Design procedure for linear unknown input functional observers

Imane SAKHRAOUI, Baptiste TRAJIN, Frédéric ROTELLA

***Abstract*—This paper considers an iterative procedure to design a minimum possible order unknown input functional observer for linear time-invariant (LTI) systems. Necessary and sufficient conditions for the existence of the observer are given. The proposed procedure is simple and is based neither on the solution of a generalized Sylvester equation nor canonical forms. This procedure can be easily implemented due to its incremental form. The main feature of our procedure is then the simplicity of the design. Moreover, in the case of a single functional to be observed, a minimum order stable linear unknown input functional observer is obtained.**

***Index Terms*—Linear systems, time-invariant systems, unknown input functional observers.**

I. Introduction

For a dynamic system, the observation problem consists in the estimation of some internal variables, that are not measured, from measurements given by sensors. From the seminal works of Luenberger [1], [2], solutions of the observation problem for a linear linear time-invariant system have been conducted in two main ways. On the one hand, methods have been developed for the design of linear functional observers (LFO) of the state vector of the system. This way of designing LFO, which has been considered in the first beginning of the observers theory, leads to reduced order observers. On the other hand, especially for diagnosis [3] or for robustness purposes, the observation problem of the whole state with unknown inputs or disturbances has led to design methods for unknown input linear observers (UILO). Existence conditions and algorithms of these observers have been reported in [4], [5], [6], [7], [8], [9] and in the books [10], [11]. The cited papers are only milestones and the interested reader can find more details in the books and in the references therein.

Functional observers are a generalized version of ordinary full order or reduced order Luenberger observers that aim at reconstructing single or multiple functions of the states of the system [4], [11], [12], [13]. These observers offer the advantage to be of lower order compared to the ordinary reduced order observers of the whole state. Nevertheless, unknown or unmeasured inputs or disturbances acting on the system have been recently considered in the design of functional observers [14], [15], [16], [17], [18] and latest developments can be found in [19], [20]. Minimality of the order of these observers is yet an open problem. Indeed, it depends on the wanted objective with respect to the desired eigenvalues of the observer. It is well known that the minimum order required for a functional observer cannot be lower than the number of functions that have to be estimated [11]. In fact, there might exist some mechanisms to increase the order of a functional observer or a UIFO, such that an asymptotically stable observer can be obtained even under nondetectability of the system [21], [22]. However, to cope with the design of minimum possible order functional observer a methodology to obtain a UIFO through an increase of the number of functional to be observed is presented in [22]. However, between the fixed-pole observer problem where the poles are specified at the outset and the stable observer problem where the poles are permitted to lie anywhere in the left half-plane, finding the lowest possible order linear unknown input observer (LUIFO) is one of the main challenges in this field of research that is still not fully addressed [22].

Rewriting the necessary and sufficient conditions for the existence of a stable UIFO for a linear system expressed in [23], a direct and iterative procedure to get a minimal order for a single stable linear functional observer has been proposed in [24] and for multi-functional observers in [25]. The herein presented paper extends these results towards minimality of LUIFO. The proposed procedure is simple, iterative and is not based on the solution of so-called Sylvester equation or on the use of canonical state space forms. The term iterative indicates that an increasing sequence for the order of the observer is tested to obtain a possible minimal order. The main feature of the proposed design procedure is the highlighting of some degrees of freedom to place some poles of the obtained observer.

The paper is organized as follows. In Section II, the well known necessary and sufficient conditions for the existence of a LUIFO are outlined. It is underlined that these conditions are focussed on the existence of a linear relationship between the state of the system and the state of the observer. To overcome this difficulty, which has been the keypoint in the design of linear observers over many years, Darouach has proposed seminal criteria in [26] for LFO and in [23] for LUIFO. In Section III the main results are claimed. A necessary and sufficient condition is detailed for the existence of an asymptotic observer. This necessary and sufficient condition has been stated for linear functional observer where the inputs are completely measured in [7], [27]. The sufficient condition is proved here in the unknown input case. As a specific feature, these conditions can be tested on the observation problem. Section IV is devoted to the constructive proof of the main result and the design procedure of the LUIFO. From the use of

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successive derivatives of state functionals, an usual realization method leads to obtain the observer. It can be remarked that the presented results extend the results in [28] in the case where unknown inputs are present. Finally, an example illustrates the procedure in Section V and points out the advantage of the proposed design regarding the previous ones. Moreover, this example offers the opportunity to illustrate the notion of degrees of freedom to, eventually, place some of the eigenvalues of the observer.

In all the following the following notations are used. $M(m \times n)$ is a constant matrix with m rows and n columns. I_n is the identity matrix of size n. $0(m \times n)$ is a null matrix with m rows and n columns and, shortly, 0_n is the square null matrix of size n. $M^{[1]}$ is any of the generalized inverses of the matrix M, it fulfills the matrix equation $MXM = M$, and, $M^\dagger$ is the pseudo-inverse, namely it is the unique generalized inverse of M that fulfills the Moore-Penrose set of equations ($XMX = X$, $MXM = M$, $(XM)^\top = XM$, and, $(MX)^\top = MX$) [29]. A matrix is a Hurwitz matrix if all its eigenvalues have strictly negative real part. Finally, $f^{(i)}$ denotes the i-th derivative of the function f, $\mathbb{N}$ is the set of natural integers including 0 and $\mathbb{N}^*$ is the set of natural integers excluding 0 and $[\![k_1 \; ; \; k_2]\!]$ is the closed set of natural integers from k_1 to k_2.

II. Linear unknown input functional observers

Let us consider a system described by the linear state space equations with unknown inputs:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dd(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where, $\forall t \in \mathbb{R}^+$, $x(t)$ is the n-dimensional state vector, $u(t)$ is a p-dimensional control vector supposed to be measured, $y(t)$ is a m-dimensional measured output vector, and, $d(t)$ is a r-dimensional unknown input vector. $A(n \times n)$, $B(n \times p)$, $C(m \times n)$ and $D(n \times r)$ are constant matrices. Without loss of generality C and D are respectively of full row and of full column ranks.

The objective of the paper is to propose a simple design of a linear observer to estimate the vector $v(t)$ related to the state $x(t)$ by:

$$v(t) = Lx(t), \tag{2}$$

where $L(l \times n)$ is a constant full row rank matrix.

To design a functional observer, the triplet (A, C, L) has to be functionally observable [28], i.e.:

$$\operatorname{rank}\left(\begin{bmatrix} \mathcal{O}_{A,C,n} \\ L \end{bmatrix}\right) = \operatorname{rank}\left(\mathcal{O}_{A,C,n}\right).$$

Moreover, in order to avoid a trivial algebraic part in the observer (where observed functionals can be estimated through linear combination of measured outputs), it is supposed without loss of generality that:

$$\operatorname{rank}\left(\begin{bmatrix} C \\ L \end{bmatrix}\right) = m + l.$$

The observation of $v(t)$ can be carried out by a linear unknown input functional observer (LUIFO) which is a Luenberger observer described by the following state space equation [1], [2]:

$$\begin{cases} \dot{z}(t) = Fz(t) + Gu(t) + Hy(t), \\ \hat{v}(t) = Pz(t) + Vy(t), \end{cases} \tag{3}$$

where, $\forall t \in \mathbb{R}^+$, $z(t)$ and $\hat{v}(t)$ are, respectively, the ν-dimensional state vector and the l-dimensional output vector of the observer. The constant matrices $F(\nu \times \nu)$, $G(\nu \times \nu)$, $H(\nu \times m)$, $P(l \times \nu)$, $V(l \times m)$ and the order ν have to be determined such that $\lim\limits_{t \to +\infty} (v(t) - \hat{v}(t)) = 0$. Moreover, it must be kept in mind that a possible minimal order observer is looked for. It is well known that the necessary and sufficient conditions for the existence of an asymptotic observer (3) are given by the following result [10], [11]. Furthermore, a LUIFO cannot be designed if there are unstable transmission zeros from the unknown input to the output [17].

Theorem 1. *The observable system (3) is a LUIFO of (2) for the system (1) if and only if F is Hurwitz and there exists a matrix $T(\nu \times n)$ such that:*

$$FT + HC - TA = 0, \tag{4}$$

$$L - PT - VC = 0, \tag{5}$$

$$G - TB = 0, \tag{6}$$

$$TD = 0. \tag{7}$$

Proof: The proof of the sufficiency is well known and is summed up below for completeness. Let us denote the errors $\epsilon(t) = z(t) - Tx(t)$ and $e(t) = v(t) - \hat{v}(t)$. We get from (1) and (3):

$$\dot{\epsilon}(t) = F\epsilon(t) + (G - TB)u(t) + (FT + HC - TA)x(t) - TDd(t). \tag{8}$$

Thus, when (4), (6) and (7) are fulfilled and F is a Hurwitz matrix we have $\lim\limits_{t \to +\infty} \epsilon(t) = 0$. The estimation error $e(t) = v(t) - \hat{v}(t)$, can be written using (2) and (3):

$$e(t) = (L - PT - VC)x(t) - P\epsilon(t).$$

Thus, when $L - PT - VC = 0$, we obtain $\lim\limits_{t \to +\infty} e(t) = 0$.

We insist afterwards on the proof of the necessity. Note that the observer must be observable. Conversely, when (3) is a LUIFO of (2) for the system (1), we have from linearity, $\forall i \in \mathbb{N}^*$, $\lim\limits_{t \to +\infty} e^{(i)}(t) = 0$. With $M = L - VC$, when $t \to +\infty$, the following relationships are deduced:

$$\begin{aligned} \lim_{t \to +\infty} e(t) &= \lim_{t \to +\infty} \left(Mx(t) - Pz(t)\right) = 0, \\ \lim_{t \to +\infty} \dot{e}(t) &= \lim_{t \to +\infty} \big((MA - PHC)\, x(t) \\ &\qquad + (MB - PG)\, u(t) \\ &\qquad + MDd(t) - PFz(t)\big) = 0. \end{aligned}$$

As the last relation must be fulfilled for every $u(t)$ and $d(t)$, we get $MB - PG = 0$ and $MD = 0$. Consequently, $\lim\limits_{t \to +\infty} \dot{e}(t) = (MA - PHC)\, x(t) - PFz(t) = 0$. Continuing this procedure with the successive derivative of $e(t)$ we get, for $i \in [\![0 \; ; \; \nu - 1]\!]$:

$$\lim_{t \to +\infty} e^{(i)}(t) = \lim_{t \to +\infty} \left(M_i x(t) - PF^i z(t)\right) = 0, \tag{9}$$

where the matrices M_i are recursively defined as $M_0 = M$ and:

$$M_{i+1} = M_i A - PF^i HC.$$

Moreover we have also $M_i B = PF^i G$ and $M_i D = 0$. Gathering the relationships (9) we get, for $i \in [\![0 \; ; \; \nu-1]\!]$:

$$\lim_{t\to+\infty}\left(\begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{\nu-1}\end{bmatrix} x(t) - \begin{bmatrix} P \\ PF \\ \vdots \\ PF^{\nu-1}\end{bmatrix} z(t)\right) = 0.$$

As the observer (3) is supposed to be observable, $\begin{bmatrix} P \\ \vdots \\ PF^{\nu-1}\end{bmatrix}$ is a full column rank matrix. Thus, there exists a unique matrix T , written:

$$T = \begin{bmatrix} P \\ PF \\ \vdots \\ PF^{\nu-1}\end{bmatrix}^{[1]} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{q-1}\end{bmatrix},$$

such that $\lim_{t\to+\infty}(Tx(t) - z(t)) = 0$. On the one hand, this property must be fulfilled for every $u(t)$, $x(t)$ and $d(t)$. Thus, from (8), we get the relationships (4), (6) and (7) and the fact that F is a Hurwitz matrix. On the other hand, $\lim_{t\to+\infty} e(t) = 0$ for every $x(t)$ leads to (5). ■

Remark 2. From Darouach work [23], [26], several works (see for instance [11]) have studied or used a particular case of Luenberger observers (3) where $P = I_\nu$, which in the following are named Darouach observers (10), are observable. In the case of unknown inputs:

$$\begin{cases} \dot{z}(t) = Fz(t) + Gu(t) + Hy(t), \\ \hat{v}(t) = z(t) + Vy(t), \end{cases} \tag{10}$$

is an asymptotic observer of (2) for the system (1) if and only if:

1) $\operatorname{rank}\left(\begin{bmatrix} C & 0 \\ L & 0 \\ CA & CD\end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} C & 0 \\ L & 0 \\ CA & CD \\ LA & LD\end{bmatrix}\right)$.
2) $\forall s$, such as $\Re(s) \geq 0$,

$$\operatorname{rank}\left(\begin{bmatrix} C & 0 \\ CA & CD \\ sL - LA & -LD\end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} C & 0 \\ L & 0 \\ CA & CD\end{bmatrix}\right).$$

On the one hand, when fulfilled, the first condition ensures the existence of the Darouach structure (10), i.e. Equations (4) - (7) have a solution considering $P = I_\nu$. On the second hand, the second condition leads to the asymptotic tracking of the functional $v(t)$. The second condition, which is a Hautus type condition [30], can be only used for linear time-invariant systems. In the case of $D = 0$, expressed in [24], the second condition has the following form:

$$LA = \Gamma_0 C + \Lambda_0 L + \Gamma_1 CA.$$

The system (10) is an observer of (2) if and only if Λ_0 is a $(l \times l)$ Hurwitz matrix. This standpoint has been extended to linear time-varying systems in [31], [32]. In the following, this criterion is extended for the unknown input case.

Remark 3. Darouach's method [23] has been used in the majority of the papers dealing with the design of unknown input functional observers. Indeed, a lot of design procedures are based on the extension of the functional to observe. Namely, these methods consist in looking for a $R(\rho \times n)$ matrix with $\nu = l + \rho$ and $P = I_\nu$ such that (3) is a Darouach observer of $\overline{v}(t) = \begin{bmatrix} R \\ L \end{bmatrix}$. Nevertheless, as it is underlined in [11], to find R is an *"intriguing and challenging problem"* and some attempts have been proposed in [19], [20], [22], [33], [34], [35]. As it will be explained in the next section, our design principle differs a lot from this standpoint.

Remark 4. Equation (4) is the so-called "constrained Sylvester equation" of the observer problem. Let us notice that it is a nonlinear equation to solve due to the fact that F, T and H are unknown. Nevertheless, until [23], [26], a great number of design methods have been devoted to the determination of the matrix T from canonical form or by imposing F. Let us mention that our procedure doesn't need to determine T as a prerequisite of the method. For the completeness of proof, T will be given afterwards.

III. Main result

This section is devoted to our claim. Firstly, let us define recursively the matrices K_q and Σ_q, $q \in \mathbb{N}$:

- $K_0 = I_n$ and for $q \geq 1$, $K_q = \begin{bmatrix} AK_{q-1} & D \end{bmatrix}$;
- $\Sigma_0 = C$ and for $q \geq 1$,

$$\Sigma_q = \left[\begin{array}{c|c} \begin{matrix}\Sigma_{q-1} \\ LK_{q-1}\end{matrix} & 0(q(m+l) \times r) \\ \hline \multicolumn{2}{c}{CK_q} \end{array}\right].$$

More explicitly, we get:

$$\Sigma_q = \begin{bmatrix} C & 0 & \cdots & 0 & 0 & 0 \\ L & 0 & \cdots & 0 & 0 & 0 \\ CA & CD & \cdots & 0 & 0 & 0 \\ LA & LD & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & 0 & 0 & 0 \\ CA^{q-1} & CA^{q-2}D & \cdots & CAD & CD & 0 \\ LA^{q-1} & LA^{q-2}D & \cdots & LAD & LD & 0 \\ CA^q & CA^{q-1}D & \cdots & CA^2D & CAD & CD \end{bmatrix}.$$

where the "0" blocks are of adapted dimensions.

In the following we use the notation:

$$\Sigma_q = \begin{bmatrix} C_{q,0} \\ L_{q,0} \\ C_{q,1} \\ L_{q,1} \\ \vdots \\ C_{q,q-1} \\ L_{q,q-1} \\ C_{q,q} \end{bmatrix}, \tag{11}$$

where the matrices $C_{q,i}$, for $i \in [\![0;q]\!]$, and $L_{q,i}$, for $i \in [\![0;q-1]\!]$, are respectively of dimensions $(m \times (n+rq))$ and $(l \times (n+rq))$.

Theorem 5. *There exists an observable unknown input functional observer (3) for the system (1) to asymptotically observe the functional (2), if there exist matrices Γ_i for $i \in [\![0;q]\!]$, and Λ_i for $i \in [\![0;q-1]\!]$, such that:*

$$LK_q = \sum_{i=0}^{q} \Gamma_i C_{q,i} + \sum_{i=0}^{q-1} \Lambda_i L_{q,i}, \tag{12}$$

and the matrix:

$$\begin{bmatrix} 0_l & \cdots & \cdots & 0_l & \Lambda_0 \\ I_l & \ddots & & \vdots & \Lambda_1 \\ 0_l & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0_l & \Lambda_{q-2} \\ 0_l & \cdots & 0_l & I_l & \Lambda_{q-1} \end{bmatrix}, \tag{13}$$

is a Hurwitz one.

The proof of this result is constructive and leads to the design of the LUIFO. Thus, it will be detailed in the following section.

Remark 6. On the one hand (12), that can be written as:

$$\text{rank}\left(\Sigma_q\right) = \text{rank}\left(\begin{bmatrix} \Sigma_q \\ LK_q \end{bmatrix}\right), \tag{14}$$

leads to the design of the state space equations of the observer (3). Indeed, as indicated in the Remark 10 of Section IV the knowledge of matrices F, G H, P and V may lead to determine a matrix T that solves Equations (4) - (7). On the other hand, the Hurwitz condition ensures that $\lim_{t\to+\infty}(v(t) - \hat{v}(t)) = 0$. Thus, the previous theorem appears as an extension of the reformulated Darouach criterion for LUIFO.

Remark 7. We insist here that the observable condition of (3) ensures the necessity in the Theorem 5. The proposed design procedure leads to an observable observer.

In the following section, the sufficient condition for the existence of a minimum order UIFO is used in order to develop a method for designing a LUIFO. When the obtained observer structure does not lead to an asymptotically stable observer, the procedure will consist in increasing the integer q. This standpoint is detailed in the example in the Section V.

IV. Design of an unknown input functional observer

This section deals with the design of an observable LUIFO for the observation problem defined by (1) and (2) when condition (12) is fulfilled. Moreover, the procedure is given to get a possible stable observer of least order.

A. First step

This step consists in the determination of the integer q candidate for the structure (3). Firstly, q is given by the condition (14) which must be fulfilled. Let us suppose that the relation (14) is fulfilled for the smallest q. Then, the matrices Γ_i for $i \in [\![0;q]\!]$ and Λ_i for $i \in [\![0;q-1]\!]$ are obtained from the solution of the consistent linear system $LK_q = X\Sigma_q$:

$$X = LK_q\Sigma_q^{[1]} + Z\left(I_{r_q} - \Sigma_q\Sigma_q^{[1]}\right),$$

where $r_q = m(q+1) + lq$ is the number of rows of Σ_q, Z is any matrix with the same size than X, and, $\Sigma_q^{[1]}$ is any of the generalized inverses of Σ_q. Due to numerical properties and implementations in numerical software, we can chose as a particular case $\Sigma_q^{[1]} = \Sigma_q^{\dagger}$. In the case where Σ_q is of full row rank, the solution is unique and, consequently, the matrices Γ_i and Λ_i also. Partitioning X with respect to the row partition of Σ_q in (11) ($X = [\Gamma_0\,\Lambda_0\,\Gamma_1\,\Lambda_1\,\ldots\,\Gamma_{q-1}\,\Lambda_{q-1}\,\Gamma_q]$), we are led to the matrices Γ_i for $i \in [\![0;q]\!]$ and Λ_i for $i \in [\![0;q-1]\!]$ in (12) that can then be explicitly written as:

$$\begin{aligned} LA^q &= \sum_{i=0}^{q} \Gamma_i CA^i + \sum_{i=0}^{q-1} \Lambda_i LA^i, \\ LA^{q-1}D &= \sum_{i=1}^{q} \Gamma_i CA^{i-1}D + \sum_{i=1}^{q-1} \Lambda_i LA^{i-1}D, \\ &\vdots \\ LA^{q-k}D &= \sum_{i=k}^{q} \Gamma_i CA^{i-k}D + \sum_{i=k}^{q-1} \Lambda_i LA^{i-k}D, \\ &\vdots \\ LAD &= \Gamma_q CAD + \Gamma_{q-1}CD + \Lambda_{q-1}LD, \\ LD &= \Gamma_q CD. \end{aligned} \tag{15}$$

Secondly, the design of the observer can be considered only when the matrix (13) is an Hurwitz matrix, that can be tested from the previous decomposition for LK_q on Σ_q. So, when the conditions of the Theorem 5 are satisfied, we can go further to the determination of the matrices in (3).

B. Second step

The design of the observer uses the successive derivations of $v(t)$. After q derivations of $v(t) = Lx(t)$, we obtain:

$$\begin{array}{rcl} \dot{v}(t) &=& LAx(t) + LBu(t) + LDd(t), \\ v^{(2)}(t) &=& LA^2x(t) + LABu(t) + LB\dot{u}(t) + LADd(t) \\ && +LD\dot{d}(t), \\ &\vdots& \\ v^{(q-1)}(t) &=& LA^{q-1}x(t) + \sum_{i=0}^{q-2} LA^iBu^{(q-2-i)}(t) \\ && + \sum_{i=0}^{q-2} LA^iDd^{(q-2-i)}(t), \\ v^{(q)}(t) &=& LA^qx(t) + \sum_{i=0}^{q-1} LA^iBu^{(q-i-1)}(t) \\ && + \sum_{i=0}^{q-1} LA^iDd^{(q-i-1)}(t). \end{array} \tag{16}$$

From the last equation of (16), $LA^kx(t)$ is expressed, for $q \in \mathbb{N}^*$ as:

$$\begin{array}{rl} LA^kx(t) = & v^{(k)}(t) - \sum_{i=0}^{k-1} LA^iBu^{(k-i-1)}(t) \\ & - \sum_{i=0}^{k-1} LA^iDd^{(k-i-1)}(t). \end{array} \tag{17}$$

Moreover, the last equation of (16) can also be written, using the first equation of (15), as:

$$\begin{array}{rl} v^{(q)}(t) = & \sum_{i=0}^{q} \Gamma_iCA^ix(t) + \sum_{i=0}^{q-1} \Lambda_iLA^ix(t) \\ & + \sum_{i=0}^{q-1} LA^iBu^{(q-i-1)}(t) + \sum_{i=0}^{q-1} LA^iDd^{(q-i-1)}(t). \end{array} \tag{18}$$

In order to eliminate $x(t)$ and $d(t)$ from (18) we use the derivative of $v(t)$ and $y(t)$. Indeed, from $y(t) = Cx(t)$ we get, for $k \in \mathbb{N}^*$:

$$\begin{array}{rl} CA^kx(t) = & y^{(k)}(t) - \sum_{i=0}^{k-1} CA^iBu^{(k-1-i)}(t) \\ & - \sum_{i=0}^{k-1} CA^iDd^{(k-1-i)}(t). \end{array}$$

Moreover, taking into account the expression of the matrices LA^iD from (15) and the terms $LA^ix(t)$ from (17), for $i \in [\![0; q-1]\!]$, it leads to:

$$v^{(q)}(t) = \sum_{i=0}^{q} \Gamma_iy^{(i)}(t) + \sum_{i=0}^{q-1} \Lambda_iv^{(i)}(t) + \sum_{i=0}^{q-1} \Phi_iu^{(i)}(t), \tag{19}$$

where, for $i \in [\![0; q-2]\!]$:

$$\begin{array}{rl} \Phi_i = & \Bigg[LA^{q-1-i} - \sum_{j=i+1}^{q} \Gamma_jCA^{j-i-1} \\ & - \sum_{j=i+1}^{q-1} \Lambda_jLA^{j-i-1} \Bigg] B, \end{array} \tag{20}$$

and $\Phi_{q-1} = [L - \Gamma_qC]\, B$.

A realization of the input-output differential Equation (19) [24], [28], [36] leads to the observer:

$$\left\{ \begin{array}{rl} \dot{z}(t) = & Fz(t) + \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_{q-1} \end{bmatrix} u(t) \\ & + \begin{bmatrix} \Gamma_0 + \Lambda_0\Gamma_q \\ \Gamma_1 + \Lambda_1\Gamma_q \\ \vdots \\ \Gamma_{q-1} + \Lambda_{q-1}\Gamma_q \end{bmatrix} y(t), \\ \hat{v}(t) = & \begin{bmatrix} 0_l & \cdots & 0_l & I_l \end{bmatrix} z(t) + \Gamma_qy(t), \end{array} \right. \tag{21}$$

with:

$$F = \begin{bmatrix} 0_l & \cdots & \cdots & 0_l & \Lambda_0 \\ I_l & \ddots & & \vdots & \Lambda_1 \\ 0_l & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0_l & \Lambda_{q-2} \\ 0_l & \cdots & 0_l & I_l & \Lambda_{q-1} \end{bmatrix},$$

and the observer design is complete.

Remark 8. When the Hurwitz condition is satisfied, it is demonstrated that (21) is an asymptotic observer of the linear functional $Lx(t)$. Otherwise, it becomes necessary to increase the integer q and to do again the design procedure with a higher order [13], [37]. Note that in case of detectable systems, by increasing step by step the integer q, we are led, at least, to the well-known reduced order state observer.

Remark 9. When Σ_q is of full row rank, the eigenvalues of the matrix F are fixed. In the opposite, when $\mathrm{rank}\,(\Sigma_q) = \rho_q < r_q$, $r_q - \rho_q$ degrees of freedom can be defined to place some of the eigenvalues of F. It leads to the possibility of increasing the integer q step by step to choose the decay rate of the designed observer.

Remark 10. From the expressions (20) for the matrices Φ_i and the relationship $G = TB$ (6), it can be deduced, as it has been indicated for LFO in [28], that

$$T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{q-1} \\ T_q \end{bmatrix},$$

where, for $j \in [\![0; q-1]\!]$:

$$T_j = LA^{q-j} - \sum_{i=j}^{q-1} \Lambda_iLA^{i-j} - \sum_{i=j}^{q} \Gamma_iCA^{i-j},$$

and $T_q = L - \Gamma_q C$. From (15), it is easy to verify that $TD = 0$.

V. Illustrative example

According to [19], let us consider the system (1) and the single functional (3) defined by:

$$
\begin{aligned}
A &= \begin{bmatrix} -2.51 & 0.33 & 0.68 & 1.12 & -0.25 \\ 0.14 & -0.23 & -0.31 & 0.91 & 0.36 \\ 0.51 & -1.18 & 0.41 & 0.63 & -0.77 \\ 0.22 & 0.33 & 0.46 & 0.65 & -0.77 \\ 0.23 & 0.33 & 3.97 & 0.06 & 0.69 \end{bmatrix}, \\
B &= \begin{bmatrix} 0.43 \\ 0.00 \\ 0.92 \\ 1.20 \\ -1.27 \end{bmatrix}, \ D = \begin{bmatrix} 1.00 & 0.00 \\ -3.00 & -1.00 \\ 0.00 & 0.50 \\ 0.45 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}, \\
C &= \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.60 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \end{bmatrix}, \\
L &= \begin{bmatrix} 2.00 & 0.00 & 0.00 & 9.00 & 0.30 \end{bmatrix}.
\end{aligned} \tag{22}
$$

In [19] it is remarked that the system matrix has two detectable invariant zeros at $-2.16 \pm 2.02i$. Consequently, they will appear as poles of the observer. The following cases illustrate our proposed design of LUIFO. In a first attempt, a minimal order observer will be obtained. The second attempt illustrates the introduction of degrees of freedom to change some of the poles of the observer. From the invariant zeros, it is assumed that the minimal order of an observer is 2. Let us mention here that in [19], a third-order observer whose eigenvalues are $\{-5,\ -2.16 \pm 2.02i\}$ has been designed.

A. Design of a minimal-order observer

As $\text{rank}(\Sigma_1) = 5$ and $\text{rank}\left(\begin{bmatrix} \Sigma_1 & \\ LA & LD \end{bmatrix}\right) = 6$, as expected, a first-order minimum observer cannot be designed.

We have:

$$
\Sigma_2 = \left[\begin{array}{ccccc|cccc} 1.00 & 0.00 & 0.00 & 0.00 & 0.60 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 2.00 & 0.00 & 0.00 & 9.00 & 0.30 & 0.00 & 0.00 & 0.00 & 0.00 \\ -2.37 & 0.53 & 3.06 & 1.16 & 0.16 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.22 & 0.33 & 0.46 & 0.65 & -0.77 & 0.45 & 0.00 & 0.00 & 0.00 \\ -2.97 & 3.73 & 6.69 & 8.11 & -7.22 & 6.05 & 0.00 & 0.00 & 0.00 \\ 7.88 & -4.08 & 0.66 & 0.51 & -2.35 & -3.44 & 1.00 & 1.00 & 0.00 \\ -0.31 & -0.59 & -2.52 & 1.21 & -1.32 & -0.48 & -0.10 & 0.45 & 0.00 \end{array}\right],
$$

and:

$$
LK_2 = \begin{bmatrix} 11.51 & -9.44 & -25.38 & 9.12 & -14.29 & -10.51 & -0.38 & 6.05 & 0.00 \end{bmatrix}.
$$

We get $\text{rank}(\Sigma_2) = 8$ and $\text{rank}\left(\begin{bmatrix} \Sigma_2 \\ LA_2 \end{bmatrix}\right) = 8$. Thus, $q = 2$ is the smallest integer that has to be considered. Consequently, as $l = 1$, a minimal second-order observer can be considered as a candidate observer. From the unique solution $X = LK_2\Sigma_2^{\dagger}$, we obtain $\Lambda_0 = -8.77$ and $\Lambda_1 = -4.32$. That yields to:

$$
F = \begin{bmatrix} 0 & -8.77 \\ 1 & -4.32 \end{bmatrix}.
$$

As expected the eigenvalues of F are $(-2.16 \pm 2.02i)$.

Moreover, to design the second-order observer we get, from $X = LK_2\Sigma_2^{\dagger}$:

$$
\begin{aligned}
\Gamma_0 &= \begin{bmatrix} 13.24 & 75.14 \end{bmatrix}, \\
\Gamma_1 &= \begin{bmatrix} 3.96 & 44.38 \end{bmatrix}, \\
\Gamma_2 &= \begin{bmatrix} 0.78 & 11.70 \end{bmatrix}.
\end{aligned}
$$

Using (21), the following matrices of the observer are deduced:

$$
G = \begin{bmatrix} -8.25 \\ -2.50 \end{bmatrix}, \ H = \begin{bmatrix} 6.36 & -27.55 \\ 0.57 & -6.21 \end{bmatrix}
$$

$$
P = \begin{bmatrix} 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 0.78 & 11.70 \end{bmatrix}.
$$

The observer design is complete. A simulation result with different initial conditions is displayed in figure 1. Initial conditions of the system (resp. the observer) are set to $x(t = 0) = [0\,0\,0\,0\,0]^\top$ (resp. $z(t = 0) = [500\,200]^\top$). Identical inputs than those in [19] are applied for $t \geq 0$: $u(t) = 0.2 + e^{-0.4t}\cos(2t)$, $d_1(t) = 0.1 + 0.2e^{-0.1\sin(t)}\tanh(2t)$ and $d_2(t) = 2$. It can be seen that the observer output converges to $v(t) = Lx(t)$ with the decay rate given by the the eigenvalues of F. It can be noticed that in [19], it is found that the order of unknown input functional observer is q=3. With our procedure we have obtained a minimal second-order LUIFO.

B. Design of a third-order observer

In this section the design of a LUIFO of a third-order observer with $q = 3$ is detailed. The appearance of a degree of freedom in the design procedure is here illustrated.

The size of Σ_3 is (11,11) and we obtain $\text{rank}(\Sigma_3) = \text{rank}\left(\begin{bmatrix} \Sigma_3 \\ LK_3 \end{bmatrix}\right) = 10$. Consequently, there exist 10 rows in Σ_3 that are independent. Let us consider Σ_3^*, where the component $L_{3,2}$ of Σ_3 has been eliminated:

$$
\Sigma_3^* = \begin{bmatrix} C & 0 & 0 & 0 \\ L & 0 & 0 & 0 \\ CA & CD & 0 & 0 \\ LA & LD & 0 & 0 \\ CA^2 & CAD & CD & 0 \\ CA^3 & CA^2D & CAD & CD \end{bmatrix}.
$$

Thus, Σ_3^* has 10 rows and $\text{rank}(\Sigma_3^*) = 10$. On the one hand, (12) can be written as:

$$
LK_3 = \begin{bmatrix} \Gamma_0 & \Lambda_0 & \Gamma_1 & \Lambda_1 & \Gamma_2 & \Gamma_3 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \Sigma_3^* \\ L_{3,2} \end{bmatrix}. \tag{23}
$$

that is non unique. As the row $L_{3,2}$ has been eliminated from Σ_3, Λ_2 appears as a degree of freedom. On the other hand, we get:

$$LK_3 = \begin{bmatrix} \Gamma_{30} & \Lambda_{30} & \Gamma_{31} & \Lambda_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix} \Sigma_3^*, \quad (24)$$

$$L_{3,2} = \begin{bmatrix} \Pi_0 & \Delta_0 & \Pi_1 & \Delta_1 & \Pi_2 & \Pi_3 \end{bmatrix} \Sigma_3^*, \quad (25)$$

where the matrices Γ_{ij}, Λ_{ij}, Π_i and Δ_i are unique. We have, from (23) and (25):

$$\begin{aligned} LK_3 = & \begin{bmatrix} \Gamma_0 & \Lambda_0 & \Gamma_1 & \Lambda_1 & \Gamma_2 & \Gamma_3 \end{bmatrix} \Sigma_3^* \\ & + \Lambda_2 \begin{bmatrix} \Pi_0 & \Delta_0 & \Pi_1 & \Delta_1 & \Pi_2 & \Pi_3 \end{bmatrix} \Sigma_3^*. \end{aligned} \quad (26)$$

Consequently, identifying (24) and (26), it yields:

$$\begin{cases} \Gamma_0 = \Gamma_{30} - \Lambda_2 \Pi_0, \\ \Lambda_0 = \Lambda_{30} - \Lambda_2 \Delta_0, \\ \Gamma_1 = \Gamma_{31} - \Lambda_2 \Pi_1, \\ \Lambda_1 = \Lambda_{31} - \Lambda_2 \Delta_1, \\ \Gamma_2 = \Gamma_{32} - \Lambda_2 \Pi_2, \\ \Gamma_3 = \Gamma_{33} - \Lambda_2 \Pi_3, \end{cases} \quad (27)$$

where Λ_2 is arbitrary. In this example, we get:

$$\begin{aligned} \Gamma_{30} &= \begin{bmatrix} -57.28 & -324.98 \end{bmatrix}, \Lambda_{30} = 37.95, \\ \Gamma_{31} &= \begin{bmatrix} -3.88 & -116.79 \end{bmatrix}, \Lambda_{31} = 9.92, \\ \Gamma_{32} &= \begin{bmatrix} 0.57 & -6.21 \end{bmatrix}, \Gamma_{33} = \begin{bmatrix} 0.78 & 11.70 \end{bmatrix}, \\ \Pi_0 &= \begin{bmatrix} 13.24 & 75.14 \end{bmatrix}, \Delta_0 = -8.77, \\ \Pi_1 &= \begin{bmatrix} 3.96 & 44.38 \end{bmatrix}, \Delta_1 = -4.32, \\ \Pi_2 &= \begin{bmatrix} 0.78 & 11.70 \end{bmatrix}, \Pi_3 = \begin{bmatrix} 0.00 & 0.00 \end{bmatrix}. \end{aligned}$$

Then, the matrix F expressed as:

$$F = \begin{bmatrix} 0 & 0 & \Lambda_{30} - \Lambda_2 \Delta_0 \\ 1 & 0 & \Lambda_{31} - \Lambda_2 \Delta_1 \\ 0 & 1 & \Lambda_2 \end{bmatrix}.$$

The eigenvalues of the matrix F are the roots of the characteristic polynomial $p_F(\lambda) = \lambda^3 - \Lambda_2 \lambda^2 - \Lambda_1 \lambda - \Lambda_0$ which depends on the parameter Λ_2. Here we obtain, with $\Lambda_0 = 37.95 + 8.77\Lambda_2$ and $\Lambda_1 = 9.92 + 4.32\Lambda_2$:

$$p_F(\lambda) = (\lambda^2 + 4.32\lambda + 8.77)(\lambda - 4.32 - \Lambda_2).$$

Due to invariant zeros, Λ_2 can place only one eigenvalue for F. In order to compare with [19] where the roots of $p_F(\lambda)$ are $(-2.16 \pm 2.02i, -5)$, we take $\Lambda_2 = -9.32$, which leads to $\Lambda_1 = -30.34$ and $\Lambda_0 = -43.78$. For these values we get the third-order Luenberger observer defined by

$$\begin{aligned} F &= \begin{bmatrix} 0 & 0 & -43.78 \\ 1 & 0 & -30.34 \\ 0 & 1 & -9.32 \end{bmatrix}, G = \begin{bmatrix} -41.77 \\ -26.04 \\ -2.50 \end{bmatrix} \\ H &= \begin{bmatrix} 31.81 & -137.79 \\ 9.21 & -58.64 \\ 0.57 & -6.22 \end{bmatrix}, \\ P &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, V = \begin{bmatrix} 0.78 & 11.70 \end{bmatrix}. \end{aligned}$$

Simulation results are displayed in figure 1 where the second and third order observer outputs are compared to the functional to be observed. Inputs and initial conditions of the system are identical for the two simulations. Initial conditions of the third-order observer are set to $z(t=0) = [500 \ -300 \ 200]^\top$.

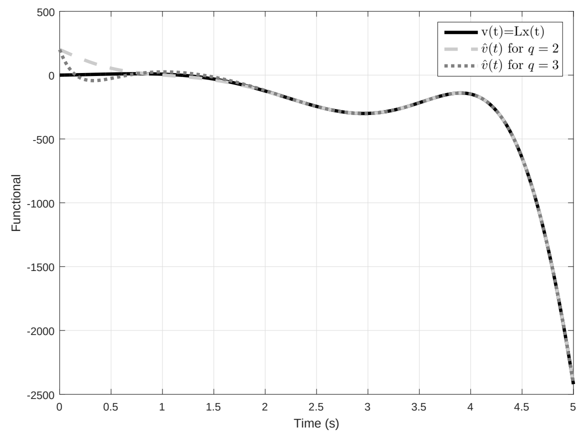

Figure 1. Simulation results for the second and third-order observers.

As shown in figure 2, the tracking property of the observers is ensured by the proposed design methods.

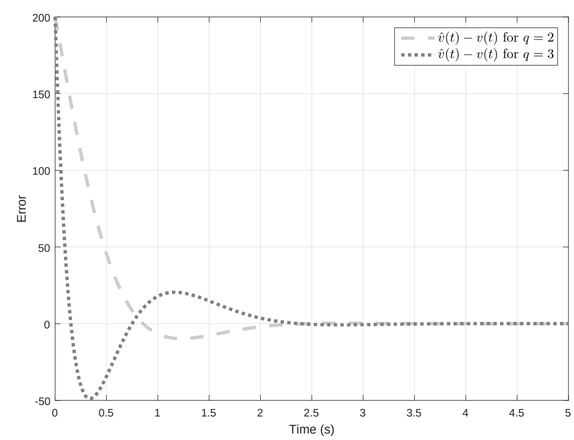

Figure 2. Simulation results of the estimation error for the second and third-order observers.

As it can be seen in figure 3, the value of the parameter Λ_2 influences the decay rate of the error. A comparison is performed between third-order observer with $\Lambda_2 = -9.32$ and $\Lambda_2 = -24.32$ leading to eigenvalues of the third-order observer of $(-2.16 \pm 2.02i, -20)$. As predicted, the response time of the estimation error is reduced by choosing a parameter Λ_2 leading to an eigenvalue with a lower negative real part.

VI. Conclusion

This paper presents a new and direct design procedure to obtain a possible minimal unknown input functional observer

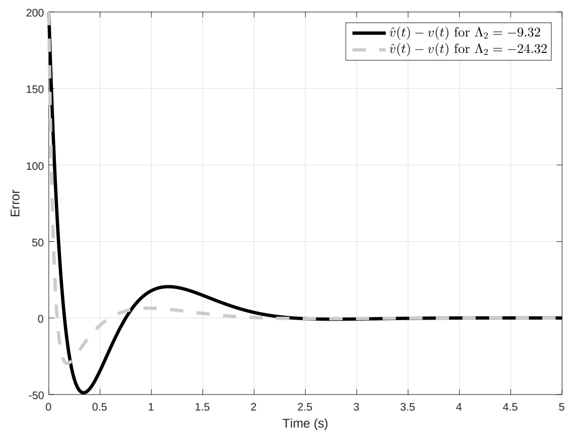

Figure 3. Simulation results of the estimation error for the third-order observers.

for LTI systems. Necessary and sufficient conditions for the existence of a stable LUIFO have been detailed. The proposed procedure is based on linear algebraic operations in a state space setting and only needs to solve linear equations. The order of the obtained observer can be increased with some degrees of freedom to choose the dynamic response of the observed functional. Finally, the proposed procedure could be extended to linear time-varying systems.

We demonstrated in this paper that for the observation of a multi-functional $Lx(t)$, q is the smallest integer that fulfils the Theorem 5. From the proposed design procedure, we can assert that the minimal order ν of the observer is such that $\nu \leq ql$. Nevertheless, to obtain a minimal stable observer and determining ν as small as possible is a challenging problem largely beyond the scope of the proposed paper. This specific point will be the topics of our future works.

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