \in \eta_{\mathscr{X}}(x)\}}$.

For the left-to-right inclusion, note that

$$\eta_{\mathscr{X}}(x) = \{-C \mid C \textit{ downwards-closed clopen and } x \in C\}$$

holds; that this is the case is a consequence of $-$ being continuous and the frame axiom Dual. Now suppose we have $-C$ such that C downwards-closed and clopen and $x \notin C$. Then $x \in \overline{C}$ and we claim that $-C = \overline{C} \mathrel{-\!\!\bullet}_{\mathscr{X}} U$. First assume $-y \in -C$. Suppose $y' \succcurlyeq -y$ and $z \in \overline{C}$ such that $w \in y' \circ z$ and assume for contradiction that $w \notin U$. Then $-w \in E$. By Compatibility, $-y' \in z \circ -w$, and by Coherence $-y' \succcurlyeq z$. By Dual and our assumption, $-z \succcurlyeq y' \succcurlyeq -y$, and by Dual again $y \succcurlyeq z$. Thus by upwards-closure of $\overline{C}$ we have $y \in \overline{C}$, but $y \in C$ by assumption; a contradiction.

Hence $w \in U$ and $-y \in \overline{C} \multimap\!\bullet_{\mathscr{X}} U$. Now suppose $-y \notin -C$. Then $y \in \overline{C}$. By Unit Existence there is $e \in E$ such that $y \in y \circ e$, and by Compatibility $-e \in -y \circ y$. We have $y \in \overline{C}$ with $-e \notin U$, so $-y \notin \overline{C} \multimap\!\bullet_{\mathscr{X}} U$. □

CBI duality is obtained as the particular case of DMBI duality when we restrict to CBI algebras.

Theorem 6.44. *θ and η form a dual equivalence of categories between* CBIAlg *and* CBISp. □

6.4 Other Variants

We finish this chapter by considering the three other variants of propositional bunched logic that were introduced in Chapter 4: subclassical bunched logics, concurrent Kleene bunched logic and separating modal logics. Given our analysis of the separating modalities in terms of $\Diamond$, $-\!\!*$ and $\neg$ we can immediately obtain the respective results for separating logics by a straightforward combination of BBI duality and modal duality (see, for example, [27, 217]). More work is required for the other variants however.

6.4.1 Subclassical Bunched Logics

We begin with the subclassical bunched logics introduced in Chapter 4 Section 4.2. These are obtained by extending (B)BI with primitive multiplicative disjunction and falsum. We first give the algebraic structures corresponding to the basic subclassical bunched logics.

Definition 6.45 (Basic Bi(B)BI Algebra)**.** *A* basic Bi(B)BI algebra *is an algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathbin{⅋}, -\!\!*, \backslash^*, \top^*, \bot^*)$ *such that* $(A, \wedge, \vee, \to, \top, \bot, *, -\!\!*, \top^*)$ *is a (B)BI algebra,* $⅋$ *a commutative binary operation,* $\backslash^*$ *a binary operation,* $\bot^*$ *a constant, such that, for all* $a, b, c \in \mathbb{A}$, $a \leq b \mathbin{⅋} c$ *iff* $a \mathbin{\backslash^*} b \leq c$.

We note that the the generalisation of this structure with $⅋$ non-commutative yields a second associated implication $\mathbin{⅌}$, but we do not consider this for simplicity. The results for those structures hold as a simple generalisation of what follows however. The residuation property of $⅋$ and $\backslash^*$ ensures $⅋$ is monotone, as well as a number of useful properties dual to those of Proposition 6.2.

Proposition 6.46. *Let $\mathbb{A}$ be a basic Bi(B)BI algebra. Then, for all $a, b, a', b' \in A$ and $X, Y \subseteq A$, we have the following:*

1. *If $a \leq a'$ and $b \leq b'$ then $a \mathbin{⅋} b \leq a' \mathbin{⅋} b'$;*

Property	Axiom	Frame Correspondent
Associativity	$a \mathbin{\mathring{\vee}} (b \mathbin{\mathring{\vee}} c) \leq (a \mathbin{\mathring{\vee}} b) \mathbin{\mathring{\vee}} c$	$t' \preccurlyeq t \in x \mathbin{\triangledown} y \wedge w \in t' \mathbin{\triangledown} z \to \exists s, s', w'$ $(s' \preccurlyeq s \in y \mathbin{\triangledown} z \wedge w \preccurlyeq w' \in x \mathbin{\triangledown} s')$
$\bot^*$ Weakening	$a \leq a \mathbin{\mathring{\vee}} \bot^*$	$u \in U \wedge x \in y \mathbin{\triangledown} u \to x \preccurlyeq y$
$\bot^*$ Contraction	$a \mathbin{\mathring{\vee}} \bot^* \leq a$	$\exists u \in U (w \in w \mathbin{\triangledown} u)$
$\mathring{\vee}$ Contraction	$a \mathbin{\mathring{\vee}} a \leq a$	$x \in x \mathbin{\triangledown} x$
Weak Distributivity	$a * (b \mathbin{\mathring{\vee}} c) \leq (a * b) \mathbin{\mathring{\vee}} c$	$t' \succcurlyeq t \in x_1 \circ x_2 \wedge t' \preccurlyeq t'' \in y_1 \mathbin{\triangledown} y_2 \to$ $\exists w (y_1 \in x_1 \circ w \wedge x_2 \in w \mathbin{\triangledown} y_2)$

Figure 6.1: Bi(B)BI properties and axioms (cf. [45]). The BiBBI variants replace $\succcurlyeq$ with $=$.

2. *If* $\bigwedge X$ *and* $\bigwedge Y$ *exist then* $\bigwedge_{x \in X, y \in Y} x \mathbin{\mathring{\vee}} y$ *exists and* $(\bigwedge X) \mathbin{\mathring{\vee}} (\bigwedge Y) = \bigwedge_{x \in X, y \in Y} x \mathbin{\mathring{\vee}} y$;

3. *If* $a = \top$ *or* $b = \top$ *then* $a \mathbin{\mathring{\vee}} b = \top$;

4. *If* $\bigwedge X$ *exists then for any* $z \in A$: $\bigvee_{x \in X} (x \mathbin{\backslash^*} z)$ *exists with* $\bigvee_{x \in X} (x \mathbin{\backslash^*} z) = (\bigwedge X) \mathbin{\backslash^*} z$;

5. *If* $\bigvee X$ *exists then for any* $z \in A$: $\bigvee_{x \in X} (z \mathbin{\backslash^*} x)$ *exists with* $\bigvee_{x \in X} (z \mathbin{\backslash^*} x) = z \mathbin{\backslash^*} (\bigvee X)$; *and*

6. $a \mathbin{\backslash^*} \top = \bot \mathbin{\backslash^*} a = \bot$. □

We recall the definition of Bi(B)BI frame. A *basic Bi(B)BI frame* is a structure $\mathscr{X} = (X, \succcurlyeq, \circ, E, \triangledown, U)$ such that $(X, \succcurlyeq, \circ, E)$ is a (B)BI frame, $\triangledown : X^2 \to \mathscr{P}(X)$ and $U \subseteq X$, satisfying

(Commutativity) $z \in x \mathbin{\triangledown} y \to z \in y \mathbin{\triangledown} x$; (U-Closure) $u \in U \wedge u \succcurlyeq u' \to u' \in U$.

Definition 6.47 (Bi(B)BI Morphism)**.** *A* Bi(B)BI morphism *is a map* $f : \mathscr{X} \to \mathscr{X}'$ *such that* f *is a (B)BI morphism satisfying the following additional properties:*

7. $x \in y \mathbin{\triangledown} z$ *implies* $g(x) \in g(y) \mathbin{\triangledown} g(z)$;

8. $g(x) \preccurlyeq' s' \in t' \mathbin{\triangledown}' u'$ *implies there exists* s, t, u *such that* $x \preccurlyeq s \in t \mathbin{\triangledown} u$, $t' \succcurlyeq' g(t)$ *and* $u' \succcurlyeq g(u)$;

9. $g(x) \succcurlyeq' s', u' \in t' \mathbin{\triangledown}' s'$ *implies there exists* s, t, u *such that* $x \succcurlyeq s, u \in t \mathbin{\triangledown} s$, $g(u) \succcurlyeq' u'$ *and* $t' \succcurlyeq' g(t)$.

Figure 6.1 gives algebraic axioms directly corresponding to the defining axioms of subclassical bunched logics specified in Chapter 4, as well as the frame properties that correspond to them. For any collection of subclassical axioms Σ, we denote by $\mathrm{Bi(B)BIAlg}_\Sigma$ the category of Bi(B)BI algebras satisfying Σ and $\mathrm{Bi(B)BI}_\Sigma$ the category of Bi(B)BI frames satisfying the frame correspondents of Σ.

The algebraic interpretation of Bi(B)BI + Σ extends that for (B)BI. We now additionally interpret $\mathbin{\overset{*}{\triangledown}}$, $\overset{*}{\setminus}$ and $\bot^*$ as follows:

$$[\![\phi \mathbin{\overset{*}{\triangledown}} \psi]\!] = [\![\phi]\!] \mathbin{\overset{*}{\triangledown}} [\![\psi]\!] \quad [\![\phi \overset{*}{\setminus} \psi]\!] = [\![\phi]\!] \overset{*}{\setminus} [\![\psi]\!] \quad [\![\bot^*]\!] = \bot^*.$$

Soundness follows once again by a Lindenbaum-Tarski construction.

Theorem 6.48 (Algebraic Soundness of Subclassical Bunched Logics)**.** *If $\varphi \vdash \psi$ is provable in the Hilbert system for Bi(B)BI + Σ, then for all interpretations $[\![-]\!]$ on Bi(B)BI + Σ algebras, $[\![\varphi]\!] \leq [\![\psi]\!]$.* □

We now set up the basic duality theory for these structures.

Definition 6.49 (Complex Bi(B)BI Algebra)**.** *Given a Bi(B)BI frame $\mathscr{X}$ the complex algebra of $\mathscr{X}$, $\mathrm{Com}^{\mathrm{Bi(B)BI}}(\mathscr{X})$, is given by extending $\mathrm{Com}^{\mathrm{(B)BI}}(\mathscr{X})$ with $\overline{U}$, together with $\blacktriangledown_{\mathscr{X}}$ and $\blacktriangledown\!\!\!\backslash_{\mathscr{X}}$ defined*

$$\begin{aligned} A \blacktriangledown_{\mathscr{X}} B &= \{x \mid \textit{for all } s,t,u, x \preccurlyeq s \in t \triangledown u \textit{ implies } t \in A \textit{ or } u \in B\} \\ A \blacktriangledown\!\!\!\backslash_{\mathscr{X}} B &= \{x \mid \textit{there exists } s,t,u \textit{ s.t. } x \succcurlyeq s, u \in t \triangledown s, u \in A \textit{ and } t \notin B\} \end{aligned}$$

Lemma 6.50. *1. Given a basic Bi(B)BI frame $\mathscr{X}$, $\mathrm{Com}^{\mathrm{Bi(B)BI}}(\mathscr{X})$ is a basic Bi(B)BI algebra.*

2. If $\mathscr{X}$ satisfies any frame property of Figure 6.1, $\mathrm{Com}^{\mathrm{Bi(B)BI}}(\mathscr{X})$ satisifies the corresponding axiom.

Proof. 1. is straightforward. For 2. we focus on the case of weak distributivity for BiBI frames, which collapses to the BiBBI variant when $\succcurlyeq$ is $=$. Let $t' \in A \bullet_{\mathscr{X}} (B \blacktriangledown_{\mathscr{X}} C)$. Then $t' \succcurlyeq t \in x_1 \circ x_2$ for some $x_1 \in A$ and $x_2 \in B \blacktriangledown_{\mathscr{X}} C$. Suppose $t' \preccurlyeq t'' \in y_1 \triangledown y_2$. We must show $y_1 \in A \bullet_{\mathscr{X}} B$ or $y_2 \in C$. Suppose $y_2 \notin C$. By the weak distributivity frame property, there exists w such that $y_1 \in x_1 \circ w$ and $x_2 \in w \triangledown y_2$. Since $y_2 \notin C$ and $x_2 \in B \blacktriangledown_{\mathscr{X}} C$ it follows that $w \in B$. Hence $y_1 \in A \bullet_{\mathscr{X}} B$ as required, and so $t' \in (A \bullet_{\mathscr{X}} B) \blacktriangledown_{\mathscr{X}} C$. □

Definition 6.51 (Prime Filter Bi(B)BI Frame)**.** *Given a Bi(B)BI algebra $\mathbb{A}$, the prime filter frame of $\mathbb{A}$, $\mathrm{Pr}^{\mathrm{Bi(B)BI}}(\mathbb{A})$ is given by extending $\mathrm{Pr}^{\mathrm{(B)BI}}(\mathbb{A})$ with the op-*

eration $\nabla_{\mathbb{A}}$*, defined*

$$F \nabla_{\mathbb{A}} F' = \{F'' \mid \forall a,b \in A : a \mathbin{\mathring{\vee}} b \in F'' \text{ implies } a \in F \text{ or } b \in F'\}$$

and the set $U_{\mathbb{A}} = \{F \mid \bot^* \notin F\}$.

Lemma 6.52. *1. Given a basic Bi(B)BI algebra* $\mathbb{A}$*,* $Pr^{Bi(B)BI}(\mathbb{A})$ *is a basic Bi(B)BI frame.*

2. If $\mathbb{A}$ *satisfies any axiom of Figure 6.1,* $Pr^{Bi(B)BI}(\mathbb{A})$ *satisfies the corresponding frame property.*

Proof. Once again we restrict ourselves to the non-trivial 2. We focus on the Weak Distributivity property for BiBI. Suppose $F_{t'} \supseteq F_t \in F_{x_1} \circ_{\mathbb{A}} F_{x_2}$ and $F_{t'} \subseteq F_{t''} \in F_{y_1} \nabla_{\mathbb{A}} F_{y_2}$. We show that

$$P(F) = \begin{cases} 1 & \text{if } F_{y_1} \in F_{x_1} \circ_{\mathbb{A}} F \text{ and } F_{x_2} \in F \nabla_{\mathbb{A}} F_{y_2} \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. First suppose $P(F_\alpha) = 1$ for all α in a $\subseteq$-chain $(F_\alpha)_{\alpha<\lambda}$. Then clearly $F_{y_1} \in F_{x_1} \circ_{\mathbb{A}} \bigcup_\alpha F_\alpha$. Suppose $a \mathbin{\mathring{\vee}} b \in F_{x_2}$ and $b \notin F_{y_2}$. Then necessarily $a \in F_\alpha$ for all α, so $F_{x_2} \in \bigcup_\alpha F_\alpha \nabla_{\mathbb{A}} F_{y_2}$. Now let $P(F \cap F') = 1$. If $F_{x_2} \in (F \cap F') \nabla_{\mathbb{A}} F_{y_2}$ it follows that $F_{x_2} \in F \nabla_{\mathbb{A}} F_{y_2}$ and $F_{x_2} \in F' \nabla_{\mathbb{A}} F_{y_2}$, so assume $F_{y_1} \notin F_{x_1} \circ_{\mathbb{A}} F, F_{x_1} \circ_{\mathbb{A}} F'$. Then there exists $a, a' \in F_{x_1}$, $b \in F$ and $b' \in F'$ such that $a * b, a' * b' \notin F_{y_1}$. We have that $a'' = a \wedge a' \in F_{x_1}$ and $b \vee b' \in F \cap F'$ so $a'' * (b \vee b') = (a'' * b) \vee (a'' * b') \in F_{x_1}$. F_{x_1} is prime so $a'' * b \in F_{x_1}$ or $a'' * b' \in F_{x_1}$. By monotonocity of $*$ and upwards-closure of F_{x_1}, $a * b \in F_{x_1}$ or $a' * b' \in F_{x_1}$, a contradiction. Hence either $P(F) = 1$ or $P(F') = 1$.

Now consider the set $F = \{b \mid \exists y \notin F_{y_2}(b \mathbin{\mathring{\vee}} d \in F_{x_2})\}$. We prove F is a proper filter. It is upwards-closed because $\mathbin{\mathring{\vee}}$ is monotonic: if $b \in F$ and $b' \geq b$ we have $d \notin F_{y_2}$ such that $b \mathbin{\mathring{\vee}} d \in F_{x_2}$ and $b \mathbin{\mathring{\vee}} d \leq b' \mathbin{\mathring{\vee}} d \in F_{x_2}$. To see it is closed under meets, suppose $b, b' \in F$. Then there exist $d, d' \notin F_{y_2}$ such that $b \mathbin{\mathring{\vee}} d$, $b' \mathbin{\mathring{\vee}} d' \in F_{x_2}$. F_{y_2} is prime so $d \vee d' \notin F_{y_2}$ and by montonocity of $\vee$, $b \mathbin{\mathring{\vee}} (d \vee d'), b' \mathbin{\mathring{\vee}} (d \vee d') \in F_{x_2}$. Let $d'' := d \vee d'$. By Proposition 6.46, $(b \wedge b') \mathbin{\mathring{\vee}} d'' = (b \mathbin{\mathring{\vee}} d'') \wedge (b' \mathbin{\mathring{\vee}} d'') \in F_{x_2}$. Finally, to see that F is proper, suppose $\bot \in F$. Then there exists $d \notin F_{x_2}$ such that $\bot \mathbin{\mathring{\vee}} d \in F_{x_2}$. Letting $a \in F_{x_1}$ be arbitrary, by Weak Distributivity and our assumption we have $a * (\bot \mathbin{\mathring{\vee}} d) \leq (a * \bot) \mathbin{\mathring{\vee}} d = \bot \mathbin{\mathring{\vee}} d \in F_t \subseteq F_{t'}$. Thus $\bot \mathbin{\mathring{\vee}} d \in F_{t'} \subseteq F_{t''}$ but $\bot \notin F_{y_1}$ and $d \notin F_{y_2}$, contradicting that $F_{t''} \in F_{y_1} \nabla_{\mathbb{A}} F_{y_2}$.

We finish the proof by showing that $P(F) = 1$, yielding the existence of a prime F_w satisfying the requirements of the frame property by the prime extension lemma.

First let $a \in F_{x_1}$ and $b \in F$. Then there exists $d \notin F_{y_2}$ such that $b \mathbin{\mathbb{V}^*} d \in F_{x_2}$. By Weak Distributivity $a * (b \mathbin{\mathbb{V}^*} d) \leq (a * b) \mathbin{\mathbb{V}^*} d \in F_t \subseteq F_{t'} \subseteq F_{t''}$, and since $d \notin F_{y_2}$ we necessarily have that $a * b \in F_{y_1}$. Now let $b \mathbin{\mathbb{V}^*} c \in F_{x_2}$ and suppose $c \notin F_{y_2}$. Then $b \in F$ by definition. □

Theorem 6.53 (Representation Theorem for Bi(B)BI + Σ Algebras)**.** *Every Bi(B)BI + Σ algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given an Bi(B)BI algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{Bi(B)BI}(Pr^{(B)BI}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr(\mathbb{A}) \mid a \in F\}$ is an embedding.*

Proof. The remaining verifications are that $\theta_{\mathbb{A}}$ respects $\mathbb{V}^*, \backslash^*$ and $\bot^*$. $\bot^*$ follows straightforwardly because $\theta_{\mathbb{A}}(\bot^*) = \overline{U_{\mathbb{A}}}$, and we verify $\mathbb{V}^*$ leaving the similar $\backslash^*$ to the reader. We must show $\theta_{\mathbb{A}}(a \mathbin{\mathbb{V}^*} b) = \theta_{\mathbb{A}}(a) \blacktriangledown_{Pr^{Bi(B)BI}(\mathbb{A})} \theta_{\mathbb{A}}(a)$. First suppose $a \mathbin{\mathbb{V}^*} b \in F$. Then $F \subseteq F_s \in F_t \triangledown_{\mathbb{A}} F_u$ means $a \mathbin{\mathbb{V}^*} b \in F_s$ and so either $a \in F_t$ or $b \in F_u$ as required.

In the other direction, suppose $a \mathbin{\mathbb{V}^*} b \notin F$. We show that

$$P(I, I') = \begin{cases} 1 & \text{if } F \in \bar{I} \triangledown_{\mathbb{A}} \overline{I'}, a \in I \text{ and } b \in I' \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate for proper ideals I, I'. First suppose we have a $\subseteq$-chain $(I_\alpha, I'_\alpha)_\alpha$ such that $P(I_\alpha, I'_\alpha) = 1$ for all α. Clearly $a \in \bigcup_\alpha I_\alpha$ and $b \in \bigcup_\alpha I'_\alpha$. Suppose $c \mathbin{\mathbb{V}^*} d \in F$ with $c \notin \overline{\bigcup_\alpha I_\alpha}$ and $d \notin \overline{\bigcup_\alpha I'_\alpha}$. Then there exists β, β' such that $c \in I_\beta$ and $d \in I'_{\beta'}$. By assumption we must have $c \in \overline{I_{\beta'}}$ and $d \in \overline{I'_\beta}$, and *wolog* we may assume $\beta \leq \beta'$. Then, because $I_\beta \subseteq I_{\beta'}$ we have $c \in \overline{I_{\beta'}} \subseteq \overline{I_\beta}$, a contradiction.

Now suppose $P(I_0 \cap I_1, I') = 1$. We have that $a \in I_0, I_1$ and $b \in I'$ so suppose both $P(I_0, I') = 0$ and $P(I_1, I') = 0$. Then there exists $c \mathbin{\mathbb{V}^*} d, c' \mathbin{\mathbb{V}^*} d' \in F$ such that $c \notin \overline{I_0}$, $d \notin \overline{I'}$, $c' \notin \overline{I_1}$ and $d' \notin \overline{I'}$. It folows that $d'' := d \vee d' \in I'$ and $c \wedge c' \in I_0 \cap I_1$. By upwards-closure and monotonicity of $\mathbb{V}^*$, $c \mathbin{\mathbb{V}^*} d''$, $c' \mathbin{\mathbb{V}^*} d'' \in F$. Hence by Proposition 6.46 $(c \wedge c') \mathbin{\mathbb{V}^*} d'' = (c \mathbin{\mathbb{V}^*} d'') \wedge (c' \mathbin{\mathbb{V}^*} d'') \in F$. However $c \wedge c' \notin \overline{I_0 \cap I_1}$ and $d'' \notin \overline{I'}$, contradicting that $F \in \overline{I_0 \cap I_1} \triangledown_{\mathbb{A}} \overline{I'}$. Thus P is a prime predicate.

Now consider the ideals $(a]$ and $(b]$. These must be proper as if $\top = a$ or b then $a \mathbin{\mathbb{V}^*} b = \top \in F$, contradicting our assumption. We also have that for any $c \mathbin{\mathbb{V}^*} d \in F$, if $c \leq a$ and $d \leq b$ we have $c \mathbin{\mathbb{V}^*} d \leq a \mathbin{\mathbb{V}^*} b \in F$, a contradiction. Hence $F \in \overline{(a]} \triangledown_{\mathbb{A}} \overline{(b]}$ and $P((a], (b]) = 1$, yielding the necessary prime filters by taking the complements of the prime ideals guaranteed to exist by the prime extension lemma. □

The assignment of objects by $Pr^{\text{Bi(B)BI}}$ and $Com^{\text{Bi(B)BI}}$ lifts to functors in a way that is now standard. That the assignment of morphisms gives morphisms in the respective categories is proved in a similar way to previous cases.

Lemma 6.54. *The functors $Pr^{Bi(B)BI}$ and $Com^{Bi(B)BI}$ is well-defined.* □

We introduce topology with the following definitions. As in the case for the topological separation property for $\circ$, property 5. of these definitions is specified by Bímbo & Dunn's topological separation properties for gaggles [25].

Definition 6.55 ($BiBI_\Sigma$ Space)**.** *Let Σ be a set of subclassical bunched logic axioms. A $BiBI_\Sigma$* space *is a structure $\mathscr{X} = (X, \mathscr{O}, \succcurlyeq, \circ, E, \triangledown, U)$ such that*

1. *$(X, \mathscr{O}, \succcurlyeq, \circ, E)$ is a BI space,*
2. *$(X, \succcurlyeq, \circ, E, \triangledown, U)$ is a basic BiBI frame satisfying the frame correspondents of Σ;*
3. *The upwards-closed clopen sets of $(X, \mathscr{O}, \succcurlyeq)$ are closed under $\blacktriangledown_{\mathscr{X}}$ and $\blacktriangledown\!\!\!\backslash_{\mathscr{X}}$,*
4. *U is clopen; and*
5. *If $x \notin y \triangledown z$ then there exists upwards-closed clopen sets C_1, C_2 such that $y \notin C_1$, $z \notin C_2$ and $x \in C_1 \blacktriangledown_{\mathscr{X}} C_2$.*

Definition 6.56 ($BiBBI_\Sigma$ Space)**.** *Let Σ be a set of subclassical bunched logic axioms. A $BiBBI_\Sigma$* space *is a structure $\mathscr{X} = (X, \mathscr{O}, \circ, E, \triangledown, U)$ such that*

1. *$(X, \mathscr{O}, \circ, E)$ is a BBI space,*
2. *$(X, \circ, E, \triangledown, U)$ is a basic BiBI frame satisfying the frame correspondents of Σ;*
3. *The clopen sets of $(X, \mathscr{O})$ are closed under $\blacktriangledown_{\mathscr{X}}$ and $\blacktriangledown\!\!\!\backslash_{\mathscr{X}}$;*
4. *U is clopen; and*
5. *If $x \notin y \triangledown z$ then there exists clopen sets C_1, C_2 such that $y \notin C_1$, $z \notin C_2$ and $x \in C_1 \blacktriangledown_{\mathscr{X}} C_2$.*

Taking continuous Bi(B)BI morphisms as morphisms we obtain the categories $\mathrm{Bi(B)BISp}_\Sigma$ for each axiom set Σ. To see that the prime filter space associated to a Bi(B)BI algebra $\mathbb{A}$ is a Bi(B)BI space it is sufficient to note that the topological separation property 5. holds: if $F_x \notin F_y \triangledown_{\mathbb{A}} F_z$ we have that there exists $a \mathbin{\overset{*}{\triangledown}} b \in F_x$ such that $a \notin F_y$ and $b \notin F_z$. Then the upwards-closed clopen sets $\theta_{\mathbb{A}}(a)$ and $\theta_{\mathbb{A}}(b)$ suffice to show the property holds. In the other direction, conditions 1. to 4. ensure that the upwards-closed clopen sets carry the structure of a basic Bi(B)BI algebra satisfying the subclassical axioms of Σ. Defining η as before we obtain the duality theorem for BiBI algebras satisfying Σ.

Theorem 6.57 (Duality for BiBI + Σ Algebras)**.** *θ and η form a dual equivalence of categories between* $\mathrm{BiBIAlg}_\Sigma$ *and* BiBISp_Σ.

Proof. Once again, showing the components of η are isomorphisms is all that remains, given our previous work. We simply have to show $\eta_{\mathscr{X}}$ is a relational isomorphism for $\triangledown$. Let $x \in y \triangledown z$ and suppose $x \in C \blacktriangledown_{\mathscr{X}} C'$ for upward-closed clopen sets C and C'. Since $x \in y \triangledown z$ it follows that $y \in C$ or $z \in C'$. Hence $\eta_{\mathscr{X}}(x) \in \eta_{\mathscr{X}}(y) \triangledown_{Clop^{\mathrm{Bi(B)BI}}_{\preccurlyeq}(\mathscr{X})} \eta_{\mathscr{X}}(z)$. If $x \notin y \triangledown z$, by the separation property 5. we have that $\eta_{\mathscr{X}}(x) \notin \eta_{\mathscr{X}}(y) \triangledown_{Clop^{\mathrm{Bi(B)BI}}_{\preccurlyeq}(\mathscr{X})} \eta_{\mathscr{X}}(z)$. □

As a special case of this duality we obtain duality for BiBBI + Σ algebras.

Theorem 6.58 (Duality for BiBBI + Σ Algebras)**.** *θ and η form a dual equivalence of categories between* $\mathrm{BiBBIAlg}_\Sigma$ *and* $\mathrm{BiBBISp}_\Sigma$. □

6.4.2 Concurrent Kleene Bunched Logic

We finish by attending to CKBI, introduced in Chapter 4 Section 4.4. We begin with the algebraic structures suitable for interpreting CKBI, which unsurprisingly share some resemblence to concurrent Kleene algebra.

Definition 6.59 (CKBI Algebra)**.** *A* CKBI algebra *is an algebra* $\mathbb{A} = (A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*, ;, \mathrel{-\!\!\triangleright}, \mathrel{\triangleright\!\!-})$ *such that* $(A, \wedge, \vee, \to, \top, \bot, *, \mathrel{-\!\!*}, \top^*)$ *is a BBI algebra and* $(A, ;, \top^*)$ *a monoid, satisfying, for all* $a, b, c, d \in \mathbb{A}$*,*

1. $a ; b \leq c$ *iff* $a \leq b \mathrel{-\!\!\triangleright} c$ *iff* $b \leq a \mathrel{\triangleright\!\!-} c$*, and*
2. *Exchange:* $(a * b) ; (c * d) \leq (a ; c) * (b ; d)$.

In effect, a CKBI algebra is a BBI algebra in which there are *two* coexisting monoidal residuated structures sharing a unit: one commutative (corresponding to concurrent execution) and one non-commutative (corresponding to sequential execution). As such the corresponding properties of Proposition 6.2 hold for $;, \mathrel{-\!\!\triangleright}$ and $\mathrel{\triangleright\!\!-}$. In the terminology of O'Hearn et al. [178], a CKBI algebra is a Boolean CKA extended with the residuals corresponding to $*$ and $;$. Using the obvious extension of interpretations on BBI algebras to CKBI algebras, we have the following algebraic soundness theorem.

Theorem 6.60 (Algebraic Soundness of CKBI)**.** *If $\varphi \vdash \psi$ is provable in the Hilbert system for CKBI, then for all interpretations $[\![-]\!]$ on CKBI algebras, $[\![\varphi]\!] \leq [\![\psi]\!]$.* □

We recall the definition of CKBI frame from Chapter 4: a *CKBI frame* is a structure $\mathscr{X} = (X, \circ, E, \triangleright)$ such that $(X, \circ, E)$ is a BBI frame and $\triangleright : X^2 \to \mathscr{P}(X)$ a

binary operation satisfying (with outermost quantification omitted for readability):

(Unit Existence$_{\mathrm{L}}$)	$\exists e \in E(x \in e \rhd x)$;
(Unit Existence$_{\mathrm{R}}$)	$\exists e \in E(x \in x \rhd e)$;
(Coherence$_{\mathrm{L}}$)	$e \in E \wedge x \in e \rhd y \to x = y$;
(Coherence$_{\mathrm{R}}$)	$e \in E \wedge x \in y \rhd e \to x = y$;
(Associativity)	$\exists t(t \in x \rhd y \wedge w \in t \rhd z) \leftrightarrow \exists t'(t' \in y \rhd z \wedge w \in x \rhd t')$
(Exchange)	$t \in w \circ y \wedge s \in x \circ z \wedge u \in t \rhd s \to$ $\exists r, v(r \in w \rhd x \wedge v \in y \rhd z \wedge u \in r \circ u)$

Let $\mathbb{A}$ be a CKBI algebra. Then the prime filter frame of $\mathbb{A}$, $Pr^{\mathrm{CKBI}}(\mathbb{A})$, is given by extending the prime filter frame of the underlying BBI algebra with the operation $\rhd_{\mathbb{A}}$, defined $F \rhd_{\mathbb{A}} F' = \{F'' \mid \forall a \in F, \forall b \in F' : a \ ; \ b \in F''\}$. In the other direction, the complex algebra of a CKBI frame $\mathscr{X}$, $Com^{\mathrm{CKBI}}(\mathscr{X})$, is given by extending the complex algebra of the underlying BBI frame with the operation $A \ ;_{\mathscr{X}} B = \{z \mid \exists x \in A, y \in B(z \in x \rhd y)\}$ and its associated adjoints. The respective results for CKBI algebras follow straightforwardly from the case for BBI: the key remaining step is the correspondence between the algebraic Exchange axiom and the frame property Exchange.

Lemma 6.61.

1. *Given a CKBI algebra $\mathbb{A}$, the prime filter frame $Pr^{CKBI}(\mathbb{A})$ is a CKBI frame.*
2. *Given a CKBI frame $\mathscr{X}$, the complex algebra $Com^{CKBI}(\mathscr{X})$ is a CKBI algebra.*

Proof. We focus on the correspondence between the Exchange properties of the respective structures.

1. Suppose we have prime filters of $\mathbb{A}$ satisfying $F_{wy} \in F_w \circ_{\mathbb{A}} F_y, F_{xz} \in F_x \circ_{\mathbb{A}} F_z$ and $F_t \in F_{wy} \rhd_{\mathbb{A}} F_{xz}$. Using similar arguments to those given in previous results, it can be seen that

$$P(F,G) = \begin{cases} 1 & \text{if } F \in F_w \rhd_{\mathbb{A}} F_x, G \in F_y \rhd_{\mathbb{A}} F_z \text{ and } F_t \in F \circ_{\mathbb{A}} G \\ 0 & \text{otherwise} \end{cases}$$

   is a prime predicate on proper filters F and G. Consider the sets $F = \{c \mid \exists a \in F_w, b \in F_x(a \ ; \ b \leq c)\}$ and $G = \{c \mid \exists a \in F_y, b \in F_z(a \ ; \ b \leq c)\}$. Both sets are obviously upwards-closed and closed under meets as monotonicity of $;$ gives that $a \ ; \ b \leq c$ and $a' \ ; \ b' \leq c'$ implies $(a \wedge a') \ ; \ (b \wedge b') \leq c \wedge c'$. Hence F and G are filters. They are also proper: suppose for contradiction

that $\bot \in F$. Then there exists $a \in F_w$ and $b \in F_x$ such that $a \mathbin{;} b = \bot$. Let $c \in F_y$ and $d \in F_z$ be arbitrary. By assumption we have that $a * c \in F_{wy}$, $b * d \in F_{xz}$ and so $(a * c) \mathbin{;} (b * d) \in F_t$. By Exchange and upwards-closure of filters, $(a \mathbin{;} b) * (c \mathbin{;} d) = \bot * (c \mathbin{;} d) = \bot \in F_t$, a contradiction. The same argument suffices to show G is proper.

Clearly $F \in F_w \rhd_{\mathbb{A}} F_x$ and $G \in F_y \rhd_{\mathbb{A}} F_z$. Further, $F_t \in F \circ_{\mathbb{A}} G$: let $c \geq a \mathbin{;} b$ and $c' \geq a' \mathbin{;} b'$ for $a \in F_w, b \in F_x, a' \in F_y$ and $b' \in F_z$. By monotonicity of $*$ and Exchange, $(a * a') \mathbin{;} (b * b') \leq (a \mathbin{;} b) * (a' \mathbin{;} b') \leq c * c'$. It then follows that $c * c' \in F_t$, since by assumption $a * a' \in F_{wy}$ and $b * b' \in F_{xz}$, so $(a * a') \mathbin{;} (b * b') \in F_t$. Hence by the prime extension lemma there exist prime F and G satisfying these properties, and so the frame property Exchange is satisfied on $Pr^{\mathrm{CKBI}}(\mathbb{A})$.

2. Suppose $t \in (A \bullet_{\mathscr{X}} C) \mathbin{;}_{\mathscr{X}} (B \bullet_{\mathscr{X}} D)$. Then there exist w, x, y, z, wy, xz such that $wy \in w \circ y, xz \in x \circ z$ and $t \in wy \rhd xz$. The frame property Exchange then ensures there are witnesses to the fact that $t \in (A \mathbin{;}_{\mathscr{X}} B) \bullet_{\mathscr{X}} (C \mathbin{;}_{\mathscr{X}} D)$.

□

We immediately obtain the following representation theorem from the representation theorem for BBI algebras.

Theorem 6.62 (Representation Theorem for CKBI Algebras)**.** *Every CKBI algebra is isomorphic to a subalgebra of a complex algebra. Specifically, given a CKBI algebra $\mathbb{A}$, the map $\theta_{\mathbb{A}} : \mathbb{A} \to Com^{CKBI}(Pr^{CKBI}(\mathbb{A}))$ defined $\theta_{\mathbb{A}}(a) = \{F \in Pr^{CKBI}(\mathbb{A}) \mid a \in F\}$ is an embedding.* □

We also immediately obtain the lifting of these assignments to functors, by defining a CKBI morphism to be a BBI morphism that also satisfies the corresponding LGL morphism properties for the sequential composition $\rhd$. To obtain a duality we specify CKBI spaces and conclude from BBI duality and Lemma 6.61.

Definition 6.63 (CKBI Space)**.** *A* CKBI space *is a structure $\mathscr{X} = (X, \mathscr{O}, \circ, \rhd, E)$ such that*

1. *$(X, \mathscr{O}, \circ, E)$ is a BBI space,*
2. *$(X, \circ, \rhd, E)$ is a CKBI frame,*
3. *The clopen sets of $(X, \mathscr{O})$ are closed under $\mathbin{;}_{\mathscr{X}}$, $\mathrel{-\!\!\rhd}_{\mathscr{X}}$ and $\rhd\!\!\mathrel{-}_{\mathscr{X}}$, and*
4. *If $x \notin y \rhd z$ then there exists clopen sets C_1, C_2 such that $y \in C_1$, $z \in C_2$ and $x \notin C_1 \mathbin{;}_{\mathscr{X}} C_2$.*

Theorem 6.64 (CKBI Duality)**.** *θ and η form a dual equivalence of categories between* CKBIAlg *and* CKBISp. □

Chapter 7

Metatheory for Propositional Bunched Logics

In this chapter we apply some of the theory of Chapter 6 to prove metatheory for propositional bunched logics. First, we establish the completeness of the algebraic and frame semantics of all the bunched logics treated in the thesis simultaneously using the representation theorems and complex algebra constructions. The argument utilises the fact that the algebraic and frame semantics are equivalent, and so we are able to freely transfer between proof theoretic, algebraic and semantic arguments in what follows. A first application is the decidability of the layered graph logics using an algebraic construction that produces finite countermodels to invalid entailments. Next we give a characterisation of the classes of bunched logic frames that can be defined by bunched logic formulae by using the duality theory of Chapter 6 to prove an analogue of the Goldblatt-Thomason theorem. Using this we prove that a number of interesting classes of bunched logic frames are undefinable in the corresponding bunched logics. Finally we consider the question of Craig interpolation for bunched logics.

Sections 7.1 and 7.2 are comprised of material from the journal papers *Stone-Type Dualities for Separation Logics* [83] and *Intuitionistic Layered Graph Logic: Semantics and Proof Theory* [82].

7.1 Completeness

We begin with completeness of algebraic and frame semantics for bunched logics. Throughout we use the notation $\mathscr{L}$ as a stand in for any of the propositional bunched logics under consideration in this thesis. We first note that the constructions of prime filter frame and complex algebra relate the two semantic approaches we have considered in a particularly strong way.

Theorem 7.1 (Equivalence of Algebraic and Frame Semantics)**.** *Let* $\mathbb{A}$ *be a* $\mathscr{L}$

algebra, $\mathscr{X}$ a $\mathscr{L}$ frame, $[\![-]\!]$ an algebraic interpretation on $\mathbb{A}$ and $\mathscr{V}$ a persistent valuation on $\mathscr{X}$. Define $\mathscr{V}_{[\![-]\!]} : \mathrm{Prop} \to \mathscr{P}(Pr(A))$ by $\mathscr{V}_{[\![-]\!]}(\mathrm{p}) = \theta_{\mathbb{A}}([\![\mathrm{p}]\!])$ and $[\![-]\!]_{\mathscr{V}}$ as the algebraic interpretation on $Com^{\mathscr{L}}(\mathscr{X})$ generated by $\mathscr{V}$. For all $\mathscr{L}$ formulae φ the following hold.

1. *$x \vDash_{\mathscr{V}} \varphi$ iff $x \in [\![\varphi]\!]_{\mathscr{V}}$;*

2. *$[\![\varphi]\!] \in F$ iff $F \vDash_{\mathscr{V}_{[\![-]\!]}} \varphi$.*

Proof. To see that 1. holds, note that the defining clauses of the complex algebra operations correspond precisely to the semantic clauses of $\mathscr{L}$. The result then immediately obtains, using the definition of $\mathscr{V}$ and by applying the inductive hypothesis appropriately. For 2., note that the interpretation $[\![-]\!]_{\mathscr{V}_{[\![-]\!]}}$ on $Com^{\mathscr{L}} Pr^{\mathscr{L}}(\mathbb{A})$ generated by $\mathscr{V}_{[\![-]\!]}$ is precisely $\theta_{\mathbb{A}}$. We thus have that $[\![\varphi]\!] \in F$ iff $F \in \theta_{\mathbb{A}}([\![\varphi]\!]) = [\![\varphi]\!]_{\mathscr{V}_{[\![-]\!]}}$ iff $F \vDash_{\mathscr{V}_{[\![-]\!]}} \varphi$. □

We are able to prove completeness of both semantics for all $\mathscr{L}$ simultaneously by proving completeness of the algebraic semantics. This is done by utilising a *Lindenbaum-Tarski* construction on the formuale of $\mathscr{L}$. We define an equivalence relation $\equiv$ on $\mathscr{L}$ formulae built from by $\varphi \equiv \psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are provable. Call the set of equivalence classes $[\varphi]_{\equiv}$ of $\mathscr{L}$ formulae $\mathscr{L} - LT(\mathrm{Prop})$. This can be given the structure of a $\mathscr{L}$ algebra by setting $\mathrm{I} = [\mathrm{I}]_{\equiv}$ for all constants I, $\Diamond[\varphi]_{\equiv} = [\Diamond\varphi]_{\equiv}$ for all unary connectives $\Diamond$ and $[\varphi]_{\equiv} \heartsuit [\psi]_{\equiv} = [\varphi \heartsuit \psi]_{\equiv}$ for all binary connectives $\heartsuit$. That these operations are well-defined on the equivalence classes and that the axioms of the respective $\mathscr{L}$ algebras are satisfied can be verified by direct examination of the Hilbert system rules.

Theorem 7.2 (Algebraic Completeness (cf. [189]))**.** *For all bunched logics $\mathscr{L}$ and $\mathscr{L}$ formulae φ and ψ, if $[\![\varphi]\!] \leq [\![\psi]\!]$ holds for all algebraic interpretations then $\varphi \vdash \psi$ is provable in the Hilbert system for $\mathscr{L}$.*

Proof. We reason contrapositively: suppose $\varphi \vdash \psi$ is not provable in the Hilbert system for $\mathscr{L}$. Consider the Lindenbaum-Tarski algebra over the propositional variables occurring in φ and ψ with the interpretation given by sending each formula to its equivalence class. $[\varphi]_{\equiv} \leq [\psi]_{\equiv}$ iff $[\varphi \to \psi]_{\equiv} = [\top]_{\equiv}$ iff $\top \vdash \varphi \to \psi$ provable iff $\varphi \vdash \psi$ (using the definition of Heyting implication and the deduction theorem for bunched logics), and so $[\varphi]_{\equiv} \not\leq [\psi]_{\equiv}$. □

Theorem 7.3 (Frame Completeness)**.** *For all bunched logics $\mathscr{L}$ and $\mathscr{L}$ formulae φ and ψ, if $\varphi \vDash \psi$ then $\varphi \vdash \psi$ is provable in the Hilbert system for $\mathscr{L}$.*

Proof. Suppose $\varphi \vDash \psi$ and suppose for contradiction that $\varphi \vdash \psi$ is not provable in the Hilbert system for $\mathscr{L}$. Then in the Lindenbaum-Tarski algebra over the propositional variables occuring in φ and ψ we have $[\varphi]_{\equiv} \not\leq [\psi]_{\equiv}$. There thus exists a prime filter F such that $[\varphi]_{\equiv} \in F$ and $[\psi]_{\equiv} \notin F$ by the prime filter theorem. Consider the interpretation sending formulae to their equivalence class. From this we obtain a valuation $\mathscr{V}$ on the prime filter frame, and by Theorem 7.1 we have that $F \vDash \varphi$ and $F \nvDash \psi$, contradicting our assumption. □

While completeness theorems have been given for the frame semantics of LGL [136], (B)BI [101, 99], CBI [39] and the subclassical bunched logics with classical additives [45] the completeness theorems for ILGL, DMBI, the separating modal logics and the subclassical bunched logics with intuitionistic additives are new. The completeness arguments for CBI and subclassical bunched logic with classical additives proceeded by translations into equivalent Sahlqvist-definable [202] modal logics in a manner that used Boolean negation in an essential way and as such could not be relativised to the evident intuitionistic variants. The method utilised here gives an argument suitable for either variants of bunched logic and as such should streamline the production of completeness theorems for future extensions.

7.2 Decidability

In this section we prove the decidability of layered graph logics using algebraic methods. To do so we exhibit a finite countermodel of a bounded size for every consequence that does not hold in each logic. We can therefore decide the logic by exhibiting every finite (I)LGL algebra (itself a finitely axiomatised structure) up to a given size and checking if each interpretation on each algebra witnesses the corresponding inequality or not. To prove the existence of such finite algebras we utilise a method that is widespread in the study of substructural logic and universal algebra: verification of the *finite embeddability property* [28].

Definition 7.4 (Finite Embeddability Property)**.** *A class of algebras $\mathscr{K}$ has the* finite embeddability property *(FEP) if, for any algebra $\mathbb{A} \in \mathscr{K}$ and any finite subset $B \subseteq A$ there exists a finite algebra $\mathbb{C} \in \mathscr{K}$ and an injective map $g : B \to C$ such that for all algebraic operations f, if $b_1, \ldots, b_n \in B$ and $f_{\mathbb{A}}(b_1, \ldots, b_n) \in B$ then $g(f_{\mathbb{A}}(b_1, \ldots, b_n)) = f_{\mathbb{C}}(g(b_1), \ldots, g(b_n))$.* □

Intuitively, the FEP states that every finite *partial* subalgebra can be completed as a finite algebra. If the class of (I)LGL algebras has the FEP then (I)LGL has a finite model property and is thus decidable by the following argument. Suppose the algebra $\mathbb{A}$ with interpretation $[\![-]\!]$ witnesses that $\varphi \vdash \psi$ does not hold: that is, $[\![\varphi]\!] \not\leq_{\mathbb{A}} [\![\psi]\!]$. Set $B = \{[\![\chi]\!] \mid \chi \text{ a subformula of } \varphi \to \psi\} \cup \{\top_{\mathbb{A}}, \bot_{\mathbb{A}}\}$. By the FEP,

we obtain a finite algebra $\mathbb{C}$ and injective map $g : B \to \mathbb{C}$, yielding an interpretation $\widetilde{[\![-]\!]}$ generated by setting $\widetilde{[\![\mathrm{p}]\!]} = g([\![\mathrm{p}]\!])$ for all propositional atoms p occurring in $\varphi \to \psi$. As g is injective and preserves existing algebraic operations in B this gives that $\widetilde{[\![\varphi \to \psi]\!]} <_{\mathbb{C}} \top_{\mathbb{C}}$. Since in (I)LGL algebras $a \leq b$ iff $a \to b = \top$, we have a finite algebra $\mathbb{C}$ and interpretation $\widetilde{[\![-]\!]}$ such that $\widetilde{[\![\varphi]\!]} \not\leq \widetilde{[\![\psi]\!]}$, witnessing that $\varphi \vdash \psi$ does not hold. As (I)LGL algebras are finitely axiomatized this yields decidability of the logic.

We adapt an argument given by Haniková & Horčík [113] to prove the class of bounded residuated distributive-lattice ordered groupoids – that is, the class of partially ordered sets with a binary operation, such that the partial order is a distributive lattice and the binary operation has left and right residuals with respect to the order – has the FEP. If such an algebra supports Heyting implication, then it is a ILGL algebra. Thus we simply have to additionally account for Heyting implication to make the desired proof go through. Independently of this work, the finite model property has recently been proved for (I)LGL in the guise of *Lambek calculus extended with (intuitionistic) classical propositional logic* by Kaminski and Francez [136, 137] by a filtration method. We also recently discovered that the FEP has also been shown for (I)LGL algebras in the guise of *(Heyting) Boolean residuated algebras* by Buszkowski [47]. That proof uses a long and complex proof theoretic argument of which the Boolean and Heyting variants are not considered as the primary case; in contrast, our proof is extremely simple and direct.

Theorem 7.5 (cf. [113])**.** *The class of ILGL algebras has the FEP.*

Proof. Let $\mathbb{A}$ be a ILGL algebra and $B \subseteq \mathbb{A}$ a finite subset that, *wlog*, contains $\top_{\mathbb{A}}$ and $\bot_{\mathbb{A}}$. Denote by $(C, \wedge_{\mathbb{C}}, \vee_{\mathbb{C}}, \top_{\mathbb{C}}, \bot_{\mathbb{C}})$ the distributive sublattice of the distributive lattice reduct of $\mathbb{A}$ generated by B. As B was finite, so too is C. Define $a \to_{\mathbb{C}} b = \bigvee_{\mathbb{C}} \{c \in C \mid a \wedge_{\mathbb{C}} c \leq_{\mathbb{C}} b\}$. Since each join is finite this is well-defined, and this makes $(C, \wedge_{\mathbb{C}}, \vee_{\mathbb{C}}, \to_{\mathbb{C}}, \top_{\mathbb{C}}, \bot_{\mathbb{C}})$ a Heyting algebra. It can be shown that if $b, b', b \to_{\mathbb{A}} b' \in B$ then $b \to_{\mathbb{A}} b' = b \to_{\mathbb{C}} b'$.

The rest of the proof now proceeds as in [113]. Define operations $\lambda, \sigma : A \to A$ by $\lambda(a) = \bigwedge_{\mathbb{C}} \{c \in C \mid a \leq_{\mathbb{A}} c\}$ and $\sigma(a) = \bigvee_{\mathbb{C}} \{c \in C \mid c \leq_{\mathbb{A}} a\}$. These are both well-defined because C is finite. It follows that, for $c \in C$, $\lambda(c) = c = \sigma(c)$. We then define $*_{\mathbb{C}}, \twoheadrightarrow_{\mathbb{C}}, \twoheadleftarrow_{\mathbb{C}}$ on C by $c *_{\mathbb{C}} c' = \lambda(c *_{\mathbb{A}} c')$, $c \twoheadrightarrow_{\mathbb{C}} c' = \sigma(c \twoheadrightarrow_{\mathbb{A}} c')$ and $c \twoheadleftarrow_{\mathbb{C}} c' = \sigma(c \twoheadleftarrow_{\mathbb{A}} c')$. The fact that λ is a closure operator and σ an interior operator can be used to show that the required residuation properties hold for these operations. We thus have a finite ILGL algebra $\mathbb{C} = (C, \wedge_{\mathbb{C}}, \vee_{\mathbb{C}}, \to_{\mathbb{C}}, \top_{\mathbb{C}}, \bot_{\mathbb{C}}, *_{\mathbb{C}}, \twoheadrightarrow_{\mathbb{C}}, \twoheadleftarrow_{\mathbb{C}})$, with the inclusion map of B into C satisfying the defining property of the FEP. □

Theorem 7.6 (Decidability of ILGL)**.** *The consequence relation $\vdash$ for ILGL is decidable.* □

The finite countermodel for an invalid consequence is bounded in size: for $\varphi \to \psi$ with n subformulae, the cardinality of the finite algebra is bounded by 2^{2^n}. Haniková & Horčík improve upon this by showing such an algebra can be represented by a poset of join-irreducibles of cardinality $m \leq 2^n - 2$ and a composition $\circ$ whose graph has cardinality m^3, where join-irreducibles are those elements that are not equal to $\bot$ and cannot be represented as the join of two distinct, non-$\bot$ elements.

The same argument also applies to LGL algebras, using the Boolean subalgebra generated from B rather than the generated distributive sublattice.

Theorem 7.7. *The class of LGL algebras has the FEP.* □

Theorem 7.8 (Decidability of LGL)**.** *The consequence relation $\vdash$ for LGL is decidable.* □

How much further can this argument be applied? It is noted by Haniková & Horčík [113] that their argument works for commutative $*$ but fails for associative $*$, ruling out an extension to the other bunched logics we consider. This is necessarily so, by the following result of Galatos & Jipsen [97].

Theorem 7.9 ([97])**.** *The class of BI algebras does not have the FEP.* □

Intriguingly, BI is known decidable by a finite tableau countermodel argument given by Galmiche et al [101]. An algebraic proof of decidability is claimed by Galatos & Jipsen [97] although a counterexample to their argument has been given by Ramanayake [191]. As such, it appears the existence of a syntactic/algebraic proof of BI remains an open problem. BBI is known to be undecidable: an algebraic proof was given by Kurucz et al [144] which remained unknown to the bunched logic community until the publication of Brotherston & Kanovich's [42] and Larchey-Wendling & Galmiche's [149] undecidability proofs. Kurucz et al's argument also highlights associativity of $*$ as a boundary for decidability. Brotherston & Kanovich went a step further than BBI undecidability, however, also showing CBI and validity in a number of propositional models of Separation Logic to be undecidable. As a corollary of the undecidability of BBI we have that the separating modal logics (which are conservative extensions of BBI) are necessarily undecidable. We collect these results in the following theorem.

Theorem 7.10 (Decidability for Bunched Logics)**.**

1. *ILGL, LGL and BI are decidable.*
2. *BBI is undecidable.*
3. *CBI is undecidable.*
4. *The separating modal logics are undecidable.* □

We also note that various fragments of separation logic have been shown to be decidable (for a survey, see [76]); in all cases this is by placing restrictions on permitted valuations, the number of variables and/or the connectives involved. One particularly important decidable fragment is the *symbolic heap* fragment, which has just enough of the separation logic connectives to specify heaps by formulae. This fragment has been utilised in program verification tools like Smallfoot [20], SpaceInvader [220] and SLAyer [21].

Returning to bunched logics, given the decidability of BI, decidability of DMBI is plausible given the weakness of the extended structure. It is not clear how to extend the existing proof of BI's finite model property to DMBI, however, so a syntactic/algebraic proof is even more desirable for the purpose of examining DMBI. Even more precise methods will be required to understand decidability of the subclassical bunched logics.

7.3 Expressivity

In this section we examine the expressivity of bunched logics via a variant of the celebrated Goldblatt-Thomason theorem for modal logic [109, 108]. That theorem gives sufficient and necessary conditions for a given first-order property of frames to be *definable* by a set of formulae; that is, any frame on which that set of formulae is valid satisfies the first-order property, and vice versa. To get there we first need a quick detour through universal algebra to collect the necessary notions.

Definition 7.11 (Equational Class)**.** *A class V of algebras of the same type is an* equational class *if there exists a set of equational axioms Σ (the* equational basis *for V) such that $\mathbb{A} \in V$ iff every equation in Σ holds in $\mathbb{A}$.*

For any bunched logic $\mathscr{L}$ the class of $\mathscr{L}$ algebras is an equational class. This is entirely clear except for the residuation property of $*$, $-\!\!*$ and $*\!\!-$. It can be seen that this can be expressed equationally, however, via the equational basis for residuated lattices [29]. The defining equations for the residuation property are

$$\begin{array}{lll} a = a \wedge (b -\!\!* ((a * b) \vee c)) & a * (b \vee c) = (a * b) \vee (a * c) & a = a \vee ((b -\!\!* a) * b) \\ b = b \wedge (a *\!\!- ((a * b) \vee c)) & (b \vee c) * a = (b * a) \vee (a * c) & a = a \vee (b * (b *\!\!- a)). \end{array}$$

This can also be straightforwardly adapted for the residuation property of $\mathbin{\overset{*}{\vee}}$ and $\overset{*}{\setminus}$ in the case of subclassical bunched logics.

Lemma 7.12 (*cf.* Proposition 4.1 [29])**.** *For each bunched logic $\mathscr{L}$, the class of $\mathscr{L}$ algebras is an equational class.*

Birkhoff [26] proved a powerful classification theorem for equational classes known as the HSP theorem, where H stands for Homomorphic image, S for Subalgebra, and P for Product.

Definition 7.13 (Variety)**.** *A class V of algebras of the same type is a* variety *if it is closed under taking homomorphic images, subalgebras and products.*

Theorem 7.14 (HSP Theorem [26])**.** *For a class V of algebras of the same type, V is a variety iff V is an equational class.* □

Of particular interest to us are varieties of complex algebras, a topic first investigated in the context of distributive lattices with operators by Goldblatt [108]. Given a class of $\mathscr{L}$ frames C, we can define a variety of complex algebras $V(C)$ using the equational basis $\Sigma_C = \{s = t \mid \text{ for all } \mathscr{X} \in C, Com^{\mathscr{L}}(\mathscr{X}) \vDash s = t\}$. By the HSP and representation theorems, this is closed under homomorphic images, subalgebras and products. Crucially, $V(C)$ witnesses the bunched logic formulae φ that are valid in C, since necessarily $\varphi = \top \in \Sigma_C$ iff $\mathscr{X} \vDash \varphi$ for all $\mathscr{X} \in C$.

Using this, together with the duality theory of $\mathscr{L}$ frames, we can use the HSP theorem to give necessary and sufficient conditions for a class of $\mathscr{L}$ frames to be definable by $\mathscr{L}$-formulae. This directly corresponds to the analogous theorem of modal logic [109], which can be relativised to the case of bunched logics using what we've proved thus far together with a few extra definitions and results. We first give a precise formulation of the notion of $\mathscr{L}$-definability.

Definition 7.15 ($\mathscr{L}$-definable (cf. [27])**.** *For a class of $\mathscr{L}$ frames C, C is* $\mathscr{L}$-definable *if there is a set of $\mathscr{L}$-formulae Γ such that $\mathscr{X} \in C$ iff $\mathscr{X} \vDash \varphi$ for all $\varphi \in \Gamma$.*

As an example of the kinds of classes of frames we have in mind, we might consider the class of (B)BI frames that correspond to particular memory models utilised in Separation Logic, with properties like deterministic and cancellative composition. We begin with some definitions that are adapted from Blackburn et al. [27]. We first require the auxillary notion of an ultraproduct. First note that for sets X_i indexed by I, the direct product $\prod_i X_i$ is defined $\prod_i X_i = \{f : I \to \bigcup_i X_i \mid \forall i : f(i) \in X_i\}$.

Definition 7.16 (Ultraproduct)**.** *Let I be a non-empty set and F a prime filter on $\mathscr{P}(I)$. Let X_i be non-empty sets indexed by I. The equivalence relation $\sim_F$ on $f, g \in \prod_i X_i$ is defined by $f \sim_F g$ iff $\{i \mid f(i) = g(i) \in F\}$. Then the* ultraproduct of X_i modulo F $\prod_F X_i$ *is the set of equivalence classes of $\sim_F$: $\prod_F X_i = \{[f]_{\sim_F} \mid f \in \prod_i X_i\}$. It is an* ultrapower *if all X_i are identical.*

Given $\mathscr{L}$ frames $\mathscr{X}_i$ indexed by I we can define their ultraproduct. This can straightforwardly be seen as precisely the same construction as the ultraproduct of first-order models in classical model theory. The basic idea that an ultraproduct of models $\mathscr{X}_i$ is an 'averaging' of the $\mathscr{X}_i$ using the prime filter F as a measure: the ultraproduct witnesses the properties of the first-order models that occur 'often enough' according to F.

Definition 7.17 (Ultraproduct Frame (cf. Definition [27])**.** *Let $\mathscr{X}_i$ be $\mathscr{L}$ frames indexed by I and F a prime filter on $\mathscr{P}(I)$. The* ultraproduct $\mathscr{L}$ frame modulo F $\prod_F \mathscr{X}_i$ *is specified as follows (here we cover all of the frame structure an $\mathscr{L}$ frame might have)*

- *The carrier of $\prod_F \mathscr{X}_i$ is $\prod_F X_i$.*
- *For $\mathscr{L}$ with intuitionistic additives, $[f]_{\sim_F} \succcurlyeq_F [g]_{\sim_F}$ iff $\{i \in I \mid f(i) \succcurlyeq_i g(i)\} \in F$.*
- *For any binary frame operations $\heartsuit$, $[f]_{\sim_F} \in [g]_{\sim_F} \heartsuit_F [h]_{\sim_F}$ iff $\{i \in I \mid f(i) \in g(i) \heartsuit_i h(i)\} \in F$.*
- *$[e]_{\sim_F} \in E_{\sim_F}$ iff $\{i \in I \mid e(i) \in E_i\} \in F$.*
- *$-[f]_{\sim_F} = [g]_{\sim F}$ where $g(i) = -f(i)$ for all $i \in I$.*
- *For any binary relations R, $R_F[f]_{\sim_F}[g]_{\sim_F}$ iff $\{i \in I \mid R_i f(i) g(i)\} \in F$.*

It is an ultrapower frame of $\mathscr{X}$ modulo F *if $\mathscr{X}_i = \mathscr{X}$ for all i.*

That ultraproduct frames of $\mathscr{L}$ frames are themselves $\mathscr{L}$ frames follows from Łos' famous theorem.

Theorem 7.18 (Łos' Theorem)**.** *Let $\prod_F \mathscr{M}_i$ be the ultraproduct of first-order models $\mathscr{M}_i$, $\varphi(x_1, \ldots, x_n)$ a first-order formula and $f_1, \ldots, f_n \in \prod_i \mathscr{M}_i$. Then $\varphi([f_1]_{\sim_F}, \ldots, [f_n]_{\sim_F})$ is true in $\prod_F \mathscr{M}_i$ iff the set of i such that $\varphi(f_1(i), \ldots, f_n(i))$ is true in $\mathscr{M}_i$ is a member of F.* □

Since each of the first-order axioms $\varphi(x_1,\ldots,x_n)$ defining an $\mathscr{L}$ frame is satisfied by each $\mathscr{L}$ frame contributing to the ultraproduct, necessarily the set of i such that $\varphi(f_1(i),\ldots,f_n(i))$ is true in $\mathscr{M}_i$ is $I \in F$. Thus ultraproduct frames are $\mathscr{L}$ frames. The other immediate corollary of Łos' theorem is that ultrapowers of a first-order model $\mathscr{M}$ satisfy the same first-order theory as $\mathscr{M}$. We collect these results in the following corollary.

Corollary 7.19.

1. *Ultrapowers of a first-order model $\mathscr{M}$ satisfy the same first-order theory as $\mathscr{M}$.*

2. *Ultraproduct frames of $\mathscr{L}$ frames are themselves $\mathscr{L}$ frames.*

Definition 7.20 ($\mathscr{L}$ Frame Constructions, Reflections and Closures)**.**

1. *For $\mathscr{L}$ frames $\mathscr{X}$ and $\mathscr{X}'$ we say $\mathscr{X}'$ is a* bounded morphic image *of $\mathscr{X}$ if there exists a surjective $\mathscr{L}$ morphism $g : \mathscr{X} \to \mathscr{X}'$.*

2. *For $\mathscr{L}$ frames $\mathscr{X}$ and $\mathscr{X}'$ we say $\mathscr{X}$ is a* generated subframe *of $\mathscr{X}'$ if there exists an injective $\mathscr{L}$ morphism $g : \mathscr{X} \to \mathscr{X}'$.*

3. *Given disjoint $\mathscr{L}$ frames $\mathscr{X}_i$ the* disjoint union $\biguplus_i \mathscr{X}_i$ *is given by the disjoint union $\biguplus_i X_i$ together with the union of the frame operations of each $\mathscr{X}_i$.*

4. *Given a $\mathscr{L}$ frame $\mathscr{X}$, the* prime extension *of $\mathscr{X}$ is $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X})$. A class of $\mathscr{L}$ frames C* reflects prime extensions *if $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X}) \in C$ implies $\mathscr{X} \in C$.*

5. *A class of $\mathscr{L}$ frames C is* closed under taking ultraproducts *if, for any $\mathscr{L}$ frame $\mathscr{X} \in C$ and any ultrapower $\prod_F \mathscr{X}$ of $\mathscr{X}$, $\prod_F \mathscr{X} \in C$.*

The duality theory on $\mathscr{L}$ frames connects up some of these notions to algebraic ones.

Lemma 7.21 (*cf.* [199, 27])**.** *For disjoint $\mathscr{L}$ frames $(\mathscr{X}_i)_{i\in I}$,*

$$Com^{\mathscr{L}}(\bigsqcup_i \mathscr{X}_i) \cong \prod_{i\in I} Com^{\mathscr{L}}(\mathscr{X}_i).$$

Proof. This follows immediately from categorical duality, but we can construct the isomorphism explicitly. The isomorphism is given by noting that every element A of $Com^{\mathscr{L}}(\biguplus_i \mathscr{X})$ is specified by an element $\{i\} \times A_i$ for $A_i \in Com^{\mathscr{L}}(\mathscr{X}_i)$ corresponding to the part of A that intersects with X_i in the disjoint union. The isomorphism is thus given by sending each X to the element $(X_i)_{i\in I}$ in $\prod_{i\in I} Com^{\mathscr{L}}(\mathscr{X}_i)$. It is a homomorphism because each of the operations is calculated component-wise in the product algebra and the frames are disjoint. □

The next lemma is straightforward to prove, with 3, 4, 5 and 6 immediate corollaries of 1 and 2.

Lemma 7.22 (cf. [27] Theorem 5.47)**.** *For any homomorphism of $\mathscr{L}$ algebras $f : \mathbb{A} \to \mathbb{A}'$ and $\mathscr{L}$ morphism $g : \mathscr{X} \to \mathscr{X}'$*

1. *If f is injective (surjective) then $f^{-1} : Pr^{\mathscr{L}}(\mathbb{A}') \to Pr^{\mathscr{L}}(\mathbb{A})$ is surjective (injective).*
2. *If g is injective (surjective) then $g^{-1} : Com^{\mathscr{L}}(\mathscr{X}') \to Com^{\mathscr{L}}(\mathscr{X})$ is surjective (injective).*
3. *If $\mathscr{X}$ is a generated subframe of $\mathscr{X}'$ then $Com^{\mathscr{L}}(\mathscr{X})$ is the homomorphic image of $Com^{\mathscr{L}}(\mathscr{X}')$.*
4. *If $\mathscr{X}'$ is the bounded morphic image of $\mathscr{X}$ then $Com^{\mathscr{L}}(\mathscr{X})$ is (isomorphic to) a subalgebra of $Com^{\mathscr{L}}(\mathscr{X}')$.*
5. *If $\mathbb{A}$ is a subalgebra of $\mathbb{A}'$ then $Pr^{\mathscr{L}}(\mathbb{A})$ is the bounded morphic image of $Pr^{\mathscr{L}}(\mathbb{A}')$.*
6. *If $\mathbb{A}'$ is the homomorphic image of $\mathbb{A}$ then $Pr^{\mathscr{L}}(\mathbb{A}')$ is a generated subframe of $Pr^{\mathscr{L}}(\mathbb{A})$.* □

We now lift the Goldblatt-Thomason theorem of modal logic to bunched logics; in both the modal and bunched cases the machinery used to prove it is essentially the same. This generalises the results of Brotherston & Villard [44] showing that failure to be closed under bounded morphic images and disjoint unions entails the undefinability of a BBI frame property by a BBI formula. The basic idea is to leverage the connections of the various frame constructions to the operations of homomorphic image, subalgebra and product in the varieties generated by classes of $\mathscr{L}$ frames.

Theorem 7.23 (Goldblatt-Thomason Theorem for Bunched Logics (cf. [108, 27]))**.** *Given a class C of $\mathscr{L}$ frames closed under taking ultrapowers, C is $\mathscr{L}$-definable iff C is closed under bounded morphic images, generated subframes and disjoint unions, and reflects prime extensions.*

In particular, all classes of first-order definable $\mathscr{L}$ frames are closed under taking ultrapowers. This theorem has the key application of demonstrating that some classes of frames relevant to applications of bunched logics are *not* $\mathscr{L}$ definable by showing that these classes of $\mathscr{L}$ frames fail one of the closure/reflection criteria. One direction can be obtained directly from the following proposition, which is the

bunched logic analogue to the same result for modal logic (see Blackburn et al [27] Proposition 5.53).

Proposition 7.24 (Preservation of Validity for $\mathscr{L}$ Frames)**.** *For all $\mathscr{L}$ formulae φ, $\mathscr{L}$ frames $\mathscr{X}$ and $\mathscr{X}'$ and disjoint $\mathscr{L}$ frames $(\mathscr{X}_i)_{i\in I}$:*

1. *If $\mathscr{X}'$ is a bounded morphic image of $\mathscr{X}$ then $\mathscr{X} \vDash \varphi$ implies $\mathscr{X}' \vDash \varphi$;*
2. *If $\mathscr{X}$ is a generated subframe of $\mathscr{X}'$ then $\mathscr{X}' \vDash \varphi$ implies $\mathscr{X} \vDash \varphi$;*
3. *If $\mathscr{X} = \biguplus_i \mathscr{X}_i$ and for all $i \in I$, $\mathscr{X}_i \vDash \varphi$ then $\mathscr{X} \vDash \varphi$;*
4. *If $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X}) \vDash \varphi$ then $\mathscr{X} \vDash \varphi$.*

Proof. 1. to 3. are essentially immediate consequences of the HSP theorem. For 1. consider the variety of complex algebras $V(\mathscr{X})$ generated by $\mathscr{X}$. Since $\mathscr{X} \vDash \varphi$ we have that $\varphi = \top$ is in the equational basis for $V(\mathscr{X})$. By Lemma 7.22 and the HSP theorem, $Com^{\mathscr{L}}(\mathscr{X}') \in V(\mathscr{X})$. Hence $\varphi = \top$ is valid in $Com^{\mathscr{L}}(\mathscr{X})$ and we can thus conclude that $\mathscr{X} \vDash \varphi$. Similarly for 2., $\mathscr{X}' \vDash \varphi$ implies $\varphi = \top$ is in the equational basis for $V(\mathscr{X}')$. By Lemma 7.22 and the HSP theorem $Com^{\mathscr{L}}(\mathscr{X}) \in V(\mathscr{X}')$ so $\mathscr{X} \vDash \varphi$. For 3. consider the variety $V(\{\mathscr{X}_i \mid i \in I\})$ generated by the frames $\mathscr{X}_i$. $\varphi = \top$ is in the equational basis for $V(\{\mathscr{X}_i \mid i \in I\})$ and since $Com^{\mathscr{L}}(\mathscr{X}) \cong \prod_{i\in I} Com^{\mathscr{L}}(\mathscr{X}_i)$ by Lemma 7.21, the HSP theorem dictates that $Com^{\mathscr{L}}(\mathscr{X}) \in V(\{\mathscr{X}_i \mid i \in I\})$ and thus $\mathscr{X} \vDash \varphi$.

Finally, for 4. suppose that $\mathscr{X} \nvDash \varphi$. Then there exists some valuation $\mathscr{V}$ and element $x \in \mathscr{X}$ such that $x \nVdash_{\mathscr{V}} \varphi$. Thus for $Com^{\mathscr{L}}(\mathscr{X})$ with induced interpretation $[\![-]\!]_{\mathscr{V}}$ we have $x \notin [\![\varphi]\!]_{\mathscr{V}}$ by Theorem 7.1. There thus exists a prime filter of $Com^{\mathscr{L}}(\mathscr{X})$ for which $[\![\varphi]\!] \notin F$. The induced valuation $\mathscr{V}_{[\![-]\!]_{\mathscr{V}}}$ then necessarily gives $F \nVdash_{\mathscr{V}_{[\![-]\!]_{\mathscr{V}}}} \varphi$ by Theorem 7.1 once more, so $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X}) \nvDash \varphi$. □

For the other direction we need some auxillary results. First we will need the following model theoretic notion. For a first-order language L, a L-model $\mathscr{M}$ and a subset $A \subseteq \mathscr{M}$ we define the expanded language $L[A]$ to be L expanded with constants corresponding to the elements in A. This makes $\mathscr{M}$ an $L[A]$ model when these new constants are interpreted by their counterparts in A.

Definition 7.25 (Countably Saturated Model)**.** *Given a first-order language L, a first-order L-model $\mathscr{M}$ is* countably saturated *if, for every subset $A \subseteq \mathscr{M}$ with $|A| < \aleph_0$ and every set of $L[A]$-formulae $\Gamma[x_0,\ldots,x_n]$ (in which only the variables $x_0,\ldots,x_n$ are free) that is consistent with the first-order theory of $\mathscr{M}$, there exists $a_0,\ldots,a_n \in \mathscr{M}$ such that $\Gamma[a_0/x_0,\ldots,a_n/x_n]$ is true in $\mathscr{M}$.*

We will require a ultrapower frame that is countably saturated when viewed as a first-order model for our next lemma. Usefully, these are guaranteed to exist: the details of the proof are beyond the scope of this thesis and can be found in Chang & Keisler [52] Section 6.1.

Lemma 7.26. *For any first-order language L and L-model $\mathscr{M}$, there exists a countably saturated ultrapower of $\mathscr{M}$.* □

In particular, when we consider $\mathscr{L}$ frames $\mathscr{X}$ as L-structures for the first-order language of $\mathscr{L}$ frames this construction yields a countably saturated ultrapower frame of $\mathscr{X}$. In this next crucial lemma, we extend an argument for intuitionistic frames given by Rodenburg [199] (based on the work of Van Benthem [19] for modal logic) that states that the prime extension of any intuitionistic frame $\mathscr{X}$ is the bounded morphic image of an ultrapower of $\mathscr{X}$. The proof essentially proceeds by extending the signature of the first-order language of $\mathscr{L}$ frames with predicates witnessing every element of $Com^{\mathscr{L}}(\mathscr{X})$. We can then use the model theory of first-order logic (in particular, Lemma 7.26 and Łos' theorem) to ensure an ultrapower frame and suitable surjective $\mathscr{L}$ morphism exist.

Lemma 7.27. *Let $\mathscr{X}$ be an $\mathscr{L}$ frame. Then $\mathscr{X}$ has an ultrapower $\prod_F \mathscr{X}$ such that $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X})$ is the bounded morphic image of $\prod_F \mathscr{X}$.*

Proof. For a given $\mathscr{X}$, consider the first-order language $L_{\mathscr{X}}$ given by extending the signature of the language of $\mathscr{L}$ frames with a predicate P_A for each $A \in Com^{\mathscr{L}}(\mathscr{X})$. By interpreting the $\mathscr{L}$ frame signature with the structure of $\mathscr{X}$ and each P_A as A, we obtain a first-order model $\mathscr{A}$ of the language $L_{\mathscr{X}}$. By Lemma 7.26 there exists a countably saturated ultrapower $\mathscr{B}$ of $\mathscr{A}$. In particular, $\mathscr{B}$ is an ultrapower frame of $\mathscr{X}$ together with an interpretation of the predicates P_A, and by Łos' theorem, $\mathscr{A}$ and $\mathscr{B}$ satisfy the same first-order theory. We define a map $\eta : \mathscr{B} \to Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X})$ by $\eta(x) = \{X \mid P_X x \text{ is true in } \mathscr{B}\}$. We claim this η is the required surjective bounded morphism.

Lemma 15.2 of Rodenburg [199] gives that η is well-defined, surjective and, in the case for $\mathscr{L}$ with intuitionistic additives, satisfies the conditions of an intuitionsitic morphism. We show that when $\mathscr{L}$ is DMBI, η is a DMBI morphism: this characteristic example is enough to show the same holds for (I)LGL and (B)BI, and the argument can be extended in an obvious way to the other bunched logics we have considered.

First, suppose $x \in y \circ_{\mathscr{B}} z$. We must show $\eta(x) \in \eta(y) \circ_{Com^{\text{DMBI}}(\mathscr{X})} \eta(z)$. Suppose $Y \in \eta(y)$ and $Z \in \eta(z)$. Then $P_Y y$ and $P_Z z$ hold in $\mathscr{B}$. It is easy to see that the formula

$\forall x,y,z((P_Y y \wedge P_Z \wedge x \in y \circ z) \to P_{Y \bullet_{\mathscr{X}} Z} x)$ is true in $\mathscr{A}$. Thus it is also true in $\mathscr{B}$. We thus obtain that $P_{Y \bullet_{\mathscr{X}} Z} x$ holds in $\mathscr{B}$, so $Y \bullet_{\mathscr{X}} Z \in \eta(x)$ as required.

Next, suppose $\eta(x) \supseteq F_w$ and $F_w \in F_y \circ_{Com^{\mathrm{DMBI}}(\mathscr{X})} F_z$. Consider the set of formulae $\Gamma := \{x \succcurlyeq w, w \in y \circ z\} \cup \{P_Y y \mid Y \in F_y\} \cup \{P_Z z \mid Z \in F_z\}$: we claim this is consistent with the theory of $\mathscr{B}$ extended with constant x. Suppose for contradiction it is not: then there is a finite $\Gamma_0 \subseteq \Gamma$ witnessing this inconsistency. For any $Y \in F_y$ and $Z \in F_z$ we have that $Y \bullet_{\mathscr{X}} Z \in F_w \subseteq \eta(x)$ so it follows that Γ_0's inconsistency amounts to the sentence $\forall w,y,z((x \succcurlyeq w \wedge w \in y \circ z \wedge P_Y y) \to \neg P_Z z)$ holding in $\mathscr{B}$ for some $Y \in F_y$ and $Z \in F_z$. By assumption, for this Y and Z we have $Y \bullet_{\mathscr{X}} Z \in \eta(x)$, so $P_{Y \bullet_{\mathscr{X}} Z} x$ is true in $\mathscr{B}$. The sentence $\forall x(P_{Y \bullet_{\mathscr{X}} Z} x \to \exists w,y,z(x \succcurlyeq w \wedge w \in y \circ z \wedge P_Y y \wedge P_Z z))$ is true in $\mathscr{A}$, and thus is also true in $\mathscr{B}$. We thus have $w,y,z \in \mathscr{B}$ with $x \succcurlyeq w \wedge w \in y \circ z \wedge P_Y y \wedge P_Z z$: a contradiction. Hence Γ is consistent with the theory of $\mathscr{B}$ extended with constant x. Since $\mathscr{B}$ is countably saturated Γ is realised for some $w,y,z \in \mathscr{B}$, and we thus obtain $x \succcurlyeq_{\mathscr{B}} w$ with $w \in y \circ_{\mathscr{X}} z$, $\eta(y) \supseteq F_y$ and $\eta(z) \supseteq F_z$.

Now we suppose that $F_w \supseteq \eta(x)$ and $F_z \in F_w \circ_{Com^{\mathrm{DMBI}}(\mathscr{X})} F_y$. Consider the set of formulae $\Gamma := \{w \succcurlyeq x, z \in w \circ y\} \cup \{P_Y y \mid Y \in F_y\} \cup \{\neg P_Z z \mid Z \notin F_z\}$. We assume again for contradiction that this is inconsistent with the theory of $\mathscr{B}$ extended with constant x. This amounts to the sentence $\varphi := \forall w,y,z(w \succcurlyeq x \wedge z \in w \circ y \wedge P_Y y \to \bigvee_{1 \leq j \leq m} P_{Z_j} z)$ being true in $\mathscr{B}$, where $Y \in F_y$ and $Z_1, \ldots, Z_m \in F_z$. Thus it is also true in $\mathscr{A}$. Straightforwardly, $\forall x(\varphi(x) \to P_{Y \mathrel{-\!\!\bullet}_{\mathscr{X}} \bigcup_j Z_j} x)$ is true in $\mathscr{A}$. Hence it is also so in $\mathscr{B}$, and we obtain that $P_{Y \mathrel{-\!\!\bullet}_{\mathscr{X}} \bigcup_j Z_j} x$ is true in $\mathscr{B}$. Thus $Y \mathrel{-\!\!\bullet}_{\mathscr{X}} \bigcup_j Z_j \in F_w$, so $Y \bullet_{\mathscr{X}} (Y \mathrel{-\!\!\bullet}_{\mathscr{X}} \bigcup_j Z_j) \in F_z$ by assumption. By upwards-closure of F_z, $\bigcup_j Z_j \in F_z$, and by primeness, for one of the j ($1 \leq j \leq m$) we have $Z_j \in F_z$: a contradiction. Hence Γ is consistent with the theory of $\mathscr{B}$ extended with constant x and is once again realised by some $w,y,z \in \mathscr{B}$. Thus $w \succcurlyeq_{\mathscr{B}} x$ and $z \in w \circ_{\mathscr{B}} y$ with $\eta(y) \supset F_y$ and $\eta(z) \subseteq F_z$ as required.

For the condition $e \in E_{\mathscr{B}}$ iff $\eta(e) \in E_{Pr^{\mathrm{DMBI}} Com^{\mathrm{DMBI}}(\mathscr{X})}$, note that $\forall x(x \in E \leftrightarrow P_E x)$ is true in $\mathscr{A}$, and hence in $\mathscr{B}$. Thus $e \in E_{\mathscr{B}}$ iff $P_E e$ holds in $\mathscr{B}$ iff $E \in \eta(e)$ as required.

Finally we consider the condition $\eta(-x) = -_{Pr^{\mathrm{DMBI}} Com^{\mathrm{DMBI}}(\mathscr{X})} \eta(x)$. First note that the sentence $\varphi := \forall x(P_C x \leftrightarrow \neg P_{C \mathrel{-\!\!\bullet}_{\mathscr{X}} \overline{-E}} - x)$ is true in $\mathscr{A}$ for any $C \in Com^{\mathrm{DMBI}}(\mathscr{X})$; this is proved in the same way as similar arguments in the duality theory of DMBI in Chapter 6. Hence this is also true in $\mathscr{B}$. The required identity amounts to showing $P_X - x$ is true in $\mathscr{B}$ iff $X \neq C \mathrel{-\!\!\bullet}_{\mathscr{X}} \overline{-E}$ for any C such that $P_C x$ is true in $\mathscr{B}$—but this is precisely ensured by the truth of φ in $\mathscr{B}$. □

We're now ready to prove our version of the Goldblatt-Thomason Theorem for

bunched logics. Given what we have proved thus far, this follows by an identical argument to that given in Blackburn et al [27] for modal logic, itself derived from Goldblatt's [108] algebraic reconstruction of the theorem. For completeness we include the argument here.

Proof of the Goldblatt-Thomason Theorem for Bunched Logics. Let C be a class of $\mathscr{L}$ frames that is closed under ultraproducts, bounded morphic images, generated subframes and disjoint unions, and reflects prime extensions. It is sufficient to show that for any frame $\mathscr{X}$ satisfying the theory of this class of models ($\{\varphi \mid \forall \mathscr{X} \in C(\mathscr{X} \vDash \varphi)\}$), $\mathscr{X} \in C$. Let $\mathscr{X}$ be such a frame. Then $Com^{\mathscr{L}}(\mathscr{X})$ is in the variety of complex algebras generated by C, $V(C)$. Since $V(C)$ is a variety, it is closed under HSP. There thus exist $\mathscr{L}$ frames $\mathscr{X}_i$ and $\mathscr{L}$ algebras $\mathbb{A}$ and $\mathbb{B}$ such that $\mathbb{B} = \prod_i Com^{\mathscr{L}}(\mathscr{X}_i)$, $\mathbb{A}$ is a subalgebra of $\mathbb{B}$ and $Com^{\mathscr{L}}(\mathscr{X})$ a homomorphic image of $\mathbb{A}$.

By Theorem 7.21, $\mathbb{B} \cong Com^{\mathscr{L}}(\biguplus_{i\in I} \mathscr{X}_i)$ and by closure under disjoint unions $\biguplus_{i\in I} \mathscr{X}_i \in C$. Taking the duals of these structures and homomorphisms, we have that $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X})$ is a generated subframe of $Pr^{\mathscr{L}}(A)$ and $Pr^{\mathscr{L}}(\mathbb{A})$ is the bounded morphic image of $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\biguplus_{i\in I} \mathscr{X}_i)$. By Lemma 7.27, $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\biguplus_{i\in I} \mathscr{X}_i)$ is the bounded morphic image of an ultrapower of $\biguplus_{i\in I} \mathscr{X}_i$, so by closure under ultraproducts $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\biguplus_{i\in I} \mathscr{X}_i) \in C$. Then $Pr^{\mathscr{L}}(\mathbb{A}) \in C$ by closure under bounded morphic images and so $Pr^{\mathscr{L}} Com^{\mathscr{L}}(\mathscr{X}) \in C$ by closure under generated subframes. Finally C reflects prime extensions so $\mathscr{X} \in C$ as required. □

Let's pause to take account of what we've proved here. While satisfaction of these conditions for a particular class C guarantees $\mathscr{L}$-definability, the set of defining formulae is possibly infinite: in the proof the set is taken to be the $\mathscr{L}$ formulae that are valid for every member of C. This does *not*, therefore, give us a correspondending $\mathscr{L}$ axiom we can use to distinguish membership in C and add in a straightforward way to a proof system for $\mathscr{L}$-validity on frames in C: to find such correspondents requires further work [202]. In Part III we will show how to sidestep this issue and define proof calculi for a large number of well-behaved classes of $\mathscr{L}$ frames.

We can demonstrate the applicability of the theorem with some simple examples. First a negative (although expected) result. Recall that a $\mathscr{L}$ frame is Downwards Closed if $\forall x,y,z(z \in x \circ y \wedge x \succcurlyeq x' \wedge y \succcurlyeq y' \rightarrow \exists z'(z \succcurlyeq z' \wedge z' \in x' \circ y')$ holds and Upwards Closed if $\forall x,y,z(z \in x \circ y \wedge z' \succcurlyeq z \rightarrow \exists x', y'(z' \in x' \circ y' \wedge x' \succcurlyeq x \wedge y' \succcurlyeq y)$ holds. These classes of frames are not $\mathscr{L}$-definable, which of course must be the case following our discussion in Chapter 3: these classes all define the same set of valid formulae as ordinary $\mathscr{L}$ frames.

Proposition 7.28. *For $\mathscr{L}$ with intuitionistic additives, the class of Upwards (Downwards) [Upwards and Downwards] Closed $\mathscr{L}$ frames is not $\mathscr{L}$-definable.*

Proof. This follows immediately from the observation (*cf.* Cao et al. [51]) that for any $\mathscr{L}$ frame $\mathscr{X}$, the prime extension $Pr^{\mathscr{L}}Com^{\mathscr{L}}(\mathscr{X})$ is both upwards and downwards closed. If $F_z \in F_x \circ_{Com^{\mathscr{L}}(\mathscr{X})} F_y$ with $F_x \supseteq F_{x'}$ and $F_y \supseteq F_{y'}$ then straightforwardly $F_z \in F_{x'} \circ_{Com^{\mathscr{L}}(\mathscr{X})} F_{y'}$. Similarly, if $F_z \in F_x \circ_{Com^{\mathscr{L}}(\mathscr{X})} F_y$ with $F_{z'} \supseteq F_z$ then $F_{z'} \in F_x \circ_{Com^{\mathscr{L}}(\mathscr{X})} F_y$. Since there exist $\mathscr{L}$ frames which are not Upwards and/or Downwards Closed, these classes of frames fail to reflect prime extensions. □

Now let's look at a positive result. A $\mathscr{L}$ frame is *Total* if for all $x, y \in \mathscr{X}$ there exists $z \in \mathscr{X}$ such that $z \in x \circ y$.

Proposition 7.29. *The class of Total $\mathscr{L}$ frames is $\mathscr{L}$-definable for $\mathscr{L}$ with classical additives.*

Proof. We show the class of Total $\mathscr{L}$ frames is closed under bounded morphic images, generated subframes and disjoint unions while reflecting prime extensions. First suppose $\mathscr{X}'$ is the bounded morphic image of a Total $\mathscr{L}$ frame, say by surjective $\mathscr{L}$ morphism g. Suppose $g(x), g(y) \in \mathscr{X}'$. Since $\mathscr{X}$ is total, there exists $z \in x \circ y$. We thus obtain $g(z) \in g(x) \circ' g(y)$. Now suppose $\mathscr{X}$ is a generated subframe of a Total $\mathscr{L}$ frame $\mathscr{X}'$, say by the injective $\mathscr{L}$ morphism g. For $g(x), g(y)$ we have $z' \in g(x) \circ' g(y)$. Since g is an $\mathscr{L}$ morphism, there exists $z, y' \in \mathscr{X}$ such that $z \in x \circ y'$ with $g(z) = z'$ and $g(y') = g(y)$. By injectivity, $y = y'$ so $g(z) \in g(x) \circ g(y)$. Closure under disjoint unions is straightforward, so we move to reflection of prime extensions. Suppose $Pr^{\mathscr{L}}Com^{\mathscr{L}}(\mathscr{X})$ is total. Then for any prime filters F_x, F_y there exists $F_z \in F_x \circ_{Com^{\mathscr{L}}(\mathscr{X})} F_y$. Let $x, y \in \mathscr{X}$ and consider the prime filters of $Com^{\mathscr{L}}(\mathscr{X})$ given by $F_x = \{A \mid x \in A\}$ and $F_y = \{A \mid y \in A\}$. Then there exists $F_z \in F_x \circ_{Com^{\mathscr{L}}(\mathscr{X})} F_y$. We have $\{x\} \in F_x$ and $\{y\} \in F_y$ so $\{x\} \bullet_{\mathscr{X}} \{y\} \in F_z$. Since F_z is prime, $\{x\} \bullet_{\mathscr{X}} \{y\} \neq \emptyset$, so $\mathscr{X}$ is a Total $\mathscr{L}$ frame as required. □

This same argument breaks down if we attempt it for $\mathscr{L}$ with intuitionistic additives as we can't use the same trick to prove reflection of prime extension: the sets $\{x\}$ and $\{y\}$ are not necessarily upwards-closed so won't always be in the prime filters F_x and F_y.

Next we recall the *separation theories* collected by Brotherston & Villard [44] on their work on BBI expressivity, which correspond to common properties of memory models of (B)BI (we will go into separation theories in greater detail in Chapter 10). These are

Partial Functional:	$w, w' \in w_1 \circ w_2 \to w = w'$
Cancellativity:	$w \circ w_1 \cap w \circ w_2 \to w_1 = w_2$
Single Unit:	$e, e' \in E \to e = e'$
Indivisible Units:	$(w \circ w') \cap E \neq \emptyset \to w \in E$
Disjointness:	$w \circ w \neq \emptyset \to w \in E$
Divisibility:	$w \notin E \to \exists w_1, w_2 \notin E (w \in w_1 \circ w_2)$
Cross Split:	$(t \circ u) \cap (v \circ w) \neq \emptyset \to \exists tv, tw, uv, uw$
	$(t \in tv \circ tw \wedge u \in uv \circ uw \wedge v \in tv \circ uv \wedge w \in tw \circ uw)$.

Brotherston & Villard proved the following undefinability results by exhibiting the failure of closure under bounded morphic images and disjoint unions. As an immediate corollary (exploiting the fact that all BBI frames (morphisms) are also BI/LGL/ILGL frames (morphisms)) the analogous results obtain for BI/LGL/ILGL.

Theorem 7.30 ([44])**.** *For $\mathscr{L} \in \{BI, BBI, LGL, ILGL\}$:*

1. *The class of Partial Functional $\mathscr{L}$ frames is not $\mathscr{L}$-definable;*
2. *The class of Partial Functional and Cancellative $\mathscr{L}$ frames is not $\mathscr{L}$-definable;*
3. *The class of Single Unit (B)BI frames is not (B)BI-definable (see also [99, 150]).* □

Extending this result to CBI and DMBI is complicated by Brotherston & Calcagno's [39] observation that any extension of a partial functional BBI model with a CBI frame's dual operator $-$ will necessarily force the new model's $\circ$ to be nondeterministic in order to satisfy the CBI frame property $x \in y \circ z \to -y \in -x \circ z$. We thus can't extend the existing counterexample for BBI to obtain one for CBI (and thus DMBI).

Let us consider the Cross Split property. Cross Split is an interesting property because most meaningful bunched logic models seem to satisfy it, including most memory models. It is also used in an essential way to show that memory models with intersection operations form models of subclassical bunched logics that don't collapse into CBI models [45]. We can consider it for any bunched logic $\mathscr{L}$ as $\circ$ is in the signature of all $\mathscr{L}$ frames. Brotherston & Villard conjecture Cross Split is BBI-undefinable and state that for BBI it is "seemingly preserved by bounded morphic images, disjoint unions and by generated sub[frames]". We can strengthen this observation to a proof.

Lemma 7.31. *For any bunched logic $\mathscr{L}$ with classical additives, Cross Split is preserved by bounded morphic images, disjoint unions and generated subframes.*

Proof. First suppose $\mathscr{X}$ is a $\mathscr{L}$ frame satisfying Cross Split and $\mathscr{X}'$ a bounded morphic image of $\mathscr{X}$; say by the surjective $\mathscr{L}$ morphism g. Let $g(x_0) \in g(t) \circ' g(u)$ and $g(x_1) \in g(v) \circ' g(w)$ for $g(x_0) = g(x_1)$. By surjectivity of g, if a cross split exists for $g(t), g(u), g(v), g(w)$ then $\mathscr{X}'$ satisfies Cross Split. Since g is a $\mathscr{L}$ morphism we can find $t_0, u_0, v_0, w_0 \in \mathscr{X}$ such that $x_0 \in t_0 \circ u_0$, $x \in v_0 \circ w_0$ and $g(t_0) = g(t), g(u_0) = g(u), g(v_0) = g(v), g(w_0) = g(w)$. Then there exists a cross split in $\mathscr{X}$: there exists tv, tw, uv, uw such that $t_0 \in tv \circ tw$, $u_0 \in uv \circ uw$, $v_0 \in tv \circ uv$ and $w_0 \in tw \circ uw$. Since g is a $\mathscr{L}$ morphism, $g(t) = g(t_0) \in g(tv) \circ' g(tw), g(u) = g(u_0) \in g(uv) \circ' g(uw), g(v) = g(v_0) \in g(tv) \circ' g(uv)$ and $g(w) = g(w_0) \in g(tw) \circ' g(uw)$.

Now suppose $\mathscr{X}$ is a generated subframe of a $\mathscr{L}$ frame $\mathscr{X}'$ satisfying Cross Split, say by the injective $\mathscr{L}$ morphism g. Suppose $g(x) \in g(t) \circ' g(u)$ and $g(x) \in g(v) \circ g(w)$. By the cross split property in $\mathscr{X}'$ there exist tv, tw, uv, uw such that $g(t) \in tv \circ' tw, g(u) \in uv \circ' uw, g(v) \in tv \circ' uv$ and $g(w) \in uw \circ' tw$. Since g is a $\mathscr{L}$ morphism, there exist tv_i, tw_i, uv_i, uw_i ($i \in \{0,1\}$) such that $t \in tv_0 \circ tw_0, u \in uv_0 \circ uw_0, v \in tv_1 \circ uv_1$ and $w \in uw_1 \circ tw_1$ with $g(tv_i) = tv, g(tw_i) = tw, g(uv_i) = uv$ and $g(uw_i) = uw$ for $i \in \{0,1\}$. By injectivity of g we have $z_0 = z_1$ for each $z \in \{tv, tw, uv, uw\}$ and so Cross Split is satisfied.

Finally, preservation by disjoint unions is a trivial consequence of the fact that any formation $x \in t \circ u, x \in v \circ w$ is necessarily contained in one of the disjoint frames of the disjoint union, and thus the cross split can be found in that same frame. □

Corollary 7.32. *For $\mathscr{L}$ with classical additives, the class C of $\mathscr{L}$ frames satisfying Cross Split is $\mathscr{L}$ definable iff C reflects prime extensions.* □

Hence proving the undefinability of Cross Split requires us to find a $\mathscr{L}$ frame $\mathscr{X}$ that doesn't satisfy Cross Split despite its prime extension satisfying it: a highly non-trivial task! We can rule out one kind of $\mathscr{L}$ frame: if $\mathscr{X}$ is finite, then the elements of the prime extension are all principal filters—that is, the set of all subsets A such that $x \in A$ for some $x \in \mathscr{X}$. It is straightforward to then show that the satisfaction of Cross Split in the prime extension of a finite frame $\mathscr{X}$ entails it in $\mathscr{X}$. This example shows that although we now have the criteria for definability in bunched logics, verifying that criteria is still a complex matter. We leave this problem open.

Problem 7.33. *Is Cross Split $\mathscr{L}$-definable for any bunched logic $\mathscr{L}$?*

We end this section with a class of $\mathscr{L}$ frames that we can't investigate with the Goldblatt-Thomason theorem. Define the relation D by xDy iff there exists z such that $y \in x \circ z$ or $y \in z \circ x$. We say a $\mathscr{L}$ frame has *well-founded decomposition* if the relation D is well-founded: there is no infinite sequence $(x_n)_{n \in \mathbb{N}}$ of distinct x_n

such that $x_{n+1}Dx_n$ for all n. Consider memory models of (B)BI: a heap is given as a finite partial function defined on, say, n values, which can thus be decomposed into at most n disjoint pointers $x_i \mapsto y_i$. This property is essential to the application of bunched logic in program verification as it entails a heap can be directly identified with a formula of Separation Logic: $x_1 \mapsto y_1 * \cdots * x_n \mapsto y_n$. In using bunched logics to model complex systems (say with layered graph logics), the property of being able to directly represent the decomposition of the system as a formula seems to be crucial. However we *cannot* apply our Goldblatt Thomason theorem to investigate if such classes of models are $\mathscr{L}$-definable: this property is not first-order definable and not preserved by ultrapowers. We defer a full investigation of (un)definable classes of bunched logic frames to another occasion.

7.4 Interpolation

We end this chapter by considering Craig interpolation for bunched logics. We begin with a precise statement of the Craig Interpolation property. First some notation: for a $\mathscr{L}$ formula φ, $\mathrm{Prop}(\varphi)$ is the set of propositional variables occuring in φ.

Definition 7.34 (Craig Interpolation)**.** *$\mathscr{L}$ has the* Craig interpolation property *(CIP) if, whenever $\varphi \vdash \psi$ is provable, there exists an* interpolant *ξ such that $\varphi \vdash \xi$ and $\xi \vdash \psi$ are provable and* $\mathrm{Prop}(\xi) \subseteq \mathrm{Prop}(\varphi) \cap \mathrm{Prop}(\psi)$.

Craig interpolation goes back to the landmark work of Craig [69], who first formulated the property and proved it held for classical logic. Following Maksimova's pioneering work [156, 157, 96] on superintuitionistic and modal logics, it has become well-known (cf. [160]) that various interpolation properties (CIP among them) can be deduced or refuted for non-classical logics by the satisfaction or refutation of *amalgamation properties* on the varieties of algebras that interpret them. The key property related to the CIP is the *super amalgamation property*.

Definition 7.35 ((Super) Amalgamation Property)**.** *A class K of algebras of the same type has the* amalgamation property *(AP) if, for any $\mathbb{A}$, $\mathbb{B}_0$, $\mathbb{B}_1 \in K$ with embeddings $e_0 : \mathbb{A} \to \mathbb{B}_0$ and $e_1 : \mathbb{A} \to \mathbb{B}_1$ (we call this a* V-formation*), there exists $\mathbb{C} \in K$ together with embeddings $m_0 : \mathbb{B}_0 \to \mathbb{C}$ and $m_1 : \mathbb{B}_1 \to \mathbb{C}$ such that $m_0 \circ e_0 = m_1 \circ e_1$.*

If K is a class of partially ordered algebras, K has the super amalgamation property *(SAP) if it has both the AP and the additional property that for all $b_i \in \mathbb{B}_i$, $b_j \in \mathbb{B}_j$ ($\{i, j\} = \{0, 1\}$), if $m_i(b_i) \leq m_j(b_j)$ then there exists $a \in \mathbb{A}$ such that $b_i \leq e_i(a)$ and $b_j \leq e_j(a)$.*

For many logics $\mathscr{L}$ with a suitable algebraic semantics, the satisfaction of the SAP for the algebras interpeting $\mathscr{L}$ is equivalent to $\mathscr{L}$ having the CIP [160]. Given

a bunched logic $\mathscr{L}$ we can consider amalgamation for the category of $\mathscr{L}$ algebras: in particular, we can show that the AP (and hence the SAP) fails for the category of $\mathscr{L}$ algebras where $\mathscr{L} \in \{BI, BBI, CBI, DMBI\}$. That this is the case can be deduced from Urquhart's work on the failure of interpolation for relevant logics [214]. We begin with the semantic structures suitable for interpreting the relevant logic KR.

Definition 7.36 (KR Frame [214])**.** *A* KR frame *is a triple $K = (X, R, e)$ where $e \in X$ and R a ternary relation on X satisfying*

1. *$Reab$ iff $a = b$;*
2. *$Raaa$;*
3. *$Rabc$ implies $Rbac$ and $Racb$;*
4. *$Rabc$ and $Rcde$ implies there exists f such that $Radf$ and $Rfbe$.*

Given that the semantics of bunched logics has its roots in relevant logic, it may not be surprising that KR frames are also models of bunched logic. In particular, a KR frame induces a CBI frame (and thus also a (B)BI and DMBI frame). Given a KR frame K we define $\mathscr{X}(K)$ by setting $a \circ b = \{c \mid Rabc\}$, $E = \{e\}$ and $-x = x$.

Lemma 7.37. *Given a KR frame K, $\mathscr{X}(K)$ is a CBI frame.*

Proof. Commutativity of $\circ$ can be deduced from 3. Closure, Unit Existence and Coherence can be deduced by the definition and 1. For Associativity, suppose $t \in x \circ y$ and $w \in t \circ z$. Then by definition $Rxyt$ and $Rtzw$ hold. By 3. this gives $Ryxt$ and $Rtzw$, so applying 4. there exists f such that $Ryzf$ and $Rfxw$. This gives $f \in y \circ z$ and $w \in x \circ f$ as desired. Finally, $--x = x$ trivially, and the Compatibility property $z \in x \circ y$ implies $-x \in -z \circ y$ follows from the definition of $-$ and the symmetry condition 3. □

Urquhart also gives a notion of KR morphism that yields a CBI morphism through our construction.

Definition 7.38 (KR Morphism)**.** *A map $g : K \to K'$ is a* KR morphism *if*

1. *$g(a) = e'$ iff $a = e$;*
2. *$Rabc$ implies $R'g(a)g(b)g(c)$;*
3. *$R'g(a)b'c'$ implies there exists b, c such that $Rabc$, $g(b) = b'$ and $g(c) = c'$.*

Lemma 7.39. *If $f : K \to K'$ is a KR morphism, then $f : K(\mathscr{X}) \to K(\mathscr{X}')$ is a CBI morphism.*

Proof. Given the conditions in the definition and the fact that $g(-x) = g(x) = -g(x)$ we only have one condition to check: $z' \in g(x) \circ' y'$ implies there exists $y, z \in X$ such that $z \in x \circ y$, $g(y) = y'$ and $g(z) = z'$. If $z' \in g(x) \circ' y'$ it follows that $g(x) \in z' \circ' y'$. Now using 3. we obtain $x \in z \circ y$ with $g(z) = z'$ and $g(y) = y'$. This yields $z \in x \circ y$, and so the condition. holds. □

Urquhart shows that KR frames can be constructed from projective planes, and using this construction proves the following.

Theorem 7.40 (Urquhart [214])**.** *(S)AP fails for the category of distributive lattice-ordered monoids.* □

The proof is too complex to give in full detail, but essentially works by giving three KR frames $\mathscr{X}_0, \mathscr{X}_1$ and $\mathscr{X}_2$ based on projective geometries such that $\mathscr{X}_0$ is the bounded morphic image of $\mathscr{X}_1$ and $\mathscr{X}_2$. Taking complex algebras, this obtains an embedding of the complex algebra of $\mathscr{X}_0$ in the complex algebras of $\mathscr{X}_1$ and $\mathscr{X}_2$. This V formation cannot be amalgamated however: a notion of non-associative composition, $\cdot$, internal to projective geometries can be encoded in the complex algebras over these KR frames. Any amalgamation would necessarily identify the elements corresponding to compositions $A \cdot (B \cdot C)$ in the complex algebra of $\mathscr{X}_1$ and the elements corresponding to compositions $(A \cdot B) \cdot C$ in the complex algebra of $\mathscr{X}_2$. However, this identification cannot hold, as this $\cdot$ fails to be associative.

In sum, any variety of distributive lattice ordered monoid that contains this V-formation fails to have the AP. However, it is easily seen that this V-formation exists in the categories of $\mathscr{L}$ algebras for $\mathscr{L} \in \{BI, BBI, DMBI, CBI\}$: the complex algebras Urquhart takes also carry the structure of a CBI algebra (and thus a BI/BBI/DMBI algebra) by Lemma 7.37 and by Lemma 7.39, the embeddings are CBI (and thus BI/BBI/DMBI) morphisms. Any amalgam found for this V-formation would also be a V-formation in the category of distributive lattice-ordered monoids and so AP (and thus SAP) fails for $\mathscr{L}$ algebras.

Theorem 7.41. *For $\mathscr{L} \in \{BI, BBI, DMBI, CBI\}$, (S)AP fails for the category of $\mathscr{L}$ algebras.* □

Does this entail the failure of CIP for these bunched logics? This would confirm Brotherston & Goré's [41] conjecture that interpolation fails for BI. Unfortunately we haven't quite shown that yet. The key remaining step is to show that the SAP holds for $\mathscr{L}$ algebras iff $\mathscr{L}$ has CIP. A partial solution can be given. Madarász [155] proved a more general version of the following theorem.

Theorem 7.42 (cf. Madarász [155] Theorem 3.7)**.** *Let $\mathscr{L}$ be an algebraizable logic such that*

1. *$\mathscr{L}$ has a Boolean reduct;*
2. *$Alg(\mathscr{L})$ forms a variety;*
3. *$\mathscr{L}$ has the local deduction property: $\varphi \vdash \psi$ provable implies $\varphi \to \psi$ is provable.*

Then $\mathscr{L}$ has the CIP iff the variety of $\mathscr{L}$ algebras has the SAP. □

This is the case for all bunched logics with classical additives, and so we obtain the following result for BBI and CBI.

Theorem 7.43. *BBI and CBI fail to have the CIP.* □

The condition of $\mathscr{L}$ having a Boolean reduct can be weakened (*cf.* Madarász Theorem 3.9 [155]) to the variety of $\mathscr{L}$ algebras being a *discriminator variety*. Hence to obtain the result for (DM)BI, it would be sufficient to show that the variety of (DM)BI algebras is a discriminator variety. Another option would be to attack the problem directly by considering (DM)BI as a substructural logic extending the full Lambek calculus and applying the techniques of Kihara & Ono [140]. We defer such an investigation to another occasion and leave this as an open problem.

Problem 7.44. *For $\mathscr{L}$ with intuitionistic additives, is it true that the category of $\mathscr{L}$ algebras has the SAP iff $\mathscr{L}$ has the CIP?*

Chapter 8

Dualities for Predicate Bunched Logics

In this chapter we utilise the dualities given in Chapter 6 to give duality theorems for the categorical structures associated with predicate extensions of bunched logics. The key structures we are concerned with here are *BI hyperdoctrines* [24], a categorical structure formulated to give models of separation logic with abstract predicates.

For the purpose of this thesis a general notion is given that is agnostic of the particular bunched logic, which is a possibility because the BI hyperdoctrine structure interpreting quantification does not explicitly interact with the multiplicative structure of the logic in question. It is instead defined with respect to the Heyting or Boolean structure of the given logic. That this is all that is required for separation logic attests to this being no major restriction when it comes to applications of bunched logic, but we believe the work of this chapter lays a foundation for an examination of *multiplicative* notions of quantification obtained by requiring the structure in question to cohere with the multiplicative structure in appropriate ways. Collinson et al [62] give an example of such a notion of quantification with a corresponding definition of hyperdoctrine in their work on bunched polymorphism. Another notion of quantification given in the context of BI hyperdoctrines is the dependence-friendly quantifiers of dependence logic that Abramsky & Väänänen [7] show can be interpreted in models of predicate BI as quantification guarded by a dependence predicate.

It is nonetheless non-trivial to prove the duality theorems even when restricting to an additive notion of quantification. Coumans [65] provides the foundational work for this by describing how to extend Stone duality to classical existential quantification by characterising the required dual properties on the topological side. This work extends that of Coumans by additionally considering intuitionistic existential

and universal quantification, and the identity predicate. By slotting the propositional bunched logic dualities into the right place, dualities are obtained for structures that interpret predicate bunched logic; including BI hyperdoctrines as a particular case.

In Section 8.1 categorical structures that extend the algebras and frames that interpret propositional bunched logics—hyperdoctrines and indexed frames—are defined, together with the interpretation of predicate bunched logics upon them. In Section 8.2 it is shown that the standard model of Separation Logic is given by an indexed (B)BI frame (*cf.* Biering et al's [24] demonstration that the standard model of Separation Logic forms a (B)BI hyperdoctrine). A model of predicate ILGL based on bigraphs is also given. In Section 8.3 the representation and duality theorems of Chapter 6 are extended to these categorical structures.

This chapter is based on material from the conference paper *A Stone-Type Duality Theorem for Separation Logic via its Underlying Bunched Logics* [80] as well as the journal papers *Intuitionistic Layered Graph Logic: Semantics and Proof Theory* [82] and *Stone-Type Dualities for Separation Logics* [83].

8.1 Categorical Structures for Predicate Bunched Logics

The first step is to specify the semantic structures of interest for predicate bunched logics. As in the previous chapter, $\mathscr{L}$ will be used to refer to an arbitrary propositional bunched logic from Part I. Crucially, as only additive quantification is being considered, the key aspect of $\mathscr{L}$ that affects what follows is the presence of classical or intuitionistic additives. As has been the case previously, generally speaking the case for classical additives can be obtained as a corollary from the case for intuitionistic additives by replacing order with equality and making an argument with regards the maximality of prime filters on Boolean algebras. Where it guides comprehension, the differences between $\mathscr{L}$ with classical additives and $\mathscr{L}$ with intuitionistic additives will be explicitly spelled out.

We start on the algebraic side with $\mathscr{L}$ hyperdoctrines.

Definition 8.1 ($\mathscr{L}$ Hyperdoctrine (cf. [183, 24])**.** *A* $\mathscr{L}$ hyperdoctrine *is a tuple*

$$(\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}, (=_X)_{X \text{ in } Ob(\mathrm{C})}, (\exists X_\Gamma, \forall X_\Gamma)_{X,\Gamma \text{ in } Ob(\mathrm{C})})$$

such that

1. C *is a category with finite products;*
2. $\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}$ *is a functor such that, for each object* X *in* C, $\mathbb{P}(X)$ *is a* $\mathscr{L}$ *algebra, and, for each morphism* f *in* C, $\mathbb{P}(f)$ *is a* $\mathscr{L}$ *algebra homomorphism;*

3. *For each object X in* C *and each diagonal morphism $\Delta_X : X \to X \times X$ in* C*, the element $=_X \in \mathbb{P}(X \times X)$ is adjoint at $\top_{\mathbb{P}(X)}$. That is, for all $a \in \mathbb{P}(X \times X)$,*

$$\top_{\mathbb{P}(X)} \leq \mathbb{P}(\Delta_X)(a) \textit{ iff } =_X \leq a;$$

4. *For each pair of objects Γ, X in* C *and each projection $\pi_{\Gamma,X} : \Gamma \times X \to \Gamma$ in* C*, $\exists X_\Gamma$ and $\forall X_\Gamma$ are left and right adjoint to $\mathbb{P}(\pi_{\Gamma,X})$. That is, they are monotone maps $\exists X_\Gamma : \mathbb{P}(\Gamma \times X) \to \mathbb{P}(\Gamma)$ and $\forall X_\Gamma : \mathbb{P}(\Gamma \times X) \to \mathbb{P}(\Gamma)$ such that, for all $a, b \in \mathbb{P}(\Gamma)$,*

$$\begin{array}{lll} \exists X_\Gamma(a) \leq b & \textit{iff} & a \leq \mathbb{P}(\pi_{\Gamma,X})(b) \text{ and} \\ \mathbb{P}(\pi_{\Gamma,X})(b) \leq a & \textit{iff} & b \leq \forall X_\Gamma(a). \end{array}$$

This assignment of adjoints is additionally natural in Γ: given a morphism $s : \Gamma \to \Gamma'$, the following diagrams commute:

$$\begin{array}{ccc} \mathbb{P}(\Gamma' \times X) & \xrightarrow{\mathbb{P}(s \times id_X)} & \mathbb{P}(\Gamma \times X) \\ {\scriptstyle \exists X_{\Gamma'}}\downarrow & & \downarrow{\scriptstyle \exists X_\Gamma} \\ \mathbb{P}(\Gamma') & \xrightarrow[\mathbb{P}(s)]{} & \mathbb{P}(\Gamma) \end{array} \qquad \begin{array}{ccc} \mathbb{P}(\Gamma' \times X) & \xrightarrow{\mathbb{P}(s \times id_X)} & \mathbb{P}(\Gamma \times X) \\ {\scriptstyle \forall X_{\Gamma'}}\downarrow & & \downarrow{\scriptstyle \forall X_\Gamma} \\ \mathbb{P}(\Gamma') & \xrightarrow[\mathbb{P}(s)]{} & \mathbb{P}(\Gamma) \end{array}$$

(B)BI hyperdoctrines were first formulated by Biering et al. [24] to prove the existence of models of higher-order variants of Separation Logic. There it was shown that the standard model of Separation Logic could be seen as a BBI hyperdoctrine, and thus safely extended with additional structure in the domain C^{op} to directly define abstract predicates like lists, trees, finite sets and relations inside the logic. The present work strengthens this result to a dual equivalence of categories. Other algebraic models of Separation Logic, like those based on Boolean quantales [71] or formal power series [85], can be seen as particular instantiations of BBI hyperdoctrines. The general definition of hyperdoctrine for classical and intuitionistic predicate logic that BI hyperdoctrines are derived from was formulated by Lawvere [151] based on his insight that quantifiers are adjoints to substitution, with our simplified presentation due to Pitts [183].

As these semantic structures support it, we consider many-sorted predicate logics. Intuitively, the category $\mathscr{C}$ gives a category of types, which can be built into contexts through the use of finite products. The functor assigns an indexing of algebraic models of propositional $\mathscr{L}$ over contexts of variables, allowing an interpretation of formulae in a context using the structure of the $\mathscr{L}$ algebra assigned to that context. Quantification is interpreted by $\exists X_\Gamma$ and $\forall X_\Gamma$, with the coherence

conditions in the definition ensuring that the move between contexts required of quantification works as one would expect. Finally, the equality predicate $=_X$ interprets equality for terms of type X.

Formally, an interpretation $[\![-]\!]$ of predicate $\mathscr{L}$ in a $\mathscr{L}$ hyperdoctrine is specified as follows. An object $[\![X]\!]$ of C is assigned to each type X, and for each context $\Gamma = \{v_1 : X_1, \ldots, v_n : X_n\}$ the object $[\![\Gamma]\!] = [\![X_1]\!] \times \cdots \times [\![X_n]\!]$ is assigned. Each function symbol $f : X_1 \times \cdots \times X_n \to X$ has a morphism $[\![f]\!] : [\![X_1]\!] \times \cdots [\![X_n]\!] \to [\![X]\!]$ assigned to it. With this data, each term of type X in context Γ can be inductively given a morphism $[\![t]\!] : [\![\Gamma]\!] \to [\![X]\!]$ assigned to it in the standard way (for example, [183]).

For each m-ary predicate symbol P of type $X_1, \ldots, X_m$, an assignment $[\![P]\!] \in \mathbb{P}([\![X_1]\!] \times \cdots \times [\![X_m]\!])$ is specified. The structure of the hyperdoctrine then permits an extension of $[\![-]\!]$ to predicate $\mathscr{L}$ formulae φ in context Γ by first setting $[\![Pt_1 \ldots t_m]\!] = \mathbb{P}(\langle [\![t_1]\!], \ldots, [\![t_m]\!] \rangle)([\![P]\!])$ and $[\![t =_X t']\!] = \mathbb{P}(\langle [\![t]\!], [\![t']\!] \rangle)(=_{[\![X]\!]})$. Elements of the $\mathscr{L}$ algebra $\mathbb{P}([\![\Gamma]\!])$ are then assigned inductively to formulae built from the non-quantifier connectives in the same way as the propositional case. Finally $[\![\exists v : X.\varphi]\!] = \exists [\![X]\!]_{[\![\Gamma]\!]}([\![\varphi]\!])$ and $[\![\forall v : X.\varphi]\!] = \forall [\![X]\!]_{[\![\Gamma]\!]}([\![\varphi]\!])$. Substitution of terms is given by $[\![\varphi(t/x)]\!] = \mathbb{P}([\![t]\!])([\![\varphi]\!])$.

φ in context Γ is satisfied by an interpretation $[\![-]\!]$ if $[\![\varphi]\!] = \top_{\mathbb{P}([\![\Gamma]\!])}$, and φ is valid if it is satisfied by all interpretations. A standard Lindenbaum-Tarski construction suffices to prove soundness and completeness of predicate $\mathscr{L}$ with respect to interpretations on $\mathscr{L}$ hyperdoctrines.

Theorem 8.2 (cf. [183, 24])**.** *For all predicate $\mathscr{L}$ formulas φ, ψ in context Γ, $\varphi \vdash^{\Gamma} \psi$ is provable iff, for all $\mathscr{L}$ hyperdoctrines $\mathbb{P}$ and all interpretations $[\![-]\!]$, $[\![\varphi]\!] \leq_{\mathbb{P}([\![\Gamma]\!])} [\![\psi]\!]$.* □

It is also worth stating a simple lemma that can be obtained as an immediate consequence of the adjointness properties of $\exists X_\Gamma$ and $\forall X_\Gamma$ as it will be used frequently in what follows.

Lemma 8.3. *Given a $\mathscr{L}$ hyperdoctrine $\mathbb{P} : \mathrm{C}^{op} \to$ Poset, for all $a, b \in \mathbb{P}(\Gamma)$ the following hold:*

1. *$a \leq \mathbb{P}(\pi_{\Gamma,X})(\exists X_\Gamma(a))$ and $\exists X_\Gamma(\mathbb{P}(\pi_{\Gamma,X})(b)) \leq b$;*
2. *$b \leq \forall X_\Gamma(\mathbb{P}(\pi_{\Gamma,X})(b))$ and $\mathbb{P}(\pi_{\Gamma,X})(\forall X_\Gamma(a)) \leq a$;*
3. *$\exists X_\Gamma(\bot) = \bot$ and $\forall X_\Gamma(\top) = \top$.* □

While hyperdoctrines are well researched structures, the predicate analogues of $\mathscr{L}$ frames—*indexed $\mathscr{L}$ frames*—are new. This definition is adapted from the

notion of indexed Stone space presented by Coumans [65] as a topological dual for Boolean hyperdoctrines. In contrast to the duality presented there, we prove the duality for the more general intuitionistic case and additionally consider (typed) equality and universal quantification. Indexed $\mathscr{L}$ frames may also be seen as a generalisation of Shirasu's *metaframes* [207], another type of indexed frame introduced to interpret predicate superintuitionistic and modal logics.

Definition 8.4 (Indexed $\mathscr{L}$ Frame)**.** *An* indexed $\mathscr{L}$ frame *is a functor* $\mathscr{R} : \mathrm{C} \to \mathscr{L}$ *such that*

1. C *is a category with finite products;*
2. *For all objects* Γ, Γ' *and* X *in* C*, all morphisms* $s : \Gamma \to \Gamma'$ *and all product projections* $\pi_{\Gamma,X}$*, for the following commutative square*

$$\begin{array}{ccc} \mathscr{R}(\Gamma \times X) & \xrightarrow{\mathscr{R}(\pi_{\Gamma,X})} & \mathscr{R}(\Gamma) \\ \Big\downarrow{\scriptstyle \mathscr{R}(s \times id_X)} & & {\scriptstyle \mathscr{R}(s)}\Big\downarrow \\ \mathscr{R}(\Gamma' \times X) & \xrightarrow[\mathscr{R}(\pi_{\Gamma',X})]{} & \mathscr{R}(\Gamma') \end{array}$$

   (a) *(for* $\mathscr{L}$ *with intuitionistic additives) the Pseudo Epi property holds:* $\mathscr{R}(\pi_{\Gamma',X})(y) \preccurlyeq \mathscr{R}(s)(x)$ *implies there exists* z *such that:* $\mathscr{R}(\pi_{\Gamma,X})(z) \preccurlyeq x$ *and* $y \preccurlyeq \mathscr{R}(s \times id_X)(z)$*;*

   (b) *(for* $\mathscr{L}$ *with classical additives) the quasi-pullback property holds: the induced map* $\mathscr{R}(\Gamma \times X) \to \mathscr{R}(\Gamma) \times_{\mathscr{R}(\Gamma')} \mathscr{R}(\Gamma' \times X)$ *is an epimorphism.*

Indexed $\mathscr{L}$ frames work similarly to $\mathscr{L}$ hyperdoctrines, with frames substituted for algebras: C once again acts as a category of contexts, with $\mathscr{R}$ assigning a $\mathscr{L}$ frame to each context. Although it may not look like it yet, condition 2. ensures that an interpretation of quantifiers based on the projections coheres correctly with the appropriate changes in context. The relation between the definition for intuitionistic additives and classical additives may not seem entirely clear at first, but unpacking what it means for the square to be a quasi-pullback should clarify: if $\mathscr{R}(\pi_{\Gamma',X})(y) = \mathscr{R}(s)(x)$ then there exists z such that: $\mathscr{R}(\pi_{\Gamma,X})(z) = x$ and $y = \mathscr{R}(s \times id_X)(z)$. This then fits in with our past practice of treating bunched logics with classical additives as the special case where the order in the frame semantics is equality.

A Kripke-style semantics can be given for predicate $\mathscr{L}$ on indexed $\mathscr{L}$ frames. For $\mathscr{L}$ with intuitionistic additives, an interpretation $[\![-]\!]$ is defined in precisely the same way as for $\mathscr{L}$ hyperdoctrines, except for the key difference that each

$x, [\![-]\!] \models^{\Gamma} \quad Pt_1 \dots t_m \quad \text{iff } \mathscr{R}(\langle [\![t_1]\!], \dots, [\![t_m]\!] \rangle)(x) \in [\![P]\!]$

$x, [\![-]\!] \models^{\Gamma} \quad t =_X t' \quad \text{iff } \mathscr{R}(\langle [\![t]\!], [\![t']\!] \rangle)(x) \in Ran(\mathscr{R}(\Delta_{[\![X]\!]}))$

$x, [\![-]\!] \models^{\Gamma} \exists v_{n+1} : X\, \varphi$ iff there exists $x' \in \mathscr{R}([\![\Gamma]\!] \times [\![X]\!])$ s.t. $\mathscr{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(x') = x$ and $x', [\![-]\!] \models^{\Gamma \cup \{v_{n+1}:X\}} \varphi$

$x, [\![-]\!] \models^{\Gamma} \forall v_{n+1} : X\, \varphi$ iff for all $x' \in \mathscr{R}([\![\Gamma]\!] \times [\![X]\!])$, $\mathscr{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(x') \succcurlyeq_{\mathscr{R}([\![\Gamma]\!])} x$, implies $x', [\![-]\!] \models^{\Gamma \cup \{v_{n+1}:X\}} \varphi$

Figure 8.1: Satisfaction on indexed $\mathscr{L}$ frames. For $\mathscr{L}$ with classical additives, $\succcurlyeq$ is $=$.

m-ary predicate symbol P of type $X_1, \dots, X_m$ an upwards closed subset $[\![P]\!] \in \mathscr{P}_{\succcurlyeq}(\mathscr{R}([\![X_1]\!] \times \cdots \times [\![X_m]\!]))$ is assigned. Similarly, an interpretation $[\![-]\!]$ for predicate $\mathscr{L}$ with classical additives is given in the same way as it is for $\mathscr{L}$ hyperdoctrines, except that, for every m-ary predicate symbol P of type $X_1, \dots, X_m$, a subset $[\![P]\!] \in \mathscr{P}(\mathscr{R}([\![X_1]\!] \times \cdots \times [\![X_m]\!]))$ is assigned to P. Then for formulas φ of $\mathscr{L}$ in context Γ with $x \in \mathscr{R}([\![\Gamma]\!])$ the satisfaction relation $\models^{\Gamma}$ is inductively defined using the clauses in Fig 8.1, together with the usual satisfaction clauses for $\mathscr{L}$ for the non-quantifier connectives, using the $\mathscr{L}$ frame structure of $\mathscr{R}([\![\Gamma]\!])$. There, $Ran(\mathscr{R}(\Delta_{[\![X]\!]})) = \{y \mid \exists z(\mathscr{R}(\Delta_{[\![X]\!]})(z) = y)\}$. We note that bound variables are renamed to be fresh throughout, in an order determined by quantifier depth.

Lemma 8.5. *For $\mathscr{L}$ with intuitionistic additives, the satisfaction relation $\models^{\Gamma}$ on indexed $\mathscr{L}$ frames is persistent.*

Proof. For atomic predicate formulas this is by design, with the assignment of upwards closed subsets to predicate symbols akin to a persistent valuation. For formulas of the form $t =_X t'$ this follows from the fact that $\mathscr{R}(\Delta_X)$ is a $\mathscr{L}$ morphism and hence order preserving. The rest of the clauses follow by an inductive argument, the most involved of which is for formulae of the form $\exists v_{n+1} : X\, \varphi$.

Suppose $x, [\![-]\!] \models^{\Gamma} \exists v_{n+1} : X\, \varphi$ and $y \succcurlyeq_{\mathscr{R}([\![\Gamma]\!])} x$. Then by definition there exists x' such that $\mathscr{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(x') = x \preccurlyeq_{\mathscr{R}([\![\Gamma]\!])} y$ and $x', [\![-]\!] \models^{\Gamma \cup \{v_{n+1}:X\}} \varphi$. Since $\mathscr{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})$ is a $\mathscr{L}$ morphism, and thus a intuitionistic morphism, there exists y' such that $y' \succcurlyeq_{\mathscr{R}([\![\Gamma]\!] \times [\![X]\!])} x'$ and $\mathscr{R}(\pi_{[\![\Gamma]\!],[\![X]\!]})(y') = y$. By the inductive hypothesis, $y', [\![-]\!] \models^{\Gamma \cup \{v_{n+1}:X\}} \varphi$ and so $y, [\![-]\!] \models^{[\![\Gamma]\!]} \exists v_{n+1} : X\, \varphi$. □

8.2 Bunched Logic Models as Indexed Frames

Although at first sight it may not seem so, indexed frames and the semantics based upon them are a generalisation of the standard store–heap semantics of Separation Logic. A similar construction can also guide the definition of predicate models of layered graph logic.

8.2.1 Separation Logic as an Indexed (B)BI Frame

Recall the BI frame $\text{Heap}^{\text{BI}} = (H, \cdot, \succcurlyeq, H)$, where H is the set of heaps, $\succcurlyeq$ is heap extension, and $\cdot$ is disjoint union. This is the BI frame corresponding to the partial monoid of heaps. We define an indexed BI frame $\text{Store}^{\text{BI}} : \text{Set} \to \text{BIFr}$ on objects by $\text{Store}^{\text{BI}}(X) = (X \times H, \cdot_X, \sqsubseteq_X, X \times H)$, where $(x_2, h_2) \in (x_0, h_0) \cdot_X (x_1, h_1)$ iff $x_0 = x_1 = x_2$ and $h_2 \in h_0 \cdot h_1$, and $(x_0, h_0) \succcurlyeq_X (x_1, h_1)$ iff $x_0 = x_1$ and $h_0 \succcurlyeq h_1$. On morphisms, set $\text{Store}^{\text{BI}}(f : X \to Y)(x, h) = (f(x), h)$. It is straightforward to see this defines a functor: for arbitrary X, $\text{Store}(X)$ inherits the BI frame properties from Heap and for arbitrary $f : X \to Y$, $\text{Store}(f)$ is trivially a BI morphism as it is identity on the structure that determines the back and forth conditions. The property (Pseudo Epi) is also trivially satisfied so this defines an indexed BI frame.

For Separation Logic with classical additives, we instead start with the BBI frame $\text{Heap}^{\text{BBI}} = (H, \cdot, \{[]\})$ where $[]$ is the empty heap. Then $\text{Store}^{\text{BBI}}$ is defined in essentially the same way, with $\text{Store}^{\text{BBI}}(X) = (X \times H, \cdot_X, X \times \{[]\})$ and $\text{Store}^{\text{BBI}}(f)(x, h) = (f(x), h)$. This defines an indexed BBI frame.

We now describe the interpretations $[\![-]\!]$ on $\text{Store}^{\text{(B)BI}}$ that yield the standard models of Separation Logic. We have one type Val and we set $[\![\text{Val}]\!] = \mathbb{Z}$, with the arithmetic operations $[\![+]\!], [\![-]\!] : [\![\text{Val}]\!]^2 \to [\![\text{Val}]\!]$ defined as one would expect. Term morphisms $[\![t]\!] : [\![\text{Val}]\!]^n \to [\![\text{Val}]\!]$ in context $\Gamma = \{v_1, \ldots v_n\}$ are then defined as usual, with each constant n assigned the morphism $[\![n]\!] : [\![\Gamma]\!] \longrightarrow \{*\} \xrightarrow{n} [\![\text{Val}]\!]$. As one would expect, the key difference between the two interpretations is in the interpretation of the points-to predicate. For Intuitionistic Separation Logic, the points-to predicate $\mapsto$ is assigned

$$[\![\mapsto]\!] = \{((a, a'), h) \mid a \in dom(h) \text{ and } h(a) = a'\} \in \mathscr{P}_{\succcurlyeq_{[\![\text{Val}]\!]^2}}(\text{Store}^{\text{BI}}([\![\text{Val}]\!]^2)).$$

This set is clearly upwards closed with respect to the order $\succcurlyeq_{[\![\text{Val}]\!]^2}$ so this is a well-defined interpretation. For Classical Separation Logic, $\mapsto$ is instead assigned

$$[\![\mapsto]\!] = \{((a, a'), h) \mid \{a\} = dom(h) \text{ and } h(a) = a'\} \in \mathscr{P}(\text{Store}^{\text{BBI}}([\![\text{Val}]\!]^2)).$$

In the indexed (B)BI frame $\text{Store}^{\text{(B)BI}} : \text{Set} \to \text{ResFr}$ with the interpretations just defined, a store is represented as an n-place vector of values over $[\![\text{Val}]\!]$. That is, the store $s = \{(v_1, a_1), \ldots, (v_n, a_n)\}$ is given by the element $(a_1, \ldots, a_n) \in [\![\text{Val}]\!]^n$. By a simple inductive argument we have the following result:

Theorem 8.6. *For all formulas φ of (B)BI pointer logic, all stores $s = \{(v_1, a_1), \ldots, (v_n, a_n)\}$ and all heaps h, $s, h \vDash \phi$ iff $((a_1, \ldots, a_n), h), [\![-]\!] \vDash^{\Gamma} \phi$.* □

After verifying that terms are evaluated to the same elements as the standard

model in both representations, the equivalence of the clauses for atomic formulas can be computed directly. The equivalence of the quantifier clauses is down to the representation of stores as vectors and the action of the product projections under the functor Store. The notions of indexed (B)BI frame and its associated semantics are therefore a natural generalization of the standard Separation Logic model.

8.2.2 Bigraphs as Indexed ILGL Frames

We now give a bigraph model of predicate ILGL, inspired by Separation Logic. It is based on the bigraph scaffolds defined in Chapter 2. These are ordered scaffolds in which the order represents a place graph (interpreted spatially), non-$\mathscr{E}$ edges represent the path information of the link graph and the layering structure represents the composition of bigraphs.

We consider the following two predicates: an unary predicate $\mathit{Contains}(-)$ and a binary predicate $- \mapsto -$. Informally, $\mathit{Contains}(r)$ designates that a subgraph contains a resource r and $r \mapsto r'$ designates that a subgraph contains a path from a resource r to a resource r'. Let $\mathscr{X}$ be a bigraph scaffold of the sort defined in Chapter 2 Section 2.2.2. Denote by G the union of the place graphs of the system. A *resource assignment* s is a finite partial function $s : \mathit{Res} \to \mathscr{P}_{\preccurlyeq}(\mathscr{V}(G))$. As the order is determined by the spatial containment encoded by the place graphs, that a resource assignment maps resources to upwards closed sets just means that if a location x contains a resource r, and x is contained in y then y also contains r. Given a resource assignment s, together with $r \in \mathit{Res}$, we define $s[r \mapsto A]$ to be the resource assignment that is equal to s everywhere except r, where it assigns r to the upward-closed set of vertices A. We define a semantics on pairs (s,H) for this extended language in Figure 8.2.

There are many design choices possible here. For example, we could additionally assign weights or permissions to edges, with satisfaction of the $\mapsto$ predicate mediated by conditions on the path. Coupled with a notion of dynamics that evolves assignments and the underlying graph theoretic structure we would have a rich environment within which to model a variety of distributed systems. We defer to another occasion an in-depth investigation of such a framework.

Similarly to the previous example, what we have defined corresponds to an indexed ILGL frame. First, let $(X, \circ, \succcurlyeq)$ be the ILGL frame corresponding to the bigraph scaffold $\mathscr{X}$: $L \in H \circ K$ iff $H \mathbin{@}_{\mathscr{E}} K \downarrow$ and $H \mathbin{@}_{\mathscr{E}} K = L$. Set $\mathscr{R} : \mathrm{Set} \to \mathrm{ILGL}$ by $\mathscr{R}(A) = (A \times X, \circ_A, \succcurlyeq_A)$ where $(z,L) \in (x,H) \circ_A (y,K)$ iff $x = y = z$, $H \mathbin{@}_{\mathscr{E}} K \downarrow$ and $H \mathbin{@}_{\mathscr{E}} K = L$, and $(x,H) \succcurlyeq_A (y,K)$ iff $x = y$ and $H \succcurlyeq K$. For functions $f : A \to B$ set $\mathscr{R}(f)(a,H) = (f(a),H)$. This defines a functor $\mathscr{R}$ which trivially satisfies (Psuedo Epi). Hence $\mathscr{R}$ is an indexed ILGL frame.

$s,H \vDash$	$\top$	always
$s,H \vDash$	$\bot$	never
$s,H \vDash$	$\mathrm{Contains}(r)$	iff H contains a $s(r)$-vertex
$s,H \vDash$	$r \mapsto r'$	iff there exists a path from a $s(r)$-vertex to a $s(r')$-vertex in H
$s,H \vDash$	$\varphi \wedge \psi$	iff $s,H \vDash \varphi$ and $s,H \vDash \psi$
$s,H \vDash$	$\varphi \vee \psi$	iff $s,H \vDash \varphi$ or $s,H \vDash \psi$
$s,H \vDash$	$\varphi \to \psi$	iff for all $K \succcurlyeq H$, $s,K \vDash \varphi$ implies $s,K \vDash \psi$
$s,H \vDash$	$\varphi * \psi$	iff there exists $K_0 @_{\mathscr{E}} K_1 \downarrow$ s.t. $H \succcurlyeq K_0 @_{\mathscr{E}} K_1$, $s,K_0 \vDash \varphi$ and $s,K_1 \vDash \psi$
$s,H \vDash$	$\varphi \mathrel{-\!\!*} \psi$	iff for all K and $L \succcurlyeq H$ s.t. $L @_{\mathscr{E}} K \downarrow$: $s,L \vDash \varphi$ implies $s,L @_{\mathscr{E}} K \vDash \psi$
$s,H \vDash$	$\varphi \mathrel{*\!\!-} \psi$	iff for all K and $L \succcurlyeq H$ s.t. $K @_{\mathscr{E}} L \downarrow$: $s,L \vDash \varphi$ implies $s,K @_{\mathscr{E}} L \vDash \psi$
$s,H \vDash$	$\exists r\varphi$	iff there exists A s.t. $s[r \mapsto A],H \vDash \varphi$
$s,H \vDash$	$\forall r\varphi$	iff for all A, $H \preccurlyeq K$ implies $s[r \mapsto A],K \vDash \varphi$

Figure 8.2: Satisfaction for bigraph models of predicate ILGL.

We define the following interpretation. The single sort is interpreted as $\mathscr{P}_{\succcurlyeq}(V(G))$ where G is the graph union of the place graph vertices of the system of bigraphs. The predicate symbols are interpreted as $[\![\mathit{Contains}(-)]\!] = \{(A,H) \mid \exists x \in A : x \in V(H)\}$ and

$$[\![\mapsto]\!] = \{((A_1,A_2),H) \mid \exists x_1 \in A_1 \text{ and } x_2 \in A_2 : H \text{ contains a path } x_1 \text{ to } x_2\}.$$

Let r_i be an enumeration of resources in *Res* and let φ be a formula with free variables amongst $r_1,\ldots,r_n$. Then

$$\{(r_1,A_1),\ldots,(r_n,A_n)\},G \vDash \varphi \text{ iff } ((A_1,\ldots,A_n),G),[\![-]\!] \vDash^{\{r_1,\ldots,r_n\}} \varphi.$$

8.3 Duality for Bunched Logic Hyperdoctrines

We now extend the duality results given for $\mathscr{L}$ algebras to $\mathscr{L}$ hyperdoctrines. For this to make sense, both $\mathscr{L}$ hyperdoctrines and indexed $\mathscr{L}$ frames need to be equipped with a notion of morphism to yield categories. The definition of $\mathscr{L}$ hyperdoctrine morphism adapts that for *coherent* hyperdoctrines given by Coumans [66].

Definition 8.7 ($\mathscr{L}$ Hyperdoctrine Morphism)**.** *Given a pair of $\mathscr{L}$ hyperdoctrines* $\mathbb{P} : \mathrm{C}^{op} \to \mathrm{Poset}$ *and* $\mathbb{P}' : \mathrm{D}^{op} \to \mathrm{Poset}$*, a $\mathscr{L}$* hyperdoctrine morphism $(\mathrm{K},\tau) : \mathbb{P} \to \mathbb{P}'$

is a pair (K, τ) *satisfying the following properties:*

1. $\mathrm{K} : \mathrm{C} \to \mathrm{D}$ *is a finite product preserving functor;*
2. $\tau : \mathbb{P} \to \mathbb{P}' \circ K$ *is a natural transformation;*
3. *For all objects* X *in* C: $\tau_{X\times X}(=_X) = \ ='_{K(X)}$;
4. *For all objects* Γ *and* X *in* C, *the following squares commute:*

$$\begin{array}{ccc} \mathbb{P}(\Gamma \times X) & \xrightarrow{\tau_{\Gamma\times X}} & \mathbb{P}'(K(\Gamma)\times K(X)) \\ {\scriptstyle \exists X_\Gamma}\downarrow & & \downarrow{\scriptstyle \exists' K(X)_{K(\Gamma)}} \\ \mathbb{P}(\Gamma) & \xrightarrow{\tau_\Gamma} & \mathbb{P}'(K(\Gamma)) \end{array} \qquad \begin{array}{ccc} \mathbb{P}(\Gamma \times X) & \xrightarrow{\tau_{\Gamma\times X}} & \mathbb{P}'(K(\Gamma)\times K(X)) \\ {\scriptstyle \forall X_\Gamma}\downarrow & & \downarrow{\scriptstyle \forall' K(X)_{K(\Gamma)}} \\ \mathbb{P}(\Gamma) & \xrightarrow{\tau_\Gamma} & \mathbb{P}'(K(\Gamma)) \end{array}$$

The composition of $\mathscr{L}$ *hyperdoctrine morphisms* $(K, \tau) : \mathbb{P} \to \mathbb{P}'$ *and* $(K', \tau') : \mathbb{P}' \to \mathbb{P}''$ *is given by* $(K' \circ K, \tau'_{K(-)} \circ \tau)$. *This forms a category* $\mathscr{L}$Hyp.

For indexed $\mathscr{L}$ frames the definition of morphism splits because of the weakening of equality to a preorder on the intuitionistic side. It is straightforward to show that the notion of indexed $\mathscr{L}$ frame morphism when $\mathscr{L}$ has intuitionistic additives collapses to that for $\mathscr{L}$ with classical additives when the preorders $\succcurlyeq$ are substituted for $=$.

Definition 8.8 (Indexed $\mathscr{L}$ Frame Morphism for intuitionistic $\mathscr{L}$)**.** *Given indexed* $\mathscr{L}$ *frames* $\mathscr{R} : \mathrm{C} \to \mathscr{L}$ *and* $\mathscr{R}' : \mathrm{D} \to \mathscr{L}$ *for* $\mathscr{L}$ *with intuitionistic additives, an* indexed $\mathscr{L}$ frame morphism $(L, \lambda) : \mathscr{R} \to \mathscr{R}'$ *is a pair* (L, λ) *such that:*

1. $L : D \to C$ *is a finite product preserving functor;*
2. $\lambda : \mathscr{R} \circ L \to \mathscr{R}'$ *is a natural transformation;*
3. *(Lift Property) If there exists* x *and* y *such that* $\mathscr{R}'(\Delta_X)(y) \preccurlyeq \lambda_{X\times X}(x)$ *then there exists* y' *such that* $\mathscr{R}(\Delta_{L(X)})(y') \preccurlyeq x$;
4. *(Morphism Pseudo Epi) If there exists* x *and* y *with* $\mathscr{R}'(\pi_{\Gamma,X})(x) \preccurlyeq \lambda_\Gamma(y)$ *then there exists* z *such that* $x \preccurlyeq \lambda_{\Gamma\times X}(z)$ *and* $\mathscr{R}(\pi_{L(\Gamma),L(X)})(z) \preccurlyeq y$.

The composition of indexed $\mathscr{L}$ *frame morphisms* $(L', \lambda') : \mathscr{R}' \to \mathscr{R}''$ *and* $(L, \lambda) : \mathscr{R} \to \mathscr{R}'$ *is given by* $(L \circ L', \lambda' \circ \lambda_{L'(-)})$. *This yields a category* Ind$\mathscr{L}$.

Definition 8.9 (Indexed $\mathscr{L}$ Frame Morphism for classical $\mathscr{L}$)**.** *For indexed* $\mathscr{L}$ *frames* $\mathscr{R} : \mathrm{C} \to \mathscr{L}$ *and* $\mathscr{R}' : \mathrm{D} \to \mathscr{L}$ *for* $\mathscr{L}$ *with classical additives, an* indexed $\mathscr{L}$ frame morphism $(L, \lambda) : \mathscr{R} \to \mathscr{R}'$ *is a pair* (L, λ) *satisfying 1. and 2. of the previous definition as well as*

(3′) (Lift Property′) if there exist x and y such that $\lambda_{X\times X}(x) = \mathscr{R}'(\Delta_X)(y)$, then there exists y' such that $\mathscr{R}((\Delta_{L(X)}))(y') = x$, and

(4′) (Quasi-Pullback) for all objects Γ and X in C*, the following square is a quasi-pullback:*

$$\begin{array}{ccc} \mathscr{R}(L(\Gamma)\times L(X)) & \xrightarrow{\lambda_{\Gamma\times X}} & \mathscr{R}(\Gamma\times X) \\ \downarrow{\scriptstyle \mathscr{R}(\pi_{L(\Gamma),L(X)}))} & & \downarrow{\scriptstyle \mathscr{R}'(\pi_{\Gamma,X})} \\ \mathscr{R}(L(\Gamma)) & \xrightarrow{\lambda_\Gamma} & \mathscr{R}(\Gamma) \end{array}$$

The composition of indexed $\mathscr{L}$ frame morphisms $(L',\lambda') : \mathscr{R}' \to \mathscr{R}''$ and $(L,\lambda) : \mathscr{R} \to \mathscr{R}'$ is given by $(L\circ L', \lambda'\circ\lambda_{L'(-)})$. This yields a category Ind$\mathscr{L}$.

Next we lift the complex algebra and prime filter frame constructions to the level of indexed frames and hyperdoctrines. This can straightforwardly be achieved by composing hyperdoctrines and indexed frames with the prime filter and complex algebra functors respectively, and most of the required properties follow immediately from the results of Chapter 6. As was the case for the propositional bunched logics, this yields a representation theorem for hyperdoctrines that proves completeness of the indexed frame semantics, as well as assignments on objects that can be made functorial.

Definition 8.10 (Complex $\mathscr{L}$ Hyperdoctrine)**.** *Given an indexed $\mathscr{L}$ frame $\mathscr{R} : \mathrm{C} \to \mathscr{L}$ for $\mathscr{L}$ with intuitionistic additives, the* complex hyperdoctrine *of $\mathscr{R}$, $ComHyp^{\mathscr{L}}(\mathscr{R})$, is given by $Com^{\mathscr{L}}(\mathscr{R}(-)) : \mathrm{C}^{op} \to \mathscr{L}$*Alg*, together with $Ran(\mathscr{R}(\Delta_X))$ as $=_X$, $\mathscr{R}(\pi_{\Gamma,X})^*$ as $\exists X_\Gamma$, and $\mathscr{R}(\pi_{\Gamma,X})_*$ as $\forall X_\Gamma$, where*

$$\begin{aligned} \mathscr{R}(\pi_{\Gamma,X})^*(A) &= \{x \mid \text{there exists } y\in A : \mathscr{R}(\pi_{\Gamma,X})(y) \preccurlyeq x\} \text{ and} \\ \mathscr{R}(\pi_{\Gamma,X})_*(A) &= \{x \mid \text{for all } y, \text{ if } x \preccurlyeq \mathscr{R}(\pi_{\Gamma,X})(y) \text{ then } y\in A\}. \end{aligned}$$

For $\mathscr{L}$ with classical additives, the definitions of $\mathscr{R}(\pi_{\Gamma,X})^*$ and $\mathscr{R}(\pi_{\Gamma,X})_*$ are as above, except with $\preccurlyeq$ replaced with $=$.

Given that the complex algebra operations thus far have matched the corresponding semantic clauses on frames, one might have expected $\exists X_\Gamma$ to be given by the direct image $\mathscr{R}(\pi_{\Gamma,X})$. Using the fact that $\mathscr{R}(\pi_{\Gamma,X})$ is an $\mathscr{L}$ morphism—and thus an intuitionistic morphism—it can be shown that $\mathscr{R}(\pi_{\Gamma,X})^*$ is in fact identical to $\mathscr{R}(\pi_{\Gamma,X})$ so this is indeed the case. We use its presentation as $\mathscr{R}(\pi_{\Gamma,X})^*$ as it simplifies some proofs that follow.

Lemma 8.11. *Given an indexed $\mathscr{L}$ frame $\mathscr{R} : \mathrm{C} \to \mathscr{L}$, the complex hyperdoctrine $ComHyp^{\mathscr{L}}(\mathscr{R})$ is a $\mathscr{L}$ hyperdoctrine.*

Proof. We concentrate on the verifications relating to $\mathscr{R}(\pi_{\Gamma,X})^*$ and $\mathscr{R}(\pi_{\Gamma,X})_*$ for intuitionistic $\mathscr{L}$. It is straightforward to see these map upwards-closed sets to upwards-closed sets and are monotone with respect to the subset ordering $\subseteq$, meaning they are well-defined. The adjointness properties follow from the definitions so it just remains to prove naturality.

We give the case for $\exists X_\Gamma$. Given a morphism $s : \Gamma \to \Gamma'$ in C and an element $A \in Com^{\mathscr{L}}(\mathscr{R}(\Gamma' \times X))$, we must show $\mathscr{R}(\pi_{\Gamma,X})^*(\mathscr{R}(s \times id_X)^{-1}(A)) = \mathscr{R}(s)^{-1}(\mathscr{R}(\pi_{\Gamma',X})^*(A))$. Suppose $x \in \mathscr{R}^*(\pi_{\Gamma,X})(\mathscr{R}(s \times id_X)^{-1}(A))$: then there exists y such that $\mathscr{R}(\pi_{\Gamma,X})(y) \preccurlyeq x$ and $\mathscr{R}(s \times id_X)(y) \in A$. We have $\mathscr{R}(\pi_{\Gamma',X})(\mathscr{R}(s \times id_X)(y)) = \mathscr{R}(s)(\mathscr{R}(\pi_{\Gamma,X})(y)) \preccurlyeq \mathscr{R}(s)(x)$. Hence $x \in \mathscr{R}(s)^{-1}(\mathscr{R}(\pi_{\Gamma',X})^*(A))$, as required.

Conversely, assume $x \in \mathscr{R}(s)^{-1}(\mathscr{R}(\pi_{\Gamma',X})^*(A))$. Then there exists $y \in A$ such that $\mathscr{R}(\pi_{\Gamma',X})(y) \preccurlyeq \mathscr{R}(s)(x)$. Then by (Psuedo Epi), there exists z such that $\mathscr{R}(\pi_{\Gamma,X})(z) \preccurlyeq x$ and $y \preccurlyeq \mathscr{R}(s \times id_X)(z)$. By upwards-closure of A, $\mathscr{R}(s \times id_X)(z) \in A$. Hence we have $x \in \mathscr{R}(\pi_{\Gamma,X})^*(\mathscr{R}(s \times id_X)^{-1}(A))$, as required.

The proof for $\mathscr{L}$ with classical additives follows immediately by substituting every instance of $\preccurlyeq$ with $=$ in the above argument, where the quasi pullback property allows us to assume the existence of z such that $\mathscr{R}(\pi_{\Gamma,X})(z) = x$ and $y = \mathscr{R}(s \times id_X)(z)$ from $\mathscr{R}(\pi_{\Gamma',X})(y) = \mathscr{R}(s)(x)$. □

Definition 8.12 (Indexed Prime Filter Frame)**.** *Given a $\mathscr{L}$ hyperdoctrine $\mathbb{P}$, the* indexed prime filter frame *of $\mathbb{P}$, $IndPr^{\mathscr{L}}(\mathbb{P})$ is given by $Pr^{\mathscr{L}}(\mathbb{P}(-))$.*

Lemma 8.13. *Given a $\mathscr{L}$ hyperdoctrine $\mathbb{P} : \mathrm{C}^{op} \to$ Poset, the indexed prime filter frame $IndPr^{\mathscr{L}}(\mathbb{P})$ is an indexed $\mathscr{L}$ frame.*

Proof. We first show the Pseudo Epi property is satisfied when $\mathscr{L}$ has intuitionistic additives. Assume we have objects Γ, Γ' and X in C and a morphism $s : \Gamma \to \Gamma'$. Let prime filters F_x and F_y be such that $\mathbb{P}(\pi_{\Gamma',X})^{-1}(F_y) \subseteq \mathbb{P}(s)^{-1}(F_x)$. It is easy to see that

$$P(F) = \begin{cases} 1 & \text{if } \mathbb{P}(\pi_{\Gamma,X})^{-1}(F) \subseteq F_x \text{ and } F_y \subseteq \mathbb{P}(s \times id_X)^{-1}(F) \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. The only non-trivial verification is showing $P(F \cap F') = 1$ implies $P(F) = 1$ or $P(F') = 1$. Suppose $P(F \cap F') = 1, P(F) = 0$ and $P(F') = 0$. Necessarily there exists a and b such that $\mathbb{P}(\pi_{\Gamma,X})(a) \in F$, $\mathbb{P}(\pi_{\Gamma,X})(b) \in F'$ and $a, b \notin F_x$. Then $\mathbb{P}(\pi_{\Gamma,X})(a) \vee \mathbb{P}(\pi_{\Gamma,X})(b) = \mathbb{P}(\pi_{\Gamma,X})(a \vee b) \in F \cap F'$ so $a \vee b \in F_x$. However F_x is prime, so $a \in F_x$ or $b \in F_x$, a contradiction.

Consider the filter $F = [\mathbb{P}(s \times id_X)(F_y))$ and suppose for contradiction it is not proper. By Proposition 5.4 this entails there exists $a \in F_y$ such that $\mathbb{P}(s \times id_X)(a) =$

$\bot$. By adjointness, $\exists X_\Gamma(\bot) = \bot$, so $\mathbb{P}(s)(\exists X_{\Gamma'}(a)) = \exists X_\Gamma(\mathbb{P}(s \times id_X)(a)) = \bot$ by naturality. This entails $\exists X_{\Gamma'}(a) \notin \mathbb{P}(s)^{-1}(F_x)$ so $\exists X_{\Gamma'}(a) \notin \mathbb{P}(\pi_{\Gamma',X})^{-1}(F_y)$ by assumption. However, by adjointness and filterhood, $\mathbb{P}(\pi_{\Gamma',X})(\exists X_{\Gamma'}(a)) \in F_y$, a contradiction.

Clearly $\mathbb{P}(s \times id_X)^{-1}(F) \supseteq F_y$. To see that the other required inclusion holds, suppose $a \in \mathbb{P}(\pi_{\Gamma,X})^{-1}(F)$. Then there exists $b \in F_y$ such that $\mathbb{P}(s \times id_X)(b) \leq \mathbb{P}(\pi_{\Gamma,X})(a)$. By adjointness $\exists X_\Gamma(\mathbb{P}(s \times id_X)(b)) \leq a$ and so by naturality $\mathbb{P}(s)(\exists X_{\Gamma'}(b)) \leq a$. Since $\mathbb{P}(\pi_{\Gamma',X})(\exists X_{\Gamma'}(b)) \in F_y$, we have $\exists X_{\Gamma'}(b) \in \mathbb{P}(\pi_{\Gamma',X})^{-1}(F_y) \subseteq \mathbb{P}(s)^{-1}(F_x)$. Thus by filterhood, $a \in F_x$. Thus $P(F) = 1$ and by the prime extension lemma we have a prime F with $P(F) = 1$, as required.

For $\mathscr{L}$ with classical additives, we instead start with the assumption of prime filters F_x and F_y such that $\mathbb{P}(\pi_{\Gamma',X})^{-1}(F_y) = \mathbb{P}(s)^{-1}(F_x)$. This is sufficient to once again prove the existence of a prime filter F satisfying $\mathbb{P}(s \times id_X)^{-1}(F) \supseteq F_y$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(F) \subseteq F_x$. However, maximality of prime filters on Boolean algebras collapses the inclusions to equalities — $\mathbb{P}(s \times id_X)^{-1}(F) = F_y$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(F) = F_x$ — so the quasi pullback property holds. □

We now lift the representation theorem for $\mathscr{L}$ algebras to $\mathscr{L}$ hyperdoctrines, making essential use of the natural transformation θ used in the duality theorems of Chapter 6.

Theorem 8.14 (Representation Theorem for $\mathscr{L}$ Hyperdoctrines)**.** *Every $\mathscr{L}$ hyperdoctrine $\mathbb{P} : \mathrm{C}^{op} \to$ Poset can be embedded in a complex $\mathscr{L}$ hyperdoctrine. That is, $\Theta_\mathbb{P} : \mathbb{P} \to Com^{\mathscr{L}} Pr^{\mathscr{L}}(\mathbb{P}(-))$ defined $(Id_\mathrm{C}, \theta_{\mathbb{P}(-)})$ is a monomorphism.*

Proof. Clearly, by the representation theorem for $\mathscr{L}$ algebras, each component of $\Theta_\mathbb{P}$ is mono, and hence $\Theta_\mathbb{P}$ is mono. It remains to show that $\Theta_\mathbb{P}$ is a $\mathscr{L}$ hyperdoctrine morphism. That Id_C preserves finite products is immediate and that $\theta_{\mathbb{P}(-)} : \mathbb{P} \to Com^{\mathscr{L}} Pr^{\mathscr{L}}(\mathbb{P}(-))$ is a natural transformation is given by $\mathscr{L}$ duality.

First we note that property 3. of $\mathscr{L}$ hyperdoctrine morphisms holds. We must show that $\theta_{\mathbb{P}(X \times X)}(=_X) = Ran(\mathbb{P}(\Delta_X)^{-1})$ for any object X of C. First suppose $F = \mathbb{P}(\Delta_X)^{-1}(G)$ for some prime filter G. By adjointness of $=_X$ at $\top$ we have that $\mathbb{P}(\Delta_X)(=_X) = \top \in G$. Hence $=_X \in F$. Conversely, assume $=_X \in F$. Straightforwardly we have that

$$P(G) = \begin{cases} 1 & \text{if } \mathbb{P}(\Delta_X)^{-1}(G) \subseteq F \\ 0 & \text{otherwise} \end{cases}$$

defines a prime predicate. By the adjointness property of $=_X$ we have that $\mathbb{P}(\Delta_X)^{-1}(\{\top\}) \subseteq F$. Hence there exists prime G with $\mathbb{P}(\Delta_X)^{-1}(G) \subseteq F$ by the

prime extension lemma. For the case of $\mathscr{L}$ with intuitionistic additives, since $\mathbb{P}(\Delta_X)^{-1}$ is an intuitionistic morphism there then exists $G' \supseteq G$ with $\mathbb{P}(\Delta_X)^{-1}(G') = F$; for the case of $\mathscr{L}$ with classical additives, maximality of prime filters means $\mathbb{P}(\Delta_X)^{-1}(G) = F$. Either way, $F \in Ran(\mathbb{P}(\Delta_X)^{-1})$ as required.

For property 4. we verify the naturality diagram for $\exists X_\Gamma$: the verification of $\forall X_\Gamma$ is similar. The verification reduces to showing that, given a prime filter F of $\mathbb{P}(\Gamma)$ and $a \in \mathbb{P}(\Gamma \times X)$, $\exists X_\Gamma(a) \in F$ iff there exists G such that $a \in G$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(G) \subseteq F$. For $\mathscr{L}$ with intuitionistic additives this corresponds precisely to computing commutativity of the diagram, and for $\mathscr{L}$ with classical additives we can conclude $\mathbb{P}(\pi_{\Gamma,X})^{-1}(G) = F$ by maximality of prime filters, yielding commutativity of the appropriate diagram for that case.

First assume $\exists X_\Gamma(a) \in F$. It is straightforward to see that

$$P(G) = \begin{cases} 1 & \text{if } \mathbb{P}(\pi_{\Gamma,X})^{-1}(G) \subseteq F \\ 0 & \text{otherwise} \end{cases}$$

defines a prime predicate. Consider $G = [a)$. This is proper, as otherwise $a = \bot$, which would entail $\exists X_\Gamma(a) = \bot \in F$, contradicting that F is a prime (and thus proper) filter. Let $\mathbb{P}(\pi_{\Gamma,X})(b) \geq a$. Then by adjointness, $\exists X_\Gamma(a) \leq b \in F$. Hence $P(G) = 1$ and so there exists a prime filter G with $P(G) = 1$, as required. Now assume $a \in G$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(G) \subseteq F$. By adjointness $a \leq \mathbb{P}(\pi_{\Gamma,X})(\exists X_\Gamma(a)) \in G$, so $\exists X_\Gamma(a) \in \mathbb{P}(\pi_{\Gamma,X})^{-1}(G) \subseteq F$ as required. □

In much the same way as was shown in Chapter 7, the representation theorem yields completeness for the indexed frame semantics. Given any interpretation on an indexed $\mathscr{L}$ frame $[\![-]\!]$, we automatically have an interpretation for the complex hyperdoctrine as predicate symbols are interpreted as (upwards-closed) subsets; that is, elements of complex algebras of $\mathscr{L}$ frames. A simple inductive argument shows that satisfaction coincides for these models.

Proposition 8.15. *Given an indexed $\mathscr{L}$ frame $\mathscr{R}$ and an interpretation $[\![-]\!]$, for all predicate $\mathscr{L}$ formulae φ in context Γ and $x \in \mathscr{R}([\![\Gamma]\!])$, $x, [\![-]\!] \vDash^\Gamma \phi$ iff $x \in [\![\phi]\!]$.* □

Similarly, given an interpretation $[\![-]\!]$ on a $\mathscr{L}$ hyperdoctrine, we can define the interpretation $\widetilde{[\![-]\!]}$ by setting $\widetilde{[\![P]\!]} = \theta_{\mathbb{P}([\![X_1]\!] \times \cdots \times [\![X_m]\!])}([\![P]\!])$ for each predicate symbol of type $X_1, \ldots, X_m$. As a corollary of the representation theorem we obtain the following proposition.

Proposition 8.16. *Given a $\mathscr{L}$ hyperdoctrine $\mathbb{P}$ and an interpretation $[\![-]\!]$, for all predicate $\mathscr{L}$ formulae φ in context Γ and prime filters F of $\mathbb{P}([\![\Gamma]\!])$, $[\![\varphi]\!] \in F$ iff $F, \widetilde{[\![-]\!]} \vDash^\Gamma \varphi$.* □

Theorem 8.17 (Soundness and Completeness for Indexed $\mathscr{L}$ frames)**.** *For all predicate $\mathscr{L}$ formulae φ in context Γ, $\varphi \vdash^{\Gamma} \psi$ is provable iff $\varphi \vDash^{\Gamma} \psi$.* □

From here it is straightforward to set an assignment of morphisms to make the assignment of complex hyperdoctrines and indexed prime filter frames functorial. Given a $\mathscr{L}$ hyperdoctrine morphism $(K,\tau) : \mathbb{P} \to \mathbb{P}'$, $IndPr^{\mathscr{L}}(K,\tau) = (K,\tau^{-1})$. Similarly, given an indexed $\mathscr{L}$ frame morphism $(L,\lambda) : \mathscr{R} \to \mathscr{R}'$, $ComHyp^{\mathscr{L}}(L,\lambda) = (L,\lambda^{-1})$.

Lemma 8.18. *The functor $ComHyp^{\mathscr{L}}$ is well-defined.*

Proof. Let (L,λ) be a indexed $\mathscr{L}$ frame morphism. First note that by definition L is a finite product preserving functor. We also have that each component $\lambda_X : \mathscr{R}(LX) \to \mathscr{R}'(X)$ is a $\mathscr{L}$ morphism. Hence by functorality of $Com^{\mathscr{L}}$, each $\lambda_X^{-1} : Com^{\mathscr{L}}(\mathscr{R}'(X)) \to Com^{\mathscr{L}}(\mathscr{R}(LX))$ is a $\mathscr{L}$ algebra homomorphism, and naturality is inherited from λ.

Next we must verify that $\lambda_{X\times X}^{-1}(Ran(\mathscr{R}'(\Delta_X))) = Ran(\mathscr{R}(\Delta_{LX}))$. The right-to-left inclusion follows immediately from naturality of λ. For the left-to-right, suppose $\lambda_{X\times X}(x) \in Ran(\mathscr{R}'(\Delta_X))$. Then there exists y such that $\lambda_{X\times X}(x) = \mathscr{R}'(\Delta_X)(y)$. In the case for $\mathscr{L}$ with intuitionsitic additives, by the lift property, there exists y' such that $\mathscr{R}(\Delta_{LX})(y') \preccurlyeq x$. Since $\mathscr{R}(\Delta_{LX})$ is an intuitionistic morphism, there thus exists x' such that $y' \preccurlyeq x'$ and $\mathscr{R}(\Delta_{LX})(x') = x$ as required. For $\mathscr{L}$ with classical additives we are given such an x' immediately by the respective lift property.

Finally we verify the commutative diagram for $\exists X_\Gamma$, leaving the similar verification for $\forall X_\Gamma$ to the reader. We must show that $\mathscr{R}(\pi_{L\Gamma,LX})^*\lambda_{\Gamma\times X}^{-1}(A) = \lambda_\Gamma^{-1}\mathscr{R}'(\pi_{\Gamma,X})^*(A)$ for $A \in Com^{\mathscr{L}}(\mathscr{R}'(\Gamma\times X))$. First consider the case of $\mathscr{L}$ with intuitionistic additives. Suppose $x \in \mathscr{R}(\pi_{L\Gamma,LX})^*\lambda_{\Gamma\times X}^{-1}(A)$. Then there exists y with $\lambda_{\Gamma\times X}(y) \in A$ and $\mathscr{R}(\pi_{L\Gamma,LX})(y) \preccurlyeq x$. Since λ is a natural transformation and its components are order-preserving we have $\mathscr{R}'(\pi_{\Gamma,X})(\lambda_{\Gamma\times X}(y)) = \lambda_\Gamma\mathscr{R}(\pi_{L\Gamma,LX})(y) \preccurlyeq \lambda_\Gamma(x)$, so $x \in \lambda_\Gamma^{-1}\mathscr{R}'(\pi_{\Gamma,X})^*(A)$. Conversely, suppose $x \in \lambda_\Gamma^{-1}\mathscr{R}'(\pi_{\Gamma,X})^*(A)$. Then $\mathscr{R}'(\pi_{\Gamma,X})(y) \preccurlyeq \lambda_\Gamma(x)$ for $y \in A$. By the Morphism Pseudo Epi property, there exists z such that $y \preccurlyeq \lambda_{\Gamma\times X}(z)$ and $\mathscr{R}(\pi_{L\Gamma,LX})(z) \preccurlyeq x$. A is an upwards-closed set so $\lambda_{\Gamma\times X}(z) \in A$, hence $x \in \mathscr{R}(\pi_{L\Gamma,LX})^*\lambda_{\Gamma\times X}^{-1}(A)$ as required. For the case where $\mathscr{L}$ has classical additives the same argument applies, where $\preccurlyeq$ is substituted for $=$ and the other Morphism Pseudo Epi property is applied to find a sufficient z in the right-to-left direction. □

Lemma 8.19. *The functor $IndPr^{\mathscr{L}}$ is well-defined.*

Proof. Let (K,τ) be a $\mathscr{L}$ hyperdoctrine morphism. As in the previous lemma, we automatically obtain properties 1. and 2. for (K,τ^{-1}) from the definition and the

complex algebra functor $Com^{\mathscr{L}}$. For properties 3. and 4. we verify the case for $\mathscr{L}$ with intuitionistic additives an obtain the case for $\mathscr{L}$ with classical additives as a special case.

First we consider the Lift Property. Suppose $\mathbb{P}(\Delta_X)^{-1}(G) \subseteq \tau_{X\times X}^{-1}(F)$. It is simple to see that

$$P(G') = \begin{cases} 1 & \text{if } \mathbb{P}'(\Delta_{KX})^{-1}(G') \subseteq F \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. By Theorem 8.14 we have that $=_X \in \mathbb{P}(\Delta_X)^{-1}(G) \subseteq \tau_{X\times X}^{-1}(F)$ so $\tau_{X\times X}(=_X) = ='_{KX} \in F$. By adjointness it then follows that $\mathbb{P}'(\Delta_{KX})^{-1}(\{\top\}) \subseteq F$, as $\mathbb{P}(\Delta_{KX})(a) = \top$ entails $='_{KX} \leq a$. By the prime extension lemma there thus exists prime G with $P(G) = 1$, as required. For the case of $\mathscr{L}$ with classical additives, maximality entails $\mathbb{P}'(\Delta_{KX})^{-1}(G') = F$.

Next, the Morphism Pseudo Epi property. Suppose $\mathbb{P}(\pi_{\Gamma,X})^{-1}(F) \subseteq \tau_{\Gamma}^{-1}(G)$. We can once again define a prime predicate

$$P(G') = \begin{cases} 1 & \text{if } F \subseteq \tau_{\Gamma\times X}^{-1}(G') \text{ and } \mathbb{P}'(\pi_{K\Gamma,KX})^{-1}(G') \subseteq G \\ 0 & \text{otherwise} \end{cases}$$

that we can use to prove the existence of the appropriate prime filter. Consider the filter $G' = [\tau_{\Gamma\times X}(F))$. This is proper, otherwise there exists $a \in F$ such that $\tau_{\Gamma\times X}(a) = \bot$. By property 4. of $\mathscr{L}$ hyperdoctrine morphism, this would entail $\tau_{\Gamma}(\exists X_{\Gamma}(a) = \exists' KX_{K\Gamma}\tau_{\Gamma\times X}(a) = \exists' KX_{K\Gamma}(\bot) = \bot$. Since $a \in F$, by adjointness $\mathbb{P}(\pi_{\Gamma,X})(\exists X_{\Gamma}(a)) \in F$. Hence $\tau_{\Gamma}(\exists X_{\Gamma}(a) = \bot \in G$ by assumption, contradicting that G is a prime filter.

Clearly $F \subseteq \tau_{\Gamma\times X}^{-1}(G')$. Further, let $b \in \mathbb{P}'(\pi_{K\Gamma,KX})^{-1}(G')$. Then there exists $a \in F$ such that $\mathbb{P}'(\pi_{K\Gamma,KX})(b) \geq \tau_{\Gamma\times X}(a)$. By adjointness it follows that $\exists KX_{K\Gamma}(\tau_{\Gamma\times X}(a)) \leq b$, and by the property 4. of $\mathscr{L}$ hyperdoctrine morphisms we have $\tau_{\Gamma}(\exists X_{\Gamma}(a)) \leq b$. Since $a \in F$ we have that $\mathbb{P}(\pi_{\Gamma,X})\exists X_{\Gamma}(a) \in F$ by adjointness and upwards closure, hence $\exists X_{\Gamma}(a) \in \mathbb{P}(\pi_{\Gamma,X})^{-1}(F) \subseteq \tau_{\Gamma}^{-1}(G)$. It follows that $\tau_{\Gamma}(\exists X_{\Gamma}(a)) \in G$, and so $b \in G$. Thus $P(G) = 1$ and so there exists a prime filter G with $P(G) = 1$ by the prime extension lemma. In the case for $\mathscr{L}$ with classical additives, by maximality these inclusions become equalities, and this yields a witness for the Quasi-Pullback property. □

At this stage topology must be introduced to yield a duality.

Definition 8.20 (Indexed $\mathscr{L}$ Space)**.** *An* indexed $\mathscr{L}$ space *is a functor* $\mathscr{R} : \mathrm{C} \to \mathscr{L}\mathrm{Sp}$ *such that*

1. *$U \circ \mathscr{R} : \mathrm{C} \to \mathscr{L}$ is an indexed $\mathscr{L}$ frame, where $U : \mathscr{L}\mathrm{Sp} \to \mathscr{L}$ is the functor that forgets topological structure.*

2. *For each object X in C, $Ran(\mathscr{R}(\Delta_X))$ is clopen;*

3. *For each pair of objects Γ and X in C, $\mathscr{R}(\pi_{\Gamma,X})^*$ and $\mathscr{R}(\pi_{\Gamma,X})_*$ map (upwards-closed) clopen sets to (upwards-closed) clopen sets.*

In the case for $\mathscr{L}$ with classical additives it is possible to weaken condition 3. to $\mathscr{R}(\pi_{\Gamma,X})^*$ being an open map and $\mathscr{R}(\pi_{\Gamma,X})_*$ a closed map. This is because $\mathscr{R}(\pi_{\Gamma,X})$ is a continuous map between a compact and a Hausdorff space, and so the direct image $\mathscr{R}(\pi_{\Gamma,X}) = \mathscr{R}(\pi_{\Gamma,X})^*$ is a closed map automatically. We also have that $\mathscr{R}(\pi_{\Gamma,X})_*$ is an open map by definition. In the intuitionistic case the same reasoning applies for $\mathscr{R}(\pi_{\Gamma,X})^*$ (using its equivalence with the direct image) but it is not clear how to make the analogous case for $\mathscr{R}(\pi_{\Gamma,X})_*$. Nonetheless, this definition of indexed $\mathscr{L}$ space gives us what we need.

Lemma 8.21. *Given a $\mathscr{L}$ hyperdoctrine $\mathbb{P}$, the indexed prime filter space $IndPr^{\mathscr{L}}(\mathbb{P})$ is a indexed $\mathscr{L}$ space.*

Proof. Given Lemma 8.13 the only verifications left are of properties 2. and 3. We immediately obtain 2. by noting once again that $Ran(\mathbb{P}(\Delta_X)^{-1}) = \theta_{\mathbb{P}(X\times X)}(=_X)$, a clopen set by $\mathscr{L}$ duality. Utilising $\mathscr{L}$ duality once more, we have that every (upwards-closed) clopen set of $Pr^{\mathscr{L}}(\mathbb{P}(Y))$ is of the form $\theta_{\mathbb{P}(Y)}(a)$ for some $a \in \mathbb{P}(Y)$. We thus demonstrate that $(\mathbb{P}(\pi_{\Gamma,X})^{-1})^*(\theta_{\mathbb{P}(\Gamma\times X)}(a)) = \theta_{\mathbb{P}(\Gamma)}(\exists X_\Gamma(a))$ and $(\mathbb{P}(\pi_{\Gamma,X})^{-1})_*(\theta_{\mathbb{P}(\Gamma\times X)}(a)) = \theta_{\mathbb{P}(\Gamma)}(\forall X_\Gamma(a))$.

First assume $F \in (\mathbb{P}(\pi_{\Gamma,X})^{-1})^*(\theta_{\mathbb{P}(\Gamma\times X)}(a))$. Then there exists F' such that $a \in F'$ and $\mathbb{P}(\pi_{\Gamma,X})^{-1}(F') \subseteq F$. By adjointness $a \leq \mathbb{P}(\pi_{\Gamma,X})(\exists X_\Gamma(a))$ so $\exists X_\Gamma(a) \in \mathbb{P}(\pi_{\Gamma,X})^{-1}(F') \subseteq F$ so $F \in \theta_{\mathbb{P}(\Gamma)}(\exists X_\Gamma(a))$ as required. Conversely, suppose $\exists X_\Gamma(a) \in F$. It is easy to see that

$$P(G) = \begin{cases} 1 & \text{if } \mathbb{P}(\pi_{\Gamma,X})^{-1}(G) \subseteq F \text{ and } a \in G \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. Consider the filter $G = [a)$. F is proper as $a \neq \bot$ (otherwise $\exists X_\Gamma(\bot) = \bot \in F$), and by adjointness, if $a \leq \mathbb{P}(\pi_{\Gamma,X})(b)$, it follows that $\exists X_\Gamma(a) \leq b \in F$. Hence $P(G) = 1$ and by the prime extension lemma there exists a prime G with $P(G) = 1$ as required. In the case for $\mathscr{L}$ with intuitionistic additives we're done; in the case for $\mathscr{L}$ with classical additives, maximality of prime filters makes the inclusion an equality.

For the other equality, first assume we have F with $\forall X_\Gamma(a) \in F$ and let $F \subseteq \mathbb{P}(\pi_{\Gamma,X})^{-1}(G)$. Then $\mathbb{P}(\pi_{\Gamma,X})(\forall X_\Gamma(a)) \in G$, and by adjointness and upwards closure of G we have $a \in G$. In the other direction, assume $\forall X_\Gamma(a) \notin F$. We show there exists a prime filter G such that $F \subseteq \mathbb{P}(\pi_{\Gamma,X})^{-1}(G)$ and $a \notin G$. First note that for proper ideals I,

$$P(I) = \begin{cases} 1 & \text{if } F \subseteq \mathbb{P}(\pi_{\Gamma,X})^{-1}(\bar{I}) \text{ and } a \in I \\ 0 & \text{otherwise} \end{cases}$$

is a prime predicate. Consider $I = (a]$. This is proper as $a \neq \top$, as otherwise $\forall X_\Gamma(a) = \top \notin F$, contradicting that F is a filter. Suppose $b \in F$. Then $\mathbb{P}(\pi_{\Gamma,X}(b)) \not\leq a$ as otherwise by adjointness $b \leq \forall X_\Gamma(a) \in F$. Thus $P(I) = 1$ and so there exists a prime ideal I such that $P(I) = 1$. The prime filter $G = \bar{I}$ gives the required witness to the inclusion. Once again, in the case with classical additives maximality ensures the inclusion of prime filters is equality. □

In the other direction, composing an indexed $\mathscr{L}$ space $\mathscr{R}$ with the clopen algebra functor $Clop^{\mathscr{L}}_{\preccurlyeq}$ yields the clopen hyperdoctrine $ClopHyp^{\mathscr{L}}(\mathscr{R})$. Conditions 2. and 3. of indexed $\mathscr{L}$ space ensure that the assignment of $Ran(\mathscr{R}(\Delta_X))$ as $=_X$, $\mathscr{R}(\pi_{\Gamma,X})^*$ as $\exists X_\Gamma$, and $\mathscr{R}(\pi_{\Gamma,X})_*$ as $\forall X_\Gamma$ is well-defined, and Lemma 8.11 suffices to show that they satisfy the required properties. The definition of indexed $\mathscr{L}$ space morphism is given by taking that for indexed $\mathscr{L}$ frames. Then the assignment of morphisms given by the indexed prime filter frame and complex hyperdoctrine functors works the same way as before.

It remains to specify the natural isomorphisms that form the dual equivalence of categories. We already have $\Theta : Id_{\mathscr{L}\mathrm{Hyp}} \to ClopHyp^{\mathscr{L}} IndPr^{\mathscr{L}}$ from the representation theorem. We also define $\mathrm{H} : Id_{\mathrm{Ind}\mathscr{L}\mathrm{Sp}} \to IndPr^{\mathscr{L}} ClopHyp^{\mathscr{L}}$ by $\mathrm{H}_{\mathscr{R}} = (Id_C, \eta_{\mathscr{R}(-)})$, where $\eta : Id_{\mathscr{L}Sp} \to Pr^{\mathscr{L}} Clop^{\mathscr{L}}_{\preccurlyeq}$ is the natural isomorphism given by $\mathscr{L}$ duality.

Theorem 8.22 (Duality for $\mathscr{L}$ Hyperdoctrines)**.** Θ *and* H *form a dual equivalence of categories between* $\mathscr{L}$Hyp *and* Ind$\mathscr{L}$Sp.

Proof. The final verification is that Θ and H are natural isomorphisms. By Theorem 8.14 we have that each component of Θ is a $\mathscr{L}$ hyperdoctrine morphism, and as each component is a natural isomorphism by $\mathscr{L}$ duality, so too is Θ. For H we must first verify that each component is an indexed $\mathscr{L}$ space morphism.

We first attend to the Lift Property for $\mathscr{L}$ with intuitionistic additives. Suppose $(\mathscr{R}(\Delta_X)^{-1})^{-1}(F) \subseteq \eta_{\mathscr{R}(X\times X)}(x)$. Let y be such that $\eta_{\mathscr{R}(X)}(y) = F$; such a y necessarily exists by $\mathscr{L}$ duality. By naturality of η we thus have $\eta_{\mathscr{R}(X\times X)}(\mathscr{R}(\Delta_X)(y)) =$

$(\mathscr{R}(\Delta_X)^{-1})^{-1}(\eta_{\mathscr{R}(X)}(y)) \subseteq \eta_{\mathscr{R}(X\times X)}(x)$ so for any upwards-closed clopen set C, $\mathscr{R}(\Delta_X)(y) \in C$ implies $x \in C$. If $\mathscr{R}(\Delta_X)(y) \not\preccurlyeq x$ the Priestley separation axiom contradicts this, so $\mathscr{R}(\Delta_X)(y) \preccurlyeq x$ as required. For $\mathscr{L}$ with classical additives, the same argument applies, except using the fact that any distinct elements can be separated by clopens. The Pseudo Epi Property is proved by essentially the same argument.

Now, since each $\eta_{R(-)}$ is a natural isomorphism by $\mathscr{L}$ duality, we have that H is a natural isomorphism, and so the dual equivalence holds. □

Summary of Part II

In this part of the thesis we set up a duality theoretic framework for investigating the metatheory of bunched logics. After the preliminaries given in Chapter 5 (including the important concept of *prime predicate*) we gave representation and duality theorems for all of the bunched logics introduced in Part I in Chapter 6. In Chapter 7 this framework was put to use to prove a raft of results concerning propositional logics. In particular these include a simultaneous completeness theorem for all of the bunched logics of Part I, decidability theorems for the layered graph logics, a characterisation theorem for the classes of bunched logic model definable by bunched logic formulae and a resolution of the open problem of Craig interpolation for a number of logics. We finished the part by extending duality to the structures interpreting predicate bunched logic, inspired in part by the widespread use of BI hyperdoctrines in Separation Logic.

Part III

Proof Theory for Bunched Logics

Introduction to Part III

In this part of the thesis we attend to the proof theory of bunched logics. To do so we develop a modular framework of tableau calculi that gives systems that are sound and complete for each propositional bunched logic we have considered, as well as restrictions to particular classes of bunched logic model of interest: for example, classes of models satisfying *separation theories* (properties common to memory models of bunched logics) and the class of layered graph models of ILGL. This is done through the observation that bunched logic frames and satisfaction upon them can be encoded as special theories of first-order logic called *coherent theories*. We show that these precisely correspond to tableau calculi, and these systems can be proved sound and complete by utilising existing results in the proof theory of coherent logic. Our methods can be thought of as a strict generalisation of the existing tableau calculi for bunched logics: existing systems can be instantiated in our framework, but others can be expressed that were not previously possible due to implicit restrictions in the way labels were previously handled.

Chapter 9

Modular Tableaux Calculi for Bunched Logics

Thus far we have considered the proof theory of bunched logics in a very light way, utilising Hilbert systems which essentially correspond to the axiomatisation of bunched logic algebras. Although they are sound and complete for the corresponding logics, working with Hilbert systems is difficult for both humans and computers: in a given proof attempt the shape of the rules/derivation gives no real guidance on how to proceed at any given step and it follows that a high degree of trial and error is necessary. In well-behaved sequent calculi (satisfying cut elimination and the subformula property), proof search becomes tractable: one can start with the desired conclusion and apply rules backwards, safe in the knowledge that every formula that appears must be a subformula of the conclusion. A high degree of non-determinism may still hold in such systems (particularly in systems involving a multiplicative $*$, necessitating significant non-determinism in the way contexts are split up through backwards proof search [120, 188]) but it is clear this is still hugely preferable to Hilbert system proof search.

There is a reason, however, that we only mentioned sequent calculi as a motivation and did not carry it through to the body of the thesis: of the bunched logics we have considered, such well-behaved sequent calculi are not known to exist outside of ILGL and BI. The clearest way to augment BI's bunched sequent calculus to obtain a system for BBI is to add a rule corresponding to the double negation law of classical propositional logic. However, this irreparably breaks cut elimination and thus the subformula property. Adding multiplicative negation to obtain DMBI is also problematic, as it requires a system in which bunches appear in both antecedent and consequent position. Such systems are not well understood, let alone one in which both Boolean and multiplicative negation coexist.

One generalisation of the sequent calculus that has been shown suitable for

bunched logics is the display calculus [17]. Brotherston [38] has shown that display calculi with cut elimination exist for BI, BBI, DMBI and CBI. Brotherston and Villard [45] further show that such proof systems can be given for BiBBI and the axiomatic extensions we have given. Ciabattoni and Ramanayake [55] completely characterise the kinds of display calculi that can be safely extended with new rules corresponding to Hilbert-style axioms of a certain syntactic form while maintaining cut elimination, and the aforementioned display calculi satisfy all of their criteria and so can be extended modularly in a way that corresponds to a large class of axiomatic extensions of bunched logics. However there is an issue for the intended applications of bunched logic: many of the classes of bunched logic frame of interest do not correspond (cf. Chapter 7, [44]) to any Hilbert-style axioms! This is compounded by the work of Brotherston & Villard [44] and Larchey-Wendling & Galmiche [150] that shows that many simple properties like indivisibility of units, partial deterministic composition and total deterministic composition determine distinct sets of valid formulae for bunched logic: if we consider (as is done in separation logic) bunched logics in the more general sense as logics specified by validity in particular classes of models, even the extremely flexible display calculus approach is insufficient to capture everything of interest.

Is it possible to build sound and complete proof systems for these classes? Answering this question affirmatively forms the remainder of the thesis. Our work is inspired by (and generalises) a long line of work in labelled tableaux calculi for bunched logics, which began with Galmiche et al.'s [101] system for BI. A number of similar systems have been given since: Galmiche & Méry [100] define a tableau system for the standard store-heap semantics of Separation Logic, Larchey-Wendling [148] for partial monoidal BBI, and a number of modal extensions of partial monoidal (B)BI enjoy a sound and complete tableaux calculus [68, 98]. Such systems have the flavour of backwards proof search in a cut-free sequent calculus with the subformula property, with the derivation of a proof guided by the decomposition of a formula into its subformulae. This is explicitly brought out in Hoú et al.'s [127] labelled sequent calculus for BBI, which is constructed in a similar manner to these tableau systems.

Our framework departs from previous work in a key way. Existing bunched logic tableaux calculi work by explicitly representing states as labels and specify that these labels form a commutative monoid, which can thus be seen as encoding the composition and unit of a partial monoidal model of the logic. We abstract a step further though. To facilitate modularity, we do not wish to encode properties like partial functionality of composition or frames only having a single unit. We therefore utilise labels that have no algebraic structure whatsoever (similarly to the

aforementioned BBI labelled sequent calculi), and formulate a systematic way to add new tableau rules that correspond to particular properties. Our framework is also distinct in that it extends the tableau method to bunched logics with connectives other than those found in (B)BI and its modal extensions. It is also related to a range of work for generically generating labelled proof systems for logics (for a sampling, see Gabbay [95], Sernadas et al. [206] and Schmidt & Tishkovsky [204]). Where our work differs is the utilisation of a well-behaved fragment of first-order logic called coherent logic as an organising principle: our systems effectively arise as theories of a coherent logic, and this uses the syntactic shape of the first-order axiomatisation of bunched logic frames in an essential way.

In this chapter we define tableaux calculi that are sound and complete for the frame semantics of bunched logics as they have been given throughout the thesis. They are thus equipollant with the Hilbert systems which we have proved soundness and completeness with respect to thus far. In the sequel we extend this to particular classes of frames. This chapter is based on material from the paper *Modular Tableaux Calculi for Separation Theories* [81].

9.1 Logical Rules for Bunched Logic Tableaux Calculi

We begin by explaining how the tableaux calculi of our framework will work. As is standard for the tableau method, derivations in our calculi are implicit attempts to construct a countermodel for the formula φ to be proved. This is done via the derivation of syntactic expressions that give partial specifications of a model that can be realized as a real model if the formula is invalid. If every possible countermodel construction (i.e., every branch of a tableau) results in a contradiction, then we may conclude that no countermodel exists and call such a tableau a proof of φ.

To understand this informal definition we must specify what exactly these syntactic expressions are, and how exactly they supply a partial specification of a model. We begin with the fundamental entities in our calculi: labelled formulae. As before, $\mathscr{L}$ stands for any bunched logic introduced in Part I.

Definition 9.1 (Labelled $\mathscr{L}$ Formula)**.** *A* labelled $\mathscr{L}$ formula $\mathbb{S}\varphi : x$ *is given by a* sign $\mathbb{S} \in \{\mathbb{T}, \mathbb{F}\}$ *together with a* $\mathscr{L}$ *formula* φ *and a* label $x \in \{c_i \mid i \in \mathbb{N}\}$.

Labels are syntactic stand-ins for states of an $\mathscr{L}$ frame. A labelled formula $\mathbb{T}\varphi : x$ should be interpreted as stating "the formula φ is true at the state represented by x"; correspondingly, $\mathbb{F}\varphi : x$ is interpreted as stating "the formula φ is false at the state represented by x". The other syntactic entities manipulated by the calculi are *label constraints*.

Definition 9.2 (Label Constraints)**.** *A* label constraint *is an expression $Cx_0 \dots x_n$ for some symbol C, where $x_0, \dots, x_n$ are labels. We call C a* n-ary constraint symbol.

To specify a tableaux calculus in our framework we first have to specify the set of constraint symbols for the system. For the tableaux calculus for a bunched logic $\mathscr{L}$, there will be constraint symbols corresponding to the first-order signature of $\mathscr{L}$ frames together with a constraint symbol corresponding to equality.

Explicitly, in the basic systems we have the following constraint symbols: unary constraint symbols C_E and C_U; binary constraint symbols $C_=, C_{\succcurlyeq}, C_-$ and C_R; and ternary constraint symbols $C_\circ$, $C_\triangledown$ and $C_\triangleright$.We define the set of constraint symbols for the $\mathscr{L}$ tableaux calculus, $ConSym(\mathscr{L})$, as follows: for a symbol $\heartsuit$, $C_\heartsuit \in ConSym(\mathscr{L})$ iff $\heartsuit$ is $=$ or $\heartsuit$ is in the first-order signature of $\mathscr{L}$ frames. A label constraint $C_\heartsuit x_0 \dots x_n$ can be read as stating that on the countermodel the tableau is attempting to build, at the states $w_0, \dots, w_n$ corresponding to the labels $x_0, \dots x_n$, $\heartsuit(w_0, \dots, w_n)$ holds. For example, $C_\circ x_0 x_1 x_2$ is read as stating that, at the states w_0, w_1, w_2 corresponding to the labels x_0, x_1, x_2 respectively, $w_2 \in w_0 \circ w_1$. Thus the constraints form a partial specification of the structure of the countermodel.

All of these definitions come together in the notion of a constrained set of statements, which generalises the notion of a branch in standard presentations of tableaux.

Definition 9.3 (CSS (cf. [148]))**.** *A* constrained set of statements *(CSS) for $\mathscr{L}$ is a pair $\langle \mathscr{F}, \mathscr{C} \rangle$ where $\mathscr{F}$ is a set of labelled $\mathscr{L}$ formulae and $\mathscr{C}$ is a set of label constraints over $ConSym(\mathscr{L})$.*

Tableau rules in our framework dictate how CSSs can be expanded. The premiss gives a condition on CSSs, and the conclusion dictates how any CSS satisfying that condition should be expanded (possibly in multiple ways, witnessed by branching). That is, they are of the following general form:

$$\frac{Cond(\langle \mathscr{F}, \mathscr{C} \rangle)}{\langle \mathscr{F}_1, \mathscr{C}_1 \rangle \mid \dots \mid \langle \mathscr{F}_k, \mathscr{C}_k \rangle}.$$

In all of the rules we consider, $Cond(\langle \mathscr{F}, \mathscr{C} \rangle)$ will always be a statement on the existence of particular labelled $\mathscr{L}$ formulae in $\mathscr{F}$ and/or label constraints in $\mathscr{C}$. A tableau $\mathscr{T}$ for a collection of such rules $\mathscr{R}$ is a finite list of CSSs separated by semi-colons ";", $\mathscr{T} = [\langle \mathscr{F}_0, \mathscr{C}_0 \rangle; \dots; \langle \mathscr{F}_n, \mathscr{C}_n \rangle]$, constructed according to the rules, in the following precise sense. Here $\oplus$ denotes concatenation of lists: $[\langle \mathscr{F}_0, \mathscr{C}_0 \rangle; \dots; \langle \mathscr{F}_n, \mathscr{C}_n \rangle] \oplus [\langle \mathscr{F}'_0, \mathscr{C}'_0 \rangle; \dots; \langle \mathscr{F}'_n, \mathscr{C}'_m \rangle] = [\langle \mathscr{F}_0, \mathscr{C}_0 \rangle; \dots; \langle \mathscr{F}_n, \mathscr{C}_n \rangle; \langle \mathscr{F}'_0, \mathscr{C}'_0 \rangle; \dots; \langle \mathscr{F}'_n, \mathscr{C}'_m \rangle]$.

Definition 9.4 (Tableau [148])**.** *Given a set of tableau rules $\mathscr{R}$ and a finite CSSs $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle$, a $\mathscr{R}$-tableau for $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle$ is a list of CSS constructed according to the following inductive definition.*

1. *The one branch list $[\langle \mathscr{F}_0, \mathscr{C}_0 \rangle]$ is a $\mathscr{R}$-tableau for $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle$;*
2. *If the list $\mathscr{T}_m \oplus [\langle \mathscr{F}, \mathscr{C} \rangle] \oplus \mathscr{T}_n$ is a $\mathscr{R}$-tableau for $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle$ and*

$$\frac{Cond(\langle \mathscr{F}, \mathscr{C} \rangle)}{\langle \mathscr{F}_1, \mathscr{C}_1 \rangle \mid \ldots \mid \langle \mathscr{F}_k, \mathscr{C}_k \rangle}$$

is a rule from $\mathscr{R}$ for which a concrete instance of $Cond(\langle \mathscr{F}, \mathscr{C} \rangle)$ is fulfilled by $\langle \mathscr{F}, \mathscr{C} \rangle$, then the list $\mathscr{T}_m \oplus [\langle \mathscr{F} \cup \mathscr{F}_1, \mathscr{C} \cup \mathscr{C}_1 \rangle; \ldots; \langle \mathscr{F} \cup \mathscr{F}_k, \mathscr{C} \cup \mathscr{C}_k \rangle] \oplus \mathscr{T}_n$ is a $\mathscr{R}$-tableau for $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle$.

The idea here is that each of the CSSs in the list represents a 'branch' in a tree (with the semi-colon ";" indicating branching) that is inductively constructed according to the rules $\mathscr{R}$. The inductive definition guarantees the tree structure: all 'branches' extend the 'root' $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle$, and branching occurs whenever the rules dictate that a branch should be extended in multiple ways.

What are the correct rules $\mathscr{R}$ for each bunched logic? We specify this in a stepwise fashion: first, we specify the rules corresponding to the decomposition of formulae into subformulae. Then, in the next section we specify the additional rules that operate on constraints. Each primitive connective $\heartsuit$ in each logic $\mathscr{L}$ has two associated decomposition rules for formulae in which $\heartsuit$ is the outermost connective: one for such labelled $\mathscr{L}$ formulae signed with $\mathbb{T}$ and one for such labelled $\mathscr{L}$ formulae signed with $\mathbb{F}$. These rules directly correspond to the semantic clauses associated with each connective. We also have a rule for each of the constants $\top^*$ and $\bot^*$; we return to the 'missing' rule for these constants when we consider the conditions under which a tableau is deemed 'inconsistent'.

Figure 9.1 lists the tableau rules for the bunched logics with classical additives; Figure 9.2 for the bunched logics with intuitionistic additives. For each logic $\mathscr{L}$, the *logical expansion rules*, $LogRules(\mathscr{L})$, are given by the rules $\langle \mathbb{S}\heartsuit \rangle$ from the appropriate figure for each $\heartsuit$ that is a primitive symbol in the grammar of $\mathscr{L}$. The condition on fresh labels simply means labels which have not yet occurred in the tableau: this is always possible because we have an infinite set of labels and start from a finite CSS.

The rules should be understood as follows: suppose in the countermodel that one is attempting to build through the tableau procedure (corresponding to the partial specification given by the branch $\langle \mathscr{F}, \mathscr{C} \rangle$) the information corresponding to the premiss holds; then it would necessarily follow by the semantic clauses of the logic

$$\langle \mathbb{T}\wedge\rangle \quad \frac{\mathbb{T}\varphi\wedge\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x, \mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\wedge\rangle \quad \frac{\mathbb{F}\varphi\wedge\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}\vee\rangle \quad \frac{\mathbb{T}\varphi\vee\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\vee\rangle \quad \frac{\mathbb{F}\varphi\vee\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x, \mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}\to\rangle \quad \frac{\mathbb{T}\varphi\to\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\to\rangle \quad \frac{\mathbb{F}\varphi\to\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x, \mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}*\rangle \quad \frac{\mathbb{T}\varphi * \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{T}\psi : c_j\}, \{C_\circ c_i c_j x\}\rangle} \qquad \langle \mathbb{F}*\rangle \quad \frac{\mathbb{F}\varphi * \psi : x\in\mathscr{F} \text{ and } C_\circ yzx\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : z\},\emptyset\rangle}$$

$$\langle \mathbb{T}\mathrel{-\!\!*}\rangle \quad \frac{\mathbb{T}\varphi \mathrel{-\!\!*} \psi : x\in\mathscr{F} \text{ and } C_\circ xyz\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}\mathrel{-\!\!*}\rangle \quad \frac{\mathbb{F}\varphi \mathrel{-\!\!*} \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\}, \{C_\circ x c_i c_j\}\rangle}$$

$$\langle \mathbb{T}*\!\!-\rangle \quad \frac{\mathbb{T}\varphi *\!\!- \psi : x\in\mathscr{F} \text{ and } C_\circ yxz\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}*\!\!-\rangle \quad \frac{\mathbb{F}\varphi *\!\!- \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\}, \{C_\circ c_i x c_j\}\rangle}$$

$$\langle \mathbb{T}\mathbin{\dot\vee}\rangle \quad \frac{\mathbb{T}\varphi \mathbin{\dot\vee} \psi : x\in\mathscr{F} \text{ and } C_\triangledown yzx\in\mathscr{C}}{\langle\{\mathbb{T}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}\mathbin{\dot\vee}\rangle \quad \frac{\mathbb{F}\varphi \mathbin{\dot\vee} \psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : c_i \mathbb{F}\psi : c_j\}, \{C_\triangledown c_i c_j x\}\rangle}$$

$$\langle \mathbb{T}\mathbin{\dot\setminus}\rangle \quad \frac{\mathbb{T}\varphi \mathbin{\dot\setminus} \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\}, C_\triangledown c_i x c_j\rangle} \qquad \langle \mathbb{F}\mathbin{\dot\setminus}\rangle \quad \frac{\mathbb{F}\varphi \mathbin{\dot\setminus} \psi : x\in\mathscr{F} \text{ and } C_\triangledown yxz\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : z\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : y\},\emptyset\rangle}$$

$$\langle \mathbb{T}\mathbin{\dot\neg}\rangle \quad \frac{\mathbb{T}\mathbin{\dot\neg}\varphi : x\in\mathscr{F} \text{ and } C_- xy\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle} \qquad \langle \mathbb{F}\mathbin{\dot\neg}\rangle \quad \frac{\mathbb{F}\mathbin{\dot\neg}\varphi : x\in\mathscr{F} \text{ and } C_- xy\in\mathscr{C}}{\langle\{\mathbb{T}\varphi : y\},\emptyset\rangle}$$

$$\langle \mathbb{T};\rangle \quad \frac{\mathbb{T}\varphi;\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{T}\psi : c_j\}, \{C_\triangleright c_i c_j x\}\rangle} \qquad \langle \mathbb{F};\rangle \quad \frac{\mathbb{F}\varphi;\psi : x\in\mathscr{F} \text{ and } C_\triangleright yzx\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : z\},\emptyset\rangle}$$

$$\langle \mathbb{T}\mathrel{-\!\!\triangleright}\rangle \quad \frac{\mathbb{T}\varphi \mathrel{-\!\!\triangleright} \psi : x\in\mathscr{F} \text{ and } C_\triangleright xyz\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}\mathrel{-\!\!\triangleright}\rangle \quad \frac{\mathbb{F}\varphi \mathrel{-\!\!\triangleright} \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\}, \{C_\triangleright x c_i c_j\}\rangle}$$

$$\langle \mathbb{T}\triangleright\!\!-\rangle \quad \frac{\mathbb{T}\varphi \triangleright\!\!- \psi : x\in\mathscr{F} \text{ and } C_\triangleright yxz\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}\triangleright\!\!-\rangle \quad \frac{\mathbb{F}\varphi \triangleright\!\!- \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\}, \{C_\triangleright c_i x c_j\}\rangle}$$

$$\langle \mathbb{T}\diamond\rangle \quad \frac{\mathbb{T}\diamond\varphi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i\}, \{C_R x c_i\}\rangle} \qquad \langle \mathbb{F}\diamond\rangle \quad \frac{\mathbb{F}\diamond\varphi : x\in\mathscr{F} \text{ and } C_R xy}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle}$$

$$\langle \mathbb{T}\top^*\rangle \quad \frac{\mathbb{T}\top^* : x\in\mathscr{F}}{\langle\emptyset, \{C_E x\}\rangle} \qquad \langle \mathbb{F}\bot^*\rangle \quad \frac{\mathbb{F}\bot^* : x\in\mathscr{F}}{\langle\emptyset, \{C_U x\}\rangle}$$

with c_i, c_j fresh labels.

Figure 9.1: Logical expansion rules for bunched logics with classical additives.

$$\langle \mathbb{T}\wedge\rangle \quad \frac{\mathbb{T}\varphi\wedge\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x, \mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\wedge\rangle \quad \frac{\mathbb{F}\varphi\wedge\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}\vee\rangle \quad \frac{\mathbb{T}\varphi\vee\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\vee\rangle \quad \frac{\mathbb{F}\varphi\vee\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x, \mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}\to\rangle \quad \frac{\mathbb{T}\varphi\to\psi : x\in\mathscr{F} \text{ and } C_{\succcurlyeq}yx\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : y\},\emptyset\rangle} \qquad \langle \mathbb{F}\to\rangle \quad \frac{\mathbb{F}\varphi\to\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_i\},\{C_{\succcurlyeq}c_ix\}\rangle}$$

$$\langle \mathbb{T}*\rangle \quad \frac{\mathbb{T}\varphi * \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{T}\psi : c_j\},\{C_\circ c_ic_jc_k, C_{\succcurlyeq}xc_k\}\rangle} \qquad \langle \mathbb{F}*\rangle \quad \frac{\mathbb{F}\varphi * \psi : x\in\mathscr{F} \text{ and } C_\circ yzw, C_{\succcurlyeq}xw\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : z\},\emptyset\rangle}$$

$$\langle \mathbb{T}\mathrel{-\!\!*}\rangle \quad \frac{\mathbb{T}\varphi \mathrel{-\!\!*} \psi : x\in\mathscr{F} \text{ and } C_\circ wyz, C_{\succcurlyeq}wx\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}\mathrel{-\!\!*}\rangle \quad \frac{\mathbb{F}\varphi \mathrel{-\!\!*} \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\},\{C_\circ c_kc_ic_j, C_{\succcurlyeq}c_kx\}\rangle}$$

$$\langle \mathbb{T}*\!\!-\rangle \quad \frac{\mathbb{T}\varphi *\!\!- \psi : x\in\mathscr{F} \text{ and } C_\circ ywz, C_{\succcurlyeq}wx\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}*\!\!-\rangle \quad \frac{\mathbb{F}\varphi *\!\!- \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\},\{C_\circ c_ic_kc_j, C_{\succcurlyeq}c_kx\}\rangle}$$

$$\langle \mathbb{T}\overset{*}{\vee}\rangle \quad \frac{\mathbb{T}\varphi \overset{*}{\vee} \psi : x\in\mathscr{F} \text{ and } C_\triangledown yzw, C_{\succcurlyeq}wx\in\mathscr{C}}{\langle\{\mathbb{T}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : z\},\emptyset\rangle} \qquad \langle \mathbb{F}\overset{*}{\vee}\rangle \quad \frac{\mathbb{F}\varphi \overset{*}{\vee} \psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : c_i\mathbb{F}\psi : c_j\},\{C_\triangledown c_ic_jc_k, C_{\succcurlyeq}c_kx\}\rangle}$$

$$\langle \mathbb{T}\overset{*}{\wedge}\rangle \quad \frac{\mathbb{T}\varphi \overset{*}{\wedge} \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_j\},C_\triangledown c_ic_kc_j, C_{\succcurlyeq}xc_k\rangle} \qquad \langle \mathbb{F}\overset{*}{\wedge}\rangle \quad \frac{\mathbb{F}\varphi \overset{*}{\wedge} \psi : x\in\mathscr{F} \text{ and } C_\triangledown ywz, C_{\succcurlyeq}xw\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : z\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : y\},\emptyset\rangle}$$

$$\langle \mathbb{T}\mathrel{-\!\!*}_1\rangle \quad \frac{\mathbb{T} \mathrel{-\!\!*}_1 \varphi : x\in\mathscr{F} \text{ and } C_-xy\in\mathscr{C}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle} \qquad \langle \mathbb{F}\mathrel{-\!\!*}_1\rangle \quad \frac{\mathbb{F} \mathrel{-\!\!*}_1 \varphi : x\in\mathscr{F} \text{ and } C_-xy\in\mathscr{C}}{\langle\{\mathbb{T}\varphi : y\},\emptyset\rangle}$$

$$\langle \mathbb{T}\top^*\rangle \quad \frac{\mathbb{T}\top^* : x\in\mathscr{F}}{\langle\emptyset,\{C_Ex\}\rangle} \qquad \langle \mathbb{F}\bot^*\rangle \quad \frac{\mathbb{F}\bot^* : x\in\mathscr{F}}{\langle\emptyset,\{C_Ux\}\rangle}$$

with c_i, c_j, c_k fresh labels.

Figure 9.2: Logical expansion rules for bunched logics with intuitionistic additives.

that the countermodel would also satisfy one of the conclusions of the rule. Our partial specification of the model is then extended in each of the ways (possibly finitely branching) the conclusion of the rule dictates it can be extended.

9.2 Tableau Rule Generation from Coherent Axioms

The rules we have given thus far are insufficient to give sound proof systems for all of the bunched logics of interest. For example, if we tried to use just the logical rules corresponding to the grammar of (B)BI we would not be able to prove $\varphi * \psi \to \psi * \varphi$. Essentially, we are missing the mediation that the structure on $\mathscr{L}$ frames provides to ensure the correct $\mathscr{L}$ algebra identities hold when taking complex algebras. How do we lift this structure into our tableaux calculi? The solution to this is to once again take seriously $\mathscr{L}$ frames as first-order structures, somewhat similarly to what we did for the Goldblatt-Thomason theorem for bunched logics in Chapter 7. The technical overhead in this case is much lighter though: we're simply interested in the syntactic shape of the first-order axioms that define $\mathscr{L}$ frames, which leads us to *coherent logic*.

Coherent logic[1] is the fragment of first-order logic consisting of sequents $\varphi \vdash \psi$ in which φ and ψ are built solely from $\wedge, \vee, \exists, \top$ and $\bot$. These sequents can be given a normal form in first-order logic by

$$A_1(\vec{x}) \wedge \cdots \wedge A_n(\vec{x}) \to \exists \vec{y_1} B_1(\vec{x}, \vec{y_1}) \vee \cdots \vee \exists \vec{y_m} B_m(\vec{x}, \vec{y_m}),$$

for $n, m \geq 0$, where each A_i is an atomic formula involving only variables from the vector $\vec{x}$, and each B_i is the conjunction of atomic formulae involving only variables from the vectors $\vec{x}$ and $\vec{y_i}$. We will henceforth call these normal form formulae *coherent formulae*. In a coherent formula, the variables $\vec{x}$ are implicitly universally quantified (with scope the whole formula) and both $\vec{x}$ and $\vec{y_i}$ may be empty. The case $n = 0$ is a consequent that is always true:

$$\top \to \exists \vec{y_1} B_1(\vec{x}, \vec{y_1}) \vee \cdots \vee \exists \vec{y_m} B_m(\vec{x}, \vec{y_m}).$$

Similarly, the case $m = 0$ is an antecedent that is always false:

$$A_1(\vec{x}) \wedge \cdots \wedge A_n(\vec{x}) \to \bot.$$

The case $m = 1$ with empty $\vec{y_1}$ gives the *Horn clause* fragment of first-order logic

[1] What we refer to here as coherent logic is sometimes given as the definition of *geometric* logic [218]. This ambiguity is unfortunate: geometric logic commonly refers to the generalisation of coherent logic in which infinitary disjunctions are permitted in the consequent. Following prior literature we maintain this distinction.

utilised in logic programming and first-order theorem provers based on the resolution method—one might usefully think of coherent logic as a strict generalisation of the Horn fragment. Like that fragment, coherent logic has a constructive/computational flavour, as it forms a Glivenko class [170], in the sense that any coherent axiom classically derivable from a set of coherent axioms is also intuitionistically derivable.

We call a set of coherent formulae Φ a *coherent theory*. Models of coherent theories are given in a way standard for first-order logic: a *Tarskian model of* Φ is a non-empty set X together with an interpretation $\mathscr{I}$, which assigns to every n-ary relation symbol R in the signature a set $R^{\mathscr{I}} \subseteq X^n$ such that for each coherent formulae in Φ, for all $\vec{x} \in X$, the consequent $\exists\overrightarrow{y_1} \in X(B^{\mathscr{I}}(\vec{x},\overrightarrow{y_1})) \vee \cdots \vee \exists\overrightarrow{y_m} \in X(B^{\mathscr{I}}(\vec{x},\overrightarrow{y_m}))$ is true whenever the antecedent $A_1^{\mathscr{I}}(\vec{x}) \wedge \cdots \wedge A_n^{\mathscr{I}}(\vec{x})$ is true.

Many common mathematical structures are axiomatized by coherent theories. For example, algebraic structures like groups, rings, lattices and fields, as well as total, partial, and linear orders. Further examples from computer science can be found in the theory of confluence for term rewriting systems [211]. Of interest for our purposes, all bunched logic frames are axiomatised by finite coherent theories, something that can be easily verified by direct examination of the definitions.

Our aim is to generate tableau rules from coherent formulae. Generating proof rules from coherent formulae is not a new idea in proof theory: it was first considered by Simpson [208] to provide natural deduction systems for intuitionistic modal logics, an idea later adapted by Braüner [33] for natural deduction systems for hybrid logics. It has been extensively developed by Negri [169, 171] to provide labelled sequent calculi for modal and intermediate logics. In this work Negri provides a schema for extracting (systems of) sequent calculi rules from modal/intermediate Kripke frame properties axiomatised by (generalised) coherent formulae.

It would also be possible to formulate our proof systems as labelled sequent calculi in the style of Negri. However, we are interested in the application of these ideas to the tableau method: to our knowledge, this is the first time this has been done. Bezem & Coquand's [22] encode the standard tableaux system for classical first-order logic in coherent logic, but our work is strictly more general, and involves the generation of proof rules from the coherent theories defining Kripke models. Working with tableaux systems also gives us the advantage, as we will shortly see, of formalising the proof systems themselves as coherent theories, leading to a particularly elegant parametric completeness proof.

Let

$$A_1(\vec{x}) \wedge \cdots \wedge A_n(\vec{x}) \rightarrow \exists\overrightarrow{y_1}B_1(\vec{x},\overrightarrow{y_1}) \vee \cdots \vee \exists\overrightarrow{y_m}B_m(\vec{x},\overrightarrow{y_m})$$

be a coherent axiom in the first-order language of $\mathscr{L}$ frames with equality. We first straightforwardly translate this into our language of label constraints by replacing instances of operations $\heartsuit$ with the constraint symbol $C_\heartsuit$. Then, for each conjunction $B_i(\vec{x},\vec{y_m}) = B^i_0(\vec{x},\vec{y_i}) \wedge \cdots \wedge B^i_k(\vec{x},\vec{y_m})$ we define $C_{B_i}(\vec{x},\vec{y_m}) := C_{B^i_0}(\vec{x},\vec{y_i}) \wedge \cdots \wedge C_{B^i_{k_i}}(\vec{x},\vec{y_m})$. Then the coherent axiom is translated to

$$C_{A_1}(\vec{x}) \wedge \cdots \wedge C_{A_n}(\vec{x}) \rightarrow \exists\vec{y_1}C_{B_1}(\vec{x},\vec{y_1}) \vee \cdots \vee \exists\vec{y_m}C_{B_m}(\vec{x},\vec{y_m}).$$

When $n,m \neq 0$, this generates the following tableau rule

$$\frac{C_{A_1}(\vec{x}),\ldots,C_{A_n}(\vec{x}) \in \mathscr{C}}{\langle\emptyset,\{C_{B^1_0}(\vec{x},\vec{c_1}),\ldots,C_{B^1_{k_1}}(\vec{x},\vec{c_1})\}\rangle \mid \ldots \mid \langle\emptyset,\{C_{B^m_0}(\vec{x},\vec{c_1}),\ldots,C_{B^m_{k_m}}(\vec{x},\vec{c_m})\}\rangle}$$

where the $\vec{c_i}$ are fresh labels. The procedure is simple: the antecedent of the coherent axiom becomes the premiss of the tableau rule, and the consequent the conclusion; existential quantification is handled with fresh labels, and the disjunction is witnessed by branching. In the case $n = 0$ we have the coherent axiom

$$\top \rightarrow \exists\vec{y_1}C_{B_1}(\vec{x},\vec{y_1}) \vee \cdots \vee \exists\vec{y_m}C_{B_m}(\vec{x},\vec{y_m}).$$

However we wish to have some control over when the tableau rule this translates to can be triggered: it should only be applied for labels $\vec{x}$ that have already occurred in the branch. Let $\vec{x} = x_0,\ldots,x_s$. We obtain the following tableau rule

$$\frac{\mathit{Expression}(x_0),\ldots,\mathit{Expression}(x_s) \in \mathscr{F} \cup \mathscr{C}}{\langle\emptyset,\{C_{B^1_0}(\vec{x},\vec{c_1}),\ldots,C_{B^1_{k_1}}(\vec{x},\vec{c_1})\}\rangle \mid \ldots \mid \langle\emptyset,\{C_{B^m_0}(\vec{x},\vec{c_1}),\ldots,C_{B^m_{k_m}}(\vec{x},\vec{c_m})\}\rangle}$$

Here $\mathit{Expression}(x)$ refers to any labelled formula or label constraint in which the label x occurs. Essentially, the premiss holds whenever the labels $\vec{x}$ already occur on the branch. What we have described thus far is sufficient for the axioms defining $\mathscr{L}$ frames: we will describe how to handle the case $m = 0$ when we require it in Chapter 10.

9.3 Frame Rules for Bunched Logic Tableaux Calculi

We now explicitly state the *frame expansion rules* in each tableau system. First we give rules governing how equality and substitution work for the constraint symbol $C_=$. These are joined by the direct translatation of the coherent axiomatisation of $\mathscr{L}$ frames for each $\mathscr{L}$ using the method described in the previous section.

Figure 9.3 gives the rules governing the equality and order constraints. $\langle = \mathit{Ref}\rangle$, $\langle = \mathit{Sym}\rangle$ and $\langle = \mathit{Trans}\rangle$ ensure $C_=$ is an equivalence relation on labels. Further, $\langle \mathit{Sub}\rangle$ provides a mechanism for substituting $C_=$-equivalent labels occurring in label

$$\langle = Ref \rangle \quad \frac{Expression(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_= xx\} \rangle} \qquad \langle = Trans \rangle \quad \frac{C_= xy, C_= yz \in \mathscr{C}}{\langle \emptyset, \{C_= xz\} \rangle}$$

$$\langle = Sym \rangle \quad \frac{C_= xy \in \mathscr{C}}{\langle \emptyset, \{C_= yx\} \rangle} \qquad \langle Sub \rangle \quad \frac{Expression(x), C_= xy \in \mathscr{C}}{\langle \emptyset, \{Expression(y/x)\} \rangle}$$

$$\langle \succcurlyeq Ref \rangle \quad \frac{Expression(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} xx\} \rangle} \qquad \langle \succcurlyeq Trans \rangle \quad \frac{C_{\succcurlyeq} xy, C_{\succcurlyeq} yz \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} xz\} \rangle}$$

Figure 9.3: Tableau rules for equality and order.

$$\langle Commutativity \rangle \quad \frac{C_\circ xyz \in \mathscr{C}}{\langle \emptyset, \{C_\circ yxz\} \rangle} \qquad \langle Unit\ Existence \rangle \quad \frac{Expression(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_E c_i, C_\circ x c_i x\} \rangle}$$

$$\langle Coherence \rangle \quad \frac{C_E z, C_\circ yzx \in \mathscr{C}}{\langle \emptyset, C_= xy \rangle} \qquad \langle Associativity \rangle \quad \frac{C_\circ xyt, C_\circ tzw \in \mathscr{C}}{\langle \emptyset, C_\circ yzc_i, C_\circ x c_i w \rangle}$$

with c_i a fresh label.

Figure 9.4: BBI frame expansion rules.

$$\langle Commutativity \rangle \quad \frac{C_\circ xyz \in \mathscr{C}}{\langle \emptyset, \{C_\circ yxz\} \rangle} \qquad \langle Closure \rangle \quad \frac{C_E x, C_{\succcurlyeq} yx \in \mathscr{C}}{\langle \emptyset, \{C_E y\} \rangle}$$

$$\langle Unit\ Existence \rangle \quad \frac{Expression(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_E c_i, C_\circ x c_i x\} \rangle} \qquad \langle Coherence \rangle \quad \frac{C_E z, C_\circ yzx \in \mathscr{C}}{\langle \emptyset, C_{\succcurlyeq} xy \rangle}$$

$$\langle Associativity \rangle \quad \frac{C_{\succcurlyeq} t't, C_\circ xyt, C_\circ t'zw \in \mathscr{C}}{\langle \emptyset, C_{\succcurlyeq} c_{s'} c_s, C_\circ yzc_s, C_{\succcurlyeq} w c_{w'}, C_\circ x c_{s'} c_{w'} \rangle}$$

with $c_i, c_s, c_{s'}, c_{w'}$ fresh labels.

Figure 9.5: BI frame expansion rules.

$$\langle Function \rangle \quad \frac{C_{-}xy, C_{-}xy' \in \mathscr{C}}{\langle \emptyset, \{C_{=}yy'\} \rangle} \qquad \langle Total \rangle \quad \frac{Expression(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_{-}xc_i\} \rangle}$$

$$\langle Dual \rangle \quad \frac{C_{\succcurlyeq}xy, C_{-}xx', C_{-}yy' \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq}y'x'\} \rangle} \qquad \langle Involutive \rangle \quad \frac{C_{-}xx', C_{-}x'x'' \in \mathscr{C}}{\langle \emptyset, C_{=}xx'' \rangle}$$

$$\langle Compatibility \rangle \quad \frac{C_{\circ}xyz, C_{-}xx', C_{-}zz' \in \mathscr{C}}{\langle \emptyset, C_{\circ}z'yx' \rangle}$$

with c_i a fresh label.

Figure 9.6: DMBI and CBI frame expansion rules.

$$\langle Commutativity \rangle \quad \frac{C_{\triangledown}xyz \in \mathscr{C}}{\langle \emptyset, \{C_{\triangledown}yxz\} \rangle} \qquad \langle U\ Closure \rangle \quad \frac{C_{U}x, C_{\succcurlyeq}xy \in \mathscr{C}}{\langle \emptyset, \{C_{U}y\} \rangle}$$

Figure 9.7: Bi(B)BI frame expansion rules.

$$\langle \textit{Associativity} \rangle \quad \frac{C_{\triangledown} xyt, C_{\triangledown} tzw \in \mathscr{C}}{\langle \emptyset, \{C_{\triangledown} yzc_i, C_{\triangledown} xc_i w\} \rangle} \qquad \langle \bot^* \textit{ Weak} \rangle \quad \frac{C_U z, C_{\triangledown} yzx \in \mathscr{C}}{\langle \emptyset, \{C_{=} yx\} \rangle}$$

$$\langle \bot^* \textit{ Contract} \rangle \quad \frac{\textit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_U c_i, C_{\triangledown} xc_i x\} \rangle} \qquad \langle \talloblong \textit{ Contract} \rangle \quad \frac{\textit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_{\triangledown} xxx\} \rangle}$$

$$\langle \textit{Weak Dist} \rangle \quad \frac{C_{\circ} x_1 x_2 t, C_{\triangledown} y_1 y_2 t \in \mathscr{C}}{\langle \emptyset, \{C_{\circ} x_1 c_i y_1, C_{\triangledown} c_i y_2 x_2\} \rangle}$$

with c_i a fresh label.

Figure 9.8: Frame expansion rules for extensions of BiBBI.

$$\langle \textit{Associativity} \rangle \quad \frac{C_{\succcurlyeq} t't, C_{\triangledown} xyt, C_{\triangledown} t'zw \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} c_s c_{s'}, C_{\triangledown} yzc_s, C_{\succcurlyeq} c_{w'} w, C_{\triangledown} xc_{s'} c_{w'}\} \rangle} \qquad \langle \bot^* \textit{ Weak} \rangle \quad \frac{C_U z, C_{\triangledown} yzx \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} yx\} \rangle}$$

$$\langle \bot^* \textit{ Contr} \rangle \quad \frac{\textit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_U c_i, C_{\triangledown} xc_i x\} \rangle} \qquad \langle \talloblong \textit{ Contr} \rangle \quad \frac{\textit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_{\triangledown} xxx\} \rangle}$$

$$\langle \textit{Weak Dist} \rangle \quad \frac{C_{\succcurlyeq} t't, C_{\circ} x_1 x_2 t, C_{\succcurlyeq} t''t', C_{\triangledown} y_1 y_2 t'' \in \mathscr{C}}{\langle \emptyset, \{C_{\circ} x_1 c_i y_1, C_{\triangledown} c_i y_2 x_2\} \rangle}$$

with c_i a fresh label.

Figure 9.9: Frame expansion rules for extensions of BiBI.

$$\langle \textit{Unit Existence}_L \rangle \quad \frac{\textit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_E c_i, C_{\triangleright} c_i xx\} \rangle} \qquad \langle \textit{Unit Existence}_R \rangle \quad \frac{\textit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_E c_i, C_{\triangleright} xc_i x\} \rangle}$$

$$\langle \textit{Coherence}_L \rangle \quad \frac{C_E z, C_{\triangleright} zyx \in \mathscr{C}}{\langle \emptyset, C_{=} xy \rangle} \qquad \langle \textit{Coherence}_R \rangle \quad \frac{C_E z, C_{\triangleright} yzx \in \mathscr{C}}{\langle \emptyset, C_{=} xy \rangle}$$

$$\langle \textit{Associativity}_L \rangle \quad \frac{C_{\triangleright} xyt, C_{\triangleright} tzw \in \mathscr{C}}{\langle \emptyset, C_{\triangleright} yzc_i, C_{\triangleright} xc_i w \rangle} \qquad \langle \textit{Associativity}_R \rangle \quad \frac{C_{\triangleright} yzt, C_{\triangleright} xtw \in \mathscr{C}}{\langle \emptyset, C_{\triangleright} xyc_i, C_{\triangleright} c_i zw \rangle}$$

$$\langle \textit{Exchange} \rangle \quad \frac{C_{\circ} wyt, C_{\circ} xzs, C_{\triangleright} tsu \in \mathscr{C}}{\langle \emptyset, C_{\triangleright} wxc_i, C_{\triangleright} yzc_j, C_{\circ} c_i c_j u \rangle}$$

with c_i, c_j fresh labels.

Figure 9.10: CKBI frame expansion rules.

constraints. One might also expect substitution of $C_=$-equivalent labels in labelled formulae as well but (usefully for the simplicity of tableau derivation) this is not neccessary for the soundness and completeness of the calculi. Analogously, $\langle \succcurlyeq Ref \rangle$ and $\langle \succcurlyeq Trans \rangle$ ensure $C_{\succcurlyeq}$ is a preorder on labels.

Next we give the rules corresponding to the $\mathscr{L}$ frame axioms for each logic $\mathscr{L}$. We define $FrRules(\mathscr{L})$ as the rules for equality (and when $\mathscr{L}$ has intuitionistic additives, the rules for order) together with the tableau rules from the corresponding Figure. Figure 9.4 gives the rules associated with BBI and Figure 9.5 gives those associated with BI; Figure 9.6 gives those that should be added to the BI rules and BBI rules to get rules for DMBI and CBI respectively. In the case for DMBI all the rules in the figure are required, while in the case for CBI the rule $\langle Dual \rangle$ is redundant.

Each rule corresponds directly to a coherent axiom that defines the respective frame for the logic. For example, a BI frame is defined as a set-theoretic structure $(X, \circ, E)$ satisfying the axioms

$$\begin{array}{ll}
\text{(Commutativity)} \quad z \in x \circ y \to z \in y \circ x & \text{(Closure)} \quad e \in E \wedge e' \succcurlyeq e \to e' \in E \\
\text{(Unit Existence)} \quad \exists e \in E (x \in x \circ e) & \text{(Coherence)} \quad e \in E \wedge x \in y \circ e \to x \succcurlyeq y \\
\multicolumn{2}{l}{\text{(Associativity)}\ t' \succcurlyeq t \in x \circ y \wedge w \in t' \circ z \to \exists s, s', w' (s' \succcurlyeq s \in y \circ z \wedge w \succcurlyeq w' \in x \circ s')}
\end{array}$$

and the rules in Figure 9.5 are directly translated from each of these axioms in the manner previously described.

It's worth briefly mentioning the rules $\langle Function \rangle$ and $\langle Total \rangle$ that don't directly correspond to DMBI/CBI frame axioms as presented in the thesis: these ensure that C_- corresponds to a total function on labels, just like $-$ is a total function on DMBI/CBI frames. An alternative would be to directly represent $-$ as a function on labels, but this would affect the uniformity of the presentation and would complicate the soundness and completeness argument we give in the next section.

Next, Figure 9.7 gives tableau rules corresponding to basic Bi(B)BI: for BiBI we require both $\langle Commutativity \rangle$ and $\langle U\ Closure \rangle$, while for BiBBI $\langle U\ Closure \rangle$ is redundant. The rules in Figure 9.9 are rules that can be added to obtain the 'subclassical' extensions of BiBI; similarly, the rules in Figure 9.8 correspond to those that can be added for the 'subclassical' extensions of BiBBI. Finally, the rules in Figure 9.10 are those associated with CKBI.

Note that we haven't mentioned frame rules corresponding to separating modal logics. In this thesis we have left the exact axiomatisation of the modality added to BBI open; in the cases from the literature the modality is an S4 [68] or S5 [98] modality. In these cases we can add analogous rules to those from Fig 9.3, corre-

sponding to the fact that S4 is complete for frames where the accessibility relation is a preorder, and S5 for frames where the accessibility relation is an equivalence relation. Many other modal logics are axiomatised by finite coherent theories [173], and this provides a schema for defining a plethora of separating modal logics with tableaux calculi.

9.4 The Tableaux Calculi

We now specify the tableaux calculus for each logic $\mathscr{L}$. For each logic $\mathscr{L}$ CSSs are defined over labelled $\mathscr{L}$ formulae and label constraints over $ConSymb(\mathscr{L})$. The set of tableau rules for each logic is given by $\mathscr{R}(\mathscr{L}) = LogRules(\mathscr{L}) \cup FrRules(\mathscr{L})$. There is a final ingredient: the *closure conditions* that dictate when a branch is determined to be inconsistent.

Definition 9.5 (Closure Conditions)**.** *A CSS $\langle \mathscr{F}, \mathscr{C} \rangle$ is* closed *if it satisfies one of the following* closure conditions*:*

1. $\mathbb{F}\varphi : y, \mathbb{T}\varphi : x \in \mathscr{F}$ *and* $C_{=}yx \in \mathscr{C}$*;*
2. $\mathbb{F}\varphi : y, \mathbb{T}\varphi : x \in \mathscr{F}$ *and* $C_{\succcurlyeq}yx \in \mathscr{C}$*;*
3. $\mathbb{F}\top : x \in \mathscr{F}$*;*
4. $\mathbb{T}\bot : x \in \mathscr{F}$*;*
5. $\mathbb{F}\top^* : x \in \mathscr{F}$ *and* $C_E x \in \mathscr{C}$*;*
6. $\mathbb{T}\bot^* : x \in \mathscr{F}$ *and* $C_U x \in \mathscr{C}$.

This definition reveals the reason behind the asymetry in the specification of logical tableau rules pertaining to $\top^*$ and $\bot^*$: the corresponding $\mathbb{F}\top^*$ and $\mathbb{T}\bot^*$ rules actually show up as closure conditions. Closure conditions directly correspond to the ways in which a CSS can give an inconsistent partial specification of a $\mathscr{L}$ model. This allows us to define when a tableau is a *proof*.

Definition 9.6 (Tableau Proof)**.** *For a $\mathscr{L}$ formula φ, a* tableau *for φ is a $\mathscr{R}(\mathscr{L})$ tableau for $\langle \{\mathbb{F}\varphi : c_0\}, \emptyset \rangle$. A* tableau proof *of φ is a tableau for φ in which every CSS is closed.*

The existence of a tableau proof for φ witnesses that every attempt to build a countermodel for φ will fail: it will always result in an inconsistency. The idea is that this shows the formula is valid, as no models in which it doesn't hold can exist.

Tableau proofs are necessarily finite constructions. The closure conditions witnessing the proof will become satisfied at a particular stage in the inductive construction of the tableau for φ, and, by design, at each inductive step the tableau is comprised of finite CSS since we start with a finite CSS, and each rule adds only finitely many expressions and can only branch finitely. Crucially, once a tableau for φ is closed, no application of expansion rules can change that fact. In the case that the tableau cannot be closed, the construction of the tableau can go on infinitely. This entails that we don't obtain a finite model property leading to decidability, as the possible countermodels represented by the branches in a never-closing construction may be infinite.

It is instructive to now give some examples of tableau proofs in this framework. To sharpen the intuition of how tableau construction works, these are presented traditionally as finitely branching trees in which nodes are labelled with finite CSSs. To understand how they relate to our formal definition, note that by taking the union of the CSSs on each branch we obtain a *branch CSS* $\langle \mathscr{F}, \mathscr{C} \rangle$. The list of these branch CSSs directly corresponds to the list of CSSs constructed by the tableau procedure given in Definition 9.4. To the right of each step in the construction we state the rule that was applied, and from where in the branch we obtain that the premiss of that rule holds. The symbol $\otimes$ denotes that the CSS generated by a branch is closed.

We begin with a tableau proof of $(\varphi \mathrel{-\!\!*} \chi) \wedge (\psi \mathrel{-\!\!*} \chi) \to ((\varphi \vee \psi) \mathrel{-\!\!*} \chi)$ for any bunched logic $\mathscr{L}$ with classical additives. This corresponds to the algebraic equation 4. of Proposition 6.2 that determines the conversion by $\mathrel{-\!\!*}$ of meets into joins. The proof is shown in Figure 9.11. At steps 5. and 6. the rule $\langle \mathbb{T} \mathrel{-\!\!*} \rangle$ is used. Doing so requires the existence of not only a labelled formula with outermost connective $\mathrel{-\!\!*}$, but also the existence of a label constraint $C_o xyz$ that already occurs on the branch. This is provided by the label constraint $C_\circ c_0 c_1 c_2$ introduced at step 3. The closure of the left most branch is because $\mathbb{F}\varphi : c_1$ occurs at step 5. and $\mathbb{T}\varphi : c_1$ at step 7.; the center left branch is closed because $\mathbb{F}\psi : c_1$ occurs at step 6. and $\mathbb{T}\psi : c_1$ occurs at step 7.; the centre right branch is closed becuse $\mathbb{F}\chi : c_2$ occurs at step 3. and $\mathbb{T}\chi : c_2$ occurs at step 6.; finally, the rightmost branch is closed because $\mathbb{F}\chi : c_2$ occurs at step 3. and $\mathbb{T}\chi : c_2$ occurs at step 5.

Next we see a tableau proof that requires the use of a frame expansion rule. Figure 9.12 shows a CKBI tableau proof for the Exchange law $((\varphi * \chi);(\psi * \theta)) \to ((\varphi;\psi) * (\chi;\theta))$. Closure of the leftmost branch is witnessed by the occurence of the labelled formulae $\mathbb{T}\varphi : c_3$ at step 4. and $\mathbb{F}\varphi : c_3$ at step 8.; the center left branch is closed because of the occurence of $\mathbb{T}\psi : c_5$ at step 5. and $\mathbb{F}\psi : c_5$ at step 8.; the center right branch is closed because of the occurence of $\mathbb{T}\xi : c_4$ at step 4. and $\mathbb{F}\xi : c_4$ at step 8.; finally, the rightmost branch is closed because of the occurence

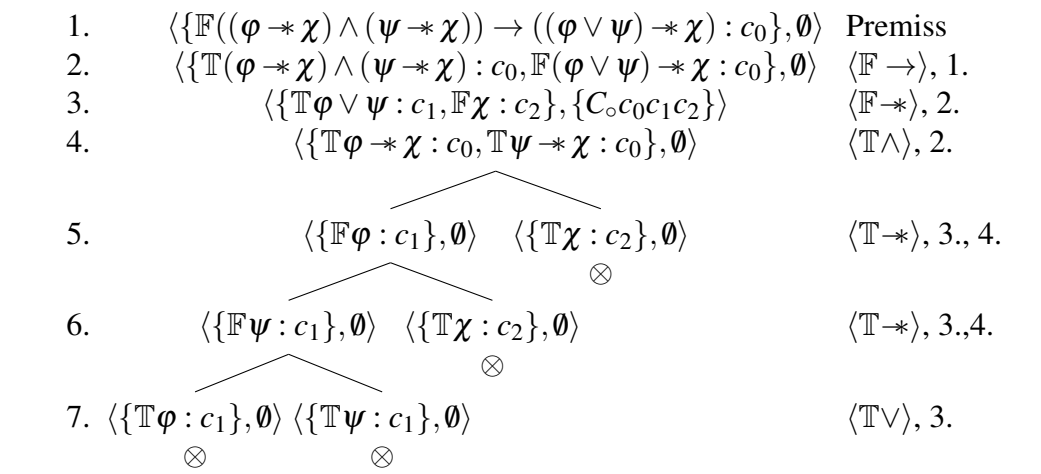


Figure 9.11: Tableau proof of $(\varphi \mathrel{-\!\!*} \chi) \wedge (\psi \mathrel{-\!\!*} \chi) \to ((\varphi \vee \psi) \mathrel{-\!\!*} \chi)$.

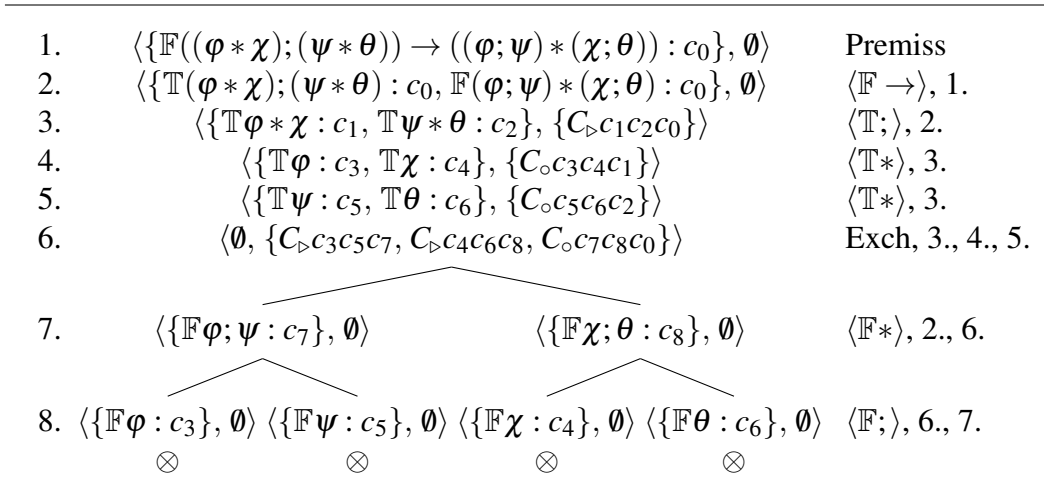


Figure 9.12: CKBI tableau proof of $((\varphi * \chi);(\psi * \theta)) \to ((\varphi;\psi) * (\chi;\theta))$.

of $\mathbb{T}\theta : c_6$ at step 5. and $\mathbb{F}\theta : c_6$ at step 8. Note the essential use of the frame expansion rule $\langle Exchange\rangle$ at step 6: this should be expected, as the frame property it corresponds to is the frame correspondent of the Exchange law. This is able to be triggered because of the introduction of the label constraints $C_\triangleright c_1 c_2 c_0$ at step 3, $C_\circ c_3 c_4 c_1$ at step 4. and $C_\circ c_7 c_8 c_0$ at step 5.

Finally we look at a tableau proof for a logic with intuitionistic additives. Figure 9.13 shows a BiBI + Weak Distributivity proof of the weak distributivity axiom $\varphi * (\psi \mathbin{\tilde{\vee}} \chi) \to (\varphi * \psi) \mathbin{\tilde{\vee}} \chi$. Here the frame expansion rule corresponding to the frame property Weak Distributivity is used at step 5., using the the label constraints that occur at steps 3. and 4. This gives the required label constraints to trigger the rules $\langle\mathbb{T}\tilde{\vee}\rangle$ at step 7. (using the labelled formula $\mathbb{T}\psi \mathbin{\tilde{\vee}} \chi : c_4$ introduced at step 3.)

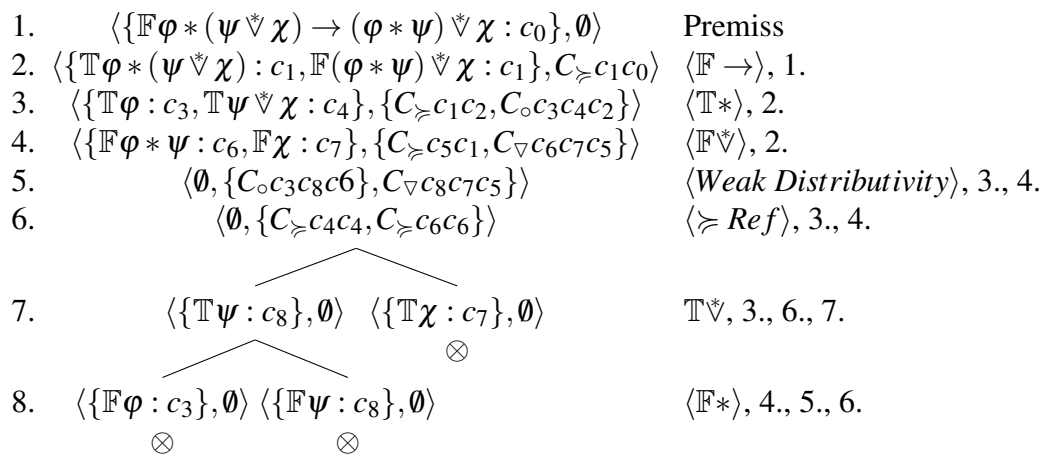


Figure 9.13: Tableau proof of the weak distributivity axiom.

and $\langle\mathbb{F}*\rangle$ at step 8. (using the labelled formula $\mathbb{F}\varphi * \psi : c_6$ introduced at step 4.). The left branch is closed because $\mathbb{T}\varphi : c_3$ occurs at step 3. and $\mathbb{F}\varphi : c_3$ occurs at step 8.; the centre branch is closed because $\mathbb{T}\psi : c_8$ occurs at step 7. and $\mathbb{F}\psi : c_8$ occurs at step 8.; finally the right branch is closed because $\mathbb{F}\chi : c_7$ occurs at step 4. and $\mathbb{T}\chi : c_7$ occurs at step 7.

9.5 Parametric Soundness and Completeness

With tableaux calculi for the breadth of bunched logics specified, we turn to proving these systems are sound and complete. Due to the uniform presentation of the systems we are able to do so parametrically in choice of calculus. We do so by utilising a novel representation of tableau systems as finite coherent theories: the key insight here is that the translation of coherent formulae into tableau rules is not one-way: tableaux rules can naturally be seen as coherent formulae in a signature augmented with special predicate symbols corresponding to labelled formulae. The parametric soundness and completeness of the framework can then be reduced to proving the soundness and completeness of Tarskian truth for coherent logic with respect to a meta-tableaux method formulated by Bezem & Coquand [22]. To our knowledge, the application of this technique to labelled tableaux calculi is new, although in the aforementioned work Bezem & Coquand show how to encode the tableau method for classical logic as a coherent theory, and trace the idea of abbreviating formulae with predicate symbols to Skolem [209].

We begin by defining the coherent theory associated with a tableaux calculus. Let $\mathscr{L}$ be a bunched logic. We extend the first-order signature $ConSym(\mathscr{L})$ with unary predicate symbols $\mathbb{T}\varphi$ and $\mathbb{F}\varphi$ for each $\mathscr{L}$ formula φ to obtain the signature

$Tab(\mathscr{L})$ for the tableau language for $\mathscr{L}$. The logical expansion rules can now be straightforwardly read as coherent formulae in this signature in effectively the same way read coherent axioms were interpreted as tableau rules: the premiss is read as a conjunction forming the antecedent of a coherent formula and the conclusion is read as the consequent; fresh labels are replaced with variables bound by existential quantification, branching is read as disjunction and the labelled formulae and label constraints in each disjunct are read as a conjunction of atomic formulae.

In Figure 9.14 we show the translation of the rules pertaining to bunched logics with classical additives; in Figure 9.15 those for bunched logics with intuitionistic additives. It should be emphasised at this point that the axioms in these figures define a schema for *infinitely* many coherent axioms, with each $\langle \mathbb{S}\heartsuit \rangle$ corresponding to the infinitely many $\mathscr{L}$ formulae which have the connective $\heartsuit$ as the outermost connective.

Next, the same translation must be done for all of the frame expansion rules associated with each $\mathscr{L}$. For the most part this consists of the pre-translation of the already-coherent axiomatisation of $\mathscr{L}$ frames into the tableau language, although we must also translate the rules for equality and order, as well as those governing the interpretation of $C_{\_}$ as a total function. For clarity we give these explicitly. In Figure 9.16 the coherent axioms corresponding to the equality and order rules is given. Note that $\langle Sub \rangle$ defines a schema for *finitely* many coherent axioms, corresponding to the finitely many ways a label can occur in a label constraint for the finitely many constraint symbols associated with each logic: this finiteness will be important in our completeness proof.

Figure 9.17 and 9.18 give the coherent axioms obtained from the frame rules for BBI and BI respectively. For CBI and DMBI, the axioms from Figure 9.19 are added to those for BBI and BI respectively, noting that $\langle Dual \rangle$ is redundant for CBI. Coherent axioms corresponding to basic Bi(B)BI are obtained by adding those of Figure 9.20 to those for (B)BI—here $\langle U - Closure \rangle$ is redundant for BiBBI. The coherent axioms corresponding to subclassical extensions of basic BiBBI are shown in Figure 9.21, whilst those for extensions of basic BiBI are shown in Figure 9.22. Finally, Figure 9.23 gives the coherent axioms that should be added to those of BBI for CKBI.

There is a final aspect of the tableaux calculi that must be captured in the coherent axiomatisation: the closure conditions. These are all coherent antecedents that are never true (i.e.; coherent formulae with conclusion $\bot$). Indeed, one might think of closure conditions as the way in which coherent axioms of this form are represented in tableaux calculi. The translations of closure conditions is shown in Figure 9.24.

$$
\begin{array}{ll}
\langle \mathbb{T}\wedge\rangle & \mathbb{T}(\varphi\wedge\psi)(x)\to\mathbb{T}\varphi(x)\wedge\mathbb{T}\psi(x)\\
\langle \mathbb{F}\wedge\rangle & \mathbb{F}(\varphi\wedge\psi)(x)\to\mathbb{F}\varphi(x)\vee\mathbb{F}\psi(x)\\
\langle \mathbb{T}\vee\rangle & \mathbb{T}(\varphi\vee\psi)(x)\to\mathbb{T}\varphi(x)\vee\mathbb{T}\psi(x)\\
\langle \mathbb{F}\vee\rangle & \mathbb{F}(\varphi\vee\psi)(x)\to\mathbb{F}\varphi(x)\wedge\mathbb{F}\psi(x)\\
\langle \mathbb{T}\to\rangle & \mathbb{T}(\varphi\to\psi)(x)\to\mathbb{F}\varphi(x)\vee\mathbb{T}\psi(x)\\
\langle \mathbb{F}\to\rangle & \mathbb{F}(\varphi\to\psi)(x)\to\mathbb{T}\varphi(x)\wedge\mathbb{F}\psi(x)\\
\langle \mathbb{T}*\rangle & \mathbb{T}(\varphi*\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{T}\psi(z)\wedge C_\circ yzx)\\
\langle \mathbb{F}*\rangle & \mathbb{F}(\varphi*\psi)(x)\wedge C_\circ yzx\to\mathbb{F}\varphi(y)\vee\mathbb{F}\psi(z)\\
\langle \mathbb{T}\mathrel{-\!\!*}\rangle & \mathbb{T}(\varphi\mathrel{-\!\!*}\psi)(x)\wedge C_\circ xyz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}\mathrel{-\!\!*}\rangle & \mathbb{F}(\varphi\mathrel{-\!\!*}\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_\circ xyz)\\
\langle \mathbb{T}*\!\!-\rangle & \mathbb{T}(\varphi*\!\!-\psi)(x)\wedge C_\circ yxz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}*\!\!-\rangle & \mathbb{F}(\varphi*\!\!-\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_\circ yxz)\\
\langle \mathbb{T}\mathbin{\dot{\vee}}\rangle & \mathbb{T}(\varphi\mathbin{\dot{\vee}}\psi)(x)\wedge C_\triangledown yzx\to\mathbb{T}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}\mathbin{\dot{\vee}}\rangle & \mathbb{F}(\varphi\mathbin{\dot{\vee}}\psi)(x)\to\exists y,z(\mathbb{F}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_\triangledown yzx)\\
\langle \mathbb{T}\mathbin{\dot{\setminus}}\rangle & \mathbb{T}(\varphi\mathbin{\dot{\setminus}}\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_\triangledown yxz)\\
\langle \mathbb{F}\mathbin{\dot{\setminus}}\rangle & \mathbb{F}(\varphi\mathbin{\dot{\setminus}}\psi)(x)\wedge C_\triangledown yxz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{T}\mathbin{\dot{\multimap}}\rangle & \mathbb{T}\mathbin{\dot{\multimap}}\varphi(x)\wedge C_- xy\to\mathbb{F}\varphi(y)\\
\langle \mathbb{F}\mathbin{\dot{\multimap}}\rangle & \mathbb{F}\mathbin{\dot{\multimap}}\varphi(x)\wedge C_- xy\to\mathbb{T}\varphi(y)\\
\langle \mathbb{T};\rangle & \mathbb{T}(\varphi\,;\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{T}\psi(z)\wedge C_\triangleright yzx)\\
\langle \mathbb{F};\rangle & \mathbb{F}(\varphi\,;\psi)(x)\wedge C_\triangleright yzx\to\mathbb{F}\varphi(y)\vee\mathbb{F}\psi(z)\\
\langle \mathbb{T}\mathrel{-\!\!\triangleright}\rangle & \mathbb{T}(\varphi\mathrel{-\!\!\triangleright}\psi)(x)\wedge C_\triangleright xyz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}\mathrel{-\!\!\triangleright}\rangle & \mathbb{F}(\varphi\mathrel{-\!\!\triangleright}\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_\triangleright xyz)\\
\langle \mathbb{T}\triangleright\!\!-\rangle & \mathbb{T}(\varphi\triangleright\!\!-\psi)(x)\wedge C_\triangleright yxz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}\triangleright\!\!-\rangle & \mathbb{F}(\varphi\triangleright\!\!-\psi)(x)\to\exists y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_\triangleright yxz)\\
\langle \mathbb{T}\diamond\rangle & \mathbb{T}\diamond\varphi(x)\to\exists y(\mathbb{T}\varphi(y)\wedge C_R xy)\\
\langle \mathbb{F}\diamond\rangle & \mathbb{F}\diamond\varphi(x)\wedge C_R xy\to\mathbb{F}\varphi(y)\\
\langle \mathbb{T}\top^*\rangle & \mathbb{T}\top^*(x)\to C_E x\\
\langle \mathbb{F}\bot^*\rangle & \mathbb{F}\bot^*(x)\to C_U x
\end{array}
$$

Figure 9.14: Logical coherent axioms for bunched logics with classical additives.

$$
\begin{array}{ll}
\langle \mathbb{T}\wedge\rangle & \mathbb{T}(\varphi\wedge\psi)(x)\to\mathbb{T}\varphi(x)\wedge\mathbb{T}\psi(x)\\
\langle \mathbb{F}\wedge\rangle & \mathbb{F}(\varphi\wedge\psi)(x)\to\mathbb{F}\varphi(x)\vee\mathbb{F}\psi(x)\\
\langle \mathbb{T}\vee\rangle & \mathbb{T}(\varphi\vee\psi)(x)\to\mathbb{T}\varphi(x)\vee\mathbb{T}\psi(x)\\
\langle \mathbb{F}\vee\rangle & \mathbb{F}(\varphi\vee\psi)(x)\to\mathbb{F}\varphi(x)\wedge\mathbb{F}\psi(x)\\
\langle \mathbb{T}\to\rangle & \mathbb{T}(\varphi\to\psi)(x)\wedge C_{\succcurlyeq}yx\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(y)\\
\langle \mathbb{F}\to\rangle & \mathbb{F}(\varphi\to\psi)(x)\to\exists y(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(y)\wedge C_{\succcurlyeq}yx\\
\langle \mathbb{T}*\rangle & \mathbb{T}(\varphi*\psi)(x)\to\exists w,y,z(\mathbb{T}\varphi(y)\wedge\mathbb{T}\psi(z)\wedge C_{\circ}yzw\wedge C_{\succcurlyeq}xw)\\
\langle \mathbb{F}*\rangle & \mathbb{F}(\varphi*\psi)(x)\wedge C_{\succcurlyeq}xw\wedge C_{\circ}yzw\to\mathbb{F}\varphi(y)\vee\mathbb{F}\psi(z)\\
\langle \mathbb{T}\mathrel{-\!\!*}\rangle & \mathbb{T}(\varphi\mathrel{-\!\!*}\psi)(x)\wedge C_{\succcurlyeq}wx\wedge C_{\circ}wyz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}\mathrel{-\!\!*}\rangle & \mathbb{F}(\varphi\mathrel{-\!\!*}\psi)(x)\to\exists w,y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_{\circ}wyz\wedge C_{\succcurlyeq}wx)\\
\langle \mathbb{T}*\!\!-\rangle & \mathbb{T}(\varphi*\!\!-\psi)(x)\wedge C_{\succcurlyeq}wx\wedge C_{\circ}ywz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}*\!\!-\rangle & \mathbb{F}(\varphi*\!\!-\psi)(x)\to\exists w,y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_{\circ}ywz\wedge C_{\succcurlyeq}wx)\\
\langle \mathbb{T}\overset{*}{\vee}\rangle & \mathbb{T}(\varphi\overset{*}{\vee}\psi)(x)\wedge C_{\succcurlyeq}wx\wedge C_{\triangledown}yzw\to\mathbb{T}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{F}\overset{*}{\vee}\rangle & \mathbb{F}(\varphi\overset{*}{\vee}\psi)(x)\to\exists w,y,z(\mathbb{F}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_{\triangledown}yzw\wedge C_{\succcurlyeq}wx)\\
\langle \mathbb{T}\overset{*}{\setminus}\rangle & \mathbb{T}(\varphi\overset{*}{\setminus}\psi)(x)\to\exists w,y,z(\mathbb{T}\varphi(y)\wedge\mathbb{F}\psi(z)\wedge C_{\triangledown}ywz\wedge C_{\succcurlyeq}xw)\\
\langle \mathbb{F}\overset{*}{\setminus}\rangle & \mathbb{F}(\varphi\overset{*}{\setminus}\psi)(x)\wedge C_{\succcurlyeq}xw\wedge C_{\triangledown}ywz\to\mathbb{F}\varphi(y)\vee\mathbb{T}\psi(z)\\
\langle \mathbb{T}\mathrel{-\!\!\!\circ}\rangle & \mathbb{T}\mathrel{-\!\!\!\circ}\varphi(x)\wedge C_{-}xy\to\mathbb{F}\varphi(y)\\
\langle \mathbb{F}\mathrel{-\!\!\!\circ}\rangle & \mathbb{F}\mathrel{-\!\!\!\circ}\varphi(x)\wedge C_{-}xy\to\mathbb{T}\varphi(y)\\
\langle \mathbb{T}\top^{*}\rangle & \mathbb{T}\top^{*}(x)\to C_{E}x\\
\langle \mathbb{F}\bot^{*}\rangle & \mathbb{F}\bot^{*}(x)\to C_{U}x
\end{array}
$$

Figure 9.15: Logical coherent axioms for bunched logics with intuitionistic additives.

$$
\begin{array}{llll}
\langle = Ref\rangle & \top\to C_{=}xx & \langle = Trans\rangle & C_{=}xy\wedge C_{=}yz\to C_{=}xz\\
\langle = Sym\rangle & C_{=}xy\to C_{=}yx & \langle Sub\rangle & C_{\heartsuit}x_0\ldots x\ldots x_n\wedge C_{=}xy\to C_{\heartsuit}x_0\ldots y\ldots x_n\\
\langle \succcurlyeq Ref\rangle & \top\to C_{\succcurlyeq}xx & \langle \succcurlyeq Trans\rangle & C_{\succcurlyeq}xy\wedge C_{\succcurlyeq}yz\to C_{\succcurlyeq}xz
\end{array}
$$

Figure 9.16: Coherent axioms for equality and order. In $\langle Sub\rangle$, $C_{\heartsuit}\in ConSymb(\mathscr{L})$.

$$
\begin{array}{ll}
\langle Commutativity\rangle & C_{\circ}xyz\to C_{\circ}yxz\\
\langle Unit\ Existence\rangle & \top\to\exists y(C_{E}y\wedge C_{\circ}xyx)\\
\langle Coherence\rangle & C_{E}z\wedge C_{\circ}yzx\to C_{=}xy\\
\langle Associativity\rangle & C_{\circ}xyt\wedge C_{\circ}tzw\to\exists s(C_{\circ}yzs\wedge C_{\circ}xsw)
\end{array}
$$

Figure 9.17: Frame coherent axioms for BBI.

$$
\begin{array}{ll}
\langle Commutativity\rangle & C_\circ xyz \to C_\circ yxz \\
\langle Unit\ Existence\rangle & \top \to \exists y(C_E y \wedge C_\circ xyx) \\
\langle Coherence\rangle & C_E z \wedge C_\circ yzx \to C_\succcurlyeq xy \\
\langle Associativity\rangle & C_\succcurlyeq t't \wedge C_\circ xyt \wedge C_\circ t'zw \\
 & \to \exists s,s',w'(C_\succcurlyeq s's \wedge C_\circ yzs \wedge C_\succcurlyeq ww' \wedge C_\circ xs'w')
\end{array}
$$

Figure 9.18: Frame coherent axioms for BI.

$$
\begin{array}{ll}
\langle Function\rangle & C_- xy \wedge C_- xy' \to C_= yy' \\
\langle Total\rangle & \top \to \exists y(C_- xy) \\
\langle Dual\rangle & C_\succcurlyeq xy \wedge C_- xx' \wedge C_- yy' \to C_\succcurlyeq y'x' \\
\langle Involutive\rangle & C_- xx' \wedge C_- x'x'' \to C_= xx'' \\
\langle Compatibility\rangle & C_\circ xyz \wedge C_- xx' \wedge C_- zz' \to C_\circ z'yx'
\end{array}
$$

Figure 9.19: Frame coherent axioms for DMBI and CBI.

$$
\langle Commutativity\rangle \quad C_\triangledown xyz \to C_\triangledown yxz \quad \langle U-Closure\rangle \quad C_U x \wedge C_\succcurlyeq xy \to C_U y
$$

Figure 9.20: Frame coherent axioms for Bi(B)BI.

$$
\begin{array}{ll}
\langle Associativity\rangle & C_\triangledown xyt \wedge C_\triangledown tzw \to \exists s(C_\triangledown yzs \wedge C_\triangledown xsw) \\
\langle \bot^* Weak\rangle & C_U z \wedge C_\triangledown yzx \to C_= yx \\
\langle \bot^* Contract\rangle & \top \to \exists y(C_U y \wedge C_\triangledown xyx) \\
\langle \divideontimes Contract\rangle & \top \to C_\triangledown xxx \\
\langle Weak\ Dist\rangle & C_\circ x_1x_2t \wedge C_\triangledown y_1y_2t \to \exists z(C_\circ x_1zy_1 \wedge C_\triangledown zy_2x_2)
\end{array}
$$

Figure 9.21: Frame coherent axioms for extensions of BiBBI.

$$
\begin{array}{ll}
\langle Assoc\rangle & C_\succcurlyeq t't \wedge C_\triangledown xyt \wedge C_\triangledown t'zw \to \exists s,s',w'(C_\succcurlyeq ss' \wedge C_\triangledown yzs \wedge C_\succcurlyeq w'w \wedge C_\triangledown xs'w') \\
\langle \bot^* Weak\rangle & C_U z \wedge C_\triangledown yzx \to C_\succcurlyeq yx \\
\langle \bot^* Contr\rangle & \top \to \exists y(C_U y \wedge C_\triangledown xyx) \\
\langle \divideontimes Contr\rangle & \top \to C_\triangledown xxx \\
\langle W.\ Dist\rangle & C_\succcurlyeq t't \wedge C_\circ x_1x_2t \wedge C_\succcurlyeq t''t' \wedge C_\triangledown y_1y_2t'' \to \exists z(C_\circ x_1zy_1 \wedge C_\triangledown zy_2x_2)
\end{array}
$$

Figure 9.22: Frame coherent axioms for extensions of BiBI.

$\langle Unit\ Existence_L \rangle$	$\top \to \exists y(C_E y \wedge C_{\triangleright} yxx)$
$\langle Unit\ Existence_R \rangle$	$\top \to \exists y(C_E y \wedge C_{\triangleright} xyx)$
$\langle Coherence_L \rangle$	$C_E z \wedge C_{\triangleright} zyx \to C_{=} xy$
$\langle Coherence_R \rangle$	$C_E z \wedge C_{\triangleright} yzx \to C_{=} xy$
$\langle Associativity_L \rangle$	$C_{\triangleright} xyt \wedge C_{\triangleright} tzw \to \exists s(C_{\triangleright} yzs \wedge C_{\triangleright} xsw)$
$\langle Associativity_L \rangle$	$C_{\triangleright} yzt \wedge C_{\triangleright} xtw \to \exists s(C_{\triangleright} xys \wedge C_{\triangleright} szw)$
$\langle Exchange \rangle$	$C_{\circ} wyt \wedge C_{\circ} xzs \wedge C_{\triangleright} tsu \to \exists v, v'(C_{\triangleright} wxv \wedge C_{\triangleright} yzv' \wedge C_{\circ} vv'u$

Figure 9.23: Frame coherent axioms for CKBI.

$\langle = \ Inconsistency \rangle$	$\mathbb{T}\varphi(x) \wedge \mathbb{F}\varphi(y) \wedge C_{=} xy \to \bot$
$\langle \succcurlyeq \ Inconsistency \rangle$	$\mathbb{T}\varphi(x) \wedge \mathbb{F}\varphi(y) \wedge C_{\succcurlyeq} xy \to \bot$
$\langle \mathbb{F}\top \rangle$	$\mathbb{F}\top(x) \to \bot$
$\langle \mathbb{T}\bot \rangle$	$\mathbb{T}\bot(x) \to \bot$
$\langle \mathbb{F}\top^* \rangle$	$\mathbb{F}\top^*(x) \wedge C_E x \to \bot$
$\langle \mathbb{T}\bot^* \rangle$	$\mathbb{T}\bot^*(x) \wedge C_U x \to \bot$

Figure 9.24: Coherent axioms for closure conditions.

For each logic $\mathscr{L}$ the coherent theory $\Phi^{\mathscr{L}}$ associated to the tableaux calculus for $\mathscr{L}$ is comprised of the logical coherent axioms for $\mathscr{L}$, together with the frame coherent axioms for $\mathscr{L}$, the closure condition axioms corresponding to the $\mathscr{L}$ tableaux calculus and the coherent axioms for equality (and in the case of $\mathscr{L}$ with intuitionistic additives, those for order as well).

The coherent theories corresponding to each calculus should be compared to the definitions of Hintikka sets that are commonly used to prove completeness for tableau systems (for bunched logic systems see [101, 148, 68, 98]). In essence what we are doing is providing a modular axiomatisation of Hintikka set for the different systems. This coheres with the observation[2] of Beckert & Goré [16] that their method for generating labelled tableaux calculi for propositional modal logics works via a "clever translation ... into first-order logic". Where we diverge from other work is that in our case we are making the metalogic—coherent logic—in which Hintikka sets are defined explicit, which allows us to leverage the metatheory of that metalogic. There is a small problem: to use that metatheory we require *finite* coherent theories, whereas each $\Phi^{\mathscr{L}}$ is infinite. We *can*, however, obtain a finite

[2]Thank you to Stéphane Demri for pointing me in the direction of his [75] in which this observation was reported.

coherent theory by looking at the part of $\Phi^{\mathscr{L}}$ that is actually relevant to each $\mathscr{L}$ formula φ.

Definition 9.7 ($\Phi^{\mathscr{L}}_{\varphi}$)**.** *Let φ be a $\mathscr{L}$ formula. The* coherent theory corresponding to *φ, $\Phi^{\mathscr{L}}_{\varphi}$, is given by*

- *Each logical coherent axiom of $\mathscr{L}$ corresponding to a subformula of φ;*
- *The frame coherent axioms for $\mathscr{L}$;*
- *The closure condition axioms for $\mathscr{L}$ (taking only the instances of $\langle = \mathit{Inconsistency}\rangle$ and $\langle \succcurlyeq \mathit{Inconsistency}\rangle$ featuring a subformula of φ);*
- *The equality and/or order coherent axioms.*

It is clear that $\Phi^{\mathscr{L}}_{\varphi}$ is finite for any $\mathscr{L}$ formula φ: since there are only finitely many subformulae of φ, only finitely many logical coherent axioms are taken from the infinitely many defined by the schema. This corresponds precisely to what happens when a tableau proof is attempted for φ: of the logical expansion rules, only instances corresponding to subformulae of φ can ever be triggered, as there is no way for a labelled formula $\mathbb{S}\psi : x$ to be introduced in the tableau if ψ is not a subformulae of φ. As there are finitely many frame coherent axioms, closure condition axioms featuring subformulae of φ and equality/order coherent axioms, the whole theory is finite.

We now use the coherent theories $\Phi^{\mathscr{L}}_{\varphi}$ to prove soundness and completeness of the systems. We first establish a connection between the existence of first-order models of $\Phi^{\mathscr{L}}_{\varphi}$ and particular Kripke $\mathscr{L}$ models. To help with the clarity of our presentation, we refer to first-order models of coherent theories (in the usual sense) as *Tarksian models*, in contrast to *Kripke models*. The key step is to establish that the existence of a Kripke $\mathscr{L}$ model with a state that does not satisfy φ is equivalent to the existence of a Tarskian model of $\Phi^{\mathscr{L}}_{\varphi} \cup \{\exists x \mathbb{F}\varphi(x)\}$.

Definition 9.8 (Induced Kripke Model of $\mathscr{M}$)**.** *Given a Tarskian model $\mathscr{M} = (X, \mathscr{I})$ of $\Phi^{\mathscr{L}}_{\varphi}$, the structure of the* induced Kripke model operations of *$\mathscr{M}$ are defined as follows. Here, $\heartsuit$ is any binary operation in the language of $\mathscr{L}$ frames.*

- $X_{\mathscr{M}} = \{[x] \mid x \in X\}$, *where* $[x] = \{y \mid C^{\mathscr{I}}_{=} yx\}$;
- $[x] \succcurlyeq_{\mathscr{M}} [y]$ *iff* $C^{\mathscr{I}}_{\succcurlyeq} xy$;
- $[x] \in [y] \heartsuit_{\mathscr{M}} [z]$ *iff* $C^{\mathscr{I}}_{\heartsuit} yzx$;
- $E_{\mathscr{M}} = \{[x] \mid C^{\mathscr{I}}_{E} x\}$;

- $-_{\mathscr{M}}[x] = \{y \mid C_{-}^{\mathscr{I}} xy\}$;
- $U_{\mathscr{M}} = \{[x] \mid C_{U}^{\mathscr{I}} x\}$;
- $R_{\mathscr{M}}[x][y]$ *iff* $C_{R}^{\mathscr{I}} xy$.
- *for* $\mathscr{L}$ *with classical additives,* $V_{\mathscr{M}}(\mathrm{p}) = \{[x] \mid \exists y \in X, \mathbb{T}\mathrm{p}^{\mathscr{I}}(y)$ *and* $C_{=}^{\mathscr{I}} xy\}$;
- *for* $\mathscr{L}$ *with intuitionistic additives,* $V_{\mathscr{M}}(\mathrm{p}) = \{[x] \mid \exists y \in X, \mathbb{T}\mathrm{p}^{\mathscr{I}}(y)$ *and* $C_{\succcurlyeq}^{\mathscr{I}} xy\}$.

The induced Kripke model of $\mathscr{M}$ *is comprised of the set* $X_{\mathscr{M}}$ *and the valuation* $\mathscr{V}_{\mathscr{M}}$ *together with the relevant* $\mathscr{L}$ *frame structure taken from the above list.*

The next lemma shows that Tarskian models of the coherent theory $\Phi_{\varphi}^{\mathscr{L}}$ capture enough of the Kripke semantics to enforce the satisfaction (or not) of subformulae ψ of φ in the induced Kripke model at particular states through the predicates $\mathbb{S}\psi$.

Lemma 9.9. *For any bunched logic* $\mathscr{L}$, $\mathscr{L}$ *formula* φ *and Tarskian model* $\mathscr{M}$ *of* $\Phi_{\varphi}^{\mathscr{L}}$:

1. *The induced Kripke model of* $\mathscr{M}$ *is a* $\mathscr{L}$ *model.*
2. *For any subformula* ψ *of* φ *and* $x \in X$, $\mathbb{T}\psi^{\mathscr{I}}(x)$ *implies* $[x] \vDash_{\mathscr{V}_{\mathscr{M}}} \psi$ *and* $\mathbb{F}\psi^{\mathscr{I}}(x)$ *implies* $[x] \nvDash_{\mathscr{V}_{\mathscr{M}}} \psi$.

Proof. 1. The axioms governing $C_{=}$ establish that it is an equivalence relation on X. Further, the axioms defined by the schema $\langle Sub \rangle$ dictate that the equivalence relation respects the $\mathscr{L}$ frame structure, and so the $\mathscr{L}$ frame structure is well-defined. The case of $-_{\mathscr{M}}$ requires some closer attention: we must show that $-_{\mathscr{M}}[x]$ is an equivalence class of $C_{=}$ and is always defined. Let $y \in -_{\mathscr{M}}[x]$ and $C_{=}yy'$. Then $C_{-}^{\mathscr{I}} xy$ and by $\langle Sub \rangle$ we have that $C^{\mathscr{I}} xy'$. Hence $y' \in -_{\mathscr{M}}[x]$. Further, $\langle Total \rangle$ dictates that $\{y \mid C_{-}^{\mathscr{I}} xy\} \neq \emptyset$.

The required frame properties are directly axiomatised by the frame axioms entailing that the structure is indeed a $\mathscr{L}$ frame. The last verification is that $\mathscr{V}_{\mathscr{M}}$ is well-defined, and further, is persistent when $\mathscr{L}$ has intuitionistic additives. Well-definedness follows straightforwardly from $\langle Sub \rangle$ so we move to persistence: suppose $[x] \in \mathscr{V}_{\mathscr{M}}(\mathrm{p})$ and $[y] \succcurlyeq_{\mathscr{M}} [x]$. Then there exists $z \in X$ such that $C_{\succcurlyeq}^{\mathscr{I}} xz$ and $\mathbb{T}\mathrm{p}^{\mathscr{I}}(z)$. By definition, we also have that $C_{\succcurlyeq}^{\mathscr{I}} yx$ so by $\langle \succcurlyeq\ Trans \rangle$, $C_{\succcurlyeq}^{\mathscr{I}} yz$. Hence $[y] \in \mathscr{V}_{\mathscr{M}}(\mathrm{p})$.

2. We proceed by structural induction on the subformulae ψ of φ. As most of the inductive steps are similar, we demonstrate a selection from the case where $\mathscr{L}$ has intuitionistic additives. For the base case, $\mathbb{T}\mathrm{p}^{\mathscr{I}}(x)$ entails $[x] \vDash_{\mathscr{V}_{\mathscr{M}}} \mathrm{p}$

by definition of $\mathscr{V}_{\mathscr{M}}$. If $\mathbb{F}\mathrm{p}^{\mathscr{I}}(x)$, it cannot be the case that there exists y with $C_{\succcurlyeq}xy$ and $\mathbb{T}\mathrm{p}^{\mathscr{I}}(y)$ as that would entail $\bot$. Hence $[x] \notin \mathscr{V}_{\mathscr{M}}(\mathrm{p})$ and $[x] \nVdash \mathrm{p}$.

Consider the case $\mathbb{T}(\psi_1 \mathrel{-\!\!*} \psi_2)^{\mathscr{I}}(x)$. Suppose $[w] \succcurlyeq_{\mathscr{M}} [x]$ and $[z] \in [w] \circ_{\mathscr{M}} [y]$. Then $C^{\mathscr{I}}_{\succcurlyeq} wx$ and $C^{\mathscr{I}}_{\circ} wyz$. Since $\psi_1 \mathrel{-\!\!*} \psi_2$ is a subformula of φ, the coherent axiom corresponding to the rule $\langle \mathbb{T}\mathrel{-\!\!*} \rangle$ for the instance $\psi_1 \mathrel{-\!\!*} \psi_2$ is in $\Phi^{\mathscr{L}}_{\varphi}$. As the antecedent of the axiom is satisfied, either $\mathbb{F}\psi_1^{\mathscr{I}}(y)$ or $\mathbb{T}\psi_2^{\mathscr{I}}(z)$. By the inductive hypothesis, either $[y] \nVdash_{\mathscr{V}_{\mathscr{M}}} \psi_1$ or $[z] \vDash_{\mathscr{V}_{\mathscr{M}}} \psi_2$. Hence $[x] \vDash \psi_1 \mathrel{-\!\!*} \psi_2$. For the case $\mathbb{F}(\psi_1 \mathrel{-\!\!*} \psi_2)^{\mathscr{I}}(x)$, since $\psi_1 \mathrel{-\!\!*} \psi_2$ is a subformula of φ, the coherent axiom corresponding to $\langle \mathbb{F}\mathrel{-\!\!*} \rangle$ for this formula is in $\Phi^{\mathscr{L}}_{\varphi}$. Hence there exist $w, y, z \in X$ such that $\mathbb{T}\psi_1^{\mathscr{I}}(y)$, $\mathbb{F}\psi_2^{\mathscr{I}}(z)$, $C^{\mathscr{I}}_{\circ} wyz$ and $C^{\mathscr{I}}_{\succcurlyeq} xw$. By the inductive hypothesis, $[y] \vDash \psi_1$ and $[z] \nVdash \psi_2$, and by definition $[w] \succcurlyeq_{\mathscr{M}} [x]$ and $[z] \in [w] \circ_{\mathscr{M}} [y]$. Hence $[x] \nVdash_{\mathscr{V}_{\mathscr{M}}} \psi_1 \mathrel{-\!\!*} \psi_2$.

The final cases we consider are those corresponding to $\mathrel{*\!\!-}$. First, suppose $\mathbb{T} \mathrel{*\!\!-} \psi^{\mathscr{I}}(x)$. By $\langle Total \rangle$, there exists $y \in X$ such that $C^{\mathscr{I}}_{-} xy$. Since $\mathrel{*\!\!-}\psi$ is a subformula of φ, the axiom instance of $\langle \mathbb{T}\mathrel{*\!\!-} \rangle$ corresponding to $\mathrel{*\!\!-}\psi$ is in $\Phi^{\mathscr{L}}_{\varphi}$. Hence $\mathbb{F}\psi^{\mathscr{I}}(y)$. By the inductive hypothesis $[y] \nVdash \psi$; we show $[y] = -_{\mathscr{M}}[x]$. Suppose $C^{\mathscr{L}}_{=} yy'$. Then by $\langle Sub \rangle$, $C^{\mathscr{I}}_{-} xy'$, hence $y' \in -_{\mathscr{M}}[x]$. Suppose instead that $C^{\mathscr{I}}_{-} xy'$. By $\langle Function \rangle$ we can conclude $C_{=} yy'$, hence $y' \in [y]$. Hence $-_{\mathscr{M}}[x] \nVdash_{\mathscr{V}_{\mathscr{M}}} \psi$, so $[x] \vDash_{\mathscr{V}_{\mathscr{M}}} \mathrel{*\!\!-}\psi$ as required. The case for $\mathbb{F} \mathrel{*\!\!-} \psi^{\mathscr{I}}(x)$ is similar. □

The other direction is from Kripke models to Tarskian models.

Definition 9.10 (Induced Tarskian Model)**.** *Let $(\mathscr{X}, \mathscr{V})$ be a Kripke $\mathscr{L}$ model. The* induced Tarskian model $\mathscr{M}_{\mathscr{X}} = (X, \mathscr{I}_{\mathscr{X}})$ *is defined as follows. Once again, $\heartsuit$ is any binary operation in the language of $\mathscr{L}$ frames.*

$$
\begin{array}{ll}
\text{The carrier of } \mathscr{M}_{\mathscr{X}} \text{ is } X; & C^{\mathscr{I}_{\mathscr{X}}}_{=} = \{(x,x) \mid x \in X\}; \\
C^{\mathscr{I}_{\mathscr{X}}}_{\succcurlyeq} = \{(y,x) \mid y \succcurlyeq x\}; & C^{\mathscr{I}_{\mathscr{X}}}_{\heartsuit} = \{(x,y,z) \mid z \in x \heartsuit y\}; \\
C^{\mathscr{I}_{\mathscr{X}}}_{E} = E; & C^{\mathscr{I}_{\mathscr{X}}}_{-} = \{(x,-x) \mid x \in X\}; \\
C^{\mathscr{I}_{\mathscr{X}}}_{U} = U; & C^{\mathscr{I}_{\mathscr{X}}}_{R} = R; \\
\mathbb{T}\varphi^{\mathscr{I}_{\mathscr{X}}} = \{x \mid x \vDash_{\mathscr{V}} \varphi\}; & \mathbb{F}\varphi^{\mathscr{I}_{\mathscr{X}}} = \{x \mid x \nVdash_{\mathscr{V}} \varphi\}.
\end{array}
$$

It is straightforward but tedious task to verify that given a Kripke $\mathscr{L}$ model $(\mathscr{X}, \mathscr{V})$, the induced Tarskian model $\mathscr{M}_{\mathscr{X}}$ is a Tarskian model of $\Phi^{\mathscr{L}}$, and hence for any $\mathscr{L}$ formula φ, a Tarskian model of $\Phi^{\mathscr{L}}_{\varphi}$. The following lemma is a simple corollary of this fact, and completes the connection between validity of φ in $\mathscr{L}$ models and the existence of Tarskian models of $\Phi^{\mathscr{L}}_{\varphi}$.

Lemma 9.11. *If $(\mathscr{X},\mathscr{V})$ is a Kripke $\mathscr{L}$ model with a state x (not) satisfying φ, then the induced Tarskian model $\mathscr{M}_{\mathscr{X}}$ is a Tarskian model of $\Phi^{\mathscr{L}}_{\varphi} \cup \{\exists x \mathbb{T}\varphi(x)\}$ ($\Phi^{\mathscr{L}}_{\varphi} \cup \{\exists x \mathbb{F}\varphi(x)\}$).* □

To summarise, if a Tarskian model $\mathscr{M}$ of $\Phi^{\mathscr{L}}_{\varphi} \cup \{\exists x \mathbb{F}\varphi(x)\}$ exists, by taking the induced Kripke model of $\mathscr{M}$ we obtain a countermodel to the validity of φ in Kripke models. If no Tarskian model of $\Phi^{\mathscr{L}}_{\varphi} \cup \{\exists x \mathbb{F}\varphi(x)\}$ exists, then in particular φ *must* be valid in Kripke models: otherwise the induced Tarskian model of any countermodel would contradict the non-existence of Tarskian models of $\Phi^{\mathscr{L}}_{\varphi} \cup \{\exists x \mathbb{F}\varphi(x)\}$.

We now connect the existence of a closed tableau to Bezem & Coquand's [22] *breadth-first forward reasoning* proof system for coherent logic. In their system, judgments of the form $X \Vdash^{\Phi} D$ are derived, where X is a set of atomic first-order sentences, Φ a finite coherent theory and D a *closed coherent disjunction*; a first-order sentence with the same syntactic shape as the consequent of a coherent formula. The derivation of the judgment $X \Vdash^{\Phi} D$ is defined inductively:

1. (Base) $X \Vdash^{\Phi} D$ holds if for one of the disjuncts $\exists \vec{y}.C$ of D, there are constants $\vec{a}$ such that all conjuncts of $C[\vec{y} := \vec{a}]$ occur in X;

2. (Inductive Step) Consider all closed instances $C_i \to D_i$ of Φ-axioms such that the conjuncts of C_i occur in X but the conjuncts of no disjunct $C_{i,j}$ of D_i do. There exist finitely many, with their consequents thus enumerated $D_0,\ldots,D_n$. Let $\exists \overrightarrow{y_{i,j}}.C_{i,j}$ denote the j-th of the m_i disjuncts of D_i, and denote by $\overline{C_{i,j}}$ the substitution of $\overrightarrow{y_{i,j}}$ with fresh constants. Infer $X \Vdash^{\Phi} D$ from $\forall j_0 \in \{1,\ldots,m_0\},\ldots,\forall j_n \in \{1,\ldots,m_n\}(X,\overline{C_{0,j_0}},\ldots,\overline{C_{n,j_n}} \Vdash^{\Phi} D)$. Importantly, if a D_i is $\bot$, then $m_i = 0$, and $X \Vdash^{\Phi} D$ is trivially inferred.

A derivation can be seen as a kind of tableau, branching at each stage by adding every possible consequence of Φ obtainable from the atomic first-order sentences at the current node. A semi-decidable procedure is given to systematically search for a derivation of $X \Vdash^{\Phi} D$. First check the base case. If it doesn't hold, apply the inductive step to any Φ-axioms fireable from X. If there are none, X forms an Herbrand countermodel of Φ against D. If the inductive step can be applied, apply the search procedure recursively to all premisses. Bezem & Coquand show that successful termination corresponds to Tarskian truth.

Theorem 9.12 ([22])**.** *$X \Vdash^{\Phi} D$ is derivable iff the search procedure successfully terminates for $X \Vdash^{\Phi} D$ iff D is true in all Tarskian models of $X \cup \Phi$.*

We're now ready to connect everything up. First, it is straightforward that the search procedure for $\{\mathbb{F}\varphi(a)\} \Vdash^{\Phi^{\mathscr{L}}_{\varphi}} \bot$ corresponds precisely to an exhaustive search

for a closed tableau for φ. The search procedure systematically applies every possible tableau rule (represented as the coherent axioms in $\Phi_{\varphi}^{\mathscr{L}}$) that can be applied in the construction of a $\mathscr{L}$ tableau for φ. The search terminates only if every possible expansion ends up resulting in $\bot$: that is, each possible branch of the tableau satisfies a closure condition.

Lemma 9.13. *There exists a closed $\mathscr{L}$ tableaux for φ iff the search procedure for $\{\mathbb{F}\varphi(a)\} \Vdash^{\Phi_{\varphi}^{\mathscr{L}}} \bot$ successfully terminates.* □

Since $\bot$ can never be true in a Tarskian model, successful termination of the search procedure in this instance corresponds to the *non-existence* of a Tarskian model of $\Phi_{\varphi}^{\mathscr{L}} \cup \{\exists x \mathbb{F}\varphi(x)\}$. Thus φ is valid in Kripke models. This direction gives the soundness of the tableau method: the $\mathscr{L}$ tableau provability of φ implies the validity of φ in $\mathscr{L}$ models. If no tableau proof exists for φ, the search procedure does not terminate. The completeness of the Bezem-Coquand system guarantees that there exists a Tarskian countermodel to $\{\mathbb{F}\varphi(a)\} \Vdash^{\Phi_{\varphi}^{\mathscr{L}}} \bot$— that is, a Tarskian model of $\Phi_{\varphi}^{\mathscr{L}} \cup \{\exists x \mathbb{F}\varphi(x)\}$ where $\bot$ does not hold (a redundant condition for a Tarskian model)—which is obtained as a Herbrand model specified by the atomic sentences on an infinite branch in the limit of the procedure. The induced Kripke model of this Tarskian model is thus a witness to the failure of validity for φ in $\mathscr{L}$ models. This takes care of the completeness of the tableau method: validity of φ in $\mathscr{L}$ models implies $\mathscr{L}$ tableau provability of φ.

Theorem 9.14 (Soundness and Completeness of $\mathscr{L}$ Tableaux Calculi)**.** *For any $\mathscr{L}$ formula φ, φ is valid in Kripke $\mathscr{L}$ models iff φ is provable in the $\mathscr{L}$ tableaux calculus.* □

One aspect we have not discussed is complexity. This highlights one downside of this method: as coherent logic is undecidable (as is easily seen by the coherent logic encoding of the tableaux system for classical first-order logic) it seems unlikely that we can analyse aspects like the decidability of any particular system through our framework. Much more work also needs to be done to add mechanisms that restrict constraint generation, something that is not controlled in our systems and is an obvious source of inefficiency in naive proof-search. We suggest that these systems should be seen from the computational point-of-view as idealised but correct systems that may yield tractable systems through refinement and the addition of suitable control processes.

Chapter 10

Tableaux Calculi for Applications of Bunched Logics

In the previous chapter we set up a modular tableaux calculus framework that yields sound and complete proof systems for each of the bunched logics under investigation in this thesis. One of the motivations to focus on the tableau method rather than essentially algebraic formalisms like the display calculus was the inexpressivity of many of the properties of models used in applications of bunched logic: inexpressivity essentially states that there is no way for such systems to recognise those properties because those properties are not witnessed algebraically. We indicated that this would not be an issue for the tableau method, which is based on the frame view of the logics rather than the algebraic one. In this chapter we make good on this claim by showing that the framework can be modularly extended to witness the vast majority of properties of interest. This is once again achieved by appealing to coherent axiomatisations of classes of frames.

We also investigate the construction of a tableaux calculus for layered graph semantics. This is somewhat more delicate than the generic frame semantics as labels must now represent graphs, with a required internal structure that may not become apparent until long after the label has been introduced in the derivation. To make this work we design a calculus from scratch, and show it is possible to extract layered graph countermodels to invalid formulae, and is thus sound and complete.

Section 10.1 of this chapter is based on material from the paper *Modular Tableaux Calculi for Separation Theories* [81], while Section 10.2 is based on the paper *Intuitionistic Layered Graph Logic* [78].

10.1 Separation Logics

The key application of bunched logic is in program verification, exemplified by Separation Logic (see Chapter 3). While the assertion language of the *standard*

model of Separation Logic is defined by the theory of the heap model of (B)BI, this idea quickly gave way to a large number of separation logic*s*, wherein assertion languages are defined by the theory of bespoke *memory models* of (B)BI.

There have been a number of attempts to capture the notion of memory model at an abstract level This was first undertaken by Calcagno, O'Hearn & Yang [50], who abstracted the details of the heap model to a structure called a *separation algebra*, a partial-deterministic and cancellative monoid model of BBI. Conflicting definitions of separation algebra have since been given by adding/removing first-order properties or strengthening/weakening the monoid properties [44, 51, 77, 84]. These mutually exclusive definitions can be encompassed in a framework of *separation theories* [44], collections of first-order axioms (*separation properties*) in the language of (B)BI frames that are common to separation logic models. All separation logics in the literature can be seen to be models of separation theories, while the verification frameworks Views [77] and Iris [135] explicitly implement the idea of generating program logics parametrically by separation theory.

A theorem prover for deriving assertions satisfied by the underlying model is a necessary component of any implementation of a separation logic, with the deployable proof theory of the standard formalism crucial for its scalability to large code bases [49, 221]. Standard implementations are model-specific, however, and only suitable for the heap model. In order to account for the large numbers of bespoke separation logics, as well as Views/Iris-style frameworks, we require tools that support parametrization by separation theory. This is somewhat complicated by the expressivity issues related to separation theories that were raised in Chapter 7 however, as many of the characteristic properties of a memory model are not necessarily being witnessed by the underlying logic.

Worse, it is known that some separation theories determine distinct classes of valid formulae. Larchey-Wendling & Galmiche [150] show that this is the case for the properties of partial functionality and total composition for BBI, while Brotherston & Calcagno [39] show that partial functionality has the same effect for CBI validity. While the effect on validity for the full space of separation theories has not been investigated, it is clear that separation theories determine distinct logics (in the semantic sense of a logic as the theory of a class of frames) in key cases and it may be expected that this phenomenon arises more broadly.

These problems are easily side stepped in our tableaux calculi framework, however: the separation theories can be directly represented as tableau rules. This is possible because virtually all of the separation properties found in the literature are given as coherent forumulae. There is one exception—Divisibility—but even this can be captured by a system of coherent axioms.

There are of course many proof theoretic treatments of the concrete heap model of Separation Logic [20, 100, 127], but very little exists for separation theories. The key exception to this is Hóu et al.'s [124] labelled sequent calculi for propositional abstract separation logic. There, a labelled sequent calculus for BBI is extended with rules corresponding to the most common separation properties – *partial determinism, cancellativity, indivisible unit* and *divisibility* – and completeness and cut elimination is proved. In Hóu's PhD dissertation [123] the properties *cross-split* and *splittability* are additionally handled, although completeness for these new rules requires 'non-trivial changes' to the previous proofs.

Recent work by Hóu et al. [125] that appeared concurrently with the writing of this thesis improves upon this by giving a schema for generating rules for their calculus based on the syntactic form of separation properties: this can be seen as a more restricted form of our method as it utilises a fragment of coherent formulae they call frame axioms. This is suitable for the properties they consider, but does not capture everything that we can. The classes of frames captured by our systems strictly extend those of Hóu et al.—in particular, by additionally considering classes of BI frames that are appropriate for intuitionistic separation logics, in addition to separation theories interpreted on frames for other bunched logics. A deficiency of our approach with respect to Hóu et al's is a lack of implementation, though we note that the representation of our systems as theories of coherent logic suggests off-the-shelf coherent logic provers (see Polonsky [184] for a survey) could be used to give naive implementations of our framework.

Brotherston & Villard [44] deal with the undefinability of separation theories by defining a conservative extension of BBI called HyBBI, extending the syntax with nominals, satisfaction operators and binders. This extra expressivity leads to the axiomatizability of the undefinable separation properties. This work is not specifically concerned with proof theory, giving only a Hilbert-style system for HyBBI, and has the defect of requiring modifications to the syntax of Separation Logic. In addition, a significant theoretical reformulation would be required to capture separation theories interpreted on BI frames this way: the satisfaction of nominals (which must be true at a single state) necessarily fails to be persistent in all but trivial models, so it is unclear if it would be possible to use them in an intuitionistic logic to axiomatise frame properties. Intuitionistic hybrid logic has been investigated by Braüner [33] but requires significant changes to the frames for intuitionistic modal logic to sidestep this issue.

In addition to this issue, the axiomatization of—and completeness argument for—HyBBI utilizes Boolean negation in an essential way to take care of the interaction between the hybrid operators and $-\!\!*$. As BI lacks Boolean negation such a

Partial Functionality	$z \in x \circ y \wedge z' \in x \circ y \to z = z'$
Total	$\exists z(z \in x \circ y)$
Cancellativity	$z \in x \circ y \wedge z \in x \circ y' \to y = y'$
Single Unit	$x \in E \wedge x' \in E \to x = x'$
Indivisible Units	$x \in y \circ z \wedge x \in E \to y \in E$
Disjointness	$x \in y \circ y \to y \in E$
Divisibility	$x \notin E \to \exists y,z(y \notin E \wedge z \notin E \wedge x \in y \circ z)$
Cross-Split	$x \in t \circ u \wedge x \in v \circ w \to \exists a,b,c,d(t \in a \circ b \wedge u \in c \circ d \wedge v \in a \circ c \wedge w \in b \circ d)$
Upwards-Closed	$z \in x \circ y \wedge z \leq z' \to \exists x',y'(z' \in x' \circ y' \wedge x \leq x' \wedge y \leq y')$
Downwards-Closed	$z \in x \circ y \wedge x' \leq x \wedge y' \leq y \to \exists z'(z' \in x' \circ y' \wedge z' \leq z)$
Non-Branching	$x \leq y \wedge x \leq y' \to y \leq y' \vee y' \leq y$
Always-Joins	$x \leq y \wedge x \leq y' \to \exists z(y \leq z \wedge y' \leq z)$
Increasing	$z \in x \circ y \to y \leq z$
Unit Self Joining	$x \in E \to x \in x \circ x$
Normal Increasing	$z \in x \circ y \wedge z \in E \to x \leq z$

Figure 10.1: Separation properties.

technique cannot be directly transferred. In contrast, in our work the necessary machinery is internalised within the proof system and both classical and intuitionistic cases are taken care of uniformly.

10.1.1 Separation Theories

We begin by collecting separation theories from the literature. Some we have already encountered in Chapter 7 and others we consider now for the first time in the thesis. In Figure 10.1 we list separation properties collected from Brotherston & Villard [44], Calcagno et al. [50], Cao et al. [51] and Dockins et al. [84]. A *separation theory* Σ is a set of these properties. Those involving $\succcurlyeq$ are of course only to be interpreted on frames for bunched logics with intuitionistic additives.

Some of these (for example, Non-Branching and Always-Joins) can be axiomatised in (B)BI (see [51]), but others like Partial Functionality and Single Unit we know cannot [44]. Separation theories containing only axiomatisable properties can be given an algebraic-style display calculus proof theory, but most separation logics are actually models of separation theories that contain some undefinable properties. We demonstrate this fact by giving a sequence of examples of actual separation logics that satisfy some separation theory obtained from this list.

Heaps. Our first example is given by the standard memory models of Separation Logic that we have seen before. Recall that a *heap* is a partial function $h : \mathbb{N} \to \mathbb{Z}$, representing an allocation of memory addresses to values. Given heaps h, h', $h\#h'$ denotes that $dom(h) \cap dom(h') = \emptyset$; $h \cdot h'$ denotes the union of functions with disjoint domains, which is defined iff $h\#h'$. The *empty heap*, $[]$, is defined nowhere.

Let H denote the set of all heaps. Then $\mathrm{Heap}_{\mathrm{BBI}} = (H, \cdot, \{[]\})$ is a BBI frame. Letting $h \succcurlyeq h'$ denote that h extends h', $\mathrm{Heap}_{\mathrm{BI}} = (H, \succcurlyeq, \cdot, H)$ defines a BI frame. These frames generate the standard classical and intuitionistic models of Separation Logic. $\mathrm{Heap}_{\mathrm{BBI}}$ satisfies Partial Determinism, Cancellativity, Single Unit, Indivisible Units, Cross-Split and Unit Self Joining; $\mathrm{Heap}_{\mathrm{BI}}$ additionally satisfies Splittability, Upwards-Closed, Downwards-Closed, Increasing and Normal Increasing while dropping Single Unit and Unit Self Joining.

Permissions. Permissions are incorporated into variants of separation logics that are designed to reason about certain kinds of concurrent algorithms and more fine-grained notions of memory disjointness: for example, disjointness modulo shared read permission. Hóu [123] reports a schema of Clouston that encompasses many such models: we recall it, with two concrete instances.

Let V be a set of values and $\star : V^2 \to V$ an associative and commutative partial function. Denote by H_V the set of V-valued heaps $h : \mathbb{N} \to V$. Then $\mathrm{Heap}_V = (H_V, \circ_\star, \{[]\})$ is a BBI frame, where $\circ_\star$ is defined by

$$h_1 \circ_\star h_2(n) = \begin{cases} h_1(n) \star h_2(n) & \text{if } n \in dom(h_1) \cap dom(h_2) \text{ and } h_1(n) \star h_2(n) \downarrow \\ h_1(n) & \text{if } n \in dom(h_1) \setminus dom(h_2) \\ h_2(n) & \text{if } n \in dom(h_2) \setminus dom(h_1) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Hóu defines Bornat et al.'s [32] *counting permissions model* with $V = \mathbb{Z}^2$ and

$$(x,i) \star (y,j) = \begin{cases} (x, i+j) & \text{if } x = y, i < 0 \text{ and } j < 0 \\ (x, i+j) & \text{if } x = y, i+j \geq 0 \text{ and } (i < 0 \text{ or } j < 0) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This frame satisfies Partial Determinism, Cancellativity, Indivisible Units, Single Unit, Cross-Split and Unit Self Joining.

Hóu defines Dockins et al.'s [84] *binary tree model* by considering the set T of non-empty binary trees with leaves labelled $\top$ or $\bot$ that are quotiented by the smallest congruence that identifies any subtree in which all leaves have the same label with a single leaf carrying that label. Then $V = \mathbb{Z} \times T$, and $\star$ is defined, where $\vee$ ($\wedge$) denotes pointwise disjunction (conjunction) of equivalent trees, by

$$(x, [t]) \star (y, [t']) = \begin{cases} (x, [t \vee t']) & \text{if } x = y \text{ and } [t \wedge t'] = [\bot] \\ \text{undefined} & \text{otherwise.} \end{cases}$$

This frame satisfies Partial Determinism, Cancellativity, Single Unit, Indivisible Units, Disjointness, Splittability, Cross-Split and Unit Self Joining.

Crash Hoare Logic. Chen et al. [53] use a separation logic to verify that the FSCQ file system meets its specification and secures its data under any sequence of crashes. Cao et. al. [51] give the underlying model as the following BI frame. Let V^+ be the set of non-empty lists over a set V and ε the empty list. Buffer heaps are defined to be heaps $h : \mathbb{N} \to V^+$. Let H_{buff} be the set of all buffer heaps. Then $Heap_{\text{buff}} = (H_{\text{buff}}, \succcurlyeq, \cdot, \{[]\})$ is a BI frame, where $\cdot$ is the usual heap composition, and $h_2 \succcurlyeq h_1$ iff $dom(h_1) = dom(h_2)$ and $\forall x \in \mathbb{N}$, $\exists l \in V^+ \cup \{\varepsilon\}$ such that $h_1(x) = l \oplus h_2(x)$. This frame satisfies Partial Determinism, Cancellativity, Single Unit, Indivisible Units, Cross-Split, Upwards-Closed, Downwards-Closed, Always-Joins, Non-Branching, Unit Self Joining, and Normal Increasing.

Typed Heaps. Cao et al. [51] give an example derived from the handling of multibyte locks in Appel's [11] Verified System Toolchain separation logic for CompCert C. Let a *typed heap* be a partial map $h : \mathbb{N} \to \{\text{char}, \text{short}_1, \text{short}_2\}$ such that $h(n) = \text{short}_1$ implies $h(n+1) = \text{short}_2$. Let H_{typ} denote the set of all typed heaps. Then $\text{Heap}_{\text{Typ}} = (H_{\text{typ}}, \succcurlyeq, \circ, H_{\text{typ}})$ is a BI frame, where $h_2 \succcurlyeq h_1$ iff, for all $n \in dom(h_1)$ either $n \in dom(h_2)$ and $h_1(n) = h_2(n)$ or $h_1(n) = \text{char}$, and $h \in h_1 \circ h_2$ iff $h_1 \cdot h_2 \leq h$. This frame satisfies Indivisible Units, Disjointness, Splittability, Cross-Split, Upwards-Closed, Downwards-Closed, Non-Branching, Increasing, and Normal Increasing.

Memory Models With Intersection Operators. Our final example is slightly more abstract. Brotherston & Villard [45] motivate the introduction of BiBBI and its subclassical extensions as a way of interpreting multiplicative disjunction $\mathbin{⅋}$ on memory models, as heaps do not form models of CBI and DMBI. On such models, the frame operation $\triangledown$ is interpreted as heap intersection. However, it is only under certain conditions that a BBI memory model can be equipped with a heap intersection operation that yields a model of all the subclassical axioms: the model must satisfy Partial Functionality, Cross-Split and Disjointness. This separation theory thus defines the class of BBI frames that are *subclassically extendable*.

10.1.2 Modular Tableaux Calculi for Separation Theories

We can now use the translation schema of Chapter 9 to transform these separation properties into tableau rules that can be added to the $\mathscr{L}$ tableaux calculus for a given bunched logic $\mathscr{L}$ to obtain the $\mathscr{L}+\Sigma$ tableaux calculus for any separation theory Σ. Note that we do not specify that $\mathscr{L}$ is (B)BI, despite the focus being on separation logic. It may be useful to investigate other bunched logics with memory interpretations extended with separation theories: for example, subclassical bunched logics

$$\langle P\,Func\rangle \quad \frac{C_\circ xyz, C_\circ xyz' \in \mathscr{C}}{\langle \emptyset, \{C_= zz'\}\rangle} \qquad \langle Total\rangle \quad \frac{Expression(x), Expression(y) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, \{C_\circ xyc_i\}\rangle}$$

$$\langle Canc\rangle \quad \frac{C_\circ xyz, C_\circ xy'z \in \mathscr{C}}{\langle \emptyset, C_= yy'\rangle} \qquad \langle U.\,Closed\rangle \quad \frac{C_\circ xyz, C_{\succcurlyeq} z'z \in \mathscr{C}}{\langle \emptyset, \{C_\circ c_i c_j z', C_{\succcurlyeq} c_i x, C_{\succcurlyeq} c_j y\}\rangle}$$

$$\langle Dis\,joint\rangle \quad \frac{C_\circ yyx \in \mathscr{C}}{\langle \emptyset, C_E y\rangle} \qquad \langle Cross-Split\rangle \quad \frac{C_\circ tux, C_\circ vwx \in \mathscr{C}}{\langle \emptyset, \{C_\circ c_i c_j t, C_\circ c_k c_l u, C_\circ c_i c_k v, C_\circ c_j c_l w\}\rangle}$$

$$\langle Single\ Unit\rangle \quad \frac{C_E x, C_E y \in \mathscr{C}}{\langle \emptyset, C_= xy\rangle} \qquad \langle D.\,Closed\rangle \quad \frac{C_\circ xyz, C_{\succcurlyeq} xx', C_{\succcurlyeq} yy' \in \mathscr{C}}{\langle \emptyset, \{C_\circ x'y'c_i, C_{\succcurlyeq} zc_i\}\rangle}$$

$$\langle Unit\ Self\ Joining\rangle \quad \frac{C_E x \in \mathscr{C}}{\langle \emptyset, \{C_\circ xxx\}\rangle} \qquad \langle Always-Joins\rangle \quad \frac{C_{\succcurlyeq} yx, C_{\succcurlyeq} y'x \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} c_i y, C_{\succcurlyeq} c_i y'\}\rangle}$$

$$\langle Increasing\rangle \quad \frac{C_\circ xyz \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} zy\}\rangle} \qquad \langle Non-Branching\rangle \quad \frac{C_{\succcurlyeq} yx, C_{\succcurlyeq} y'x \in \mathscr{C}}{\langle \emptyset, \{C_{\succcurlyeq} yy'\}\rangle \mid \langle \emptyset, \{C_{\succcurlyeq} y'y\}\rangle}$$

$$\langle N.\,Increasing\rangle \quad \frac{C_\circ xyz, C_E z \in \mathscr{C}}{\langle \emptyset, C_{\succcurlyeq} zx\rangle}$$

with c_i, c_j, c_k, c_l fresh labels.

Figure 10.2: Separation theory frame expansion rules.

which have heaps with intersection as a model, or perhaps CKBI for concurrent separation logic like models. Modularity of our framework makes this straightforward. We give these rules, with the exception of Divisibility, in Figure 10.2.

Dealing with Divisibility requires a small amount of pre-processing: unlike every other $\mathscr{L}$ frame axiom we have come across in this thesis, it is *not* a coherent axiom. Written explicitly in the first-order tableau language, it is of the form $\neg C_E x \to \exists y, z(\neg C_E y \wedge \neg C_E z \wedge C_\circ yzx)$. The presence of these negations takes us outside of coherent logic. This can be fixed by directly representing the *complement* of E with a constraint symbol, and using a coherent axiomatisation of the property of being the complement of a particular set to generate supplementary rules.

Explicitly, for $\mathscr{L} + \Sigma$ tableaux calculi for which Σ contains Divisibility, we add a new constraint symbol to $ConSym(\mathscr{L})$: $C_{\overline{E}}$. Now $C_{\overline{E}} x$ is interpreted as saying $x \notin E$. To enforce that interpretation we have the coherent axioms: $\top \to C_E x \vee C_{\overline{E}} x$ and $C_E x \wedge C_{\overline{E}} x \to \bot$. Thus Figure 10.3 gives the tableau rules corresponding to Divisibility. These correspond precisely to the first of the complement axioms and the Divisibility property itself. For the second complement axiom, we are in the translation case we did not cover in the previous chapter: a coherent formula with $\bot$ as consequent. Straightforwardly, the antecedent becomes a new closure condition:

7. $C_{\overline{E}} x, C_E x \in \mathscr{C}$.

$$\langle \mathit{Divisibility} \rangle \quad \frac{C_{\overline{E}} x \in \mathscr{C}}{\langle \emptyset, \{C_{\overline{E}} c_i, C_{\overline{E}} c_j, C_\circ c_i c_j x\} \rangle} \qquad \langle \mathit{Complement}\ 1 \rangle \quad \frac{\mathit{Expression}(x) \in \mathscr{F} \cup \mathscr{C}}{\langle \emptyset, C_E x \rangle \mid \langle \emptyset, C_{\overline{E}} x \rangle}$$

Figure 10.3: Divisibility frame expansion rules.

1.	$\langle \{\mathbb{F}\phi * \psi \to \psi : c_0\}, \emptyset \rangle$	Premiss
2.	$\langle \{\mathbb{T}\phi * \psi : c_1, \mathbb{F}\psi : c_1\}, \{C_{\succcurlyeq} c_1 c_0\} \rangle$	$\langle \mathbb{F}\to \rangle$, 1.
3.	$\langle \{\mathbb{T}\phi : c_3, \mathbb{T}\psi : c_4\}, \{C_\circ c_3 c_4 c_2, C_{\succcurlyeq} c_1 c_2\} \rangle$	$\langle \mathbb{T}* \rangle$, 2.
4.	$\langle \emptyset, \{C_{\succcurlyeq} c_2 c_4\} \rangle$	$\langle \mathit{Increasing} \rangle$, 3.
5.	$\langle \emptyset, \{C_{\succcurlyeq} c_1 c_4\} \rangle$	$\langle \succcurlyeq\ \mathit{Trans} \rangle$, 2., 3.
	$\otimes$	

Figure 10.4: Tableau proof of $\phi * \psi \to \psi$ in the BI + Increasing system.

This sheds some light on the construction of the underlying tableaux calculi for bunched logics with $\top^*$ or $\bot^*$. The labelled formulae $\mathbb{F}\top^* : x$ and $\mathbb{T}\bot^* : x$ *already* encode $x \notin E$ and $x \in U$. The closure conditions 5. and 6. are thus of the same essential form of 7. One might ask, why not just use $\mathbb{F}\top^* : x$ instead of the new constraint symbol? The main reason is that the introduction of labelled formulae is much more controlled than the introduction of label constraints, and there is no mechanism in the tableau countermodel construction procedure that ensures that every equivalence class of labels $[x]$ that ends up not being in $E_{\mathscr{M}}$ has a representative x that occurs in a labelled formula $\mathbb{F}\top^* : x$. This means Divisibility would not necessarily hold in the countermodel, breaking completeness for these calculi. This is explicitly ensured by the rule $\langle \mathit{Complement}\ 1 \rangle$ for the new label constraint method.

We now give a couple of examples of separation theory tableau proofs witnessing validity that does not hold at the most general level of $\mathscr{L}$ frames. One of the key properties distinguishing the standard heap models is that weakening for $*$ (i.e., $\phi * \psi \to \psi$) is valid in the intuitionistic heap model but not the classical. Cao et al. [51] show that this corresponds to the separation property Increasing. Figure 10.4 — again, written using the traditional representation of tableaux — shows a single branch tableaux proof of $\phi * \psi \to \psi$ for $BI +$ Increasing, closed because $\mathbb{T}\psi : c_4$, $\mathbb{F}\psi : c_1$ and $C_{\succcurlyeq} c_1 c_4$ occur.

A more interesting example can be given of a formula that is valid in frames satisfying a particular separation property that the formula nonetheless doesn't define. The formula $(\neg\top^* \mathrel{-\!\!*} \bot) \to \top^*$ is valid in BBI models satisfying Total, but not in all BBI models [149], and Figure 10.5 shows that the tableaux calculus for BBI + Total proves it. At the final step, the identity $\neg\varphi := \varphi \to \bot$ allows us to apply

1.	$\langle\{\mathbb{F}(\neg\top^* \mathrel{-\!\!*} \bot) \to \top^* : c_0\}, \emptyset\rangle$	Premiss
2.	$\langle\{\mathbb{T}\neg\top^* \mathrel{-\!\!*} \bot : c_0, \mathbb{F}\top^* : c_0\}, \emptyset\rangle$	$\langle\mathbb{F}\to\rangle$, from (1)
3.	$\langle\emptyset, \{C_\circ c_0 c_0 c_1\}\rangle$	Total, from (1)
4.	$\langle\{\mathbb{F}\neg\top^* : c_0\}, \emptyset\rangle$ $\quad$ $\langle\{\mathbb{T}\bot : c_1\}, \emptyset\rangle$	$\langle\mathbb{T}\mathrel{-\!\!*}\rangle$, from (2), (3)
5.	$\langle\{\mathbb{T}\top^* : c_0, \mathbb{F}\bot : c_0\}, \emptyset\rangle$ $\quad$ $\otimes$	$\langle\mathbb{F}\to\rangle$, from (4)
	$\otimes$	

Figure 10.5: Tableau proof in the BBI + Total system.

the $\langle\mathbb{T}\to\rangle$ rule, as our system does not have a primitive rule for $\neg$ as it is a defined connective. The left-hand branch is closed because $\mathbb{T}\top^* : c_0$ occurs at step 2. and $\mathbb{F}\top^* : c_0$ occurs at step 5., while the right branch is closed because $\mathbb{T}\bot : c_1$ occurs at step 4.

The soundness and completeness of the tableaux calculi augmented with separation theory rules is an immediate corolllary of the case for the base tableau: the only difference is the additional coherent axioms obtained from the separation theory rules enforce that the induced Kripke models of Tarskian models of $\Phi_\varphi^{\mathscr{L}+\Sigma}$ satisfy the separation theory Σ, and that every Kripke $\mathscr{L}$ model satisfying a separation theory Σ induces a Tarskian model of $\Phi_\varphi^{\mathscr{L}+\Sigma}$. The rest of the argument goes through identically.

Theorem 10.1 (Soundness and Completeness of $\mathscr{L}+\Sigma$ Tableaux Calculi)**.** *For any $\mathscr{L}$ formula φ, and any separation theory Σ in the language of $\mathscr{L}$ frames, φ is valid in Kripke $\mathscr{L}+\Sigma$ models iff φ is provable in the $\mathscr{L}+\Sigma$ tableaux calculus.* □

This exhaustively treats the separation theories of the literature. Moreover, if any new separation property is deemed to be of interest in applications of bunched logic, as long as it is axiomatisable via a finite coherent theory, tableaux calculi can be defined for the classes of frames it defines. A folklore result reconstructed by Dyckhoff & Negri [90] demonstrates just how much this captures: given *any* first-order sentence φ, there exists a finite coherent theory (involving at most finitely many new predicate symbols added to the language of φ) that conservatively extends the theory of φ. Moreover, this coherent theory can be constructed from φ. Our treatment of Divisibility is a specific case of this general result.

We should mention the limitations of this approach, however. There are some frame properties of interest that are out of reach for this technique. In Chapter 7 we defined the concept of *well-founded decomposition* and identified it as the property that allows states of the heap model to be identified with formulae of separation logic. As this property is not first-order definable, we cannot define tableaux calculi for frames satisfying well-founded decomposition in our framework.

A similar issue arises with a frame property that is essential for the higher order concurrent separation logic framework Iris [135]. That framework utilises *step-indexed* memory models, with a state essentially given by a pair (r,n) where r is a resource and $n \in \mathbb{N}$. Approximately, $(r,n) \vDash \varphi$ if φ holds of r for at least n reduction steps in an operational program semantics. This additional structure allows the interpretation of Nakano's [168] *later modality*, which is used to reason about recursion and is one of the key aspects of the framework. The later modality defines an intuitionistic modal logic known [59] to be sound and complete for Kripke frames with an accessibility relation R that is transitive and *converse well-founded*—that is, has no infinite sequences $x_0Rx_1Rx_2Rx_3\ldots$. Converse well-foundedness is not first-order definable and thus cannot be captured by rules generated from coherent formulae. Clouston & Goré [59] have given a sound and complete sequent calculus for the modal logic of the later modality, but it would take substantial work to understand how those techniques could be combined with ours.

10.2 Layered Graph Models

We finish the thesis where we began: back at the weakest bunched logic ILGL. Just as memory models are the classes of frames of interest for logics including and extending (B)BI, layered graph models are of primary interest for the layered graph logics. Using our framework to define a tableaux calculus that is sound and complete for *this* class of frames is somewhat more complicated, however.

One way forward would be similar to the treatment of separation theories: abstract from the specific internal structure of layered graph models to a class of axioms that captures salient features of the composition, in the same way that heap models are abstracted to (B)BI frames satisfying particular additional axioms. To do so we might include rules enforcing anti-reflexivity, contra-commutativity and partial functionality of composition.

We'd like to do something stronger than this though, and define a system that is genuinely complete for layered graph models themselves. This boils down to ensuring that the countermodels generated by the tableau construction procedure are genuine layered graph models, but this is tricky: states in our countermodels are equivalence classes of labels, and it is not clear how to systematically assign the necessary graph structure to each one of these states to make the resulting frame an ordered scaffold with a valuation that faithfully represents the satisfaction data encoded by labelled formulae.

Instead we define a new tableaux calculus for ILGL with layered countermodel extraction. It shares some similarities with the presentation of tableaux calculi of Chapter 9, but we are now much more careful in our treatment of labels, which will

become the vertices of the layered graph countermodels extracted for invalid ILGL formulae. The design of this system is much closer to previous tableau systems for bunched logics [101, 148, 68], with its soundness and completeness argument essentially that given for those calculi. In particular, the design philosophy is similar to Galmiche & Méry's [100] tableaux calculus for Separation Logic, which controls the introduction of labels in such a way that they can be used to define heap countermodels.

10.2.1 A Tableaux Calculus for Layered Graph Models

Definition 10.2 (Graph labels)**.** *Let* $\Sigma = \{c_i \mid i \in \mathbb{N}\}$ *be a countable set of* atomic labels*. We define the set* $\mathbb{L} = \{x \in \Sigma^\star \mid 0 < |x| \leq 2\} \setminus \{c_i c_i \mid c_i \in \Sigma\}$ *to be the set of* graph labels*. A* sub-label *y of a label x is a non-empty sub-word of x, and we denote the set of sub-labels of x by* $\mathscr{S}(x)$*.*

The graph labels are a syntactic representation of the subgraphs of a model, with labels of length 2 representing a graph that can be decomposed into two layers. Note that we are now explicitly representing the partial functional graph composition in the structure of labels. Importantly, we exclude the possibility $c_i c_i$ as layering is anti-reflexive. We also introduce constraints to represent the preorder $\succcurlyeq$. These are the only constraint symbols for the tableau system, and as we aren't going through the intermediate stage of translating to the first-order tableau language we are much more direct in our representation.

Definition 10.3 (Constraints)**.** *A* constraint *is an expression of the form* $x \succcurlyeq y$*, where x and y are graph labels.*

Let $\mathscr{C}$ be a set of constraints. The *domain* of $\mathscr{C}$, $\mathscr{D}(\mathscr{C})$, is the set of all sub-labels appearing in $\mathscr{C}$. In particular, $\mathscr{D}(\mathscr{C}) = \bigcup_{x \preccurlyeq y \in \mathscr{C}} (\mathscr{S}(x) \cup \mathscr{S}(y))$ The *alphabet* of $\mathscr{C}$ is the set of atomic labels appearing in $\mathscr{C}$. In particular, we have $\mathscr{A}(\mathscr{C}) = \Sigma \cap \mathscr{D}(\mathscr{C})$. In this system the derivation of constraints is outsourced to maintain control of the labels that are being introduced to a branch. It additionally has the computational benefit of controlling the application of the rules that pertain to constraint derivation around logical expansion rules, something that is difficult to specify at the level of genericity we were working at with the modular framework.

Definition 10.4 (Closure of constraints)**.** *Let* $\mathscr{C}$ *be a set of constraints. The closure of* $\mathscr{C}$*, denoted* $\overline{\mathscr{C}}$*, is the least relation closed under the rules of Figure 10.6 such that* $\mathscr{C} \subseteq \overline{\mathscr{C}}$*.*

This closure yields a preorder on $\mathscr{D}(\mathscr{C})$, with $\langle \mathrm{R}_1 \rangle - \langle \mathrm{R}_6 \rangle$ generating reflexivity and $\langle \mathrm{Tr} \rangle$ yielding transitivity. Crucially, taking the closure of the constraint set

$$\langle \mathrm{R}_1 \rangle \frac{x \succcurlyeq y}{x \succcurlyeq x} \qquad \langle \mathrm{R}_2 \rangle \frac{x \succcurlyeq y}{y \succcurlyeq y} \qquad \langle \mathrm{R}_3 \rangle \frac{x \succcurlyeq yz}{y \succcurlyeq y} \qquad \langle \mathrm{R}_4 \rangle \frac{x \succcurlyeq yz}{z \succcurlyeq z}$$

$$\langle \mathrm{R}_5 \rangle \frac{xy \succcurlyeq z}{x \succcurlyeq x} \qquad \langle \mathrm{R}_6 \rangle \frac{xy \succcurlyeq z}{y \succcurlyeq y} \qquad \langle \mathrm{Tr} \rangle \frac{x \succcurlyeq y \qquad y \succcurlyeq z}{x \succcurlyeq z}$$

Figure 10.6: Rules for closure of constraints.

does not cause labels to proliferate and the generation of any particular constraint from an arbitrary constraint set $\mathscr{C}$ is fundamentally a finite process.

Proposition 10.5. *Let $\mathscr{C}$ be a set of constraints.*

1. $x \in \mathscr{D}(\overline{\mathscr{C}})$ *iff* $x \succcurlyeq x \in \overline{\mathscr{C}}$.
2. $\mathscr{D}(\mathscr{C}) = \mathscr{D}(\overline{\mathscr{C}})$ *and* $\mathscr{A}(\mathscr{C}) = \mathscr{A}(\overline{\mathscr{C}})$. □

Lemma 10.6 (Compactness)**.** *Let $\mathscr{C}$ be a (possibly countably infinite) set of constraints. If $x \succcurlyeq y \in \overline{\mathscr{C}}$, then there is a finite set of constraints $\mathscr{C}_f \subseteq \mathscr{C}$ such that $x \succcurlyeq y \in \overline{\mathscr{C}_f}$.* □

We now give the definition of labelled formula and CSS for this calculi. These are essentially the same as in Chapter 9, but we impose more conditions on CSSs to ensure we can transform them into layered graph models.

Definition 10.7 (Labelled Formula / CSS)**.** *A* labelled formula *is an expression $\mathbb{S}\varphi : x$ where $\mathbb{S} \in \{\mathbb{T}, \mathbb{F}\}$, φ is a formula of ILGL and x is a graph label. A* constrained set of statements *(CSS) is a pair $\langle \mathscr{F}, \mathscr{C} \rangle$, where $\mathscr{F}$ is a set of labelled formulae and $\mathscr{C}$ is a set of constraints, satisfying the following properties: for all $x \in \mathbb{L}$ and distinct $c_i, c_j, c_k \in \Sigma$,*

1. *(Ref) if $\mathbb{S}\varphi : x \in \mathscr{F}$, then $x \preccurlyeq x \in \overline{\mathscr{C}}$,*
2. *(Contra) if $c_i c_j \in \mathscr{D}(\mathscr{C})$, then $c_j c_i \notin \mathscr{D}(\mathscr{C})$, and*
3. *(Freshness) if $c_i c_j \in \mathscr{D}(\mathscr{C})$, then $c_i c_k, c_k c_i, c_j c_k, c_k c_j \notin \mathscr{D}(\mathscr{C})$.*

A CSS $\langle \mathscr{F}, \mathscr{C} \rangle$ is finite *if $\mathscr{F}$ and $\mathscr{C}$ are finite. The relation $\subseteq$ is defined on CSSs by $\langle \mathscr{F}, \mathscr{C} \rangle \subseteq \langle \mathscr{F}', \mathscr{C}' \rangle$ iff $\mathscr{F} \subseteq \mathscr{F}'$ and $\mathscr{C} \subseteq \mathscr{C}'$. We denote by $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \subseteq_f \langle \mathscr{F}, \mathscr{C} \rangle$ when $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \subseteq \langle \mathscr{F}, \mathscr{C} \rangle$ holds and $\langle \mathscr{F}_f, \mathscr{C}_f \rangle$ is finite.*

The CSS properties ensure models can be built from the labels: (Ref) ensures we have enough data for the closure rules to generate a preorder, (Contra) ensures the contra-commutativity of graph layering is respected, and (Freshness) ensures

$$\langle \mathbb{T}\wedge\rangle \quad \frac{\mathbb{T}\varphi\wedge\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x, \mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\wedge\rangle \quad \frac{\mathbb{F}\varphi\wedge\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}\vee\rangle \quad \frac{\mathbb{T}\varphi\vee\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : x\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : x\},\emptyset\rangle} \qquad \langle \mathbb{F}\vee\rangle \quad \frac{\mathbb{F}\varphi\vee\psi : x\in\mathscr{F}}{\langle\{\mathbb{F}\varphi : x, \mathbb{F}\psi : x\},\emptyset\rangle}$$

$$\langle \mathbb{T}\to\rangle \quad \frac{\mathbb{T}\varphi\to\psi : x\in\mathscr{F} \text{ and } y\succcurlyeq x\in\overline{\mathscr{C}}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : y\},\emptyset\rangle} \qquad \langle \mathbb{F}\to\rangle \quad \frac{\mathbb{F}\varphi\to\psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{F}\psi : c_i\},\{c_i\succcurlyeq x\}\rangle}$$

$$\langle \mathbb{T}*\rangle \quad \frac{\mathbb{T}\varphi * \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_i, \mathbb{T}\psi : c_j\},\{x\succcurlyeq c_ic_j\}\rangle} \qquad \langle \mathbb{F}*\rangle \quad \frac{\mathbb{F}\varphi * \psi : x\in\mathscr{F} \text{ and } x\succcurlyeq yz\in\overline{\mathscr{C}}}{\langle\{\mathbb{F}\varphi : y\},\emptyset\rangle \mid \langle\{\mathbb{F}\psi : z\},\emptyset\rangle}$$

$$\langle \mathbb{T}{-\!\!*}\rangle \quad \frac{\mathbb{T}\varphi -\!\!* \psi : x\in\mathscr{F} \text{ and } y\succcurlyeq x, yz\succcurlyeq yz\in\overline{\mathscr{C}}}{\langle\{\mathbb{F}\varphi : z\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : yz\},\emptyset\rangle} \qquad \langle \mathbb{F}{-\!\!*}\rangle \quad \frac{\mathbb{F}\varphi -\!\!* \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_j, \mathbb{F}\psi : c_ic_j\},\{c_i\succcurlyeq x, c_ic_j\succcurlyeq c_ic_j\}\rangle}$$

$$\langle \mathbb{T}*{-}\rangle \quad \frac{\mathbb{T}\varphi *- \psi : x\in\mathscr{F} \text{ and } y\succcurlyeq y, zy\succcurlyeq zy\in\overline{\mathscr{C}}}{\langle\{\mathbb{F}\varphi : z\},\emptyset\rangle \mid \langle\{\mathbb{T}\psi : zy\},\emptyset\rangle} \qquad \langle \mathbb{F}*{-}\rangle \quad \frac{\mathbb{F}\varphi *- \psi : x\in\mathscr{F}}{\langle\{\mathbb{T}\varphi : c_j, \mathbb{F}\psi : c_jc_i\},\{c_i\succcurlyeq x, c_jc_i\succcurlyeq c_jc_i\}\rangle}$$

with c_i and c_j being fresh atomic labels.

Figure 10.7: Tableau rules for ILGL.

the layering structure of the models we construct is exactly that specified by the labels and constraints in the CSS. One might wonder why we didn't represent these conditions directly as rules in the style of the modular tableaux calculi framework. Had this been done, it would not be possible to prove soundness of the system as the freshness property if conceived as a proof rule is clearly not sound for arbitrary layered graph models when the labels are interpreted as subgraphs. We *do* need it to turn suitable CSSs into layered graph models, however, but this mismatch is benign: the design of the tableau rules will ensure that every branch CSS constructed according to the tableaux calculus satisfies these properties anyway, and the layered graph semantics will be shown sound with respect to those rules.

As with constraint closure, CSSs have a finite character.

Proposition 10.8. *For any CSS $\langle\mathscr{F}_f,\mathscr{C}\rangle$ in which $\mathscr{F}_f$ is finite, there exists $\mathscr{C}_f\subseteq\mathscr{C}$ such that $\mathscr{C}_f$ is finite and $\langle\mathscr{F}_f,\mathscr{C}_f\rangle$ is a CSS.* □

Figure 10.7 presents the rules of the tableau system for ILGL. That 'c_i and c_j are fresh atomic labels' means $c_i\neq c_j\in\Sigma\setminus\mathscr{A}(\mathscr{C})$. This means it is impossible to introduce the non-label word $c_ic_i\notin\mathbb{L}$ as a label. Tableaux in this system are defined identically to those in the modular framework. There is a small distinction in the notion of a *tableau for* φ, however. Now, a tableau for φ is a tableau for $\langle\{\mathbb{F}\varphi : c_0\},\{c_0\succcurlyeq c_0\}\rangle$: this initial constraint is required to make the root CSS satisfy the CSS properties of Definition 10.7 and begin the generation of the preorder $\succcurlyeq$. It is straightforward but tedious to verify that the tableau rules also preserve the CSS properties of Definition 10.7. Hence tableau construction produces branches satisfying (Ref), (Contra) and (Freshness), as we will require.

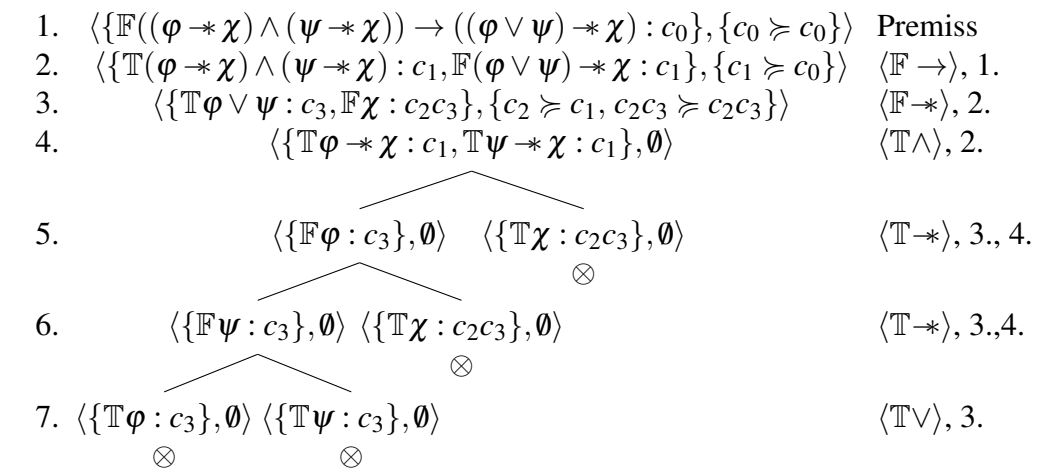


Figure 10.8: ILGL tableau proof of $(\varphi \mathrel{-\!\!*} \chi) \wedge (\psi \mathrel{-\!\!*} \chi) \to ((\varphi \vee \psi) \mathrel{-\!\!*} \chi)$.

The closure conditions specifying when a tableau is a proof are defined essentially the same as those for the ILGL calculus from the modular framework:

1. $\mathbb{T}\varphi : x, \mathbb{F}\varphi : y \in \mathscr{F}$ and $x \succcurlyeq y \in \overline{\mathscr{C}}$;
2. $\mathbb{F}\top : x \in \mathscr{F}$;
3. $\mathbb{T}\bot : x \in \mathscr{F}$.

A *tableau proof of* φ is thus a tableau for φ in which all branches are closed. An example of a tableau proof for this system is given in Figure 10.8. This revisits the formula $(\varphi \mathrel{-\!\!*} \chi) \wedge (\psi \mathrel{-\!\!*} \chi) \to ((\varphi \vee \psi) \mathrel{-\!\!*} \chi)$ previously proved for all bunched logics with classical additives in Figure 9.11. Note that at steps 5. and 6. it is possible to trigger $\langle\mathbb{T}\mathrel{-\!\!*}\rangle$ because of the occurrence of $c_2 \succcurlyeq c_1$ and $c_2c_3 \succcurlyeq c_2c_3$ at step 3. We then obtain closure of the leftmost branch because $\mathbb{F}\varphi : c_3$ occurs at step 5. and $\mathbb{T}\varphi : c_3$ occurs at step 7.; the centre-left branch is closed because $\mathbb{F}\psi : c_3$ occurs at step 6. and $\mathbb{T}\psi : c_3$ occurs at step 7.; the centre-right branch is closed because $\mathbb{F}\chi : c_2c_3$ occurs at step 3. and $\mathbb{T}\chi : c_3$ occurs at step 6.; finally, the rightmost branch is closed because $\mathbb{F}\psi : c_2c_3$ occurs at step 3. and $\mathbb{T}\chi : c_2c_3$ occurs at step 5.

10.2.2 Soundness and Completeness

We now prove this system is sound and complete for the layered graph semantics of ILGL. In contrast to the modular framework, this is done directly by constructing layered graph countermodels to invalid formulae by saturating a branch of tableau. First we prove soundness by strengthening the informal interpretation of the labelled

formulae and constraints as encodings of the structure of a layered graph model through the notion of realization.

Definition 10.9 (Realization)**.** *Let $\langle \mathscr{F}, \mathscr{C} \rangle$ be a CSS. A* realization *of $\langle \mathscr{F}, \mathscr{C} \rangle$ is a triple $\mathfrak{R} = (\mathscr{X}, \mathscr{V}, \lfloor . \rfloor)$ where $\mathscr{M} = (\mathscr{X}, \mathscr{V})$ is a layered graph model and $\lfloor . \rfloor : \mathscr{D}(\mathscr{C}) \to X$ is such that*

1. *for all $x \in \mathscr{D}(\mathscr{C})$, if $x = c_i c_j$, then $\lfloor c_i \rfloor @_{\mathscr{E}} \lfloor c_j \rfloor \downarrow$ and $\lfloor x \rfloor = \lfloor c_i \rfloor @_{\mathscr{E}} \lfloor c_j \rfloor$,*
2. *if $x \succcurlyeq y \in \mathscr{C}$, then $\lfloor x \rfloor \succcurlyeq_{\mathscr{M}} \lfloor y \rfloor$,*
3. *if $\mathbb{T}\varphi : x \in \mathscr{F}$, then $\lfloor x \rfloor \models_{\mathscr{V}} \varphi$,*
4. *if $\mathbb{F}\varphi : x \in \mathscr{F}$, then $\lfloor x \rfloor \not\models_{\mathscr{V}} \varphi$.*

We say that a CSS is *realizable* if there exists a realization of it. We say that a tableau is *realizable* if at least one of its branches is realizable. We can also show that the relevant clauses of the definition extend to the closure of the constraint set automatically.

Proposition 10.10. *Let $\langle \mathscr{F}, \mathscr{C} \rangle$ be a CSS and $\mathfrak{R} = (\mathscr{X}, \mathscr{V}, \lfloor . \rfloor)$ a realization of it. Then:*

1. *for all $x \in \mathscr{D}(\overline{\mathscr{C}})$, $\lfloor x \rfloor$ is defined;*
2. *if $x \succcurlyeq y \in \overline{\mathscr{C}}$, then $\lfloor x \rfloor \succcurlyeq \lfloor y \rfloor$.* □

As the name suggests, a realization *realizes* the partial specification of a layered graph model encoded in the labelled formulae and constraints of a CSS as an actual layered graph model. Soundness follows from the preservation of realizability by the tableau rules together with the fact that closed tableaux are not realizable. Note that a realization may map distinct labels to the same subgraphs and the layered graph model realizing the CSS may include more layering structure than that dictated by the presence of labels $c_i c_j$ in the CSS. This is how the CSS property Freshness does not affect anything for soundness: no such analogous property is required to be satisifed by the layered graph model realizing the CSS. The following lemmas are routine proofs (*cf.* [101, 148, 68]) proceeding by case analysis.

Lemma 10.11. *The tableau rules for ILGL preserve realizability.*

Proof. We give a characteristic example. Suppose a tableau $\mathscr{T}$ is realizable, via the realizable branch $\langle \mathscr{F}, \mathscr{C} \rangle$, say by $\mathfrak{R}$. Suppose $\mathbb{T}\varphi \twoheadrightarrow \psi : x \in \mathscr{F}$ and $y \succcurlyeq x, yz \succcurlyeq yz \in \overline{\mathscr{C}}$. Then by assumption, $\lfloor x \rfloor \vDash_{\mathscr{V}} \varphi \twoheadrightarrow \psi$, $y \succcurlyeq_{\mathscr{X}} x$ and $\lfloor y \rfloor @_{\mathscr{E}} \lfloor z \rfloor \downarrow$. If $\lfloor z \rfloor \nvDash_{\mathscr{V}} \varphi$,

then the branch $\langle \mathscr{F} \cup \{\mathbb{F}\varphi : z\}, \mathscr{C}\rangle$ is realizable by $\mathfrak{R}$. If $\lfloor z \rfloor \vDash_{\mathscr{V}} \varphi$, it follows from our assumption that $\lfloor y \rfloor @_{\mathscr{E}} \lfloor z \rfloor \vDash_{\mathscr{V}} \psi$. Hence the branch $\langle \mathscr{F} \cup \{\mathbb{T}\psi : yz\}, \mathscr{C}\rangle$ is realizable by $\mathfrak{R}$. □

Lemma 10.12. *Closed branches are not realizable.* □

Theorem 10.13 (Soundness)**.** *If there exists a closed tableau for the formula φ, then φ is valid in layered graph models.*

Proof. Suppose that there exists a tableau proof for φ. Then there is a closed tableau $\mathscr{T}_\varphi$ for the CSS $\mathfrak{C} = \langle \{\mathbb{F}\varphi : c_0\}, \{c_0 \succcurlyeq c_0\}\rangle$. Now suppose that φ is not valid. Then there is a model $\mathscr{M} = (\mathscr{X}, \mathscr{V})$ and a subgraph $H \in X$ such that $H \not\models_{\mathscr{M}} \varphi$. Define $\mathfrak{R} = (\mathscr{M}, \mathscr{V}, \lfloor . \rfloor)$ with $\lfloor c_0 \rfloor = H$. Note that $\mathfrak{R}$ is a realization of $\mathfrak{C}$, hence by Lemma 10.11, $\mathscr{T}_\varphi$ is realizable. By Lemma 10.12, $\mathscr{T}_\varphi$ cannot be closed. But, this contradicts the fact that $\mathscr{T}_\varphi$ is a tableau proof and therefore a closed tableau. It follows that φ is valid. □

We now proceed to establish the completeness of the labelled tableaux with respect to layered graph semantics. We begin with the notion of a Hintikka CSS, which will facilitate the construction of countermodels. This should be compared to the coherent theories $\Phi^{\mathscr{L}}$, as the definition of Hintikka CSS can essentially be obtained from the specific case of $\mathscr{L}$ being ILGL.

Definition 10.14 (Hintikka CSS)**.** *A CSS $\langle \mathscr{F}, \mathscr{C}\rangle$ is a* Hintikka CSS *if, for any ILGL formulae φ, ψ and any graph labels x, y, the following holds:*

1. $\mathbb{T}\varphi : x \notin \mathscr{F}$ *or* $\mathbb{F}\varphi : y \notin \mathscr{F}$ *or* $x \preccurlyeq y \notin \overline{\mathscr{C}}$*;*
2. $\mathbb{F}\top : x \notin \mathscr{F}$*;*
3. $\mathbb{T}\bot : x \notin \mathscr{F}$*;*
4. *if* $\mathbb{T}\varphi \wedge \psi : x \in \mathscr{F}$*, then* $\mathbb{T}\varphi : x \in \mathscr{F}$ *and* $\mathbb{T}\psi : x \in \mathscr{F}$*;*
5. *if* $\mathbb{F}\varphi \wedge \psi : x \in \mathscr{F}$*, then* $\mathbb{F}\varphi : x \in \mathscr{F}$ *or* $\mathbb{F}\psi : x \in \mathscr{F}$*;*
6. *if* $\mathbb{T}\varphi \vee \psi : x \in \mathscr{F}$*, then* $\mathbb{T}\varphi : x \in \mathscr{F}$ *or* $\mathbb{T}\psi : x \in \mathscr{F}$*;*
7. *if* $\mathbb{F}\varphi \vee \psi : x \in \mathscr{F}$*, then* $\mathbb{F}\varphi : x \in \mathscr{F}$ *and* $\mathbb{F}\psi : x \in \mathscr{F}$*;*
8. *if* $\mathbb{T}\varphi \to \psi : x \in \mathscr{F}$*, then, for all* $y \in \mathbb{L}$*, if* $y \succcurlyeq x \in \overline{\mathscr{C}}$*, then* $\mathbb{F}\varphi : y \in \mathscr{F}$ *or* $\mathbb{T}\psi : y \in \mathscr{F}$
9. *if* $\mathbb{F}\varphi \to \psi : x \in \mathscr{F}$*, then there exists* $y \in \mathbb{L}$ *such that* $y \succcurlyeq x \in \overline{\mathscr{C}}$ *and* $\mathbb{T}\varphi : y \in \mathscr{F}$ *and* $\mathbb{F}\psi : y \in \mathscr{F}$*;*

10. *if* $\mathbb{T}\varphi * \psi : x \in \mathscr{F}$, *then there are* $c_i, c_j \in \Sigma$ *such that* $x \succcurlyeq c_i c_j \in \overline{\mathscr{C}}$ *and* $\mathbb{T}\varphi : c_i \in \mathscr{F}$ *and* $\mathbb{T}\psi : c_j \in \mathscr{F}$;

11. *if* $\mathbb{F}\varphi * \psi : x \in \mathscr{F}$, *then, for all* $c_i, c_j \in \Sigma$, *if* $x \succcurlyeq c_i c_j \in \overline{\mathscr{C}}$, *then* $\mathbb{F}\varphi : c_i \in \mathscr{F}$ *or* $\mathbb{F}\psi : c_j \in \mathscr{F}$;

12. *if* $\mathbb{T}\varphi \mathrel{-\!\!*} \psi : x \in \mathscr{F}$, *then, for all* $c_i, c_j \in \Sigma$, *if* $c_i \succcurlyeq x \in \overline{\mathscr{C}}$ *and* $c_i c_j \in \mathscr{D}(\overline{\mathscr{C}})$, *then* $\mathbb{F}\varphi : c_j \in \mathscr{F}$ *or* $\mathbb{T}\psi : c_i c_j \in \mathscr{F}$;

13. *if* $\mathbb{F}\varphi \mathrel{-\!\!*} \psi : x \in \mathscr{F}$, *then there are* $c_i, c_j \in \Sigma$ *such that* $c_i \succcurlyeq x \in \overline{\mathscr{C}}$ *and* $c_i c_j \in \mathscr{D}(\overline{\mathscr{C}})$ *and* $\mathbb{T}\varphi : c_j \in \mathscr{F}$ *and* $\mathbb{F}\psi : c_i c_j \in \mathscr{F}$;

14. *if* $\mathbb{T}\varphi \mathrel{*\!\!-} \psi : x \in \mathscr{F}$, *then, for all* $c_i, c_j \in \Sigma$, *if* $c_i \succcurlyeq x \in \overline{\mathscr{C}}$ *and* $c_j c_i \in \mathscr{D}(\overline{\mathscr{C}})$, *then* $\mathbb{F}\varphi : c_j \in \mathscr{F}$ *or* $\mathbb{T}\psi : c_j c_i \in \mathscr{F}$

15. *if* $\mathbb{F}\varphi \mathrel{*\!\!-} \psi : x \in \mathscr{F}$, *then there are* $c_i, c_j \in \Sigma$ *such that* $c_i \succcurlyeq x \in \overline{\mathscr{C}}$ *and* $c_j c_i \in \mathscr{D}(\overline{\mathscr{C}})$ *and* $\mathbb{T}\varphi : c_j \in \mathscr{F}$ *and* $\mathbb{F}\psi : c_j c_i \in \mathscr{F}$.

We now give the definition of a function Ω that extracts a layered graph model from a Hintikka CSS.

Definition 10.15 (Function Ω)**.** *Let* $\langle \mathscr{F}, \mathscr{C} \rangle$ *be a Hintikka CSS. The function* Ω *associates to* $\langle \mathscr{F}, \mathscr{C} \rangle$ *a tuple* $\Omega(\langle \mathscr{F}, \mathscr{C} \rangle) = (\mathscr{G}, \mathscr{E}, X, \succcurlyeq, \mathscr{V})$, *defined*

1. $V(\mathscr{G}) = \mathscr{A}(\mathscr{C})$,

2. $E(\mathscr{G}) = \{(c_i, c_j) \mid c_i c_j \in \mathscr{D}(\mathscr{C})\} = \mathscr{E}$, $X = \{x^\Omega \mid x \in \mathscr{D}(\mathscr{C})\}$, *where* $V(c_i^\Omega) = \{c_i\}$, $E(c_i^\Omega) = \emptyset$, $V((c_i c_j)^\Omega) = \{c_i c_j\}$, *and* $E((c_i c_j)^\Omega) = \{(c_i, c_j)\}$,

3. $x^\Omega \succcurlyeq y^\Omega$ *iff* $x \succcurlyeq y \in \overline{\mathscr{C}}$, *and*

4. $x^\Omega \in \mathscr{V}(p)$ *iff there exists* $y \in \mathscr{D}(\mathscr{C})$ *such that* $x \succcurlyeq y \in \overline{\mathscr{C}}$ *and* $\mathbb{T}p : y \in \mathscr{F}$.

The next lemma shows that there is a precise correspondence between the structure that the Hintikka CSS properties impose on the labels and the layered structure specified by the construction of the model.

Lemma 10.16. *Let* $\langle \mathscr{F}, \mathscr{C} \rangle$ *be a Hintikka CSS and* $\Omega(\langle \mathscr{F}, \mathscr{C} \rangle) = (\mathscr{G}, \mathscr{E}, X, \succcurlyeq, \mathscr{V})$.

1. *If* $c_i, c_j \in \mathscr{A}(\mathscr{C})$, *then* $c_i c_j \in \mathscr{D}(\mathscr{C})$ *iff* $c_i^\Omega \mathbin{@}_{\mathscr{E}} c_j^\Omega \downarrow$.

2. *If* $c_i c_j \in \mathscr{D}(\mathscr{C})$, *then* $(c_i c_j)^\Omega = c_i^\Omega \mathbin{@}_{\mathscr{E}} c_j^\Omega$.

3. $x^\Omega \mathbin{@}_{\mathscr{E}} y^\Omega \downarrow$ *iff there exist* $c_i, c_j \in \mathscr{A}(\mathscr{C})$ *s.t.* $x = c_i$, $y = c_j$ *and* $c_i c_j \in \mathscr{D}(\mathscr{C})$.

Proof. 1. Immediate from CSS property (Contra).

2. Immediate from 1. and the definition of Ω.

3. The right-to-left direction is trivial, so assume $x^\Omega @_\mathscr{E} y^\Omega \downarrow$. There are three possible cases for x and y other than $x = c_i$ and $y = c_j$: we attend to one as the others are similar. Suppose $x = c_i c_j$ and $y = c_k$. Then $x^\Omega @_\mathscr{E} y^\Omega \downarrow$ must hold because of either $(c_i, c_k) \in \mathscr{E}$ or $(c_j, c_k) \in \mathscr{E}$. That is, $c_i c_k \in \mathscr{D}(\mathscr{C})$ or $c_j c_k \in \mathscr{D}(\mathscr{C})$. In both cases the *CSS* property (Freshness) is contradicted so neither can hold. It follows that only the case $x = c_i$ and $y = c_j$ is non-contradictory, and so by 1. $c_i c_j \in \mathscr{D}(\mathscr{C})$. □

We can now show that Ω turns Hintikka CSSs into layered graph models. Crucially, these models still reflect the satisfaction requirements encoded by labelled formulae.

Lemma 10.17. *Let $\langle \mathscr{F}, \mathscr{C} \rangle$ be a Hintikka CSS. $\Omega(\langle \mathscr{F}, \mathscr{C} \rangle)$ is a layered graph model.*

Proof. $\mathscr{G}$ is clearly a graph and the fact that $\succcurlyeq$ is a preorder on X can be read off of the rules for the closure of constraint sets. Thus the only non-trivial aspects of the proof are that X is admissible and that $\mathscr{V}$ is persistent.

First we show that X is an admissible subgraph set. Let $H, K \in Sg(\mathscr{G})$ with $H @_\mathscr{E} K \downarrow$. First we assume $H, K \in X$. Then $H = x^\Omega$ and $K = y^\Omega$ for labels x, y. By the previous lemma it follows that $x = c_i$ and $y = c_j$ and $c_i c_j \in \mathscr{D}(\mathscr{C})$. Thus $H @_\mathscr{E} K = c_i^\Omega @_\mathscr{E} c_j^\Omega = (c_i c_j)^\Omega \in X$. Now suppose $H @_\mathscr{E} K \in X$. Then $H @_\mathscr{E} K = x^\Omega$ for some $x \in \mathscr{D}(\mathscr{C})$. The case $x = c_i$ is clearly impossible as $E(c_i^\Omega) = \emptyset$ so necessarily $x = c_i c_j$. Then we have $c_i, c_j \in \mathscr{D}(\mathscr{C})$ as sub-labels of $c_i c_j$ and $c_i^\Omega @_\mathscr{E} c_j^\Omega \downarrow$ with $c_i^\Omega @_\mathscr{E} c_j^\Omega$ the only possible composition equal to $(c_i c_j)^\Omega$. It follows that $H = c_i^\Omega \in X$ and $K = c_j^\Omega \in X$ as required.

Finally we must show $\mathscr{V}$ is a persistent valuation. Let $H \in \mathscr{V}(\mathrm{p})$ with $K \succcurlyeq H$. Then $H = x^\Omega$ and $K = y^\Omega$ for some $x, y \in \mathscr{D}(\mathscr{C})$ with $y \succcurlyeq x \in \overline{\mathscr{C}}$. By definition of $\mathscr{V}$ there exists $z \in \mathscr{D}(\mathscr{C})$ with $x \succcurlyeq z \in \overline{\mathscr{C}}$ and $\mathbb{T}p : z \in \mathscr{F}$. By the closure rule $\langle Tr \rangle$ we have $y \succcurlyeq z \in \overline{\mathscr{C}}$ so $K = y^\Omega \in \mathscr{V}(p)$. □

Lemma 10.18. *Let $\langle \mathscr{F}, \mathscr{C} \rangle$ be a Hintikka CSS and $\mathscr{M} = \Omega(\langle \mathscr{F}, \mathscr{C} \rangle) = (\mathscr{G}, \mathscr{E}, X, \preccurlyeq, \mathscr{V})$. For all formulae $\varphi \in$ Form, and all $x \in \mathscr{D}(\mathscr{C})$. we have*

1. *if $\mathbb{F}\varphi : x \in \mathscr{F}$, then $x^\Omega \not\models_\mathscr{V} \varphi$, and*

2. *if $\mathbb{T}\varphi : x \in \mathscr{F}$, then $x^\Omega \models_\mathscr{V} \varphi$.*

Hence, if $\mathbb{F}\varphi : x \in \mathscr{F}$, then φ is not valid and $\Omega(\langle \mathscr{F}, \mathscr{C} \rangle)$ is a countermodel of φ.

Proof. We proceed by a simultaneous structural induction on φ, concentrating on characteristic cases.

- Base cases.
  - Case $\mathbb{F}\mathrm{p} : x \in \mathscr{F}$. We suppose that $x^\Omega \vDash_{\mathscr{V}} \mathrm{p}$. Then $x^\Omega \in \mathscr{V}(\mathrm{p})$. By the definition of $\mathscr{V}$, there is a label y such that $x \succcurlyeq y \in \overline{\mathscr{C}}$ and $\mathbb{T}\mathrm{p} : y \in \mathscr{F}$. Then by condition (1) of Definition 10.14, $\langle \mathscr{F}, \mathscr{C} \rangle$ is not a Hintikka CSS, a contradiction. It follows that $x^\Omega \nvDash_{\mathscr{M}} \mathrm{p}$.
  - Case $\mathbb{T}\mathrm{p} : x \in \mathscr{F}$. By CSS property (Ref), $x \succcurlyeq x \in \overline{\mathscr{C}}$. Thus, by definition of $\mathscr{V}$ we have $x^\Omega \in \mathscr{V}(\mathrm{p})$. Thus $x^\Omega \vDash_{\mathscr{V}} \mathrm{p}$.
- Inductive step. We now suppose that (1) and (2) hold for formulae φ and ψ (IH).
  - Case $\mathbb{T}\varphi \to \psi : x \in \mathscr{F}$. Suppose $x^\Omega \preccurlyeq y^\Omega$. Then $x \preccurlyeq y \in \overline{C}$ and by Definition 10.14 property 8. it follows that $\mathbb{F}\varphi : y \in \mathscr{F}$ or $\mathbb{T}\psi : y \in \mathscr{F}$. By (IH) it follows that if $y^\Omega \vDash_{\mathscr{V}} \varphi$ then $y^\Omega \vDash_{\mathscr{V}} \psi$ as required.
  - Case $\mathbb{T}\varphi * \psi : x \in \mathscr{F}$. By Definition 10.14 property 10. there exist labels $c_i, c_j \in \mathscr{D}(\mathscr{C})$ such that $x \succcurlyeq c_i c_j \in \overline{\mathscr{C}}$ and $\mathbb{T}\varphi : c_i \in \mathscr{F}$ and $\mathbb{T}\psi : c_j \in \mathscr{F}$. By (IH) we have $c_i^\Omega \vDash_{\mathscr{V}} \varphi$ and $c_j^\Omega \vDash_{\mathscr{V}} \psi$. Further, by definition of Ω we have that $(c_i c_j)^\Omega = c_i^\Omega \mathbin{@_{\mathscr{E}}} c_j^\Omega \preccurlyeq x^\Omega$, so $x^\Omega \vDash_{\mathscr{V}} \varphi * \psi$.
  - Case $\mathbb{T}\varphi \mathrel{-\!\!*} \psi : x \in \mathscr{F}$. Suppose $x^\Omega \preccurlyeq y^\Omega$ with $y^\Omega \mathbin{@_{\mathscr{E}}} z^\Omega \downarrow$ and $z^\Omega \vDash_{\mathscr{V}} \varphi$. By Lemma 10.16 we know $y = c_i, z = c_j \in \mathscr{A}(\mathscr{C})$ with $c_i c_j \in \mathscr{D}(\mathscr{C})$. Hence by Definition 10.14 property 12., either $F\varphi : c_j \in \mathscr{F}$ or $T\psi : c_i c_j \in \mathscr{F}$. By (IH) it follows either $c_j^\Omega \models_{\mathscr{V}} \varphi$ or $(c_i c_j)^\Omega = c_i^\Omega \mathbin{@_{\mathscr{E}}} c_j^\Omega \vDash_{\mathscr{V}} \psi$. As we know the former cannot be true, it must be the latter. Hence $x^\Omega \vDash_{\mathscr{V}} \varphi \mathrel{-\!\!*} \psi$ as required. □

This construction of a countermodel would fail in the analogous labelled tableaux system for LGL. We would require a systematic way to construct the internal structure of each subgraph in the model, as the classical semantics for $*$ demands strict equality between the graph under interpretation and the decomposition into layers. This is complicated by the fact that the internal structure required may not be known until much later than the introduction of the label in the tableau construction This issue is sidestepped for ILGL since each time the tableaux rules require a decomposition of a graph into layers we can move to a 'fresh' layered graph further down the ordering. Thus we can safely turn each graph label into the simplest instantiation of the kind of graph it represents: either a single vertex (indecomposable) or two vertices and an edge (layered). A well-foundedness condition

on the constraints of CSSs may make this method adaptable to LGL, but it is clear from our previous discussions that this is not straightforward and may not be possible in this setting: the techniques we have used utilise the first-order definability of the semantics of the logic, and well-foundedness is not a first-order property.

It now remains to give a procedure that constructs a Hintikka CSS containing $\mathbb{F}\varphi : c_0$ for every φ that does not have a tableau proof. This is done in identical fashion to that of previous bunched logic tableau systems [148, 68]: given a CSS $\mathfrak{C}$ that cannot be closed, consistent labelled formulae and constraints are added systematically in accordance to the tableau rules to saturate $\mathfrak{C}$. In particular, we must try to apply every tableau rule infinitely often, as the conditions allowing it to be triggered in a derivation may not be satisfied the first time it is attempted. This requires the concept of a fair strategy.

Definition 10.19 (Fair strategy)**.** *A* fair strategy *for ILGL is a sequence of labelled formulae* $(\mathbb{S}_i\varphi_i : x_i)_{i\in\mathbb{N}}$ *such that* $\{i \in \mathbb{N} \mid \mathbb{S}_i\varphi_i : x_i \equiv \mathbb{S}\varphi : x\}$ *is infinite for any labelled formula* $\mathbb{S}\varphi : x$*.*

Simple considerations of the countability of each component of labelled formulae justifies the existence of such a strategy.

Proposition 10.20 (cf. [148])**.** *There exists a fair strategy.* □

Next we need the concept of an oracle, which allows Hintikka sets to be constructed inductively by testing the required consistency properties at each stage.

Definition 10.21 (Oracle)**.** *Let* $\mathscr{P}$ *be a set of CSSs.*

1. $\mathscr{P}$ *is* $\subseteq$-closed *if* $\langle \mathscr{F}, \mathscr{C} \rangle \in \mathscr{P}$ *holds whenever* $\langle \mathscr{F}, \mathscr{C} \rangle \subseteq \langle \mathscr{F}', \mathscr{C}' \rangle$ *and* $\langle \mathscr{F}', \mathscr{C}' \rangle \in \mathscr{P}$ *holds.*

2. $\mathscr{P}$ *is of* finite character *if* $\langle \mathscr{F}, \mathscr{C} \rangle \in \mathscr{P}$ *holds whenever* $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \in \mathscr{P}$ *holds for every* $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \subseteq_f \langle \mathscr{F}, \mathscr{C} \rangle$*.*

3. $\mathscr{P}$ *is* saturated *if, for any* $\langle \mathscr{F}, \mathscr{C} \rangle \in \mathscr{P}$ *and any instance*

$$\frac{\mathit{cond}(\mathscr{F}, \mathscr{C})}{\langle \mathscr{F}_1, \mathscr{C}_1 \rangle \mid \ldots \mid \langle \mathscr{F}_k, \mathscr{C}_k \rangle}$$

of a rule of Figure 10.7, if $\mathit{cond}(\mathscr{F}, \mathscr{C})$ *is fulfilled, then* $\langle \mathscr{F} \cup \mathscr{F}_i, \mathscr{C} \cup \mathscr{C}_i \rangle \in \mathscr{P}$*, for at least one* $i \in \{1, \ldots, k\}$*.*

An oracle *is a set of open CSSs which is* $\subseteq$*-closed, of finite character, and saturated.*

Definition 10.22 (Consistency)**.** *Let $\langle \mathscr{F}, \mathscr{C} \rangle$ be a CSS. We say $\langle \mathscr{F}, \mathscr{C} \rangle$ is* consistent *if it is finite and has no closed tableau. We say $\langle \mathscr{F}, \mathscr{C} \rangle$ is* finitely consistent *if every finite sub-CSS $\langle \mathscr{F}_f, \mathscr{C}_f \rangle$ is consistent.*

Proposition 10.23 (cf. [148])**.** *1. Consistency is $\subseteq$-closed.*

2. A finite CSS is consistent iff it is finitely consistent.

We denote the set of finitely consistent CSS by $\mathscr{P}_{\text{fincon}}$.

Lemma 10.24. *$\mathscr{P}_{\text{fincon}}$ is an oracle.*

Proof. For $\subseteq$-closure and finite character see [148]. For saturation we show the case $\langle \mathbb{T} \mathrel{-\!\!*} \rangle$: the rest are similar.

Let $\langle \mathscr{F}, \mathscr{C} \rangle \in \mathscr{P}_{\text{fincon}}$, $\mathbb{T}\varphi \mathrel{-\!\!*} \psi : x \in \mathscr{F}$ and $y \succcurlyeq x, yz \succcurlyeq yz \in \overline{\mathscr{C}}$. Suppose neither $\langle \mathscr{F} \cup \{\mathbb{F}\varphi : z\}, \mathscr{C} \rangle \in \mathscr{P}_{\text{fincon}}$ nor $\langle \mathscr{F} \cup \{\mathbb{T}\psi : yz\}, \mathscr{C} \rangle \in \mathscr{P}_{\text{fincon}}$. Then there exist $\langle \mathscr{F}_f^A, \mathscr{C}_f^A \rangle \subseteq_f \langle \mathscr{F} \cup \{\mathbb{F}\varphi : z\}, \mathscr{C} \rangle$ and $\langle \mathscr{F}_f^B, \mathscr{C}_f^B \rangle \subseteq_f \langle \mathscr{F} \cup \{\mathbb{T}\psi : yz\}, \mathscr{C} \rangle$ that are inconsistent. By compactness (Lemma 10.6), there exist finite $\mathscr{C}_0, \mathscr{C}_1 \subseteq \mathscr{C}$ such that $z \preccurlyeq z \in \mathscr{C}_0$ and $yz \preccurlyeq yz \in \mathscr{C}_1$. Thus we define $\mathscr{F}'_f = (\mathscr{F}_f^A \setminus \{\mathbb{F}\varphi : z\}) \cup (\mathscr{F}_f^B \setminus \{\mathbb{T}\psi : yz\}) \cup \{\mathbb{T}\varphi \mathrel{-\!\!*} \psi : x\}$ and $\mathscr{C}'_f = \mathscr{C}_f^A \cup \mathscr{C}_f^B \cup \mathscr{C}_0 \cup \mathscr{C}_1$. Then $\langle \mathscr{F}'_f, \mathscr{C}'_f \rangle$ is a finite CSS and $[\langle \mathscr{F}'_f \cup \{\mathbb{F}\varphi : z\}, \mathscr{C}'_f \rangle; \langle \mathscr{F}'_f \cup \{\mathbb{T}\psi : yz\}, \mathscr{C}'_f \rangle]$ is a tableau for it. We have $\langle \mathscr{F}_f^A, \mathscr{C}_f^A \rangle \subseteq_f \langle \mathscr{F}'_f \cup \{\mathbb{F}\varphi : z\}, \mathscr{C}'_f \rangle$ and $\langle \mathscr{F}_f^B, \mathscr{C}_f^B \rangle \subseteq_f \langle \mathscr{F}'_f \cup \{\mathbb{T}\psi : yz\}, \mathscr{C}'_f \rangle$, so by $\subseteq$-closure of consistency $\langle \mathscr{F}_f^A, \mathscr{C}_f^A \rangle$ and $\langle \mathscr{F}_f^B, \mathscr{C}_f^B \rangle$ are inconsistent: let $\mathscr{T}_A$ and $\mathscr{T}_B$ be closed tableaux for them respectively. Then $\mathscr{T}_A \oplus \mathscr{T}_B$ is a closed tableau for $\langle \mathscr{F}'_f, \mathscr{C}'_f \rangle$ and the CSS is inconsistent, contradicting $\langle \mathscr{F}'_f, \mathscr{C}'_f \rangle \subseteq_f \langle \mathscr{F}, \mathscr{C} \rangle \in \mathscr{P}_{\text{fincon}}$. □

We can now show completeness of the tableaux calculus. Consider a formula φ for which there exists no closed tableau. We show there is a countermodel to φ. We start with the initial tableau $\mathscr{T}_0$ for φ. Then, we have that $\mathscr{T}_0 = [\langle \{\mathbb{F}\varphi : c_0\}, \{c_0 \preccurlyeq c_0)\} \rangle]$ and $\mathscr{T}_0$ cannot be closed. By Proposition 10.20, there exists a fair strategy, which we denote by $\mathscr{S}$, with $\mathbb{S}_i\varphi_i : x_i$ the i^{th} formula of $\mathscr{S}$. As $\mathscr{T}_0$ cannot be closed, $\langle \{\mathbb{F}\varphi : c_0\}, \{c_0 \succcurlyeq c_0\} \rangle \in \mathscr{P}_{\text{fincon}}$. We build a sequence $\langle \mathscr{F}_i, \mathscr{C}_i \rangle_{i \geqslant 0}$ as follows:

- $\langle \mathscr{F}_0, \mathscr{C}_0 \rangle = \langle \{\mathbb{F}\varphi : c_0\}, \{c_0 \succcurlyeq c_0\} \rangle$;
- if $\langle \mathscr{F}_i \cup \{\mathbb{S}_i\varphi_i : x_i\}, \mathscr{C}_i \rangle \notin \mathscr{P}_{\text{fincon}}$, then we have $\langle \mathscr{F}_{i+1}, \mathscr{C}_{i+1} \rangle = \langle \mathscr{F}_i, \mathscr{C}_i \rangle$; and
- if $\langle \mathscr{F}_i \cup \{\mathbb{S}_i\varphi_i : x_i\}, \mathscr{C}_i \rangle \in \mathscr{P}_{\text{fincon}}$, then we have $\langle \mathscr{F}_{i+1}, \mathscr{C}_{i+1} \rangle = \langle \mathscr{F}_i \cup \{\mathbb{S}_i\varphi_i : x_i\} \cup F_e, \mathscr{C}_i \cup \mathscr{C}_e \rangle$ where F_e and $\mathscr{C}_e$ are determined by

$\mathbb{S}_i$	φ_i	F_e	$\mathscr{C}_e$
$\mathbb{F}$	$\varphi \to \psi$	$\{\mathbb{T}\varphi : c_{\mathfrak{I}+1}, \mathbb{F}\psi : c_{\mathfrak{I}+1}\}$	$\{c_{\mathfrak{I}+1} \succcurlyeq x_i\}$
$\mathbb{T}$	$\varphi * \psi$	$\{\mathbb{T}\varphi : c_{\mathfrak{I}+1}, \mathbb{T}\psi : c_{\mathfrak{I}+2}\}$	$\{x_i \succcurlyeq c_{\mathfrak{I}+1}c_{\mathfrak{I}+2}\}$
$\mathbb{F}$	$\varphi -\!\!* \psi$	$\{\mathbb{T}\varphi : c_{\mathfrak{I}+2}, \mathbb{F}\psi : c_{\mathfrak{I}+1}c_{\mathfrak{I}+2}\}$	$\{c_{\mathfrak{I}+1 \succcurlyeq x_i}, c_{\mathfrak{I}+1}c_{\mathfrak{I}+2} \succcurlyeq c_{\mathfrak{I}+1}c_{\mathfrak{I}+2}\}$
$\mathbb{F}$	$\varphi *\!- \psi$	$\{\mathbb{T}\varphi : c_{\mathfrak{I}+2}, \mathbb{F}\psi : c_{\mathfrak{I}+2}c_{\mathfrak{I}+1}\}$	$\{c_{\mathfrak{I}+1 \succcurlyeq x_i}, c_{\mathfrak{I}+2}c_{\mathfrak{I}+1} \succcurlyeq c_{\mathfrak{I}+2}c_{\mathfrak{I}+1}\}$
Otherwise		$\emptyset$	$\emptyset$

with $\mathfrak{I} = \max\{j \mid c_j \in \mathscr{A}(\mathscr{C}_i) \cup \mathscr{S}(x_i)\}$.

Proposition 10.25. *For any $i \in \mathbb{N}$, the following properties hold:*

1. *$\mathscr{F}_i \subseteq \mathscr{F}_{i+1}$ and $\mathscr{C}_i \subseteq \mathscr{C}_{i+1}$;*
2. *$\langle \mathscr{F}_i, \mathscr{C}_i \rangle \in \mathscr{P}_{\text{fincon}}$.*

Proof. Only 2 is non-trivial. and we prove it by induction on i. The base case $i = 0$ is given by our initial assumption. Now for the inductive hypothesis (IH) we have that $\langle \mathscr{F}_i, \mathscr{C}_i \rangle \in \mathscr{P}_{\text{fincon}}$. Then the inductive step is an immediate consequence of Lemma 10.24 for the non-trivial cases. □

We now define the limit $\langle \mathscr{F}_\infty, \mathscr{C}_\infty \rangle = \langle \bigcup_{i \geqslant 0} \mathscr{F}_i, \bigcup_{i \geqslant 0} \mathscr{C}_i \rangle$ of the sequence $\langle \mathscr{F}_i, \mathscr{C}_i \rangle_{i \geqslant 0}$.

Proposition 10.26. *The following properties hold:*

1. *$\langle \mathscr{F}_\infty, \mathscr{C}_\infty \rangle \in \mathscr{P}_{\text{fincon}}$;*
2. *For all labelled formulae $\mathbb{S}\varphi : x$, if $\langle \mathscr{F}_\infty \cup \{\mathbb{S}\varphi : x\}, \mathscr{C}_\infty \rangle \in \mathscr{P}_{\text{fincon}}$, then $\mathbb{S}\varphi : x \in \mathscr{F}_\infty$.*

Proof. 1. First note that $\langle \mathscr{F}_\infty, \mathscr{C}_\infty \rangle$ is a CSS since each stage of construction satisfies (Ref) and by our choice of constants throughout the construction (Contra) and (Freshness) are satisfied. Further, it is open since otherwise there would be some stage $\langle \mathscr{F}_k, \mathscr{C}_k \rangle$ at which the offending closure condition is satisfied, which would contradict that each $\langle \mathscr{F}_i, \mathscr{C}_i \rangle$ is consistent. Now let $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \subseteq_f \langle \mathscr{F}_\infty, \mathscr{C}_\infty \rangle$. Then there exists $k \in \mathbb{N}$ such that $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \subseteq_f \langle \mathscr{F}_k, \mathscr{C}_k \rangle$. By Proposition 10.25 $\langle \mathscr{F}_k, \mathscr{C}_k \rangle \in \mathscr{P}_{\text{fincon}}$ so it follows $\langle \mathscr{F}_f, \mathscr{C}_f \rangle \in \mathscr{P}_{\text{fincon}}$. As $\mathscr{P}_{\text{fincon}}$ is of finite character, we thus have $\langle \mathscr{F}_\infty, \mathscr{C}_\infty \rangle \in \mathscr{P}_{\text{fincon}}$.

2. First note that $\langle \mathscr{F}_\infty \cup \{\mathbb{S}\varphi : x\}, \mathscr{C}_\infty \rangle$ is a CSS so (Contra) and (Freshness) are satisfied when the label x is introduced. By compactness, there exists finite $\mathscr{C}_{fin} \subseteq \mathscr{C}_\infty$ such that $x \succcurlyeq x \in \overline{\mathscr{C}_{fin}}$. As it is finite, there exists $k \in \mathbb{N}$ such that $\mathscr{C}_{fin} \subseteq \mathscr{C}_k$ and by fairness there exists $l \geq k$ such that $\mathbb{S}_l\varphi_l : x_l \equiv \mathbb{S}\varphi : x$. Since

(Freshness) and (Contra) are fufilled with respect to $\mathscr{F}_\infty$ they are also fulfilled with respect to $\mathscr{F}_l \cup \{\mathbb{S}\varphi : x\}$ so $\langle \mathscr{F}_{l+1}, \mathscr{C}_{l+1}\rangle = \langle \mathscr{F}_l \cup \{\mathbb{S}\varphi : x\}, \mathscr{C}_l\rangle \in \mathscr{P}_{\text{fincon}}$ and $\langle \mathscr{F}_{l+1}, \mathscr{C}_{l+1}\rangle = \langle \mathscr{F}_l \cup \{\mathbb{S}\varphi : x\} \cup \mathscr{F}_e, \mathscr{C}_l \cup \mathscr{C}_e\rangle$. Hence $\mathbb{S}\varphi : x \in \mathscr{F}_\infty$. □

Lemma 10.27. *The limit CSS is a Hintikka CSS.*

Proof. For properties $1. - 3.$ we have that $\langle \mathscr{F}_\infty, \mathscr{C}_\infty\rangle$ is open. For the other conditions, the saturation property of the oracle $\mathscr{P}_{\text{fincon}}$ and Proposition 10.26 item 2. suffice. □

We immediately obtain completeness.

Theorem 10.28 (Completeness)**.** *If φ is valid in layered graph models, then there exists a closed tableau for φ.* □

Proof. Suppose there exists no tableau proof for the formula φ. Then by Lemma 10.27 we can construct the Hintikka CSS $\langle \mathscr{F}_\infty, \mathscr{C}_\infty\rangle$ from $\mathscr{T}_0 = [\langle \{\mathbb{F}\varphi : c_0\}, \{c_0 \preccurlyeq c_0)\}\rangle]$ as outlined above, with $\mathbb{F}\varphi : c_0 \in \mathscr{F}_\infty$. Then by Lemma 10.18, $\Omega(\langle \mathscr{F}_\infty, \mathscr{C}_\infty\rangle)$ is a layered graph countermodel for φ. That is, φ is not valid. □

Summary of Part III

In this part of the thesis we set up a uniform and modular labelled tableaux proof theory for propositional bunched logics, as well as interesting classes of bunched logic model that are incomplete for standard sequential systems. In Chapter 9 we set up a framework to define systems that are sound and complete for the logics introduced in Part I. The key concept utilised to do so is coherent logic, a fragment of first-order logic containing just the goal-directed implications. Not only can coherent formulae be transformed into proof rules (used here to encode bunched logic frames in proof theoretic form), tableaux calculi themselves can be seen as coherent theories. This enabled us to prove soundness and completeness by embedding the calculi in a proof system for coherent logic. In Chapter 10 we produced proof systems for application-inspired classes of models in two ways. First, we used the coherent logic based translation to give new proof rules representing the separation theories used to axiomatise memory models of bunched logic. This gives the first uniform and modular proof theory for the breadth of abstract separation logic models. Second, we built a tableaux calculus from scratch for layered graph models of ILGL, carefully controlling the introduction of labels during derivation to enable the transformation of branches into countermodels for invalid formulae.

Part IV

Conclusions & Further Work

Conclusions

In this thesis we have specified a family of bunched logics and investigated it through two uniform frameworks. The first is a duality theoretic framework that relates the algebraic and Kripke-style interpretations of the logics, the second is a modular tableaux calculi framework, sound and complete for both validity of the logics and validity in classes of model of interest for the principal applications of the logics. In doing so we have made a number of contributions to the literature on bunched logic.

- The formulation of the intuitionistic variants of a number of bunched logics that had previously not been well investigated (if at all). This includes Kripke-style layered graph and resource interpretations as well as proof theory.
- The formulation of a schema for defining separating modal bunched logics, strictly generalising those found in the literature.
- The formulation of concurrent Kleene bunched logic, a formalism connecting concurrent Kleene algebra to bunched logic.
- Systematic duality theorems for the structures interpreting bunched logics, both propositional and predicate.
- Uniform soundness and completeness theorems for bunched logics. For many bunched logics with intuitionistic additives these are new; for those with classical additives that have previously been investigated these greatly simplify existing proofs.
- Decidability theorems for layered graph logics.
- A characterisation theorem for the classes of bunched logic models that can be defined by bunched logic formulae, analogous to the Goldblatt-Thomason theorem for modal logic.
- The resolution of the open problem of Craig interpolation for BBI and CBI, as well as the reduction of the same problem for BI and DMBI to a simpler one.
- Uniform labelled tableaux proof theory for the breadth of bunched logics, extending to particular classes of bunched logic model of interest in applications. Of interest beyond our particular systems is the new insight that labelled tableaux systems can be seen as representations of theories of coherent logic, opening up a new direction in their study.

More abstractly, this work highlights the viability of algebraic approaches to the study of bunched logic, something long neglected in the field. Our representation and duality theorems witness the fact that the resource semantics that has made bunched logic so impactful in computer science arise from the algebras interpreting bunched logic, while the results of Chapter 7 show that algebraic techniques are capable of resolving significant open problems. This may not be so surprising for logicians working with logics adjacent to bunched logic (e.g., modal logic or the substructural logics extending the full Lambek calculus), but it is nonetheless an advance on the previous state of the art.

Another perspective previously foreign to the study of bunched logic is the significance of classical model theory. Our bunched logic version of the Goldblatt-Thomason theorem relies on lemmas that take the first-order definition of bunched logic frames seriously, while our tableaux calculi effectively arise and are proven sound and complete through a careful examination of the fragment of first-order logic in which bunched logic models can be defined.

We believe both of these perspectives will be important to future work on bunched logics, and if there is a particular takeaway from this work that we would urge beyond our specific results, it is that these techniques should be taken seriously by practitioners in the field.

Further Work

There are many interesting directions future work based on this thesis could take, and we outline a few now.

Modelling with bunched logics. While the use of (B)BI's resource semantics to model computer memory has generated impact at an industrial level through Separation Logic, applications of other models of bunched logics have not been pursued in nearly as much detail.

We suggest two applications that could fruitfully be investigated. The first is quantum mechanics, through both the resource theory models of BI and the effect algebra models of DMBI and CBI. Fritz [94] suggests that linear logic might be used to reason about resource theories, but their formalisation as ordered commutative monoids is much more suggestive of BI, the semantics of which directly reflect Fritz's conceptual development, in contrast to linear logic's phase semantics. Insights relating to the use of partiality in BI may also be useful for reasoning about resource theories in which resources are not assumed to be universally composable with each other. The second is complex systems modelling of the sort motivating the formulation of layered graph logics. Preliminary work in this direction has

been done [64], but nothing with a suitable notion of dynamics. We suggest the bigraph-style models of ILGL could form the basis of an formalism analogous to Separation Logic, with program execution replaced by the dynamics given by bigraphical reactive systems.

Transition systems and coalgebraic logic. One application of resource semantics that we have not talked about much in this thesis is the resource-sensitive process algebra of Pym & Tofts [190], Collinson & Pym [61] and Anderson & Pym [10]. Inspired by BI's semantics, these calculi provide executable formalisations of resource-sensitive transition systems that are suitable for discrete-event modelling.

It is now well known that coalgebra [201] provides an excellent mathematical framework for specifying transition systems, with duality theory providing machinery to automatically output coalgebraic logics that can be used to reason about them [30]. We believe the duality theory for bunched logics outlined in this thesis provides the foundation to lift that work to the aforementioned *resource-sensitive* transition systems, through which our approach to bunched logic and the process algebra approach can be unified. Furthermore, this generic approach could additionally be used to specify calculi that also capture the spatial aspects of systems like those modelled by bunched logics like (I)LGL.

Further resolution of open problems. There are many more logical properties that remain open for bunched logics. Although in this thesis we were able to show that Craig interpolation fails for BBI and CBI, our argument fell just short of proving the same for BI and DMBI, though it seems likely that it is also the case for them given the property we reduced it to typically holds of well-behaved logics. Beyond these, the problems remain open for the other bunched logics we've examined in the thesis. Given that bunched logics characteristically have more than one implication, it would also be of interest to examine interpolation properties based on the multiplicative implications $-\!\!*$ and $*\!\!-$ as well as $\rightarrow$. We might also investigate other logical properties through algebraic means: for example, the Beth definability property is known to correspond to the property of epimorphisms being surjective in the category of algebras interpreting a logic.

Another problem that remains open in the bunched logic literature is the decidability of DMBI. We conjecture it *is* decidable, based on the decidability of BI, but it is not clear how to resolve the problem one way or the other. The algebraic methodology of Galatos & Jipsen [97] seems promising, but given Ramanayake's [191] discovery of a flaw in their proof of BI's decidability, some caution must be exercised. A problem with a much clearer path to resolution is the characterisation

of a Sahlqvist-like [202] fragment of bunched logic for which validity corresponds precisely to the satisfaction of first-order axioms on bunched logic frames. As with the Goldblatt-Thomason theorem, in the case of modal logic this can be proved through duality theoretic means [203], and we believe a similar argument should work here.

Multiplicative quantification revisited. In this thesis we have looked at logics featuring multiplicative versions of every standard connective except for the quantifiers. In Pym's [187] monograph on BI a sequent calculus and Kripke semantics was given for an extension of BI with additive and multiplicative quantifiers, but Biering [23] found a number of flaws that revealed the system to be ill-defined. Multiplicative quantification is nonetheless a well-defined idea, as evidenced by its sound usage in the aforementioned bunched process calculi. There their utility is emphasised by the elegant definition of multiplicative modalities that is induced by multiplicative quantification.

Multiplicative quantifiers were revisited by Collinson et al. [62] in the context of BI's type theory, the $\alpha\lambda$ calculus, and interpreted by hyperdoctrines with additional coherence conditions to the ones we considered. We believe the duality theoretic approach to BI hyperdoctrines given in Chapter 8 could be extended to these structures to resurrect the semantic approach to multiplicative quantification. It may also be possible to use the structure of these extended hyperdoctrines to define a proof system without the defects of Pym's.

Duality-theoretic approaches to Separation Logic. In Chapter 8 we gave duality theorems that subsumed the semantics of Separation Logic's assertion language. However, what we didn't consider was a semantics of program execution, extending the assertion language to Hoare triples. Examples of duality theoretic approaches to Hoare logic can be found in the work of Abramsky [1] and Brink & Rewitzky [34], and we believe they may be reconfigured with our bunched logic dualities to give a duality theoretic framework for the entirety of the Separation Logic formalism, subsuming the *local predicate transformer* approach to the semantics of commands of O'Hearn & Yang [221] . Recent work by Hino et al. [116] has examined healthiness properties of program logics through a duality theoretic framework, and it would be interesting to investigate if this can be extended to Separation Logic through our work.

Implementations and generalisations of the tableaux calculi. Our final suggestion for further work is a programme of research based on the tableaux calculi of

Part III. The first and perhaps most obvious task is the implementation of those systems. Given they can all be given as theories of coherent logic, an immediate possibility is the use of off-the-shelf coherent theorem provers, of which Polonsky [184] gives a summary. It has been reported [22] that many tasks for which traditional first-order provers are typically used are more efficiently tackled by coherent provers, and it would be interesting to see if that efficiency transfers at all to our tableaux calculi. It would also be fruitful to see if parametric separation logic tools can be implemented using the tableaux calculi for separation theories. This is a more complex task than the implementation of the calculi for bunched logics, as the tableaux calculi would need to be embedded in a system that also captures the Hoare logic component of Separation Logic, as well as typical control processes like bi-abduction [49] that enable scalability to large code bases.

Beyond this, the extension of our techniques to logics interpreted on structures that lie outside of the scope of coherent logic is of clear interest. Labelled tableaux systems in which the labels are automata [110], or can be transformed into structures with fixed points [56] abound in the literature: if our techniques can be seen as giving a foundation for typical labelled tableaux calculi in which labels represent possible worlds, perhaps such an extension could give a foundation for these more esoteric systems. This would also facilitate the extension of our systems with features like cyclic proof [37, 40], a formalism typically used for separation logics that include inductive predicates.

It is also clear that the techniques that yield the proof systems of Part III can be applied more broadly than just bunched logics. Essentially any logic which is interpreted on Kripke structures with a finite coherent axiomatisation can also be given a tableaux calculus using a coherent logic translation. It would thus be of interest to generalise this as far as possible, raising the intriguing possibility of a semantic (as opposed to type-theoretic) logical framework that uses coherent logic as its metalogic. Once again, off-the-shelf coherent logic provers may be put to use for the implementation of such a framework.

Appendix A

Category Theory

In this appendix we recount the category theoretic definitions used throughout the thesis. This should primarily be used as a reference: there are many good introductory category theory textbooks (e.g., Awodey [15] or Mac Lane [154]) if one requires greater depth.

A *category* $\mathscr{C}$ is a collection of *objects* $Ob(\mathscr{C})$ and a collection of *arrows* $Arr(\mathscr{C})$ together with maps $dom, cod : Arr(\mathscr{C}) \to Ob(\mathscr{C})$ assigning to each arrow f a domain $dom(f)$ and codomain $cod(f)$ in $Ob(\mathscr{C})$, such that the following axioms are satisfied:

- **Composition**: Given arrows f, g with $cod(f) = dom(g)$ there exists a unique arrow
$$g \circ f : dom(f) \to cod(g)$$
- **Associativity**: Given arrows f, g, h with $cod(f) = dom(g)$ and $cod(g) = dom(h)$ we have
$$h \circ (g \circ f) = (h \circ g) \circ f$$
- **Identity**: For every object X there exists an identity arrow Id_X such that, given arrows f, g with $dom(f) = X = cod(g)$ we have
$$f \circ id_X = f \qquad\qquad id_X \circ g = g$$

We refer to arrows interchangeably as morphisms, and use $f : X \to Y$ or $X \xrightarrow{f} Y$ to denote that $dom(f) = X$ and $cod(f) = Y$. We can distinguish some special species of arrow that generalise familiar set theoretic properties of functions. We call an arrow $f : X \to Y$ a *monomorphism* if it is *left cancellable*: for any pair of arrows $g, h : Z \to X$ such that $fg = fh$ we have $g = h$. Dually we have the notion of an *epimorphism*; a *right cancellable* morphism. An *isomorphism* $f : X \to Y$ is

an invertible arrow: that is, there exists an arrow $g : Y \to X$ such that $gf = id_X$ and $fg = id_Y$.

The appropriate notion of morphism between categories is called a *functor*. Given categories $\mathscr{C}$ and $\mathscr{D}$, a functor $F : \mathscr{C} \to \mathscr{D}$ is given by a pair of maps $F_0 : Ob(\mathscr{C}) \to Ob(\mathscr{D})$ and $F_1 : Arr(\mathscr{C}) \to Arr(\mathscr{C})$ that interacts coherently with the category structure:

- For all arrows $f : X \to Y$, $F_1(f) : F_0(X) \to F_0(Y)$;
- For all objects X, $F_1(id_X) = id_{F_0(X)}$;
- For all arrows $f : X \to Y$, $g : Y \to Z$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$.

It is an *endofunctor* if $\mathscr{C} = \mathscr{D}$. If $F : \mathscr{C}^{op} \to \mathscr{D}$ is a functor—where $\mathscr{C}^{op}$ is the category $\mathscr{C}$ with arrows reversed—we call it a *contravariant functor* $F : \mathscr{C} \to \mathscr{D}$.

Next we give a notion of morphism between functors. Given functors $F, G : \mathscr{C} \to \mathscr{D}$, a *natural transformation* $\mu : F \Rightarrow G$ is given by a collection of $\mathscr{C}$-indexed maps

$$(\mu_X : F(X) \to G(X) \mid X \text{ in } Ob(\mathscr{C}))$$

satisfying the following *naturality* condition: given any arrow $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\mu_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\mu_Y]{} & G(Y) \end{array}$$

A *natural isomorphism* is a natural transformation μ in which each μ_X is an isomorphism. A *dual adjunction* is a pair of contravariant functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and a pair of natural transformations $\eta : Id_{\mathscr{C}} \to GF$ and $\theta : Id_{\mathscr{D}} \to FG$ such that $F(\eta) \circ \theta F = Id_F$ and $G(\theta) \circ \eta G = Id_G$. It is a *dual equivalence* if η and θ are natural isomorphisms.

A *monoidal structure* $(\otimes, 1, \varepsilon, \iota, a)$ on a category $\mathscr{C}$ consists of the following data:

- **Tensor Product**: A functor $\mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$;
- **Unit Object**: An object 1 in $\mathscr{C}$;

- **Unitors**: Natural isomorphisms

$$\varepsilon : 1 \otimes - \Rightarrow id_{\mathscr{C}}$$

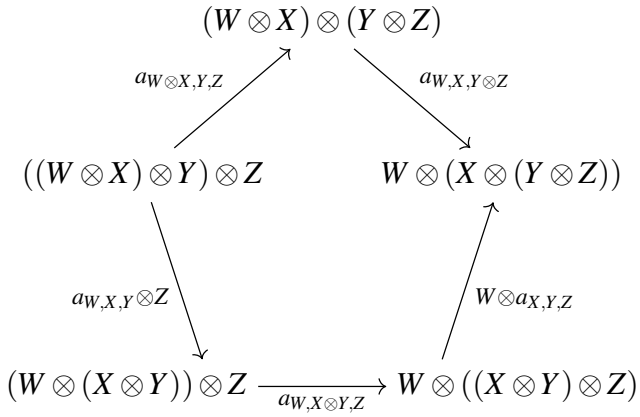
$$\iota : - \otimes 1 \Rightarrow id_{\mathscr{C}}$$

- **Associator**: A natural isomorphism

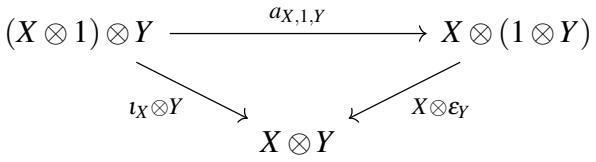
$$a : (- \otimes -) \otimes - \Longrightarrow - \otimes (- \otimes -)$$

Making the following diagrams commute

- **Pentagon Identity**

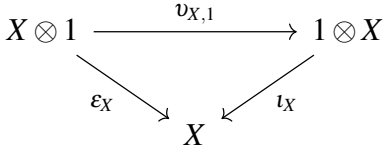


- **Triangle Identity**:

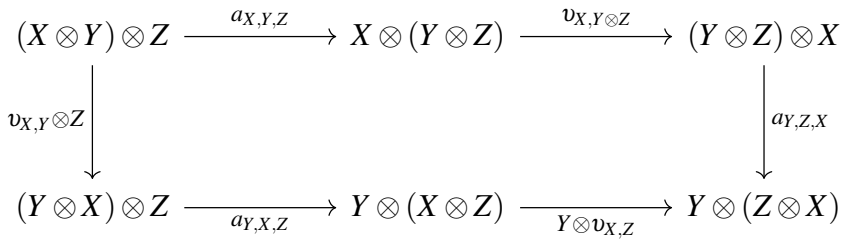


We call a category equipped with a monoidal structure a *monoidal category*. It is *symmetric monoidal category* if there exists natural isomorphisms $\upsilon_{X,Y} : X \otimes Y \simeq Y \otimes X$ such that the following diagrams commute:

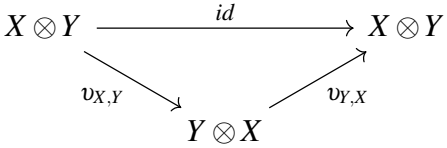
- **Unit Coherence**:



- **Associativity Coherence**:



- **Inverse Law**:



Finally, an object X is the *product* of objects $(X_i)_{i\in I}$ iff there exist morphisms $\pi_i : X \to X_i$ such that for every object Y and every indexed family of morphisms $f_i : X_i \to Y$ there exists a unique morphism f such that $\pi_i \circ f = f_i$ for all i. A category $\mathscr{C}$ has *finite products* if the product exists for every finite family of objects $(X_i)_{i\in I}$ in $\mathscr{C}$.

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