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A Measurement Error Approach for Modeling Consumer Risk Preference

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Abstract

Von Neumann-Morgenstern (vN-M) utility theory is the dominant theoretical model of risk preference. Recently, market researchers have adapted vN-M theory to model consumer risk preference. But, most applications assess utility functions by asking just *n* questions to specify *n* parameters. However, any questioning format, especially under market research conditions, introduces measurement error. This paper explores the implications of measurement error on the estimation of the unknown parameters in vN-M utility functions and provides procedures to deal with measurement error.

We assume that the functional form of the utility function, but not its parameters, can be determined a priori through qualitative questioning. We then model measurement error as if question format and other influences cause the consumer to choose the unknown "risk parameter" from a probability distribution and to make his decisions accordingly. We provide procedures to estimate the unknown parameters when the measurement error is either (a) Normal or (b) Exponential.

Uncertainty in risk parameters induces uncertainty in utility and expected utility, and hence uncertainty in choice outcomes. Thus, we derive the induced probability distributions of the consumer's utility and the estimators for the implied probability that an alternative is chosen.

Results are obtained for both the standard decision analysis "preference indifference" question format and for a "revealed preference" format in which the consumer is asked simply to choose between two risky alternatives.

Since uniattribute functions illustrate the essential risk preference properties of vN-M functions, we emphasize uniattribute results. We also provide multiattribute estimation procedures. Numerical examples illustrate the analytical results.

Keywords
marketing, utility theory, risk modeling

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A Measurement Error Approach for Modeling
Consumer Risk Preference
by

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ABSTRACT

Von Neumann-Morgenstern (vN-M) utility theory is the dominant theoretical model of risk preference. Recently, market researchers have adapted vN-M theory to model consumer risk preference. But, most applications assess utility functions by asking just n questions to specify n parameters. However, any questioning format, especially under market research conditions, introduces measurement error. This paper explores the implications of measurement error on the estimation of the unknown parameters in vN-M utility functions and provides procedures to deal with measurement error.

We examine two measurement error distributional assumptions, Normal and Exponential. For each error assumption we provide (1) maximum likelihood estimators for the risk parameters (or their distributions), (2) induced probability distributions of the utility functions, and (3) estimates of the implied probability that an alternative is chosen. (Uncertainty in risk parameters induces uncertainty in utility and expected utility, and hence uncertainty in choice outcomes.)

We provide results for the standard preference indifference questions in which the consumer provides either a "certainty equivalent" or a probability such that he is indifferent among two alternative lotteries. We also provide results for revealed preference questions in which the consumer simply chooses between two alternatives. For prediction, outcomes can be specified by lotteries or by continuous distributions.

Since uniattribute functions illustrate the essential risk preference properties of vN-M functions, we emphasize uniattributed results. We also provide multiattribute procedures and an example.

Numerical examples illustrate the results.

1. PERSPECTIVE

The measurement and modeling of how consumers form preferences among risky alternatives is becoming an important problem in marketing science as researchers begin to focus on purchases of durable goods such as automobiles, home heating systems, home computers, and major appliances. An integral part of such consumer decisions is the choice of a specific product, say a gas furnace, when the attributes of the product, say annual cost and reliability, are not known with certainty.

A number of procedures have been proposed to model consumer risk. For example, Pras and Summers (1978) include the standard deviation of an attribute as a risk measure. Among these procedures is explicit risk assessment with von Neumann-Morgenstern (vN-M) utility functions. VN-M utility functions have the advantages that they are:

(1) Theoretically derived from an axiomatic base (von Neumann and Morgenstern 1947, Friedman and Savage 1952, Herstein and Milnor 1953, Jensen 1967, Marshak 1950, and others),

(2) provide a set of practical functional forms derived from testable behavioral assumptions (see review in Keeney and Raiffa 1976), and

(3) have been applied extensively to model managers' decisions (see extensive reviews in Farquhar 1977 and Keeney and Raiffa 1976).

However, until recently, vN-M utility functions have not achieved widespread use in marketing. This reluctance by marketing academics and practitioners stems in part because the question formats can be difficult and because the consumer modeling has not acknowledged measurement error as have more widely accepted techniques such as conjoint analysis (Green and Srinivasan 1978) and logit analysis (McFadden 1980). For example, both Hauser and Urban (1979) and

Eliashberg (1980) have successfully modeled consumer preferences and have forecast reasonably well with vN-M theory, but both studies use the decision analysis procedure which requires complex questions to first test behavioral assumptions and then obtain exactly n observations to fit n parameters.

The consumer preference modeling task is different from the decision analysis task. Market research interviews are usually severely limited in time, hence, tradeoffs must be made among interviewee training, assumption testing, complexity of questions, and the number of questions. Marketing researchers/scientists often prefer to ask more simpler questions to statistically infer properties and estimate parameters. Such procedures must acknowledge potential measurement error.

More recently, marketing scientists have recognized these issues and have begun to adapt vN-M theory to marketing problems. For example, Ingene (1981) uses a Taylor series expansion to obtain simpler functional forms which are estimatable with linear regression; Currim and Sarin (1982) adapt conjoint analysis to vN-M utility functions. Both approaches have practical merit and indicate the renewed interest in vN-M utilty modeling.

In this paper, we take a different approach to the marketing problem. We explicitly acknowledge measurement error, but retain the axiomatic base and powerful, practical functional forms of vN-M theory. In the face of measurement error, we develop procedures to estimate unknown parameters for vN-M utility functions and we examine the implications of such measurement error on the utility functions and choice outcomes.

In approaching this problem, we make a number of tradeoffs with respect to (1) how we model measurement error, (2) what probability distributions we assume for measurement error, (3) which functional forms we investigate, (4) what type of attribute uncertainty we analyze, and (5) which question formats

we consider. No doubt other researchers will have different tastes and make other tradeoffs. Our goal is not to exhaust the possibilities, but to analyze in depth the issues we feel are most relevant. We are hopeful that other researchers will choose to extend our results, address related issues (with different tradeoffs), and undertake empirical investigations that will provide evidence for or against our error distribution assumptions. We feel that this paper provides an important first step.

2. REVIEW OF VN-M CONCEPTS

This section briefly reviews some aspects of vN-M utility theory that are necessary for our analyses. For greater detail see Keeney and Raiffa (1976).

The primary advantage of vN-M utility theory is its ability to model risk preferences. Basically, products are represented by their attributes and uncertainty (risk) is modeled as a probability distribution over the attributes. The vN-M function assigns a scalar value to every possible outcome of the uncertain attributes such that the consumer will prefer the product which has the maximum expected utility. The axioms imply that such a utility function exists and is unique (subject to a scaling change). The market research task is to obtain an estimate of this function such that expected utility is a reasonable predictor of the consumer's behavior.[1]

In general, a vN-M utility function can be an arbitrary function, but research in the last 20 years has identified a set of parametered functions based on reasonable behavioral assumptions. These functions are valuable for market research because they allow us to parameterize, and hence simplify, the

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[1][In marketing research, measurement error exists. Thus, we rarely can predict with certainty and instead forecast choice probabilities. Predictions of choice probabilities require modification of the vN-M axiom system. For one set of axioms see Hauser (1978).]

estimation problem and because they focus our attention on functional forms

that can be justified a priori with a qualitative analysis of the consumer's

risk preferences.

Uniattributed functions

Uniattributed functions are derived from assumptions about how a

consumer's risk preference changes as his "assets" increase. For example, we

might expect a consumer to be less concerned about uncertainty of $\pm\$100$ in

heating bills if his base heating bill were $3000 than he would be if his base

heating bill were $300. Pratt (1964) proposed a measure, called local risk

aversion, $R(x)$, of how a consumer's risk attitude varies with his asset level,

x. If $u(x)$ is the utility function, $R(x)$ is given by:

$$R(x) = -\frac{d^2u(x)}{dx^2} \bigg/ \frac{du(x)}{dx} \qquad (1)$$

If $R(x)$ is positive the consumer is risk averse, if $R(x)$ is negative, risk

prone, and if $R(x)$ is zero, risk neutral. Larger absolute values of $R(x)$

imply greater risk aversion (proneness).

A related concept is proportional risk aversion, $S(x)$, which measures a

consumer's risk preference when consequences are measured in proportion to

assets. For example, if the uncertainty in heating bills was $\pm 10\%$ of the

base bill. If x_o is the minimum (reference) value of x, then $S(x)$ is given

by:

$$S(x) = (x - x_o) R(x). \qquad (2)$$

For clarity we refer to $R(x)$ as "absolute" risk aversion measure to distinguish it from the proportional risk aversion measure.

The most common uniattributed functional forms are based on constant $R(x)$ or $S(x)$. As Table 1 indicates, constant $R(x)$ implies an exponential function and constant $S(x)$ implies a power function. The third functional form, linear utility, is a special case when $R(x) = S(x) = 0$. This is the risk neutral form which applies when risk does not affect the consumer's decision.

(In Table 1 we have restricted $r \geq 0$ in both functional forms. This is for simplicity of exposition. Our analyses can be modified for $r < 0$, but to do so would unnecessarily complicate the exposition. We leave these straightforward extensions to the reader.)

TABLE 1

COMMON UNIATTRIBUTED vN-M UTILITY FUNCTIONS

Behavioral Assumption	Functional Form		Range of Attribute
1. Constant absolute risk averse ($R(x)=r$)	$1 - e^{-r(x - x_0)}$	$r > 0$	$x_0 \leq x \leq \infty$
	$\dfrac{(1 - e^{-r(x - x_0)})}{(1 - e^{-r(x_* - x_0)})}$,	$r > 0$	$x_0 \leq x \leq x_*$
2. Constant proportional risk averse or prone ($S(x) = 1 - r$)	$\dfrac{(x - x_0)^r}{(x_* - x_0)^r}$,	$r \geq 0$	$x_0 \leq x \leq x_*$
3. Risk neutral (special case of (1) when $r \to 0$ and (2) when $r \to 1$.)	$\dfrac{(x - x_0)}{(x_* - x_0)}$		$x_0 \leq x \leq x_*$

Note: Functional forms also exist for $r < 0$. For ease of exposition we restrict our analyses to $r > 0$. For constant proportional risk attitude, the utility function is risk averse for $0 < r < 1$ and risk prone for $r > 1$.

Other uniattributed functional forms are possible, for example, a logarithmic form or a sum of exponential forms, but the three functions in Table 1 are the functional forms that have dominated applications in decision analysis and marketing science. Furthermore, in reviewing 30 applications, Fishburn and Kochenberger (1979) found the constant R(x) and constant S(x) functional forms fit the data quite well and substantially better than the linear form.

Multiattributed functions

Multiattributed functions are derived from assumptions about utility and preference independence (or dependence) among attributes. Empirical experience in decision analysis and marketing science has found them to be feasible and useful. Rather than review the most common functional forms here, we return to the multiattributed issue in Section 5 where we provide an example based on the commonly used multilinear form.

Empirical experience

Neither decision analysts nor marketing scientists have explicity approached vN-M utility measures as error-laden measures. Meyer and Pratt (1968) provide a procedure for "fairing" deterministically a smooth function through a set of points, Fishburn and Kochenberger (1979) use a minimum mean squared error procedure, and Currim and Sarin (1982) use a conjoint-like procedure, but none of these authors explicitly model measurement error statistically or examine its implications. We know of no systematic empirical study quantifying measurement error at the individual level.

It is useful, however, to examine how risk parameters have varied across applications. Table 2 summarizes the Fishburn and Kochenberger data for "above target" risk averse individuals.[2] Although the intervals are coarse and unequal, Table 2 might suggest a normal distribution across applications for the parameter in the constant proportional risk aversion utility function and an exponential distribution for the constant absolute risk averse utility function. Results vary by whether utility is assessed for gains or losses and whether or not the individual is risk averse or seeking, but in 7 of 8 cases the distribution "looks" either normal or exponential.

TABLE 2

VARIATION IN RISK PARAMETERS ACROSS APPLICATIONS

(Risk averse individuals from review by Fishburn and Kochenberger, 1979)

Constant Proportional Risk Aversion		Constant Absolute Risk Aversion	
r	no. of applications	r	no. of applications
[0, 1/2]	3	[0, 1/4)	10
[1/2, 4/5]	10	[1/4, 1]	7
[4/5, 1]	4	[1, ∞]	0

[2] [Fishburn and Kochenberger assess functions for utility "above target" and "below target." They find interesting variation. Our Table 2 is but part of their extensive study. Our purpose is to illustrate the type of results obtained.]

3. SINGLE PARAMETER UNIATTRIBUTED UTILITY FUNCTIONS

VN-M utility theory applies to single attributes and to multiple attributes. We begin our analyses with single-parameter uniattributed utility functions because the essential risk modeling capability of vN-M theory is embodied in uniattributed functions and because the most commonly used functional forms are single-parameter functional forms. Section 4 examines multiple-parameter uniattributed forms and Section 5 examines multiattributed utility functions.

Following Fishburn and Kochenberger (1979) we assume that separate parameters are estimated "above target" and "below target", thus we can assume that the utility function is either concave throughout the region, $x_o \leq x \leq x_*$, or convex throughout the region. Without loss of generality we assume that the attribute of interest, x, has been scaled such that preference is monotonically increasing in x over the region of estimation. For example, if there were a finite ideal point, say length of an automobile, we either (1) assess separately for the range above and the range below the ideal point or (2) assess with respect to a rescaled attribute such as distance from the ideal point.

Our results are derived at the level of the individual consumer, that is, we assume that any variation in the unknown parameter, r, represents uncertainty in measuring that parameter and/or uncertainty across time and situations. We note, however, that our results can be interpreted for variation across consumers with proper modification in definitions. Before we present the conceptual framework that we have developed, we provide an example that illustrates the nature of the problem of interest, and its essential characteristics.

An Illustrative Example

Suppose that a consumer is considering replacing his antiquated home heating system with a new oil, gas, electric, or solar system. He is uncertain about unit fuel cost, about heating efficiency, and about weather, thus, the annual savings, x, of the new system over the present system is an uncertain outcome. Suppose that he has some prior beliefs about the savings due to each system and that these prior beliefs can be characterized by a probability distribution over the range of $200 \leq x \leq $1200. We want to estimate his utility for values of x and to predict his future choices.

Using a standard decision analysis lottery questioning format, we ask the lottery question in Figure 1. The consumer is given a choice between two heating systems. Heating system A, a solar system, has a known savings of x_i dollars. The savings of heating system B, an oil system, are less certain and depend upon the price of oil. If conditions are favorable, the savings are $1200, and if they are unfavorable, the savings are only $200. The consumer is asked to specify the likelihood (probability), p_i, of favorable conditions such that he would be indifferent between system A and system B.

Solar Heating	Oil Heating

x_i dollars saved $\sim$

p_i — $1200 saved
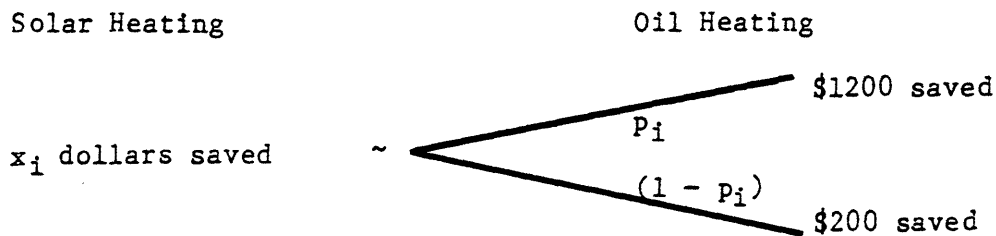
$(1 - p_i)$ — $200 saved

Figure 1: Schematic of Lottery Measurement

From discussions with the consumer we believe a constant proportional risk averse utility function is appropriate. For our problem, the function is:

$$u(x_i, r) = (x_i - 200)^r / (1000)^r \tag{3}$$

If the vN-M axioms hold, then Figure 1 implies:

$$u(x_i, r) = p_i u(1200, r) + (1 - p_i) u(200, r) \qquad (4)$$

Substituting equation (3) in equation (4) yields:

$$(x_i - 200)^r / 1000^r = p_i (1) + (1-p_i)(0) = p_i \qquad (5)$$

Finally, if there were no errors and we know x_i and p_i, we could obtain r by solving the algebraic relationship in equation (5).

The practice in marketing research is to ask multiple questions as illustrated in Table 3 and to utilize all the information obtained. That is, we could vary x_i and have the consumer specify a p_i for each x_i. We would then solve equation (5) for each x_i. However, as Table 3 indicates, we are likely to get a different value of r for each question since it is quite unlikely that the consumer will be perfectly consistent in responding to the various questions.

TABLE 3
EXAMPLE ASSESSMENT FOR THE ANNUAL SAVINGS
OF A HOME HEATING SYSTEM

i Measurement	x_i (dollars)	p_i	$r(x_i, p_i)$ (constant proportional risk averse)
1	300	.30	.52
2	400	.45	.50
3	500	.55	.50
4	600	.65	.47
5	700	.70	.51
6	800	.75	.56
7	900	.85	.46
8	1000	.90	.47
9	1100	.95	.49

Conceptualization of Measurement

We can conceptualize this measurement as shown in Figure 2. We, the experimenter, choose a set of questions. The type of question chosen as well as other factors could well induce errors in the measurement process. For example, Hershey, Kunreuther and Schoemaker (1982) found that the domain of outcomes (e.g., pure loss versus mixed lottery) and the decision context (e.g., abstract versus concrete formulation) may be influential in the observation of the consumer risk attitude. In our framework, for a given utility function, there is some true risk parameter, r^T, but our question format induces error. We describe this error by a distribution, $f(r|\lambda)$, of the risk parameter.

We then model the consumer's response <u>as if</u> he chooses a utility function, $u(x, r)$, draws a risk parameter, r_i, from $f(r|\lambda)$ independently for each question,[3] and provides an answer to the question, say p_i, such that p_i is consistent with $u(x, r_i)$. When we obtain I observations, it is our task to estimate $f(r|\lambda)$. If errors are unbiased (zero mean) or if the bias is known, we can then obtain an estimator of r^T.

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[3] [Or for each product which he evaluates in answering the question.]

Parameterized utility function $u(x, r^T)$ describes consumer. We know form but not the parameter, r^T.

Experimenter chooses question format which induces error. Error can also be induced by other sources.

"truth" "error"

Consumer is now modeled by utility form, $u(x, r)$, and distribution, $f(r|\lambda)$ of risk parameter.

For each question*, consumer chooses r_i, from $f(r|\lambda)$.

Consumer evaluates alternatives in question with $u(x, r_i)$.

Consumer provides answer to question.

When I observations are obtained, experimenter estimates $\hat{\lambda}$ and hence $f(r|\hat{\lambda})$.

*Or each alternative
in the question.

Figure 2: Conceptualizaition of Error Modeling

The assumption of error being induced by question format or by other sources such as temporal variation, approximation, etc., and its modeling through random draws of the risk parameter is similar to "random utility" error theories such as Thurstone (1927) or Luce and Suppes (1965), but modified to emphasize the strength of vN-M theory -- risk preference.

We note that our model of the consumer's response (dotted box in Figure 2) is a paramorphic model, that is, we assume that the consumer responds as if he follows the postulated procedure. Such details of cognitive response are inherently unobservable (without introducing new observation errors), but serve to provide a modeling framework with which to represent measurement error.

For the question format in Figure 1, the experimenter specifies x_i, the consumer draws r_i from $f(r|\lambda)$, and provides us with p_i. Figure 2 implies that p_i then must satisfy the equation

$$p_i = u(x_i, r_i) \tag{6}$$

We can then solve this equation for r_i. That is, for a constant proportional risk averse utility function,

$$r_i = r(x_i, p_i) = \log(p_i)/\log[(x_i - 200)/1000] \tag{7}$$

This is the value given in the third column of table 3. It will be useful in estimating $f(r|\lambda)$.

To analyze the implications of Figure 2 we investigate a number of issues. (1) We obtain methods to estimate $\hat{\lambda}$, and hence $f(r|\hat{\lambda})$, from data obtained from indifference questions such as that shown in Figure 1. (We allow λ to be vector valued.) (2) We obtain methods to estimate $\hat{\lambda}$ from revealed preference questions where the consumer is given two alternatives and asked to choose his most preferred. (3) Since uncertainty in r induces uncertainty in $u(x, r)$, we derive the distribution of utility from the estimated distribution of r. (4) Since uncertainty in utility induces uncertainty in expected utility and hence uncertainty in choice outcomes, we derive expressions for the probability a given alternative is chosen by the consumer. We investigate these issues for alternatives represented by discrete (Bernoulli) distributions of the attribute, x, and for alternatives represented by continuous distributions (e.g., Normal) of the attribute x.

We begin with maximum likelihood estimators, $\hat{\lambda}$, for λ, when questions are asked in the format of an indifference question. We address revealed preference questions after we derive the necessary analytic tools, i.e., expressions for the distribution of utility and for choice probabilities.

Estimation for Preference Indifference Question Formats

A preference indifference question is a question such as Figure 1 where the experimenter provides x_*, x_o, and either x_i or p_i. The consumer answers with a value of p_i (or x_i) such that he is indifferent between the two alternatives.

The experimenter's task is to estimate $\hat{\lambda}$ from I indifference questions. Before we can proceed further, we must make an assumption about the family of distributions, $f(r|\lambda)$. In this paper, we investigate two error distributions: (1) Normal distributions and (2) Exponential distributions.

Normal error theory has the advantage that it is the natural assumption usually made in statistical theory. Its drawback is that r_i can take on any value in the range $(-\infty, \infty)$. However, if the mean is significantly larger than the standard deviation, then negative values of r_i will be extremely rare.

Exponential error is not subject to this problem since we can restrict $r \geq r_o$, i.e., $f(r|\lambda) = (\lambda - r_o)^{-1} \exp[-(r - r_o)/(\lambda - r_o)]$ for $r \geq r_o$. However, exponential error theory does imply an assymetric distribution with its peak at $r = r_o$ and zero probability for $r < r_o$.

Normal error and Exponential error are clearly quite different theories. Each has its advantages and its disadvantages and, a priori, each reader will have his own favorite theory. We investigate both assumptions in this paper in the belief that these two assumptions are each flexible and together span a broad range of potential shapes for $f(r|\lambda)$.

As it turns out, it is quite simple to obtain the maximum likelihood estimator (MLE) for λ, once an error assumption is made about the shape of $f(r|\lambda)$. (MLE's are important for applied statistics because they are consistent, efficient, and a function of minimal statistics.)

Suppose we ask I questions of the format of Figure 1. That is, for a vector of "certain outcomes," $\underline{x} = (x_1, x_2, \ldots, x_I)$, we obtain a vector of corresponding "answers," $\underline{p} = (p_1, p_2, \ldots, p_I)$. Because successive questions are independent, the joint probability, $F(\underline{p}|\underline{x}, \lambda)$, of observing $\underline{p}$ given $\underline{x}$ and λ is:

$$F(\underline{p}|\underline{x}, \lambda) = \prod_i f(r(x_i, p_i)|\lambda) \, |\partial r(x_i, p_i)/\partial p_i| \tag{8}$$

where $|\partial r(x_i, p_i)/\partial p_i|$ is the absolute value of the Jacobian transformation (Mood and Greybill 1963) and $r(x_i, p_i)$ is defined as the solution to equation (6).

To obtain the MLE's for λ, $\hat{\lambda}$, we maximize $F(\underline{p}|\underline{x},\lambda)$ with respect to λ. Since the Jacobian is independent of λ and maximizing log $F(\underline{p}|\underline{x},\lambda)$ is equivalent to maximizing $F(\underline{p},|\underline{x},\lambda)$ we can also obtain $\hat{\lambda}$ by maximizing the following log likelihood function:

$$L(\lambda|\underline{x}, \underline{p}) = \Sigma_i \log f(r_i|\lambda) \tag{9}$$

where

$$r_i = r(x_i, p_i)$$

In other words, we simply treat the r_i as data points. This has a number of very practical advantages.

Estimators

If we treat the r_i as data, the MLE's are well known for both Normal error and Exponential error. If μ and σ^2 are the mean and variance of the Normal distribution, then[4]:

$$\hat{\mu} = (1/I)\Sigma_i r_i \tag{10}$$

$$\hat{\sigma}^2 = (1/I)\Sigma_i (r_i - \hat{M})^2 \tag{11}$$

For exponential errors:

$$\hat{\lambda} = (1/I)\Sigma_i r_i \tag{12}$$

Furthermore, $\hat{\mu}$ and $\hat{\lambda}$ can be interpreted as the expected ("true") values of r for Normal and Exponential error, respectively, if we assume induced error is zero-mean.

MLE's are invariant under transformation, that is, if $\hat{\Theta}$ is an MLE for Θ, then $g(\hat{\Theta})$ is the MLE for $g(\Theta)$ (Giri 1977, Lemma 5.1.3). Thus, if we interpret $\hat{\mu}$ or $\hat{\lambda}$ as the "true" values of r, then $u(x_i, \hat{\mu})$ or $u(x_i, \hat{\lambda})$ are estimators of the "true" values of $u(x_i)$ for Normal and Exponential error, respectively.

Note that equations (10), (11), and (12) apply for both the constant absolute and constant proportional risk averse forms in Table 1. For that matter, they apply for any uniattributed utility function for which a unique r_i can be computed for each consumer question. For the generalized version of the binary preference comparison question shown in Figure 1 ($x_o \leq x_i \leq \infty$ for absolute risk aversion and $x_o \leq x_i \leq x_*$ for proportional risk aversion), the inverse functions are given by:

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[4][Equation (11) is the MLE for σ^2, but it is biased for finite I. The more commonly used estimator is $(I/I-1))\hat{\sigma}^2$. Also, if we want to estimate r_o, its MLE is $\hat{r}_o = \min_i \{r_i\}$.]

$$\text{Constant Absolute risk: } r(x_i, p_i) = -\frac{\log (1 - p_i)}{(x_i - x_o)} \tag{13}$$

$$\text{Constant Proportional risk: } r(x_i, p_i) = \frac{\log (p_i)}{\log [(x_i - x_o)/(x_* - x_o)]} \tag{14}$$

When $x_o \leq x \leq x_*$ for absolute risk aversion, the inverse function

can be obtained numerically and, for a few special cases, analytically.

Question Format

We derived equation (9) for the case when the x_i's were specified by

the experimenter such that the consumer's answers were the p_i's. But, by

symmetry, it is clear that equation (9) applies if the probabilities, p_i's

are specified and the consumer supplies the certain outcomes, x_i's. In

fact, a modified equation (9) will apply for any question format for which

one can obtain an observation of r_i. See Farquahar (1982) for a review of

alternative question formats. However, equation (9) does not imply that the

experimenter's choice of question format is free from systematic bias.

Different formats can induce different magnitudes of error, e.g., different

σ^2 for Normal errors, or, for that matter, different types of error,

e.g., different $f(r|\lambda)$. But equation (9) does state that once the error

assumption is made, equations (10) and (11) or (12) apply independent of the

question format.

Statistical Inference

One can test a hypothesis about the "true" value of r. For example, if

normal error theory applies and the researcher wishes to test whether the

"true" value of r is significantly different from some hypothesized value,

r_H, he can use a t-test with (I - 1) degrees of freedom based on the

statistic, $(r_H - \hat{\mu})(I - 1)^{1/2}/\hat{\sigma}$. Similar results apply for exponential

error theory, except that the sampling distribution for $\hat{\lambda}$ has a gamma

density with mean, λ_H and variance λ_H^2/n.

An Illustration

Consider the problem in Table 3. Using equations (10), (11), and (12) we

estimate $\hat{\mu} = .50$ and $\hat{\sigma} = .03$ for Normal errors and $\hat{\lambda} = .50$ for Exponential

error. A chi-square goodness-of-fit test suggests that the data are more

likely to be generated from a Normal distribution. A utility function based

on $r^T = \hat{\mu} = .5$ is shown in Figure 3. For normal error theory, a 95%

confidence interval for $\hat{\mu}$ is [.48, .52] and for σ it is [.02, .06].

Distribution of Utility

If the risk parameter, r, were known with certainty, we could compute u(x, r)

for any x and compute directly the expected utility of a product. However, even

with an MLE for the "true" parameter, λ, our knowledge about r is still

represented by a random variable with distribution, $f(r|\lambda)$. This uncertainty

in r induces uncertainty in u(x, r) for any x. Hence, the expected utility and,

ultimately, the choice outcome are random variables. We begin by computing the

probability density function of u(x, r). We then examine its implications. For

simplicity of analytic exposition we restrict our results to the infinite range

$(0 \leq x \leq \infty)$ constant absolute risk averse utility function and (for
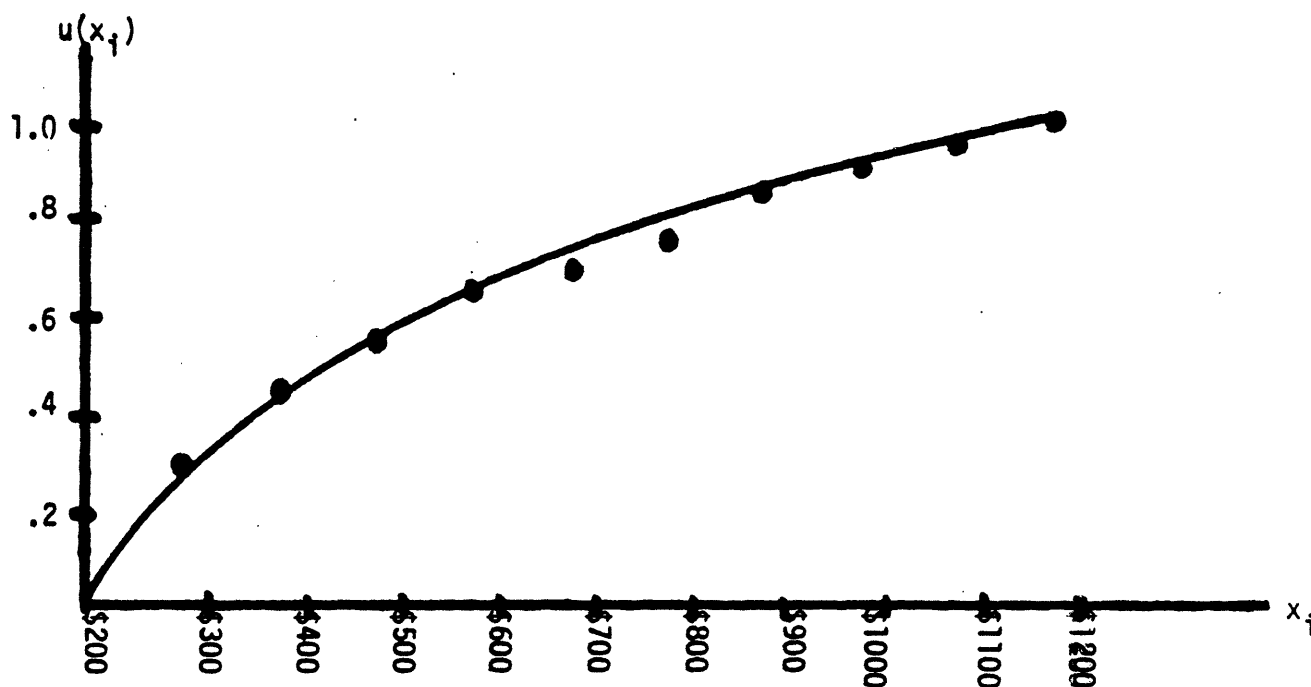
exponential errors) to $r \geq 0$.

Figure 3: Maximum-likelihood Estimate of Assessed Utility Function

Proposition 1: If measurement error is modeled as NORMAL, then the utility functions have lognormal distributions. In particular:

$$u(x, r) \sim \Lambda(-k\hat{\mu}, k\hat{\sigma}^2) \text{ for constant proportional risk aversion}$$

$$1 - u(x, r) \sim \Lambda(-x\hat{\mu}, x^2\hat{\sigma}^2) \text{ for constant absolute risk aversion}$$

$$\text{where } k = \log \left[(x_* - x_0)/(x - x_0) \right]$$

$$\Lambda(a, b) = \text{a lognormal distribution with parameters } a, b.$$

Proof. By definition, if z is a normal random variable with mean, μ, amd variance, σ^2, and if $z = \log y$, then y is a lognormal random variable with parameters μ and σ^2, designated $y \sim \Lambda(\mu, \sigma^2)$. See Aitchison and Brown (1969). For constant proportional risk aversion,

$r(x, u) = -k^{-1}\log u$ or $\log u = -kr(x, u)$. If $r(x, u) \sim N(\hat{\mu}, \hat{\sigma}^2)$,

-19-

then $-kr(x, u) \sim N(-k\hat{\mu}, k^2\hat{\sigma}^2)$ which is our first result. For

constant absolute risk aversion $\log (1 - u) = -xr(x, u)$ yielding the second

result.

Proposition 2: If measurement error is modeled as EXPONENTIAL, then the

utility functions have Beta distributions. In particular,

$u(x, r) \sim Beta(1/\hat{\lambda}k, 1)$ **for constant proportional risk aversion**

$u(x, r) \sim Beta(1, 1/\hat{\lambda}x)$ **for constant absolute risk aversion**

where Beta(c, d) = a Beta distribution with parameters c, d.

Proof. Restriction to $r \geq 0$ implies $r_o = 0$. $f(r|\lambda)$ induces a

distribution on u, $g(u|\lambda)$ according to the following transformation

formula: $g(u|\lambda) = f(r(x, u)|\lambda)|\partial r(x,u)/\partial u|$. See Mood and Greybill

(1963, p. 224). For constant proportional risk aversion $f(r|\hat{\lambda}) = \hat{\lambda}^{-1}$

$\exp(-r/\hat{\lambda})$, $r(x, u) = k^{-1} \log u$, and $|\partial r/\partial u| = 1/ku$. Substituting

in the transformation formula yields $g(u|\hat{\lambda}) = (\hat{\lambda}k)^{-1}u^{(1/\hat{\lambda}k)-1}$

which we recognize as a Beta distribution with parameters $(1/\hat{\lambda}k)$ and 1. A

beta distribution with parameters c, d is proportional to $u^{c-1} (1-u)^{d-1}$.

For constant absolute risk aversion $r(x, u) = -(1/x) \log(1 - u)$ and

$|\partial r/\partial u| = 1/[x(1 - u)]$. Substituting the transformation formula

yields $g(u|\hat{\lambda}) = (\hat{\lambda}x)^{-1}(1 - u)^{(1/\hat{\lambda}x)-1}$ which we recognize again as

a Beta distribution with parameters 1 and $(1/\hat{\lambda}x)$.

Propositions 1 and 2 are useful for practical applications involving

risky and riskless alternatives. For both error theories, the induced

distributions on $u(x, r)$ are recognizable distributions with known

properties similar to those that arise in quantal choice problems. This

will become key as we proceed to forecasts of choice probabilities. Because

lognormal and Beta distributions compound nicely with conjugate

distributions (DeGroot 1970) it is possible to obtain analytic results for

important distributions of outcomes.

Probability of Choice

If r were known with certainty, the expected utility of each product

could be computed and we would simply forecast that the consumer would

choose the product with the maximum expected utility. In which case, our

forecasting statement would be made categorically, that is, with probability

zero or one. Instead, u(x, r) is a random variable with distributions given

by Propositions 1 and 2. Hence, the best we can forecast is the

probability, P_j $(0 \leq P_j \leq 1)$ that the consumer will choose product

j. That is,

$$P_j = Prob[\int u(x, r)h_j(x)dx \geq \int u(x, r)h_k(x)dx \text{ for } k = 1, 2, \ldots J] \quad (15)$$

where $h_j(x)$ is the probability distribution of outcomes for Alternative j.

If we were evaluating riskless alternatives, then equation 15 becomes a

quantal choice problem similar to logit or probit analysis (McFadden 1980)

except that we use lognormal or Beta distributions rather than the double

exponential and normal distributions used in logit and probit analyses,

respectively. Related quantal choice problems for riskless alternatives have

been studied. For example, Boyd and Mellmon (1980) estimated a quantal choice

model for automobiles in which the distribution of utility was a lognormal

mixture of double exponential distributions. Since the key contribution of

vN-M theory is modeling of uncertain outcomes and since the details of quantal

choice models are discussed in at least two separate books (Manski and

McFadden, 1981, and Daganzo, 1979), we do not dwell on riskless alternatives

here.

Instead, we examine in detail two important cases of equation (15). We examine binary choices among:

(1) Products whose outcomes are specified by lotteries possessing discrete (Bernoulli) probabilities, and

(2) Products whose outcomes are specified by continuous probability density functions, especially normal distributions.

These cases illustrate the essential ideas behind equation (15). We can obtain analytic results for both problems. We leave the problems of other uncertain outcomes and multiple choices for future research, although we point out that, in principle, one could use Propositions 1 and 2 with numerical techniques to compute P_j via equation (15). This would be analogous to the use of numerical techniques in state-of-the-art multiple choice probit analysis (see Daganzo 1979).

Binary Choice Between Lotteries

The first consumer choice situation that we consider is characterized as a binary choice problem with dichotomous outcomes illustrated in Figure 4. Without loss of generality assume $x_1 > x_2$ and $\beta > \alpha$. (If $x_1 > x_2$ and $\alpha > \beta$, then Alternative 1 would dominate Alternative 2.) This simple choice problem contains the essence of risky choice; the individual must decide among a potentially greater payoff, Alternative 1, and a greater likelihood of the payoff, Alternative 2.

x_1

$-\alpha$

$1 - \alpha$

x_0

x_2

β

$1 - \beta$

x_0
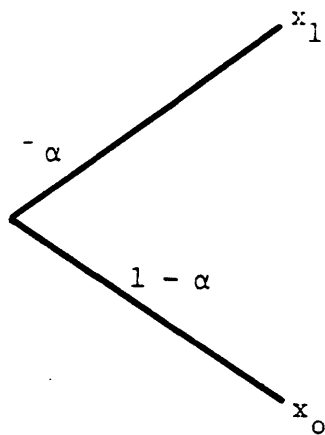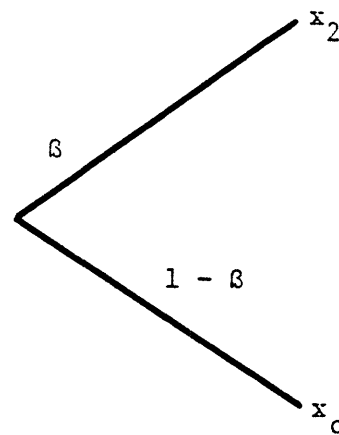
Alternative 1   Alternative 2

Figure 4: Binary Choice Problem

Our objective is to estimate the probability, $\hat{P}_1$, that the consumer will choose Alternative 1, given that we have estimated the parameters, λ, of the probability density function from which the risk parameter, r, is drawn. Before we proceed, we note that, for the binary choice problem presented in Figure 4, measurement errors may be induced once, for the question as a whole, or twice, once for each alternative. This gives rise to two viewpoints (assumptions) regarding the nature of our conceptualizations of how consumers draw r_i from $f(r|\lambda)$.

We label these assumptions as single and multiple random draws. Under the single random draw assumption, the consumer draws the corresponding risk parameter only once, and he is consistent in the sense of using the same parameter (and hence, the same utility function) to evaluate all alternatives in his choice set. Under the multiple random draw assumption, the consumer

draws the risk parameter every time he evaluates an alternative. The two assumptions imply similar, but slightly different, choice probabilities. We begin with single random draw.

Proposition 3: For the binary choice problem with discrete outcomes (Figure 4), under the single random draw assumption, if measurement error is modeled as NORMAL, then:

$$\hat{P}_1 \tilde{=} \Phi[(\hat{\mu} - \kappa \log(\beta/\alpha))/\hat{\sigma}] \text{ for constant proportional risk aversion}$$

$$\hat{P}_1 \tilde{=} \begin{cases} \Phi[(r_c - \hat{\mu})/\hat{\sigma}] \text{ if } \alpha x_1 > \beta x_2 \text{ for constant absolute risk aversion} \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{(infinite range, } 0 \leq x < \infty) \\ 0 \text{ if } \alpha x_1 \leq \beta x_2 \end{cases}$$

where

$$\kappa^{-1} = \log[(x_1 - x_0)/x_2 - x_0)],$$

and r_c solves the equation

$$\beta \exp(-r_c x_2) - \alpha \exp(-r_c x_1) = \beta - \alpha,$$

and $\Phi [\quad]$ denotes the cumulative distribution function of a normally distributed variate.

<u>Proof.</u> We scale $u(x, r)$ such that $u(x_0, r) = 0$ and $u(x_*, r) = 1$. Then, for the binary choice problem $\hat{P}_1 = \text{Prob} [\alpha u(x_1, \tilde{r}) > \beta u(x_2, \tilde{r})]$ where $\tilde{r}$ indicates random variable. Substituting for the constant proportional risk aversion utility function, $u(x, r) = (x - x_0)^r/(x_* - x_0)^r$. This yields that $\hat{P}_1 = \text{Prob}[\tilde{r} \geq \log(\beta/\alpha)/\log[(x_1 - x_0)/(x_2 - x_0)] =$ Prob $[\tilde{r} \geq \kappa \log(\beta/\alpha)]$. Recognizing that $r \sim N((\hat{\mu}, \hat{\sigma}^2)$ and $\Phi((\mu - z)/\sigma) = \text{Prob} [\tilde{r} \geq z]$ yields the result. The result is only approximate since we ignore $r < 0$ which occurs with low probability when $\mu >> \sigma$. Now substituting the infinite range constant absolute risk

aversion utility function, $u(x, r) = 1 - e^{-rx}$, into $\alpha u(x_1, r) =$ $\beta u(x_2, r)$ yields the equation for r_c. Note that as $r \to \infty$, $u(x, r) \to 1$, and Alternative 2 will be preferred since $\beta > \alpha$. As $r \to 0$, Alternative 1 will be preferred if $\alpha x_1 > \beta x_2$ since $u(x, r)$ approaches linearity. Thus, if there is only one solution to the equation for r_c, $\hat{P}_1 = \text{Prob } [0 \leq \tilde{r} \leq r_c]$. For $\alpha x_1 > \beta x_2$, there is only one solution to the equation for $r_c \geq 0$. We provide a proof of this fact in Lemma 1. See appendix. If $\alpha x_1 \leq \beta x_2$, then $\alpha u(x_1, r) < \beta u(x_2, r)$ for $r \geq 0$, hence $\hat{P}_1 = 0$.

Proposition 4: For the binary choice problem with discrete outcomes (Figure 4), under the single random draw assumption, if measurement error is modeled as EXPONENTIAL, then:

$$\hat{P}_1 = [\beta/\alpha]^{-\frac{\kappa}{\hat{\lambda}}} \quad \text{for constant proportional risk aversion}$$

$$\hat{P}_1 = \begin{cases} 1 - \exp[r_c/\hat{\lambda}] & \text{if } \alpha x_1 > \beta x_2 \text{ for constant absolute risk aversion} \\ & \hspace{2cm} \text{(infinite range, } 0 \leq x < \infty) \\ 0 & \text{if } \alpha x_1 \leq \beta x_2 \end{cases}$$

where κ and r_c are defined in Proposition 3.

Proof. The results follow the same arguments except $\text{Prob}[\tilde{r} \geq Z] = \exp(-z/\hat{\lambda})$. The result is exact since $\text{Prob}[r \leq 0] = 0$.

Propositions 3 and 4 are useful results. To illustrate their application, consider the hypothetical alternatives in Figure 5. Alternative 1 is oil heat where the high risk reflects volatile supplies. Alternative 2 is gas heat where the risk reflects only uncertainty in the heating characteristics of the home.

```
        $1200                          $600
       /                             /
      /                             /
     / .3                          / .5
    /                             /
   <                             <
    \                             \
     \ .7                          \ .5
      \                             \
       \                             \
        $200                          $200

   Oil Heat                      Gas Heat
```
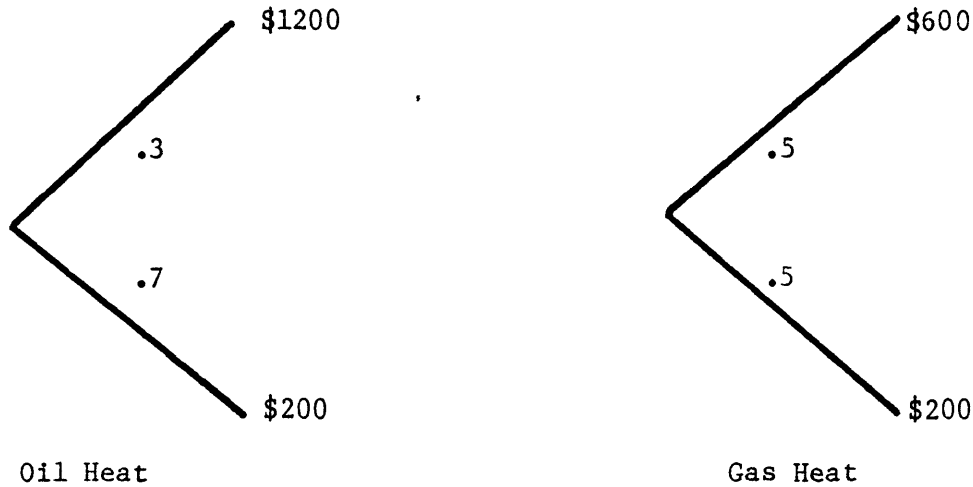
Figure 5: Hypothetical Characteristics of the Risk Involved for
Two Home Heating Systems

Using the distribution implied by the data in Table 3, $\hat{\mu}$ = .5, $\hat{\sigma}$ = .03.

From Figure 5, α = .3, β = .5, x_1 = 1200, and x_2 = 600. Assuming a

constant proportional risk averse utility function and substituting these

values in (16), yields $\hat{P}_1$ = $\Phi[\{.5 - \log(.5/.3)/\log(1000/400)\}/.03]$ $\overset{\sim}{=}$

$\Phi[-1.92]$ $\overset{\sim}{=}$ .027 for normal error theory. In a marketing forecasting

application, we would assign a .027 value to the probability that the consumer

would choose oil heat.

We now consider multiple random draws. We have been able to obtain

analytic results for constant proportional risk averse utility functions.

These results are stated in Proposition 5.

Proposition 5: For the binary choice problem with discrete outcomes

(Figure 4), under the multiple random draw assumption, for a constant

proportional risk averse utility function:

$$\hat{P}_1 = \Phi\{[\hat{\mu} - \kappa \log(\beta/\alpha)]/\kappa\eta\hat{\sigma}\} \qquad \text{for NORMAL errors}$$

where $\eta^2 = k_1^2 + k_2^2$

$$k_1 = \log[(x_* - x_0)/(x_1 - x_0)], \quad k_2 = \log[(x_* - x_0)/(x_2 - x_0)]$$

$$\hat{P}_1 = [k_2/(k_1 + k_2)][\beta/\alpha]^{-(1/\hat{\lambda}k_2)} \qquad \text{for EXPONENTIAL errors}$$

<u>Proof.</u> Alternative 1 will be chosen if $\alpha u(x_1, r_1) > \beta u(x_2, r_2)$.

Consider first normal error theory. Rearranging terms this condition

becomes $\log u(x_1, r_1) - \log u(x_2, r_2) > \log (\beta/\alpha)$. Using

Proposition 1, the left hand side of the inequality is distributed as

$N[\hat{\mu}(k_2 - k_1), \hat{\sigma}^2(k_1^2 + k_2^2)]$ and the result follows from

the recognition that $k_2 - k_1 = \log[(x_1 - x_0)/(x_2 - x_0)] = \kappa^{-1}$.

Now consider exponential error theory. Again rearranging terms indicates

that Alternative 1 will be chosen if $u(x_1, r_1)/u(x_2, r_2) > \beta/\alpha$.

Let $u_i = u(x_i, r_i)$ then by Proposition 2 and the assumption of

independent draws $g(u_1, u_2)$ is given by:

$$g(u_1, u_2) = (\hat{\lambda}^2 k_1 k_2)^{-1} (u_1)^{(1/\hat{\lambda}k_1)-1} (u_2)^{(1/\hat{\lambda}k_2)-1}$$

Define $z = u_1/u_2$ and $t = u_2$ then the p. d. f. of z and t is obtained

using a Jacobian transformation:

$$f_{zt}(z, t) = q_1 q_2 (z)^{q_1-1}(t)^{q_1 + q_2 -1}$$

$$\text{where } q_i = (1/\hat{\lambda}k_i)$$

Integrating out t yields the marginal distribution for z:

$$f_z(z) = \begin{cases} \dfrac{q_1 q_2}{q_1 + q_2} z^{q_1-1} & 0 \le z < 1 \\[2ex] \dfrac{q_1 q_2}{q_1 + q_2} z^{-q_2-1} & z \ge 1 \end{cases}$$

Since $(\beta/\alpha) > 1$, $\hat{P}_1 = \text{Prob}[z > \beta/\alpha] = \int_{\beta/\alpha}^{\infty} f_z(z)dz = [q_1/(q_1 + q_2)](\beta/\alpha)^{-q_2}$.

Finally, substituting $\hat{q}_i = (1/\lambda k_i)$ into the above expression yields the result.

It is interesting to compare Proposition 3 (Normal errors, constant proportional risk aversion) to Proposition 5 (Normal errors, constant proportional risk aversion). This comparison illustrates the impact of measurement error on our ability to estimate choice probabilities. Without error, r^T is known and alternative 1 will be chosen, $P_1 = 1$, whenever $r^T > \kappa \log(\beta/\alpha)$. This corresponds to Propositions 3 and 5

with $\hat{\sigma} \rightarrow 0$. As our uncertainty, $\hat{\sigma}$, about r increases, our ability to predict decreases, i.e., $\hat{P}_1$ decreases for $\hat{\mu} > \kappa \log(\beta/\alpha)$. If we compound that error by allowing the consumer multiple random draws from $f(r|\lambda)$, then our ability to predict is modified still further because we replace $\hat{\sigma}$ by $\kappa\eta\hat{\sigma}$.

The differences in Propositions 3 and 5 make clear the implications of our assumption about our knowledge of the consumer's risk parameter.

One can obtain similar interpretations by comparing Propositions 4 and 5 for exponential errors. The forms are the same, but the constants vary, e.g., κ vs. k_2^{-1}.

Thus, clearly, an "open-loop" prediction of probabilities, i.e., use indifference questions to estimate $f(r|\hat{\lambda})$ and Propositions 3 or 5 to estimate $\hat{P}_1$, will depend on the assumptions we make about how uncertainty in $u(x, r)$ affects uncertainty in choice outcomes. On the other hand, a "closed-loop" revealed preference prediction of probabilities will not depend on this

assumption. For example, a revealed preference estimate of $\hat{\sigma}$ will be smaller by a factor of $(\kappa\eta)^{-1}$ if we use Proposition 3 rather than Proposition 5, but $\kappa\eta$ will cancel out when we use the same proposition to forecast probabilities.

In other words, if we are most interested in estimating $\hat{P}_1$, then we should use revealed preference questions (discussed later) because they will produce estimates of $\hat{P}_1$ which are invariant with respect to the single/multiple draw assumption. If we are most interested in estimating $\hat{\lambda}$, then preference indifference questions are likely to be better because $\hat{\lambda}$ obtained by equations (10)-(12) does not depend on "solving" $\hat{\lambda} = g(\hat{P}_1)$ and hence will not depend upon the single/multiple draw assumption. (The need to "solve" $\hat{\lambda} = g(\hat{p}_1)$ will become clear when we discuss revealed preference.)

Such robustness of "closed-loop" revealed preference techniques is discussed in the econometric literature. For example, Domencich and McFadden (1975, p. 57) provide a table and discussion illustrating the similarity in probability predictions of the Logit, Probit, and Arctan probability of choice models. The Logit is based on Double-exponential errors, the Probit is based on Normal errors, and the Arctan is based on Cauchy errors.

We derive revealed preference estimators later, but first we complete this subsection with the estimates of $\hat{P}_1$ for constant absolute risk averse utility functions. Despite the fact that the lognormal and Beta distributions are well studied (e.g. Aitchison and Brown 1969, DeGroot 1970, Drake 1967) we

have been unable to obtain analytic results for $\hat{P}_1$ with constant absolute

risk averse utility. Instead Proposition 6 relies upon implied integral

equations which require numerical techniques.

Proposition 6: For the binary choice problem with discrete outcomes

(Figure 4), under the multiple random draw assumption, for a constant absolute

risk averse utility function:

$$\hat{P}_1 = \text{Prob } [\varphi_2 - \varphi_1 \geq (\beta - \alpha)] \quad \text{for NORMAL errors}$$

where $\varphi_1 \sim \Lambda(\log \alpha - x_1\hat{\mu}, \ x_1^2\hat{\sigma}^2)$

$\varphi_2 \sim \Lambda(\log \beta - x_2\hat{\mu}, \ x_2^2\hat{\sigma}^2)$

$$\hat{P}_1 = \text{Prob } [\alpha u_1 - \beta u_2 > 0] \quad \text{for EXPONENTIAL errors}$$

where $u_\ell \sim \text{Beta } (1, \ 1/\hat{\lambda}x_\ell)$

Proof. The proof is similar to that of Proposition 5. For NORMAL errors we

use the limited reproductive properties of the lognormal distribution

(Aitchison and Brown, 1969) and some algebra. See Barouch and Kaufman (1976)

for issues involving sums of lognormal random variables.

Binary Choice Among Alternatives Represented by Continuous Distributions of Outcomes

The previous subsection dealt with choices among outcomes represented by

lotteries. Such choices are important because (1) most measurement questions

take the form of lotteries and (2) lotteries represent the essential risk

problem of a choice between one alternative with greater potential reward and

another alternative with greater chance of getting the reward.

Although attributes of many consumer products can be represented by lotteries, many other attributes will be represented by continuous distributions such as the Normal distribution. For example, if we buy a new automobile, we might expect that the miles per gallon we actually obtain is best represented by a Normal distribution based on the published EPA estimate.

Proposition 7 derives the probability of choice, $\hat{P}_1$, if the two outcomes are represented by Normal distributions with means and variances, m_1, v_1^2 and m_2, v_2^2 respectively. We assume that outcomes take on mostly positive values, that is, $m_1 >> v_1$ and $m_2 >> v_2$. For simplicity we state the result only for single random draws with constant risk averse utility functions. Other results are obtainable but some require numerical techniques. Without loss of generality assume $m_1 > m_2$ and $v_1 > v_2$. (If $m_1 > m_2$ and $v_1 < v_2$ then $\hat{P}_1 = 1$.)

Proposition 7: For binary choice among Normally distributed outcomes, under the single random draw assumption, for a constant absolute risk averse utility function:

$$\hat{P}_1 = \phi[2(m_1 - m_2)/(v_1^2 - v_2^2)\hat{\sigma} - \hat{\mu}/\hat{\sigma}] \quad \text{for \textbf{NORMAL} errors}$$

$$\hat{P}_1 = 1 - \exp[2(m_1 - m_2)/\hat{\lambda}(v_1^2 - v_2^2)] \quad \text{for \textbf{EXPONENTIAL} errors}$$

Proof. First we recognize that the expected utility for a constant risk averse utility function with outcomes described by $f(x)$ is given by $E(u) = \int u(x, r)f(x)dx = 1 - \int e^{-rx}f(x)dx = 1 - M(r)$ where $M(r)$ is the

moment generating function of $f(x)$. See Keeney and Raiffa (1976, p. 201),

Drake (1967, chapter 3). For the Normal distribution, $M(r) =$

$\exp(-rm + r^2v^2/2)$. Thus, $\hat{P}_1 = \text{Prob}[1 - \exp(-rm_1 + r^2v_1^2/2) \geq 1 -$

$\exp(-rm_2 + r v_2^2/2)]$. Simplifying yields, $P_1 = \text{Prob}[r \leq 2(m_1 - m_2)/(v_1^2 - v_2^2)]$.

Finally, substituting the appropriate $f(r|\lambda)$ yields the result.

We have stated the result explicity for Normally distributed outcomes, but since the key idea of the proof is the use of moment generating functions (also known as exponential or Laplace transforms) we can obtain results for any distribution for which the moment generating function is tabled. This includes the continuous Beta, Cauchy, Chi-square, Erlang, Exponential, Gamma, Laplace, and Uniform distributions as well as some discrete distributions such as the Binomial, Geometric, and Poisson distributions. See tables in Keeney and Raiffa (1976, p. 202) and Drake (1967, pp. 271-276).

For the constant proportional risk averse utility function we can also obtain results by using the Mellin transform, $\int x^r f(x) dx$, for those distributions for which it exists. See tables in Bateman (1954).

Estimation for Revealed Preference Questions

The most commonly used question formats in decision analysis use some form of preference indifference question. However, in marketing such questions have been criticized as too complex. On the other hand, revealed preference questions, where the consumer is asked to choose among (or rank order) alternatives, are very common. For example, conjoint analysis, as reviewed by Green and Srinivasan (1978), uses this form of questioning and is one of the most widely used marketing research procedures. In fact, Currim and Sarin (1982) use a modified conjoint analysis procedure to estimate vN-M-like

utility functions. Furthermore, revealed preference is one of the most
commonly used techniques in transportation demand analysis. See Manski and
McFadden(1981).

As discussed earlier, revealed preference is a "closed loop" technique for
choice probabilities because we estimate based on observed $\hat{P}_1$'s in order to
predict for new $\hat{P}_1$'s. Intuitively, we expect such measurement techniques to be
less sensitive to the assumed distribution of measurement error when predicting
probabilities than "open-loop" techniques that estimate on indifference questions in
order to predict probabilities.[5]

Since we are addressing a market research issue, we allow the experimenter to
choose the question format much as he would choose the fractional factorial design
in conjoint analysis. For revealed preference estimation the consumer's task is
simple. He is given I pairs of alternatives. Each alternative is described by the
probability distribution of outcomes (usually lotteries, but continuous
distributions are allowable if they can be described adequately to the consumer).
For each pair of alternatives, the consumer is asked to choose the alternative which
he prefers. Propositions 3 through 7 give us the analytic tools to obtain
estimators for $f(r|\lambda)$ from the answers to such questions.

Let $\delta_i = 1$ if the consumer chooses Alternative 1 from the ith pair and let
$\delta_i = 0$ if he chooses Alternative 2. Let $\underline{\delta}$ be the vector of δ_i's. Then,
the joint probability, $F(\underline{\delta}|\lambda)$, of observing a particular set of answers, $\underline{\delta}$,
given $f(r|\lambda)$ is given by:

---

[5]If our interest is in the risk parameter, r, not the purchase probability, P_1,
then indifference questions are "closed-loop" and revealed preference is
"open-loop." Thus, for estimating r, indifference questions will be less sensitive
to the assumed form of the error distribution.

$$F(\underline{\delta}|\lambda) = \prod_{i=1}^{I} P_{1i}^{\delta i}\ (1 - P_{1i})^{1 - \delta i} \qquad (16)$$

where $P_{1i} = P_{1i}(\lambda)$ are determined for each question, i, by the appropriate proposition. (I.e., Proposition 3, 4, 5, 6 or 7 or their extensions.) For example, for lotteries under the multiple random draw assumption with exponential errors and a constant proportional risk averse utility function, P_{1i} is given by:

$$P_{1i}(\lambda) = [k_{2i}/(k_{1i} + k_{2i})][\beta/\alpha]^{-(1/\lambda k_{2i})} \qquad (17)$$

In principle we could form a log-likelihood function based on equations (16) and (17) and then maximize it by numerical techniques to obtain MLE's of λ, $\hat{\lambda}$. However, if the experimenter chooses his measurement design carefully, he can obtain practical analytic expressions for $\hat{\lambda}$. In particular, for equation (16), if (1) he chooses α, β, x_1 and x_2 for the first question (review figure 4) and if (2) for every subsequent question, i, he chooses α_i and x_{1i}. Then β_i and x_{2i} according to the following rule:

$$x_{2i} = x_o + [(x_2 - x_o)/(x_* - x_o)]^{\gamma i}\ (x_* - x_o) \qquad (18)$$

$$\beta_i = (\beta/\alpha)^{\gamma i}\ \alpha_i \qquad (19)$$

$$\gamma_i = \log[(x_{1i} - x_o)/(x_* - x_o)]/\log[(x_1 - x_o)/(x_* - x_o)] \qquad (20)$$

then he can obtain an analytic expression for $\hat{\lambda}$. (Note we have suppressed the subscript 1 on x_{11}, etc for the base question, i=1).

The analytic expression is obtained from the invariance properties of MLE's in the following way. Equations (18) through (20), ensure that for all λ, P_{1i} is constant for all questions. Define I_1 as the number of times the first

alternative is chosen, then $F(\underline{\delta}|\lambda)$ becomes a Binomial distribution function for

I_1. The MLE for a Binomial distribution is obtained simply as $\hat{P}_1 = I_1/I$. $\hat{\lambda}$

is obtained by solving the equation $\hat{P}_1(\lambda) = I_1/I$. For equation (17) we obtain

$$\hat{\lambda} = [(1/k_2) \log(\beta/\alpha)]/[\log\{k_2 I/(k_1 + k_2)I_1\}] \qquad (21)$$

where α, β, k_1, and k_2 are obtained from the reference question.

To illustrate this technique, consider a set of questions in which each alternative is a potential heating system. The attribute of interest is reliability, that is, a 0 to 10 scale indicating how likely it is that the system will not require major repairs during the next five years. One such question is illustrated in Figure 6. (For example, the reliability index might be 10 times the probability that no repair will be required.)
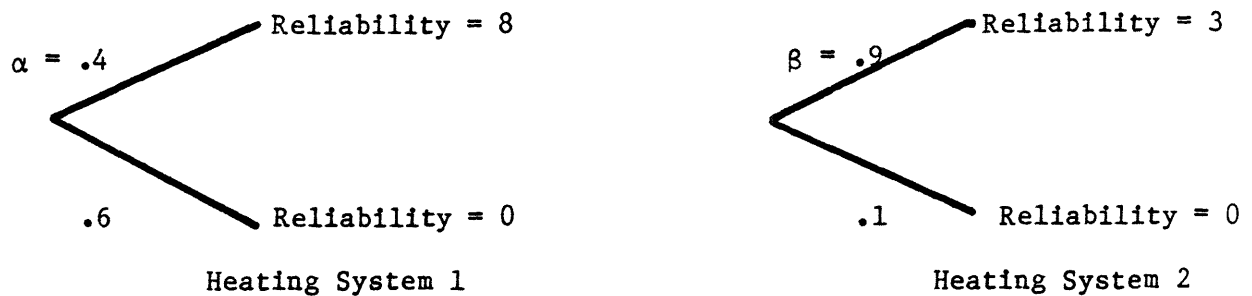
Figure 6: Schematic of Revealed Preference Question Corresponding to Proposition 5

We can then ask 10 questions of this form as indicated in Table 4. (We have rounded β to the nearest .05.) We record I_1, the number of times the consumer prefers heating system 1.

TABLE 4
Example Experimental Design for Revealed Preference Questions

Question Number	Heating System 1 x_{1i}	α_i	Heating System 2 x_{2i}	β_i
1	9.5	.5	7.5	.60
2	9.0	.5	5.5	.75
3	8.5	.5	4.0	.90
4	8.0	.4	3.0	.90
5	7.5	.3	2.0	.85
6	7.0	.2	1.5	.70
7	6.5	.2	1.0	.90
8	9.5	.6	7.5	.70
9	9.0	.6	5.5	.90
10	9.5	.7	7.5	.85

For example, if $I_1 = 2$, then $\hat{\lambda} = .44$, and if $I_1 = 3$, then $\hat{\lambda} = .61$.

We could, of course, obtain better estimates by asking more questions. For example, if $I = 100$ and $I_1 = 21$, then $\hat{\lambda} = .45$ and if $I_1 = 22$, then $\hat{\lambda} = .47$.

We constructed equations (18) through (21) and Table 4 for Proposition 5, Exponential error. It is also posible to construct experimental designs for Normal errors, for continuous distributions of outcomes, and for constant absolute risk averse utility function. The experimenter simply chooses the appropriate proposition (or its extension) and derives the conditions on α, β, x_1, and x_2 such that $\hat{P}_1$ is constant for all questions. $\hat{\lambda}$ is the solution to $P_1(\hat{\lambda}) = I_1/I$. For example, for Proposition 7, we restrict $(m_1 - m_2)/(v_1^2 - v_2^2)$ to be constant to assure $\hat{P}_1$ is constant. For Normal errors, there are two unknown parameters, hence we must either (1) assume one parameter is known, or (2) ask two sets of clustered questions to obtain two equations in two unknowns.

Summary of Single Parameter Uniattributed Utility Functions

This completes our discussion of single paramter, uniattributed utility functions. We have presented a conceptual framework that can be applied when the marketing researcher is interested in explicitly considering measurement errors. Throughout our development, we have tried to provide the researcher with flexibility by (1) specifying two error distributions; (2) analyzing two commonly accepted utility functions; and (3) deriving estimators and choice probabilities for various choice situations and assumptions about single and multiple random draws.

Our estimators give the experimenter a choice of question formats. He can choose preference indifference questions such as those used in decision analysis or revealed preference questions such as those used in conjoint analysis. For estimators of P_1, revealed preference questions are likely to be less sensitive to the form of error distribution. For estimators of $f(r|\hat{\lambda})$, preference indifference questions are likely to be best.

However, if the experimenter wishes to determine the form of $f(r|\lambda)$ empirically, we suggest he use preference indifference questions to obtain $r(x_i, p_i)$ for sufficiently large I and plot its histogram. If the histogram is symmetric and unimodal, Normal errors are likely to be the best assumption; if the histogram is unimodal and skewed with $\hat{\sigma} \simeq \hat{\mu}$, then Exponential errors are likely to be the best assumption. If the experimenter wishes to select empirically among the assumptions of single and multiple random draws, we suggest he obtain both preference indifference _and_ revealed preference questions and determine empirically whether Propositions 3 and 5 or Propositions 4 and 5 produce the best match among the alternative question formats.

To select among constant proportional or constant absolute risk aversion, we suggest the experimenter use qualitative techniques such as those discussed in Farquahar (1982) and Keeney and Raiffa (1976, pp. 188-200). Keeney and Raiffa provide numerous examples for both functional forms.

4. MULTIPLE PARAMETER UNIATTRIBUTED UTILITY FUNCTIONS

While the class of single parameter utility functions are the most commonly used because they are flexible and can accommodate a wide range of interesting problems, occasionally an experimenter may wish to estimate the parameters of a utility function that is more complex. For example, Keeney and Raiffa (1976, p. 209) report that a computer program which has been used at the Harvard Business School since 1966 is based on the decreasingly risk averse three parameter function that is a sum of constant absolute risk averse utility functions. In general, the computation of choice probabilities is not analytically tractable for multiple parameter functions, but equation 15 still applies for numerical solutions. Since a researcher choosing a multiple parameter utility function may be willing to sacrifice analytic simplicity for greater flexibility, we provide a means to estimate the parameters of the utility function recognizing numerical integration may be necessary for choice probabilities and revealed preference.

We provide two methods. The first method requires clustered questioning, but provides maximum likelihood estimates. The second method relaxes the clustering requirement, but requires a regression approximation.

Clustered Questions

Suppose an experimenter wishes to estimate an error distribution for a three-parameter function. One procedure might be to ask the consumer sets of three questions and use the information from each set to solve for r_{1i}, r_{2i}, and r_{3i}. I clusters of three questions then provide the data from which to estimate a multivariate distribution for r_1, r_2, and r_3.

Let (x_{ki}, p_{ki}) be the certainty equivalent and lottery probability associated with the ith cluster and kth question within the cluster. Let $\underline{x}_i = (x_{1i}, x_{2i}, \ldots x_{Ki})$ and $\underline{p}_i = (p_{1i}, p_{2i}, \ldots, p_{Ki})$. If a vector-valued function, $\underline{r}(\underline{x}_i, \underline{p}_i)$, exists mapping the vectors $\underline{x}_i$ and $\underline{p}_i$ onto the range of the K unknown parameters of the utility function, and if we assume that errors cause $\underline{r}(\underline{x}_i, \underline{p}_i)$ to be distributed with a multivariate normal distribution with mean, $\underline{\mu}$, and covariance matrix, Γ, then the maximum likelihood estimators, $\hat{\underline{\mu}}$ and $\hat{\Gamma}$, are simply the multivariate extensions of the univariate estimators in equations (10) and (11). That is,

$$\hat{\underline{\mu}} = (1/I) \, \Sigma_i \underline{r}(\underline{x}_i, \underline{p}_i) \tag{22}$$

$$\hat{\Gamma} = (1/I) \, \Sigma_i [\underline{r}(\underline{x}_i, \underline{p}_i) - \hat{\underline{\mu}}][\underline{r}(\underline{x}_i, \underline{p}_i) - \hat{\underline{\mu}}]^T \tag{23}$$

For a formal proof, see Giri (1977, Chapter 15). As before, we can construct confidence regions with the multivariate extension of a t-test. For example, the appropriate statistic for $\hat{\underline{\mu}}$ is Hotelling's T^2 statistic (Giri 1977, Chapter 7; and Green 1978, p. 257).

Similar results apply for multivariate exponential error.

Independent Questions

If, for whatever reasons, the experimenter feels that clustered questions are not appropriate for his situation, he may wish to ask K x I independent questions. In this case, without further specifying the interrelationships of the question formats, we cannot obtain analytic maximum likelihood estimators. However, we can obtain a practical regression approximation.

Following Pratt (1964) and Keeney and Raiffa (1976, p. 160) define a risk premium, π_i, as the amount by which the certainty equivalent, x_i, exceeds the expected value of the lottery, $\bar{x}_i$. For the measurement in Figure 1:

$$\pi_i(x_i, p_i) = x_i - p_i x_* - (1-p_i)x_o \tag{24}$$

Keeney and Raiffa (1976, p. 161) then consider variation about the expected value of the lottery and show by Taylor's series expansion that the local risk aversion, $R(x, \underline{r})$, is approximately proportional to π_i. ($R(x, \underline{r})$ is defined by equation (1) where we have added the unknown parameters, $\underline{r}$, to the notation.) In particular, Keeney and Raiffa show

$$\pi_i(x_i, p_i) = (1/2)v_i^2 R(x_i, \underline{r}) + \varepsilon \tag{25}$$

where v_i^2 is the variance of the lottery, $v_i^2 = (1/2)(x_* - \bar{x}_i)^2 + (1/2)(x_o - \bar{x}_i)^2$, and ε indicates higher order terms that are assumed to be negligible. Rearranging terms yields:

$$R_i^o(x_i, p_i) = R(x_i, \underline{r}) + \tilde{e}_i \tag{26}$$

where $R_i^o(x_i, p_i) = 4(x_i - p_i x_* - (1-p_i)x_o)/[(x_*-\bar{x}_i)^2 + (x_o-\bar{x}_i)^2]$ is a function of known data because $\bar{x}_i = p_i x_* + (1-p_i)x_o$. Note that we have incorporated the Taylor's series error, ε, in the measurement error, $\tilde{e}_i$.

Equation (26) is now in the form of a regression equation. If $R(x_i,\underline{r})$ is linear in its parameters, ordinary least-squares regression applies. Alternatively, a researcher may use non-linear techniques for non-linear $R(x_i,\underline{r})$. Once the parameters of $R(x_i,\underline{r})$ are estimated, we can recover $u(x_i,\underline{r})$ from equation (1) by integration since $u(x,\underline{r})$ = $f_1 \int \exp[-\int R(x,\underline{r})dx]dx + f_2$, where f_1 and f_2 are constants chosen to scale the utility function.

For example, we might wish to consider utility functions which combine constant absolute and proportional risk aversion in this case, $R(x_i,\underline{r})$ = $r_1 + r_2(x_i - x_o)^{-1}$ is linear in the unknown parameters. Since equation (26) does not require an inverse function, we can allow x_* and x_o to vary across measurements, i.

5. MULTIATTRIBUTED UTILITY FUNCTIONS

The analyses and propositions in Sections 3 and 4 provide us with a means to estimate and use uniattributed utility functions under conditions of measurement error. For many applications, such as decisions among alternative financial investments, a uniattributed theory suffices. However, there are many applications in marketing where it is necessary to model decisions involving multiple attributes, each of which is risky. For example, the decision to buy a home heating system might involve reliability as well as annual dollar savings. (Choffray and Lilien (1978) illustrate empirically a multiattributed preference problem for solar air-conditioning.)

We can estimate multiattributed utility functions in two ways.

Estimation Procedure 1

The first procedure is a two-step procedure which combines the results of Section 3 with commonly used methods in marketing. In Step 1, we use either preference indifference or revealed preference questions to obtain estimators for uniattributed functions for each attribute. Step 1 implicitly assumes "mutual utility independence" (Keeney 1972) among the attributes, but such an assumption is implicit in most conjoint and logit analyses. The multiattributed function, $U(\underline{x},\underline{w})$ is given by:

$$U(\underline{x},\underline{w}) = \sum_{m=1}^{M} w_m u(x_m, r_m) + \sum_{m=1}^{M} \sum_{k=m}^{M} w_{mk} u(x_m, r_m) u(x_k, r_k) \qquad (27)$$

$$+ \text{ third order and higher interaction terms.}$$

Note that if the higher order terms are zero (for conditions see Fishburn 1974), then equation (27) reduces to the commonly used forms in conjoint and logit analysis.

In Step 2, the experimenter then asks either preference indifference (or revealed preference) questions using multiattributed alternatives. Standard conjoint (or logit) analysis procedures are then used to obtain $\hat{\underline{w}}$ with $u(x_m, \hat{r}_m)$ rather than x_m as the explanatory variables. ($\hat{r}_m$ is our best estimate of r_m.)

Estimation procedure 1 is an approximation. It is a two-stage prcedure with the potential problem of compounding errors from step 1 to step 2. However, (1) if such compounding is small relative to other measurement errors, or (2) if the additive form applies and errors are independent and identically distributed (i.i.d.) across attributes, then estimation procedure 1 should work well. (If errors are i.i.d., then the w_m are equally biased which has no effect since $u(\underline{x},\underline{w})$ is only unique to a translation.)

Estimation Procedure 2

Estimation procedure 2 is a one-stage procedure, but is limited by practicality to preference indifference questions. (Revealed preference would require numerical techniques.)

Suppose we ask I x L preference indifference questions where L is the number of parameters, $\underline{w}$, to be estimated. Let $\underline{x}_{\rho i} = (x_{\rho i 1}, x_{\rho i 2}, \ldots x_{\rho i M})$ be the levels of the M attributes for the certainty equivalent in the ρth question in the ith cluster. In assessing $U(\underline{x}, \underline{w})$ we specify either (1) all of $\underline{x}_{\rho i}$ or (2) $p_{\rho i}$ and all but one of the $\underline{x}_{\rho i}$.

If we ask our questions in I clusters of L questions and if a computable vector-valued inverse function $\underline{W}(\underline{\underline{x}}_i, \underline{p}_i)$, exists mapping the question set onto the unknown parameters, then the multiattributed problem is isomorphic to the multiple parameter uniattributed problem. ($\underline{\underline{x}}_i$ is the matrix with rows $\underline{x}_{\rho i}$ and $\underline{p}_i$ is the vector with elements $p_{\rho i}$.). Equations 22 and 23 can be used to estimate the mean and covariance of a multivariate normal distribution on $\underline{w}$. Confidence intervals are computed with Hotelling's T^2 statistic.

Probability of Choice

Choice probabilities are obtained with equation (15) and numerical techniques. For example, one might use equation (15) by sampling from the multivariate normal distribution , then using the sampled $\underset{\sim}{\underline{w}}$ to compute the expected utility of each option. Choice probabilities are then the percent of times an alternative is chosen in, say 1000, draws. This computation method is similar to methods used in probit analysis, e.g., Dagonzo (1979), and has proven feasible in that context.

Estimation Example

We illustrate estimation procedure 2 with a home heating system example. Suppose that besides annual savings, x_1, the individual is concerned with reliability as measured by 10 times the probability, x_2, that no repair will be needed each year. We suppose that the individual plans to purchase a service contract (a form of insurance policy) such that only negative effect of a repair is inconvenience (not dollar cost). We wish to model the decision maker's preferences by a constant proportional risk averse, multilinear utility function. (This is a two-attribute version of equation (27).)

$$U(\underline{x},\underline{w}) = w_3 u_1(x_1) + w_4 u_2(x_2) + (1-w_3-w_4) u_1(x_1) u_2(x_2) \tag{28}$$

$$u_1(x_1) = [(x_1 - 200)/1000]^{w_1}; \quad u_2(x_2) = x_2^{w_2}$$

We estimate the four unknown parameters, $w = (w_1, w_2, w_3, w_4)$ by asking the lottery questions shown in Table 5. In each question, the decision maker is asked to give a probability, $p_{\rho i}$, such that he is indifferent between a certainty equivalent, $(x_{\rho i1}, x_{\rho i2})$ and a lottery where the system is described by $(x''_{\rho i1}, x''_{\rho i2})$ with probability $p_{\rho i}$, and by $(x'_{\rho i1}, x'_{\rho i2})$ with probability, $(1-p_{\rho i})$. In other words, the standard lottery shown in Figure 7:
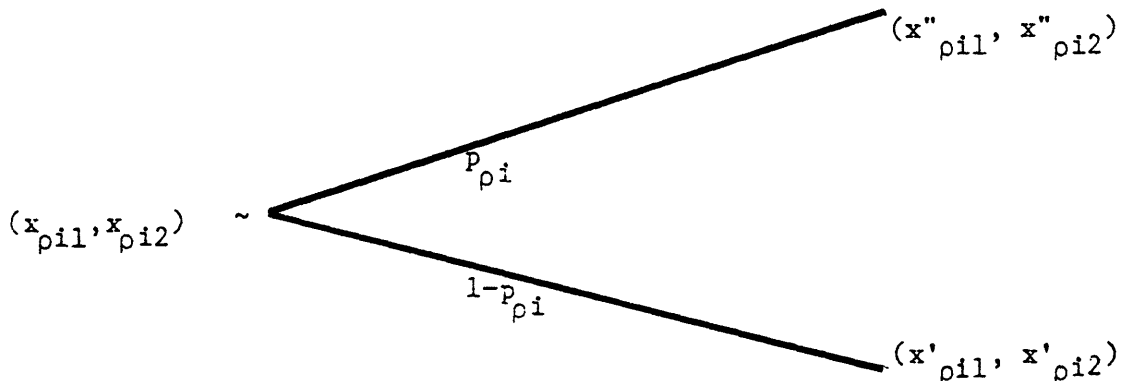

Figure 7. Schematic of Multivariate Lottery

The reader will note that we have constructed the questions in Table 4 for easy computation of the inverse function, $\underline{w}(\underline{x}_i, \underline{p}_i)$.

$$w_{1i} = \log(p_{1i})/\log[x_{1i1} - 200)/1000] \qquad (29)$$

$$w_{2i} = \log(p_{2i})/\log(x_{2i2})$$

$$w_{3i} = p_{3i}/p_{1i}$$

$$w_{4i} = p_{4i}/p_{2i}$$

This simplicity is for ease of exposition. Tradeoff questions as well as lotteries can be used and the inverse function can vary with i as long as it is computable for all i. Even with our simplification, the sixteen questions provide the experimenter with a variety of questions to be asked. The "answers", $p_{\rho i}$, to the lottery qustions were "constructed" by assuming $\underline{w} = (.50, .33, .40, .80)$ and rounding off to the nearest .05.

Examination of Table 4 reveals that the estimated parameters, $\underline{w}^T$, recover the "known" values quite well. The covariance matrix, $\hat{\Gamma}$, and the corresponding correlation matrix, $\hat{C}$, can be readily computed with equations 22 and 23.

$$\hat{\Gamma} = \begin{bmatrix} .0054 & .0102 & .0009 & .0009 \\ & .0226 & .0010 & .0020 \\ & & .0021 & .0010 \\ & & & .0009 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} 1.0 & .93 & .27 & .41 \\ & 1.00 & .15 & .44 \\ & & 1.00 & .74 \\ & & & 1.00 \end{bmatrix}$$

We note that the high off-diagonal elements in $\hat{C}$ are partially due to the small sample size, I=4, and partially due to structural correlation in equation 29. E.g., p_{1i} appears in the equations for both w_1 and w_3. Such correlations can be avoided with larger sample sizes and judicious choice of question format.

Table 5

EXAMPLE ASSESSMENT FOR THE COST AND RELIABILITY
OF A HOME HEATING SYSTEM

		Certainty Equivalent		"Win"		"Loss"		Probability	Parameters			
i	ρ	$x_{\rho i1}$	$x_{\rho i2}$	$x''_{\rho i1}$	$x''_{\rho i2}$	$x'_{\rho i1}$	$x'_{\rho i2}$	$P_{\rho i}$	w_{1i}	w_{2i}	w_{3i}	w_{4i}
1	1	400	.20	1200	2	200	2	.45	.50			
	2	400	.20	400	10	400	0	.60		.32		
	3	400	.00	1200	10	200	0	.20			.44	
	4	200	.20	1200	10	200	0	.50				.83
2	1	600	.40	1200	4	200	4	.65	.47			
	2	600	.40	600	10	600	0	.75		.31		
	3	600	.00	1200	10	200	0	.25			.38	
	4	200	.40	1200	10	200	0	.60				.80
3	1	800	.60	1200	6	200	6	.75	.56			
	2	800	.60	800	10	800	0	.80		.44		
	3	800	.00	1200	10	200	0	.30			.40	
	4	200	.60	1200	10	200	0	.65				.81
4	1	1000	.80	1200	8	200	8	.90	.47			
	2	1000	.80	1000	10	100	0	.95		.23		
	3	1000	.00	1200	10	200	0	.35			.39	
	4	200	.80	1200	10	200	0	.75				.79
								$\hat{\mu} =$	.50	.33	.40	.81

x_1 = savings in dollars

x_2 = reliability index

6. SUMMARY AND FUTURE DIRECTIONS

This completes our analysis of the implications of measurement error for
modeling consumer risk preference with vN-M utility functions. Our emphasis
is on uniattributed single parameter functions since they are the most
commonly used and illustrate the unique advantage of vN-M utility functions.

Our main results are practical and flexible. They enable the experimenter to choose among question formats, error assumptions, and functional forms for the utility function. They provide MLE's for the distributions of risk parameters and for choice probabilities. Furthermore, we have indicated (with references) how one can use numerical techniques for the cases where analytic results are unobtainable.

We have also provided practical procedures for multiple parameter uniattributed functions and for multiattribute functions. Our procedures combine the strength of vN-M uniattributed theory with practical measurement models, conjoint or logit analysis, now in use widely in market research.

We have chosen to illustrate our measurement and estimation procedures with numerical examples rather than empirical examples. We feel this is appropriate because:

(1) VN-M utility assessment has proven feasible and reasonably accurate in published market research and decision analysis studies.

(2) Published evidence seems to suggest that either Normal or Exponential errors are a reasonable error model.

(3) Analytic reasoning suggests when each question format is likely to be best; revealed preference for $\hat{P}_1$ and preference indifference for $\hat{\lambda}$.

(4) An extensive literature exists on the type of errors introduced when assessing vN-M functions. E.G., see Hershey, Kunreuther and Schoemaker (1982), Schoemaker and Waid (1982), and Schoemaker (1981). Our propositions provide the analytic tools with which to model that error and with which to obtain MLE's in the presence of error.

(5) The effects of measurement are best illustrated when the measurement error is "known."

Nonetheless, we posit that a measurement error approach will prove useful both for practical market research problems and for scientific research into the type and magnitude of errors introduced by alternative question formats. Our propositions provide the means by which to quantify that error.

Such empirical issues are extremely important and represent non-trivial future directions. We hope that our analyses provide a beginning.

APPENDIX

Lemma 1: Assume $\beta > \alpha$ and $x_1 > x_2$, then the equation $\beta \exp(-r_c x_2) - \alpha \exp(-r_c x_1) = \beta - \alpha$, has at most one solution for $r_c > 0$.

Proof: First, rewrite the equation in functional form:

$$E(r) = \alpha[1-\exp(-rx_1)] - \beta[1-\exp(-rx_2)] \qquad (A1)$$

recognizing $x_1 > x_2$ and $\beta > \alpha$. Alternative 1 will be chosen whenever $E(r) \geq 0$.

By a Taylor expansion $E(r) \simeq \alpha x_1 - \beta x_2$ as $r \to 0$. Let $E(0) = \lim_{r_c \to 0} E(r_c)$ and let $E(\infty) = \lim_{r_c \to \infty} E(r_c)$. Then $E(0) > 0$ if $\alpha x_1 > \beta x_2$ and $E(0) \leq 0$ if $\alpha x_1 \leq \beta x_2$. By direct substitution $E(\infty) = \alpha - \beta > 0$ since $\alpha > \beta$.

Now differentiate $E(r)$ yielding

$$E'(r) = dE(r)/dr = \alpha x_1 \exp(-rx_1) - \beta x_2 \exp(-rx_2) \qquad \text{Setting the}$$

derivative equal to zero yields $r^* = (\log \alpha x_1 - \log \beta x_2)/(x_1 - x_2)$.

Since $x_1 > x_2$, $r^* > 0$ iff $\alpha x_1 > \beta x_2$. Furthermore $E'(0) = \alpha x_1 - \beta x_2$ thus $E'(0) > 0$ iff $\alpha x_1 > \beta x_2$

Assume $\alpha x_1 < \beta x_2$, then $E(0) < 0$, $E(\infty) < 0$, and $E(r)$ is monotonic in the range $(0,\infty)$. Thus there is no solution to A1 for $r_c > 0$. If $\alpha x_1 = \beta x_2$, the only solution for $r \geq 0$ is $r_c = 0$.

Assume $\alpha x_1 > \beta x_2$, then $E(0) > 0$, $E'(0) > 0$, and $r^* > 0$. Thus $E(r) > 0$ for $r \leq r^*$. Now $E(r^*) > 0$, $E(\infty) < 0$, and $E(r)$ is monotonic in the range (r^*,∞). Thus, there is exactly one solution to A1 for $r_c > 0$ and it occurs in the range (r^*, ∞). Note that we have also proven that $E(r) > 0$ for $r \epsilon (0,r_c)$ and $E(r) < 0$ for $r \epsilon (r_c,\infty)$, thus alternative 1 will only be chosen in the range $(0,r_c)$.

REFERENCES

1. Aitchison, J. and J.A.C. Brown, The Lognormal Distribution with Special Reference to its Uses in Economics (England: Cambridge University Press, 1969).

2. Barouch, E. and G.M. Kaufman, "On Sums of Lognormal Random Variables", Working Paper #831-76, Sloan School of Management, M.I.T., Cambridge, MA February 1976.

3. Bateman, H. (Bateman Manuscript Project) Tables of Integral Transforms,(McGraw Hill Book Co., Inc., New York) 1954.

4. Boyd, J.H. and R.E. Mellman, "The Effect of Fuel Economy Standards on the U.S. Automobile Market: A Hedonic Demand Analysis," Transportation Research, Vol. 14A, No. 5-6 (October-December 1980), pp. 367-378.

5. Choffray, J.J. and G.L. Lilien, "The Market for Solar Cooling: Perceptions, Response, and Strategy Implications," Studies in Management Science, Vol. 10, 1978, pp. 209-226.

6. Currim, I. and R. Sarin, "A Comparative Evaluation of Multiattribute Consumer Preference Models", Working Paper, University of California at Los Angeles, Los Angeles, Calif., Revised February 1983. Forthcoming Management Science.

7. Daganzo, C., Multinomial Probit: The Theory and its Applications to Demand Forecasting (New York: Academic Press, 1979).

8. Degroot, M.H., Optimal Statistical Decisions, (McGraw Hill, New York), 1970.

9. Domencich, T.A. and D. McFadden, Urban Travel Demand: A Behavioral Analysis, American Elsevier Publishing Co., New York, 1975.

10. Drake, A.W., Fundamentals of Applied Probability Theory, McGraw Hill Book Co., New York, 1967.

11. Eliashberg, J., "Consumer Preference Judgments: An Exposition with Empirical Applications," Management Science, Vol. 26, No. 1 (January 1980), pp. 60-77.

12. Farquhar, P.H., "A Survey of Multiattribute Utility Theory and Applications," Studies in Management Science, Vol. 6 (1977), pp. 59-89,

13. Farquhar, P.H., "Utility Assessment Methods", Working Paper, University of California, Davis, Calif., November 1982. And to appear in Management Science.

14. Fishburn, P.C., "von Neumann-Morgenstern Utility Functions on Two Attributes," Operations Research, Vol. 22, No. 1 (January-February 1974), pp. 35-45.

15. Fishburn, P.C., and G.A. Kochenberger, "Two-Piece von Neumann-Morgenstern Utility Functions," Decision Sciences, Vol. 10 (1979), pp. 503-518.

16. Friedman, M. and L.J. Savage, "The Expected-Utility Hypothesis and the Measurability of Utility," Journal of Political Economy, Vol. 60 (1952), pp. 463-474.

17. Giri, N.C., Multivariate Statistical Inference, (Academic Press, NY, 1977).

18. Green, P.E., Analyzing Multivariate Data (Hinsdale, IL: The Dryden Press, 1978).

19. Green, P.E. and V. Srinivasan, "Conjoint Analysis in Consumer Research: Issues and Outlook," Journal of Consumer Research, Vol. 5, No. 2 (September 1978), pp. 103-123.

20. Hauser, J.R., "Consumer Preference Axioms: Behavioral Postulates for Describing and Predicting Stochastic Choice," Management Science, Vol. 24, No. 13 (September 1978), pp. 1131-1341.

21. Hauser, J.R. and G.L. Urban, "Direct Assessment of Consumer Utility Functions: von Neumann-Morgenstern Theory Applied to Marketing," Journal of Consumer Research, Vol. 5 (March 1979), pp. 251-262.

22. Hershey, J.C., H.C. Kunreuther, and P.J.H. Schoemaker, "Sources of Bias in Assessment Procedures for Utility Functions," Management Science, Vol. 28 (August 1982), pp 936-954.

23. Herstein, I.N. and J. Milnor, "An Axiomatic Approach to Measureable Utility," Econometrica, Vol. 21 (1953), pp. 291-297.

24. Ingene, C.A., "Risk Management by Consumers", Working Paper, University of Texas at Dallas, Richardson, Texas, October 1981.

25. Jensen, N.E., "An Introduction to Bernoullian Utility Theory. I. Utility Functions," Swedish Journal of Economics, Vol. 69 (1967), pp. 163-183.

26. Keeney, R.L., "Utility Functions for Multiattributed Consequences," Management Science, 18 (1972), pp. 276-287.

27. Keeney, R.L., and H. Raiffa, Decisions with Multiple Objectives: Preferences and Value Tradeoffs, (New York: John Wiley & Sons, 1976).

28. Luce, R.D. and P. Suppes, "Preference, Utility, and Subjective Probability," in R.D. Luce, R.R. Bush, and E. Galanter (eds.) Handbook of Mathematical Psychology, Vol. 3 (New York: John Wiley & Sons, 1965), pp. 249-410.

29. Marschak, J., "Rational Behavior, Uncertain Prospects, and Measureable Utility," Econometrica, Vol. 18 (1950), pp. 111-141.

30. Manski, C.F. and D. McFadden, Structural Analysis of Discrete Data with Econometric Applications, MIT Press, Cambridge, MA, 1981.

31. McFadden, D., "Quantal Choice Analysis: A Survey," Annals of Economics and Social Measurement, (1976), pp. 363-390.

32. McFadden, D. "Econometric Models for Probabilistic Choice Among Products," <u>The Journal of Business</u>, Vol. 53, No. 3, Pt. 2, July 1980, pp. 513-530.

33. Meyer, R.F. and J.W. Pratt, "The Consistent Assessment and Fairing of Preference Functions," <u>IEEE Transactions on Systems Science and Cybernetics,</u>" Vol. SSC-4, No. 3 (September 1968), pp. 270-278.

34. Mood, A.M. and F.A. Graybill, <u>Introduction to the Theory of Statistics</u>, (New York: McGraw-Hill Book Co., 1963).

35. Pras, B. and J.O. Summers, "Perceived Risk and Composition Models for Multiattribute Decisions", <u>Journal of Marketing Research</u>, Vol. 15, August 1978, pp. 429-427.

36. Pratt, J.W., "Risk Aversion in the Small and the Large," <u>Econometrica</u>, Vol. 32 (1964), pp. 122-136.

37. Schoemaker, P.J.H., "Behavioral Issues in Multiattribute Utility Modeling and Decision Analysis," <u>Organizations: Multiple Agents with Multiple Criteria</u>, J. Morse, ed., Springer-Verlag, New York, 1981, pp 447-464.

38. Schoemaker, P. and C.C. Waid, "An Experimental Comparison of Different Approaches to Determining Weights in Additive Utility Models", <u>Management Science</u>, Vol. 28, (February 1982), pp 182-196.

39. Thurstone, L., "A Law of Comparataive Judgment," <u>Psychological Review</u>, 34 (1927), pp. 273-286.

40. von Neumann, J. and O. Morgenstern, <u>Theory of Games and Economic Behavior</u>, Princeton, N.J.: Princeton University Press, 1947.