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# Irreversible Investment and the Cross-Section of Stock Returns in General Equilibrium

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# Irreversible Investment and the Cross-Section of Stock Returns in General Equilibrium

## **Abstract**

A general equilibrium production economy with heterogeneous firms and irreversible investment generates the value premium. Investment irreversibility prevents unprofitable value firms from optimally scaling down their capital stock. In contrast, profitable and fast growing - growth - firms can optimally use investment to provide consumption insurance. Value firms are riskier and have higher expected returns than growth firms, especially in bad times when consumption volatility is high. The value premium is larger for small stocks as small value firms are more severely affected by irreversibility. Firms' investment and capital predict the cross-section of stock returns much like book-to-market and market equity both in the model and data. The model can replicate the failure of the unconditional CAPM. Multifactor models, including the Fama and French (1993) factor model, and to a lesser extent, conditional versions of the CAPM, outperform the unconditional CAPM.

## **Disciplines**

Finance and Financial Management

# Irreversible Investment and the Cross-Section of Stock Returns in General Equilibrium

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\*London Business School. Correspondence to [vgala@london.edu](mailto:vgala@london.edu). This paper is based on my doctoral dissertation at The University of Chicago, GSB. I am very grateful to my dissertation committee members John Cochrane (chair), George Constantinides, Lars Hansen, John Heaton, Monika Piazzesi, and Pietro Veronesi for constant support and guidance. I acknowledge valuable suggestions from Nick Barberis, Darrell Duffie, Bernard Dumas, Ken French, Adlai Fisher, Tim Johnson, Kenneth Judd, Ron Giammarino, Francisco Gomes, Joao Gomes, Leonid Kogan, Jonathan Lewellen, Dmitry Livdan, Lars Lochstoer, Pascal Maenhout, Igor Makarov, Rajnish Mehra, Carlson Murray, Jun Pan, Lubos Pastor, Anna Pavlova, Stephen Ross, Tano Santos, Raman Uppal, Dimitri Vayanos, Jessica Wachter, Amir Yaron, Moto Yogo and Lu Zhang. I thank seminar participants at the University of Chicago, AFA 2006 Boston Meetings, Dartmouth College, Columbia Business School, MIT Sloan, Wharton, Duke University, University of British Columbia, London Business School, London School of Economics, Bocconi University, INSEAD, University of Lausanne, CEPR Summer Symposium and Lehman Brothers for their comments. Research support from Fondazione IRI and the Sanford J. Grossman Fellowship in Honor of Arnold Zellner are gratefully acknowledged; any opinions expressed herein are the author's and not necessarily those of Fondazione IRI or Sanford J. Grossman or Arnold Zellner.

## ABSTRACT

A general equilibrium production economy with heterogeneous firms and irreversible investment generates the value premium. Investment irreversibility prevents unprofitable value firms from optimally scaling down their capital stock. In contrast, profitable and fast growing - growth - firms can optimally use investment to provide consumption insurance. Value firms are riskier and have higher expected returns than growth firms, especially in bad times when consumption volatility is high. The value premium is larger for small stocks as small value firms are more severely affected by irreversibility. Firms' investment and capital predict the cross-section of stock returns much like book-to-market and market equity both in the model and data. The model can replicate the failure of the unconditional CAPM. Multifactor models, including the Fama and French (1993) factor model, and to a lesser extent, conditional versions of the CAPM, outperform the unconditional CAPM.

Value stocks with high book-to-market ratios have earned higher average returns than growth stocks with low book-to-market ratios as documented in Fama and French (1992). While it seems natural to interpret this difference in average stock returns - namely, the value premium - as compensation for fundamental risk, the well-known Capital Asset Pricing Model (CAPM) fails to explain this pattern in average returns, as market betas have, if anything, the opposite pattern. These facts have motivated a large number of empirically successful asset-pricing models that extend the CAPM in various ways (Cochrane (2006) contains a detailed survey). Despite their empirical success, however, little is known about whether and how differences in firm fundamentals and exposure to fundamental risk can account for such cross-sectional heterogeneity in stock returns.

I develop a general equilibrium model that produces value and size effects. Firms are subject to aggregate and firm-specific productivity shocks. Through optimal capital investment, these shocks generate endogenous cross-sectional variation in firm characteristics such as market equity, book-to-market, investment and capital. Firms face adjustment costs and irreversibility in investment. Adjustment costs are lower for firms with low capital relative to the average firm capital. Small firms invest more and grow faster than otherwise identical firms, a model implication I verify empirically. The adjustment-cost specification makes a firm marginal  $q$  different from Tobin's (average)  $Q$ , thus generating expected profitability and size effects in both a firm's investment and its stock returns. Investors' preferences are power utility. The model aggregates so that a single moment of the joint cross-sectional distribution of firm-specific productivity and capital, along with aggregate productivity, are sufficient state variables for aggregate quantities and prices.

I investigate the properties of the model through its analytic solution, and I simulate a calibrated version to study the model's ability to match empirical facts quantitatively. While with power utility the model cannot satisfactorily address the equity premium and risk-free rate puzzles (Mehra and Prescott (1985), Weil (1989)), it does capture several familiar features of the data. First, value firms with high book-to-market ratios and small firms with low market equity have higher average stock returns, and the value effect relating book-to-market ratios with returns is weaker for large firms. Second, the unconditional CAPM fails to capture this variation in average returns. Third, multifactor models, such

as the Fama-French (1993) three-factor model, and to a lesser extent conditional versions of the CAPM, do capture the pattern of expected returns.

Most importantly, I relate patterns in the cross section of stock returns to the real side of the economy. I find in the model, and confirm in the data, that firms with low book-to-market ratios have persistently high profitability and investment rates to go with their low stock returns. I also find that firms with low investment rates and small capital stock earn on average higher stock returns. These variables capture expected profitability and scale effects similar to those captured by book-to-market and market equity, respectively.

The general equilibrium analysis provides an endogenous consumption insurance explanation for the relation between risk and expected returns. Irreversible investment and more generally capital adjustment costs are the main impediments to such consumption insurance as they limit a firm's ability to use investment to mitigate productivity shocks.

In bad times - after a sequence of poor aggregate productivity shocks - the marginal benefit of investment is low. Most firms are up against the investment irreversibility constraint - value firms. In the face of a further adverse productivity shock, there is nothing value firms can do to mitigate a further decline in their output and dividend, so must pass along the productivity shock and their stock returns must fully adjust. In contrast, growth firms with persistently high profitability can easily lower investment and maintain or even increase their dividends. Growth stocks have a relatively high payoff when consumption is low. Hence, investors demand a lower premium to hold growth stocks, which are in high demand as valuable providers of consumption insurance and low supply because most firms in the economy become value. With most firms on the irreversibility constraint, there is less flexibility in using investment to insure consumption against productivity shocks. Consumption growth becomes particularly volatile and generates a higher equity premium.

The difference between the systematic risk of value and growth firms gets smaller and even reverses in good times as the composition of the economy shifts towards growth firms facing steeper adjustment costs and value firms move away from the irreversibility constraint. This difference widens for small firms as the spread in investment elasticities gets larger for firms with small capital and their contribution to consumption insurance is only marginal given their size.

The spread in conditional market betas between value and growth stocks is high in bad times when the market premium is also high, and low or even negative in good times when the market premium is low or even negative. This endogenous covariation of conditional market betas and equity premium, which is consistent with the empirical findings in Petkova and Zhang (2005), determines the failure of the unconditional CAPM to explain average returns. In a single-factor consumption based model with power utility the Consumption CAPM holds unconditionally. However, the heteroskedasticity in consumption growth and stock market returns makes it possible for the unconditional CAPM to fail as the two series are unconditionally imperfectly correlated. Recently, Jagannathan and Wang (2007) provide empirical evidence in support of the Consumption CAPM and the failure of the CAPM.

In simulations, I find that multifactor models, including the Fama and French (1993) factor model, and to a lesser extent, conditional versions of the CAPM, outperform the unconditional CAPM. These models are more successful because they better capture the information for time-varying betas and market risk premium. To interpret the success of the Fama-French model, I note that firms' market betas can be represented as an average of betas for assets in place and betas for growth options. The Fama-French factors, HML and SMB, provide natural proxies to account for the covariation of each market betas' component and the market premium.

## I. Comparison to the Literature

A growing literature has explored the implications of production and investment on the cross section of stock returns. In their seminal contribution, Berk, Green, and Naik (1999) consider a firm as a collection of past projects and options to make profitable investments in the future. All projects have the same scale, but different risk-return profiles randomly drawn from an exogenous distribution. Over its life cycle, a firm's risk-return profile changes as past projects may die off, and new investment opportunities become available. Book-to-market and market value serve as state variables for a firm's exposure to the different systematic risk of assets-in-place and growth options, respectively. Their framework can thus provide a partial equilibrium explanation for several empirical regularities in the cross-section of stock returns.

Gomes, Kogan and Zhang (2003) extend the theoretical approach of Berk, Green and Naik (1999) to general equilibrium. They focus on a single-factor model in which the conditional CAPM holds. Instead of appealing to multiple sources of risk as in Berk et al. (1999), they emphasize the role of beta mismeasurement in generating the observed cross-sectional relations among book-to-market, market value and stock returns.

While these models have provided intuitive economic explanations for various aspects of the cross section of returns, a challenge for this class of models is the relation of the cross section of returns to fundamentals such as profitability and investment. In their economies, projects are randomly allocated to firms and are ex-ante identical across firms and over time. Once adopted, variation in the project-specific productivity affects only that project's capital. In this paper, where I model firm-specific rather than project-specific productivity, variation in firm-productivity affects current investment decisions and the entire stock of a firm's capital as in the standard Q-theory of investment. Allowing a firm to choose the optimal scale of investment according to its profitability can then generate: i) the positive relation between profitability and investment as observed in the data; ii) cross-sectional predictability of stock returns by a firm's investment and capital as confirmed in the data; iii) a new state variable in general equilibrium - namely the distribution of capital among firms with different productivity - which affects the dynamics of aggregate and firm-level variables and contributes to explain the success of the Fama and French (1993) model.

Carlson, Fisher and Giammarino (2004) study the investment decisions of a monopolistic firm over its life-cycle (juvenile, adolescent and mature). They show that growth opportunities and operating leverage represent two important factors for linking firm characteristics to market betas. The size effect arises because of limited growth opportunities as in Berk, Green, and Naik (1999), while the book-to-market effect captures the exposure to the risk of assets-in-place through operating leverage.

Cooper (2006) relates the value premium to nonconvex adjustment costs and investment irreversibility in a real option model. Value firms with excessive capital suffer during economic downturn and may benefit during economic expansions, thus they have higher systematic risk and command higher risk premium.

Zhang (2005) uses a neoclassical industry equilibrium model with adjustment costs and



costly reversibility to generate the value premium. An exogenous countercyclical market price of risk amplifies the value premium generated by costly reversibility to better fit the data on stock returns. The model is solved numerically to approximate for the equilibrium price dynamics depending on the entire industry distribution of productivity and capital.

In contrast to the above mentioned literature, I provide a full general equilibrium model, solved analytically, in which risk premia derive from investor's risk aversion and the equilibrium consumption stream. The endogenous time-varying composition of value versus growth firms in the economy generates a new channel affecting stock returns. The lower return premium on growth stocks during bad times stems not only because these stocks are in high demand as providers of consumption insurance, but also because they are in low supply as most firms become value. The latter effect shows up in the properties of the equilibrium pricing kernel through endogenous consumption. This internally consistent mechanism, which is absent from existing partial equilibrium models, is not only theoretically relevant, but also contributes to quantitatively match empirical facts. Beyond the model's ability to generate the value premium, and differently from the existing literature, I focus further on the model replication of the empirical success of corporate investments and capital in predicting stock returns, and the empirical performance of several asset pricing models including the CAPM and the Fama and French (1993) model.

## **II. The Economy**

I consider an economy populated by a continuum of heterogeneous firms that produce a single nondurable consumption good, which is the numeraire. Firms differ in their level of productivity and in the stock of capital they own. The flow of output can either be used for capital investment or it can be paid out as dividends. The representative household derives income from accumulated financial wealth, which consists of a riskless bond in zero net supply and risky assets in positive net supply. The risky assets represent claims to firms' dividends. Agents are perfectly competitive in that they formulate optimal policies taking economy wide state variables as given. In the rest of the section, I describe the environment where the interaction of households and firms takes place.

## A. Firms

Firms are infinitely-lived and all-equity financed. Given their production and investment technologies, they formulate optimal investment policies to maximize the value of equity. I assume that the set of firms  $F$  is exogenously fixed and I use subscript  $i$  to index an individual firm.

### A.1. Production

A typical competitive firm uses capital,  $K_i$ , to produce a nonstorable output flow,  $Y_i$ , according to a constant return to scale technology:

$$Y_{it} = (e^{a_t} x_{it} + f) K_{it} \quad (1)$$

where  $a$  and  $x$  denote economy-wide and firm-specific stochastic productivity, respectively. The parameter  $f$  represents a common time-invariant component of the firm marginal productivity of capital. Depending on its sign,  $f$  might be interpreted as a constant operating cost ( $f < 0$ ) or revenue ( $f > 0$ ) per unit of installed capital.

The productivity index  $a$  is common to all firms and evolves stochastically according to a mean reverting process:

$$da_t = \kappa_a (\bar{a} - a_t) dt + \sigma_a dW_{at} \quad (2)$$

where  $W_a$  is a standard Brownian motion, and  $\kappa_a$ ,  $\bar{a}$ , and  $\sigma_a$  are strictly positive.<sup>1</sup> The stochastic nature of the economy-wide productivity introduces aggregate uncertainty, thus ensuring the existence of an ex-ante equity premium, which would otherwise equal zero.

The firm-specific productivity  $x$  evolves according to a square root process:

$$dx_{it} = \kappa_x (\bar{x} - x_{it}) dt + \sigma_x \sqrt{x_{it}} dW_{it} \quad (3)$$

where  $\kappa_x$ ,  $\bar{x}$ , and  $\sigma_x$  are strictly positive, and  $W_i$  is standard Brownian process. Firm-specific productivity shocks are idiosyncratic: they are independent of each other and of

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<sup>1</sup>The process in (2) is chosen to have a stationary distribution with constant instantaneous volatility so that aggregate output growth exhibits no heteroskedasticity.

the economy-wide productivity shock.<sup>2</sup> The mean reversion property in firms' productivity not only prevents the growth rate of aggregate output from exploding, but also ensures an equilibrium nondegenerate cross-sectional distribution of firms' productivity and capital. The heterogeneity and persistence in firms' productivity is also consistent with the empirical findings in Bartelsman and Doms (2000).

### A.2. Investment

The stock of capital  $K_i$  depreciates at a common fixed rate  $0 \leq \delta \leq 1$  and it increases by undertaking gross investment at a rate  $I_i$ . Hence, the stock of capital accumulates according to the law of motion:

$$dK_{it} = (I_{it} - \delta K_{it}) dt; \quad K_{it} \geq 0 \quad \forall t. \quad (4)$$

A non-negative minimum investment,  $\widehat{I}_i = \widehat{i}K_i$ , is required each period to partially replace worn out equipment and keep a firm's installed capital productive. Investment in excess of the minimum level is irreversible and costly to adjust. The adjustment cost has the following functional form:

$$c(I_i, K_i, k_i) = \alpha k_i^{\frac{1}{n-1}} \left( \frac{I_i - \widehat{I}_i}{K_i} \right)^{\frac{n}{n-1}} K_i \quad (5)$$

where  $\alpha$  is a strictly positive adjustment parameter and  $n \in \{2, 4, 6, \dots\}$  controls the degree of curvature in (5).<sup>3</sup> The adjustment-cost technology departs from the traditional formulation in that adjustment costs are scaled by a firm relative capital,  $k_i \equiv K_i/K$ , where  $K \equiv \int_{i \in F} K_i di$ .<sup>4</sup>

Adjustment costs exhibit increasing return to scale in  $I_i$  and  $K_i$ , which makes firm growth less costly for firms with low capital relative to the average firm capital. This property leads to a conditional (on firm profitability) inverse growth-size relation: everything else equal,

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<sup>2</sup>The process in (3) is chosen to have a stationary distribution so that a law of large numbers can be applied for aggregation. Among the class of stationary processes, it has the advantage that the time- $t$  conditional expectation of  $x_{t+s}$  is linear in  $x_t$ , which facilitates the model aggregation. Furthermore, the conditional heteroskedasticity does not affect stock returns' systematic risk, which is the focus of the paper.

<sup>3</sup>Abel and Eberly (1997) use a similar adjustment cost specification in a model of firm investment decisions.

<sup>4</sup>Throughout the paper, I use the symbol  $\int_{i \in F} [\cdot] di$  to denote aggregation over the set of firms  $F$ .

smaller firms grow faster. Among others, Evans (1987) and Hall (1987) provide empirical evidence of an inverse growth-size relation robust to alternative size measures and econometric issues. In Appendix D, I provide further direct evidence of a conditional inverse growth-size relation focusing on physical capital as a measure of firm size. Furthermore, the adjustment-cost function is linearly homogeneous in  $I_i$ ,  $K_i$  and  $K$ , which restores the independence of growth and size at the aggregate level as empirically documented in Appendix D.

[Figure 1 here]

Figure 1 provides a graphical illustration of the adjustment-cost specification in (5). The irreversibility constraint makes the cost of adjusting the capital downward practically infinite, thus preventing value maximizing firms from setting their investment below the minimum level  $\hat{i}$ . This restriction, while ensuring the positivity of a firm's capital, prevents firms from partitioning into smaller parts and guarantees the existence of a well-defined competitive equilibrium.

The adjustment-cost function in (5) has an important analytic advantage: it reduces the number of economy-wide state variables to be only the economy-wide productivity and the second moment of the (joint) cross-sectional distribution of firm-specific productivity and stock of capital. This result is the by-product of the linear production technology and the independence of a firm's excess investment from its capital stock. Under the traditional adjustment-cost formulation - 1 in place of  $k_i$  in (5) - the knowledge of the entire (joint) cross-sectional distribution of firm-specific productivity and stock of capital would be necessary to compute aggregate quantities and prices. Therefore, the adjustment-cost specification in (5) allows me to focus on an *exact* general equilibrium solution rather than resorting to *approximate* solutions.

Finally, firms' equity represent claims on the stream of future dividends, which equal operating profits net of investment costs:

$$D_i = (e^a x_i + f) K_i - I_i - \alpha k_i^{\frac{1}{n-1}} \left( \frac{I_i - \hat{I}_i}{K_i} \right)^{\frac{n}{n-1}} K_i. \quad (6)$$

Taking economy wide state variables as given, firms choose the optimal investment strategy so as to maximize the expected present value of future dividends.

### B. Households

The representative household has standard time-separable preferences over consumption,  $C$ :

$$U \equiv \mathbf{E}_0 \left[ \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right], \quad (7)$$

where  $\rho > 0$  denotes the subjective discount rate and  $\gamma > 0$  is the Arrow-Pratt coefficient of relative risk-aversion. The representative household derives income from accumulated financial wealth,  $W$ . Financial markets are complete and there are no frictions or constraints in trading financial securities.

The representative household chooses paths of consumption  $\{C_t\}_{t \geq 0}$  to maximize its lifetime utility (7) subject to the static budget constraint:

$$\mathbf{E}_t \left[ \int_0^\infty \frac{\Lambda_{t+s}}{\Lambda_t} C_{t+s} ds \right] \leq W_t. \quad (8)$$

Standard optimality conditions imply the well-know relation between consumption and the pricing kernel:

$$\frac{\Lambda_{t+s}}{\Lambda_t} = e^{-\rho s} \left( \frac{C_{t+s}}{C_t} \right)^{-\gamma}. \quad (9)$$

The pricing kernel determining prices of all financial assets equals the marginal rate of intertemporal consumption substitution.

## III. Equilibrium

I characterize the equilibrium of the model in two steps. First, I determine the partial equilibrium optimal investment policy and the dynamics of the economy-wide state variables. Second, I provide the general equilibrium allocations and prices. Proofs and technical details are in Appendix A.

To solve for the equilibrium it is necessary to identify the state variables characterizing the dynamics of the aggregate state of the economy. As shown in the propositions below, the key aggregate state variables are the economy-wide productivity,  $a$ , and the variable,  $\omega$ , which represents a capital-weighted average of firm-specific productivities:

$$\omega_t \equiv \int_{i \in F} x_{it} k_{it} di. \quad (10)$$

This variable quantifies the conditional cross-sectional covariation between the firm-specific productivity,  $x$ , and its relative capital,  $k$ . The persistence of a firm-specific productivity and the path-dependent nature of its capital stock imply a nonzero endogenous cross-sectional covariation. I conjecture and verify that  $\omega$  follows the process:

$$d\omega_t = \mu_\omega(a_t, \omega_t) dt \quad (11)$$

whose drift is only a function of  $a$  and  $\omega$  itself. Let  $I_t$  denote aggregate investment defined as  $I_t \equiv \int_{i \in F} I_{it} di$ . Since firms' capital depreciates at a common rate, it follows that the aggregate stock of capital,  $K_t$ , accumulates according to:

$$dK_t = (I_t - \delta K_t) dt; \quad K_t \geq 0 \quad \forall t. \quad (12)$$

Then, a firm's relative capital evolves according to:

$$dk_{it} = k_{it} (i_{it} - i_t) dt, \quad (13)$$

where I conjecture and verify that a firm's investment rate depends on  $x_i$ ,  $k_i$ ,  $a$  and  $\omega$ , and the aggregate investment rate is a function only of the aggregate state variables  $a$  and  $\omega$ . The conjecture that  $a$  and  $\omega$  are the only aggregate state variables implies that the pricing kernel evolves stochastically according to:

$$\frac{d\Lambda_t}{\Lambda_t} = -r(a_t, \omega_t) dt - \lambda(a_t, \omega_t) dW_{at}, \quad (14)$$

with the risk-free rate,  $r$ , and the market price of risk,  $\lambda$ , depending on  $a$  and  $\omega$ . As shown in the propositions below, all the relevant information about the aggregate state of the economy contained in the (joint) cross-sectional distribution of  $x$  and  $k$  can be sufficiently summarized by its second moment,  $\omega$ . While solving for the equilibrium, I verify that (i)  $a$  and  $\omega$  are the only state variables sufficient to describe the aggregate state of the economy; and (ii) the equilibrium dynamics of  $\omega$ ,  $k$  and  $\Lambda$  satisfy the conjectured laws of motion given in (11), (13) and (14), respectively.

The following proposition states the equilibrium optimal firm investment policy.

**Proposition 1** *Given the dynamics of  $\omega$ ,  $k$ , and  $\Lambda$  described in (11), (13) and (14), the optimal investment policy of each firm  $i_i^* \equiv I_i^*/K_i$  is*

$$i_i^* = \hat{i} + \left( \frac{n-1}{\alpha n} \right)^{n-1} (q_i - 1)^{n-1} k_i^{-1} \mathbf{1}_{\{q_i \geq 1\}}, \quad (15)$$

with marginal  $q$  given by

$$q_i \equiv q(a, \omega, x_i) = \bar{x}\bar{q}(a, \omega) + [x_i - \bar{x}]\hat{q}(a, \omega) \quad (16)$$

and  $\bar{q}(a, \omega)$ ,  $\hat{q}(a, \omega)$  defined as

$$\bar{q}(a_t, \omega_t) = \mathbf{E}_t \left[ \int_0^\infty e^{-(\delta - \hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} \left[ e^{a_{t+s}} + \bar{x}^{-1} (f - \hat{i}) \right] ds \right] \quad (17)$$

$$\hat{q}(a_t, \omega_t) = \mathbf{E}_t \left[ \int_0^\infty e^{-(\kappa_x + \delta - \hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} e^{a_{t+s}} ds \right]. \quad (18)$$

The optimal investment policy originates from the first-order condition requiring a firm to invest till the marginal benefit of investment as measured by its marginal  $q$  equals its marginal opportunity cost. A firm's investment rate equals its minimum level whenever the irreversibility constraint is binding, and exceeds the minimum investment rate otherwise. The excess investment depends positively on a firm's marginal  $q$  and negatively on its cost of capital adjustment.

A firm's marginal  $q$  is increasing in the state variables  $a$ ,  $\omega$  and  $x$ . High values of  $a$  and  $x$  make the current stock of capital persistently more productive and hence more valuable. When the aggregate stock of capital is more efficiently distributed across firms - high  $\omega$  - the overall aggregate productivity increases, which in turn raises the marginal rate of intertemporal consumption substitution and affects positively the value of capital via the pricing kernel.

While  $\bar{q}$  represents the component of the marginal  $q$  common to all firms,  $\hat{q}$  captures the sensitivity of marginal  $q$  to firm-specific productivity.  $\hat{q}$  quantifies the extra contribution to the market value of a firm's capital attributable to a firm's relative competitive advantage as measured by its firm-specific productivity in excess to the market average,  $x^i - \bar{x}$ . The factor  $\exp(-(\delta - \hat{i}))$  in (17) and (18) captures the fact that productive capital depreciates at a rate  $\delta$  and increases at a rate  $\hat{i}$  because of a firm's investment commitment. Given the absence of arbitrage or equivalently the strict positivity of the pricing kernel, the parameter restriction  $f \geq \hat{i}$  suffices to ensure the positivity of a firm's marginal  $q$ .<sup>5</sup>

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<sup>5</sup>Since the utility function in (7) satisfies the Inada conditions, the equilibrium aggregate consumption is always strictly positive, which ensures the absence of arbitrage.

The cost of capital adjustment is increasing in the adjustment parameter,  $\alpha$ , the degree of convexity in the adjustment cost function,  $n$ , and a firm's relative capital,  $k_i$ . Ceteris paribus, a firm's investment rate declines with its relative capital.

The minimum investment rate,  $\hat{i}$ , along with the adjustment parameter,  $\alpha$ , control for the magnitude of scale effects in a firm's investment rate. While the adjustment-cost specification in (5) might induce scale effects in a firm's investment larger than their empirical counterparts, the endogenous positive cross-sectional correlation between a firm's productivity and capital, along with high values of  $\hat{i}$  and  $\alpha$ , reduce the elasticity of a firm's investment rate to its relative size.

[Figure 2 here]

Figure 2 provides a graphical representation of a firm's optimal investment rate as a function of the marginal benefit and cost of investment. With quadratic adjustment cost, the marginal cost of investment is linearly increasing in a firm's investment rate. Small firms face lower marginal adjustment costs. Firms with high productivity - marginal  $q$  above one - find it profitable to invest, and the optimal investment is at the intersection between marginal  $q$  and marginal cost of investment. Small firms invest more, ceteris paribus.

Firms with low productivity - marginal  $q$  below one - are better off in selling the capital rather than using it. In the absence of irreversibility, the optimal policy would lead to disinvestment. However, in the presence of irreversibility firms are prevented from disinvesting and are constrained to set their optimal investment at the minimum investment rate. Therefore, they will be burdened with a higher (than optimal) stock of unproductive capital. This is especially true for small firms: everything else equal, small firms would like to disinvest more than big firms, but are prevented from doing so.

#### *A. Heterogeneity and Aggregation*

Proposition 2 determines the aggregate (average) quantities by aggregation of their firm-level counterparts. To compute aggregate quantities I appeal to a law of large numbers for a continuum of i.i.d. random variables.<sup>6</sup> According to the optimal investment policy in

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<sup>6</sup>Aside from technicalities, models of law of large numbers for large economies have been formalized in Judd (1985), Feldman and Gilles (1985), Uhlig (1990), Anderson (1991) and Green (1994).



equation (15), a firm faces a binding irreversibility constraint whenever its marginal  $q$  falls below one, or equivalently using equation (16),  $x_i$  falls below the threshold  $\tilde{x}_t \equiv \tilde{x}(a_t, \omega_t) = [1 - \bar{x}(\bar{q}_t - \hat{q}_t)]/\hat{q}_t$ , where  $\tilde{x} \in [0, 1/\hat{q}]$ .

**Proposition 2** Define  $\theta \equiv 2\kappa_x/\sigma_x^2$  and  $v \equiv 2\kappa_x\bar{x}/\sigma_x^2$ . Then, the equilibrium aggregate output  $Y$  can be represented as

$$Y \equiv \int_{i \in F} Y_i di = (e^a \omega + f) K, \quad (19)$$

aggregate investment,  $I^*$ , can be characterized as

$$I^* \equiv \int_{i \in F} I_i^* di = \left[ \hat{i} + \left( \frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right] K, \quad (20)$$

and similarly the aggregate dividend,  $D^*$ , can be written as

$$D^* \equiv \int_{i \in F} D_i^* di = Y - I^* - \alpha^{-(n-1)} \left( \frac{n-1}{n} \right)^n g(a, \omega; n, 1) K \quad (21)$$

where

$$g(a, \omega; m_1, m_2) \equiv \hat{q}^{m_1} \sum_{k=0}^{m_1} \frac{\Gamma(m_1+1) \Gamma_U(k+v, \theta\tilde{x})}{\Gamma(m_1+1-k) \Gamma(k+m_2) \Gamma(v)} (-\tilde{x})^{m_1-k} \theta^{-k} \quad (22)$$

and  $\Gamma$  and  $\Gamma_U$  denote the gamma and the upper incomplete gamma function, respectively.

The stochastic component of the aggregate marginal productivity of capital can be represented as the product of two terms: the *exogenous* productivity index,  $a$ , and the *endogenous* productivity index,  $\omega$ . This last one accounts for the endogenous distribution of capital among firms with different productivity.

Aggregate investment is increasing in  $g(a, \omega; n-1, 1)$ , which measures the average marginal  $q$  among firms with positive excess investments, and is linearly homogeneous in aggregate capital. This last property makes the growth of aggregate capital independent of the economy scale.

The aggregate dividend is also homogeneous in aggregate capital, thus generating an “Ak” model of stochastic growth with capital adjustment costs. To ensure nonstochastic perpetual growth, it is sufficient to impose the condition  $(e^{\bar{a}\bar{x}} + f - \delta) > \rho$ .

[Figure 3 here]

Figure 3 shows the behavior of equilibrium aggregate quantities using parameters calibrated below. In bad times - low aggregate productivity - capital is overall less productive and the marginal benefit of investment in the economy is low. Most firms find it profitable to scale down their capital stock, but they cannot because investment is irreversible. As a result, aggregate investment is small and approaches the minimum investment rate,  $\hat{i}$ . Only investments in excess of the minimum rate incur adjustment costs, which are small and approach zero. Most of aggregate output is allocated to minimum investments and the excess is used for consumption. As aggregate productivity rises, capital becomes persistently more productive and more firms find it profitable to invest. Aggregate investment and adjustment costs rise. Consumption also rises as the increase in output exceeds the increase in investment and adjustment costs.

In good times - high aggregate productivity - more and more firms invest. Aggregate investment and adjustment costs are large. An increasing portion of output is now allocated to investment and lost for capital adjustment. A further increase in aggregate productivity decreases consumption as the increase in investment and adjustment costs now exceeds the increase in output.

When the stock of capital in the economy is more efficiently distributed across firms -  $\omega$  is high - the overall aggregate productivity increases. The most productive firms also have a large stock of capital, which directly increases aggregate output. Consumption decreases as an increasing fraction of the additional output is used for investment and capital adjustment. The marginal rate of intertemporal substitution raises and affects positively a firm's marginal  $q$ , consistently with the increase in aggregate investment.

Proposition 3 characterizes the dynamics of the state variable  $\omega$ .

**Proposition 3** *The endogenous component of aggregate productivity,  $\omega$ , evolves according to the stochastic process:*

$$d\omega = \left\{ \left[ \kappa_x + i - \hat{i} \right] (\bar{x} - \omega) + \left( i - \hat{i} \right) \theta^{-1} \frac{g(a, \omega; n - 1, 0)}{g(a, \omega; n - 1, 1)} \right\} dt \quad (23)$$

with the function  $g(\cdot)$  defined in (22).

The irreversibility of a firm's investment prevents a firm's capital from being negative and thus ensures the positivity of  $\omega$ . The positive relation between firm investment and

productivity generates a nonnegative cross-sectional covariation between a firm's capital stock and its firm-specific productivity. This endogenous cross-sectional covariation makes  $\bar{x}$  a lower bound for the state variable  $\omega$ .<sup>7</sup>

Since firm-specific productivities are cross-sectionally i.i.d., a law of large numbers implies that the cross-sectional distribution of firm-specific productivity  $x$  is time-invariant and equals its steady-state distribution. Therefore, the state variable  $\omega$  tracks the evolution of capital allocation among firms with different productivity. When the capital is uniformly distributed across firms - each firm has a capital stock equal to the aggregate (average) capital -  $\omega$  is equal to the steady-state mean of the firm-specific productivity  $\bar{x}$ . The higher the concentration of capital among more productive firms, the higher the value of  $\omega$ .

The instantaneous change in  $\omega$  is driven by two terms. First, the term in square brackets in (23) is positive and tend to pull  $\omega$  back to its lower bound  $\bar{x}$ . This reverting effect stems from the inverse growth-size relation: small firms tend to grow faster than big firms thus attenuating the cross-sectional dispersion in firm relative capital. This effect is stronger during economic booms as all firms benefit from the higher shadow value of capital. The second term in (23) is always positive and increases  $\omega$  as more productive firms tend to have also a larger stock of capital. This effect is stronger the higher the cross-sectional dispersion of firm-specific productivity and the better the current state of the economy. For any given relative capital distribution, the higher the cross-sectional dispersion of firm-specific productivity, the higher the value of  $\omega$ , since more firms are concentrated on the right tail of the  $x$  distribution.

### *B. Equilibrium Allocations*

I now state the definition of the competitive general equilibrium and determine the equilibrium allocation of economy-wide resources.

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<sup>7</sup>With a nonnegative cross-sectional covariation between  $k_{it}$  and  $x_{it}$ , the second moment of their joint cross-sectional distribution can be bounded from below by the product between the first moments of each cross-sectional distribution:

$$\omega \equiv \int_{i \in F} x_{it} k_{it} di \geq \int_{i \in F} x_{it} di \times \int_{i \in F} k_{it} di = \bar{x}$$

where the last equality follows from a LLNs for a continuum of i.i.d. random variables and  $\int_{i \in F} k_{it} di = 1$ .

**Definition 1** *A competitive general equilibrium is summarized by stochastic processes for the pricing kernel  $\Lambda$ , the optimal consumption policy  $C^*$ , and the optimal firm investment policy  $I_i^*$ , such that: (i) taking asset returns as given, the representative household maximizes its expected utility (7), subject to the budget constraint (8); (ii) taking the pricing kernel and aggregate capital as given, producers make investment decisions according to (15); (iii) consumption good market clears,  $C^* = D^*$ .*

The following proposition establishes the general equilibrium consumption and investment policies as the solution to a system of two partial differential equations and two algebraic equations.

**Proposition 4** *The competitive equilibrium is characterized by the optimal firm's investment policy  $i^*(a, \omega, x_i, k_i)$  described in (15) - (16), and the household's consumption policy  $C^*(a, \omega, K)$ , which satisfy:*

$$C^*(a, \omega, K) = c^*(a, \omega) K \quad (24)$$

where

$$c^* \equiv e^a \omega + f - \hat{i} - \left( \frac{n-1}{\alpha n} \right)^{n-1} \left[ g(a, \omega; n-1, 1) + \left( \frac{n-1}{n} \right) g(a, \omega; n, 1) \right] \quad (25)$$

with the function  $g(\cdot)$  defined in (22), and

$$\bar{q} = c^*(a, \omega)^\gamma \bar{\Phi}(a, \omega) \quad (26)$$

$$\hat{q} = c^*(a, \omega)^\gamma \hat{\Phi}(a, \omega) \quad (27)$$

where  $\bar{\Phi}(a, \omega)$  and  $\hat{\Phi}(a, \omega)$  satisfy the partial differential equations (A33) - (A34) in Appendix A.

The endogeneity of consumption and pricing kernel provides a complete description of the economic mechanisms affecting equilibrium investment and returns.

### C. Equilibrium Asset Prices

I now characterize equilibrium asset prices and returns.

**Proposition 5** *The equilibrium risk-free rate is:*

$$r(a, \omega) = \rho + \gamma \frac{\mathcal{D}^{a, \omega, K} [C^*]}{C^*} - \frac{1}{2} \gamma (\gamma + 1) \sigma_a^2 \left[ \frac{\partial_a C^*}{C^*} \right]^2 \quad (28)$$

*and the equilibrium market price of productivity risk:*

$$\lambda(a, \omega) = \gamma \sigma_a \frac{\partial_a C^*}{C^*} \quad (29)$$

where  $\mathcal{D}^{a, \omega, K} [\cdot]$  denotes the infinitesimal generator of the stochastic processes  $a$ ,  $\omega$  and  $K$  :

$$\mathcal{D}^{a, \omega, K} [M(\cdot)] = \kappa_a (\bar{a} - a) \partial_a M(\cdot) + \frac{\sigma_a^2}{2} \partial_{aa}^2 M(\cdot) + \mu_\omega(a, \omega) \partial_\omega M(\cdot) + (I^* - \delta K) \partial_K M(\cdot). \quad (30)$$

The risk-free rate and the market price of risk are standard results from power utility. As conjectured in (14), both variables are functions only of the economy-wide productivity  $a$  and the state variable  $\omega$ .

Firms' equity represent claims on the dividends paid out to shareholders.

**Proposition 6** *A firm market value,  $V_i$ , is:*

$$V_i = \mathbf{E}_t \left[ \int_0^\infty \frac{\Lambda_{t+s}}{\Lambda_t} D_{t+s}^i ds \right] = q(a, \omega, x_i) K_i + h(a, \omega, x_i) K, \quad (31)$$

where a firm's marginal  $q$  (i.e.  $\partial V_i / \partial K_i$ ) is described in (16) and a firm's marginal  $h$  (i.e.  $\partial V_i / \partial K$ ) is determined by:

$$h(a, \omega, x_i) = c^*(a, \omega)^\gamma H(a, \omega, x_i) \quad (32)$$

where  $H(a, \omega, x_i)$  satisfies the partial differential equation (A42) in Appendix A.

A firm market value is characterized as the sum of two components: value of assets in place,  $V_i^A$ , and value of growth opportunities,  $V_i^O$ . The value of assets in place is the present value of future operating profits accruing to the stock of capital currently in place, and equals a firm's marginal  $q$  times its stock of capital. The value of growth opportunities is the present value of rents accruing to a firm because of the adjustment technology, and equals a firm's marginal  $h$  times the aggregate stock of capital.

A firm's marginal  $h$  quantifies the marginal contribution to a firm's market value of a reduction in the capital adjustment cost. The higher the average capital in the economy, the lower a firm's relative size, the lower the capital adjustment costs, the more valuable the option to use investment to take advantage of the current economic conditions. A firm's marginal  $h$  is positive and increasing in the state variables  $a$ ,  $\omega$  and  $x_i$ : a decrease in the marginal cost of investment is more valuable when a firm is more productive.

The non-homogeneity property of a firm's dividends translates one-to-one into its market value, thus creating a wedge between marginal  $q$  (i.e.  $\partial V_i/\partial K_i$ ) and Tobin's  $Q$  (i.e.  $V_i/K_i$ ): the value of growth opportunities per unit of installed capital,  $h_i K/K_i$ . The higher a firm's profitability and the lower its relative size, the larger this wedge.

The stock market value is the price of a claim to consumption, which in equilibrium equals aggregate dividends. The law of one price and the absence of arbitrage ensure that its value can be computed by aggregating the market value of all firms.

**Proposition 7** *The stock market value,  $V$ , is:*

$$V = [q_m(a, \omega) + h_m(a, \omega)] K, \quad (33)$$

where  $q_m$  denotes the average of firms' marginal  $q$ :

$$q_m(a, \omega) = \bar{x}\bar{q}(a, \omega) + [\omega - \bar{x}]\hat{q}(a, \omega),$$

where the functions  $\bar{q}$  and  $\hat{q}$  are described in (26) - (27), and the function  $h_m$  represents the average of firms' marginal  $h$ :

$$h_m(a, \omega) = c^*(a, \omega)^\gamma H_m(a, \omega), \quad (34)$$

where the function  $H_m$  satisfies the partial differential equation (A48) in Appendix A.

The aggregate stock market value is the sum of aggregate assets in place  $V^A$  - average of firms' marginal  $q$  times average stock of capital - and aggregate growth opportunities  $V^O$  - average of firms' marginal  $h$  times average stock of capital.

### C.1. Stock Returns and Conditional CAPM

The single-factor nature of the model, where the only source of systematic risk is aggregate productivity uncertainty, implies conditional perfect correlation between the instantaneous stock market return and the pricing kernel. Therefore, the market portfolio is instantaneously conditionally mean-variance efficient, and the cross-sectional distribution of expected returns can be fully determined by the distribution of firms' conditional market betas. The next proposition establishes the risk-return relation as a Conditional Capital Asset Pricing Model (CCAPM).

**Proposition 8** *The instantaneous risk and expected return of individual firms can be characterized by a conditional CAPM:*

$$\mu_{R_i,t} = r_t + \beta_{it} [\mu_{R,t} - r_t], \quad (35)$$

with the conditional market beta given by

$$\beta_{it} = \frac{K_{it}}{V_{it}} q_{it} \beta_{it}^A + \frac{K_t}{V_{it}} h_{it} \beta_{it}^O, \quad (36)$$

where  $\beta_{it}^A$  and  $\beta_{it}^O$  measure the risk of a firm's assets in place and growth opportunities as described in Appendix A.

The decomposition of stock prices into the value of assets in place and the value of growth opportunities provides a convenient framework to relate the riskiness of firms' stock returns and observable firm characteristics. A firm's market beta depends on both aggregate and firm-specific state variables. Since firms' dividends and prices are not homogeneous in the stock of capital, both variables  $x_i$  and  $k_i$  generate cross-sectional differences in market betas.

Any pair of observable firm characteristics function of these variables can potentially be used to identify cross-sectional differences in firm market betas and expected returns. Two natural candidates are market equity and book-to-market. These observable firm characteristics show up directly in the market beta representation as value-weighted average of betas for assets in place and growth opportunities. These betas measure the elasticity of a firm's value of assets in place and growth options to changes in the stock market value, respectively.

They are positive as both components of a firm's market value increase when also the stock market value increases - namely, after positive productivity shocks. Therefore, market betas and expected stock returns depend positively on book-to-market and negatively on market equity. These relations are investigated further in the empirical section.

#### *D. Economic Mechanisms*

Before proceeding to the quantitative evaluation of the model ability to match empirical facts, I underline the economic mechanisms behind the model main results. Figure 4 provides a graphical representation using parameters calibrated below.

**[Figure 4 here]**

Growth firms accumulate capital to take advantage of their persistently high profitability and incur convex capital adjustment costs. As a consequence, their output and dividends move in opposite directions in response to aggregate productivity shocks. In contrast, value firms, which have persistently low productivity, are unprofitable and willing to cut back their stock of capital, but cannot because investment is irreversible. Thus, their dividends share the properties of output in response to productivity shocks.

This differential response of dividends to aggregate shocks, which is in line with the empirical evidence in Xing and Zhang (2004), affect the properties of consumption, whose behavior depends on the endogenous composition of value versus growth firms in the economy and the distribution of capital among them.

When aggregate productivity is low - bad times - the marginal benefit of investment is low. Most firms become value - against the irreversibility constraint - and own most of the capital stock in the economy. Then, consumption inherits mainly the properties of value firms' dividends: an adverse productivity shock decreases consumption as the reduction in output cannot be offset by disinvestment. The lack of flexibility in using investment to insure consumption against productivity shocks generates a more volatile consumption stream, and hence a higher equity premium.

Growth stocks face less steep adjustment costs. Their elastic investment attenuates the response of stock returns to productivity shocks, thus reducing their systematic risk. An adverse shock causes their dividends to increase, while consumption decreases. As valuable



providers of consumption insurance, growth stocks are in high demand, but low supply because most firms become value. Therefore, in equilibrium investors demand a lower premium to hold growth stocks. In contrast, the investment of value firms is inelastic to changes in productivity, thus their stock returns must fully adjust to absorb productivity shocks. An adverse shock causes their dividends to decrease, while also consumption decreases. As such they are in low demand and high supply, which leads investors to demand a higher premium.

In good times - aggregate productivity is high - the composition of the economy shifts towards growth firms and output and consumption move in opposite directions in response to productivity shocks. Investment can now be used to insure consumption, which reduces the volatility of consumption growth. The equity premium can be even negative because market returns provide a hedge against productivity shocks: the stock market has a high payoff when consumption is low.

The systematic risk of growth stocks increases as the economy becomes more productive because the investment of growth firms becomes less elastic as they now face steeper adjustment costs. In contrast, the market beta of value firms decreases because their stock of capital becomes progressively less unprofitable - they face less severe irreversibility constraint. As a result, the spread in betas between value and growth stocks is positive in bad times and becomes even negative in good times. Although the risk spread becomes negative when the economy is particularly productive, growth stocks still earn lower expected returns than value stocks because the equity premium also becomes negative. Growth stocks co-vary more with market returns, which now become negatively correlated with consumption growth.

The value spread in market betas widens for small firms because the spread in investment elasticities gets larger for firms with small capital. The investment of small firms is more elastic to changes in productivity as they face less steep adjustment costs. While small growth firms benefit from the higher investment elasticity, small value firms become more adversely affected by the irreversibility of investment. They would like to disinvest more than big value firms, but they cannot do so. As a result, their stock returns has to adjust more to absorb aggregate productivity shocks. In addition, given the small size of their capital stock, they can only contribute marginally to insure consumption. Therefore, the

value premium gets larger for small stocks.

The endogenous covariation of conditional market betas and equity premium, which is consistent with the empirical findings in Petkova and Zhang (2005), determines the failure of the unconditional CAPM to explain average returns. While unconditional market betas might account for the high average betas of value stocks and the low average beta of growth stocks, they do not capture their positive and negative covariation with the conditional equity premium, respectively. As a result, the unconditional CAPM underestimates average returns on value stocks - positive alphas - and overestimates average returns on growth stocks - negative alphas. In contrast, multifactor models such as conditional versions of the CAPM and the Fama-French (1993) model can be more successful to the extent they better capture the information for time-varying betas and market premium.

#### IV. Empirical Analysis

In this section I conduct a simulation study to evaluate the model's ability to reproduce the main empirical properties of firm investments and stock returns. The empirical analysis is based on a panel of firms drawn from the CRSP-COMPUSTAT merged database for the years 1962 - 2002. The description of the data is provided in Appendix C.

##### *A. Calibration and Estimation*

The simulation study is based on the parameters summarized in Table I. The subjective discount rate  $\rho$  is set to be 0.01%, which is a typical value in the macro and asset pricing literature. I set the degree of curvature in the adjustment cost function  $n$  equal to 2 so that a firm's investment is linear in marginal  $q$ . In addition, the empirical evidence in the investment literature does not favor a convex relation between investment and marginal  $q$  - see Abel and Eberly (2002). I choose the minimum investment rate  $\hat{i}$  to be 0.12, which approximately matches the average investment rate among slow growing firms - value firms in the 90th percentile of the book-to-market distribution. The constant component of firm productivity  $f$  equals the minimum investment rate  $\hat{i}$ . This restriction ensures the strict positivity of marginal  $q$ . All firms are always productive enough to meet the minimum investment commitment, which allows me to isolate the impact of investment irreversibility

as a source for the value premium and abstract from an operating leverage mechanism. The long-run mean of idiosyncratic productivity  $\bar{x}$  is set to 1.00.

The values of the remaining model parameters used in the simulation study are as follows: risk aversion coefficient,  $\gamma$ , 15.57; adjustment cost parameter,  $\alpha$ , 2.27; depreciation rate,  $\delta$ , 0.11; long-run mean of the aggregate productivity,  $\bar{a}$ , -2.59; speed of mean reversion of aggregate productivity,  $\kappa_a$ , 0.27; volatility of aggregate productivity,  $\sigma_a$ , 0.05; speed of mean reversion of idiosyncratic productivity,  $\kappa_x$ , 0.17; and volatility of idiosyncratic productivity,  $\sigma_x$ , 0.42. I choose these values to match key unconditional aggregate and cross-sectional moments: unconditional mean and standard deviation of consumption growth, aggregate investment rate and equity premium; unconditional mean of value and size premia, average cross-sectional volatility of stock returns and average cross-sectional correlation between (the logarithms of) size and book-to-market.

[Table I here]

Table II compares the model implied target moments with their empirical counterparts. I simulate 100 artificial panels each with 200 firms and 5,000 years. I calculate aggregate and cross-sectional moments for each artificial panel and then I report cross-sample averages, standard deviations and 95 percent confidence intervals.

[Table II here]

The model captures reasonably well the historical level and volatility of the equity premium, while maintaining reasonably low values for the first two moments of aggregate consumption growth. Specifically, the mean and volatility of the equity premium are 6.0% and 26.1% versus 7.7% and 20.4% in historical data, respectively. Consumption growth averages about 2.5% and has a volatility of 4.9% versus 2.2% and 2.5% in historical data, respectively. With power utility and low historical volatility of consumption growth this can be achieved only with a sizeable value for the risk-aversion coefficient, 15.57. The time separable nature of preferences also implies that most of the variation in the equity premium is due to variation in the risk-free rate, whose volatility averages about 22.0% versus 4.0% in historical data.

In the model, economic growth occurs via capital accumulation, which implies that the average consumption growth equals approximately the average net investment rate, 2.1%. In addition, the endogenous determination of aggregate consumption implies a negative autocorrelation for consumption growth (ranging from -0.05 over one year to -0.07 over five years) at odds with the near random walk pattern or even positive autocorrelation observed in historical data.

The model matches well the value and size premia, 6.6% and 3.7% versus 6.2% and 4.4% in historical data, respectively. However, this is possible at the expense of an aggregate investment rate which is on average smaller, 12.8%, and less volatile, 1.0%, than its historical counterpart, whose mean and volatility are at 17.8% and 3.7%, respectively. In fact, the model can generate substantial equity, value and size premia when most of the firms are on average up against the investment irreversibility constraint, which reduces the overall level and volatility of investment in the economy despite the low adjustment cost parameter, 2.27. Adjustment costs average about 1.1% of aggregate output consistently with previous empirical estimate - among others, Cochrane (1991) also reports a fraction of output lost for adjustment of about 1.0%. Despite the low volatility of aggregate investment, the model can still generate substantial variation in the aggregate Tobin's Q (whose volatility averages about 24.0%) because aggregate investment depends only on the average marginal  $q$ .

The average cross-sectional volatility of stock returns, which is a direct measure of the degree of dispersion in the cross-sectional distribution of firms, is approximately 21.5%. Although lower than its historical counterpart in the sample under consideration, 34.7%, it is close to the average of 25.0% reported by Campbell et al. (2001). The average cross-sectional correlation between (the logarithms of) size and book-to-market is about -0.58 versus -0.31 in historical data. Nonetheless, the model simulations can still disentangle both value and size effects in the analysis of cross-sectional predictability of stock returns as shown below.

While it is unlikely that a model with power utility provides an accurate description of the empirical relation between aggregate consumption and equity premium, this seems overall an acceptable approximation given the focus of the paper on the cross-sectional properties of investment and stock returns.

### *B. Predictability of Market Returns*

I now investigate the model implications about the predictability of stock market returns. Table III reports standard deviations and autocorrelations of aggregate log dividend yield and log book-to-market (Panel A) and the results of predictability regressions of market returns on these variables (Panel B). To facilitate the comparison with historical data, I simulate 100 artificial panels each with 50 years, which is comparable to the sample size used in the empirical asset pricing literature and approximately equal to the length of the historical sample under consideration. I then report cross-sample averages of summary statistics.

[Table III here]

The model captures well the historical standard deviations and autocorrelations of both log dividend yield and log book-to-market. The volatility of the log dividend yield is about 31% versus 37% in historical data. Similarly, the log book-to-market is only slightly less volatile, 26%, than its historical counterpart, 29%. Both series are highly persistent and positively autocorrelated. The autocorrelation of log book-to-market decreases from 0.79 over one year to 0.27 over five years versus 0.85 and 0.48 in historical data. The log dividend yield shares a similar decreasing pattern in autocorrelations both in simulated and historical data.

Consistently with historical data, the coefficients of cumulative market returns on log dividend yield or log book-to-market are positive: low prices relative to dividend or book value imply high expected returns. The coefficients on log dividend yield rise with the forecast horizon ranging from 0.22 over one year to 1.60 over five years versus 0.15 and 0.81 in historical data, respectively. Adjusted  $R^2$ s also build up with the forecast horizon increasing from 0.06 to 0.44 over one and five years versus 0.04 and 0.18 in historical data, respectively. The statistical significance of the log dividend yield as a predictor of future market returns also increases with the horizon.

I find a similar pattern in the regression coefficients, statistical significance and adjusted  $R^2$  when using the log book-to-market as a predictor of market returns in both simulated and historical data. While both log dividend yield and log book-to-market have similar low

predictability at short horizons, the log book-to-market has a higher predictability power over the long horizon as suggested by its higher adjusted  $R^2$  in both datasets.

The predictability of market returns stems from the persistence in the dividend yield and book-to-market ratio, and the countercyclical behavior of consumption growth and market returns volatilities. These business cycle properties are consistent with the empirical findings in Kandel and Stambaugh (1990), Bansal and Yaron (2004), Bekaert and Liu (2004), and Lettau, Ludvigson, and Wachter (2007).

**[Figure 5 here]**

Figure 5 provides a graphical illustration of the economic mechanism behind the predictability of market returns. In bad times, the stock market value decreases relative to its book value and dividend because the stock of capital becomes persistently less productive. More firms become unprofitable and are up against the investment irreversibility constraint. The economy become riskier as aggregate consumption growth and hence stock market returns become more volatile. Thus, investors require higher expected market returns to hold claims on consumption. The reverse holds true in good times.

### *C. Cross-Sectional Predictability of Returns*

This section establishes the key quantitative results. After examining the relation between firm characteristics and stock returns, I investigate the performance of alternative asset pricing models including unconditional and conditional versions of the CAPM and the Fama and French (1993) model. To facilitate the comparison with historical data, I simulate 100 artificial panels each with 200 firms and 50 years. I then report cross-sample averages of the relevant statistics.

#### *C.1. Stock Returns and Firm Characteristics*

In Tables IV and V I report average excess returns and firm characteristics of portfolios sorted on book-to-market and market equity, respectively. Panel A shows summary statistics based on historical data, and Panel B those based on simulated data.

**[Table IV here]**

According to Table IV, the pattern of excess stock returns and firm characteristics in simulated data matches the empirical evidence well. Average returns on book-to-market sorted portfolios fall from 14.2% per year for the highest book-to-market portfolio (95th percentile) to 5.7% for the lowest (5th percentile) versus 13.7% and 3.0% in historical data, respectively. The portfolio's Sharpe ratios share the same decreasing pattern, ranging from 0.43 to 0.19 versus 0.58 and 0.12 in historical data.

The spread in average profitability between growth and value is about 22%, which is close to its historical counterpart, 27%. While the spread in profitability is large, the spread in relative capital averages only about 0.42 versus -0.35 in historical data. The small spread in relative capital confirms that the cross-sectional variation in book-to-market is mostly profitability driven.

Average investment rates correlate positively with profitability both in historical and simulated data. The spread in investment rates between growth and value portfolios is about 11% versus 23% in historical data. The model produces virtually no spread among high book-to-market portfolios (above 50th percentile). The stocks in these portfolios are on average up against the investment irreversibility constraint with their investment rate flat at the minimum investment rate, 12%, and their book-to-market well above one. The joint characterization of aggregate and cross-sectional dynamics requires most firms to face on average investment irreversibility in order to generate substantial equity and value premia.

[Table V here]

The pattern of excess stock returns and firm characteristics for portfolios sorted on market equity is also consistent with historical data. As shown in Table V, average stock returns and Sharpe ratios decrease with market equity. Average returns fall from 10.8% for the smallest size portfolio (5th percentile) to 5.1% for the largest versus 16.4% and 3.8% in historical data, respectively. Similarly, Sharpe ratios decrease from 0.33 to 0.17 versus 0.63 and 0.23 in historical data.<sup>8</sup>

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<sup>8</sup>Differently from the historical samples commonly used in the empirical asset pricing literature, the more stringent sample selection criteria used to form this historical sample generate a larger spread in average returns among portfolios sorted on book-to-market and market equity.

The profitability of stocks in the smallest size portfolio averages about 4.4% and monotonically increases to 20.3% for the largest size portfolio versus 0.5% and 13.3% in historical data, respectively. Relative capital also increases monotonically from the smallest to the largest size portfolios with values ranging from 0.19 to 2.54 consistently with historical data. The large spread in relative capital contributes substantially to generate the cross-sectional variation in market equity.

Although the historical sample under consideration does not show a clear pattern in investment rates, the model generates a slight increase from 12.2% for the smallest portfolio to 15.2% for the highest. The large variations in both profitability and relative capital have offsetting effects on investment rates as induced by the inverse growth-size relation.

To disentangle book-to-market and scale effects on average returns and investment rates, I report in Table VI average excess returns and firm characteristics across 3x3 portfolios formed by a two-dimensional sort of stocks on firm market equity and book-to-market. Panel A shows summary statistics based on historical data, and Panel B those computed on simulated data.

[Table VI here]

In simulated data, a double sort of stocks on book-to-market and market equity disentangles successfully the variation in average returns and investment rates attributable to difference in profitability and relative capital. Book-to-market captures most of the variation in profitability as there is little variation across market equity percentiles, and market equity mainly captures the variation in capital as there is only a modest variation across book-to-market percentiles.

Both in historical and simulated data, the size of the value premium varies with market equity: the value premium is larger for small stocks. In simulated data, the value premium falls from about 8.0% per year for the small size portfolios to about 3.5% for the big size portfolios versus 8.0% and 5.8% in historical data, respectively.

Consistent with the inverse growth-size relation, small growth firms have lower profitability, smaller relative capital, and higher investment rates than big growth firms. This pattern is also confirmed in historical data. In the model, small growth firms also earn lower average



returns than big growth firms as they face less steep adjustment costs and their investment is more elastic to changes in aggregate productivity. In contrast, small value firms earn higher average returns than big value firms while being both affected by investment irreversibility. Small value firms have a slightly lower profitability and a much smaller relative capital than big value firms.

To establish quantitatively the relation between stock returns and firm characteristics, I report in Table VII the results of Fama-MacBeth regressions of individual excess stock returns on firm characteristics such as book-to-market and market equity, and firm fundamentals such as investment rates and relative capital. Panel A and B report statistics based on historical and simulated data, respectively. All dependent variables are in logs.

**[Table VII here]**

The first two univariate regressions (line 1 and 2) confirm that book-to-market and size have useful information about the cross-section of stock returns. The relation between returns and book-to-market is significantly positive, while the relation with market equity is significantly negative. The regression coefficients on book-to-market, 5.3%, and market equity, -1.5%, are close to their historical counterparts, 4.4% and -1.4%, respectively.

When both book-to-market and market equity are used as dependent variables (line 3), their coefficients become about 5.2% and -0.04%, respectively. Both coefficients are statistically significant at conventional levels. While the coefficient on book-to-market is in line with historical data, 3.1%, the market equity coefficient is much lower than its historical counterpart, -1.1%. The lower coefficient on market equity is partially due to the relatively high cross-sectional correlation between book-to-market and size in simulated data. In addition, since book-to-market captures cross-sectional variation in profitability, which determines whether a firm faces investment irreversibility, the control for book-to-market leaves to market equity, which captures variation in relative capital, only a residual effect. However, book-to-market effects are economically more significant than size effects in both historical and simulated data.

The addition of an interaction term between market equity and book-to-market (line 4) results in a negative coefficient of about -1.4% versus -0.6% in historical data. The nega-

tive and statistically significant coefficient on the interaction term confirms the empirically observed negative relation between the value premium and market equity.

In the second bivariate regression (line 5), I regress excess stock returns on investment rate and relative capital. The average slopes confirm the negative relation of average stock returns with both these variables. Both coefficients are more than two standard errors from zero. Controlling for firm relative capital, doubling a firm's investment rate decreases on average its stock returns by about 4.1% per year versus 1.8% in historical data. Similarly, a doubling of relative capital leads to a reduction in average returns of about 0.4% per year versus 1.1% in historical data. Moreover, the effect of market equity on average stock returns is of a similar order of magnitude of that generated by a firm relative capital, both in simulated and historical data.

When I consider book-to-market and relative capital (in place of market equity) as dependent variables (line 6), the statistically significant coefficients still confirm the positive and negative relation of average returns with book-to-market and relative capital, respectively. Controlling for a firm relative capital, doubling a firm book-to-market rises average returns by about 5.3% versus 3.5% in historical data. Similarly, a double of relative capital leads to a reduction in average returns of about -0.04% versus -1.0% in historical data. These values are the same order of magnitude of the coefficients on book-to-market and market equity (line 3), respectively. The similar magnitudes confirm that, controlling for book-to-market, market equity mainly captures cross-sectional variation in stock returns driven by differences in capital.

In line 7, I run excess stock returns on market equity and investment (in place of book-to-market): both coefficients are negative and statistically significant. Market equity enters with a coefficient of about -1.1% as in historical data. Investment has coefficient of -4.0% versus -1.7% in historical data. These values suggest that, controlling for market equity, investment rates mostly capture cross-sectional variation in stock returns driven by differences in profitability or similarly in market-to-book.

Therefore, book-to-market and market equity on one hand, and investment rate and relative capital on the other hand, capture similar expected profitability and scale effects in average stock returns in both simulated and historical data. Moreover, expected profitability

effects - captured by book-to-market or investment rate - are economically more important than scale effects - captured by market equity or relative capital - in predicting cross-sectional variation of stock returns.

[Figure 6 here]

[Figure 7 here]

In the model, book-to-market and investment rate are related to expected returns because they proxy for firm profitability: firms with high book-to-market and low investment rate are less profitable and therefore less valuable to investors looking for consumption insurance. Figure 6 and 7 plot the average profitability and investment rate of value and growth portfolios for 11 years around portfolio formation and in the time series based on simulated and historical data, respectively. The figures show that book-to-market is associated with persistent differences in profitability and investment rates. Growth firms are on average more profitable and faster growing than value firms for five years before and after portfolio formation. The profitability of growth firms improves prior to portfolio formation and deteriorate thereafter. The opposite is true for value firms. Investment rates follow a similar pattern. Both patterns are driven by the mean-reverting behavior of firm productivity and the endogeneity of firm investment. These patterns are confirmed in historical data. The persistent difference in profitability and investment rate between value and growth is also confirmed when examined chronologically, though in historical data these series are more volatile. In sum, firm profitability and investment rate are what determines value or growth characteristics.

### *C.2. Asset Pricing Models*

A central finding in the asset pricing literature is the failure of the CAPM to explain cross-sectional differences in average stock returns. In this section, I investigate the extent to which the model is consistent with the failure of the CAPM and the relative success of alternative asset pricing models such as conditional versions of the CAPM and the Fama and French (1993) model. I use as test assets the twelve book-to-market sorted portfolios, which provide a sizeable spread in average returns.

[Table VIII here]

In Table VIII I report descriptive statistics for the equity premium and the Fama and French (1993) factors, SMB and HML. Statistics based on historical data are from the CRSP-COMPUSTAT merged database for the years 1962 - 2002. The size of the premia are comparable in both historical and simulated data. In historical data the equity premium averages about 5.0% per year versus 7.5% in simulated data. The average returns on SMB and HML are 2.5% and 5.1% versus 0.5% and 5.0% in simulated data, respectively. While both equity and value premia are statistically significant at conventional levels, average returns on SMB are insignificant in both datasets. Correlations among the series are also comparable in both datasets, with the exception of the correlation between the equity and value premia, which is negative in this historical sample and positive in simulated data.

[Table IX here]

Table IX shows the results of time-series regressions of excess returns on each of the twelve portfolios on the excess returns on the market portfolio. I report the results based on historical and simulated data in Panel A and B, respectively. Each panel shows the intercepts,  $\alpha$ , and the market betas,  $\beta_M$ , along with their corresponding standard errors. Standard errors starred with an asterisk are statistically significant at the five percent level. I also report summary statistics such as a  $\chi^2$ -test for the null hypothesis of alphas jointly equal to zero, the root mean squared alphas (RMSA) and the Hansen-Jagannathan distance (HJD).

Both in historical and simulated data, the alphas are large and mostly statistically significant. The  $\chi^2$ -tests strongly reject the null hypothesis of alphas jointly equal to zero with p-values almost indistinguishable from zero. The CAPM mispricing is conveniently summarized by the RMSA of about 3.1% per year versus 4.3% in historical data. The alphas share the same increasing pattern in both simulated and historical data: growth stocks have large negative alphas whereas value stocks have large positive ones. This pattern is consistent with the model predictions: the unconditional CAPM overestimates average returns on growth stocks and underestimates average returns on value stocks as it does not account for their negative and positive covariation with the conditional equity premium, respectively.

The market betas are all statistically significant in both datasets. While they exhibit mostly a pattern decreasing with average excess returns in historical data - value premium puzzle - they increase slightly from 0.93 for the growth portfolio to 1.19 for the value portfolio in simulated data. Therefore, unconditional market betas partially account for the low average beta of growth stocks and the high average betas of value stocks, though the small spread in unconditional betas can only account marginally for the large spread in average returns. The failure of the CAPM is also represented graphically in Figure 8, where I plot the model predicted versus actual mean excess returns for both historical (Panel A) and simulated (Panel C) data. In both cases, mean excess returns line up almost vertically rather than on the 45 degree line.

**[Figure 8 here]**

Table X shows the results of Fama-MacBeth cross-sectional regressions in historical (line 1) and simulated (line 6) data. The cross-sectional intercept is statistically significant in both datasets. Although statistically significant only at the ten percent level, the estimated market premium in historical data is -14.4%, whose negative sign is indeed consistent with the decreasing pattern in unconditional market betas. In contrast, the estimated market premium in simulated data is 36.8%, whose sign is consistent with the slightly positive cross-sectional covariation between unconditional betas and average returns. Based on statistical and economic considerations such as the size of the estimated market premium and the significance of the intercept, the unconditional CAPM fails to price book-to-market sorted portfolios in both historical and simulated data.

**[Table X here]**

In contrast, when I use the model implied market beta as dependent variable (line 11), the cross-sectional intercept becomes small and statistically insignificant. The estimated market premium is statistically significant and exceeds its time-series counterpart only by about 1%. Additionally, the adjusted  $R^2$  increases to 98%. Thus, the failure of the unconditional CAPM stems from its inability to properly account for the time variation in betas and market premium.

In the effort to improve on the unconditional CAPM performance, I test its conditional version based on empirically observable variables able to forecast future returns. In addition to excess market returns, I include as a factor the interaction term of a conditioning variable and excess market returns.

Line 2 and 7 show the results of a conditional CAPM with aggregate log book-to-market as conditioning variable in historical and simulated data, respectively. In historical data, this version of the conditional CAPM does not improve on the CAPM performance. The cross-sectional intercept is slightly larger than its CAPM counterpart and statistically significant. The estimated market premium is also wrong signed. Both the estimated market premium and the coefficient on the interaction term are statistically insignificant.

In simulated data, there is some improvement on the CAPM performance. Both the cross-sectional intercept and the estimated market premium are smaller than their CAPM counterparts and statistically significant. The coefficient on the instrumented market premium is positive as in historical data and statistically significant. The adjusted  $R^2$  increases up to 74%. While performing better than its unconditional counterpart, this conditional version of the CAPM is still far from correctly pricing book-to-market sorted portfolios.

In line 3 and 8 I report the results of a conditional CAPM with aggregate log dividend yield as conditioning variable in historical and simulated data, respectively. In both historical and simulated data, there is some improvement on the CAPM performance. The adjusted  $R^2$  increases up to 25% in historical data versus 74% in simulated data. The coefficient for the instrumented market is positive and statistically significant in both datasets. However, the cross-sectional intercept is still statistically significant and the size of the estimated market premium is an order of magnitude different from its time-series counterpart. Therefore, based on economic and statistical considerations also this conditional version of the CAPM does not properly account for the time variation in betas and market premium. While both book-to-market and dividend yield can forecast market returns at long horizon, their low predictability power at the short horizon weakens their performance as conditioning variables.

Finally, I test the performance of a two factor model including excess market returns and HML, and the Fama-French (1993) model. In both simulated and historical data, includ-

ing only excess market returns and HML makes a substantial improvement over the CAPM performance in both its unconditional and conditional versions. This substantial improvement can be seen in line 4 and 9 of Table X. The estimated market premium is positive and smaller than its CAPM counterparts in both datasets. The estimated value premia is slightly higher than the average return on HML and statistically significant in both simulated and historical data. The cross-sectional intercept is statistically indistinguishable from zero consistently with economic restrictions. Additionally, the adjusted  $R^2$  rises up to 77% and 79% in historical and simulated data, respectively.

The Fama and French (1993) model outperforms all the above mentioned empirically based asset pricing models. In historical data, the cross-sectional intercept is insignificant, the estimated market premium and the loadings on HML and SMB are only slightly higher than their time-series counterparts. In addition, the adjusted  $R^2$  rises up to 81%.

In simulated data, the estimated market premium is an order of magnitude smaller than the CAPM estimates, but still higher than its time-series counterpart. The estimated value and size premia are 5.5% and 1.4% versus 5.0% and 0.5% as time-series averages, respectively. While being statistically insignificant, the inclusion of SMB brings the size of the equity and value premia closer to their time-series counterparts. The cross-sectional intercept is also indistinguishable from zero. The adjusted  $R^2$  also rises to about 84%.

[Table XI here]

To investigate further the success of the Fama-French model, I report in Table XI the results from time-series regressions. Both in historical and simulated data, the alphas are lower by an order of magnitude relative to the CAPM alphas, and only few of them - namely, value portfolios 10A and 10B - remain statistically significant in the model generated data. The market beta flattens out across portfolios (a common finding in the asset pricing literature) and the loadings on SMB and HML share similar patterns: from negative values for growth portfolios they increase to positive values for high book-to-market portfolios. While only few loadings on SMB are significant - mostly for value portfolios - most of HML loadings are statistically significant. Most importantly, the  $\chi^2$ -tests cannot reject the null hypothesis of alphas jointly equal to zero at conventional significance levels.

The success of the Fama and French (1993) model is summarized graphically in Figure 8. Mean excess returns line up much better on the 45 degree line. In simulated data, the root mean squared alphas decreases from 3.1% per year for the CAPM to 1.6% for the Fama and French model. Similarly, it falls from 4.3% to 1.7% per year, in historical data.

To interpret the success of the Fama-French model, note that firms' market betas can be represented as a value-weighted average of betas for assets in place and betas for growth options as shown in (36). The weights on the beta for assets in place and beta for growth options depend directly on the firm book-to-market and market equity, respectively. According to its definition, HML is the difference between returns on high and low book-to-market portfolios with about the same market equity. This difference should be largely free of the influence of market equity and the risk of HML should stem mostly from changes in the value of assets in place. Therefore, unconditionally its average returns should approximately mimic the empirically unobservable covariation of the systematic component of betas for assets in place and the market risk premium.

Similarly, SMB is the difference between the returns on small and big stock portfolios with about the same book-to-market. This difference should be largely free of the influence of book-to-market and the risk of SMB should stem mostly from changes in the value of growth options. Therefore, unconditionally its average returns should mostly mimic the empirically unobservable covariation of the systematic component of betas for growth options and the market risk premium.

The correlation between HML and SMB in simulated data is only about -0.29, which supports the ability of HML and SMB to isolate market equity and book-to-market effects, respectively. In addition, from the market beta decomposition in (36), firms with high book-to-market and similar market equity derive most of their riskiness from changes in the value of assets in place. Similarly, firms with small market equity and similar book-to-market derive most of their riskiness from changes in the value of growth options. This is confirmed in Table XI by the increasing pattern of the loadings on HML from growth to value. Similarly, the loadings on SMB also increase from growth to value as their market equity decreases.

Although not directly implied by the model, the success of the Fama and French (1993) model in simulated data stems from the ability of HML and SMB to account for the covari-



ation of each market betas' component and the market risk premium.

## V. Conclusion

In a general equilibrium framework, investment irreversibility and the endogenous composition of unprofitable versus profitable firms in the economy play a key role in determining the cross-sectional properties of stock returns. Profitable big and growth firms can optimally slow down their growth in response to adverse productivity shocks, maintain and even increase their dividends, particularly in bad times, when consumption is low. In contrast, unprofitable small and value firms up against the irreversibility constraint must pass along adverse productivity shocks, delivering a riskier payout stream, and hence generating higher average returns. The greater risk of these firms shows up in a conditional beta that is high in bad times when also the market premium is high, but not necessarily in a high unconditional beta. This endogenous covariation of conditional market betas and equity premium is responsible for the failure of the CAPM and the relative success of the Fama and French (1993) model, and to a lesser extent, conditional versions of the CAPM.

The model can generate many empirical regularities in the cross section of stock returns and its relation with fundamentals. However, its general equilibrium nature seems to require an economy where most firms face binding irreversibility constraint. This excess concentration of value firms might be relaxed in an economy with preference shocks or more sophisticated preferences able to describe the time series behavior of aggregate quantities and prices better than the simple power utility. Nonetheless, I view this effort as a contribution towards a better understanding of how risk, returns and firm characteristics can be endogenously related to the real side of the economy within a tractable general equilibrium framework.

# Appendix A: Proofs and Technical Details

In this section I provide all the proofs and technical details. In the following, I omit the time subscript where unnecessary.

## Proof of Proposition 1

Let  $V_{it} \equiv V(a_t, \omega_t, x_{it}, k_{it}, K_{it})$  be the value function of the firm:

$$V_{it} = \max_{\{i_{it+s} \geq \hat{i}; s \in \mathbb{R}^+\}} \mathbf{E}_t \left[ \int_0^\infty \frac{\Lambda_{t+s}}{\Lambda_t} \left( e^{a_{t+s}} x_{it+s} + f - i_{it+s} - \alpha k_{it+s}^{\frac{1}{n-1}} \left( i_{it+s} - \hat{i} \right)^{\frac{n}{n-1}} \right) K_{it+s} ds \right] \quad (\text{A1})$$

subject to the evolution of the economy wide productivity  $a$  in (2), the law of motion of the idiosyncratic productivity  $x_i$  in (3), the firm capital accumulation with its non-negativity constraint in (4), the evolution of the firm relative capital  $k_i$  in (13), and the conjectured dynamics of the equilibrium pricing-kernel  $\Lambda$  and the state variable  $\omega$  described in (14) and (11), respectively.

Then, the firm value function  $V_i$  satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \max_{i_i \geq \hat{i}} \left\{ \Lambda K_i \left[ e^a x_i + f - i_i - \alpha k_i^{\frac{1}{n-1}} \left( i_i - \hat{i} \right)^{\frac{n}{n-1}} \right] + \mathcal{D}[\Lambda V_i] \right\} \quad (\text{A2})$$

with  $\mathcal{D}[\Lambda V_i]$  denoting the infinitesimal generator of the Markov processes  $a$  and  $x_i$ , and the processes  $K_i$ ,  $k_i$  and  $\omega$ , applied to the discounted firm value function  $\Lambda V_i$ , along with the transversality (“no bubble”) condition:

$$\lim_{T \rightarrow \infty} \mathbf{E}_t [|\Lambda_{t+T} V_{it+T}|] = 0. \quad (\text{A3})$$

Conjecture that the value function takes the form:

$$V(a, \omega, x_i, k_i, K_i) = [q(a, \omega, x_i) + h(a, \omega, x_i) k_i^{-1}] K_i \quad (\text{A4})$$

Then, the HJB equation in (A2) reads:

$$0 = \max_{i_i \geq \hat{i}} \left\{ e^a x_i + f - i_i - \alpha k_i^{\frac{1}{n-1}} \left( i_i - \hat{i} \right)^{\frac{n}{n-1}} + \frac{\mathcal{D}[\Lambda q_i K_i]}{\Lambda K_i} + \frac{\mathcal{D}[\Lambda h_i K]}{\Lambda K_i} \right\} \quad (\text{A5})$$

where

$$\begin{aligned} \mathcal{D}[\Lambda q_i K_i] &= \Lambda q_i (I - \delta K_i) + K^i \mathcal{D}[\Lambda q_i] \\ \mathcal{D}[\Lambda h_i K] &= \Lambda h_i (I - \delta K) + K \mathcal{D}[\Lambda h_i]. \end{aligned}$$

Rearranging terms in (A5) leads to:

$$0 = \max_{i_i \geq \hat{i}} \left\{ [q_i - 1] i_i - \alpha k_i^{\frac{1}{n-1}} \left( i_i - \hat{i} \right)^{\frac{n}{n-1}} \right\} + e^a x_i + f - q_i \delta + \frac{\mathcal{D}[\Lambda q_i]}{\Lambda} + k_i^{-1} \left[ h_i (i - \delta) + \frac{\mathcal{D}[\Lambda h_i]}{\Lambda} \right] \quad (\text{A6})$$

Given that the maximand in (A6) is strictly concave and everywhere differentiable in  $i_i$ , the first order condition uniquely determining the optimal investment policy  $i_i^*$  is given by

$$[q_i - 1] - \frac{\alpha n}{n-1} k_i^{\frac{1}{n-1}} \left( i_i - \hat{i} \right)^{\frac{1}{n-1}} \leq 0 \quad (\text{A7})$$

along with the complementary slackness condition

$$\left[ [q_i - 1] - \frac{\alpha n}{n-1} k_i^{\frac{1}{n-1}} \left( i_i - \hat{i} \right)^{\frac{1}{n-1}} \right] \left( i_i - \hat{i} \right) = 0. \quad (\text{A8})$$

According to equations (A7) - (A8) the firm optimal investment policy can be summarized as

$$i_i^* = \hat{i} + \left( \frac{n-1}{\alpha n} \right)^{n-1} (q_i - 1)^{n-1} k_i^{-1} \mathbf{1}_{\{q_i \geq 1\}}. \quad (\text{A9})$$

Hence, evaluating equation (A6) at the optimal investment policy (A9) leads to:

$$\left[ q_i \left( \delta - \hat{i} \right) - e^a x_i - f + \hat{i} - \frac{\mathcal{D}[\Lambda q_i]}{\Lambda} \right] = k_i^{-1} \left[ \eta (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}} + h_i (i - \delta) + \frac{\mathcal{D}[\Lambda h_i]}{\Lambda} \right] \quad (\text{A10})$$

where  $\eta = (n-1)^{n-1} \alpha^{-(n-1)} n^{-n} > 0$ . Since the left-hand side of equation (A10) is independent of the value  $k_i \in \mathbb{R}^+$ , in order for (A10) to hold for all  $k_i \in \mathbb{R}^+$  the term in square brackets on the left-hand side must equal zero:

$$\Lambda \left( e^a x_i + f - \hat{i} \right) - \Lambda q_i \left( \delta - \hat{i} \right) + \mathcal{D}[\Lambda q_i] = 0 \quad (\text{A11})$$

and the right-hand side must also equal zero:

$$\Lambda \eta (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}} - \Lambda h_i (\delta - i) + \mathcal{D}[\Lambda h_i] = 0. \quad (\text{A12})$$

The Feynman-Kac Theorem<sup>9</sup> implies that the partial differential equation (A11) admits the following probabilistic solution for  $q \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ :

$$q(a_t, \omega_t, x_{it}) = \mathbf{E}_t \left[ \int_0^\infty e^{-(\delta - \hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} \left( e^{a_{t+s}} x_{it+s} + f - \hat{i} \right) ds \right]. \quad (\text{A13})$$

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<sup>9</sup>See, for example, Duffie (Appendix E, 2001), Karatzas and Shreve (Theorem 7.6, 1991), Krylov (Theorem 4, pag. 198, 1995), Yong and Zhou (Theorem 4.1-3, pag. 373-5, 1999).

The probabilistic solution to the partial differential equation (A13) can be further represented as

$$\begin{aligned}
q(a_t, \omega_t, x_{it}) &\stackrel{(1)}{=} \int_0^\infty e^{-(\delta-\hat{i})s} \mathbf{E}_t \left[ \frac{\Lambda_{t+s}}{\Lambda_t} e^{a_{t+s}} \right] \mathbf{E}_t [x_{it+s}] ds + \mathbf{E}_t \left[ \int_0^\infty e^{-(\delta-\hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} (f - \hat{i}) ds \right] \\
&\stackrel{(2)}{=} \bar{x} \bar{q}(a_t, \omega_t) + [x_{it} - \bar{x}] \hat{q}(a_t, \omega_t)
\end{aligned} \tag{A14}$$

where (1) follows from the application of Tonelli's Theorem and the independence of  $x_{it}$  from  $a_t$  and  $\omega_t$ , (2) follows from the Strong Markov property of  $x_{it}$  and  $E[x_{it+s}|x_{it}] = \bar{x} + [x_{it} - \bar{x}] e^{-\kappa_x s}$ , and from

$$\begin{aligned}
\bar{q}(a_t, \omega_t) &= \mathbf{E}_t \left[ \int_0^\infty e^{-(\delta-\hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} \left[ e^{a_{t+s}} + \bar{x}^{-1} (f - \hat{i}) \right] ds \right] \\
\hat{q}(a_t, \omega_t) &= \mathbf{E}_t \left[ \int_0^\infty e^{-(\kappa_x + \delta - \hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} e^{a_{t+s}} ds \right].
\end{aligned}$$

Given the strict positivity of the pricing-kernel  $\Lambda$ , which is inherited from the strict positivity of aggregate consumption ensured by the Inada conditions, it is sufficient to restrict  $f \geq \hat{i}$  to ensure the positivity of the firm "marginal  $q$ ". *Q.E.D.*

## Proof of Proposition 2

Let  $\int_{i \in F} [\cdot] di$  denote the aggregation operator over firms, and define the aggregate (average) capital stock as  $K \equiv \int_{i \in F} K_i di$ . In order to facilitate the representation of aggregate quantities, let  $g(a, \omega; m_1, m_2)$  denote a function of the state variables  $a$  and  $\omega$  defined as

$$g(a, \omega; m_1, m_2) \equiv \hat{q}^{m_1} \sum_{k=0}^{m_1} \frac{\Gamma(m_1 + 1) \Gamma_U(k + \nu, \theta \tilde{x})}{\Gamma(m_1 + 1 - k) \Gamma(k + m_2) \Gamma(\nu)} (-\tilde{x})^{m_1 - k} \theta^{-k} \tag{A15}$$

where  $m_1$  and  $m_2$  represent constant parameters.

Aggregate output is defined as  $Y \equiv \int_{i \in F} Y_i di$  and can be represented as

$$Y = \int_{i \in F} (e^a x_i K_i + f K_i) di \stackrel{(1)}{=} \left[ e^a \int_{i \in F} x_i k_i di + f \right] K \stackrel{(2)}{=} (e^a \omega + f) K \tag{A16}$$

where (1) follows from the definition of  $K$  and the firm relative capital  $k_i \equiv K_i/K$ , and (2) from the definition of the endogenous aggregate productivity component  $\omega$  in (10).

Similarly, aggregate (average) investment is defined as  $I \equiv \int_{i \in F} I_i^* di$  and can be characterized as follows:

$$\begin{aligned}
I &\stackrel{(1)}{=} \int_{i \in F} \left[ \widehat{i}K_i + \left( \frac{n-1}{\alpha n} \right)^{n-1} (q_i - 1)^{n-1} K \mathbf{1}_{\{q_i - 1 \geq 0\}} \right] di \\
&\stackrel{(2)}{=} \widehat{i}K + \left( \frac{n-1}{\alpha n} \right)^{n-1} \widehat{q}^{n-1} K \int_{i \in F} (x_i - \widetilde{x})^{n-1} \mathbf{1}_{\{x_i - \widetilde{x} \geq 0\}} di \\
&\stackrel{(3)}{=} \widehat{i}K + \left( \frac{n-1}{\alpha n} \right)^{n-1} \widehat{q}^{n-1} K \int_0^\infty (x - \widetilde{x})^{n-1} \mathbf{1}_{\{x - \widetilde{x} \geq 0\}} f_x(x; \theta, \nu) dx \\
&\stackrel{(4)}{=} \widehat{i}K + \left( \frac{n-1}{\alpha n} \right)^{n-1} \widehat{q}^{n-1} K \int_{\widetilde{x}^+}^\infty (x - \widetilde{x})^{n-1} \frac{\theta^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\theta x} dx
\end{aligned} \tag{A17}$$

where (1) follows from the firm optimal investment policy in (A9), (2) from the definition of aggregate (average) capital  $K \equiv \int_{i \in F} K_i di$  and from the fact that the firm marginal  $q$  can be rewritten as  $q_i = 1 + \widehat{q}(x_i - \widetilde{x})$ , where  $\widetilde{x} \equiv [1 - \bar{x}(\bar{q} - \widehat{q})] / \widehat{q}$  denotes the investment irreversibility threshold with reference to the firm-specific productivity and it is expressed in terms of the aggregate values  $\bar{q}$  and  $\widehat{q}$ . The third equality follows from the Glivenko-Cantelli Theorem<sup>10</sup>, according to which the cross-sectional distribution of the i.i.d. firm-specific productivity  $x$  equals its stationary distribution  $f_x(x; \theta, \nu)$ , and (4) from the fact that the stationary distribution of the stochastic process  $x$  whose dynamics is given in (3) is a gamma distribution:<sup>11</sup>

$$f_x(x; \theta, \nu) = \frac{\theta^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\theta x} \mathbf{1}_{\{0 \leq x < \infty\}}; \quad \theta, \nu > 0 \tag{A18}$$

with  $\theta \equiv 2\kappa_x / \sigma_x^2$  and  $\nu \equiv 2\kappa_x \bar{x} / \sigma_x^2$  ( $\kappa_x, \bar{x} > 0$  and  $\sigma_x \neq 0$ ). The value  $\widetilde{x}^+$  in the lower limit of integration in (A17) stands for  $\max(0, \widetilde{x})$  and results from the product of the two indicator functions  $\mathbf{1}_{\{x \geq \widetilde{x}\}} \times \mathbf{1}_{\{0 \leq x < \infty\}}$ .

<sup>10</sup>See, for example, Billingsley (Theorem 20.6, 1979) and Parthasarathy (Theorem II.7.1, 1967).

<sup>11</sup>See, for example, Cox, Ingersoll, Ross (1985).

The integral in (A17) can be further represented as:

$$\begin{aligned}
& \int_{\tilde{x}^+}^{\infty} \frac{\theta^v}{\Gamma(v)} (x - \tilde{x})^{n-1} x^{v-1} e^{-\theta x} dx \stackrel{(1)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n) (-\tilde{x})^{n-1-k} \theta^{-(k-1)}}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} \int_{\tilde{x}^+}^{\infty} (\theta x)^{k+v-1} e^{-\theta x} dx \\
& \stackrel{(2)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n) (-\tilde{x})^{n-1-k} \theta^{-k}}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} \int_{\theta \tilde{x}^+}^{\infty} y^{k+v-1} e^{-y} dy \\
& \stackrel{(3)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n) \Gamma_U(k+v, \theta \tilde{x}^+)}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} (-\tilde{x})^{n-1-k} \theta^{-k} \tag{A19}
\end{aligned}$$

where (1) follows from the Binomial Theorem  $(x - \tilde{x})^{n-1} = \sum_{k=0}^{n-1} \frac{\Gamma(n) x^k (-\tilde{x})^{n-1-k}}{\Gamma(n-k) \Gamma(k+1)}$ , (2) from the change of variable  $y = \theta x$ , and (3) from the definition of the upper incomplete gamma function  $\Gamma_U(\alpha, z) \equiv \int_z^{\infty} x^{\alpha-1} e^{-x} dx$ .

In order to ensure the existence of  $q \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ , I assume throughout the following analysis that the investment threshold never falls below zero, i.e.  $\tilde{x} \geq 0$ . Under standard integrability conditions, it is sufficient to appropriately restrict the model parameters such that  $\sup |\bar{q} - \hat{q}| \leq 1/\bar{x}$  in order to meet this restriction. Furthermore, the strict positivity of the firm marginal  $q$  implies that  $\tilde{x} < 1/\hat{q}$ . Hence, the existence of a strictly positive firm marginal  $q \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$  implies that the investment threshold  $\tilde{x} \in [0, 1/\hat{q})$ .

Therefore, aggregate investment can be characterized as:

$$I = \left[ \hat{i} + \left( \frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right] K \tag{A20}$$

with the function  $g$  in (A15) evaluated at  $m_1 = n-1$  and  $m_2 = 1$ .

Aggregate dividend is defined as  $D \equiv \int_{i \in F} D_i^* di$  and can be characterized as:

$$\begin{aligned}
D & \stackrel{(1)}{=} \int_{i \in F} \left[ (e^a x_i + f) K_i - I_i - \alpha k_i^{\frac{1}{n-1}} \left( \frac{I_i - \hat{I}_i}{K_i} \right)^{\frac{n}{n-1}} K_i \right] di \\
& \stackrel{(2)}{=} Y - I - \alpha \left( \frac{n-1}{\alpha n} \right)^n K \int_{i \in F} (q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}} di \\
& \stackrel{(3)}{=} Y - I - \alpha \left( \frac{n-1}{\alpha n} \right)^n K \hat{q}^n \int_{i \in F} (x_i - \tilde{x})^n \mathbf{1}_{\{x_i - \tilde{x} \geq 0\}} di \\
& \stackrel{(4)}{=} Y - I - \alpha \left( \frac{n-1}{\alpha n} \right)^n K \hat{q}^n \int_{\tilde{x}}^{\infty} (x - \tilde{x})^n \frac{\theta^v}{\Gamma(v)} x^{v-1} e^{-\theta x} dx \tag{A21}
\end{aligned}$$

where (1) follows from the definition of firm dividends in (6), (2) from the firm optimal investment policy in (A9) and definition of aggregate output and investment in (A16) and

(A20), respectively. The third equality results from the fact that the firm marginal  $q$  can be rewritten as  $q_i = 1 + \widehat{q}(x_i - \tilde{x})$ , where  $\tilde{x} \equiv [1 - \bar{x}(\bar{q} - \widehat{q})] / \widehat{q}$  denotes the investment irreversibility threshold. The last equality follows from the Glivenko-Cantelli Theorem, according to which the cross-sectional distribution of the i.i.d. firm specific productivity  $x$  equals its stationary distribution in (A18), and the restriction on the investment threshold  $\tilde{x} \in [0, 1/\widehat{q})$ .

The integral in (A21) can be further represented as:

$$\begin{aligned}
& \int_{\tilde{x}}^{\infty} (x - \tilde{x})^n \frac{\theta^v}{\Gamma(v)} x^{v-1} e^{-\theta x} dx \stackrel{(1)}{=} \sum_{k=0}^n \frac{\Gamma(n+1) (-\tilde{x})^{n-k} \theta^{-(k-1)}}{\Gamma(n-k+1) \Gamma(k+1) \Gamma(v)} \int_{\tilde{x}}^{\infty} (\theta x)^{k+v-1} e^{-\theta x} dx \\
& \stackrel{(2)}{=} \sum_{k=0}^n \frac{\Gamma(n+1) (-\tilde{x})^{n-k} \theta^{-k}}{\Gamma(n-k+1) \Gamma(k+1) \Gamma(v)} \int_{\theta \tilde{x}}^{\infty} (y)^{k+v-1} e^{-y} dy \\
& \stackrel{(3)}{=} \sum_{k=0}^n \frac{\Gamma(n+1) \Gamma_U(k+v, \theta \tilde{x})}{\Gamma(n-k+1) \Gamma(k+1) \Gamma(v)} (-\tilde{x})^{n-k} \theta^{-k} \tag{A22}
\end{aligned}$$

where (1) follows from the Binomial Theorem  $(x - \tilde{x})^n = \sum_{k=0}^n \frac{\Gamma(n+1) x^k (-\tilde{x})^{n-k}}{\Gamma(n-k+1) \Gamma(k+1)}$ , (2) from the change of variable  $y = \theta x$ , and (3) from the definition of the upper incomplete gamma function  $\Gamma_U(\alpha, z) \equiv \int_z^{\infty} x^{\alpha-1} e^{-x} dx$ .

Therefore, aggregate dividend can be represented as:

$$\frac{D}{K} = e^a \omega + f - \widehat{i} - \left( \frac{n-1}{\alpha n} \right)^{n-1} \left[ g(a, \omega; n-1, 1) + \frac{n-1}{n} g(a, \omega; n, 1) \right] \tag{A23}$$

where the function  $g$  is given in (A15). *Q.E.D.*

### Proof of Proposition 3

The second moment of the joint cross-sectional distribution of  $x_i$  and  $k_i$  is defined as  $\omega \equiv \int_{i \in F} x_i k_i di$ . I now, derive its law of motion following two steps. First, I characterize the law of motion of the weighted firm specific productivity  $x_i k_i$  as:

$$\begin{aligned}
& dx_i k_i \stackrel{(1)}{=} k_i [\kappa_x (\bar{x} - x_i) dt + \sigma_x \sqrt{x_i} dW_i] + x_i k_i [i_i - i] dt \\
& \stackrel{(2)}{=} \left\{ \kappa_x (\bar{x} - x_i) k_i + \left( \frac{n-1}{\alpha n} \right)^{n-1} [\widehat{q}^{n-1} (x_i - \tilde{x})^{n-1} x_i \mathbf{1}_{\{x_i \geq \tilde{x}\}} - g(a, \omega; n-1, 1) x_i k_i] \right\} dt \\
& + k_i \sigma_x \sqrt{x_i} dW_i \tag{A24}
\end{aligned}$$

where (1) follows from the application of Ito's Formula to  $x_i k_i$  with the processes  $x_i$  and  $k_i$  evolving as in (3) and (13), respectively, and (2) from the optimal firm investment policy and aggregate investment rate in (A9) and (A20), respectively. Then, it follows that  $\omega$  evolves according to:

$$d\omega \stackrel{(1)}{=} \int_{i \in F} dx_i k_i di \stackrel{(2)}{=} \left\{ \begin{aligned} & \kappa_x (\bar{x} - \omega) + \left( \frac{n-1}{\alpha n} \right)^{n-1} \\ & \times \left[ \hat{q}^{n-1} \int_{i \in F} x_i (x_i - \tilde{x})^{n-1} \mathbf{1}_{\{x_i \geq \tilde{x}\}} di - g(a, \omega; n-1, 1) \omega \right] \end{aligned} \right\} dt \quad (\text{A25})$$

where (1) follows from Fubini's Theorem under the assumption of joint measurability, (2) from the definition of  $\omega \equiv \int_{i \in F} x_i k_i di$  and the fact that  $\int_{i \in F} k_i di = 1$ . The independence of  $k_i \sqrt{x_i}$  and  $dW_i$  and the law of large numbers applied to  $dW_i$ 's, which are cross-sectionally i.i.d. with zero mean and finite variance, ensures that  $\sigma_x \int_{i \in F} k_i \sqrt{x_i} dW_i di = 0$ .

The integral in (A25) can be computed as:

$$\begin{aligned} & \int_{i \in F} x_i (x_i - \tilde{x})^{n-1} \mathbf{1}_{\{x_i \geq \tilde{x}\}} di \stackrel{(1)}{=} \int_{\tilde{x}}^{\infty} x (x - \tilde{x})^{n-1} \frac{\theta^v}{\Gamma(v)} x^{v-1} e^{-\theta x} dx \\ & \stackrel{(2)}{=} \theta^{-1} \sum_{k=0}^{n-1} \frac{\Gamma(n) (-\tilde{x})^{n-1-k} \theta^{-k}}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} \int_{\theta \tilde{x}}^{\infty} y^{k+v} e^{-y} dy \\ & \stackrel{(3)}{=} \theta^{-1} \sum_{k=0}^{n-1} \frac{\Gamma(n) \Gamma_U(k+v+1, \theta \tilde{x})}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} (-\tilde{x})^{n-1-k} \theta^{-k} \\ & \stackrel{(4)}{=} \theta^{-1} \sum_{k=0}^{n-1} \frac{\Gamma(n) (-\tilde{x})^{n-1-k} \theta^{-k}}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} \left[ (k+v) \Gamma_U(k+v, \theta \tilde{x}) + (\theta \tilde{x})^{k+v} e^{-\theta \tilde{x}} \right] \\ & \stackrel{(5)}{=} \sum_{k=0}^{n-1} \frac{\Gamma(n) \Gamma_U(k+v, \theta \tilde{x})}{\Gamma(n-k) \Gamma(k+1) \Gamma(v)} (-\tilde{x})^{n-1-k} \theta^{-k} \left( \frac{k}{\theta} + \bar{x} \right) \end{aligned} \quad (\text{A26})$$

where (1) follows from the Glivenko-Cantelli Theorem, according to which the cross-sectional distribution of the i.i.d. firm-specific productivity  $x$  equals its stationary distribution in (A18), and the restriction on the investment threshold  $\tilde{x} \in [0, 1/\hat{q})$ . The second equality follows from the Binomial Theorem,  $(x - \tilde{x})^{n-1} = \sum_{k=0}^{n-1} \frac{\Gamma(n) x^k (-\tilde{x})^{n-1-k}}{\Gamma(n-k) \Gamma(k+1)}$ , and the change of variable  $y = \theta x$ , (3) from the definition of the upper incomplete gamma function  $\Gamma_U(\alpha, z) \equiv \int_z^{\infty} x^{\alpha-1} e^{-x} dx$ . The fourth equality results from the property of the upper incomplete gamma function (integration by parts),  $\Gamma_U(k+v+1, \theta \tilde{x}) = (k+v) \Gamma_U(k+v, \theta \tilde{x}) + (\theta \tilde{x})^{k+v} e^{-\theta \tilde{x}}$ , and (5) from the fact that  $\sum_{k=0}^{n-1} \frac{\Gamma(n)}{\Gamma(n-k) \Gamma(k+1)} (-1)^{n-1-k} = (1-1)^{n-1} = 0$  and  $v/\theta = \bar{x}$ .



Therefore, the endogenous component of aggregate productivity evolves according to:

$$\begin{aligned}
d\omega &\stackrel{(1)}{=} \left\{ \left[ \kappa_x + \left( \frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right] (\bar{x} - \omega) + \left( \frac{n-1}{\alpha n} \right)^{n-1} \theta^{-1} g(a, \omega; n-1, 0) \right\} dt \\
&\stackrel{(2)}{=} \left\{ \left[ \kappa_x + (i - \hat{i}) \right] (\bar{x} - \omega) + (i - \hat{i}) \theta^{-1} \frac{g(a, \omega; n-1, 0)}{g(a, \omega; n-1, 1)} \right\} dt
\end{aligned} \tag{A27}$$

where (1) results from the property of the gamma function,  $\Gamma(k+1) = k\Gamma(k)$ , and the definition of the function  $g$  in (A15), and (2) from the characterization of aggregate investment in (A20). *Q.E.D.*

#### Proof of Proposition 4

The market clearing condition for the consumption good requires aggregate consumption  $C^*$  to equal aggregate dividend  $D^*$ , hence equations (24) - (25) correspond to the aggregate dividend in (21).

The equilibrium prices and quantities depend on the firm marginal  $q$  as characterized in equations (16) - (18), which is now evaluated at the equilibrium pricing-kernel  $\Lambda$ . In equilibrium, the representative household intertemporal marginal rate of substitution between consumption at time  $t+s$  and consumption at time  $t$  is given by:

$$\begin{aligned}
\frac{\Lambda_{t+s}}{\Lambda_t} &\stackrel{(1)}{=} e^{-\rho s} \left( \frac{C_{t+s}^*}{C_t^*} \right)^{-\gamma} \stackrel{(2)}{=} e^{-\rho s} \left( \frac{c^*(a_{t+s}, \omega_{t+s})}{c^*(a_t, \omega_t)} \right)^{-\gamma} \left( \frac{K_{t+s}}{K_t} \right)^{-\gamma} \\
&\stackrel{(3)}{=} e^{-\int_t^{t+s} \left\{ \rho + \gamma \left[ \hat{i} + \left( \frac{n-1}{\alpha n} \right)^{n-1} g(a_u, \omega_u; n-1, 1) - \delta \right] \right\} du} \left( \frac{c^*(a_{t+s}, \omega_{t+s})}{c^*(a_t, \omega_t)} \right)^{-\gamma}
\end{aligned} \tag{A28}$$

where (1) follows from the representative household's first-order optimality condition, (2) from the equilibrium consumption policy (24) - (25), and (3) from the dynamics of the aggregate stock of capital evaluated at the equilibrium aggregate investment in (A20). From the characterization of  $\bar{q}(a, \omega)$  and  $\hat{q}(a, \omega)$  in (17) - (18), after applying some straightforward algebra it follows that

$$\bar{q}_t = \mathbf{E}_t \left[ \int_0^\infty e^{-(\delta - \hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} \left[ e^{a_{t+s}} + \bar{x}^{-1} (f - \hat{i}) \right] ds \right] = c^*(a_t, \omega_t)^\gamma \bar{\Phi}(a_t, \omega_t) \tag{A29}$$

$$\hat{q}_t = \mathbf{E}_t \left[ \int_0^\infty e^{-(\kappa_x + \delta - \hat{i})s} \frac{\Lambda_{t+s}}{\Lambda_t} e^{a_{t+s}} ds \right] = c^*(a_t, \omega_t)^\gamma \hat{\Phi}(a_t, \omega_t) \tag{A30}$$

where

$$\bar{\Phi}_t \equiv \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} \{\rho + (1-\gamma)(\delta - \hat{i}) + \gamma \left(\frac{n-1}{\alpha n}\right)^{n-1} g(a_u, \omega_u; n-1, 1)\} du} \frac{e^{a_{t+s}} + (f - \hat{i})}{\bar{c}^*(a_{t+s}, \omega_{t+s})^\gamma} ds \right] \quad (\text{A31})$$

$$\hat{\Phi}_t \equiv \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} \{\rho + \kappa_x + (1-\gamma)(\delta - \hat{i}) + \gamma \left(\frac{n-1}{\alpha n}\right)^{n-1} g(a_u, \omega_u; n-1, 1)\} du} \frac{e^{a_{t+s}}}{c^*(a_{t+s}, \omega_{t+s})^\gamma} ds \right] \quad (\text{A32})$$

The Feynman-Kac Theorem implies that  $\bar{\Phi}, \hat{\Phi} \in C^2(\mathbb{R} \times \mathbb{R}^+)$  satisfy the following partial differential equations:

$$\left\{ \rho + (1-\gamma)(\delta - \hat{i}) + \gamma \left(\frac{n-1}{\alpha n}\right)^{n-1} g(a, \omega; n-1, 1) \right\} \bar{\Phi} - \mathcal{D}[\bar{\Phi}] = \frac{e^a + (f - \hat{i})}{\bar{c}^*(a, \omega)^\gamma} \quad (\text{A33})$$

and

$$\left\{ \rho + \kappa_x + (1-\gamma)(\delta - \hat{i}) + \gamma \left(\frac{n-1}{\alpha n}\right)^{n-1} g(a, \omega; n-1, 1) \right\} \hat{\Phi} - \mathcal{D}[\hat{\Phi}] = \frac{e^a}{c^*(a, \omega)^\gamma} \quad (\text{A34})$$

provided that standard integrability conditions are satisfied, i.e.  $\bar{\Phi}, \hat{\Phi} < \infty$ . Notice that the existence of no arbitrage is ensured by the strict positivity of the aggregate consumption process resulting from the fact that the marginal utility of consumption satisfies the Inada conditions. The stationarity and strict positivity of the aggregate output-to-capital ratio and aggregate consumption-to-capital ratio imply that the aggregate investment rate is bounded. This in turn implies that  $\bar{q}$  and  $\hat{q}$  are also bounded, since the aggregate investment rate is an increasing function of  $\bar{q}$  and  $\hat{q}$ . *Q.E.D.*

### Proof of Proposition 5

The equilibrium pricing kernel dynamics can be computed by applying Ito's Formula to the representative household marginal utility of consumption  $\Lambda_t = e^{-\rho t} (C_t^*)^{-\gamma}$  as

$$\frac{d\Lambda}{\Lambda} = -\rho dt - \frac{\gamma}{C^*} dC^* + \frac{1}{2} \frac{\gamma(\gamma+1)}{(C^*)^2} \langle dC^*, dC^* \rangle = -r(a, \omega) dt - \lambda(a, \omega) dW_a \quad (\text{A35})$$

where

$$r(a, \omega) \equiv \rho + \gamma \frac{\mathcal{D}^{a, \omega, K}[C^*(a, \omega, K)]}{C^*(a, \omega, K)} - \frac{1}{2} \gamma(\gamma+1) \sigma_a^2 \left[ \frac{\partial_a C^*(a, \omega, K)}{C^*(a, \omega, K)} \right]^2 \quad (\text{A36})$$

$$\lambda(a, \omega) \equiv \gamma \sigma_a \frac{\partial_a C^*(a, \omega, K)}{C^*(a, \omega, K)}. \quad (\text{A37})$$

and

$$\begin{aligned} \frac{\mathcal{D}^{a,\omega,K} [C^*(a, \omega, K)]}{C^*(a, \omega, K)} &\equiv \kappa_a (\bar{a} - a) \frac{\partial_a C^*(a, \omega, K)}{C^*(a, \omega, K)} + \frac{1}{2} \sigma_a^2 \frac{\partial_{aa}^2 C^*(a, \omega, K)}{C^*(a, \omega, K)} \\ &\quad + \mu_\omega (a_t, \omega_t) \frac{\partial_\omega C^*(a, \omega, K)}{C^*(a, \omega, K)} + (I^* - \delta K) \frac{\partial_K C^*(a, \omega, K)}{C^*(a, \omega, K)} \end{aligned} \quad (\text{A38})$$

The independence of the risk-free rate and the market price of risk from the stock of aggregate capital follows from the linear homogeneous property of the aggregate consumption (24).

*Q.E.D.*

### Proof of Proposition 6

The market value of firm equity can be represented as in (A4). The firm marginal  $q$  can be characterized as in (16), with  $\bar{q}$  and  $\hat{q}$  having the representation in (26) - (27), and satisfying the system of partial differential equations (A33) - (A34). The function  $h(a, \omega, x_i)$  can be characterized as the probabilistic solution to the partial differential equation (A12), which according to the Feynman-Kac Theorem admits the following representation for  $h \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ :

$$h(a_t, \omega_t, x_{it}) = \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} (\delta - i_u) du} \frac{\Lambda_{t+s}}{\Lambda_t} \eta(q_{it+s} - 1)^n \mathbf{1}_{\{q_{it+s} \geq 1\}} ds \right] \quad (\text{A39})$$

Notice that the strict positivity of pricing-kernel  $\Lambda$  ensures the positivity of the function  $h$ . Evaluating (A39) at the equilibrium pricing-kernel in (A28), the function  $h$  can be represented as

$$h(a_t, \omega_t, x_{it}) = c^*(a_t, \omega_t)^\gamma H(a_t, \omega_t, x_{it}) \quad (\text{A40})$$

where

$$H_t \equiv \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} \{\rho + (1-\gamma)(\delta - i_u)\} du} \frac{\eta(q_{it+s} - 1)^n \mathbf{1}_{\{q_{it+s} \geq 1\}}}{c^*(a_{t+s}, \omega_{t+s})^\gamma} ds \right]. \quad (\text{A41})$$

The Feynman-Kac Theorem implies that  $H \in C^2(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$  satisfies the following partial differential equation:

$$\left\{ \rho + (1 - \gamma) \left( \delta - \hat{i} - \left( \frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right) \right\} H - \mathcal{D}^{a,\omega,x} [H] = \frac{\eta(q_i - 1)^n \mathbf{1}_{\{q_i \geq 1\}}}{c^*(a, \omega)^\gamma} \quad (\text{A42})$$

where  $\mathcal{D}^{a,\omega,x} [H]$  denotes the infinitesimal generator of the stochastic processes  $a$ ,  $\omega$  and  $x$ , applied to the function  $H$ :

$$\mathcal{D}^{a,\omega,x} [H] = \kappa_a (\bar{a} - a) \partial_a H + \frac{\sigma_a^2}{2} \partial_{aa}^2 H + \mu_\omega (a, \omega) \partial_\omega H + \kappa_x (\bar{x} - x) \partial_x H + \frac{\sigma_x^2 x}{2} \partial_{xx}^2 H. \quad (\text{A43})$$

*Q.E.D.*

### Proof of Proposition 7

The aggregate stock market value can be computed by aggregating individual firm market values as

$$\begin{aligned} V &= \int_{i \in F} V_i di \stackrel{(1)}{=} \int_{i \in F} q(a, \omega, x_i) K_i + h(a, \omega, x_i) K di \\ &\stackrel{(2)}{=} \{\bar{x}q(a, \omega) + [\omega - \bar{x}]\hat{q}(a, \omega) + h_m(a, \omega)\} K \end{aligned} \quad (\text{A44})$$

where (1) follows from the firm market value representation in (31), and (2) from the definition of  $K \equiv \int_{i \in F} K_i di$  and  $\omega \equiv \int_{i \in F} x_i k_i di$ . The function  $h_m(a, \omega) \equiv \int_{i \in F} h(a, \omega, x_i) di$  can be computed as

$$\begin{aligned} h_m(a, \omega) &\stackrel{(1)}{=} \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} (\delta - i_u) du} \frac{\Lambda_{t+s}}{\Lambda_t} \eta \left[ \int_{i \in F} (q_{it+s} - 1)^n \mathbf{1}_{\{q_{it+s} \geq 1\}} \right] dids \right] \\ &\stackrel{(2)}{=} \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} (\delta - i_u) du} \frac{\Lambda_{t+s}}{\Lambda_t} \eta \left[ \hat{q}_{t+s}^n \int_{i \in F} (x_{it+s} - \tilde{x}_{t+s})^n \mathbf{1}_{\{x_{it+s} \geq \tilde{x}_{t+s}\}} di \right] ds \right] \\ &\stackrel{(3)}{=} \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} (\delta - i_u) du} \frac{\Lambda_{t+s}}{\Lambda_t} \eta \left[ \hat{q}_{t+s}^n \int_{\tilde{x}_{t+s}}^\infty (x_{t+s} - \tilde{x}_{t+s})^n \frac{\theta^v}{\Gamma(v)} x_{t+s}^{v-1} e^{-\theta x_{t+s}} dx_{t+s} \right] ds \right] \\ &\stackrel{(4)}{=} \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} (\delta - i_u) du} \frac{\Lambda_{t+s}}{\Lambda_t} \eta g(a_{t+s}, \omega_{t+s}; n, 1) ds \right] \end{aligned} \quad (\text{A45})$$

where (1) follows from the definition of  $h_m(a, \omega, x_i)$  in (A39) and Fubini's Theorem under the assumption of joint measurability, (2) from the representation of the firm marginal  $q$  as  $q_i = 1 + \hat{q}(x_i - \tilde{x})$ . The third equality follows from the Glivenko-Cantelli Theorem, according to which the cross-sectional distribution of the i.i.d. firm specific productivity  $x$  equals its stationary distribution in (A18), and the restriction on the investment threshold  $\tilde{x} \in [0, 1/\hat{q}]$ . The last equality follows from (A22) and the definition of the function  $g$  in (A15) evaluated at  $m_1 = n$  and  $m_2 = 1$ .

Evaluating (A45) at the equilibrium pricing-kernel in (A28), the function  $h_m$  can be represented as

$$h_m(a_t, \omega_t) = c^*(a_t, \omega_t)^\gamma H_m(a_t, \omega_t) \quad (\text{A46})$$

where

$$H_{m,t} \equiv \mathbf{E}_t \left[ \int_0^\infty e^{-\int_t^{t+s} \{\rho + (1-\gamma)(\delta - i_u)\} du} \frac{\eta g(a_{t+s}, \omega_{t+s}; n, 1)}{c^*(a_{t+s}, \omega_{t+s})^\gamma} ds \right] \quad (\text{A47})$$

The Feynman-Kac Theorem implies that  $H_m \in C^2(\mathbb{R} \times \mathbb{R}^+)$  satisfies the following partial differential equation:

$$\left\{ \rho + (1 - \gamma) \left( \delta - \hat{i} - \left( \frac{n-1}{\alpha n} \right)^{n-1} g(a, \omega; n-1, 1) \right) \right\} H_m - \mathcal{D}^{a, \omega} [H_m] = \frac{\eta g(a, \omega; n, 1)}{c^*(a, \omega)^\gamma} \quad (\text{A48})$$

where  $\mathcal{D}^{a, \omega} [H_m]$  denotes the infinitesimal generator of the stochastic processes  $a$  and  $\omega$  applied to the function  $H_m$ :

$$\mathcal{D}^{a, \omega} [H_m] = \kappa_a (\bar{a} - a) \partial_a H_m + \frac{\sigma_a^2}{2} \partial_{aa}^2 H_m + \mu_\omega(a, \omega) \partial_\omega H_m. \quad (\text{A49})$$

*Q.E.D.*

### Proof of Proposition 8

The equilibrium cumulative aggregate stock return dynamics can be computed as  $dR = \frac{dV}{V} + \frac{D}{V} dt$ , where the aggregate stock market return dynamics  $\frac{dV}{V}$  is obtained by applying Ito's Formula to the function  $V(a, \omega, K)$  defined in (33). It follows that

$$dR = \mu_R(a, \omega) dt + \sigma_R(a, \omega) dW_a \quad (\text{A50})$$

whose drift and diffusion are given by

$$\mu_R(a, \omega) = \frac{V^A \mathcal{D}^{a, \omega, K} [V^A]}{V \frac{V^A}{V^A}} + \frac{V^O \mathcal{D}^{a, \omega, K} [V^O]}{V \frac{V^O}{V^O}} + \frac{D}{V} \quad (\text{A51})$$

$$\sigma_R(a, \omega) = \left[ \frac{V^A \partial_a V^A}{V \frac{V^A}{V^A}} + \frac{V^O \partial_a V^O}{V \frac{V^O}{V^O}} \right] \sigma_a. \quad (\text{A52})$$

Similarly, from (31), the cumulative firm stock return evolves according to:

$$dR_i = \mu_{R_i}(a, \omega, x_i, k_i) dt + \sigma_{R_i, a}(a, \omega, x_i, k_i) dW_a + \sigma_{R_i, x}(a, \omega, x_i, k_i) dW_i \quad (\text{A53})$$

whose drift and diffusions are determined by

$$\mu_{R_i} = \frac{V_i^A \mathcal{D}^{a, \omega, x, K_i} [V_i^A]}{V_i \frac{V_i^A}{V_i^A}} + \frac{V_i^O \mathcal{D}^{a, \omega, x, K} [V_i^O]}{V_i \frac{V_i^O}{V_i^O}} + \frac{D_i}{V_i} \quad (\text{A54})$$

$$\sigma_{R_i, a} = \left[ \frac{V_i^A \partial_a V_i^A}{V_i \frac{V_i^A}{V_i^A}} + \frac{V_i^O \partial_a V_i^O}{V_i \frac{V_i^O}{V_i^O}} \right] \sigma_a \quad (\text{A55})$$

$$\sigma_{R_i, x} = \left[ \frac{V_i^A \partial_x V_i^A}{V_i \frac{V_i^A}{V_i^A}} + \frac{V_i^O \partial_x V_i^O}{V_i \frac{V_i^O}{V_i^O}} \right] \sigma_x \sqrt{x_i} \quad (\text{A56})$$

The optimality condition of the producer's optimization problem described by the HJB equation (A2) implies that at the optimum the following relation must hold:

$$0 = \Lambda D_i + \mathcal{D}[\Lambda V_i]. \quad (\text{A57})$$

Rewriting the infinitesimal generator of the discounted firm value  $\mathcal{D}[\Lambda V_i]$  as  $\mathbf{E}_t[d\Lambda V_i]/dt$ , and dividing both sides of equation (A57) by  $\Lambda V_i$  yields the more familiar relation:

$$0 = \frac{D_i}{V_i} dt + \mathbf{E}_t \left[ \frac{d\Lambda V_i}{\Lambda V_i} \right]. \quad (\text{A58})$$

A straightforward application of Ito's Formula to the discounted firm value  $\Lambda V_i$  leads to the fundamental asset pricing relation:

$$E_t[dR_i] = r_t dt - E_t \left[ \frac{d\Lambda}{\Lambda} \frac{dV_i}{V_i} \right] \quad (\text{A59})$$

where  $E_t[dR_i] = E_t \left[ \frac{dV_i}{V_i} \right] + \frac{D_i}{V_i} dt$  denotes the cumulative stock expected return and  $r_t = -\frac{1}{dt} E_t \left[ \frac{d\Lambda}{\Lambda} \right]$  the instantaneous risk-free rate. The asset pricing relation (A59) must hold for any return including the aggregate stock market return. From the aggregate stock market return dynamics (A50) and the equilibrium pricing kernel dynamics (A35) it follows that the aggregate stock market return is instantaneously perfectly conditionally correlated with the pricing-kernel, that is the aggregate market portfolio is conditionally mean-variance efficient. Therefore, standard asset pricing results imply that the risk-return trade-off of any traded asset admits a beta-representation<sup>12</sup>, which takes the form of a conditional CAPM:

$$\mu_{R_i,t} = r_t + \beta_{it} [\mu_{R,t} - r_t] \quad (\text{A60})$$

where the instantaneous conditional market beta  $\beta_{it} \equiv \frac{\text{cov}_t(dR_i, dR)}{\text{var}_t(dR)}$ , and  $\mu_{R_i,t}$  and  $\mu_{R,t}$  represent the instantaneous expected return on firm  $i$  stock and aggregate market portfolio as characterized in (A54) and (A51), respectively.

The conditional market beta can then be decomposed as:

$$\beta_{it} \equiv \frac{\text{cov}_t(dR_{it}, dR_t)}{\text{var}_t(dR_t)} \stackrel{(1)}{=} \frac{\sigma_{R_i,a}}{\sigma_R} \stackrel{(2)}{=} \frac{\partial [\ln(V_{it}/K_{it})]}{\partial [\ln(V_t/K_t)]} \stackrel{(3)}{=} \frac{K_{it} q_{it}}{V_{it}} \beta_{it}^A + \frac{K_t h_{it}}{V_{it}} \beta_{it}^O \quad (\text{A61})$$

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<sup>12</sup>See, for instance, Cochrane (2001, Chapter 6) and Duffie (2001, Section 6D).

where (1) follows from the characterization of stock returns in (A50) and (A53), (2) from the representation of  $\sigma_{R_i,a} = \sigma_a \partial_a [\ln(V_{it}/K_{it})]$  and  $\sigma_R = \sigma_a \partial_a [\ln(V_t/K_t)]$ , and (3) from the definition of  $\beta_{it}^A \equiv \partial \ln q(a_t, \omega_t, x_{it}) / \partial \ln(V_t/K_t)$  and  $\beta_{it}^O \equiv \partial \ln h(a_t, \omega_t, x_{it}) / \partial \ln(V_t/K_t)$ .  
*Q.E.D.*

## Appendix B: Computation of Competitive Equilibrium

I solve for the competitive equilibrium iteratively. I approximate the system of partial differential equations for  $\bar{q}(a, \omega)$  and  $\hat{q}(a, \omega)$  upon discretizing the state-space of  $a$  and  $\omega$ . Let  $i = 1, 2, \dots, I$  and  $j = 1, 2, \dots, J$  index the value of  $a \in \mathbb{R}$  and  $\omega \in \mathbb{R}^{++}$  on the two-dimensional state-space, respectively. At each node  $i \times j$ , I can rewrite the discretized system of algebraic equations (A29) - (A30) as

$$\bar{q}_{i,j} = (c_{i,j})^\gamma \bar{\Phi}_{i,j} \quad (\text{A62a})$$

$$\hat{q}_{i,j} = (c_{i,j})^\gamma \hat{\Phi}_{i,j} \quad (\text{A62b})$$

along with the system of partial differential equations (A33) - (A34) that  $\bar{\Phi}_{i,j}, \hat{\Phi}_{i,j} \in C^2(\mathbb{R} \times \mathbb{R}^{++})$  must satisfy:

$$\left\{ \rho + (1 - \gamma) (\delta - \hat{i}) + \gamma \left( \frac{n-1}{\alpha n} \right)^{n-1} g_{i,j} \right\} \bar{\Phi}_{i,j} - \hat{\mathcal{D}}[\bar{\Phi}_{i,j}] = \frac{e^{a_i} + (f - \hat{i})}{\bar{x} (c_{i,j})^\gamma} \quad (\text{A63})$$

and

$$\left\{ \rho + \kappa_x + (1 - \gamma) (\delta - \hat{i}) + \gamma \left( \frac{n-1}{\alpha n} \right)^{n-1} g_{i,j} \right\} \hat{\Phi}_{i,j} - \hat{\mathcal{D}}[\hat{\Phi}_{i,j}] = \frac{e^{a_i}}{(c_{i,j})^\gamma} \quad (\text{A64})$$

where  $\hat{\mathcal{D}}[\Phi_{i,j}]$  is the finite-difference approximation to the infinitesimal generator  $\mathcal{D}[\Phi]$  evaluated at the node  $i \times j$ :

$$\hat{\mathcal{D}}[\Phi_{i,j}] = \kappa_a (\bar{a} - a_i) [\partial_a \Phi]_{i,j} + \frac{1}{2} \sigma_a^2 [\partial_{aa}^2 \Phi]_{i,j} + \mu_\omega (a_i, \omega_j) [\partial_\omega \Phi]_{i,j}. \quad (\text{A65})$$

$$\begin{aligned} [\partial_a \Phi]_{i,j} &= \frac{\Phi_{i+1,j} - \Phi_{i-1,j}}{2h_a}; \quad [\partial_\omega \Phi]_{i,j} = \frac{\Phi_{i,j+1} - \Phi_{i,j-1}}{2h_\omega}; \\ [\partial_{aa}^2 \Phi]_{i,j} &= \frac{\Phi_{i+1,j} - 2\Phi_{i,j} + \Phi_{i-1,j}}{h_a^2}. \end{aligned} \quad (\text{A66})$$

with  $h_a$  and  $h_\omega$  being the increments of  $a$  and  $\omega$  on the discrete two-dimensional state-space. The approximated system of partial differential equations (A63) - (A64) can be rewritten for  $i = 2, \dots, I - 1$  and  $j = 2, \dots, J - 1$  as a system of linear algebraic equations:

$$\bar{A}_{i,j} \bar{\Phi}_{i,j} + B_i \bar{\Phi}_{i+1,j} + C_i \bar{\Phi}_{i-1,j} + D_{i,j} \bar{\Phi}_{i,j+1} + E_{i,j} \bar{\Phi}_{i,j-1} = \bar{F}_{i,j} \quad (\text{A67})$$

$$\hat{A}_{i,j} \hat{\Phi}_{i,j} + B_i \hat{\Phi}_{i+1,j} + C_i \hat{\Phi}_{i-1,j} + D_{i,j} \hat{\Phi}_{i,j+1} + E_{i,j} \hat{\Phi}_{i,j-1} = \hat{F}_{i,j} \quad (\text{A68})$$

where

$$\begin{aligned} \bar{A}_{i,j} &\equiv \left\{ \rho + (1 - \gamma) (\delta - \hat{i}) + \gamma \left( \frac{n-1}{\alpha n} \right)^{n-1} g_{i,j} + \frac{\sigma_a^2}{h_a^2} \right\} \\ \hat{A}_{i,j} &\equiv \left\{ \rho + \kappa_x + (1 - \gamma) (\delta - \hat{i}) + \gamma \left( \frac{n-1}{\alpha n} \right)^{n-1} g_{i,j} + \frac{\sigma_a^2}{h_a^2} \right\} \\ B_i &\equiv - \left[ \frac{\kappa_a (\bar{a} - a_i)}{2h_a} + \frac{\sigma_a^2}{2h_a^2} \right]; \quad C_i \equiv \left[ \frac{\kappa_a (\bar{a} - a_i)}{2h_a} - \frac{\sigma_a^2}{2h_a^2} \right] \\ D_{i,j} &\equiv - \frac{\mu_\omega (a_i, \omega_j)}{2h_\omega}; \quad E_{i,j} \equiv \frac{\mu_\omega (a_i, \omega_j)}{2h_\omega} \\ \bar{F}_{i,j} &\equiv \left[ e^{a_i} + \bar{x}^{-1} (f - \hat{i}) \right] (c_{i,j})^{-\gamma}; \quad \hat{F}_{i,j} \equiv e^{a_i} (c_{i,j})^{-\gamma}. \end{aligned}$$

Including the zero-gradient boundary conditions, we can rewrite equation (A67) in matrix form as:

$$\bar{M} \bar{\Phi} = \bar{F} \quad (\text{A69})$$

where  $M$  is a  $[(I-2) \times (J-2)] \times [(I-2) \times (J-2)]$ -dimensional five-diagonal matrix, the column vector  $\bar{\Phi}$  is structured as

$$\bar{\Phi}_{[[I-2] \times [J-2]] \times 1} = \begin{bmatrix} \bar{\Phi}_2 \\ \dots \\ \bar{\Phi}_{J-1} \end{bmatrix}, \quad \bar{\Phi}_j = \begin{bmatrix} \bar{\Phi}_{2,j} \\ \dots \\ \bar{\Phi}_{I-1,j} \end{bmatrix},$$

and the column vector  $\bar{F}$  is

$$\bar{F}_{[[I-2] \times [J-2]] \times 1} = \begin{bmatrix} \bar{F}_2 \\ \dots \\ \bar{F}_{J-1} \end{bmatrix}, \quad \bar{F}_j = \begin{bmatrix} \bar{F}_{2,j} \\ \dots \\ \bar{F}_{I-1,j} \end{bmatrix}.$$

Similarly, we can rewrite equation (A68) in matrix form as

$$\hat{M} \hat{\Phi} = \hat{F} \quad (\text{A70})$$



where the matrix  $\widehat{M}$  and the column vectors  $\widehat{\Phi}$  and  $\widehat{F}$  preserve the same structure and dimensionality as for (A69). Then, I solve the system of linear equations (A69) - (A70) along with equations (A62a) - (A62b) by using the following iterative procedure. At each iteration  $n$ , given candidate solutions for  $\bar{q}^{(n)}$  and  $\widehat{q}^{(n)}$ , we can compute the corresponding value of  $\bar{\Phi}^{(n)}$  and  $\widehat{\Phi}^{(n)}$  as

$$\begin{aligned}\bar{\Phi}^{(n)} &= [\overline{M}(\bar{q}^{(n)}, \widehat{q}^{(n)})]^{-1} \overline{F}(\bar{q}^{(n)}, \widehat{q}^{(n)}) \\ \widehat{\Phi}^{(n)} &= [\widehat{M}(\bar{q}^{(n)}, \widehat{q}^{(n)})]^{-1} \widehat{F}(\bar{q}^{(n)}, \widehat{q}^{(n)}).\end{aligned}$$

With those values at hand, we can solve for the equilibrium  $\bar{q}^{(n)}$  and  $\widehat{q}^{(n)}$  by using the Newton-Raphson iterative procedure on the system:

$$\begin{bmatrix} \bar{q}_{i,j}^{(n+1)} \\ \widehat{q}_{i,j}^{(n+1)} \end{bmatrix} = \begin{bmatrix} \bar{q}_{i,j}^{(n)} \\ \widehat{q}_{i,j}^{(n)} \end{bmatrix} - \Delta [I_2 - J_{i,j}^{(n)}]^{-1} \begin{bmatrix} \bar{q}_{i,j}^{(n)} - (c_{i,j}^{(n)})^\gamma \bar{\Phi}_{i,j}^{(n)} \\ \widehat{q}_{i,j}^{(n)} - (c_{i,j}^{(n)})^\gamma \widehat{\Phi}_{i,j}^{(n)} \end{bmatrix}$$

where  $J_{i,j}^{(n)}$  denotes the  $2 \times 2$  Jacobian matrix evaluated at  $a_i$  and  $\omega_j$

$$J_{i,j}^{(n)} = \begin{bmatrix} \left\{ J_{i,j}^{(n)} \right\}_{11} & \left\{ J_{i,j}^{(n)} \right\}_{12} \\ \left\{ J_{i,j}^{(n)} \right\}_{21} & \left\{ J_{i,j}^{(n)} \right\}_{22} \end{bmatrix} = \gamma (c_{i,j}^{(n)})^{\gamma-1} \begin{bmatrix} \bar{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \bar{q}} & \bar{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \widehat{q}} \\ \widehat{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \bar{q}} & \widehat{\Phi}_{i,j}^{(n)} \frac{\partial c_{i,j}^{(n)}}{\partial \widehat{q}} \end{bmatrix}$$

and the step-size  $0 < \Delta \leq 1$  is adjusted to ensure convergence.

## Appendix C: Data Description

The empirical analysis is based on an unbalanced panel of firms drawn from the CRSP-COMPUSTAT merged database for the years 1962 - 2002. The data include only publicly traded firms in NYSE, NASDAQ and AMEX. I study only December fiscal year-end firms to eliminate the problem caused by the use of overlapping observations. I require a firm to have at least three years of valid observations to be included in the sample. I ignore firms with negative accounting numbers for book equity, capital and investment. I trim the values of extreme observations at the 0.5th and 99.5th percentiles or I use logs (where possible) to reduce the impact of extreme values which are common for ratios in firm panels drawn from accounting data.

Market equity is price times shares outstanding. Price is from CRSP (if available) or COMPUSTAT #199, shares outstanding are from CRSP (if available) or COMPUSTAT

#25. Book-equity is computed as the sum of stockholders' equity and deferred taxes and investment tax credit minus book value of preferred stock. Negative or zero book values are treated as missing. Stockholders' equity is COMPUSTAT #216 (if available), or COMPUSTAT #60 plus COMPUSTAT #130, or COMPUSTAT #6 minus COMPUSTAT #181. Deferred taxes and investment tax credit is COMPUSTAT #35. Book value of preferred stock is COMPUSTAT #56 (if available), or COMPUSTAT #10, or COMPUSTAT #130. Investment is capital expenditure (COMPUSTAT #128). Capital is net property, plant and equipment (COMPUSTAT #8). Stock returns are calculated from the beginning of July to the end of June of the following year. Profitability (ROE) is the ratio of common equity income to the book value of common equity at the beginning of fiscal year. Common equity income is the sum of end-of-year earnings before extraordinary items (COMPUSTAT #18) and depreciation (COMPUSTAT #14). Relative capital is the value of a firm net property, plant and equipment (COMPUSTAT #8) divided by the cross-sectional average value of net property, plant and equipment of all firms.

Aggregate series are obtained by averaging firm-level data. Aggregate ratios are computed as ratios of sums (for example, the aggregate investment rate is the ratio of the sum of firm investments over the sum of firm capital). Aggregate consumption is total consumption expenditures of nondurables plus services from NIPA for the period 1929 - 2004. Value weighted stock market returns and T-Bill rates are from CRSP for the period 1929 - 2004. All variables are in real terms. I use the implicit price deflator for non residential investment to deflate investment and capital. All other variables are deflated using the personal consumption expenditures deflator. Both price indexes are obtained from NIPA.

## Appendix D: Empirical Properties of Investments

This section provides empirical evidence of (i) a conditional (on firm profitability) inverse growth-size relation, and (ii) the independence of growth and size at the aggregate level.

The following empirical analysis is based on historical data described in Appendix C. I measure firm size by its relative capital. Since marginal  $q$  is not directly observable, I use Tobin's  $Q$  and cash flow rate as proxies for future investment profitability.

[Table D.I. here]

Table D.I. summarizes the variation of capital-weighted average investment rates across the intersection of five size and Tobin's  $Q$  quintiles. Across all Tobin's  $Q$  quintiles the pattern in average investment rates shows a monotonic inverse relation with size. This relation is statistically and economically significant. The hypotheses of average investment rates being equal to zero or being all the same across quintiles is strongly rejected with p-values of the  $\chi^2$ - statistic very close to zero.

[Table D.II. here]

In Table D.II. I provide descriptive statistics of investment rates across size and cash flow rate quintiles. I use the cash flow rate - cash flow over beginning of period capital - as alternative proxy for future investment profitability.

Across all cash flow rate quintiles the pattern in average investment rates confirms a monotonic negative relation with size. This relation is statistically and economically significant. The hypotheses of average investment rates being equal to zero or being all the same across quintiles is also strongly rejected with p-values of the  $\chi^2$ - statistic much less than 1%.

The descriptive statistics of investment rates across size and Tobin's  $Q$  or cash flow rate quintiles clearly show that (i) controlling for size, investment rates are positively related to future investment profitability (as proxied by Tobin's  $Q$  or cash flow rate), and (ii) controlling for future investment profitability, investment rates are inversely related to firm size.

I consider, but do not report for brevity, a number of robustness tests: (1) I split the full sample in manufacturing and non-manufacturing firms to control for the capital intensity of different industries; (2) I use market-to-book to proxy for Tobin's  $Q$ ; and (3) I measure cash flow using EBITDA. In all cases the results are qualitatively and statistically robust.

To summarize how quantitatively important is the inverse growth-size relation, I decompose the aggregate investment rate as

$$i_t \equiv E^i [i_{it}k_{it}] = Cov^i (i_{it}, k_{it}) + E^i [i_{it}] E^i [k_{it}] = Cov^i (i_{it}, k_{it}) + E^i [i_{it}]$$

where the superscript  $i$  denotes empirical cross-sectional moments. The last equality follows from the definition of relative capital, which implies  $E^i [k_{it}] = 1$ . There is a negative relation between  $i_{it}$  and  $k_{it}$  if and only if their cross-sectional covariance is negative,  $Cov^i (i_{it}, k_{it}) < 0$ .

**[Figure D1 here]**

Figure D1 plots the time-series of the aggregate investment rate,  $i_t$ , against the cross-sectional mean of firm investment rates,  $E^i [i_{it}]$ . This last one is always above the aggregate investment rate, which implies a negative cross-sectional covariance over the entire period. The spread between the two series widens substantially after the mid-80s. The cross-sectional mean of firm investment rates exceeds on average the aggregate investment rate by about 9.24% with a t-statistics of about 9.04. Therefore the average cross-sectional covariance is negative and strongly statistically significant. This empirical evidence shows that there is a strong economic and statistical significant inverse relation between firm investment rates and relative capital.

**[Table D.III. here]**

To establish the independence of growth and size at the aggregate level, I report the results of aggregate investment regressions in Table D.III. The dependent variable is the first difference of the aggregate investment rate. The independent variables are first differences of aggregate Tobin's  $Q$ , cash flow rate and capital. The first difference specification reflects a well-known feature of empirical aggregate investment equations: the presence of a highly serially correlated disturbance term when run in levels. Aggregate size has no economic or statistical significance in both univariate and multivariate regressions with Tobin's  $Q$  and/or aggregate cash flows. There is a statistically significant relation between investment with Tobin's  $Q$  and aggregate cash flows. These results confirm the independence of growth and size at the aggregate level.

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**Table I**  
**Parameter Values**

The table reports the parameter values used in the model simulations. The values reported in columns denoted “A Priori” correspond to a subset of parameters with a priori restrictions based on economic considerations explained in the main text. The values reported in columns denoted “Calibration” correspond to a subset of parameters restricted to match key unconditional aggregate and cross-sectional moments: unconditional mean and standard deviation of consumption growth, aggregate investment rate and equity premium; unconditional mean of value and size premia, average cross-sectional volatility of stock returns and average cross-sectional correlation between (the logarithms of) size and book-to-market. The calibration of model parameters is based on 100 artificial panels each with 200 firms and 5,000 years.

A Priori					Calibration							
$\rho$	$n$	$\hat{i}$	$f$	$\bar{x}$	$\gamma$	$\alpha$	$\delta$	$\kappa_a$	$\bar{a}$	$\sigma_a$	$\kappa_x$	$\sigma_x$
0.01%	2.00	0.12	0.12	1.00	15.57	2.27	0.11	0.27	-2.59	0.05	0.17	0.42

**Table II**  
**Target Moments**

This table reports unconditional moments for variables used in the calibration of the model. Panel A, aggregate moments: mean and standard deviations of consumption growth, investment-to-capital ratio and equity premium. Panel B, cross-sectional moments: mean of value and size premia, average cross-sectional volatility of stock returns and average cross-sectional correlation between (the logarithms of) size and book-to-market. The value (size) premium is the difference in returns on value (small) and growth (big) portfolios corresponding to the bottom 20 and top 20 percentiles of the book-to-market (market capitalization) distribution, respectively. Historical data are from NIPA and CRSP-COMPUSTAT merged database. Consumption growth and equity premium are computed using annual data from NIPA and CRSP for the period 1929 - 2004. The aggregate investment-to-capital ratio is computed as capital weighted average of firm investment rates from COMPUSTAT for the period 1962 - 2002. All series are in real terms. More details are provided in Appendix C. Simulated data are based on 100 artificial panels each with 200 firms and 5,000 years. I calculate aggregate and cross-sectional moments for each artificial panel and then I report cross-sample averages, standard deviations (in parenthesis) and 95 percent confidence intervals (in brackets). All numbers except those in the last row are annual percentages.

	Historical Data		Simulated Data	
	Mean	Std. Dev.	Mean	Std. Dev.
<b>A: Aggregate Moments</b>				
$dC/C$	2.14	2.54	2.53 (0.11) [2.30, 2.76]	4.85 (0.21) [4.42, 5.28]
$I/K$	17.75	3.66	12.77 (0.05) [12.67, 12.88]	0.96 (0.056) [0.85, 1.06]
$R_{Market} - R_f$	7.74	20.4	6.03 (0.75) [4.63, 7.48]	26.08 (2.41) [21.65, 30.36]
<b>B: Cross-Sectional Moments</b>				
$R_{Value} - R_{Growth}$	6.22		6.63 (0.07) [6.52, 6.80]	
$R_{Small} - R_{Big}$	4.42		3.70 (0.06) [3.60, 3.83]	
$\sigma^i(R_i)$	34.73		21.46 (2.93) [15.82, 27.02]	
$\rho^i(\ln \frac{K_i}{V_i}, \ln V_i)$	-0.31		-0.58 (0.01) [-0.59, -0.57]	

**Table III**  
**Market Returns Predictability**

Panel A: Standard deviations and autocorrelations of log dividend yield,  $\ln(D/V)$ , and log book-to-market,  $\ln(K/V)$ , in historical and simulated data. Panel B: predictability regressions of market returns,  $R^M$ , at the 1, 3 and 5 year horizon on log dividend yield and log book-to-market:

$$R_{t,t+k}^M = a_k + b_k x_t + \varepsilon_{t+k} \quad \text{for } k = 1, 3, 5.$$

I report Hansen-Hodrick corrected standard errors (in parenthesis),  $\sigma(b)$ . Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively.  $\bar{R}^2$  denotes adjusted  $R^2$ . Historical data are from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. Simulated data are based on 100 artificial panels each with 50 years. I calculate returns and characteristics for each sample and then report the cross-sample averages of coefficients, standard errors, and adjusted  $R^2$ .

<b>A: Standard Deviations and Autocorrelations</b>							
	Source	Std. Dev.	1Y	2Y	3Y	4Y	5Y
$\ln(D/V)$	Data	0.37	0.97	0.95	0.95	0.92	0.79
	Model	0.31	0.79	0.62	0.48	0.37	0.27
$\ln(K/V)$	Data	0.29	0.85	0.74	0.76	0.67	0.48
	Model	0.26	0.79	0.61	0.47	0.37	0.27
<b>B: Predictability Regressions</b>							
<b>Historical Data</b>				<b>Simulated Data</b>			
Horizon ( $k$ )	1Y	3Y	5Y	1Y	3Y	5Y	
	$\ln(D/V)$			$\ln(D/V)$			
$b$	0.15	0.34	0.81	0.22	0.86	1.60	
$\sigma(b)$	(0.09)	(0.20)	(0.37)**	(0.11)*	(0.20)***	(0.29)***	
$\bar{R}^2$	0.04	0.07	0.18	0.06	0.31	0.44	
	$\ln(K/V)$			$\ln(K/V)$			
$b$	0.16	0.45	0.77	0.26	1.00	1.85	
$\sigma(b)$	(0.13)	(0.26)*	(0.33)**	(0.14)*	(0.23)***	(0.30)***	
$\bar{R}^2$	0.02	0.12	0.19	0.06	0.33	0.47	

**Table IV**  
**Properties of Portfolios Sorted on Book-to-Market**

This table reports time-series averages of portfolios characteristics formed yearly on the basis of ranked values of book-to-market. Portfolios 2-9 cover corresponding book-to-market deciles. The bottom and top two portfolios (1A, 1B, 10A and 10B) split the bottom and top deciles in half. The portfolio excess return,  $R_p^e$ , investment-to-capital ratio,  $I_p/K_p$ , and profitability,  $ROE_p$ , are in percentage terms.  $SR_p$  denotes the portfolio Sharpe Ratio.  $K_p/V_p$  and  $V_p/V$  are the portfolio book-to-market and relative market value (portfolio market value relative to the aggregate market value), respectively.  $ROE_p$  is the portfolio profitability computed as portfolio common equity income to beginning-of-year portfolio book value of equity.  $K_p/K$  is the portfolio relative capital (portfolio capital relative to the aggregate capital). The portfolio value of  $R_p^e$  is a value-weighted average of excess returns for all firms in the portfolio. The portfolio values of  $K_p/V_p$ ,  $I_p/K_p$ ,  $ROE_p$  and  $K_p/K$  are computed as ratios of sums of the corresponding values of each firm characteristic for all firms in the portfolio. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years. I calculate returns and firm characteristics for each sample and then report cross-sample averages.

	<b>Portfolio</b>											
	<b>1A</b>	<b>1B</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10A</b>	<b>10B</b>
	<b>A: Historical Data</b>											
$R_p^e$	2.99	3.64	4.53	5.45	5.25	6.02	6.76	8.16	7.25	10.74	12.99	13.66
$SR_p$	0.12	0.19	0.27	0.31	0.30	0.35	0.41	0.47	0.44	0.62	0.65	0.58
$K_p/V_p$	0.21	0.29	0.42	0.54	0.66	0.79	0.92	1.08	1.29	1.65	2.05	4.58
$V_p/V$	2.40	2.12	1.54	1.33	1.04	0.96	0.76	0.72	0.64	0.49	0.33	0.19
$I_p/K_p$	32.91	28.63	26.03	23.34	21.38	19.69	18.00	15.84	14.09	13.51	12.23	9.99
$ROE_p$	27.39	20.42	17.12	14.88	12.42	11.33	9.63	8.86	7.67	6.29	5.00	0.50
$K_p/K$	0.46	0.64	0.72	0.96	0.97	1.10	1.07	1.25	1.32	1.18	0.95	0.81
	<b>B: Simulated Data</b>											
$R_p^e$	5.70	5.29	5.41	5.84	6.75	7.72	8.39	9.49	10.90	12.02	13.16	14.21
$SR_p$	0.19	0.19	0.19	0.21	0.24	0.27	0.29	0.32	0.36	0.38	0.41	0.43
$K_p/V_p$	0.74	0.93	1.07	1.27	1.61	1.77	1.94	2.35	2.66	2.91	3.28	3.36
$V_p/V$	3.06	2.30	1.77	1.42	1.10	0.91	0.82	0.71	0.60	0.54	0.49	0.46
$I_p/K_p$	22.65	18.77	15.88	13.93	12.54	12.08	12.01	12.00	12.00	12.00	12.00	12.00
$ROE_p$	24.19	20.58	17.11	14.18	10.79	8.45	7.22	5.61	4.07	3.14	2.36	1.76
$K_p/K$	1.34	1.21	1.10	1.03	0.97	0.94	0.93	0.92	0.91	0.91	0.90	0.92

**Table V**  
**Properties of Portfolios Sorted on Size**

This table reports time-series averages of portfolios characteristics formed yearly on the basis of ranked values of market equity. Portfolios 2-9 cover corresponding market equity deciles. The bottom and top two portfolios (1A, 1B, 10A and 10B) split the bottom and top deciles in half. The portfolio excess return,  $R_p^e$ , investment-to-capital ratio,  $I_p/K_p$ , and profitability,  $ROE_p$ , are in percentage terms.  $SR_p$  denotes the portfolio Sharpe Ratio.  $K_p/V_p$  and  $V_p/V$  are the portfolio book-to-market and relative market value (portfolio market value relative to the aggregate market value), respectively.  $ROE_p$  is the portfolio profitability computed as portfolio common equity income to beginning-of-year portfolio book value of equity.  $K_p/K$  is the portfolio relative capital (portfolio capital relative to the aggregate capital). The portfolio value of  $R_p^e$  is a value-weighted average of excess returns for all firms in the portfolio. The portfolio values of  $K_p/V_p$ ,  $I_p/K_p$ ,  $ROE_p$  and  $K_p/K$  are computed as ratios of sums of the corresponding values of each firm characteristic for all firms in the portfolio. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years. I calculate returns and firm characteristics for each sample and then report cross-sample averages.

	<b>Portfolio</b>											
	<b>1A</b>	<b>1B</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10A</b>	<b>10B</b>
	<b>A: Historical Data</b>											
$R_p^e$	16.37	12.38	10.57	9.70	8.84	8.36	7.22	8.58	6.88	6.14	5.59	3.77
$SR_p$	0.63	0.49	0.43	0.40	0.37	0.38	0.36	0.44	0.36	0.37	0.37	0.23
$K_p/V_p$	1.33	1.25	1.11	1.05	0.99	0.91	0.83	0.79	0.75	0.73	0.67	0.50
$V_p/V$	0.01	0.02	0.04	0.07	0.11	0.18	0.31	0.56	1.10	2.48	4.44	32.03
$I_p/K_p$	18.60	18.41	20.31	19.12	18.38	18.60	19.56	18.54	17.72	16.23	17.31	18.87
$ROE_p$	0.47	0.92	3.03	3.82	4.64	6.69	8.03	9.10	9.66	10.02	10.95	13.33
$K_p/K$	0.01	0.02	0.04	0.08	0.13	0.19	0.29	0.52	0.97	2.09	3.72	7.64
	<b>B: Simulated Data</b>											
$R_p^e$	10.82	10.42	10.08	9.70	9.13	8.67	8.35	7.70	6.95	6.37	5.77	5.16
$SR_p$	0.33	0.34	0.33	0.32	0.31	0.29	0.29	0.27	0.24	0.22	0.20	0.17
$K_p/V_p$	2.47	2.36	2.27	2.17	2.03	1.91	1.82	1.65	1.44	1.27	1.09	0.90
$V_p/V$	0.15	0.24	0.32	0.44	0.66	0.79	0.94	1.42	1.94	2.51	3.97	5.11
$I_p/K_p$	12.22	12.27	12.30	12.34	12.47	12.62	12.73	13.02	13.51	14.03	14.60	15.16
$ROE_p$	4.39	4.64	5.00	5.42	6.15	6.86	7.41	8.66	10.67	12.77	15.73	20.31
$K_p/K$	0.19	0.29	0.39	0.50	0.67	0.84	0.96	1.16	1.46	1.71	2.01	2.54

**Table VI**  
**Properties of Portfolios Sorted on Book-to-Market and Size**

This table reports time-series averages of portfolios characteristics formed yearly on the basis of ranked values of market equity and book-to-market. In particular, each year stocks are allocated to three size groups based on the breakpoints for the bottom and top 20 percent of the ranked values of market equity. Similarly, each year stocks are allocated in an independent sort to three book-to-market groups based on the breakpoints for the bottom and top 20 percent of the ranked values of book-to-market. The nine portfolios are the intersection of the three size and the three book-to-market groups. The numbers reported in columns denoted  $|3Q-1Q|$  represent the absolute value of the difference in the values of a variable between the highest and lowest book-to-market portfolios. The portfolio excess return,  $R_p^e$ , investment-to-capital ratio,  $I_p/K_p$ , and profitability,  $ROE_p$ , are in percentage terms.  $ROE_p$  is the portfolio profitability computed as portfolio common equity income to beginning-of-year portfolio book value of equity.  $K_p/K$  is the portfolio relative capital (portfolio capital relative to the aggregate capital). The portfolio value of  $R_p^e$  is a value-weighted average of excess returns for all firms in the portfolio. The portfolio values of  $I_p/K_p$ ,  $ROE_p$  and  $K_p/K$  are computed as ratios of sums of the corresponding values of each firm characteristic for all firms in the portfolio. Panel A reports statistics based on historical data from the CRSP-COMPUSTAT merged database. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years. I calculate returns and firm characteristics for each sample and then report cross-sample averages.

	<b>A: Historical Data</b>				<b>B: Simulated Data</b>			
	Book-to-Market				Book-to-Market			
Size	1Q	2	3Q	$ 3Q-1Q $	1Q	2	3Q	$ 3Q-1Q $
	$R_p^e$				$R_p^e$			
Small	5.83	11.38	13.94	8.11	4.03	8.62	11.97	7.94
2	4.27	8.00	11.47	7.20	5.92	8.06	11.65	5.73
Big	3.83	5.53	9.66	5.83	5.45	7.38	8.93	3.48
	$I_p/K_p$				$I_p/K_p$			
Small	33.87	22.23	16.98	16.89	19.08	12.22	12.00	7.08
2	33.33	19.16	13.12	20.21	16.28	12.19	12.00	4.28
Big	26.74	17.57	12.63	14.11	15.34	12.18	12.00	3.34
	$ROE_p$				$ROE_p$			
Small	3.98	4.59	0.39	3.59	13.83	6.85	3.18	10.65
2	16.60	9.69	3.36	13.23	15.06	7.81	3.42	11.64
Big	20.24	10.92	6.14	14.10	17.63	9.20	3.80	13.83
	$K_p/K$				$K_p/K$			
Small	0.01	0.02	0.04	0.03	0.24	0.37	0.49	0.25
2	0.13	0.36	0.60	0.47	0.66	0.90	1.16	0.50
Big	1.72	4.41	7.50	5.79	1.49	1.87	2.58	1.08

**Table VII**  
**Excess Returns and Firm Characteristics**

The table reports coefficients and standard errors (in parenthesis) of Fama-MacBeth cross-sectional regressions. The dependent variable is individual firm excess stock return. The independent variables are the (logarithms of) investment-to-capital ratio,  $I_i/K_i$ , relative capital,  $K_i/K$ , book-to-market,  $K_i/V_i$ , relative market equity,  $V_i/V$ , and their interaction. Standard errors are adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively. Panel A reports statistics based on historical data. Details are provided in Appendix C. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years. I calculate returns and firm characteristics for each sample and then report cross-sample averages of coefficients and standard errors. All numbers are annual percentages.

	$\ln(K_i/V_i)$	$\ln(V_i/V)$	$\ln(I_i/K_i)$	$\ln(K_i/K)$	$\ln(K_i/V_i) \times \ln(V_i/V)$
<b>A: Historical Data</b>					
1	4.36 (1.02)***				
2		-1.42 (0.53)***			
3	3.14 (1.17)***	-1.06 (0.53)**			
4	2.67 (1.31)**	-1.43 (0.61)**			-0.60 (0.30)**
5			-1.80 (0.85)**	-1.08 (0.53)**	
6	3.48 (1.11)***			-0.96 (0.54)**	
7		-1.09 (0.53)**	-1.69 (0.76)**		
<b>B: Simulated Data</b>					
1	5.29 (0.37)***				
2		-1.46 (0.11)***			
3	5.23 (0.37)***	-0.04 (0.01)**			
4	5.17 (0.37)***	0.48 (0.05)***			-1.35 (0.08)***
5			-4.10 (1.24)***	-0.35 (0.07)***	
6	5.27 (0.37)***			-0.04 (0.01)**	
7		-1.08 (0.15)***	-4.02 (1.25)***		

**Table VIII**  
**Summary Statistics of Factors**

The table reports summary statistics of excess stock market returns, *SMB* and *HML* in both historical and simulated data. *SMB* and *HML* are returns on the “small minus big” portfolio and “high minus low” portfolio constructed as in Fama and French (1993), respectively. Panel A reports means and standard errors (in parenthesis) adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively. Panel B reports the correlation matrix. Historical data are from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Simulated data are based on 100 artificial panels each with 200 firms and 50 years. I calculate returns for each sample and then report cross-sample averages of summary statistics. All numbers except correlations are in percentage terms.

	<b>Historical Data</b>			<b>Simulated Data</b>		
<b>A: Means and Standard Errors</b>						
	$R_M^e$	<i>SMB</i>	<i>HML</i>	$R_M^e$	<i>SMB</i>	<i>HML</i>
$\mu$	5.04	2.52	5.05	7.49	0.44	4.95
$\sigma(\mu)$	(2.06)**	(2.25)	(2.04)**	(3.57)**	(0.51)	(0.85)***
<b>B: Correlations</b>						
	$R_M^e$	<i>SMB</i>	<i>HML</i>	$R_M^e$	<i>SMB</i>	<i>HML</i>
$R_M^e$	1.00	0.28	-0.31	1.00	0.04	0.40
<i>SMB</i>		1.00	-0.16		1.00	-0.29
<i>HML</i>			1.00			1.00



**Table IX**  
**CAPM - Time Series Regressions**

The table reports summary statistics of time series regressions of book-to-market sorted portfolios' excess stock returns,  $R_{t+1}^{ep}$ , on the excess stock market returns,  $R_{t+1}^{eM}$ :

$$R_{t+1}^{ep} = \alpha^p + \beta_M^p R_{t+1}^{eM} + \varepsilon_{t+1}^p \quad \text{for } p = 1, \dots, 12.$$

The time-series intercepts,  $\alpha$ , and standard errors,  $\sigma(\alpha)$ , are in percentage terms. The coefficients,  $\beta_M$ , denote CAPM -  $\beta$ s. Standard errors (in parenthesis) are adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one asterisk are statistically significant at the five percent level.  $\chi^2$ -test statistics and associated  $p$ -value (in percentage terms) for the null of alphas jointly equal to zero. RMSA is the root mean squared alphas in percentage terms. HJD is the Hansen-Jagannathan distance. Panel A reports statistics based on historical data from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years. I calculate returns for each sample and then report cross-sample averages of regression coefficients, standard errors and summary statistics.

	<b>Portfolio</b>											
	<b>1A</b>	<b>1B</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10A</b>	<b>10B</b>
	<b>A: Historical Data</b>											
$\alpha$	-3.61	-1.68	-0.57	0.11	0.06	1.21	2.25	3.08	2.97	6.52	8.24	8.56
$\sigma(\alpha)$	(1.15)*	(0.86)*	(0.61)	(0.88)	(1.19)	(1.49)	(1.12)*	(1.22)*	(1.72)	(2.08)*	(2.36)*	(3.17)*
$\beta_M$	1.31	1.06	1.01	1.06	1.03	0.96	0.90	1.01	0.85	0.84	0.94	1.01
$\sigma(\beta_M)$	(0.19)*	(0.08)*	(0.06)*	(0.11)*	(0.06)*	(0.06)*	(0.09)*	(0.06)*	(0.07)*	(0.14)*	(0.12)*	(0.16)*
<i>Statistics</i>		$\chi^2$	46.98		p.v.	0.00		RMSA	4.33		HJD	0.67
	<b>B: Simulated Data</b>											
$\alpha$	-1.58	-1.99	-1.88	-1.54	-0.61	0.25	0.86	1.81	3.07	3.99	5.03	5.98
$\sigma(\alpha)$	(1.46)	(0.88)*	(0.88)*	(0.65)*	(0.47)	(0.67)	(0.65)	(0.53)*	(0.74)*	(0.88)*	(0.90)*	(1.31)*
$\beta_M$	0.93	0.94	0.95	0.97	0.99	1.01	1.02	1.05	1.09	1.13	1.16	1.19
$\sigma(\beta_M)$	(0.07)*	(0.04)*	(0.04)*	(0.03)*	(0.02)*	(0.03)*	(0.03)*	(0.02)*	(0.03)*	(0.04)*	(0.04)*	(0.06)*
<i>Statistics</i>		$\chi^2$	156.69		p.v.	0.00		RMSA	3.06		HJD	0.82

**Table X**  
**Asset Pricing Models - Fama-MacBeth Regressions**

The table reports summary statistics of Fama-MacBeth cross-sectional regressions. The dependent variable is excess stock return on book-to-market sorted portfolios. The independent variables are a constant and betas from time-series regression of excess returns on the factors. Standard errors (in parenthesis) are adjusted for heteroskedasticity, serial correlation (one lag) and sampling variation in estimated betas using GMM formulas. Standard errors starred with one, two and three asterisks are statistically significant at the ten, five and one percent level, respectively.  $\bar{R}^2$  denotes adjusted  $R^2$ . Coefficients and standard errors are in percentage terms. Lines 1 and 6, CAPM, where  $MKT$  represents the average excess stock market return. Line 2, 7: conditional CAPM with the aggregate log book-to-market,  $\ln(K/V)$ . Line 3, 8: conditional CAPM with the aggregate log dividend yield,  $\ln(D/V)$ . Line 4, 9: two-factor model ( $MKT$  and  $HML$ ), where  $HML$  denotes the average return on the “high minus low” portfolio. Line 5, 10: Fama and French (1993) model, where  $SMB$  is the average return on the “small minus big” portfolio. Line 11: conditional CAPM with model implied conditional  $\beta$ . Panel A: statistics based on historical data. Details are provided in Appendix C. Panel B: cross-sample averages of statistics based on 100 artificial panels each with 200 firms and 50 years.

	<i>Intercept</i>	<i>MKT</i>	$\ln(K/V) \times MKT$	$\ln(D/V) \times MKT$	<i>HML</i>	<i>SMB</i>	$\beta$	$\bar{R}^2$
<b>A: Historical Data</b>								
1	21.64 (7.71)***	-14.39 (7.70)*						0.18
2	23.82 (8.51)***	-16.65 (8.65)*	8.16 (4.56)*					0.11
3	23.87 (8.14)***	-17.24 (8.30)**		61.35 (28.84)**				0.25
4	-10.17 (6.46)	15.28 (6.33)**			7.01 (2.65)**			0.77
5	-2.33 (5.22)	7.10 (5.51)			4.62 (2.73)*	8.68 (4.19)**		0.81
<b>B: Simulated Data</b>								
6	-29.02 (5.84)***	36.84 (6.76)***						0.67
7	-16.52 (6.47)**	24.17 (7.41)***	12.72 (2.95)***					0.74
8	-15.82 (6.52)**	23.48 (7.46)***		47.54 (20.73)**				0.74
9	-12.25 (6.79)	19.85 (7.70)**			6.27 (1.11)***			0.79
10	-9.36 (7.31)	16.89 (8.16)**			4.50 (1.35)***	1.68 (1.34)		0.83
11	-1.13 (2.09)						8.51 (2.82)***	0.98

**Table XI**  
**Fama and French (1993) Model - Time Series Regressions**

The table reports summary statistics of time series regressions of book-to-market sorted portfolios' excess stock returns,  $R_{t+1}^{ep}$ , on the excess stock market returns,  $R_{t+1}^{eM}$ , the returns on  $SMB$ ,  $R_{t+1}^{SMB}$ , and  $HML$ ,  $R_{t+1}^{HML}$ :

$$R_{t+1}^{ep} = \alpha^p + \beta_M^p R_{t+1}^{eM} + \beta_{SMB}^p R_{t+1}^{SMB} + \beta_{HML}^p R_{t+1}^{HML} + \varepsilon_{t+1}^p \quad \text{for } p = 1, \dots, 12.$$

The time-series intercepts,  $\alpha$ , and standard errors,  $\sigma(\alpha)$ , are in percentage terms. Standard errors (in parenthesis) are adjusted for heteroskedasticity and serial correlation using Newey-West formula with one lag. Standard errors starred with one asterisk are statistically significant at the five percent level.  $\chi^2$ -test statistics and associated  $p$ -value (in percentage terms) for the null of alphas jointly equal to zero. RMSA is the root mean squared alphas in percentage terms. HJD is the Hansen-Jagannathan distance. Panel A reports statistics based on historical data from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Panel B shows the results based on 100 artificial panels each with 200 firms and 50 years. I calculate returns for each sample and then report cross-sample averages of regression coefficients, standard errors, and summary statistics.

	<b>Portfolio</b>											
	<b>1A</b>	<b>1B</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10A</b>	<b>10B</b>
	<b>A: Historical Data</b>											
$\alpha$	0.29	0.64	0.74	-0.71	-0.77	-0.55	-0.24	0.08	-0.97	2.18	2.96	4.23
$\sigma(\alpha)$	(1.95)	(1.40)	(0.65)	(0.90)	(1.27)	(1.38)	(1.29)	(0.95)	(1.35)	(1.28)	(1.91)	(3.16)
$\beta_M$	1.13	0.98	1.00	1.08	1.05	0.96	0.94	1.08	0.93	0.95	1.01	1.04
$\sigma(\beta_M)$	(0.18)*	(0.07)*	(0.05)*	(0.10)*	(0.06)*	(0.07)*	(0.07)*	(0.04)*	(0.07)*	(0.14)*	(0.10)*	(0.16)*
$\beta_{SMB}$	0.11	-0.05	-0.13	0.02	0.03	0.21	0.19	0.14	0.27	0.21	0.47	0.48
$\sigma(\beta_{SMB})$	(0.16)	(0.09)	(0.06)*	(0.06)	(0.08)	(0.09)*	(0.10)*	(0.07)*	(0.09)*	(0.08)*	(0.15)*	(0.16)*
$\beta_{HML}$	-0.64	-0.36	-0.18	0.13	0.12	0.24	0.36	0.45	0.57	0.65	0.74	0.59
$\sigma(\beta_{HML})$	(0.13)*	(0.13)*	(0.04)*	(0.08)	(0.10)	(0.09)*	(0.10)*	(0.05)*	(0.07)*	(0.09)*	(0.11)*	(0.16)*
<i>Statistics</i>		$\chi^2$	10.31		p.v.	58.9		RMSA	1.70	HJD		0.57
	<b>B: Simulated Data</b>											
$\alpha$	-0.47	-0.59	-0.28	-0.21	0.15	-0.31	0.01	-0.25	-0.33	0.98	2.19	2.75
$\sigma(\alpha)$	(1.97)	(1.15)	(1.14)	(0.81)	(0.63)	(0.90)	(0.88)	(0.58)	(0.65)	(0.86)	(0.89)*	(1.37)*
$\beta_M$	0.95	0.98	1.00	1.01	1.01	0.99	0.99	0.99	0.99	1.04	1.08	1.10
$\sigma(\beta_M)$	(0.08)*	(0.04)*	(0.04)*	(0.03)*	(0.02)*	(0.03)*	(0.03)*	(0.02)*	(0.03)*	(0.03)*	(0.04)*	(0.06)*
$\beta_{SMB}$	-0.21	-0.25	-0.23	-0.18	-0.11	0.06	0.14	0.27	0.48	0.60	0.62	0.68
$\sigma(\beta_{SMB})$	(0.44)	(0.24)	(0.24)	(0.17)	(0.13)	(0.19)	(0.18)	(0.12)*	(0.14)*	(0.18)*	(0.19)*	(0.32)*
$\beta_{HML}$	-0.22	-0.30	-0.33	-0.28	-0.16	0.12	0.17	0.43	0.72	0.63	0.58	0.66
$\sigma(\beta_{HML})$	(0.27)	(0.15)*	(0.15)*	(0.11)*	(0.08)*	(0.12)	(0.11)	(0.07)*	(0.09)*	(0.12)*	(0.12)*	(0.20)*
<i>Statistics</i>		$\chi^2$	31.40		p.v.	6.08		RMSA	1.58	HJD		0.80

**Table D.I.**  
**Investment Rate, Size and Tobin's Q**

This table reports the time-series average of portfolios' investment rates across the intersection of five size and Tobin's  $Q$  quintiles over the period, 1962 - 2002. A portfolio's investment rate,  $I/K$ , for year  $t$  is the sum of end-of-year  $t$  capital expenditure,  $I$ , for the firms in the portfolio in the end-of-year  $t - 1$ , divided by the sum of their net property, plant and equipment,  $K$ , in the end-of-year  $t - 1$ . The 25 size - Tobin's  $Q$  portfolios are formed as follows. Each end-of-year  $t$  from 1962 to 2002 size and Tobin's  $Q$  quintile breakpoints are used to allocate firms to 5x5 groups resulting from the intersection of the five size and the five Tobin's  $Q$  quintiles. Size,  $k_{it}$ , is the value of firm  $i$  net property, plant and equipment (COMPUSTAT # 8) in the end-of-year  $t$ , divided by the cross-sectional average value of net property, plant and equipment for all the firms in the end-of-year  $t$ . Tobin's  $Q$ ,  $Q_i$  is the market value of assets divided by the book value of assets. A firm's market value of assets equals the book value of assets (COMPUSTAT # 6) plus the market value of common stock less the sum of book value of common stock (COMPUSTAT # 60) and balanced sheet deferred taxes (COMPUSTAT # 74). The time-series average of each quintile breakpoint is reported in parenthesis next to each portfolio quintile. The row and column labeled "All" contain values of descriptive statistics computed by aggregating across size and Tobin's  $Q$  quintiles, respectively. Robust standard errors of mean investment rates are reported in parenthesis. All numbers except quintile breakpoints are in percentages.

<sup>(1)</sup> p-value of the  $\chi^2$ -test for the null of mean investment rates jointly equal to zero (across quintiles).

<sup>(2)</sup> p-value of the  $\chi^2$ -test for the null of mean investment rates being all the same (across quintiles).

		Tobin's Q ( $Q_i$ )						
Size ( $k_i$ )	1Q (0.92)	2Q (1.08)	3Q (1.30)	4Q (1.81)	5Q (9.29)	All	p <sup>(1),(2)</sup>	
1Q (0.03)	24.2 (1.43)	27.1 (0.84)	31.7 (1.33)	36.6 (1.43)	52.8 (1.98)	35.2 (1.09)	< 1%	
2Q (0.09)	19.8 (0.50)	23.5 (0.96)	26.7 (0.93)	32.2 (0.99)	44.3 (1.41)	29.3 (0.74)	< 1%	
3Q (0.25)	18.2 (0.49)	20.7 (0.69)	24.0 (0.67)	30.0 (0.78)	37.5 (1.33)	26.1 (0.60)	< 1%	
4Q (0.97)	15.0 (0.45)	17.6 (0.50)	20.1 (0.56)	24.9 (0.86)	30.2 (0.91)	21.6 (0.50)	< 1%	
5Q (26.37)	13.1 (0.76)	14.6 (0.58)	18.1 (0.59)	21.5 (0.62)	25.8 (0.79)	17.0 (0.60)	< 1%	
All	13.2 (0.63)	15.0 (0.57)	18.5 (0.57)	22.3 (0.59)	27.7 (0.74)	17.8 (0.58)		
p <sup>(1),(2)</sup>	< 1%	< 1%	< 1%	< 1%	< 1%	< 1%		

**Table D.II.**  
**Investment Rate, Size and Cash Flow Rate**

This table reports the time-series average of portfolios' investment rates across the intersection of five size and cash flow rate quintiles over the period, 1962 - 2002. A portfolio's investment rate,  $I/K$ , for year  $t$  is the sum of end-of-year  $t$  capital expenditure,  $I$ , for the firms in the portfolio in the end-of-year  $t - 1$ , divided by the sum of their net property, plant and equipment,  $K$ , in the end-of-year  $t - 1$ . The 25 size - cash flow rate portfolios are formed as follows. Each end-of-year  $t$  from 1962 to 2002 size and cash flow rate quintile breakpoints are used to allocate firms to 5x5 groups resulting from the intersection of the five size and the five cash flow rate quintiles. Size,  $k_{it}$ , is the value of firm  $i$  net property, plant and equipment (COMPUSTAT # 8) in the end-of-year  $t$ , divided by the cross-sectional average value of net property, plant and equipment for all the firms in the end-of-year  $t$ . Cash flow rate,  $CF_i/K_i$ , is the sum of end-of-year earnings before extraordinary items (COMPUSTAT # 18) and depreciation (COMPUSTAT # 14) over beginning-of-year net property, plant and equipment. The time-series average of each quintile breakpoint is reported in parenthesis next to each portfolio quintile. The row and column labeled "All" contain values of descriptive statistics computed by aggregating across size and cash flow rate quintiles, respectively. Robust standard errors of mean investment rates are reported in parenthesis. All numbers except quintile breakpoints are in percentages.

(<sup>1</sup>) p-value of the  $\chi^2$ -test for the null of mean investment rates jointly equal to zero (across quintiles).

(<sup>2</sup>) p-value of the  $\chi^2$ -test for the null of mean investment rates being all the same (across quintiles).

Size ( $k_i$ )	Cash Flow Rate ( $CF_i/K_i$ )					All	p <sup>(1),(2)</sup>
	1Q (0.12)	2Q (0.22)	3Q (0.34)	4Q (0.56)	5Q (6.03)		
1Q (0.03)	26.9 (1.53)	23.4 (1.14)	25.8 (1.31)	32.8 (1.50)	46.8 (1.61)	35.2 (1.09)	< 1%
2Q (0.09)	21.4 (1.07)	20.8 (1.12)	24.4 (0.82)	29.2 (0.79)	42.3 (1.22)	29.3 (0.74)	< 1%
3Q (0.25)	19.2 (0.68)	21.1 (0.84)	25.4 (0.73)	28.6 (0.77)	35.7 (1.09)	26.1 (0.60)	< 1%
4Q (0.97)	14.5 (0.41)	18.0 (0.69)	22.9 (0.60)	26.1 (0.63)	31.5 (1.15)	21.6 (0.50)	< 1%
5Q (26.37)	12.1 (0.62)	15.9 (0.72)	22.1 (0.58)	25.9 (0.73)	32.7 (1.28)	17.0 (0.60)	< 1%
All	12.4 (0.56)	16.0 (0.71)	22.2 (0.52)	26.2 (0.61)	33.3 (0.87)	17.8 (0.58)	
p <sup>(1),(2)</sup>	< 1%	< 1%	< 1%	< 1%	< 1%	< 1%	

**Table D.III.**  
**Aggregate Investment Regressions**

The dependent variable is aggregate investment rate,  $I_t/K_{t-1}$ . The independent variables are aggregate Tobin's  $Q$ , cash flow rate and size. Data are from the CRSP-COMPUSTAT merged database for the period 1962 - 2002. All aggregate series are obtained by aggregating firm-level data. For each regression, OLS coefficient estimates based on first-differencing of the variables are reported in the first line. The standard errors reported in parenthesis are corrected for heteroskedasticity and serial correlation using Newey-West methodology with 3 lags. Standard errors starred with one asterisk are statistically significant at one percent level. A constant term is included, but not reported. All numbers except p-values are in percentages.

(<sup>1</sup>) p-value of the  $\chi^2$ -test for the null of the coefficients (excluding the intercept) jointly equal to zero.

$Q_{t-1}$	$CF_{t-1}/K_{t-2}$	$K_{t-1}$	p <sup>(1)</sup>
		0.00	0.90
		(0.01)	
6.11		0.00	< 1%
(1.30)*		(0.01)	
	65.62	0.00	< 1%
	(6.28)*	(0.01)	
5.33	62.92	0.00	< 1%
(0.98)*	(6.00)*	(0.00)	

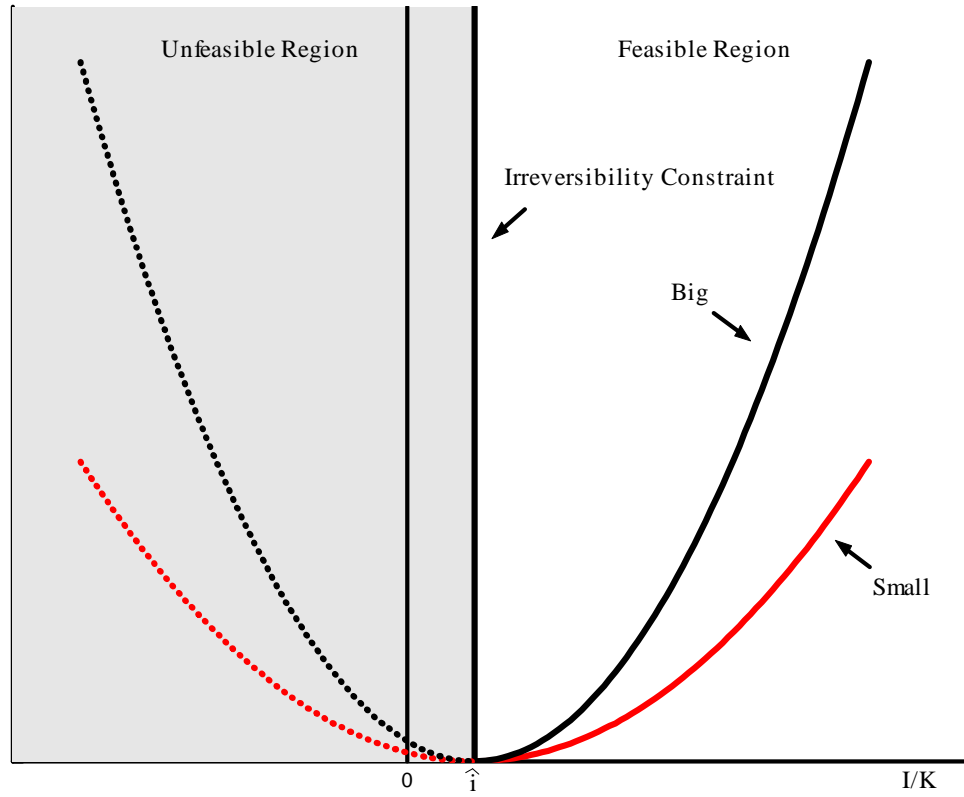


Figure 1. Adjustment Cost. The figure illustrates the specification of capital adjustment costs for a firm with small relative capital (“Small”) and a firm with large relative capital (“Big”). The investment rate,  $I/K$ , is on the x-axis and the adjustment costs are on the y-axis. The minimum investment rate,  $\hat{i}$ , separate the plane into a feasible investment region and an unfeasible investment region (shaded area). The parameters are reported in Table 1.

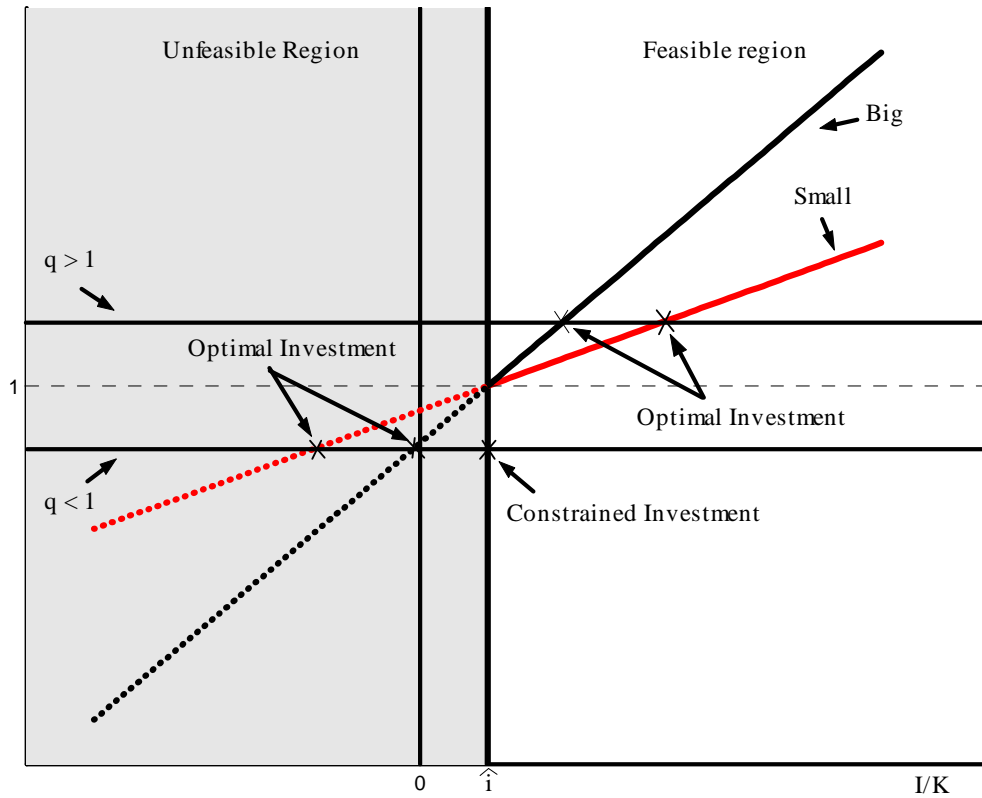


Figure 2. Optimal Investment Policy. The figure illustrates the optimal investment policy for a firm with small relative capital (“Small”) and a firm with large relative capital (“Big”). The investment rate,  $I/K$ , is on the x-axis and the marginal benefit and cost of investment are on the y-axis. Marginal benefits of investment (marginal  $q$ ) are the horizontal lines. The marginal costs of investment are the upward sloping lines. The minimum investment rate,  $\hat{i}$ , separate the plane into a feasible investment region and an unfeasible investment region (shaded area). The parameters are reported in Table 1.



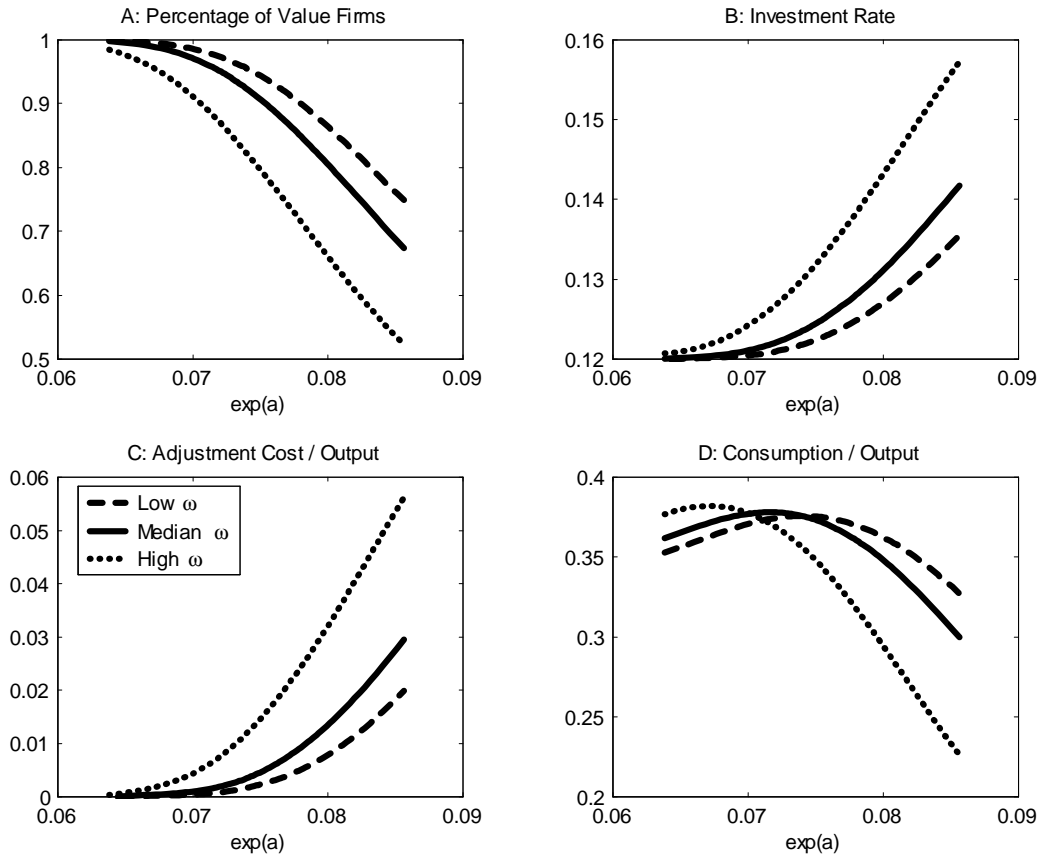


Figure 3. Aggregate Quantities in Equilibrium. The figure plots some relevant aggregate variables in competitive equilibrium as a function of the aggregate productivity,  $exp(a)$ , and three values of  $\omega$  corresponding to the 10th (Low), 50th (Median) and 90th (High) percentiles of its unconditional distribution, respectively. Panel A: percentage of firms up against the investment irreversibility constraint. Panel B: investment as a fraction of capital. Panel C: adjustment costs as a fraction of output. Panel D: consumption as a fraction of output. The parameters are reported in Table 1.

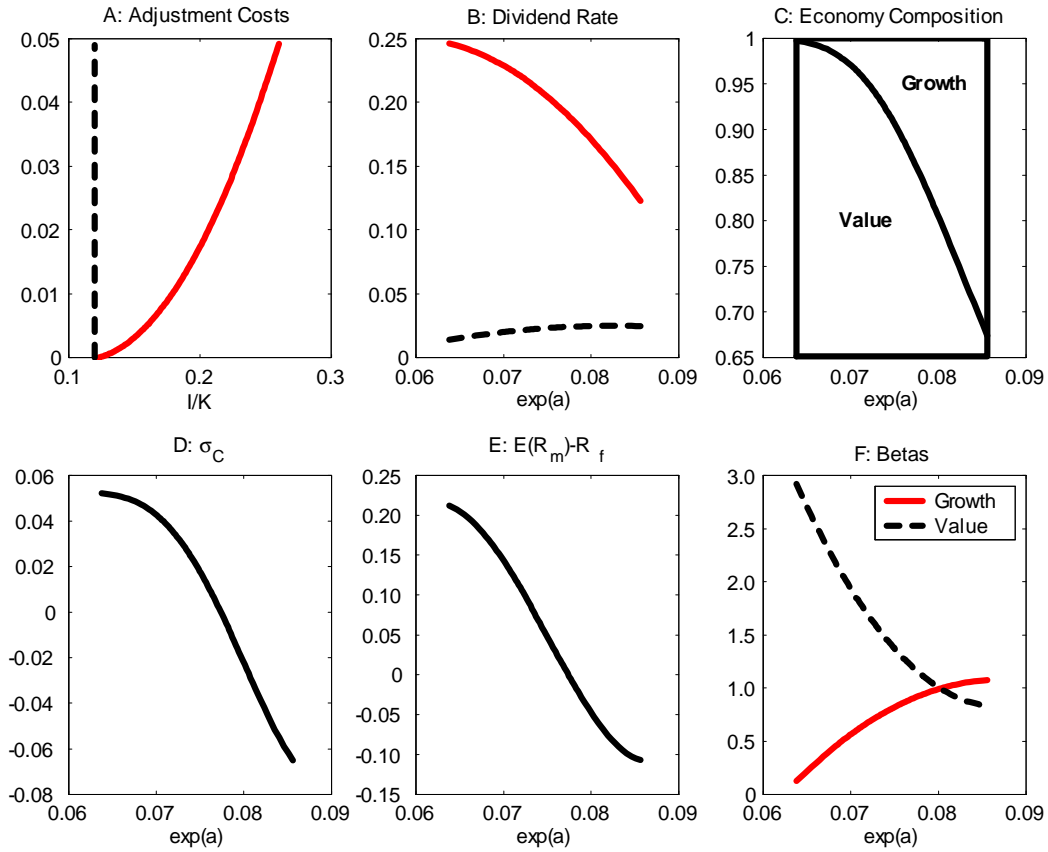


Figure 4. Economic Mechanisms. The figure plots some relevant aggregate and cross-sectional variables in competitive equilibrium as a function of the aggregate productivity,  $exp(a)$ , and the median value of  $\omega$ . Panel A: adjustment costs as a fraction of firm capital for growth (solid line) and value (dashed line) firms. Panel B: dividends as fraction of capital for growth and value firms. Panel C: percentage of firms up against the investment irreversibility constraint. Panel D: diffusion component of consumption growth. Panel E: conditional equity premium. Panel F: conditional market betas for growth and value firms. Growth and value firms belong to the 10th and 90th percentiles of the book-to-market distribution, respectively. The parameters are reported in Table 1.

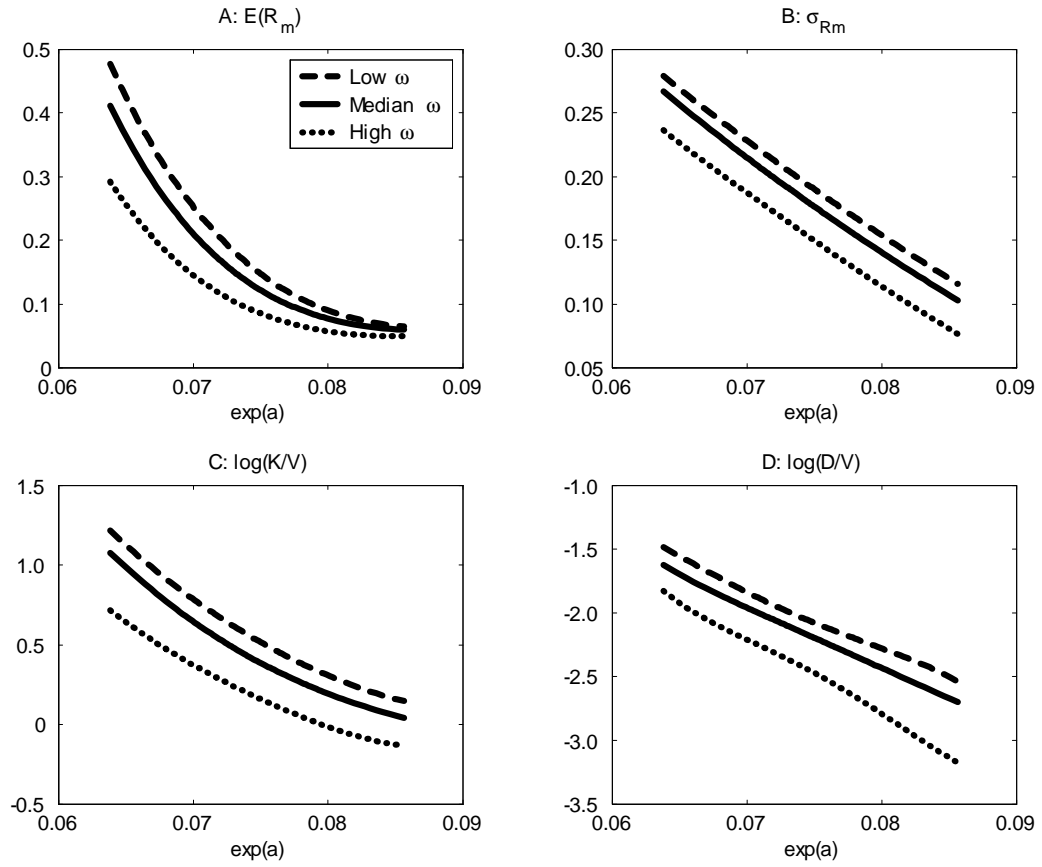


Figure 5. Market Returns and Aggregate Ratios. The figure plots conditional moments of market returns and aggregate ratios as a function of the aggregate productivity,  $\exp(a)$ , and three values of  $\omega$  corresponding to the 10th (Low), 50th (Median) and 90th (High) percentiles of its unconditional distribution, respectively. Panel A: expected market returns. Panel B: conditional volatility of market returns. Panel C: log book-to-market ratio. Panel D: log dividend yield. The parameters are reported in Table 1.

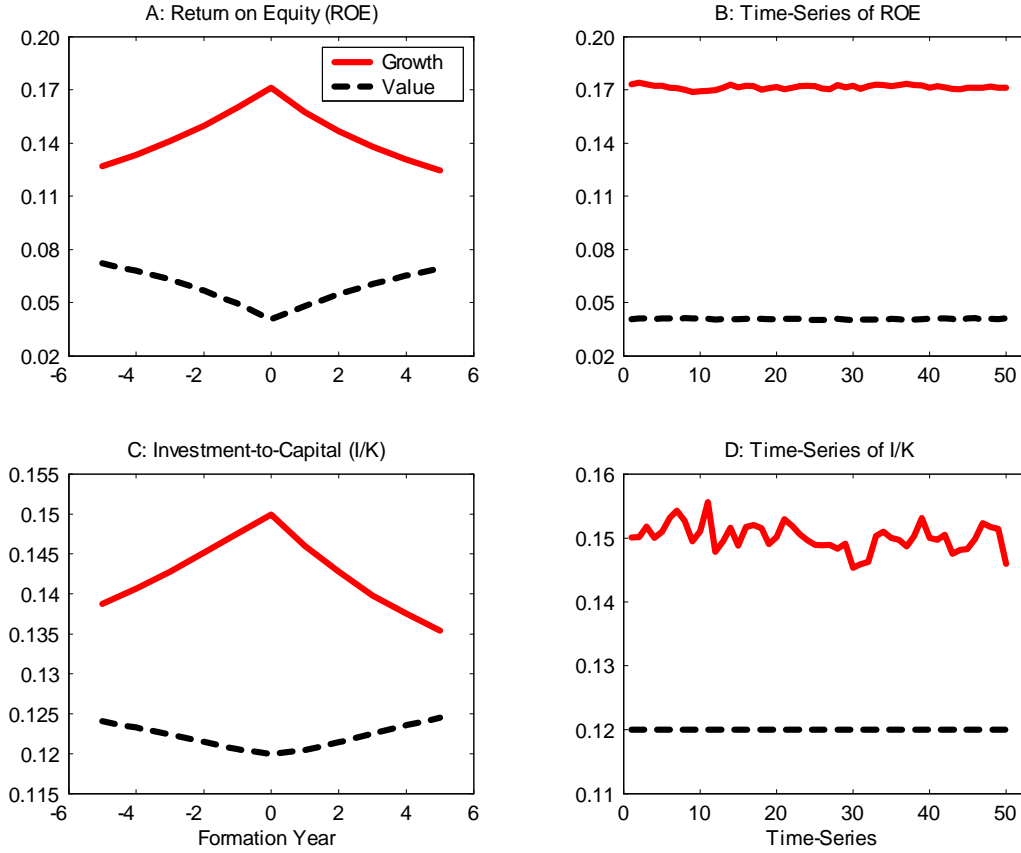


Figure 6. Value versus Growth in Simulated Data. The figure illustrates the relation between profitability and investment-to-capital ratio for growth and value portfolios in simulated data. Growth (value) indicates the portfolio containing firms in the bottom (top) 20 percent of the values of book-to-market ratios. I measure profitability by return on equity (ROE) as  $[\Delta K_t + D_t] / K_{t-1}$ , where  $K_{t-1}$  denotes the book value of equity and  $D_t$  is the dividend payout. The profitability of a portfolio is defined as the sum of  $[\Delta K_{it} + D_{it}]$  for all firms  $i$  in the portfolio divided by the sum of  $K_{it-1}$ . The investment-to-capital ratio of a portfolio is defined as the sum of  $I_{it}$  for all firms  $i$  in the portfolio divided by the sum of  $K_{it-1}$ . For each portfolio formation year  $t$ , the ratios of  $[\Delta K_{t+k} + D_{t+k}] / K_{t+k-1}$  and  $I_{t+k} / K_{t+k-1}$  are calculated for year  $t+k$ , where  $k = -5, \dots, 5$ . The ratio for year  $t+k$  is then averaged across portfolio formation years. Panel A and C show the 11 - year evolution of profitability and investment-to-capital ratio for growth and value portfolios, respectively. Panel B and D show the time-series of profitability and investment-to-capital ratio for growth and value portfolios, respectively. The figure is based on 100 artificial panels each with 200 firms and 50 years. I calculate profitability and investment-to-capital ratio for value and growth portfolios for each sample, and then report cross-sample averages.

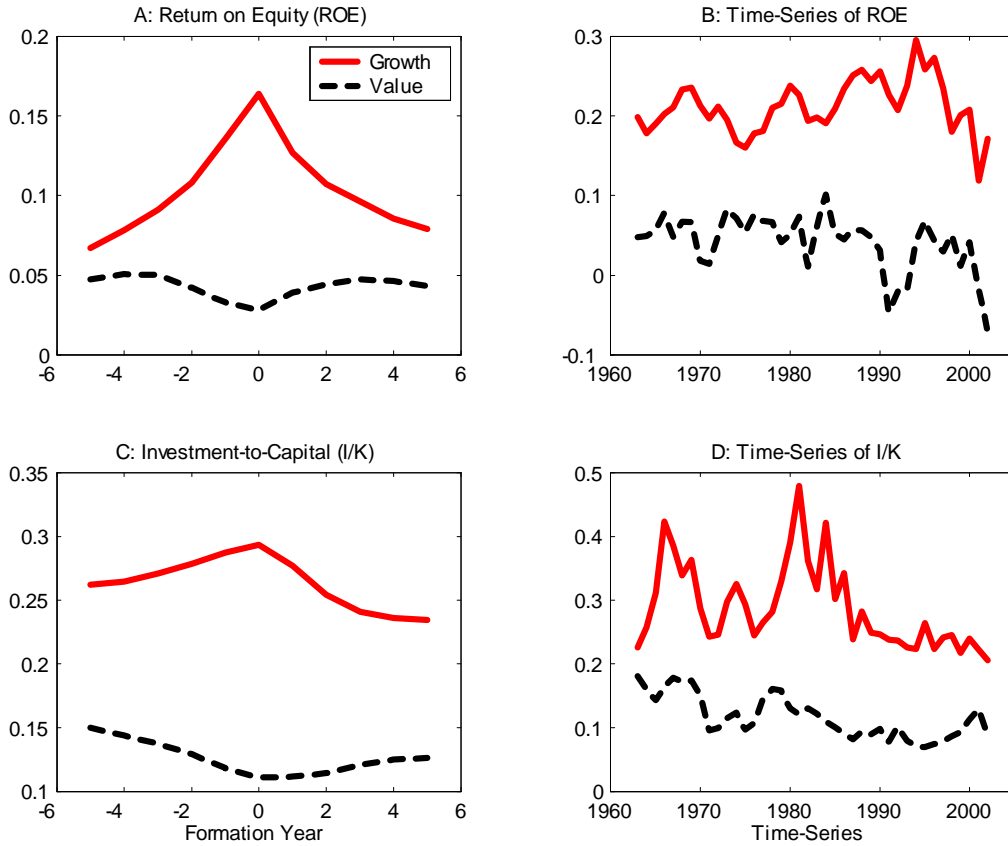


Figure 7. Value versus Growth in Historical Data. The figure illustrates the relation between profitability and investment-to-capital ratio for growth and value portfolios in historical data. Growth (value) indicates the portfolio containing firms in the bottom (top) 20 percent of the values of book-to-market ratios. I measure profitability by return on equity (ROE) as the ratio of common equity income for the fiscal year ending in calendar year  $t$  and the book value of equity for year  $t - 1$ . The profitability of a portfolio is defined as the sum of common equity income for all firms in the portfolio divided by the sum of book value of equity. The investment-to-capital ratio of a portfolio is defined as the sum of capital expenditures for the fiscal year ending in calendar year  $t$  for all firms in the portfolio divided by the sum of net property, plant and equipment for year  $t - 1$ . For each portfolio formation year  $t$ , the  $ROE_{t+k}$  and  $I_{t+k}/K_{t+k-1}$  are calculated for year  $t + k$ , where  $k = -5, \dots, 5$ . The ratio for year  $t + k$  is then averaged across portfolio formation years. Panel A and C show the 11 - year evolution of profitability and investment-to-capital ratio for growth and value portfolios, respectively. Panel B and D show their time-series dynamics. The figure is based on historical data from the CRSP-COMPUSTAT merged database for the period 1962 - 2002. More details are provided in Appendix C.

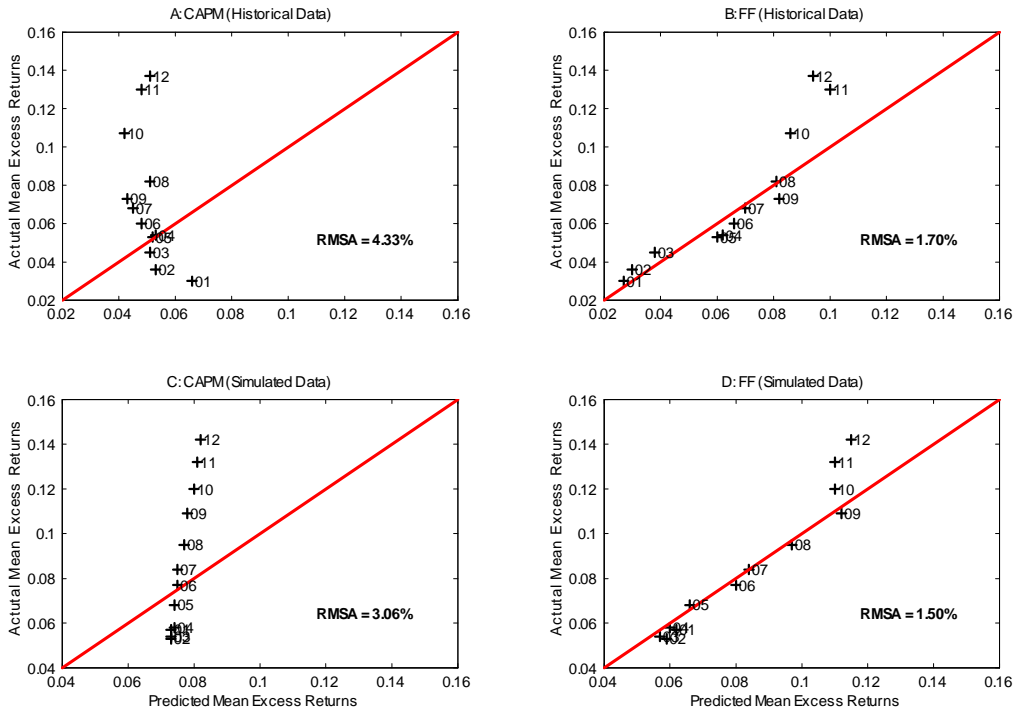


Figure 8. Predicted versus Actual Excess Returns. The figure shows model predicted versus actual annual mean excess returns on book-to-market sorted portfolios in historical and simulated data. Panel A and C: CAPM. Panel B and D: Fama and French (1993) three factor model. RMSA is the root mean squared alpha. Historical data are from CRSP. Stock returns are calculated from the beginning of July to the end of June of the following year for the period 1962 - 2002. More details are provided in Appendix C. Simulated data are based on 100 artificial panels each with 200 firms and 50 years. I calculate portfolios returns for each sample and then report cross-sample averages.

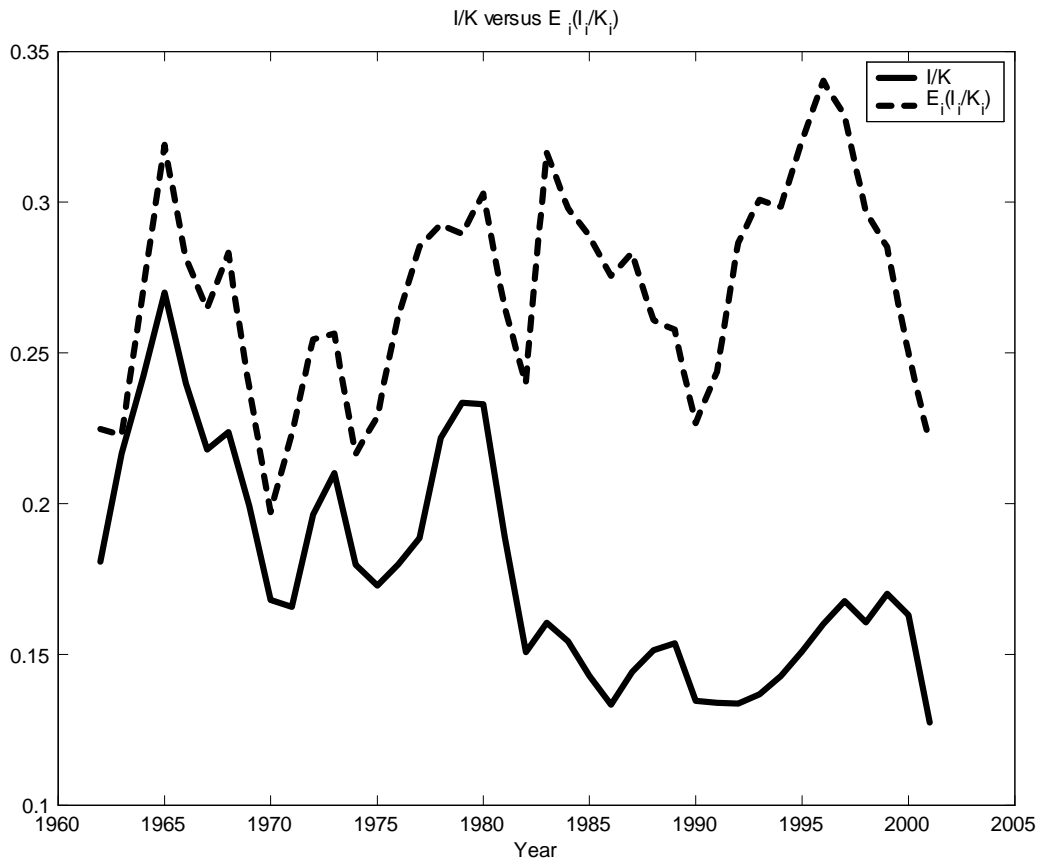


Figure D.1. Inverse Growth-Size Relation. The figure plots aggregate investment rate,  $i_t$ , versus the cross-sectional mean of firm investment rates,  $E^i [i_{it}]$ , during the period 1962 - 2002. Aggregate investment rate is the capital-weighted average of firm investment rate. The cross-sectional mean of firm investment rates is the equally-weighted average of firm investment rates. Data are from the CRSP-COMPUSTAT merged database. More details are provided in Appendix C.