Technical Reports (ESE)

Department of Electrical & Systems Engineering

9-1-2016

Technical Report on: Comparative Design, Scaling, and Control of Appendages for Inertial Reorientation

Thomas Libby
tlibby@berkeley.edu

Aaron Johnson
amj1@cmu.edu

Evan Chang-Siu

R J. Full
rjfull@berkeley.edu

Daniel E. Koditschek
kod@seas.upenn.edu

Technical Report on: Comparative Design, Scaling, and Control of Appendages for Inertial Reorientation

Disciplines
Electrical and Computer Engineering | Engineering | Systems Engineering

Technical Report on: Comparative Design, Scaling, and Control of Appendages for Inertial Reorientation

Thomas Libby, *Student Member, IEEE,* Aaron M. Johnson, *Member, IEEE,*
Evan Chang-Siu, *Member, IEEE,* Robert J. Full, and D. E. Koditschek, *Fellow, IEEE*

This technical report provides full derivations and definitions for the paper, [1].

I. Analytic solution of template dynamics

Integration of the system dynamics, given in [1, Eqn. 8] as,

$$\ddot{\theta}_b = \begin{cases} \dfrac{4P}{\omega_m I_d}\left(1 - \dfrac{\dot{\theta}_b}{\xi\omega_m}\right), & \text{for } 0 \leq t < t_s, \\ -\dfrac{4P}{\omega_m I_d}, & \text{for } t \geq t_s. \end{cases}$$

is easier in the rescaled coordinates introduced in [1, Eqn. 13],

$$\tilde{t}_s = \gamma t_s, \quad \tilde{t}_f = \gamma t_f, \quad \tilde{\theta}_h = \frac{\theta_h}{\theta_{b,f}}, \quad \tilde{\omega}_m = \frac{\xi\omega_m}{\gamma\theta_{b,f}}, \tag{1}$$

where [1, Eqn. 14],

$$\gamma := \left(\frac{4P\xi}{I_d\theta_{b,f}^2}\right)^{\frac{1}{3}}. \tag{2}$$

We will use prime notation instead of a dot to denote time derivatives with respect to $\tilde{t}$, i.e. $()' := d/d\tilde{t}$,

$$\dot{\theta}_b = \gamma\theta_{b,f}\tilde{\theta}_b', \qquad \ddot{\theta}_b = \gamma^2\theta_{b,f}\tilde{\theta}_b''. \tag{3}$$

In the rescaled system, the dynamics are simply,

$$\tilde{\theta}'' = \begin{cases} \frac{1}{\tilde{\omega}_m}\left(1 - \frac{\tilde{\theta}'}{\tilde{\omega}_m}\right), & \text{for } 0 \leq \tilde{t} < \tilde{t}_s \\ -\frac{1}{\tilde{\omega}_m}, & \text{for } \tilde{t} \geq \tilde{t}_s. \end{cases} \tag{4}$$

Integrating from the initial conditions $\tilde{\theta}'(0) = 0$ and $\tilde{\theta}(0) = 0$ yields the flow over the acceleration phase,

$$\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}) = \tilde{\omega}_m\left(1 - \exp\left(\frac{-\tilde{t}}{\tilde{\omega}_m^2}\right)\right) \tag{5}$$

$$\tilde{\theta}(\tilde{\omega}_m, \tilde{t}) = \tilde{\omega}_m\tilde{t} - \tilde{\omega}_m^3\left(1 - \exp\left(\frac{-\tilde{t}}{\tilde{\omega}_m^2}\right)\right), \tag{6}$$

for $\tilde{t} < \tilde{t}_s$. The flow over deceleration is,

$$\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}) = \tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s) - \frac{1}{\tilde{\omega}_m}(\tilde{t} - \tilde{t}_s), \tag{7}$$

$$\tilde{\theta}(\tilde{\omega}_m, \tilde{t}) = (\tilde{t} - \tilde{t}_s)\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s) - \frac{1}{2\tilde{\omega}_m}(\tilde{t} - \tilde{t}_s)^2 + \tilde{\theta}(\tilde{\omega}_m, \tilde{t}_s,), \tag{8}$$

for $\tilde{t} > \tilde{t}_s$. The maneuver ends at a halting time $\tilde{t}_h = \tilde{t}_s + \tilde{t}_r$, when the body comes to rest. The duration of the braking phase, $\tilde{t}_r$, is the zero of (7), or equivalently the speed at the switch divided by the braking acceleration $(1/\tilde{\omega}_m)$,

$$\tilde{t}_r = \tilde{\omega}_m\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s). \tag{9}$$

The final body angle is thus an explicit function of the switching time and $\tilde{\omega}_m$, and can be written out by combining (5)–(9),

$$\begin{aligned} \tilde{\theta}_h &= \tilde{g}_\theta(\tilde{\omega}_m, \tilde{t}_s) := \tilde{\theta}(\tilde{t}_s + \tilde{g}_r(\tilde{\omega}_m, \tilde{\omega}_m, \tilde{t}_s)) \\ &= \tilde{t}_r\tilde{\theta}'(\tilde{\omega}_m, t_s,) - \frac{1}{2\tilde{\omega}_m}\tilde{t}_r^2 + \tilde{\theta}(\tilde{\omega}_m, t_s) \\ &= \frac{\tilde{\omega}_m}{2}(\tilde{\theta}'(\tilde{\omega}_m, t_s))^2 + \tilde{\theta}(\tilde{\omega}_m, t_s) \\ &= \frac{\tilde{\omega}_m}{2}\left(\tilde{\omega}_m - \tilde{\omega}_m\exp\left(\frac{-\tilde{t}_s}{\tilde{\omega}_m^2}\right)\right)^2 \\ &\qquad + \tilde{\omega}_m\tilde{t}_s - \tilde{\omega}_m^3 + \tilde{\omega}_m^3\exp\left(\frac{-\tilde{t}_s}{\tilde{\omega}_m^2}\right) \\ &= \tilde{\omega}_m\tilde{t}_s - \frac{\tilde{\omega}_m^3}{2}\left(1 - \exp\left(\frac{-2\tilde{t}_s}{\tilde{\omega}_m^2}\right)\right). \end{aligned} \tag{10}$$

The halting time is simply the sum of the switching time and the braking time,

$$\begin{aligned} \tilde{t}_h &= \tilde{g}_h(\tilde{\omega}_m, \tilde{t}_s) := \tilde{t}_s + \tilde{t}_r \\ &= \tilde{t}_s + \tilde{\omega}_m^2\left(1 - \exp\left(\frac{-\tilde{t}_s}{\tilde{\omega}_m^2}\right)\right) \end{aligned} \tag{11}$$

This work was supported in part by the ARL/GDRS RCTA and in part by the NSF under the CiBER-IGERT Award DGE-0903711 and the CABiR Award CDI-II 1028237.

T. Libby is with the Departments of Mechancial Engineering and Integrative Biology, University of California Berkeley, Berkeley, CA 94720 USA email: tlibby@berkeley.edu.

R.J. Full is with the Department of Integrative Biology, University of California Berkeley, Berkeley, CA 94720 USA email: rjfull@berkeley.edu.

A. M. Johnson is with the Department of Mechanical Engineering, Carnegie Mellon University, Pittsburgh, PA 15232 USA email: amj1@andrew.cmu.edu.

E. Chang-Siu is with the Department of Engineering Technology, California State University Maritime Academy, Vallejo, CA 94590 USA email: echangsiu@csum.edu.

D. E. Koditschek is with the Electrical and Systems Engineering Department, University of Pennsylvania, Philadelphia, PA 19104 USA email: kod@seas.upenn.edu.

Substituting the definitions of the rescaled coordinates, (1) and (2), into (10) and (11),

$$t_h = g_h(\mathbf{p}) := \frac{1}{\gamma}\tilde{g}_h\left(\frac{\xi\omega_m}{\gamma\theta_{b,f}}, \gamma t_s\right) \tag{12}$$
$$= t_s + \frac{I_b\xi\omega_m^2}{4P}\left(1 - \exp\left(-\frac{4P}{I_b\xi\omega_m^2}t_s\right)\right), \tag{13}$$

$$\theta_h = g_\theta(\mathbf{p}) := \theta_{b,f}\tilde{g}_\theta\left(\frac{\xi\omega_m}{\gamma\theta_{b,f}}, \gamma t_s\right) \tag{14}$$
$$= \xi\omega_m t_s - \frac{I_b\xi^2\omega_m^3}{8P}\left(1 - \exp\left(-\frac{8P}{I_b\xi\omega_m^2}t_s\right)\right). \tag{15}$$

as given in [1, Eqn. 9] and [1, Eqn. 10].

A. Analytic solution of template dynamics with a current limit

If the maximum allowable torque is limited to some factor $\beta \in (0,1)$ less than the stall torque of the motor, $\tau_\ell = \beta\tau_m$, the optimal reorientation consists of three phases: a constant torque phase until a time $\tilde{t}_\ell$ when the acceleration becomes voltage-limited, then a phase following the speed–torque curve of the motor until the controlled switch at $\tilde{t}_s$, followed by a constant braking torque phase of duration $\tilde{t}_r$ until $\tilde{t}_h$. In this case, the time-switched dynamics of (4) become instead,

$$\tilde{\theta}'' = \begin{cases} \frac{\beta}{\tilde{\omega}_m}, & \text{for } 0 \le \tilde{t} < \tilde{t}_\ell \\ \frac{1}{\tilde{\omega}_m}\left(1 - \frac{\tilde{\theta}'}{\tilde{\omega}_m}\right), & \text{for } \tilde{t}_\ell \le \tilde{t} < \tilde{t}_s \\ -\frac{\beta}{\tilde{\omega}_m}, & \text{for } \tilde{t} \ge \tilde{t}_s \end{cases} \tag{16}$$

The current limited acceleration flow is,

$$\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}, \beta) = \frac{\beta}{\tilde{\omega}_m}\tilde{t}$$
$$\tilde{\theta}(\tilde{\omega}_m, \tilde{t}, \beta) = \frac{\beta}{2\tilde{\omega}_m}\tilde{t}^2 \tag{17}$$

for $\tilde{t} < \tilde{t}_\ell$.

The transition to voltage-limited acceleration occurs at a time t_ℓ, when the current-limited torque equals the back-EMF limited torque,

$$\tilde{t}_\ell = \inf\left\{\tilde{t} > 0 \mid \frac{\beta}{\tilde{\omega}_m} = \frac{1}{\tilde{\omega}_m}\left(1 - \frac{\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}, \beta)}{\tilde{\omega}_m}\right)\right\} \tag{18}$$
$$= \frac{1-\beta}{\beta}\tilde{\omega}_m^2, \tag{19}$$

The transition state is thus an explicit function of β and $\tilde{\omega}_m$,

$$\tilde{\theta}'_\ell := \tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_\ell, \beta) = (1-\beta)\tilde{\omega}_m \tag{20}$$
$$\tilde{\theta}_\ell := \tilde{\theta}(\tilde{\omega}_m, \tilde{t}_\ell, \beta) = \frac{(1-\beta)^2}{2\beta}\tilde{\omega}_m^3. \tag{21}$$

With these initial conditions, (20)–(21), the voltage-limited dynamics admit the solution,

$$\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}, \beta) = \tilde{\omega}_m\left(1 - \beta\exp\left(\frac{-(\tilde{t} - \tilde{t}_\ell)}{\tilde{\omega}_m^2}\right)\right), \tag{22}$$
$$\tilde{\theta}(\tilde{\omega}_m, \tilde{t}, \beta) = \tilde{\theta}_\ell + \tilde{\omega}_m(\tilde{t} - \tilde{t}_\ell) - \beta\tilde{\omega}_m^3\left(1 - \exp\left(\frac{-(\tilde{t} - \tilde{t}_\ell)}{\tilde{\omega}_m^2}\right)\right), \tag{23}$$

for $\tilde{t}_\ell \le \tilde{t} \le \tilde{t}_s$. Finally, the flow over deceleration is,

$$\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}, \beta) = \tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s, \beta) - \frac{\beta}{\tilde{\omega}_m}(\tilde{t} - \tilde{t}_s), \tag{24}$$
$$\tilde{\theta}(\tilde{\omega}_m, \tilde{t}, \beta) = \tilde{\theta}(\tilde{\omega}_m, \tilde{t}_s, \beta) + (\tilde{t} - \tilde{t}_s)\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s, \beta) - \frac{\beta}{2\tilde{\omega}_m}(\tilde{t} - \tilde{t}_s)^2, \tag{25}$$

for $\tilde{t} > \tilde{t}_s$. The analysis follows similarly to the previous section, with the return time given by the function,

$$\tilde{t}_r = \tilde{g}_r(\tilde{\omega}_m, \tilde{t}_s, \beta) := \frac{\tilde{\omega}_m}{\beta}\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s, \beta), \tag{26}$$

the final time given by,

$$\tilde{t}_h = \tilde{g}_h(\tilde{\omega}_m, \tilde{t}_s, \beta) := \tilde{t}_s + \tilde{t}_r \tag{27}$$
$$= \tilde{t}_s + \frac{\tilde{\omega}_m^2}{\beta}\left(1 - \beta\exp\left(\frac{-(\tilde{t}_s - \tilde{t}_\ell)}{\tilde{\omega}_m^2}\right)\right) \tag{28}$$

and the explicit form of $\tilde{g}_\theta$,

$$\tilde{\theta}_h = \tilde{g}_\theta(\tilde{\omega}_m, \tilde{t}_s, \beta) := \tilde{\theta}(\tilde{\omega}_m, \tilde{t}_s + \tilde{g}_r(\tilde{\omega}_m, \tilde{t}_s), \beta) \tag{29}$$
$$= \tilde{\theta}(\tilde{\omega}_m, \tilde{t}_s, \beta) + \frac{\tilde{\omega}_m}{2\beta}(\tilde{\theta}'(\tilde{\omega}_m, \tilde{t}_s, \beta))^2 \tag{30}$$
$$= \tilde{\omega}_m\tilde{t}_s + \tilde{\omega}_m^3(\beta - 1)\exp\left(\frac{1-\beta}{\beta} - \frac{\tilde{t}_s}{\tilde{\omega}_m^2}\right) + \frac{\beta\tilde{\omega}_m^3}{2}\left(1 - \exp\left(\frac{2(1-\beta)}{\beta} - \frac{2\tilde{t}_s}{\tilde{\omega}_m^2}\right)\right) \tag{31}$$

Note that if $\tilde{\omega}_m$ is very large, the acceleration will be so slow that the system never reaches the speed-limited phase and the critical switching time $\tilde{t}_c \ge \tilde{t}_\ell$. In this case, the acceleration and braking phases are symmetric with equal durations and $\tilde{t}_h = 2\tilde{t}_s$. The condition for this behavior can be found by taking $\tilde{t}_s = \tilde{t}_\ell$ in (30) and is,

$$\tilde{\omega}_m \ge \left(\frac{\beta}{(1-\beta)^2}\right)^{\frac{1}{3}}. \tag{32}$$

The optimal gearing can be found by using the critical switching time, $\tilde{t}_c = \tilde{g}_c(\tilde{\omega}_m)$ and minimizing the final time $\tilde{t}_h = \tilde{g}_\theta(\tilde{\omega}_m, \tilde{g}_c(\tilde{\omega}_m))$, with no constraints on $\tilde{\omega}_m$ other than non-negative real. The halting time $\tilde{t}_h$ varies with β, thus varying the power constant k_p and speed constant k_s, as defined in [1, Sec. II-C1]. Optimal gear ratio is only weakly sensitive to current limit, varying less than 5% over the possible values of β (Fig. 1, middle). The required nominal power with the optimal gear ratio grows rapidly with decreasing current limit; limiting torque to 50% increases required nominal power by 53%, while a current limit of 25% nearly triples the required nominal power (Fig. 1, bottom).

The current-limited versions of the template behavior, [1, Eqn. 9] and [1, Eqn. 10], can be derived by substituting the definitions of the rescaled coordinates, (1), into (28) and (30) as in the previous subsection.

II. Alternate Controller Formulations

A. Event-based switching

The time-switched bang-bang controller of the previous section can be replaced by an event-based switch or guard

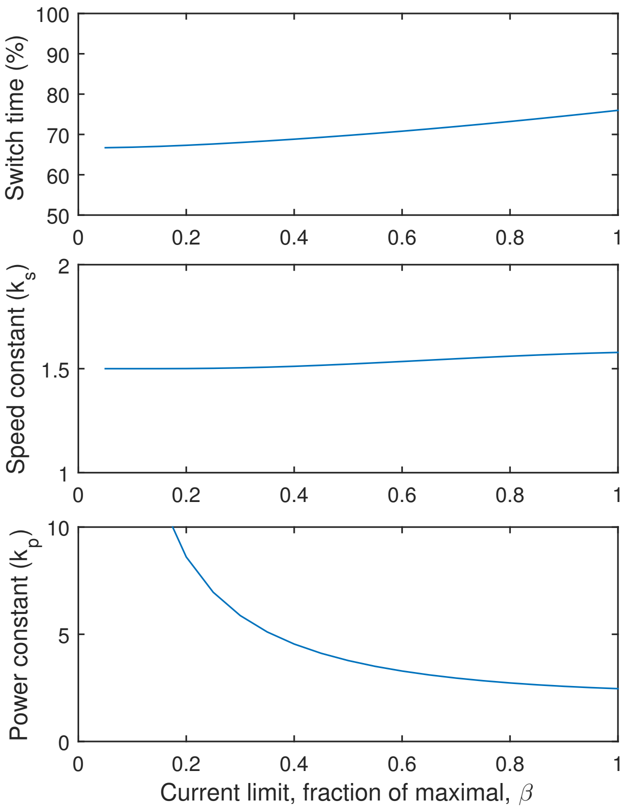

Fig. 1. Constrained switching time fraction, $\tilde{t}_c/\tilde{t}_f$, no-load speed ratio k_s, and power constant k_p for submaximal current limitation.

condition $G(\tilde{\theta}, \tilde{\theta}') = 0$. For example,

$$G_\theta := \tilde{\theta} - \tilde{\theta}_s, \tag{33}$$

where the value of $\tilde{\theta}_s$ is found easily from (6) (or (23) for the current-limited case). For the optimally geared case with $\beta = 1$, $\tilde{\theta}_s \approx 0.7$; this value changes for suboptimal gearing or $\beta < 1$ (Fig. 1, top).

B. Feedback controllers regulating body angle

For additional robustness, the template controller may use proportional-derivative (PD) feedback on the body angle (relative to the desired final position, $\theta_{b,f}$, and velocity, $\dot{\theta}_b = 0$). The controller torque takes the form,

$$\tau = K_p(\theta_{b,f} - \theta_b) + K_d(0 - \dot{\theta}_b), \tag{34}$$

subject to the limits imposed by the motor model. Given high enough gains, the torque will saturate, producing speed-limited acceleration and current-limited braking as in the switched case; the effective switching time (when $\tau = 0$) depends on the ratio of controller gains. Finding the ratio of gains corresponding to a particular value of $\mathbf{p}$ is easily done in the dimensionless system coordinates. Substituting $\tau = \ddot{\theta}_b/I_d$ and applying the spatiotemporal rescaling of (1) to (34), the dimensionless closed-loop dynamics are,

$$\theta'' = \tilde{K}_p(1 - \tilde{\theta}) + \tilde{K}_d(0 - \tilde{\theta}'), \tag{35}$$

where $\tilde{K}_p = K_p/(\gamma^2 I_d)$ and $\tilde{K}_d = K_d/(\gamma I_d)$. The closed loop dynamics are subject to the motor-imposed acceleration limits,

$$-\frac{\beta}{\tilde{\omega}_m} \leq \tilde{\theta}'' \leq \frac{\beta}{\tilde{\omega}_m}\left(1 - \frac{\tilde{\theta}'}{\tilde{\omega}_m}\right), \tag{36}$$

for $\tilde{\theta}' \geq 0$ (the condition for negative body velocity is found by multiplying the inequality by -1, but will never occur during the optimal reorientation).

Substituting expressions (5)–(6) for the state at the time of switch (where $\theta'' = 0$),

$$\begin{aligned} 0 &= \tilde{K}_p(1 - \tilde{\theta}(\tilde{t}_s, \tilde{\omega}_m)) - \tilde{K}_d\tilde{\theta}'(\tilde{t}_s, \tilde{\omega}_m) \\ \frac{\tilde{K}_d}{\tilde{K}_p} &= \frac{1 - \tilde{\theta}(\tilde{t}_s, \tilde{\omega}_m)}{\tilde{\theta}'(\tilde{t}_s, \tilde{\omega}_m)} \\ \frac{\tilde{K}_d}{\tilde{K}_p} &= \frac{1 - \tilde{\omega}_m\tilde{t}_c + \tilde{\omega}_m^3\left(1 - \exp\left(\frac{-\tilde{t}_c}{\tilde{\omega}_m^2}\right)\right)}{\tilde{\omega}_m\left(1 - \exp\left(\frac{-\tilde{t}_c}{\tilde{\omega}_m^2}\right)\right)}, \end{aligned} \tag{37}$$

for $\beta = 1$ (the expression for the gain ratio follows the above, substituting (24)–(25) instead). The critical value that produces the optimal switch is found by substituting the optimal no-load speed, $\tilde{\omega}_m^* \approx 0.74$, and corresponding switching time, $\tilde{t}_c^* \approx 1.63$ and has a value of $\tilde{K}_d/\tilde{K}_p \approx 0.26$. For current-limited dynamics, the ratio of gains increases with decreasing β.

When scaling back to physical torques, the gains will scale with I_d and γ as defined above, so the optimal ratio is,

$$\frac{K_d}{K_p} = \frac{\tilde{K}_d}{\gamma\tilde{K}_p} \approx 0.26\frac{1}{\gamma} \tag{38}$$

III. Derivation of tail connection field

The angular momentum of the system of rigid bodies can be found by adding the angular momentum of each body with respect to some point, O,

$$\mathbf{H_O} = \mathbf{H_{b,O}} + \mathbf{H_{t,O}}.$$

Let $\{\mathbf{E}_1, \mathbf{E}_2\}$ be the world reference frame in the plane, and define $\mathbf{E}_3 := \mathbf{E}_1 \times \mathbf{E}_2$ which exits the page. Let $\mathbf{r}_b$ and $\mathbf{r}_t$ be the position vectors relative to O of the body and tail, respectively (as in Fig. 2). Each link's angular momentum is the sum of its angular momentum about its own COM and its moment of linear momentum about O,

$$\mathbf{H_{i,O}} = I_i\dot{\theta}_i\mathbf{E}_3 + \mathbf{r}_i \times m_i\mathbf{v_i},$$

where the subscripts $i \in \{b, t\}$ denote the body and tail, respectively, and $\mathbf{v_i} := \mathbf{v_O} + \dot{\mathbf{r}}_i$ is the absolute velocity of each link. The total angular momentum of the two body system is,

$$\begin{aligned} \mathbf{H_O} &= \mathbf{H_{b,O}} + \mathbf{H_{t,O}} \\ &= (m_b\mathbf{r}_b + m_t\mathbf{r}_t) \times \mathbf{v_O} + \sum_{i\in\{b,t\}}\left(I_i\dot{\theta}_i\mathbf{E}_3 + \mathbf{r}_i \times m_i\dot{\mathbf{r}}_i\right). \end{aligned} \tag{39}$$

The centroid of the combined tailed-body mechanism with respect to O, denoted $\mathbf{r}_{com}$, is a weighted sum of the link positions,

$$\mathbf{r}_{com} = \frac{m_b\mathbf{r}_b + m_t\mathbf{r}_t}{m_b + m_t}. \tag{40}$$

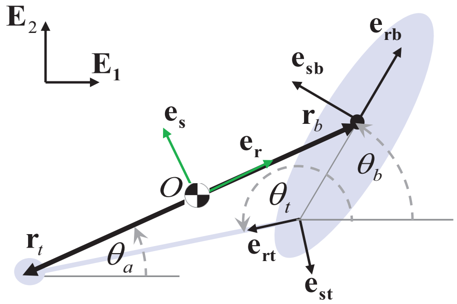

Fig. 2. Reference frames and coordinates.

Note that the first term in (39) is eliminated by choosing $\mathbf{r}_{com} = \mathbf{0}$ by placing O at the system COM; in this case, the angular momentum about O is invariant to system velocity.

Let $\{\mathbf{e}_r, \mathbf{e}_s\}$ be an orthonormal reference frame with $\mathbf{e}_r$ aligned with the vector connecting the tail COM to the body COM, and let θ_a be the angle of $\mathbf{e}_r$ with respect to the world reference frame, i.e. the frame is defined by a rotation of θ_a about $\mathbf{E}_3$,

$$\mathbf{e}_r := \cos\theta_a \mathbf{E}_1 + \sin\theta_a \mathbf{E}_2, \tag{41}$$
$$\mathbf{e}_s := -\sin\theta_a \mathbf{E}_1 + \cos\theta_a \mathbf{E}_2. \tag{42}$$

This frame enables a simple definition of the vectors from the system COM to the segment COMs,

$$\mathbf{r}_b = r\mathbf{e}_r; \qquad \mathbf{r}_t = -(l-r)\mathbf{e}_r. \tag{43}$$

The definition of the center of mass fixes r,

$$-m_t(l-r) + m_b r = 0 \qquad \Rightarrow \qquad r = \frac{m_t}{m_b+m_t} l. \tag{44}$$

Hence the body and tail vectors are related by,

$$\mathbf{r}_t = \frac{r-l}{r}\mathbf{r}_b = (1 - \frac{m_b+m_t}{m_t})\mathbf{r}_b = -\frac{m_b}{m_t}\mathbf{r}_b. \tag{45}$$

We can now simplify (39), the expression for total angular momentum,

$$\begin{aligned}\mathbf{H_O} &= (I_b\dot\theta_b + I_t\dot\theta_t)\mathbf{E}_3 + \mathbf{r}_b \times (m_b\dot{\mathbf{r}}_b) + \mathbf{r}_t \times (m_t\dot{\mathbf{r}}_t)\\ &= (I_b\dot\theta_b + I_t\dot\theta_t)\mathbf{E}_3 + \left(m_b + \frac{m_b^2}{m_t}\right)\mathbf{r}_b \times \dot{\mathbf{r}}_b.\end{aligned} \tag{46}$$

The last term of (46) describes the component of angular momentum due to the two point masses orbiting the COM. This cross product, derived below, is always perpendicular to the plane and has a relatively simple expression for its magnitude in terms of the body-fixed reference frame, $\{\mathbf{e}_{rb},\ \mathbf{e}_{sb}\}$, and the tail-fixed reference frame, $\{\mathbf{e}_{rt},\ \mathbf{e}_{st}\}$ (defined analogously to (41)–(42) as a rotation about $\mathbf{E}_3$ of θ_b and θ_t, respectively).

Equating two expressions for the vector from the pivot to the system COM,

$$\begin{aligned} l_b\mathbf{e}_{rb} - \mathbf{r}_b &= l_t\mathbf{e}_{rt} - \mathbf{r}_t\\ -\frac{m_b+m_t}{m_t}\mathbf{r}_b &= l_t\mathbf{e}_{rt} - l_b\mathbf{e}_{rb}\\ \mathbf{r}_b &= -\frac{m_t}{m_b+m_t}(l_t\mathbf{e}_{rt} - l_b\mathbf{e}_{rb}). \end{aligned} \tag{47}$$

The vector $\dot{\mathbf{r}}_b$ follows from time differentiation of $\mathbf{r}_b$,

$$\dot{\mathbf{r}}_b = -\frac{m_t}{m_b+m_t}(l_t\dot{\mathbf{e}}_{rt} - l_b\dot{\mathbf{e}}_{rb}) \tag{48}$$
$$= -\frac{m_t}{m_b+m_t}(l_t\dot\theta_t\mathbf{e}_{st} - l_b\dot\theta_b\mathbf{e}_{sb}). \tag{49}$$

Hence the final term in (46) becomes,

$$\frac{m_b m_t}{(m_b+m_t)}(l_t\mathbf{e}_{rt} - l_b\mathbf{e}_{rb}) \times (l_t\dot\theta_t\mathbf{e}_{st} - l_b\dot\theta_b\mathbf{e}_{sb}).$$

The mass coefficient is also known as the reduced mass,

$$m_r := \frac{m_b m_t}{(m_b+m_t)}. \tag{50}$$

Using the following identities,

$$(\mathbf{e}_{rt}\times\mathbf{e}_{sb}) = \cos\theta_r\mathbf{E}_3; \quad (\mathbf{e}_{rb}\times\mathbf{e}_{st}) = \cos\theta_r\mathbf{E}_3, \tag{51}$$

we can now evaluate the remaining cross product,

$$\begin{aligned}&(l_t\mathbf{e}_{rt} - l_b\mathbf{e}_{rb}) \times (l_t\dot\theta_t\mathbf{e}_{st} - l_b\dot\theta_b\mathbf{e}_{sb})\\ &= (l_t^2\dot\theta_t - l_b l_t\dot\theta_b\cos\theta_r - l_b l_t\dot\theta_t\cos\theta_r + l_b^2\dot\theta_b)\mathbf{E}_3\\ &= \big((l_t^2 - l_b l_t\cos\theta_r)\dot\theta_t + (l_b^2 - l_b l_t\cos\theta_r)\dot\theta_b\big)\mathbf{E}_3.\end{aligned}$$

As all terms of $\mathbf{H_O}$ are perpendicular to the plane, we drop the vector notation and simply examine the magnitude of the total angular momentum in this tail anchor, $H_{O,t}$, where $H_{O,t}\mathbf{E}_3 = \mathbf{H}_O$. With the coordinate substitution, $\theta_t = \theta_b + \theta_r$, and the simplification of the cross product,

$$\begin{aligned}H_{O,t} &= (I_b + m_r(l_b^2 - l_b l_t\cos\theta_r))\dot\theta_b\\ &\quad + (I_t + m_r(l_t^2 - l_b l_t\cos\theta_r))\dot\theta_t\\ &= (I_b + I_t + m_r(l_b^2 + l_t^2 - 2l_b l_t\cos\theta_r))\dot\theta_b\\ &\quad + (I_t + m_r(l_t^2 - l_b l_t\cos\theta_r))\dot\theta_r,\end{aligned}$$

as stated for the tail template kinematics, [1, Eqn. 30].

A. Restriction on domain of dimensionless parameters

Because of coupling between the dimensionless constants and the requirement of non-negativity of the dimensioned parameters, only a subset of the dimensionless parameter space is physically realizable. By definition, [1, Eqn. 32], ξ_t is restricted to the interval $[0,1]$, as the denominator is no smaller than the numerator and both are strictly positive. Furthermore, for a given value of ξ_t there is a maximum value of η. Starting with positivity of physical parameters,

$$\begin{aligned} 0 &< I_b I_t + I_b m_r l_t^2 + I_t m_r l_b^2\\ \frac{m_r^2 l_b^2 l_t^2}{(I_t + m_r l_t^2)^2} &< \frac{I_b I_t + I_b m_r l_t^2 + I_t m_r l_b^2 + m_r^2 l_b^2 l_t^2}{(I_t + m_r l_t^2)^2}\\ \eta^2 &< \frac{1-\xi_t}{\xi_t}, \qquad \eta < \sqrt{\frac{1-\xi_t}{\xi_t}}, \end{aligned} \tag{52}$$

that is η is bounded above as shown in gray in [1, Fig. 5]. Another bound used in the paper, ensuring positivity of the denominator of [1, Eqn. 35], may be found by starting with the positivity of physical parameters and of squared values,

$$\begin{aligned} 0 &< \frac{I_b + I_t + m_r(l_b - l_t)^2}{I_b + I_t + m_r(l_b^2 + l_t^2)}\\ 0 &< 1 - \frac{2m_r l_b l_t}{I_b + I_t + m_r(l_b^2 + l_t^2)}, \qquad 2\xi_t\eta < 1 \end{aligned} \tag{53}$$

B. Integration of the connection field

The total inertial effect of the tail over a given tail sweep, $\theta_r \in [\theta_{r1}, \theta_{r2}]$, is the integral of the connection magnitude $A(\theta_r)$, [1, Eqn. 35], over that stroke range. This integral can be written in closed form by first factoring the connection,

$$\begin{aligned} A(\theta_r) &= \xi \frac{1-\eta\cos\theta_r}{1-2\xi\eta\cos\theta_r} \\ &= \frac{1}{2}\frac{2\xi - 2\xi\eta\cos\theta_r + 1 - 1}{1-2\xi\eta\cos\theta_r} \\ &= \frac{1}{2}\left(\frac{2\xi-1}{1-2\xi\eta\cos\theta_r}+1\right) \\ &= \frac{1}{2} + \frac{2\xi-1}{4\xi\eta}\frac{1}{a-\cos\theta_r}, \quad a := \frac{1}{2\xi\eta}. \end{aligned}$$

The change in body angle, $\Delta\theta_b := \theta_{b,f} - \theta_{b,0}$, over a given tail sweep, $\Delta\theta_r := \theta_{r2} - \theta_{r1}$, is then,

$$\begin{aligned} \Delta\theta_b &= -\int_{\theta_{r1}}^{\theta_{r2}} A(\theta_r)\, d\theta_r & (54) \\ &= -\frac{\Delta\theta_r}{2} + \frac{1-2\xi}{4\xi\eta}\int_{\theta_{r1}}^{\theta_{r2}} \frac{d\theta_r}{a-\cos\theta_r}. & (55) \end{aligned}$$

The remaining integral can be simplified by way of the substitution, $t := \tan\frac{\theta_r}{2}$, and the following identities,

$$\begin{aligned} 2\arctan t &= \theta_r \\ d\theta_r &= \frac{2}{1+t^2}dt \\ \cos\frac{\theta_r}{2} &= \frac{1}{\sqrt{1+t^2}} \\ \sqrt{\frac{1+\cos\theta_r}{2}} &= \frac{1}{\sqrt{1+t^2}} \\ \cos\theta_r &= \frac{2}{1+t^2} - 1 \\ \cos\theta_r &= \frac{2}{1+t^2} - \frac{1+t^2}{1+t^2} \\ \cos\theta_r &= \frac{1-t^2}{1+t^2}. \end{aligned}$$

Making the substitutions, the integral in (55) simplifies to,

$$\begin{aligned} \int \frac{d\theta_r}{a-\cos\theta_r} &= \int \frac{1}{a - \frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt \\ &= \int \frac{2}{a(1+t^2)-(1-t^2)}dt \\ &= \int \frac{2}{(a-1)+(a+1)t^2}dt \\ &= \frac{2}{(a+1)}\int \frac{1}{b^2+t^2}dt, \quad b^2 := \frac{a-1}{a+1} \\ &= \frac{2}{a+1}\left(\frac{1}{b}\arctan\frac{t}{b} + C_1\right) \\ &= \frac{2}{b(a+1)}\arctan\left(\frac{\tan\left(\frac{\theta_r}{2}\right)}{b}\right) + C_2. \end{aligned}$$

For the sake of space, define the function,

$$\begin{aligned} R(\theta_i) &:= \arctan\left(\frac{\tan\left(\frac{\theta_i}{2}\right)}{b}\right) & (56) \\ &= \arctan\left(\sqrt{\frac{a+1}{a-1}}\tan\left(\frac{\theta_i}{2}\right)\right) \\ &= \arctan\left(\sqrt{\frac{1+2\xi\eta}{1-2\xi\eta}}\tan\left(\frac{\theta_i}{2}\right)\right). \end{aligned}$$

Returning to the expression for body stroke, (55),

$$\Delta\theta_b = -\frac{\Delta\theta_r}{2} + \frac{1-2\xi}{\sqrt{1-(2\xi\eta)^2}}\big(R(\theta_{r2}) - R(\theta_{r1})\big). \quad (57)$$

IV. Derivation of Equations of Motion for a Tailed System

Equipped with the kinematic results of [1, Sec. II], the balance of angular momentum for a general tailed system about the COM of each body is (see Fig. 3),

$$\begin{aligned} \dot{\mathbf{H}}_b &= \tau\mathbf{E}_3 + (-l_b\mathbf{e}_{rb}) \times \mathbf{F}_p, & (58) \\ \dot{\mathbf{H}}_t &= -\tau\mathbf{E}_3 + (-l_t\mathbf{e}_{rt}) \times (-\mathbf{F}_p), & (59) \end{aligned}$$

where τ denotes the torque output of the power train, and $\mathbf{F}_p$ is the pin constraint force (see Fig. 3). Since both the body and the COM frame are subject to the same gravitational acceleration, the force of gravity does not appear in the pin force, which is simply $\mathbf{F}_p = m_b\ddot{\mathbf{r}}_b$. The body acceleration relative to the COM is found by differentiating (49),

$$\ddot{\mathbf{r}}_b = -\frac{m_t}{m_b+m_t}(l_t\ddot{\theta}_t\mathbf{e}_{st} - l_t\dot{\theta}_t^2\mathbf{e}_{rt} - l_b\ddot{\theta}_b\mathbf{e}_{sb} + l_b\dot{\theta}_b^2\mathbf{e}_{rb}).$$

Substituting into (58) yields,

$$\begin{aligned} \dot{\mathbf{H}}_b &= \tau\mathbf{E}_3 - l_b\mathbf{e}_{rb} \times m_b\ddot{\mathbf{r}}_b \\ I_b\ddot{\theta}_b\mathbf{E}_3 &= \tau\mathbf{E}_3 + m_r l_b\mathbf{e}_{rb}\times \\ &\quad (l_t\ddot{\theta}_t\mathbf{e}_{st} - l_t\dot{\theta}_t^2\mathbf{e}_{rt} - l_b\ddot{\theta}_b\mathbf{e}_{sb} + l_b\dot{\theta}_b^2\mathbf{e}_{rb}). \end{aligned}$$

Using the identities (51) from Section III, above, along with

$$(\mathbf{e}_{rb} \times \mathbf{e}_{rt}) = \sin\theta_r\mathbf{E}_3; \quad (60)$$

to evaluate the cross products, collecting terms and dropping the vector notation (as all terms are aligned with $\mathbf{E}_3$) we arrive at the equation of motion for the body link,

$$(I_b + m_r l_b^2)\;\ddot{\theta}_b = \tau + m_r l_b l_t(\cos\theta_r\ddot{\theta}_t - \sin\theta_r\dot{\theta}_t^2) \quad (61)$$

Following the same procedure for the tail,

$$\begin{aligned} \dot{\mathbf{H}}_t &= -\tau\mathbf{E}_3 + l_t\mathbf{e}_{rt} \times m_b\ddot{\mathbf{r}}_b \\ I_t\ddot{\theta}_t\mathbf{E}_3 &= -\tau\mathbf{E}_3 - m_r l_t\mathbf{e}_{rt}\times \\ &\quad (l_t\ddot{\theta}_t\mathbf{e}_{st} - l_t\dot{\theta}_t^2\mathbf{e}_{rt} - l_b\ddot{\theta}_b\mathbf{e}_{sb} + l_b\dot{\theta}_b^2\mathbf{e}_{rb}) \\ (I_t + m_r l_t^2)\ddot{\theta}_t &= -\tau + m_r l_b l_t(\cos\theta_r\ddot{\theta}_b + \sin\theta_r\dot{\theta}_b^2). \quad (62) \end{aligned}$$

the equations of motion for the full nonlinear system are,

$$\mathbf{M}(\theta_r)\begin{bmatrix} \ddot{\theta}_b \\ \ddot{\theta}_t \end{bmatrix} + \begin{bmatrix} m_r l_b l_t \sin\theta_r\dot{\theta}_t^2 \\ -m_r l_b l_t \sin\theta_r\dot{\theta}_b^2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\tau \quad (63)$$

with a mass matrix,

$$\mathbf{M}(\theta_r) = \begin{bmatrix} I_b + m_r l_b^2 & -m_r l_b l_t\cos\theta_r \\ -m_r l_b l_t\cos\theta_r & I_t + m_r l_t^2 \end{bmatrix}, \quad (64)$$

as claimed in [1, Eqn. 38] and [1, Eqn. 39].

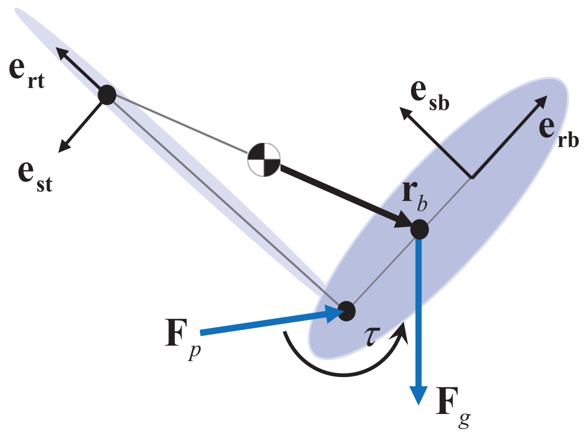

Fig. 3. Free body diagram for derivation of equations of motion.

A. Nondimensionalization of nonlinear tail dynamics

The equations of motion [1, Eqn. 38] and [1, Eqn. 39] can be written in the generalized coordinates (θ_b, θ_r) by substituting for $\theta_t = \theta_b + \theta_r$ and applying the change of basis to [1, Eqn. 38],

$$\mathbf{M}(\theta_r)\begin{bmatrix}\ddot{\theta}_b\\ \ddot{\theta}_r\end{bmatrix} + \begin{bmatrix} m_r l_b l_t \sin\theta_r(2\dot{\theta}_b\dot{\theta}_r + \dot{\theta}_r^2)\\ -m_r l_b l_t \sin\theta_r \dot{\theta}_b^2\end{bmatrix} = \begin{bmatrix}0\\ -1\end{bmatrix}\tau, \tag{65}$$

with a mass matrix,

$$\mathbf{M}(\theta_r) = \begin{bmatrix} I_b + I_t + m_r(l_t^2 + l_b^2 - 2l_b l_t\cos\theta_r) & I_t + m_r l_t^2 - m_r l_b l_t \cos\theta_r \\ I_t + m_r l_t^2 - m_r l_b l_t\cos\theta_r & I_t + m_r l_t^2\end{bmatrix}.$$

Following the process of [1, Sec. II-B], we substitute the template motor model for the torque and the scaling factors from the template, [1, Eqn. 14], along with a new scaling for the relative angle, $\theta_r' := \dot{\theta}_r/\gamma$ (note that unlike for θ_b, we do not normalize for final position). Normalizing by $\frac{\xi_t}{1-\xi_t}(I_b + m_r l_b^2)$, we define the dimensionless mass matrix,

$$\widetilde{\mathbf{M}}(\theta_r) = \begin{bmatrix} \frac{1-\xi_t}{\xi_t} + 1 - 2\eta\cos\theta_r & 1 - \eta\cos\theta_r \\ 1-\eta\cos\theta_r & 1\end{bmatrix},$$

and the dimensionless Coriolis terms,

$$\widetilde{C}(\theta_r, \tilde{\theta}', \theta_r') = \eta\theta_{b,f}\sin\theta_r\begin{bmatrix}\frac{2\tilde{\theta}'\theta_r'}{\theta_{b,f}} + (\frac{\theta_r'}{\theta_{b,f}})^2 \\ -(\tilde{\theta}')^2\end{bmatrix}, \tag{66}$$

resulting in dimensionless system dynamics,

$$\mathbf{M}(\tilde{\theta}_r)\begin{bmatrix}\tilde{\theta}'' \\ \frac{1}{\theta_{b,f}}\theta_r''\end{bmatrix} + \widetilde{C}(\theta_r, \tilde{\theta}', \theta_r') = \begin{bmatrix}0\\ -1\end{bmatrix}\frac{(1-\xi_t)\tilde{\tau}}{\xi_t}, \tag{67}$$

with

$$\tilde{\tau} = \frac{1}{\tilde{\omega}_m}\left(1 - \frac{\xi_t\theta_r'}{\theta_{b,f}\tilde{\omega}_m}\right) \tag{68}$$

during acceleration, and $\tilde{\tau} = 1/\tilde{\omega}_m$ during braking.

V. Derivation of the connection for assemblage of limbs

Here we consider a simplified case, where all appendages are parallel (but potentially out of phase by 180°, as in RHex's alternating tripod gait), and the N limbs are arranged with pivots along the centerline of the robot's body (along which the body's COM also falls). Again, the limbs are driven by a high-gain synchronizing control such that all N legs share the same angle θ_t, modulo the phasing noted above.

Using the same reference frames from the tail case, Section III, above, let $\mathbf{e}_{rb}$ be the vector parallel to the body axis, and $\mathbf{e}_{rt}$ be the vector to which all limbs are parallel. Denote the vector from body COM to the ith pivot by,

$$\mathbf{p}_i := \ell_i\mathbf{e}_{rb}, \tag{69}$$

and the vector from pivot to appendage COM by,

$$\mathbf{t}_i := s_i l_i \mathbf{e}_{rt}, \tag{70}$$

where ℓ_i is the position of the pivot along the body (ℓ is negative for pivots behind the body COM), l_i is the length of the ith limb, and $s_i := \pm 1$ is negative for legs out of phase with $\mathbf{e}_{rt}$ by π. The vector from system COM to appendage COM is,

$$\mathbf{r}_i := \mathbf{r}_b + \mathbf{p}_i + \mathbf{t}_i = \mathbf{r}_b + \ell_i\mathbf{e}_{rb} + s_i l_i\mathbf{e}_{rt}, \tag{71}$$

and the relation between system COM and segment COMs is,

$$m_{tot}\mathbf{r}_{com} = m_b\mathbf{r}_b + \sum_{i=1}^{N} m_i\mathbf{r}_i, \tag{72}$$

where m_i is the mass of the ith appendage, and $m_{tot} := m_b + \sum_{i=1}^{N} m_i$ is the total system mass. Placing the origin at the system COM ($\mathbf{r}_{com} = \mathbf{0}$) and solving for $\mathbf{r}_b$,

$$\mathbf{0} = m_b\mathbf{r}_b + \sum_{i=1}^{N} m_i(\mathbf{r}_b + \ell_i\mathbf{e}_{rb} + s_i l_i\mathbf{e}_{rt}) \tag{73}$$

$$m_{tot}\mathbf{r}_b = -\sum_{i=1}^{N} m_i(\ell_i\mathbf{e}_{rb} + s_i l_i\mathbf{e}_{rt}) \tag{74}$$

$$\mathbf{r}_b = -\frac{1}{m_{tot}}\left(\mathbf{e}_{rb}\sum_{i=1}^{N} m_i\ell_i + \mathbf{e}_{rt}\sum_{i=1}^{N} m_i s_i l_i\right). \tag{75}$$

If $\sum_{i=1}^{N} m_i\ell_i = 0$ (that is, the mass-weighted pivot distances from body COM are symmetric), then $\mathbf{r}_b$ is strictly parallel to $\mathbf{e}_{rt}$,

$$\mathbf{r}_b = c\,\mathbf{e}_{rt}; \qquad c := -\frac{1}{m_{tot}}\sum_{i=1}^{N} m_i s_i l_i, \tag{76}$$

and the vector to the ith appendage COM simplifies to,

$$\mathbf{r}_i = \ell_i\mathbf{e}_{rb} + (c + s_i l_i)\mathbf{e}_{rt}. \tag{77}$$

The connection can be derived from the total angular momentum; extending (39) to multiple appendages,

$$\mathbf{H}_{O,l} = I_b\dot{\theta}_b\mathbf{E}_3 + \mathbf{r}_b \times (m_b\dot{\mathbf{r}}_b) + \sum_{i=1}^{N}\left(I_i\dot{\theta}_t\mathbf{E}_3 + \mathbf{r}_i \times (m_i\dot{\mathbf{r}}_i)\right). \tag{78}$$

The moment of linear momentum due to the body mass can be simplified using (76),

$$\begin{aligned}\mathbf{r}_b \times m_b\dot{\mathbf{r}}_b &= c\,\mathbf{e}_{rt} \times m_b c\,\mathbf{e}_{st}\\ &= m_b c^2\dot{\theta}_t\mathbf{E}_3.\end{aligned}$$

The moment of linear momentum due to each appendage can be simplified using (77),

$$\begin{aligned}&\mathbf{r}_i \times m_i\dot{\mathbf{r}}_i\\ &= m_i\Big(\ell_i\mathbf{e}_{rb} + (c+s_il_i)\mathbf{e}_{rt}\Big) \times \Big(\ell_i\dot{\theta}_b\mathbf{e}_{sb} + (c+s_il_i)\dot{\theta}_t\mathbf{e}_{st}\Big)\\ &= m_i\Big(\ell_i^2\dot{\theta}_b + \ell_i(c+s_il_i)(\dot{\theta}_b+\dot{\theta}_t)\cos\theta_r + (c+s_il_i)^2\dot{\theta}_t\Big)\mathbf{E}_3.\end{aligned}$$

With these simplifications, the magnitude of the angular momentum, (78), in the $\mathbf{E}_3$ direction, $H_{O,l}\mathbf{E}_3 := \mathbf{H}_{O,l}$, is,

$$\begin{aligned}H_{O,l} = I_b\dot{\theta}_b + m_bc^2\dot{\theta}_t + \sum_{i=1}^{N}\Big(I_i\dot{\theta}_t + m_i\Big(\ell_i^2\dot{\theta}_b\\ + (c+s_il_i)^2\dot{\theta}_t + \ell_i(c+s_il_i)(\dot{\theta}_b+\dot{\theta}_t)\cos\theta_r\Big)\Big),\end{aligned}$$

where the only remaining configuration dependent term is,

$$\sum_{i=1}^{N} m_i\ell_i(c+s_il_i)(\dot{\theta}_b+\dot{\theta}_t)\cos\theta_r,$$

and hence one criterion for configuration independence is,

$$\sum_{i=1}^{N} m_i\ell_i(c+s_il_i) = 0. \tag{79}$$

This is satisfied if all appendages have equal length, l_i, and phase, s_i, (as when all six of XRL's legs share the same angle) and if $\sum_{i=1}^{N} m_i\ell_i = 0$ (as required for the simplification of $\mathbf{r}_b$). Note that if an assemblage of N appendages satisfy this condition, then the addition of an appendage with $\ell_i = 0$ will result in an assemblage of $N+1$ appendages that will satisfy this condition as well.

For limb systems that satisfy (79), the magnitude of the angular momentum, (78), in the $\mathbf{E}_3$ direction simplifies to,

$$\begin{aligned}H_{O,l} = \Big(I_b + \sum_{i=1}^{N} m_i\ell_i^2\Big)\dot{\theta}_b + \qquad\qquad\qquad (80)\\ \Big(m_bc^2 + \sum_{i=1}^{N}\Big(I_i + m_i\Big(s_il_i - \frac{\sum_{j=1}^{N} m_js_jl_j}{m_{tot}}\Big)^2\Big)\Big)\dot{\theta}_t.\end{aligned}$$

If, further, all legs have identical mass, length, and inertia, which we will call m_t, l_t and I_t for comparison with the tail anchor, and the pivot locations are symmetric across the body centerline, i.e. $\sum \ell_i = 0$),

$$\begin{aligned}H_{O,l} = (I_b + m_t\sum_{i=1}^{N}\ell_i^2)\dot{\theta}_b + NI_t\dot{\theta}_t + \qquad (81)\\ m_tl_t^2\Big(\frac{m_bm_t}{m_{tot}^2}\Big(\sum_{i=1}^{N}s_i\Big)^2 + \sum_{i=1}^{N}\Big(s_i - \frac{m_t\sum_{j=1}^{N}s_j}{m_{tot}}\Big)^2\Big)\dot{\theta}_t.\end{aligned}$$

To simplify further, assume first that $\sum s_i = 0$,

$$H_{O,l} = (I_b + m_t\sum_{i=1}^{N}\ell_i^2)\dot{\theta}_b + N(I_t + m_tl_t^2)\dot{\theta}_t. \tag{82}$$

which, after a change of coordinates to (θ_b, θ_r), is as claimed in [1, Eqn. 51]. If instead $\sum s_i = N$,

$$\begin{aligned}H_{O,l} &= (I_b + m_t\sum_{i=1}^{N}\ell_i^2)\dot{\theta}_b + NI_t\dot{\theta}_t + \qquad (83)\\ &m_tl_t^2\Big(\frac{m_bm_t}{m_{tot}^2}N^2 + \sum_{i=1}^{N}\Big(\frac{m_b + Nm_t}{m_{tot}} - \frac{m_tN}{m_{tot}}\Big)^2\Big)\dot{\theta}_t.\\ &= (I_b + m_t\sum_{i=1}^{N}\ell_i^2)\dot{\theta}_b + NI_t\dot{\theta}_t + \qquad (84)\\ &\frac{m_tm_b}{m_{tot}}l_t^2N\Big(\frac{Nm_t + m_b}{m_{tot}}\Big)\dot{\theta}_t.\end{aligned}$$

which, after a change of coordinates to (θ_b, θ_r) and substituting the definition of m_{rt}, is as claimed in [1, Eqn. 52].

References

[1] T. Libby, A. M. Johnson, E. Chang-Siu, *et al.*, "Comparative design, scaling, and control of appendages for inertial reorientation," *IEEE Transactions on Robotics*, vol. 32, no. 6, pp. 1380–1398, 2016.