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Abstract
In this paper we obtain the first non-trivial lower bound on the number of disjoint empty convex pentagons in planar points sets. We show that the number of disjoint empty convex pentagons in any set of n points in the plane, no three on a line, is at least $\lfloor 5n/47 \rfloor$. This bound can be further improved to $(3n-1)/28$ for infinitely many n.

Keywords
convex hull, discrete geometry, empty convex polygons, Erdős-Szekeres theorem, pentagons

Disciplines
Applied Mathematics | Business | Mathematics | Statistics and Probability

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Disjoint Empty Convex Pentagons in Planar Point Sets

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Abstract. Harborth [*Elemente der Mathematik*, Vol. 33 (5), 116–118, 1978] proved that every set of 10 points in the plane, no three on a line, contains an empty convex pentagon. From this it follows that the number of disjoint empty convex pentagons in any set of n points in the plane is least $\lfloor \frac{n}{10} \rfloor$. In this paper we prove that every set of 19 points in the plane, no three on a line, contains two disjoint empty convex pentagons. We also show that any set of $2m+9$ points in the plane, where m is a positive integer, can be subdivided into three disjoint convex regions, two of which contains m points each, and another contains a set of 9 points containing an empty convex pentagon. Combining these two results, we obtain non-trivial lower bounds on the number of disjoint empty convex pentagons in planar points sets. We show that the number of disjoint empty convex pentagons in any set of n points in the plane, no three on a line, is at least $\lfloor \frac{5n}{47} \rfloor$. This bound has been further improved to $\frac{3n-1}{28}$ for infinitely many n.

Keywords. Convex hull, Discrete geometry, Empty convex polygons, Erdős-Szekeres theorem, Pentagons.

1 Introduction

The origin of the problems concerning the existence of empty convex polygons goes back to the famous theorem due to Erdős and Szekeres [10]. It states that for every positive integer $m \geq 3$, there exits a smallest integer $ES(m)$, such that any set of n points ($n \geq ES(m)$) in the plane, no three on a line, contains a subset of m points which lie on the vertices of a convex polygon. Evaluating the exact value of $ES(m)$ is a long standing open problem. A construction due to Erdős [11] shows that $ES(m) \geq 2^{m-2}+1$, which is also conjectured to be sharp. It is known that $ES(4) = 5$ and $ES(5) = 9$ [18]. Following a long computer search, Szekeres and Peters [28] recently proved that $ES(6) = 17$. The value of $ES(m)$ is unknown for all $m > 6$. The best known upper bound for $m \geq 7$ is due to Tóth and Valtr [29] - $ES(m) \leq \binom{2m-5}{m-3}+1$. For a more detailed description of the Erdős-Szekeres theorem and its numerous ramifications see the surveys by Bárány and Károlyi [3] and Morris and Soltan [24].

In 1978, Erdős [9] asked whether for every positive integer k, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains k points which lie on the vertices of a convex polygon whose interior contains no points of the set. Such a subset is called an *empty convex k-gon* or a *k-hole*. Esther Klein showed $H(4) = 5$ and Harborth [13] proved that $H(5) = 10$. Horton [14] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that $H(k)$ does not exist for $k \geq 7$. Recently, after a long wait, the existence of $H(6)$ has been proved by Gerken [12] and independently by Nicolás [25]. Later Valtr [32] gave a simpler version of Gerken's proof. For results regarding the number of k-holes in planar point sets and other related problems see [2–4, 8, 27]. Existence of a hole of any fixed size in sufficiently large point sets, with some additional restrictions on the point sets, has been studied by Károlyi et al. [19, 20], Kun and Lippner [22], and Valtr [31].

Two empty convex polygons are said to be *disjoint* if their convex hulls do not intersect. For positive integers $k \leq \ell$, denote by $H(k, \ell)$ the smallest integer such that any set of $H(k, \ell)$ points in the plane, no three on a line, contains both a k-hole and a ℓ-hole which are disjoint. Clearly, $H(3, 3) = 6$ and Horton's result [14] implies that $H(k, \ell)$ does not exist for all $\ell \geq 7$. Urabe [30] showed that $H(3, 4) = 7$, while Hosono and Urabe [17] showed that $H(4, 4) = 9$. Hosono and Urabe [15] also proved that $H(3, 5) = 10$, $12 \leq H(4, 5) \leq 14$, and $16 \leq H(5, 5) \leq 20$. The results $H(3, 4) = 7$ and $H(4, 5) \leq 14$ were later reconfirmed by Wu and Ding [33]. Using the computer-aided order-type enumeration method, Aichholzer et al. [1] proved that every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole. Recently, this result was proved geometrically by Bhattacharya and Das [5, 6]. Using this Ramsey-type result, Hosono and Urabe [16] proved that $H(4, 5) \leq 13$, which was later tightened to $H(4, 5) = 12$ by Bhattacharya and Das [7]. Hosono and Urabe [16] have also improved the lower bound on $H(5, 5)$ to 17.

The problems concerning disjoint holes was, in fact, first studied by Urabe [30] while addressing the problem of partitioning of planar point sets. For any set S of points in the plane, denote by $CH(S)$ the *convex hull* of S. Given a set S of n points in the plane, no three on a line, a *disjoint convex partition* of S is a partition of S into subsets $S_1, S_2, \ldots S_t$, with $\sum_{i=1}^{t} |S_i| = n$, such that for each $i \in \{1, 2, \ldots, t\}$, $CH(S_i)$ forms a $|S_i|$-gon and $CH(S_i) \cap CH(S_j) = \emptyset$, for any pair of indices i, j. Observe that in any disjoint convex partition of S, the set S_i forms a $|S_i|$-hole and the holes formed by the sets S_i and S_j are disjoint for any pair of distinct indices i, j. If $F(S)$ denote the minimum number of disjoint holes in any disjoint convex partition of S, then $F(n) = \max_S F(S)$, where the maximum is taken over all sets S of n points, is called the *disjoint convex partition number* for all sets of fixed size n. The disjoint convex partition number $F(n)$ is bounded by $\lceil \frac{n-1}{4} \rceil \leq F(n) \leq \lceil \frac{5n}{18} \rceil$. The lower bound is by Urabe [30] and the upper bound by Hosono and Urabe [17]. The proof of the upper bound uses the fact that every set of 7 points in the plane contains a 3-hole and a disjoint 4-hole. Later, Xu and Ding [34] improved the lower bound to $\lceil \frac{n+1}{4} \rceil$. Recently, Aichholzer et al. [1] introduced the notion pseudo-convex partitioning of planar point sets, which extends the concept partitioning, in the sense, that they allow both convex polygons and pseudo-triangles in the partition.

Urabe [17] also defined the function $F_k(n)$ as the minimum number of pairwise disjoint k-holes in any n-element point set. If $F_k(S)$ denotes the number of k-holes in a disjoint partition of S, then $F_k(n) = \min_S\{\max_{\pi_d} F_k(S)\}\}$, where the maximum is taken over all disjoint partitions π_d of S, and the minimum is taken over all sets S with $|S| = n$. Hosono and Urabe [17] proved any set of 9 points, no three on a line, contains two disjoint 4-holes. They also showed any set of $2m+4$ points can be divided into three disjoint convex regions, one containing a 4-hole and the others containing m points each. Combining these two results they proved $F_4(n) \geq \lfloor \frac{5n}{22} \rfloor$. This bound can be improved to $(3n-1)/13$ for infinitely many n.

The problem, however, appears to be much more complicated in the case of disjoint 5-holes. Harborth's result [13] implies $F_5(n) \geq \lfloor \frac{n}{10} \rfloor$, which, to the best our knowledge, is the only known lower bound on this number. A construction by Hosono and Urabe [16] shows that $F_5(n) \leq 1$ if $n \leq 16$. In general, it is known that $F_5(n) < n/6$ [3]. Moreover, Hosono and Urabe [17] states the impossibility of an analogous result for 5-holes with $2m+5$ points.

In this paper, following a couple of new results for small point sets, we prove non-trivial lower bounds on $F_5(n)$. At first, we show that every set of 19 points in the plane, no three on a line, contains two disjoint 5-holes. In other words, this implies, $F_5(19) \geq 2$ or $H(5, 5) \leq 19$. Drawing parallel from the result of Hosono and Urabe [17], we also show that any set of

$2m + 9$ points in the plane, where m is a positive integer, can be subdivided into three disjoint convex regions, two of which contains m points each, and the third one is a set of 9 points containing a 5-hole. Combining these two results, we prove $F_5(n) \geq \lfloor \frac{5n}{47} \rfloor$. This bound can be further improved to $\frac{3n-1}{28}$ for infinitely many n. The proofs rely on a series of results concerning the existence of 5-holes in planar point sets having less than 10 points.

The paper is organized as follows. The results proving the existence of 5-holes in point sets having less than 10 points, and the characterization of 9-point sets not containing any 5-hole are presented in Section 3. In Section 4, we give the formal statements of our main results and use them to prove lower bounds on $F_5(n)$. The proofs of the 19-point result and the $2m + 9$-point partitioning theorem are presented in Sections 5 and 6, respectively. In Section 2 we introduce notations and definitions and in Section 7 we summarize our work and provide some directions for future work.

2 Notations and Definitions

We first introduce the definitions and notations required for the remainder of the paper. Let S be a finite set of points in the plane in general position, that is, no three on a line. Denote the convex hull of S by $CH(S)$. The boundary vertices of $CH(S)$, and the points of S in the interior of $CH(S)$ are denoted by $\mathcal{V}(CH(S))$ and $\mathcal{I}(CH(S))$, respectively. A region R in the plane is said to be empty in S, if R contains no elements of S. A point $p \in S$ is said to be *k-redundant* in a subset T of S, if there exists a k-hole in $T \backslash \{p\}$.

By $\mathcal{P} = p_1 p_2 \ldots p_k$ we denote a convex k-gon with vertices $p_1, p_2, \ldots, p_k$ taken in the counter-clockwise order. $\mathcal{V}(\mathcal{P})$ denotes the set of vertices of $\mathcal{P}$ and $\mathcal{I}(\mathcal{P})$ the interior of $\mathcal{P}$.

The *j-th convex layer* of S, denoted by $L\{j, S\}$, is the set of points that lie on the boundary of $CH(S \backslash \{\bigcup_{i=1}^{j-1} L\{i, S\}\})$, where $L\{1, S\} = \mathcal{V}(CH(S))$. If $p,\ q \in S$ are such that pq is an edge of the convex hull of the j-th layer, then the open halfplane bounded by the line pq and not containing any point of $S \backslash \{\bigcup_{i=1}^{j-1} L\{i, S\}\}$ will be referred to as the *outer* halfplane induced by the edge pq.

For any three points $p, q, r \in S$, $\mathcal{H}(pq, r)$ (respectively $\mathcal{H}_c(pq, r)$) denotes the open (respectively closed) halfplane bounded by the line pq containing the point r. Similarly, $\overline{\mathcal{H}}(pq, r)$ (respectively $\overline{\mathcal{H}}_c(pq, r)$) is the open (respectively closed) halfplane bounded by pq not containing the point r.

Moreover, if $p, q, r \in S$ is such that $\angle rpq < \pi$, then $Cone(rpq)$ is the set of points in $\mathbb{R}^2$ which lies in the interior of the angular domain $\angle rpq$. A point $s \in Cone(rpq) \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in $Cone(rpq)$ if $Cone(spq)$ is empty in S. In general, whenever we have a convex region R, we think of R as the set of points in $\mathbb{R}^2$ which lies in the region R. Thus, for any convex region R a point $s \in R \cap S$ is called the *nearest angular neighbor* of $\overrightarrow{pq}$ in R if $Cone(spq) \cap R$ is empty in S. More generally, for any positive integer k, a point $s \in S$ is called the *k-th angular neighbor* of $\overrightarrow{pq}$ whenever $Cone(spq) \cap R$ contains exactly $k - 1$ points of S in its interior. Also, for any convex region R, the point $s \in S$, which has the shortest perpendicular distance to the line pq, $p, q \in S$, is called the *nearest neighbor* of pq in R.

3 5-Holes With Less Than 10 Points

We begin by restating a well known result regarding the existence of 5-holes in planar point sets.

Lemma 1. *[23] Any set of points in general position containing a convex hexagon, contains a 5-hole.*

From the Erdős Szekeres theorem, we know that every sufficiently large set of points in the plane in general position, contains a convex hexagon. Lemma 1 therefore ensures that every sufficiently large set of points in the plane contains a 5-hole. Harborth [13] showed that a minimum of 10 points are required to ensure the existence of a 5-hole, that is $H(5) = 10$. This means, the existence of a 5-hole is not guaranteed if we have less than 10 points in the plane [13].

In the following, we prove two lemmas where we show, if the convex hull of the point set is not a triangle, a 5-hole can be obtained in less than 10 points.

Lemma 2. *If Z is a set of points in the plane in general position, with $|\mathcal{V}(CH(Z))| = 5$ and $|\mathcal{I}(CH(Z))| \geq 2$, then Z contains a 5-hole.*

Proof. To begin with suppose there are only two points y_1 and y_2 in $\mathcal{I}(CH(Z))$. The extended straight line y_1y_2 divides the plane into two halfplanes, one of which must contain at least three points of $\mathcal{V}(CH(Z))$. These three points along with the points y_1 and y_2 forms a 5-hole (Figure 1(a)).

Next suppose, there are three points y_1, y_2, and y_3 in $\mathcal{I}(CH(Z))$. Consider the partition of the exterior of $y_1y_2y_3$ into disjoint regions R_i as shown in Figure 1(b). Let $|R_i|$ denote the number of points of $\mathcal{V}(CH(Z))$ in region R_i. If Z does not contain a 5-hole, we must have:

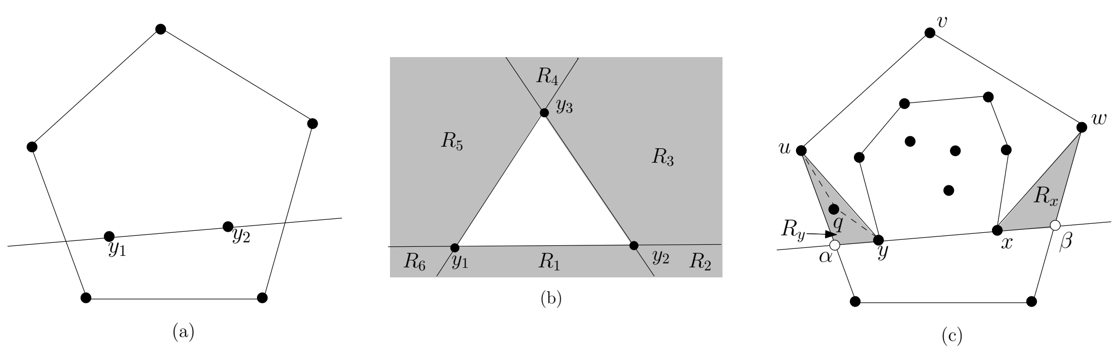


Fig. 1. Illustrations for the proof of Lemma 2.

$$|R_1| \leq 1, \qquad |R_3| \leq 1, \qquad |R_5| \leq 1, \tag{1}$$

$$\begin{aligned} |R_6| + |R_1| + |R_2| &\leq 2, \\ |R_2| + |R_3| + |R_4| &\leq 2, \\ |R_4| + |R_5| + |R_6| &\leq 2. \end{aligned} \tag{2}$$

Adding the inequalities of (2) and using the fact $|\mathcal{V}(CH(Z))| = 5$ we get $|R_2| + |R_4| + |R_6| \leq 1$. On adding this inequality with those of (1) we finally get $\sum_{i=1}^{6} |R_i| \leq 4 < 5 = |\mathcal{V}(CH(Z))|$, which is a contradiction.

Finally, suppose $|\mathcal{I}(CH(Z))| = k \geq 4$. Let $x, y \in Z$ be such that xy is an edge of $CH(\mathcal{I}(CH(Z)))$ and $z \in \mathcal{I}(CH(Z))$ be any other point. If $|\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy, z)| \geq 3$, the points x and y together with the three points of $\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy, z)$ form a 5-hole.

When $|\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy,z)| = 1$, the 4 points in $\mathcal{V}(CH(Z)) \cap \mathcal{H}(xy,z)$ along with the points x and y form a convex hexagon, which contains a 5-hole from Lemma 1. Otherwise, $|\mathcal{V}(CH(Z)) \cap \overline{\mathcal{H}}(xy,z)| = 2$. Denote by α, β the points where the extended straight line passing through the points x and y intersects the boundary of $CH(Z)$, as shown in Figure 1(c). Let $R_x = \mathcal{I}(wx\beta)$ and $R_y = \mathcal{I}(uy\alpha)$ be the two triangular regions generated inside $CH(Z)$ in the halfplane $\mathcal{H}(xy,z)$. If any one of R_x or R_y is non-empty in Z, the nearest neighbor q of the line uy (or wx) in R_y (or R_x) forms the convex hexagon $uvwxyq$ (or $xyuvwq$), which contains an 5-hole from Lemma 1. Therefore, assume that both R_x and R_y are empty in Z. Observe that the number of points of Z inside $uvwxy$ is exactly two less than the number of points of Z inside $CH(Z)$. By applying this argument repeatedly on the modified pentagon we finally get a 5-hole or a convex pentagon with two or three interior points. □

Lemma 3. *If Z is a set of points in the plane in general position, with $|\mathcal{V}(CH(Z))| = 4$ and $|\mathcal{I}(CH(Z))| \geq 5$, then Z contains a 5-hole.*

Proof. Let $CH(Z)$ be the polygon $p_1p_2p_3p_4$. If some outer halfplane induced by an edge of $CH(\mathcal{I}(CH(Z)))$ contains more than two points of $\mathcal{V}(CH(Z))$, then Z contains a 5-hole. Therefore, we assume

Assumption 1 *Every outer halfplane induced by the edges of $CH(\mathcal{I}(CH(Z)))$ contains at most two points of $\mathcal{V}(CH(Z))$.*

To begin with suppose $|\mathcal{I}(CH(Z))| = 5$. If $|\mathcal{V}(CH(\mathcal{I}(CH(Z))))| = 5$, we are done. Thus, the convex hull of the second layer of Z is either a quadrilateral or a triangle. Let $CH(\mathcal{I}(CH(Z)))$ be the polygon $z_1z_2 \ldots z_k$, where k is either 3 or 4. This means $3 \leq |L\{2,Z\}| \leq 4$, and we have the following two cases:

Case **1:** $|L\{2,Z\}| = 4$. Let $x \in L\{3,Z\}$ and w. l. o. g. assume $x \in \mathcal{I}(z_1z_3z_4) \cap Z$. Consider the partition of the exterior of the quadrilateral $z_1z_2z_3z_4$ into disjoint regions R_i as shown in Figure 2(a). Let $|R_i|$ denote the number of points of $\mathcal{V}(CH(Z))$ in the region R_i. If there exists a point $p_i \in R_3 \cap Z$, then $p_iz_2z_1z_3x$ forms a 5-hole. Therefore, assume that $|R_3| = 0$, and similarly, $|R_5| = 0$. Moreover, if $|R_1| + |R_2| \geq 2$, $((R_1 \cup R_2) \cap \mathcal{V}(CH(Z))) \cup \{z_1, z_4, x\}$ contains a 5-hole. This implies, $|R_1| + |R_2| \leq 1$ and similarly $|R_6| + |R_7| \leq 1$. Therefore, $|R_4| \geq 2$ and Assumption 1 implies that $|R_4| = 2$. This implies that the set of points in $(R_4 \cap Z) \cup \{z_1, z_3, z_4\}$ forms a convex pentagon with exactly two interior points, which then contains a 5-hole from Lemma 2.

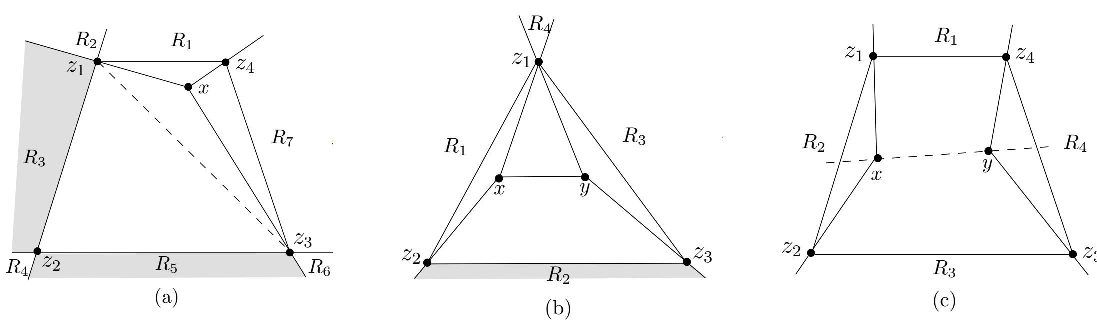


Fig. 2. Illustrations for the proof of Lemma 3: (a) $|L\{2,Z\}| = 4$, (b) $|L\{2,Z\}| = 3$, (c) Illustration for the proof of Theorem 1.

Case **2:** $|L\{2,Z\}| = 3$. Let $L\{3,Z\} = \{x,y\}$. Consider the partition of the exterior of $CH(\mathcal{I}(CH(Z)))$ as shown in Figure 2(b). Observe that Z contains a 5-hole unless $|R_2| = 0$, $|R_1| \leq 1$, and $|R_3| + |R_4| \leq 1$. This implies that $\sum_{i=1}^{4} |R_i| \leq 3 < 4 = |\mathcal{V}(CH(Z))|$, which is a contradiction.

Now, consider $|\mathcal{I}(CH(Z))| > 5$. W.l.o.g. assume that $\mathcal{I}(p_1p_2p_3) \cap Z$ is non-empty. If $|CH(Z \backslash \{p_2\})| \geq 5$, a 5-hole in $Z \backslash \{p_2\}$ is ensured from Lemma 1 and Lemma 2. Otherwise, $CH(Z \backslash \{p_2\})$ is a quadrilateral with exactly one less point of Z in its interior than $CH(Z)$. By repeating this process we finally get a convex quadrilateral with exactly 5 points in its interior, thus reducing the problem to *Case* 1 and *Case* 2. □

From the argument at the end of the proof of the previous lemma, it follows that if $|\mathcal{I}(CH(Z))| \geq 6$, then either p_1 or p_3 is 5-redundant in Z. Similarly, either p_2 or p_4 is 5-redundant in Z. Therefore, we have the following corollary:

Corollary 1. *Let Z be a set of points in the plane in general position, such that $CH(Z)$ is the polygon $z_1z_2z_3z_4$, and $|\mathcal{I}(CH(Z))| \geq 6$. Then the following statements hold:*

(i) If for some $i \in \{1,2,3,4\}$, $\mathcal{I}(z_{i-1}z_iz_{i+1}) \cap Z$ is non-empty, then z_i is 5-redundant in Z, where the indices are taken modulo 4.

(ii) At least one of the vertices corresponding to any diagonal of $CH(Z)$ is 5-redundant in Z. □

Moreover, by combining Lemmas 1, 2, and 3, the following result about the existence of 5-holes is immediate.

Corollary 2. *Any set Z of 9 points in the plane in general position, with $|\mathcal{V}(CH(Z))| \geq 4$, contains a 5-hole.* □

Two sets of points, S_1 and S_2, in general position, having the same number of points belong to the same *layer equivalence class* if the number of layers in both the point sets is the same and $|L\{k,S_1\}| = |L\{k,S_2\}|$, for all k. A set S of points with 3 different layers belongs to the layer equivalence class $L\{a,b,c\}$ whenever $|L\{1,S\}| = a$, $|L\{2,S\}| = b$, and $|L\{3,S\}| = c$, where a,b,c are positive integers.

It is known that there exist sets with 9 points without any 5-hole, belonging to the layer equivalence classes $L\{3,3,3\}$ [21] and $L\{3,5,1\}$ [13]. In the following theorem we show that any 9-point set not belonging to either of these two equivalent classes contains a 5-hole.

Theorem 1. *Any set of 9 points in the plane in general position, not containing a 5-hole either belongs to the layer equivalence class $L\{3,3,3\}$ or to the layer equivalence class $L\{3,5,1\}$.*

Proof. Let S be a set of 9 points in general position. If $|\mathcal{V}(CH(S))| \geq 4$, a 5-hole is guaranteed from Corollary 2. Thus, for proving the result is suffices to show that S contains a 5-hole if $S \in L\{3,4,2\}$.

Assume $S \in L\{3,4,2\}$ and suppose z_1, z_2, z_3, z_4 are the vertices of the second layer. Let $L\{3,S\} = \{x,y\}$. The extended straight line xy divides the entire plane into two halfplanes. If one these halfplane contains three points of $L\{2,S\}$, these three points along with the points x and y form a 5-hole.

Otherwise, both halfplanes induced by the extended straight line xy contain exactly two points of $L\{2,S\}$. The exterior of the quadrilateral $z_1z_2z_3z_4$ can now be partitioned into 4 disjoint regions R_1, R_2, R_3, and R_4, as shown in Figure 2(c). Let $|R_i|$ denote the number of points of $\mathcal{V}(CH(S))$ in the region R_i. If R_1 or R_3 contains any point of $\mathcal{V}(CH(S))$, a 5-hole is immediate. Therefore, $|R_1| = |R_3| = 0$, which implies that $|R_2| + |R_4| = |\mathcal{V}(CH(S))| = 3$. By the pigeonhole principle, either $|R_2| \geq 2$ or $|R_4| \geq 2$. If $|R_2| \geq 2$, $(R_2 \cap S) \cup \{x, z_1, z_2\}$ contains a 5-hole. Otherwise, $|R_4| \geq 2$, and $(R_4 \cap S) \cup \{y, z_3, z_4\}$ contains a 5-hole.

Thus, a set S of 9 points not containing a 5-hole, must either belong to $L\{3, 3, 3\}$ or $L\{3, 5, 1\}$. □

4 Disjoint 5-Holes: Lower Bounds

In this section we present our main results concerning the existence of disjoint 5-holes in planar point sets, which leads to a non-trivial lower bound on the number of disjoint 5-holes in planar point sets. As $H(5) = 10$, it is clear that every set 20 points in the plane in general position, contains two disjoint 5-holes. At first, we improve upon this result by showing that any set of 19 points also contains two disjoint 5-holes.

Theorem 2. *Every set of 19 points in the plane in general position, contains two disjoint 5-holes.*

Drawing parallel from the $2m + 4$-point result for disjoint 4-holes due to Hosono and Urabe [17], we prove a partitioning theorem for disjoint 5-holes for any set of $2m + 9$ points in the plane in general position.

Theorem 3. *For any set of $2m + 9$ points in the plane in general position, it is possible to divide the plane into three disjoint convex regions such that one contains a set of 9 points which contains a 5-hole, and the others contain m points each, where m is a positive integer.*

Since $H(5) = 10$, the trivial lower bound on $F_5(n)$ is $\lfloor \frac{n}{10} \rfloor$. Observe that any set of 47 points can be partitioned into two sets of 19 points each, and another set of 9 points containing a 5-hole, by Theorem 3. Hence, from Theorems 2 and 3, it follows that, $F_5(47) = 5$. Using this result, we obtain an improved lower bound on $F_5(n)$.

Theorem 4. $F_5(n) \geq \lfloor \frac{5n}{47} \rfloor$.

Proof. Let S be a set of n points in the plane, no three of which are collinear. By a horizontal sweep, we can divide the plane into $\lceil \frac{n}{47} \rceil$ disjoint strips, of which $\lfloor \frac{n}{47} \rfloor$ contain 47 points each and one remaining strip R, with $|R| < 47$. The strips having 47 points contain at least 5 disjoint 5-holes, since $F_5(47) = 5$ (Theorems 2 and 3). If $9k + 1 \leq |R| \leq 9k + 9$, for $k = 0$ or $k = 1$, there exist at least k disjoint 5-holes in R. If $19 \leq |R| \leq 28$, Theorem 2 guarantees the existence of 2 disjoint 5-holes in R. Finally, if $9k + 2 \leq |R| \leq 9k + 10$, for $k = 3$ or 4, at least k disjoint 5-holes exist in R. Thus, the total number of disjoint 5-holes in a set of n points is always at least $\lfloor \frac{5n}{47} \rfloor$. □

We can obtain a better lower bound on $F_5(n)$ for infinitely many n, of the form $n = 28 \cdot 2^{k-1} - 9$ with $k \geq 1$, by the repeated application of Theorem 3.

Theorem 5. $F_5(n) \geq (3n - 1)/28$, *for $n = 28 \cdot 2^{k-1} - 9$ and $k \geq 1$.*

Proof. Let $g(k) = 28 \cdot 2^{k-1} - 9$ and $h(k) = 3 \cdot 2^{k-1} - 1$. We need to show $F_5(g(k)) \geq h(k)$. We prove the inequality by induction on k. By Theorem 2, the inequality holds for $k = 1$. Suppose the result is true for k, that is, $F_5(g(k)) \geq h(k)$. Since, $g(k+1) = 2g(k) + 9$, any set of $g(k+1)$ points can be partitioned into three disjoint convex regions, two of which contain $g(k)$ points each, and the third a set of 9 points containing a 5-hole by Theorem 3. Hence, $F_5(g(k+1)) = F_5(2g(k)+9) \geq 2h(k) + 1 = h(k+1)$. This completes the induction step, proving the result for $n = 28 \cdot 2^{k-1} - 9$. □

5 Proof of Theorem 2

Let S be a set of 19 points in the plane in general position. We say S is *admissible* if it contains two disjoint 5-holes. We prove Theorem 2 by considering the various cases based on the size of $|\mathcal{V}(CH(S))|$. The proof is divided into two subsections. The first section considers the cases where $|\mathcal{V}(CH(S))| \geq 4$, and the second section deals with the case where $|\mathcal{V}(CH(S))| = 3$.

5.1 $|\mathcal{V}(CH(S))| \geq 4$

Let $CH(S)$ be the polygon $s_1 s_2 \ldots s_k$, where $k = |\mathcal{V}(CH(S))|$ and $k \geq 4$. A diagonal $d := s_i s_j$ of $CH(S)$, is called a *dividing* diagonal if

$$\big||\mathcal{H}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| - |\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))|\big| = c,$$

where c is 0 or 1 according as k is even or odd, and $s_m \in \mathcal{V}(CH(S))$ is such that $m \neq i, j$. Consider a dividing diagonal $d := s_i s_j$ of $CH(S)$. Observe that for any fixed index $m \neq i, j$, either $|\mathcal{H}(s_i s_j, s_m) \cap S| \geq 9$ or $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S| \geq 9$. Now, we have the following observation.

Observation 1 *If for some dividing diagonal $d = s_i s_j$ of $CH(S)$, $|\mathcal{H}(s_i s_j, s_m) \cap S| > 10$, where $m \neq i, j$, then S is admissible.*

Proof. Let $Z = \overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S$ and β and γ the first and the second angular neighbors of $\overrightarrow{s_i s_j}$ in $\mathcal{H}(s_i s_j, s_m) \cap S$, respectively. Now, $|\mathcal{V}(CH(Z))| \geq 3$, since $|\mathcal{V}(CH(S))| > 3$. We consider different cases based on the size of $CH(Z)$.

Case **1:** $|\mathcal{V}(CH(Z))| \geq 5$. This implies that $|\mathcal{V}(CH(Z \cup \{\beta\}))| \geq 6$ and so $Z \cup \{\beta\}$ contains a 5-hole by Lemma 1. This 5-hole is disjoint from the 5-hole contained in $(\mathcal{H}(s_i s_j, s_m) \cap S) \backslash \{\beta\}$.

Case **2:** $|\mathcal{V}(CH(Z))| = 4$. If $|\mathcal{I}(CH(Z))| \geq 2$, then $Z \cup \{\beta\}$ is a convex pentagon with at least two interior points. From Lemma 2, $Z \cup \{\beta\}$ contains a 5-hole which is disjoint from the 5-hole contained in $(\mathcal{H}(s_i s_j, s_m) \cap S) \backslash \{\beta\}$. Otherwise, $|\mathcal{I}(CH(Z))| \leq 1$. Let $Z' = Z \cup \{\beta, \gamma\}$. It follows from Lemmas 1 and 2 that Z' always contains a 5-hole. This 5-hole is disjoint from the 5-hole contained in $(\mathcal{H}(s_i s_j, s_m) \cap S) \backslash \{\beta, \gamma\}$, since $|(\mathcal{H}(s_i s_j, s_m) \cap S) \backslash \{\beta, \gamma\}| \geq 12$.

Case **3:** $|\mathcal{V}(CH(Z))| = 3$. If $|\mathcal{I}(CH(Z))| = 5$, $|\mathcal{V}(CH(Z \cup \{\beta\}))| = 4$ and $Z \cup \{\beta\}$ contains a 5-hole by Corollary 2, which is disjoint from the 5-hole contained in $(\mathcal{H}(s_i s_j, s_m) \cap S) \backslash \{\beta\}$. So, let $|\mathcal{I}(CH(Z))| = b \leq 4$, which implies, $|\mathcal{H}(s_i s_j, s_m) \cap S| = 16 - b$. Let η be the $(6-b)$-th angular neighbor of $\overrightarrow{s_i s_j}$ in $\mathcal{H}(s_i s_j, s_m) \cap S$. Let $S_1 = \mathcal{H}_c(\eta s_i, s_j) \cap S$ and $S_2 = \overline{\mathcal{H}}(\eta s_i, s_j) \cap S$. Now, since $|S_1| = 9$ and $|\mathcal{V}(CH(S_1))| \geq 4$, S_1 contains 5-hole, by Corollary 2. This 5-hole disjoint from the 5-hole contained in S_2. □

Observation 1 implies that for any dividing diagonal $d := s_i s_j$ and for any fixed vertex s_m, with $m \neq i, j$, S is admissible unless $|\mathcal{H}(s_i s_j, s_m) \cap S| \leq 10$ and $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S| \leq 10$. This can now be used to show the admissibility of S whenever $|\mathcal{V}(CH(S))| \geq 8$.

Lemma 4. *S is admissible whenever $|\mathcal{V}(CH(S))| \geq 8$.*

Proof. Let $d := s_i s_j$ be a dividing diagonal of $CH(S)$, and $s_m \in \mathcal{V}(CH(S))$ be such that $m \neq i, j$. Since $|\mathcal{V}(CH(S))| \geq 8$, both $|\mathcal{H}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))|$ and $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))|$ must be greater than 3. Moreover, if $|\mathcal{H}(s_i s_j, s_m) \cap S| > 10$, Observation 1 ensures that S is admissible. Thus, we have the following two cases:

Case **1:** $|\mathcal{H}(s_i s_j, s_m) \cap S| = 10$. Now, since $|\mathcal{V}(CH(\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S))| \geq 4$, $\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S$ contains a 5-hole which is disjoint from the 5-hole contained in $\mathcal{H}(s_i s_j, s_m) \cap S$.

Case **2:** $|\mathcal{H}(s_i s_j, s_m) \cap S| = 9$. As $|\mathcal{V}(CH(S))| \geq 8$ and $\overrightarrow{s_i s_j}$ is a dividing diagonal of $CH(S)$, we have $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| \geq 3$. Let $W = (\overline{\mathcal{H}}(s_i s_j, s_m) \cap S) \cup \{s_i\}$. Then from Corollary 2, W contains a 5-hole, since $|W| = 9$ and $|\mathcal{V}(CH(W))| \geq 4$. The 5-hole contained in W is disjoint from the 5-hole contained in $(\mathcal{H}(s_i s_j, s_m) \cap S) \cup \{s_j\}$. Hence S is admissible.

Case **3:** $|\mathcal{H}(s_i s_j, s_m) \cap S| \leq 8$. In this case, $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S| \geq 9$, and the problem reduces to the previous cases. □

Therefore, it suffices to show the admissibility of S whenever $4 \leq |\mathcal{V}(CH(S))| \leq 7$. Observe that S is admissible whenever $|\mathcal{H}(s_i s_j, s_m) \cap S| = 10$ and $|\mathcal{V}(CH(\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S))| \geq 4$. Moreover, *Case* 2 of Lemma 4 shows that S is admissible if $|\mathcal{H}(s_i s_j, s_m) \cap S| = 9$ and $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| \geq 3$. Thus, hereafter we shall assume,

Assumption 2 *For every dividing diagonal $s_i s_j$ of $CH(S)$, there exists $s_m \in \mathcal{V}(CH(S))$, with $m \neq i, j$, such that either $|\mathcal{H}(s_i s_j, s_m) \cap S| = 10$ and $|\mathcal{V}(CH(\overline{\mathcal{H}}_c(s_i s_j, s_m) \cap S))| = 3$, or $|\mathcal{H}(s_i s_j, s_m) \cap S| = 9$ and $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap \mathcal{V}(CH(S))| \leq 2$.*

A dividing diagonal $s_i s_j$ of $CH(S)$ is said to be an $(a, b)-$*splitter* of $CH(S)$, where $a \leq b$ are integers, if either $|\mathcal{H}(s_i s_j, s_m) \cap S \backslash \mathcal{V}(CH(S))| = a$ and $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S \backslash \mathcal{V}(CH(S))| = b$ or $|\mathcal{H}(s_i s_j, s_m) \cap S \backslash \mathcal{V}(CH(S))| = b$ and $|\overline{\mathcal{H}}(s_i s_j, s_m) \cap S \backslash \mathcal{V}(CH(S))| = a$.

The admissibility of S in the different cases which arise are now proved as follows:

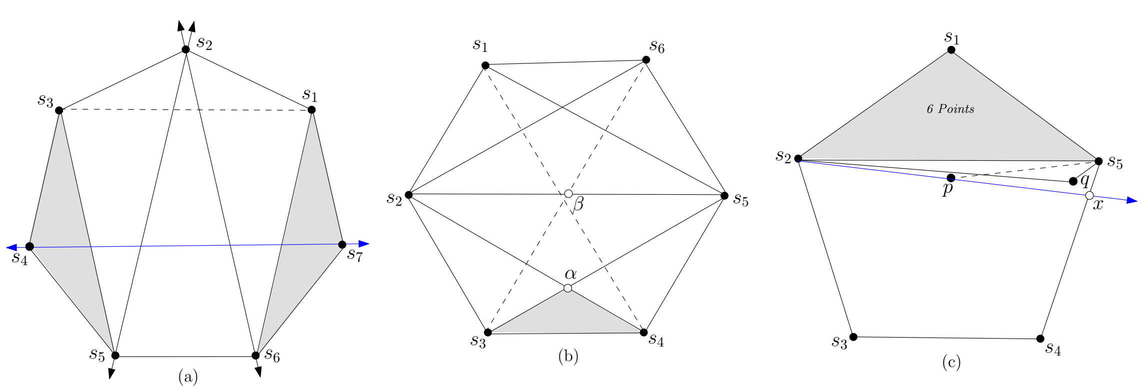


Fig. 3. Illustrations for the proof of Lemma 5: (a) $|\mathcal{V}(CH(S))| = 7$, (b) $|\mathcal{V}(CH(S))| = 6$, (c) Illustration for the proof of Lemma 6.

Lemma 5. *S is admissible whenever* $6 \leq |\mathcal{V}(CH(S))| \leq 7$.

Proof. We consider the two cases based on the size of $|\mathcal{V}(CH(S))|$ separately as follows:

Case **1:** $|\mathcal{V}(CH(S))| = 7$. Refer to Figure 3(a). From Assumption 2 it follows that every dividing diagonal of $CH(S)$ must be a $(6,6)$-splitter of $CH(S)$. As both s_2s_5 and s_2s_6 are $(6,6)$-splitters, it is clear that $\mathcal{I}(s_2s_5s_6)$ is empty in S. Now, if s_2 is 5-redundant in either $\mathcal{H}_c(s_2s_5, s_4)\cap S$ or $\mathcal{H}_c(s_2s_6, s_2)\cap S$, the admissibility of S is immediate. Therefore, assume that s_2 is not 5-redundant in either $\mathcal{H}_c(s_2s_5, s_4)\cap S$ or $\mathcal{H}_c(s_2s_6, s_2)\cap S$. This implies that $\mathcal{I}(s_2s_3s_4s_5) \cap S \subset \mathcal{I}(s_3s_4s_5)$ and $\mathcal{I}(s_2s_6s_1s_7) \cap S \subset \mathcal{I}(s_1s_6s_7)$. Therefore, $\mathcal{I}(s_1s_2s_3)$ is empty in S. Now, since s_4s_7 is also a $(6,6)$-splitter of $CH(S)$, $|\mathcal{V}(CH(\mathcal{H}(s_4s_7, s_2)\cap S))| \geq 4$ (see Figure 3(a)), and Corollary 2 implies $\mathcal{H}(s_4s_7, s_2)\cap S$ contains a 5-hole. This 5-hole disjoint from the 5-hole contained in $\mathcal{H}_c(s_4s_7, s_5) \cap S$.

Case **2:** $|\mathcal{V}(CH(S))| = 6$. Refer to Figure 3(b). Again, Assumption 2 implies that every dividing diagonal of $CH(S)$ must be a $(6,7)$-splitter of $CH(S)$. W.l.o.g. assume that $|\mathcal{I}(s_1s_2s_5s_6) \cap S| = 7$ and $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 6$. Let α be the point of intersection of the diagonals of the quadrilateral $s_2s_3s_4s_5$. If s_2 or s_5 is 5-redundant in $\mathcal{H}_c(s_2s_5, s_4)\cap S$, then the admissibility of S is immediate. Therefore, assume that neither s_2 nor s_5 is 5-redundant in $\mathcal{H}_c(s_2s_5, s_4) \cap S$. This implies that $\mathcal{I}(s_2s_3s_4s_5) \cap S \subset \mathcal{I}(s_3\alpha s_4)$. Now, if $|\mathcal{I}(s_1s_2s_3s_4)\cap S| = 6$, then s_4 is 5-redundant in $\mathcal{H}_c(s_1s_4, s_2)\cap S$ and the admissibility of S follows. Similarly, if $|\mathcal{I}(s_3s_4s_5s_6) \cap S| = 6$, then S is admissible, as s_3 is 5-redundant in $\mathcal{H}_c(s_3s_6, s_5) \cap S$. Hence, assume $|\mathcal{I}(s_1s_2s_3s_4) \cap S| = |\mathcal{I}(s_3s_4s_5s_6) \cap S| = 7$. Now, as $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 6$, $(\mathcal{I}(s_3s_4s_5s_6)\backslash\mathcal{I}(s_3s_4\alpha)) \cap S \subset \mathcal{I}(s_5s_6\beta)$, where β is the point of intersection of the diagonals s_2s_5 and s_3s_6. Therefore, $|\mathcal{V}(CH(\mathcal{H}(s_3s_6, s_5) \cap S))| \geq 4$. Therefore, the 5-hole contained in $\mathcal{H}(s_3s_6, s_5) \cap S$ is disjoint from the 5-hole contained in $\mathcal{H}_c(s_3s_6, s_1) \cap S$. □

Lemma 6. *S is admissible whenever* $|\mathcal{V}(CH(S))| = 5$.

Proof. Assumption 2 implies that a dividing diagonal of $CH(S)$ is either a $(6,8)$-splitter or a $(7,7)$-splitter of $CH(S)$. To begin with suppose, every dividing diagonal of $CH(S)$ is a $(7,7)$-splitter of $|\mathcal{V}(CH(S))|$. Then $|\mathcal{I}(s_1s_2s_3) \cap S| = |\mathcal{I}(s_1s_4s_5) \cap S| = 7$, which means that $|\mathcal{I}(s_1s_3s_4) \cap S| = 0$. Similarly, $|\mathcal{I}(s_2s_4s_5) \cap S| = |\mathcal{I}(s_3s_5s_1) \cap S| = |\mathcal{I}(s_4s_2s_1) \cap S| = |\mathcal{I}(s_5s_2s_3) \cap S| = 0$. This implies $|\mathcal{I}(CH(S))| = 0$, which is a contradiction.

Therefore, assume that there exists a $(6,8)$-splitter of $CH(S)$. W.l.o.g., assume s_2s_5 is a $(6,8)$-splitter of $CH(S)$. There are two possibilities:

Case **1:** $|\mathcal{I}(s_1s_2s_5) \cap S| = 6$ and $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 8$. Refer to Figure 3(c). Let p be the nearest neighbor of s_2s_5 in $\mathcal{H}(s_2s_5, s_4)\cap S$. W.l.o.g., assume $\mathcal{I}(s_1s_2p)\cap S$ is non-empty. Let x be the point where $\overrightarrow{s_2p}$ intersects the boundary of $CH(S)$. Then $\mathcal{H}_c(s_2x, s_1)\cap S$ contains a 5-hole, and by Corollary 1 s_2 is 5-redundant in $\mathcal{H}_c(s_2p, s_1)\cap S$. Now, if $Cone(s_5px)\cap S$ is empty, the 5-hole contained in $(\mathcal{H}_c(s_2p, s_1)\cap S)\backslash\{s_2\}$ is disjoint from the 5-hole contained in $(\overline{\mathcal{H}}(s_2p, s_1) \cap S) \cup \{s_2\}$. Otherwise, assume $Cone(s_5px)\cap S$ is non-empty. Let q be the first angular neighbor of $\overrightarrow{s_2s_5}$ in $Cone(s_5px)$. Observe that $\mathcal{I}(s_1s_2q) \cap S$ is non-empty, since $\mathcal{I}(s_1s_2p) \cap S$ is assumed to be non-empty, and $\mathcal{H}_c(s_2q, s_1) \cap S$ contains a 5-hole. Now, Corollary 1 implies that s_2 is 5-redundant in $\mathcal{H}_c(s_2q, s_1)\cap S$, and the admissibility of S follows.

Case **2:** $|\mathcal{I}(s_1s_2s_5) \cap S| = 8$ and $|\mathcal{I}(s_2s_3s_4s_5) \cap S| = 6$. Clearly, $\mathcal{H}_c(s_2s_5, s_3) \cap S$ contains a 5-hole. Now, if either s_2 or s_5 is 5-redundant in $\mathcal{H}_c(s_2s_5, s_3) \cap S$, then S is admissible. Therefore, assume $\mathcal{I}(s_2s_3s_4s_5)\cap S \subset \mathcal{I}(s_3s_4\alpha)$, where α is the point where the diagonals of the quadrilateral $s_2s_3s_4s_5$ intersect. The problem now reduces to *Case* 1 with respect to the dividing diagonal s_2s_4. $\square$

The case $|\mathcal{V}(CH(S))| = 4$ is dealt separately in the next section.

$|\boldsymbol{\mathcal{V}(CH(S))}| = \mathbf{4}$ As before, let $CH(S)$ be the polygon $s_1s_2s_3s_4$. From Observation 1, we have to consider the cases where a dividing diagonal of $CH(S)$ is either a $(6,9)$-splitter or a $(7,8)$-splitter of $CH(S)$.

Firstly, suppose some dividing diagonal of $CH(S)$, say s_2s_4, is a $(6,9)$-splitter of $CH(S)$. Assume that $|\mathcal{I}(s_1s_2s_4) \cap S| = 6$ and $|\mathcal{I}(s_2s_3s_4) \cap S| = 9$. Begin by taking the nearest neighbor p of s_2s_4 in $\mathcal{I}(s_2s_3s_4)$. Then choose the first angular neighbor q of either $\overrightarrow{s_2s_4}$ or $\overrightarrow{s_4s_2}$ in $\mathcal{I}(s_2s_3s_4)$, and proceed as in *Case* 1 of Lemma 6 to show the admissibility of S.

Therefore, it suffices to assume that

Assumption 3 *Both the dividing diagonals of the quadrilateral $s_1s_2s_3s_4$ are $(7,8)$-splitters of $CH(S)$.*

W.l.o.g., let $|\mathcal{I}(s_1s_2s_4) \cap S| = 8$ and $|\mathcal{I}(s_2s_3s_4) \cap S| = 7$. Let α be the point where the diagonals of $CH(S)$ intersect. Observe, there always exists an edge of $CH(S)$ say, s_2s_3, such that $|\mathcal{I}(s_1s_2s_3) \cap S| = |\mathcal{I}(s_2s_3s_4) \cap S| = 7$, and $|\mathcal{I}(s_1s_3s_4) \cap S| = |\mathcal{I}(s_1s_2s_4) \cap S| = 8$. This implies, $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n$, with $0 \leq n \leq 7$. We begin with the following simple observation

Lemma 7. *S is admissible whenever $n = 0$.*

Proof. Let $Z = (\mathcal{H}(s_2s_4, s_1) \cap S) \cup \{s_4\}$. Observe that $|Z| = 10$, which means Z contains a 5-hole. If $|\mathcal{V}(CH(Z))| \geq 5$, s_4 is 5-redundant in Z, and $Z \backslash \{s_4\}$ contains a 5-hole which is disjoint from the 5-hole contained in $\mathcal{H}_c(s_2s_4, s_3) \cap S$. Let r be the nearest angular neighbor of $\overrightarrow{s_1s_3}$ in $Cone(s_4s_1s_3)$. If $|\mathcal{V}(CH(Z))| = 4$, either r or s_4 is 5-redundant in Z by Corollary 1, and the admissibility of S follows. Otherwise, $|\mathcal{V}(CH(Z))| = 3$ and at least one of s_1, s_4, or r is 5-redundant in Z and the admissibility of S follows similarly. $\square$

From the previous lemma, it suffices to assume $n > 0$. Let p be the first angular neighbor of $\overrightarrow{s_2s_4}$ in $Cone(s_4s_2s_3)$ and x the intersection point of $\overrightarrow{s_2p}$ with the boundary of $CH(S)$. Let α be the point of intersection of the diagonals of the quadrilateral $s_1s_2s_3s_4$. If $Cone(s_3px) \cap S$ is non-empty, $|\mathcal{V}(CH(\mathcal{H}_c(s_2p, s_3) \cap S))| \geq 4$. From Corollary 2, $\mathcal{H}_c(s_2p, s_3) \cap S$ contains a 5-hole which is disjoint from the 5-hole contained in $(\mathcal{H}(s_2s_4, s_1) \cap S) \cup \{s_4\}$. Therefore, we shall assume that

Assumption 4 *$Cone(s_3px) \cap S$ is empty.*

Assumption 4 and the fact that $n > 0$ implies that $p \in \mathcal{I}(s_3\alpha s_4) \cap S$ (see Figure 4(a)). Let q be the first angular neighbor of $\overrightarrow{ps_2}$ in $Cone(s_2ps_1)$. The admissibility of S in the remaining cases is proved in the following two lemmas.

Lemma 8. *S is admissible whenever $n \geq 2$.*

Proof. To begin with suppose, $q \in \mathcal{I}(s_2\alpha s_1) \cap S$, as shown in Figure 4(a). By Assumption 4, there exists a point in $\mathcal{I}(s_3s_4\alpha) \cap S$, different from the point p, which belongs to $\mathcal{I}(qps_3) \cap S$. Hence, by Corollary 1, p is 5-redundant in $\mathcal{H}_c(pq, s_2) \cap S$, and the 5-hole contained in $(\mathcal{H}(pq, s_2) \cap S) \cup \{q\}$ is disjoint from the 5-hole contained in $(\mathcal{H}(pq, s_1) \cap S) \cup \{p\}$.

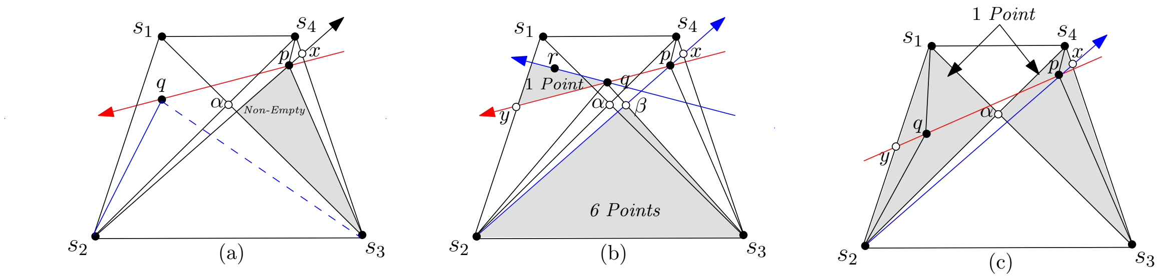


Fig. 4. $|\mathcal{V}(CH(S))| = 4$: (a) $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n \geq 2$ and $q \in \mathcal{I}(s_2\alpha s_1) \cap S$, (b) $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n \geq 2$, and $q \in \mathcal{I}(s_1\alpha s_4) \cap S$, (c) $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n = 1$.

Otherwise, assume that $q \in \mathcal{I}(s_1\alpha s_4) \cap S$ and refer to Figure 4(b). Observe that S is admissible if either p or q is 5-redundant in $\mathcal{H}_c(pq, s_2) \cap S$. Hence, assume that neither p nor q is 5-redundant in $\mathcal{H}_c(pq, s_2) \cap S$. This implies $\mathcal{I}(s_2s_3pq) \cap S \subset \mathcal{I}(s_2s_3\beta)$, where β is the point of intersection of the diagonals of the quadrilateral s_2s_3pq. Let r be the second angular neighbor of $\overrightarrow{qy}$ in $Cone(yqs_1)$, where y is the point where $\overrightarrow{pq}$ intersects the boundary $CH(S)$. Note that the point r exists because $n \geq 2$ and $q \in \mathcal{I}(s_1s_4\alpha) \cap S$. Now, the 5-hole contained in $(\mathcal{H}(qr, s_2) \cap S) \cup \{q\}$ is disjoint from the 5-hole contained in $(\mathcal{H}(qr, s_1) \cap S) \cup \{r\}$ by Corollary 2. $\square$

Lemma 9. *S is admissible whenever $n = 1$.*

Proof. To begin with let $q \in \mathcal{I}(s_1\alpha s_2)$. Refer to Figure 4(c). Assume, $\mathcal{I}(s_4pq) \cap S$ is non-empty and let $Z = (\mathcal{H}(pq, s_1) \cap S) \cup \{q\}$. Observe that $|\mathcal{V}(CH(Z))| \geq 4$, and by Corollary 1 either q or s_4 is 5-redundant in Z, and the admissibility of S follows.

Otherwise, assume that $\mathcal{I}(s_4pq) \cap S$ is empty. If either q or s_4 is 5-redundant in Z, the admissibility of S is immediate. Therefore, it suffices to assume that there exists a 5-hole in Z with qs_4 as an edge. This implies that we have a 6-hole with ps_4 and pq as edges. Observe that s_1 cannot be a vertex of this 6-hole. Hence, there exists a 5-hole with ps_4 as an edge, which does not contain s_1 and q as vertices. Thus, s_1 and q are 5-redundant in $\mathcal{H}_c(s_4q, s_1) \cap S$. This 5-hole is disjoint from the 5-hole contained in $\mathcal{H}_c(s_1s_3, s_2) \cap S$.

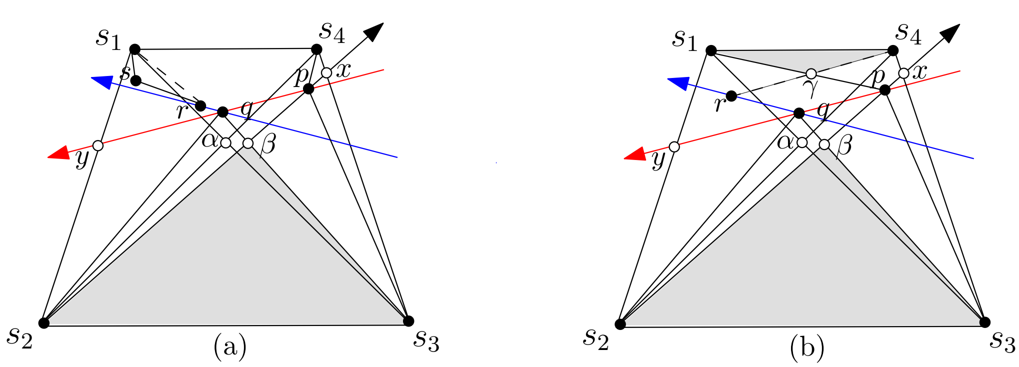


Fig. 5. $|\mathcal{V}(CH(S))| = 4$ with $|\mathcal{I}(s_1s_2\alpha) \cap S| = |\mathcal{I}(s_3s_4\alpha) \cap S| = n = 1$: (a) $q, r \in \mathcal{I}(s_1s_4\alpha) \cap S$, and (b) $q \in \mathcal{I}(s_1s_4\alpha)$ and $r \in \mathcal{I}(s_1s_2\alpha)$.

Finally, suppose $q \in \mathcal{I}(s_1s_4\alpha) \cap S$ (see Figure 5(a)). Observe that since $Cone(s_3px) \cap S$ is empty by Assumption 4, S is admissible whenever either p or q is 5-redundant in $\mathcal{H}_c(pq, s_2) \cap S$. Hence, assume that $\mathcal{I}(s_2s_3pq) \cap S \subset \mathcal{I}(s_2s_3\beta)$, where β is the point of intersection of the diagonals of the quadrilateral s_2s_3pq. Let r be the first angular neighbor of $\overrightarrow{qy}$ in $Cone(yqs_1)$, where y is the point where $\overrightarrow{pq}$ intersects the boundary $CH(S)$. If $r \in \mathcal{I}(s_1s_4\alpha) \cap S$, then $|\mathcal{V}(CH(\mathcal{H}_c(pq,s_1) \cap S))| = 6$ and both p and q are 5-redundant in $\mathcal{H}_c(pq,s_1) \cap S$ (Figure 5(a)). Thus, the partition of S given by $\mathcal{H}(pq,s_1) \cap S$ and $\mathcal{H}_c(pq,s_2) \cap S$ is admissible. Otherwise, assume that $r \in \mathcal{I}(s_1s_2\alpha) \cap S$, as shown in Figure 5(b). Let γ be the point of intersection of the diagonals of the quadrilateral s_1rps_4. From Corollary 1, it is easy to see that whenever there exists a point of $(\mathcal{H}(pq,s_1) \cap \mathcal{I}(s_1s_4\alpha)) \cap S$ outside $\mathcal{I}(s_1s_4\gamma)$, at least one of p or r is 5-redundant in $(\mathcal{H}(pq,s_1)) \cap S \cup \{p\}$, and the admissibility of S is immediate. Therefore, it suffices to assume that $(\mathcal{H}(pq,s_1) \cap \mathcal{I}(s_1s_4\alpha)) \cap S \subset \mathcal{I}(s_1s_4\gamma)$. Then $|\mathcal{V}(CH(\mathcal{H}(s_2s_4,s_1) \cap S))| \geq 4$ and $|\mathcal{H}(s_2s_4,s_1) \cap S| = 9$. Hence, the 5-hole contained in $\mathcal{H}(s_2s_4,s_1) \cap S$ (Corollary 2), is disjoint from the 5-hole contained in $\mathcal{H}_c(s_2s_4,s_3) \cap S$. □

5.2 $|\mathcal{V}(CH(S))| = 3$

Let s_1, s_2, s_3 be the three vertices of $CH(S)$. Let $\mathcal{I}(CH(S)) = \{u_1, u_2, \ldots, u_{16}\}$ be such that u_i is the i-th angular neighbor of $\overrightarrow{s_1s_2}$ in $Cone(s_2s_1s_3)$. For $i \in \{1,2,3\}$ and $j \in \{1,2,\ldots,16\}$, let p_{ij} be the point where $\overrightarrow{s_iu_j}$ intersects the boundary of $CH(S)$. For example, p_{17} is the point of intersection of $\overrightarrow{s_1u_7}$ with the boundary of $CH(S)$.

If $\mathcal{I}(u_7p_{17}s_2)$ is not empty in S, $|\mathcal{V}(CH(\mathcal{H}_c(s_1u_7,s_2) \cap S))| \geq 4$ and by Corollary 2, $\mathcal{H}_c(s_1u_7,s_2) \cap S$ contains a 5-hole which is disjoint from the 5-hole contained in $\mathcal{H}(s_1u_7,s_3) \cap S$. Therefore, $\mathcal{I}(u_7p_{17}s_2) \cap S$ can be assumed to be empty. In fact, we can make the following more general assumption.

Assumption 5 *For all $i \neq j \neq k \in \{1,2,3\}$, $Cone(p_{it}u_ts_j) \cap S$ is empty, where u_t is the seventh angular neighbor of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k) \cap S$.*

Now, we have the following observation.

Observation 2 *If for some $i \neq j \neq k \in \{1,2,3\}$, $Cone(p_{it}u_ts_j) \cap S$ is non-empty, where u_t is the eighth angular neighbor of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k)$, then S is admissible.*

Proof. W.l.o.g., let $i = 1$ and $j = 2$, which means, $t = 8$. Set $T = \mathcal{H}_c(s_1u_8,s_2) \cap S$. Suppose, there exists a point $u_r \in \mathcal{I}(s_2u_8p_{18}) \cap S$. This implies that $|\mathcal{V}(CH(T))| \geq 4$. When $|\mathcal{V}(CH(T))| \geq 5$, u_8 is 5-redundant in T and $T \backslash \{u_8\}$ contains a 5-hole which is disjoint from the 5-hole contained in $(\mathcal{H}(s_1u_8,s_3) \cap S) \cup \{u_8\}$.

Hence, it suffices to assume $|\mathcal{V}(CH(T))| = 4$. Let $\mathcal{V}(CH(T)) = \{s_1, s_2, u_r, u_8\}$, with $r \leq 7$, and α the point of intersection of the diagonals of the quadrilateral $s_1s_2u_ru_8$. By Corollary 1, it follows that unless $\mathcal{I}(s_1s_2u_ru_8) \cap S \subset \mathcal{I}(s_2\alpha u_r)$, either s_1 or u_8 is 5-redundant in T and hence S is admissible. Therefore, assume $\mathcal{I}(s_1s_2u_ru_8) \cap S \subset \mathcal{I}(s_2\alpha u_r)$, which implies $u_r = u_7$, as shown in Figure 6(a). Suppose, $Cone(s_1u_7u_8) \cap S$ is non-empty, and let u_k be the first angular neighbor of $\overrightarrow{u_7s_1}$ in $Cone(s_1u_7u_8)$. Then $\mathcal{I}(u_ku_7s_2) \cap S$ is non-empty, and u_7 is 5-redundant in $\mathcal{H}_c(u_7u_k,s_1) \cap S$. Thus, the 5-hole contained in $\mathcal{H}(u_7u_k,s_1) \cap S) \cup \{u_k\}$ is disjoint from the 5-hole contained in $(\mathcal{H}(u_7u_k,s_3) \cap S) \cup \{u_7\}$. However, if $Cone(s_1u_7u_8) \cap S$ is empty, u_7 is 5-redundant in $\mathcal{H}_c(u_7u_8,s_1) \cap S$ by Corollary 1, and the 5-hole contained in $(\mathcal{H}(u_7u_8,s_1) \cap S) \cup \{u_8\}$ is disjoint from the 5-hole contained in $(\mathcal{H}(u_7u_8,s_3) \cap S) \cup \{u_7\}$. □

Lemma 10. *If for some $i \neq j \neq k \in \{1,2,3\}$, $Cone(p_{jt}u_ts_i) \cap S$ is empty, where u_t is the seventh angular neighbor of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k)$, then S is admissible.*

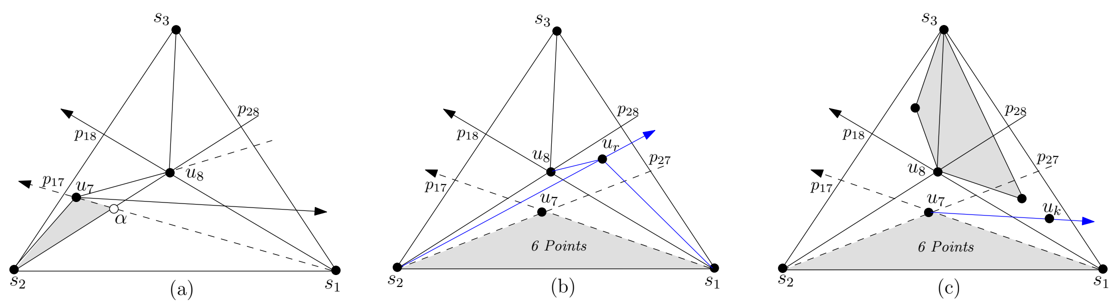


Fig. 6. (a) Proof of Observation 2, (b) Proof of Lemma 10, and (c) Proof of Lemma 11.

Proof. W.l.o.g., let $i = 1$ and $j = 2$. This means $t = 7$ and $Cone(s_1u_7p_{27})$ is empty in S. From Assumption 5, $\mathcal{I}(u_7p_{17}s_2) \cap S$ is empty. Based on Observation 2 we may suppose $Cone(s_2u_8p_{18}) \cap S$ is empty. Now, if $Cone(p_{28}u_8s_1) \cap S$ is empty, at least one of s_1, s_2, or u_8 is 5-redundant in $\mathcal{H}_c(s_1u_8, s_2) \cap S$, and admissibility of S is immediate.

Therefore, assume that $Cone(p_{28}u_8s_1) \cap S$ is non-empty, which implies that $Cone(p_{27}s_2p_{28}) \cap S$ is non-empty, since $Cone(s_1u_7p_{27}) \cap S$ is assumed to be empty. Let u_r be the first angular neighbor of $\overrightarrow{s_2u_7}$ in $Cone(p_{27}s_2p_{28}) \cap S$ (see Figure 6(b)). Now, S is admissible unless there exists a 5-hole in $\mathcal{H}_c(s_1u_8, s_2) \cap S$ with s_1u_8 as an edge. Observe that this 5-hole cannot have s_2 as a vertex. Moreover, the remaining three vertices of this 5-hole, that is, the vertices apart from s_1 and u_8, lie in the halfplane $\mathcal{H}(u_rs_2, s_1)$. Now, this 5-hole can be extended to a convex hexagon having s_1, u_8, and u_r as three consecutive vertices. Note that this convex hexagon may not be empty, and it does not contain s_2 as a vertex. From this convex hexagon, we can get a 5-hole with u_rs_1 as an edge, which does not contain u_8 as a vertex and which lies in the halfplane $\mathcal{H}(u_rs_1, s_2)$. Hence, $(\mathcal{H}(s_2u_r, s_1) \cap S) \cup \{u_r\}$ contains a 5-hole which is disjoint from the 5-hole contained in $(\mathcal{H}(s_2u_r, s_3) \cap S) \cup \{s_2\}$. □

Hereafter, in light of the previous lemma, let us assume

Assumption 6 *For all $i \neq j \neq k \in \{1, 2, 3\}$, $Cone(p_{jt}u_ts_i) \cap S$ is non-empty, where u_t is the seventh angular neighbor of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k)$.*

With this assumption we have the following two lemmas.

Lemma 11. *If for some $i \neq j \neq k \in \{1, 2, 3\}$, $Cone(s_ku_ts_j) \cap S$ is non-empty, where u_t is the eighth angular neighbor of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k) \cap S$, then S is admissible.*

Proof. It suffices to prove the result for $i = 1$ and $j = 2$, which means $t = 8$. Refer to Figure 6(c). Based on Observation 2 we may suppose S is admissible whenever $\mathcal{I}(s_2u_8p_{18}) \cap S$ is non-empty. Therefore, assume that $\mathcal{I}(s_2u_8p_{18}) \cap S$ is empty. Now, suppose $\mathcal{I}(u_8s_3p_{18}) \cap S$ is non-empty, and let $\mathcal{I}(u_8s_3p_{18}) \cap S$. Let u_k be the first angular neighbor of $\overrightarrow{u_7s_1}$ in $Cone(s_1u_7p_{27})$, which is non-empty by Assumption 6. If $Cone(u_ku_7p_{27})$ is empty, from Corollary 1, s_2 is 5-redundant in $\mathcal{H}_c(u_7u_k, s_2) \cap S$ and the admissibility of S follows. Thus, there exists some point u_m ($m \neq k$) in $Cone(u_ku_7p_{27}) \cap S$. Therefore, $|\mathcal{V}(CH((\mathcal{H}(u_7u_k, s_3) \cap S)))| \geq 4$, and by Corollary 2, $\mathcal{H}(u_7u_k, s_3) \cap S$ contains a 5-hole. This 5-hole is disjoint from the 5-hole contained in $\mathcal{H}_c(u_7u_k, s_2) \cap S$. □

Lemma 12. *If for some $i \neq j \neq k \in \{1, 2, 3\}$, $Cone(s_ku_ts_j) \cap S$ is non-empty, where u_t is the seventh angular neighbor of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k)$, then S is admissible.*

Proof. W.l.o.g., let $i = 1$ and $j = 2$, which means $t = 7$. From Assumption 5, $\mathcal{I}(u_7p_{17}s_2) \cap S$ is empty. Next, suppose there exists a point u_a in $\mathcal{I}(u_7p_{17}s_3) \cap S$. Refer to Figure 7(a). Since $Cone(s_1u_7p_{27}) \cap S$ is non-empty by Assumption 6, let u_k be the first angular neighbor of $\overrightarrow{u_7s_1}$ in $Cone(s_1u_7p_{27})$ and α the point of intersection of the diagonals of the convex quadrilateral $u_7s_2s_1u_k$. From Corollary 1, it is easy to see that S is admissible unless $\mathcal{I}(s_1s_2u_7u_k) \cap S \subset \mathcal{I}(s_1s_2\alpha)$. Now, if u_7 is the eighth angular neighbor of $\overrightarrow{s_2s_1}$ or $\overrightarrow{s_2s_3}$ in $Cone(s_1s_2s_3)$, then S is admissible from Lemma 11, since $\mathcal{I}(u_7s_3s_1) \cap S$ is not empty. Since the eighth angular neighbor of $\overrightarrow{s_2s_3}$ in $Cone(s_1s_2s_3)$ is the ninth angular neighbor of $\overrightarrow{s_2s_1}$ in $Cone(s_1s_2s_3)$, u_7 cannot be the eighth or ninth angular neighbor $\overrightarrow{s_2s_1}$ in $Cone(s_1s_2s_3)$. Thus there exist at least two points, u_m and u_n in $Cone(p_{27}u_7u_k) \cap S$, where u_m is the first angular neighbor of $\overrightarrow{u_7u_k}$ in $Cone(p_{27}u_7u_k)$. Then, the 5-hole contained in $(\mathcal{H}(u_7u_m, s_1) \cap S) \cup \{u_m\}$ is disjoint from the 5-hole contained in $(\mathcal{H}(u_7u_m, s_3) \cap S) \cup \{u_7\}$, since $|\mathcal{V}(CH((\mathcal{H}(u_7u_m, s_3) \cap S) \cup \{u_7\}))| \geq 4$ (see Figure 7(a)). □

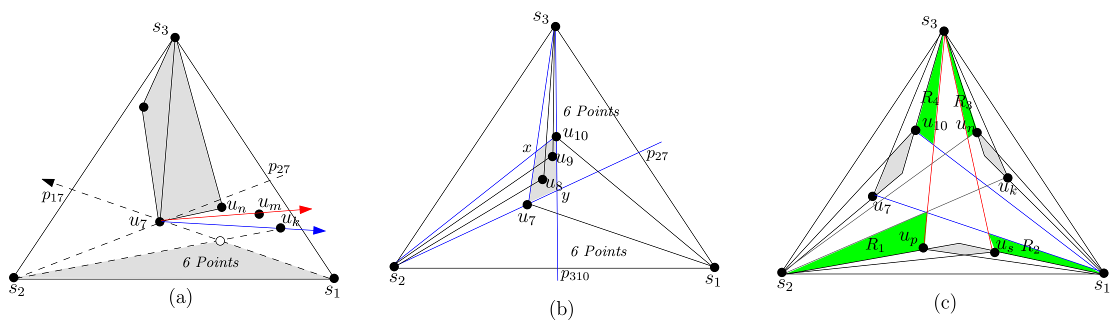


Fig. 7. (a) Illustration for the proof of Lemma 12, (b) Diamond arrangement $D\{u_7, u_{10}\}$, (c) Arrangement of diamonds $D\{u_7, u_{10}\}$, $D\{u_k, u_n\}$, and $D\{u_p, u_s\}$ in $\mathcal{I}(s_1s_2s_3)$.

The following lemma proves the admissibility of S in the remaining cases.

Lemma 13. *If for all $i \neq j \neq k \in \{1, 2, 3\}$, $Cone(s_ku_\alpha s_j) \cap S$ and $Cone(s_ku_\beta s_j) \cap S$ are empty, where u_α, u_β are the seventh and eighth angular neighbors of $\overrightarrow{s_is_j}$ in $Cone(s_js_is_k)$, respectively, then S is admissible.*

Proof. Lemmas 11 and 12 imply that S is admissible unless the interiors of $s_2u_7s_3$, $s_2u_8s_3$, $s_2u_9s_3$, and $s_2u_{10}s_3$ are empty in S. Thus, points u_7, u_8, u_9, u_{10} must be arranged inside $CH(S)$ as shown in Figure 7(b). We call such a set of 4 points a *diamond* and denote it by $D\{u_7, u_{10}\}$. Note that, $|\mathcal{I}(s_1s_2u_7) \cap S| = |\mathcal{I}(s_1s_3u_{10}) \cap S| = 6$.

Since $Cone(s_1u_7p_{27}) \cap S$ is non-empty by Assumption 6, u_7 cannot be the seventh, eighth, ninth, and tenth angular neighbors of $\overrightarrow{s_2s_1}$ in $Cone(s_1s_2s_3)$. Let u_k be the seventh angular neighbor of $\overrightarrow{s_2s_1}$ in $Cone(s_1s_2s_3)$. Suppose that $u_k \in \mathcal{I}(u_7s_2s_1)$. Then we have $|\mathcal{I}(s_1u_kp_{2k}) \cap S| \geq 1$, as $|\mathcal{I}(u_7s_1s_2) \cap S| = 6$. Hence, $|\mathcal{V}(CH(\mathcal{H}_c(s_2u_k, s_1) \cap S))| \geq 4$, and since $|\mathcal{H}_c(s_2u_k, s_1) \cap S| = 9$, the admissibility of S, in this case, follows from Corollary 2.

Therefore, it can be assumed that the seventh angular neighbor of $\overrightarrow{s_2s_1}$, that is, u_k lies in $\mathcal{I}(p_{27}u_7s_1) \cap S$. Then Lemmas 11 and 12 imply that the eighth, ninth, and tenth angular neighbors of $\overrightarrow{s_2s_1}$ are in $Cone(s_1u_7p_{27})$. Let u_l, u_m, and u_n denote the eighth, ninth, and tenth angular neighbors of $\overrightarrow{s_2s_1}$ in $Cone(s_1s_2s_3)$, respectively. From similar arguments as before, these three points along with the point u_k form a diamond, $D\{u_k, u_n\}$, which is disjoint from diamond $D\{u_7, u_{10}\}$ (see Figure 7(c)).

Let u_s be the seventh angular neighbor of $\overrightarrow{s_3s_1}$ in $Cone(s_1s_3u_{10})$ as shown in Figure 7(c). Again, Assumption 6 and the same logic as before implies S is admissible if u_{10} is the eighth, ninth or tenth angular neighbor of $\overrightarrow{s_3s_1}$ in $Cone(s_1s_3u_{10})$. Let u_r, u_q, and u_p be the eighth, ninth, and tenth angular neighbors of $\overrightarrow{s_3s_1}$ in $Cone(s_1s_3u_{10})$, respectively. As before, these three points along with the point u_s, form another diamond $D\{u_p, u_s\}$, which disjoint from both $D\{u_7, u_{10}\}$ and $D\{u_k, u_n\}$.

Let R_1, R_2, R_3, R_4 be the shaded regions inside $CH(S)$, as shown in Figure 7(c). To begin with suppose that $|R_1 \cap S| \geq 1$. Let u_z be the first angular neighbor of $\overrightarrow{u_ps_3}$ in $Cone(p_{2p}u_ps_3)$. Note that $|\mathcal{H}_c(u_pu_z, s_3) \cap S| = 10$ and $\mathcal{I}(s_2u_zu_p) \cap S$ is non-empty, as $|R_1 \cap S| \geq 1$. This implies that u_p is 5-redundant in $\mathcal{H}_c(u_pu_z, s_3) \cap S$. Therefore, the 5-hole contained in $(\mathcal{H}(u_pu_z, s_3) \cap S) \cup \{u_z\}$ is disjoint from the 5-hole contained in $(\mathcal{H}(u_pu_z, s_1) \cap S) \cap \{u_p\}$. Therefore, assume that $|R_1 \cap S| = 0$. This implies that $|R_4 \cap S| = 2$, as $|\mathcal{I}(s_2s_3u_p) \cap S| = 6$. The admissibility of S now follows from exactly similar arguments by taking the nearest angular neighbor of $\overrightarrow{u_{10}s_1}$ in $Cone(s_1u_{10}p_{310})$. □

Since all the different cases have been considered, the proof of the case $|\mathcal{V}(CH(S))| = 3$, and hence the theorem is finally completed.

6 Proof of Theorem 3

Let S be any set of $2m + 9$ points in the plane in general position, and u_1, u_2, and w_m be vertices of $CH(S)$ such that u_1u_2 and u_1w_m are edges of $CH(S)$. We label the points in the set S inductively as follows.

(i) u_i be the $(i-2)$-th angular neighbor of $\overrightarrow{u_1u_2}$ in $Cone(w_mu_1u_2)$, where $i \in \{3, 4, \ldots, m\}$.
(ii) v_i be the i-th angular neighbor of $\overrightarrow{u_1u_m}$ in $Cone(w_mu_1u_m)$, where $i \in \{1, 2, \ldots, 9\}$.
(iii) w_i be the i-th angular neighbor of $\overrightarrow{u_1v_9}$ in $Cone(w_mu_1v_9)$, where $i \in \{1, 2, \ldots, m\}$.

Therefore, $S = U \cup V \cup W$, where $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_9\}$, and $W = \{w_1, w_2, \ldots, w_m\}$.

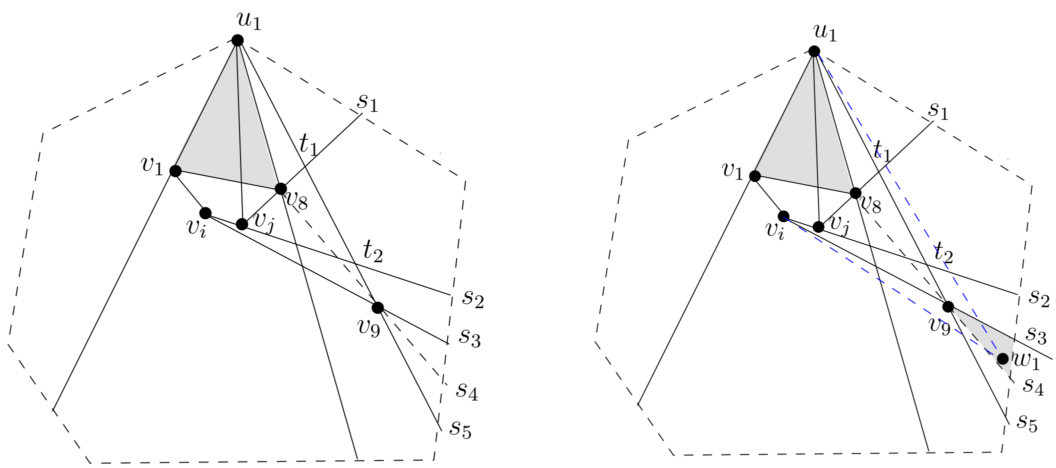
A disjoint convex partition of S into three subsets S_1, S_2, S_3 is said to be a *separable* partition of S (or *separable* for S) if $|S_1| = |S_3| = m$ and the set of 9 points S_2 contains a 5-hole. The set S is said to be *separable* if there exists a partition which is separable for S. For proving Theorem 3 we have to identify a separable partition for every set of $2m+9$ points in the plane in general position. It is clear, from Corollary 2, that S is separable whenever $|\mathcal{V}(CH(V))| \geq 4$.

Let $T = V \backslash \{v_9\} \cup \{u_1\}$. If $|\mathcal{V}(CH(T))| \geq 6$, u_1 is 5-redundant in T and $S_1 = U$, $S_2 = V$, and $S_3 = W$ is a separable partition of S.

Therefore, assume that $|\mathcal{V}(CH(T))| \leq 5$. The three cases based on the size of $|\mathcal{V}(CH(T))|$ are considered separately in the following lemmas.

Lemma 14. *S is separable whenever $|\mathcal{V}(CH(T))| = 5$.*

Proof. Let $\{u_1, v_1, v_i, v_j, v_8\}$ be the vertices of the convex hull of T. It suffices to assume that $\mathcal{I}(u_1v_1v_i)$ and $\mathcal{I}(u_1v_1v_8)$ are empty in S, otherwise either v_1 or u_1 is, respectively, 5-redundant and S is separable. Let the lines $\overrightarrow{v_jv_8}$ and $\overrightarrow{v_iv_j}$ intersect $\overrightarrow{u_1v_9}$ at the points t_1, t_2, and $CH(S)$ at the points s_1, s_2, respectively (Figure 8(a)). Now, we consider the following cases based on the location of the point v_9 on the line segment u_1s_5, where s_5 is the point where $\overrightarrow{u_1v_9}$ intersects the boundary of $CH(S)$.



(a) (b)

Fig. 8. Illustrations for the proof of Lemma 14.

Case **1:** v_9 lies on the line segment u_1t_2. This implies, $|\mathcal{V}(CH(V))| \geq 4$ and by Corollary 2, $S_1 = U$, $S_2 = V$, and $S_3 = W$ is a separable partition of S.

Case **2:** v_9 lies on the line segment t_2s_5. Let s_3 and s_4 be the points where the lines $\overrightarrow{v_iv_9}$ and $\overrightarrow{v_8v_9}$ intersects $CH(S)$, respectively. (Note that if $v_9 = s_5$, then the points s_3 and s_4 coincide with the point v_9.) If $Cone(u_1t_1s_1) \cap S$ is non-empty, let w_q be the first angular neighbor of $\overrightarrow{v_8u_1}$ in $Cone(u_1t_1s_1)$. This implies, $|\mathcal{V}(CH(V\backslash\{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 5$ and by Corollary 2 $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1, v_9\} \cup \{u_1, w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{v_9\}$ is a separable partition of S. So, assume that $Cone(u_1t_1s_1) \cap S$ empty.

Case **2.1:** $Cone(s_1v_js_2) \cap W$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_js_1}$ in $Cone(s_1v_js_2)$. Then, $|\mathcal{V}(CH(V\backslash\{v_9\} \cup \{w_q\}))| \geq 4$, and the partition, $S_1 = U$, $S_2 = V\backslash\{v_9\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{v_9\}$ is separable for S.

Case **2.2:** $Cone(s_1v_js_2) \cap W$ is empty and $Cone(s_5v_9s_4) \cap W$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_9s_5}$ in $Cone(s_5v_9s_4)$. Observe that $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 4$ and $\mathcal{I}(v_8v_9w_q) \cap S$ is empty. Now, if $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 5$, then v_1 is clearly 5-redundant in $V \cup \{w_q\}$. Otherwise, Corollary 1 now implies that v_1 is 5-redundant in $V \cup \{w_q\}$. Therefore, the partition $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{u_1\}$ is separable for S.

Case **2.3:** $Cone(s_1v_js_2) \cap W$ and $Cone(s_5v_9s_4) \cap W$ are both empty. If w_1, the nearest angular neighbor of $\overrightarrow{u_1s_5}$ in W, lies in $Cone(s_2v_is_3)$, $|\mathcal{V}(CH(V\backslash\{v_1\} \cup \{u_1, w_1\}))| = 4$ and u_1 is 5-redundant in $V\backslash\{v_1\} \cup \{u_1, w_1\}$ by Corollary 1. Therefore, $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1\} \cup \{w_1\}$, and $S_3 = W\backslash\{w_1\} \cup \{u_1\}$ is separable for S. Finally, consider that $w_1 \in Cone(s_4v_9s_3)$ and let $Z = V\backslash\{v_1\} \cup \{u_1, w_1\}$. Observe that $|\mathcal{V}(CH(Z))| = 3$ (Figure 8(b)). Now, since $|Z| = 10$, Z must contain a 5-hole. Note that since $\mathcal{I}(u_1v_1v_8)$ is assumed to be empty in S, it follows that all the four vertices of the 4-hole $u_1v_8v_9w_1$ cannot be a part of any 5-hole in Z. Moreover, there cannot be a 5-hole in Z with the points u_1, v_9, w_1 or the points u_1, v_8, v_9 as vertices, since $Cone(s_5u_1w_1)$ and $Cone(u_1w_1v_8)$ are empty in Z. Emptiness of $Cone(s_5u_1w_1) \cap Z$ and $Cone(u_1w_1v_8) \cap Z$ also implies that there cannot be a 5-hole in Z with both the points u_1 and w_1 as vertices. Thus, either u_1 or w_1 is 5-redundant in Z, and separability of S follows. □

Lemma 15. *S is separable whenever* $|\mathcal{V}(CH(T))| = 4$.

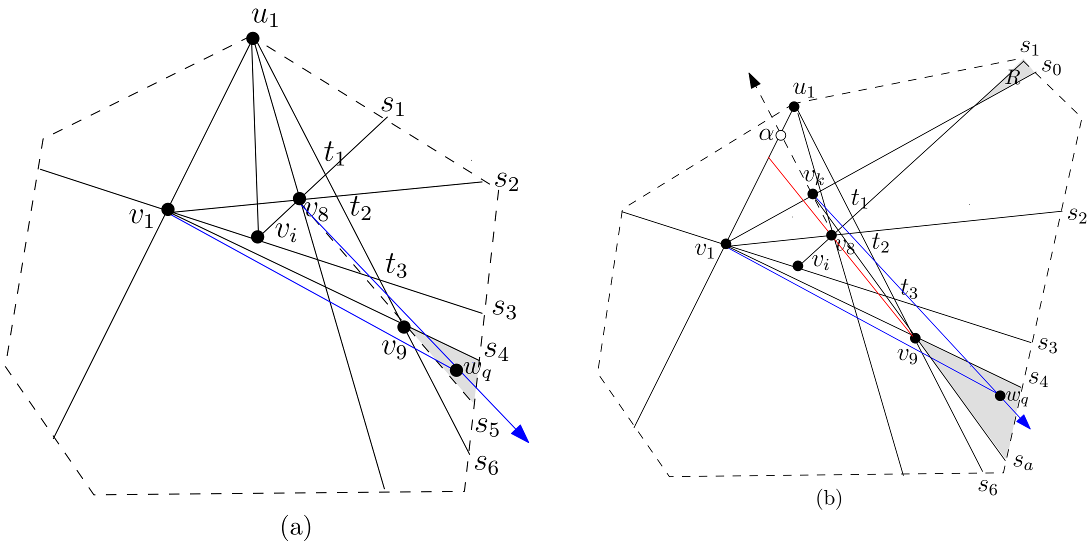


Fig. 9. Illustrations for the proof of Lemma 15: *Case* 1 and *Case* 2.

Proof. Suppose $\{u_1, v_1, v_i, v_8\}$ are the vertices of the convex hull of T. Let the lines $\overrightarrow{v_i v_8}$, $\overrightarrow{v_1 v_8}$, and $\overrightarrow{v_1 v_i}$ intersect $\overrightarrow{u_1 v_9}$ at the points t_1, t_2, t_3, and $CH(S)$ at the points s_1, s_2, s_3, respectively (see Figure 9(a)). If v_9 lies on the line segment u_1t_1 or t_2t_3, then $|\mathcal{V}(CH(V))| \geq 4$ and $S_1 = U$, $S_2 = V$, and $S_3 = W$ is separable for S. So, assume that v_9 lies on the line segment t_1t_2, or on the line segment t_3s_6, where s_6 is the point of intersection of $\overrightarrow{u_1 v_9}$ and $CH(S)$. Now, we consider the following cases.

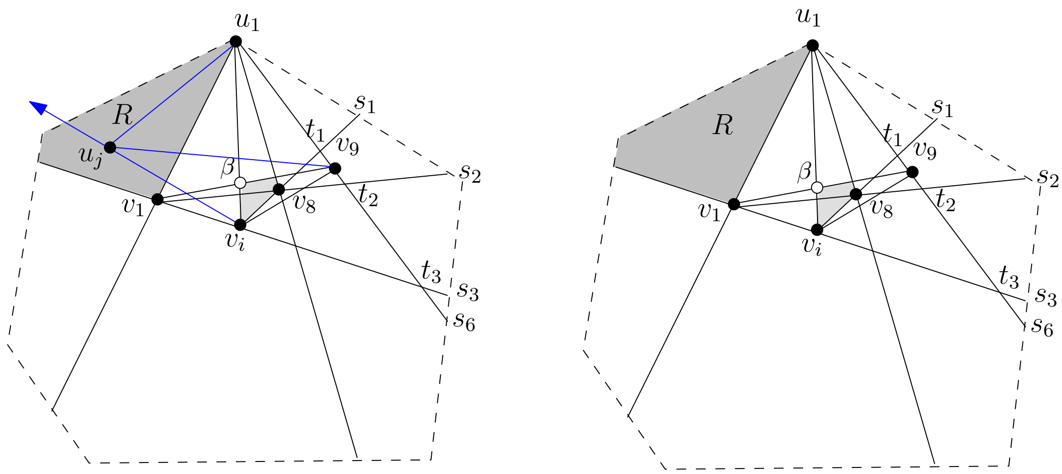
Case **1:** v_9 lies on the line segment t_3s_6, and $\mathcal{I}(u_1v_1v_8) \cap S$ is empty. Let s_4 and s_5 be the points where $\overrightarrow{v_1 v_9}$ and $\overrightarrow{v_8 v_9}$ intersect the boundary of $CH(S)$, respectively.

Case **1.1:** $Cone(u_1v_8s_1) \cap W$ is non-empty. If w_q be the first angular neighbor of $\overrightarrow{v_8 u_1}$ in $Cone(u_1v_8s_1)$, then $|\mathcal{V}(CH(V \backslash \{v_9\} \cup \{w_q\}))| = 4$. Hence, $S_1 = U$, $S_2 = V \backslash \{v_9\} \cup \{w_q\}$, and $S_3 = W \backslash \{w_q\} \cup \{v_9\}$ is a separable partition.

Case **1.2:** $Cone(u_1v_8s_1) \cap W$ is empty, and $Cone(s_6v_9s_5) \cap W$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_9 s_6}$ in $Cone(s_6v_9s_5)$. Note that $CH(V \cup \{w_q\})$ is a quadrilateral and $\mathcal{I}(v_8v_9w_q) \cap S$ is empty. This implies that v_1 is 5-redundant in $V \cup \{w_q\}$ by Corollary 1. Therefore, $S_1 = U \backslash \{u_1\} \cup \{v_1\}$, $S_2 = V \backslash \{v_1\} \cup \{w_q\}$, and $S_3 = W \backslash \{w_q\} \cup \{u_1\}$ is separable for S.

Case **1.3:** Both $Cone(u_1v_8s_1) \cap W$ and $Cone(s_6v_9s_5) \cap W$ are empty, but $Cone(s_5v_8s_2) \cap W$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_8 v_9}$ in $Cone(s_5v_8s_2)$. To begin with, assume $w_q \in Cone(s_5v_8s_2) \backslash Cone(s_5v_9s_4)$. Then $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 4$ and $V \cup \{w_q\}$ contains a 5-hole. Now, by Corollary 1, either v_1 or w_q is 5-redundant in $V \cup \{w_q\}$, and the separability of S is immediate. Otherwise, $w_q \in Cone(s_5v_9s_4)$, and $|\mathcal{V}(CH(V \cup \{w_q\}))| = 3$ (Figure 9(a)). Now, $V \cup \{w_q\}$ contains a 5-hole and at least one of v_1, v_8, and w_q is 5-redundant in $V \cup \{w_q\}$. If w_q is 5-redundant, the separability of S is immediate. If v_1 is 5-redundant, the partition $S_1 = U \backslash \{u_1\} \cup \{v_1\}$, $S_2 = V \backslash \{v_1\} \cup \{w_q\}$, and $S_3 = W \backslash \{w_q\} \cup \{u_1\}$ is a separable partition of S. Finally, if v_8 is 5-redundant, then the partition $S_1 = U$, $S_2 = V \backslash \{v_8\} \cup \{w_q\}$, and $S_3 = W \backslash \{w_q\} \cup \{v_8\}$ is a separable partition of S.

Case **1.4:** $W \subset Cone(s_1v_8s_2)$. Let w_q be the nearest angular neighbor of $\overrightarrow{v_i s_1}$ in $Cone(s_1v_is_3)$. If $\mathcal{I}(u_1v_1v_i) \cap S$ is non-empty, then $|\mathcal{V}(CH(V \backslash \{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 4$ and the partition $S_1 = U \backslash \{u_1\} \cup \{v_1\}$, $S_2 = V \backslash \{v_1, v_9\} \cup \{u_1, w_q\}$, and $S_3 = W \backslash \{w_q\} \cup \{v_9\}$ is separable for S. Otherwise, assume $\mathcal{I}(u_1v_1v_i) \cap S$ is empty. Let w_1 be the first angular neighbor of $\overrightarrow{u_1s_6}$ in W. Then, $|\mathcal{V}(CH(V\backslash\{v_1\} \cup \{w_1\}))| \geq 4$, and the partition $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1\} \cup \{w_1\}$, and $S_3 = W\backslash\{w_1\} \cup \{u_1\}$ is separable for S.



(a) (b)

Fig. 10. Illustrations for the proof of Lemma 15: *Case* 3.

Case **2:** v_9 lies on the line segment t_3s_6, and $\mathcal{I}(u_1v_1v_8) \cap S$ is non-empty. Let v_k be the first angular neighbor of $\overrightarrow{v_8u_1}$ in $Cone(u_1v_8v_1)$, and let s_0, s_4 and s_a be the points where $\overrightarrow{v_1v_k}$, $\overrightarrow{v_1v_9}$ and $\overrightarrow{v_kv_9}$ intersect $CH(S)$, respectively. Note that if $v_k \in \overline{\mathcal{H}}(v_9v_8, u_1) \cap V$, then $|\mathcal{V}(CH(V))| \geq 4$ and the separability of S is immediate. Therefore, assume that $v_k \in \mathcal{H}(v_9v_8, u_1) \cap V$ (see Figure 9(b)). Let α be the point where $\overrightarrow{v_8v_k}$ intersects $\overrightarrow{u_1v_1}$. If $\mathcal{I}(v_1v_k\alpha) \cap V$ is non-empty, then $|CH(V)| \geq 5$, and the separability of S is immediate. Therefore, assume that $\mathcal{I}(v_1v_k\alpha) \cap V$ is empty, that is, $\mathcal{I}(v_1v_ku_1) \cap V$ is empty.

Case **2.1:** $Cone(s_6v_9s_a) \cap W$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_9s_6}$ in $Cone(s_6v_9s_a)$. Then $|\mathcal{V}(CH(V \cup \{w_q\}))| = 4$ and by Corollary 1 either v_1 or v_9 is 5-redundant in $V \cup \{w_q\}$. The separability of S now follows easily.

Case **2.2:** $Cone(s_6v_9s_a) \cap W$ is empty and $Cone(s_0v_ks_a) \cap W$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_kv_9}$ in $Cone(s_0v_ks_a)$. If $w_q \in Cone(s_0v_ks_a)\backslash Cone(s_4v_9s_a)$ then $|\mathcal{V}(CH(V\backslash\{v_1\} \cup \{w_q\}))| \geq 4$, and the separability of S is immediate. Otherwise, assume $w_q \in Cone(s_4v_9s_a)$. Then $|\mathcal{V}(CH(V \cup \{w_q\}))| = 3$ and either v_1, v_k, or w_q is 5-redundant in $V \cup \{w_q\}$, and the separability of S is immediate.

Case **2.3:** Both $Cone(s_6v_9s_a) \cap W$ and $Cone(s_0v_ks_a) \cap W$ are empty, but $Cone(u_1v_ks_0) \cap W$ is non-empty. Now, if $Cone(u_1v_8s_1) \cap W$ is non-empty, the partition $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1, v_9\} \cup \{u_1, w_i\}$, and $S_3 = W\backslash\{w_i\} \cup \{v_9\}$ is separable for S, where w_i is the first angular neighbor of $\overrightarrow{v_8u_1}$ in $Cone(u_1v_8s_1) \cap W$. Therefore, assume that $Cone(u_1v_8s_1) \cap W$ is empty. This implies, $W \subset R \cap S$, where R is the shaded region as shown in Figure 9(b). Let w_q be the nearest angular neighbor of $\overrightarrow{v_iv_8}$ in $Cone(s_1v_is_3)$. If $\mathcal{I}(u_1v_1v_i) \cap S$ is non-empty, then $|\mathcal{V}(CH(V\backslash\{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 4$ and the partition $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1, v_9\} \cup \{u_1, w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{v_9\}$ is separable for S. Otherwise, assume $\mathcal{I}(u_1v_1v_i) \cap S$ is empty. Let w_1 be the first angular neighbor of $\overrightarrow{u_1s_6}$ in W. Then, $|\mathcal{V}(CH(V\backslash\{v_1\} \cup \{w_1\}))| \geq 4$, and the partition

$S_1 = U \backslash \{u_1\} \cup \{v_1\}$, $S_2 = V \backslash \{v_1\} \cup \{w_1\}$, and $S_3 = W \backslash \{w_1\} \cup \{u_1\}$ is separable for S.

Case **3:** v_9 lies on the line segment t_1t_2. Observe that if either u_1 or v_1 is 5-redundant in $V \cup \{u_1\}$, then the separability of S is immediate. Therefore, from Corollary 1, it suffices to assume that all the points inside $CH(V \cup \{u_1\})$ must lie in $\mathcal{I}(v_9v_i\beta)$, where β is the point of intersection of the diagonals of the quadrilateral $u_1v_1v_iv_9$. Next, suppose that $R \cap S$ is non-empty, where R is the shaded region inside $CH(S)$ as shown in Figure 10(a). Let $u_j \in R \cap S$ be the first angular neighbor of $\overrightarrow{v_iu_1}$ in R. Then $|\mathcal{V}(CH(V \backslash \{v_1\} \cup \{u_1, u_j\}))| = 4$ and v_i is 5-redundant in $V \backslash \{v_1\} \cup \{u_1, u_j\}$, since $\mathcal{I}(u_jv_iv_9) \cap S$ is non-empty (Corollary 1). Hence, the partition of S given by $S_1 = U \backslash \{u_1, u_j\} \cup \{v_1, v_i\}, S_2 = V \backslash \{v_1, v_i\} \cup \{u_1, u_j\}, S_3 = W$ is separable. On the other hand, if $R \cap S$ is empty, then the partition $S_1 = U \backslash \{u_1\} \cup \{v_i\}$, $S_2 = V \backslash \{v_i\} \cup \{u_1\}$, and $S_3 = W$ is separable, since v_i is 5-redundant in $V \cup \{u_1\}$ by Corollary 1 (see Figure 10(b)).

Lemma 16. *S is separable whenever $|\mathcal{V}(CH(T))| = 3$.*

Proof. Let $\mathcal{V}(CH(T)) = \{u_1, v_1, v_8\}$. Let v_i and v_j be the first angular neighbors of $\overrightarrow{v_8u_1}$ and $\overrightarrow{v_8v_1}$ respectively in $Cone(u_1v_8v_1)$. Let $\overrightarrow{v_jv_8}$ and $\overrightarrow{v_iv_8}$ intersect $\overrightarrow{u_1v_9}$ at t_1 and t_2, respectively (Figure 11(a)). If v_9 lies on the line segment u_1t_1, $|\mathcal{V}(CH(V \backslash \{v_1\} \cup \{u_1\}))| \geq 4$ and by Corollary 2, $V \backslash \{v_1\} \cup \{u_1\}$ contains a 5-hole. Thus, $S_1 = U \backslash \{u_1\} \cup \{v_1\}$, $S_2 = V \backslash \{v_1\} \cup \{u_1\}$, and $S_3 = W$ is a separable partition of S. Similarly, if v_9 lies on the line segment t_2s_4, where s_4 is the point where $\overrightarrow{u_1v_9}$ intersects the boundary of $CH(S)$, then $|\mathcal{V}(CH(V))| \geq 4$, and $S_1 = U$, $S_2 = V$, and $S_3 = W$ is separable for S.

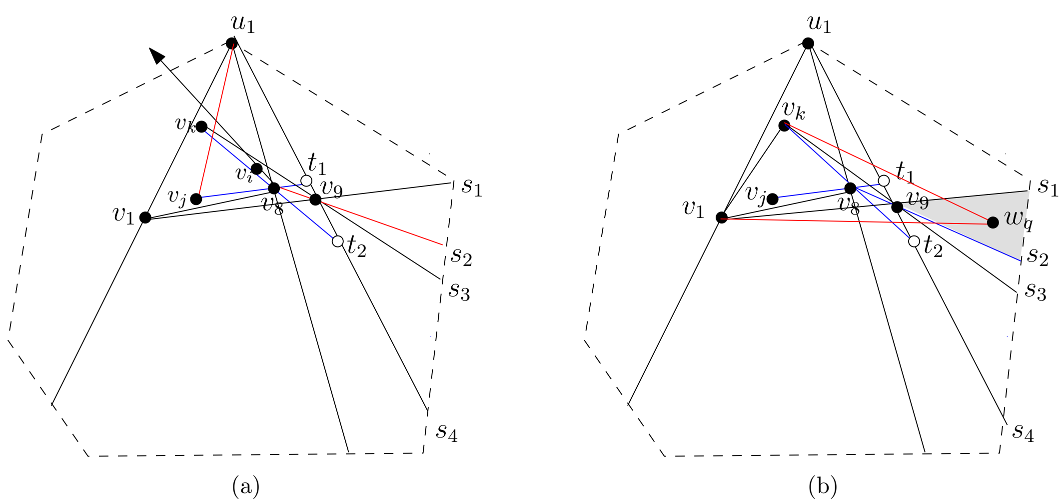


Fig. 11. Illustrations for the proof of Lemma 16.

Therefore, v_9 lies on the line segment t_1t_2. Clearly, S is separable unless $|\mathcal{V}(CH(V))| = 3$. Let $\mathcal{V}(CH(V)) = \{v_1, v_k, v_9\}$. (Note that v_k need not be the point v_i as shown in Figure 11(a)). Let s_1, s_2, and s_3 be the points where $\overrightarrow{v_1v_9}$, $\overrightarrow{v_8v_9}$, and $\overrightarrow{v_kv_9}$ intersect $CH(S)$, respectively. Now, we have the following cases:

Case **1:** $Cone(u_1v_8t_1) \cap S$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_8u_1}$, in $Cone(u_1v_8t_1)$. This implies, $|\mathcal{V}(CH(V \backslash \{v_1, v_9\} \cup \{u_1, w_q\}))| \geq 4$, and $S_1 = U \backslash \{u_1\} \cup \{v_1\}$, $S_2 = V \backslash \{v_1, v_9\} \cup \{u_1, w_q\}$, and $S_3 = W \backslash \{w_q\} \cup \{v_9\}$ is a separable partition of S.

Case **2:** $Cone(u_1v_8t_1) \cap S$ is empty and $Cone(s_4v_9s_3) \cap S$ is non-empty. Suppose, w_q is the first angular neighbor of $\overrightarrow{v_9s_4}$ in $Cone(s_4v_9s_3)$. Since $|\mathcal{V}(CH(V \cup \{w_q\}))| \geq 4$, either v_1 or v_9 is 5-redundant in $V \cup \{w_q\}$ by Corollary 1. Thus, either $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{u_1\}$ or $S_1 = U$, $S_2 = V\backslash\{v_9\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{v_9\}$ is, respectively, separable for S.

Case **3:** $Cone(u_1v_8t_1) \cap S$ and $Cone(s_4v_9s_3) \cap S$ are empty but $Cone(s_3v_9s_2) \cap S$ is non-empty. If w_q is the first angular neighbor of $\overrightarrow{v_9s_3}$ in $Cone(s_3v_9s_2)$, then $v_1v_jv_8v_9w_q$ is a 5-hole, and $S_1 = U$, $S_2 = V\backslash\{v_k\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{v_k\}$ is separable for S.

Case **4:** The three sets $Cone(u_1v_8t_1) \cap S$, $Cone(s_4v_9s_3) \cap S$, and $Cone(s_3v_9s_2) \cap S$ are all empty, but $Cone(t_1v_8s_2) \cap S$ is non-empty. Let w_q be the first angular neighbor of $\overrightarrow{v_kv_9}$ in $Cone(u_1v_kv_9)$. Clearly, $w_q \in Cone(t_1v_8s_2)$.

Case **4.1:** $w_q \in Cone(t_1v_8s_2)\backslash Cone(s_2v_9s_1)$. In this case, $|\mathcal{V}(CH(V \cup \{w_q\}))| = 4$ and v_1 is 5-redundant in $V \cup \{w_q\}$ by Corollary 1. Then the partition $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{u_1\}$ is separable for S.

Case **4.2:** $w_q \in Cone(s_2v_9s_1)$ (see Figure 11(b)). Let $Z = V \cup \{w_q\}$. Observe, $|\mathcal{V}(CH(Z))| = 3$ and Z must contain a 5-hole, since $|Z| = 10$. Now, either v_1, v_k, or w_q is 5-redundant in Z. If w_q is 5-redundant, the separability of S is immediate. If v_1 is 5-redundant, the partition $S_1 = U\backslash\{u_1\} \cup \{v_1\}$, $S_2 = V\backslash\{v_1\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{u_1\}$ is a separable partition of S. Finally, if v_k is 5-redundant, then the partition $S_1 = U$, $S_2 = V\backslash\{v_k\} \cup \{w_q\}$, and $S_3 = W\backslash\{w_q\} \cup \{v_k\}$ is a separable partition of S. □

This finishes the analysis of all the different cases, and completes the proof of Theorem 3.

7 Conclusion

In this paper we address problems concerning the existence of disjoint 5-holes in planar point sets. We prove that every set of 19 points in the plane, in general position, contains two disjoint 5-holes. Next, we show that any set of $2m+9$ points in the plane can be subdivided into three disjoint convex regions such that one contains a set of 9 points which contains a 5-hole, and the others contain m points each, where m is a positive integer. Combining these two results we show that the number of disjoint empty convex pentagons in any set of n points in the plane in general position, is at least $\lfloor \frac{5n}{47} \rfloor$. This bound has been further improved to $\frac{3n-1}{28}$ for infinitely many n.

In other words, we have shown that $H(5,5) \leq 19$. This improves upon the results of Hosono and Urabe [15, 16], where they showed $17 \leq H(5,5) \leq 20$. There is still a gap between the upper and lower bounds of $H(5,5)$, which probably requires a more complicated and detailed argument to be settled.

However, we are still quite far from establishing non-trivial bounds on $F_6(n)$ and $H(6,\ell)$, for $0 \leq \ell \leq 6$, since the exact value of $H(6) = H(6,0)$ is still unknown. The best known bounds are $H(6) \leq ES(9) \leq 1717$ and $H(6) \geq 30$ by Gerken [12] and Overmars [26], respectively.

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