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Economics Of Non-Neutrality In The Internet

Abstract

Net-neutrality on the Internet is the set of policies that prevents a paid or unpaid discrimination by Internet Service Providers (ISPs) among different types of transmitted data. The recent moves to change the net neutrality rules and the growing demand for data have driven the ISPs to provide differential treatment of traffic to generate additional revenue streams from Content Providers (CPs). In this thesis, we consider economic frameworks to investigate different questions about the departure toward a non-neutral regime and its possible consequences. In particular, we i) assess whether different entities of the market have the incentive to adopt a non-neutral pricing scheme; and if yes ii) what are the pricing strategies they choose; and iii) how these changes affect the Internet market. First, we investigate the incentives of different entities of the Internet market for migrating to a non-neutral regime. Thus, we consider early stages of a non-neutral Internet. We consider a diverse set of parameters for the market, e.g. market powers of ISPs, sensitivity of EUs and CPs to the quality of the content. The goal is to obtain founded insights on whether there exists a market equilibrium, the structure of the equilibria, and how they depend on different parameters of the market when the current equilibrium (neutral regime) is disrupted and some ISPs have switched to a non-neutral regime. Then, we seek to investigate frameworks using which ISPs and CPs select appropriate incentives for each other, and investigate the implications of these new schemes on the entities of the Internet market. We analyze two non-neutral frameworks. In the first framework, we focus on the price competition between ISPs in the presence of uncertainty in competition and demand when CPs, i.e. demand, is merely price taker, i.e. passive in equilibrium selection. Then, in the second framework, we consider the case in which CPs have an active role in the market, and decide on the number of resources they want to reserve/buy from ISPs based on the price ISPs quote. In this case, we also consider the coupling between limited resources and the quality of the content delivered to end-users and subsequently the strategies of the decision makers. We obtain strategies for ISPs and CPs under a variety of market dynamics.

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Mohammad Hassan Lotfi

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Dedicated to my parents

ABSTRACT

ECONOMICS OF NON-NEUTRALITY IN THE INTERNET

Mohammad Hassan Lotfi

Saswati Sarkar

Net-neutrality on the Internet is the set of policies that prevents a paid or unpaid discrimination by Internet Service Providers (ISPs) among different types of transmitted data. The recent moves to change the net neutrality rules and the growing demand for data have driven the ISPs to provide differential treatment of traffic to generate additional revenue streams from Content Providers (CPs). In this thesis, we consider economic frameworks to investigate different questions about the departure toward a non-neutral regime and its possible consequences. In particular, we i) assess whether different entities of the market have the incentive to adopt a non-neutral pricing scheme; and if yes ii) what are the pricing strategies they choose; and iii) how these changes affect the Internet market. First, we investigate the incentives of different entities of the Internet market for migrating to a non-neutral regime. Thus, we consider early stages of a non-neutral Internet. We consider a diverse set of parameters for the market, e.g. market powers of ISPs, sensitivity of EUs and CPs to the quality of the content. The goal is to obtain founded insights on whether there exists a market equilibrium, the structure of the equilibria, and how they depend on different parameters of the market when the current equilibrium (neutral regime) is disrupted and some ISPs have switched to a non-neutral regime. Then, we seek to investigate frameworks using which ISPs and CPs select appropriate incentives

for each other, and investigate the implications of these new schemes on the entities of the Internet market. We analyze two non-neutral frameworks. In the first framework, we focus on the price competition between ISPs in the presence of uncertainty in competition and demand when CPs, i.e. demand, is merely price taker, i.e. passive in equilibrium selection. Then, in the second framework, we consider the case in which CPs have an active role in the market, and decide on the number of resources they want to reserve/buy from ISPs based on the price ISPs quote. In this case, we also consider the coupling between limited resources and the quality of the content delivered to end-users and subsequently the strategies of the decision makers. We obtain strategies for ISPs and CPs under a variety of market dynamics.

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Chapter 1

Introduction and Overview

1.1 Network Neutrality and Pricing of the Internet

1.1.1 Net Neutrality

Net-neutrality on the Internet is the set of policies that prevents a paid or unpaid discrimination by Internet Service Providers (ISPs) among different types of transmitted data. This precludes ISPs from charging Content Providers (CPs) to carry their data to the End-Users (EUs) in the last-mile. Since January 2014, when a federal appeals court struck down parts of the Federal Communication Commission's (FCC) rules for net-neutrality [65], the net-neutrality debate has received more attention. In February 2015, the FCC reclassified the Internet as a utility [55], providing the grounds for this agency to secure even stricter net-neutrality rules. In Europe, in October 2015, the European parliament has rejected legal amendments for strict net-neutrality rules, and passed a set of rules that allow for sponsored data plans and Internet fast lanes for "specialized services" [61].

Further actions, from ISPs and Content Providers (CPs), are expected, since both

may have incentives to adopt a non-neutral regime: the growing demand for data and the saturating revenue of ISPs have driven them to provide differential treatment of traffic to generate additional revenue streams from CPs. In addition to the ISPs generating revenue from the CPs, with such a model, CPs can ensure the quality of service they provide for their end-users particularly when resources are scarce such as in wireless networks. Offering a *sponsored data plan* by At&T in 2014 is an instance of departure toward a non-neutral regime. In this plan, AT&T allows CPs to pay for the data bytes that their users consume, thereby not eating into the users' data quota.

In addition, net-neutrality rules are often considered to be vague. For example, in February 2014, Comcast and Netflix negotiated a contract in which Netflix would pay Comcast for a faster access to Comcast's subscribers [66]. Both parties announced that the contract is a peering agreement, and its goal is to resolve the traffic imbalance. However, after deploying the agreement, the average Netflix download speed improved significantly [59]. Note that a contract for resolving *aggregate* traffic imbalance at tier-1 ties (particularly between an "eyeball" ISP and one serving a CP) in which the party receiving the net traffic imbalance get paid is considered "neutral" [31, 28]. Thus, although the Netflix-Comcast deal does not violate the net-neutrality rules, it has a non-neutral outcome of a side-payment between a residential ISP and a CP. This reveals a net-neutrality loophole at tier-1 ties Service Level Agreements (SLAs).

1.1.2 Pricing of the Internet

New pricing schemes in the Internet market either target end-users or CPs. For the end-user side, different pricing schemes have been proposed to replace the traditional flat rate

pricing [58], [17], [34]. These schemes can create additional revenue for SPs and provide a more flexible data plan for end-users. However, SPs are reluctant in adopting such pricing schemes due to the fact that these schemes are typically not user-friendly. Thus, SPs mainly focus on changing the pricing structure of the CP side, for which they should deal with net-neutrality rules.

Thus, and given the potential changes in net-neutrality policies in near future, the pricing schemes on the Internet market are gradually departing from a one-sided pricing to a two-sided scheme in which local ISPs which own the last mile charge both sides of the network, i.e. CPs and end-users. Therefore, new regulations may fundamentally alter the flow of the money and services among different stakeholders of the Internet market. A schematic view of the two-sided pricing scheme on the Internet is presented in Figure 1.1.

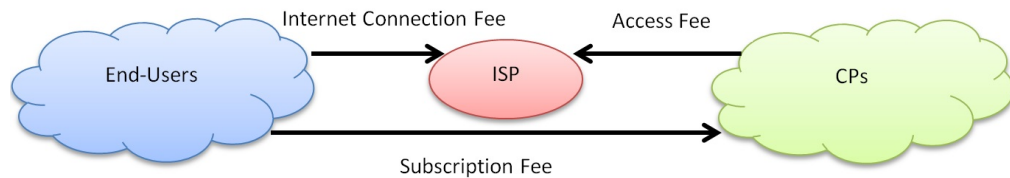


Figure 1.1: Schematic view of the two-sided pricing scheme on the Internet

1.2 Summary of Contributions

It has almost been a decade that the advantages and disadvantages of the Internet non-neutrality have been put on debate. Proponents of Net-Neutrality claim that non-neutrality kills the innovation on the CP side, decreases the competition among CPs, and undermines the so called “free Internet”. On the other hand, those who advocate relaxing

the net-neutrality rules claim that the Internet neutrality, as it is perceived commonly, is a barrier to further developments on the Internet since it decreases the incentives of ISPs, as the pipe-line holders of the Internet, to invest on their infra-structure. This debate has mostly been conducted on a qualitative level without rigorous economic and technical analyzes. A survey of the works on economic analysis of the net-neutrality debate is presented in [57].

In this dissertation, we consider economic frameworks to investigate different questions about the departure toward a non-neutral regime and its possible consequences. In particular, we i) assess whether different entities of the market have the incentive to adopt a non-neutral pricing scheme; and if yes ii) what are the pricing strategies they choose; and iii) how these changes affect the Internet market.

We first consider the migration to a non-neutral Internet and its consequences on different entities of the Internet market. Then, we consider and analyze two different non-neutral frameworks for a non-neutral Internet market. In particular:

- In Chapter 2, we investigate the incentives of different entities of the Internet market for migrating to a non-neutral regime. Thus, we consider early stages of a non-neutral Internet. We model the interaction between ISPs and CPs in a non-neutral regime in the presence of *asymmetric* competition between ISPs when some of the ISPs are non-neutral and some are neutral. In addition, we consider CPs that can differentiate between ISPs by controlling the quality of the content they are offering on each one. We consider a diverse set of parameters for the market, e.g. market powers of ISPs, sensitivity of EUs and CPs to the quality of the content. The

goal is to obtain founded insights on whether there exists a market equilibrium, the structure of the equilibria, and how they depend on different parameters of the market when the current equilibrium (neutral regime) is disrupted and some ISPs have switched to a non-neutral regime. Insights from our work can be used by the regulator in designing efficient rules for the Internet market.

Adopting non-neutrality gives rise to new distinctive environments where ISPs can increase their profit by pricing the service they provide for the CPs, and CPs can provide their users a better reception quality in exchange of monetary incentives for ISPs. We subsequently seek to investigate frameworks using which ISPs and CPs select appropriate incentives for each other (Figure 1.1), and investigate the implications of these new schemes on the entities of the Internet market. Thus, we consider a market consisting of ISPs, CPs, and end-users in which ISPs sell the bandwidth to CPs in exchange of financial incentives:

- In Chapter 3, we study the case in which ISPs compete with each other to set appropriate prices for CPs to sell/rent their bandwidth where the competition and demand are uncertain. In this case, CPs have a passive role, in the sense that they cannot alter their demand in accordance with the price set by ISPs. However, CPs have the ability to choose amongst the ISPs based on their price.
- In Chapter 4, we consider the case in which CPs have an active role in the market, and decide on the number of resources they want to reserve/buy from the ISPs based on the price ISPs quote. In addition, we consider the coupling between limited resources and the strategies of the decision makers. We obtain strategies for

ISPs and CPs under a variety of market dynamics.

We now present and motivate the problems in more detail, and summarize our contributions and related literature corresponding to each case:

1.3 Migration to a Non-Neutral Internet

1.3.1 The Research Challenges and Goals

Note that in reality, at initial stages of migration to a non-neutral regime, *some ISPs would adopt a non-neutral regime before others*. Thus, we need to consider a model in which some of the ISPs are neutral and some are non-neutral. We consider the market with two ISPs, one neutral and one non-neutral. This can represent two *groups* of ISPs, neutral and non-neutral, that are competing against each other. We also consider a “big” CP with high market power that chooses her strategies to influence the equilibrium outcome of the market. All other CPs are considered to be passive in the equilibrium selection process, and their effects are modeled using a common factor in the utility of End-Users (EUs). In addition, we consider a continuum of EUs that decide on the ISP they want to buy Internet subscription from. We assume that EUs have different levels of innate preferences for each ISP which can be because of initial set-up costs of a new service upon switching the ISP or the reluctance of EUs to change the existing ISP. These innate preferences capture the degree by which EUs are locked in with a particular ISP. Market powers of ISPs are defined as a function of these innate preferences.

In our model, both ISPs offer a free service for CPs up to a threshold on quality. In addition, the non-neutral ISP offers a premium quality in exchange of a side payment from

the CP. This side-payment can be negative or positive, where a negative side-payment means a net payment from the non-neutral ISP to the CP. For instance, a negative side payment can arise in a scenario that the non-neutral ISP wants to make sure that the monopolistic CP offers with a premium quality and exclusively for her EUs. We assume that the CP generates revenue through advertisements, and the advertisement profit of the CP is an increasing function of the quality she offers to EUs.

We formulate a four-stage sequential game and seek the Sub-game Perfect Nash Equilibrium (SPNE) of the sequential game using backward induction.

Note that the equilibrium outcome has a complex dependency on a wide range of parameters. Thus, the structure, the existence and the uniqueness of the equilibrium is not apriori clear. One can expect different equilibrium outcomes in which either (i) the CP offers her content only with a free (best effort) quality, or (ii) the CP offers her content with free quality on the neutral and with premium quality on the non-neutral ISP, or (iii) the CP offers with a premium quality only on the non-neutral ISP. Moreover, different equilibrium Internet access fees and side payments can be selected by the ISPs whose value directly affects the welfare of EUs. For example, the non-neutral ISP can select a small Internet access fee to increase the number of her EUs and generates most of her revenue through the side-payment she charges the CP. In this case, because of competition, the neutral ISP should decrease her Internet access fee. Thus, the welfare of EUs would be high. Or, the non-neutral ISP may select a small side-payment (possibly negative) to make sure that the CP offers with a premium quality, and generate her revenue by increasing Internet access fees for EUs, which enables the neutral ISP to increase her price for EUs. Thus, this scenario yields a small welfare for EUs. Note that equilibrium

outcomes determine the division of EUs between ISPs, and some divisions maybe more desirable for the CP. Thus, the CP can have an active role in choosing the desirable equilibrium outcome (as well as the division of EUs with ISPs) by controlling the quality of her content on each ISP appropriately.

1.3.2 Contributions

Analytical Results

We show that if an SPNE exists, it would be one of the five possible strategies each of which we explicitly characterize. In some of these strategies, the CP offers her content on only the non-neutral ISP, and in the rest she offers her content on both ISPs. In addition, in one of the outcomes, all EUs pay the Internet access fee to the non-neutral ISP, i.e. the neutral ISP is driven out of the market. However, in the rest, both ISPs receive a positive share of EUs, i.e. both ISPs are active. In addition, by providing specific instances, we shows that an SPNE does not always exist.

We prove that when EUs have sufficiently low *inertia* for ISPs, i.e. when the preferences are “relatively” small and do not over rule major discrepancies on price and quality, the game has a unique SPNE. In this SPNE, the CP offers her content with premium quality on the non-neutral ISP while she does not offer her content on the neutral ISP, to attract all EUs to the non-neutral ISP on which users can receive a better quality. Thus, the neutral ISP would be driven out of the market. This implies that when inertias are small, upon switching to a non-neutral regime by an ISP, the neutral ISPs are forced to either leave the market or adopt a non-neutral regime.

We also consider the case that EUs have sufficiently high inertia for at least one of the

ISPs, and EUs cannot easily switch between ISPs. This case often happens in practice in the Internet market, e.g. when ISPs bundle Internet access with other services (e.g. cable, phone). In this case, an EU may incur additional expenses for other services if she buys Internet access from another ISP. Another example of high inertia of EUs is the case in which EUs require different devices to access the Internet through different ISPs (e.g. different devices for cable and DSL services), i.e. high set up costs. We prove that there exists a unique SPNE with a non-neutral outcome, and we explicitly characterize the SPNE. In the unique SPNE, both ISPs are active, and the CP offers her content with free quality on the neutral ISP and with premium quality on the non-neutral ISP.

In addition, we consider a benchmark case in which both ISPs are neutral. In this case, we prove that there exists a unique SPNE, in which the CP offers her content over both ISPs with free quality, and both ISPs would be active. We use the results of this case as a benchmark for assessing the extent of benefit of switching to non-neutrality for different entities of the market.

Numerical Results

Numerical results confirm our theoretical results that when the inertias of EUs for ISPs are small (respectively, high) enough, then the SPNE (respectively, the SPNE with a non-neutral outcome) exists and is unique. Numerical results also help pinpoint which of the five possible SPNE strategies occurs when the inertias are between these two extreme cases (high and low inertias). More specifically, results yield that if the inertia are between these two extreme cases but still on the lower end of the region in between, the game has an SPNE outcome in which both ISPs are active, but the CP offers her content with

premium quality and only on the non-neutral ISP. Results also reveal that if the inertias are between the two extreme cases but on the upper end of the region in between, then the game has no SPNE. Results of simulation over large sets of parameters also suggest that in all scenarios, the SPNE is unique if it were to exist.

Numerical results reveal that the neutral ISP loses payoff in all SPNE outcomes in comparison to the benchmark case. In addition, for a wide range of parameters, the non-neutral ISP receives a better payoff under a non-neutral scenario. This implies that it is beneficial for ISPs to operate as non-neutral, if they have the choice. However, switching to a non-neutral regime is *not* always profitable for ISPs. If EUs or the CP are not sensitive to the quality of the content delivered and the market power of the non-neutral ISP is small, then ISPs are better off staying neutral.

Results also reveal that the sensitivity of the EUs and the CP, and the market power of ISPs substantially influences the welfare of EUs (EUW) in neutral and non-neutral scenarios. The EUW would be higher in a non-neutral setting (as compared to the neutral setting) if (i) the market power of the non-neutral ISP is low, (ii) the sensitivity of the CP to the quality is high, or (iii) EUs are not very sensitive to the quality, or a combination of these conditions. In these cases a cheaper Internet access fee would be charged to the EUs by the non-neutral ISP which yields a higher EUW. In the absence of these conditions, the EUW of the neutral scenario would be higher.

1.3.3 Related Works

This work falls in the category of economic models for a non-neutral Internet [57]. This line of work can be divided into two broad categories: those that consider a non-neutral

regime in which (a) a non-neutral ISP blocks the content of the CPs that do not pay the side-payment, examples are [11], [52], [35], and those that consider (b) a non-neutral ISP that provides quality differentiations for CPs and do not necessarily block a content, examples are [43, 44, 32, 54, 7, 48, 2, 18, 8, 6]. Note that in reality and because of FCC restrictions on blocking the content, we expect the latter scenario (differentiation in quality) to emerge. Thus, in this work, we consider the second scenario.

These works can also be further divided into two other categories: (i) those that consider monopolistic ISPs: [44, 32, 54, 7, 20, 48, 2], and (ii) those that consider competition between ISPs: [43, 11, 52, 18, 8, 6, 35]. Our work belongs to the latter case.

To the best of our knowledge, this work is one of the few works that considers the problem of migration to a non-neutral regime, i.e. when some of the ISPs are neutral and some are non-neutral. The focus of previous works is on the social welfare analysis of the market when all ISPs are neutral and/or all are non-neutral, without considering the incentives of individual ISPs to adopt a non-neutral regime. The exception is [43] in which the authors consider competition between a neutral (public option) ISP with non-neutral ISPs. They argue that the existence of a non-neutral ISP alongside of a neutral ISP increases the customer surplus in comparison to a neutral scenario in which all ISPs are neutral. However, parameters such as different market powers of ISPs and the sensitivity of EUs and CPs to the quality of the content are important in determining the welfare of EUs which are absent in the model of [43]. We consider these parameters in our model. Contrary to their results, we show that the competition between the neutral and non-neutral ISPs would not always increase the customers welfare.

In addition, in contrast to the previous works, we consider competition between ISPs

that have different market powers, i.e. an asymmetric competition (Market power is the ability of a decision maker to raise the market price for a good or service.). Moreover, in most of the previous works, CPs have a passive role, i.e. they are only price-takers. However, in our model, we consider the quality of the content that a CP offers for EUs of each ISP as the strategy by which she can influence the equilibrium of the market. For example, a CP can select a particular ISP and offer with a high quality on this ISP, and stop offering her content on other ISPs. By doing so, the CP might be able to migrate EUs of other ISPs to the selected ISP.

1.4 Non-Neutrality Framework I- Uncertain Price Competition in an Internet Market

1.4.1 The Research Challenges and Goals

We consider a market with two ISPs (henceforth denoted by sellers), where each seller offers multiple units of resources for sale to CPs (henceforth denoted by buyers or customers). We investigate the price selection strategy for sellers in presence of uncertainty in competition using Game Theory. Customers shop around for the lowest available prices. Therefore sellers seek to set prices that will ensure that their commodities are sold and also fetch adequate profit. Often times, a seller is not aware of the number of units available to her competitor before quoting her price. Thus, the competition that each seller faces is uncertain, and different sellers have different number of resources available (heterogeneous availability). Each seller selects the price per unit depending on the number of units she has available for sale, the statistics of the availability process for her competitor, and the

demand. In general, each seller chooses her price randomly using different probability distributions for different availability levels. Thus, the strategy of each player is a vector of probability distributions. For instance, if a seller can potentially offer up to three units of resources, her vector of strategies would be $(\Phi_1(\cdot), \Phi_2(\cdot), \Phi_3(\cdot))$, where $\Phi_i(\cdot)$ is the price selection probability distribution when the seller offers i units.

Due to uncertainty in competition, quoting a high price by a seller enhances the risk of not being able to sell the resources offered by that seller. On the other hand, although selecting a low price increases the chance of winning the competition, it also decreases the profit earned by the seller. Therefore, pricing in presence of uncertainty in competition is a risk-reward tradeoff. It is not apriori clear that how offering multiple number of units affects the price selection by sellers. For instance, a seller with a large number of available units may be motivated to quote a low price, since in the event of winning the competition, a small amount of profit per unit would result in a large total profit. On the other hand, a seller may also be enticed to select a high price when the availability is high to significantly increase her overall profit, even at the risk of not being able to sell the available units. We focus on investigating the impact of heterogeneous availability and uncertain competition on the aforementioned risk-reward tradeoff.

Note that uncertainty in competition is an integral feature not only a non-neutral Internet but also a diverse sets of application such as microgrid and secondary spectrum markets. We later discuss about how our model captures these applications.

1.4.2 Contributions

In our work, sellers are allowed to have different probability distributions for different availability levels (asymmetric market). Note that since the utility of sellers is not a continuous function of their strategy, classical theorems for existence and uniqueness of NE cannot be used. We identify key properties that every NE pricing strategy should satisfy when demand is greater than the maximum possible availability level (necessary conditions). The properties reveal that the sellers randomize their price using probability distributions whose support sets are mutually disjoint and in decreasing order of the number of availability. In the context of the aforementioned risk-reward tradeoff, sellers opt for low-risk pricing when they have high availability. We also prove that any strategy profile that satisfies these properties constitutes an NE regardless of the relation between the demand and the number of available units (sufficiency condition). This sufficiency result naturally leads to an algorithm for computing the strategies that satisfy the mentioned properties (If such a strategy exists, it is an NE).

In addition, We consider a symmetric market and prove that these properties are also necessary conditions for a NE regardless of the relation between the demand and the number of available units. We prove that the symmetric NE exists uniquely, and obtain an algorithm for explicitly computing it. Note that the uniqueness is specific to the symmetric market- our analysis reveals that an asymmetric market allows for multiple Nash equilibria.

Furthermore, we propose a strategy for sellers in a symmetric oligopoly that satisfies the necessary and sufficient properties identified for a symmetric NE in a symmetric

duopoly market. Numerical evaluations reveal that this strategy constitutes a fairly good approximation for the symmetric NE of a symmetric oligopoly. Finally, we generalize the results to the case of random demand.

1.4.3 Related Works

Price competition among different entities has been extensively studied in [10, 19, 13, 60, 50, 51, 69, 47, 67, 16, 64]. In economics literature as also in the context of specific applications, uncertainty in competition has been investigated when the availability level is either zero or one [23, 29, 25, 27, 26]. The strategy profile of each seller consists of only one probability distribution since sellers need to select a price only when they have one unit available for sale. We, however, characterize the Nash equilibrium pricing strategies when sellers have arbitrary and potentially different number of available units for sale (not merely zero or one). In this case, different price selection strategies may be required for different number of available units. Thus, the pricing strategy profile of each seller is a collection of probability distributions, one for each availability value. Therefore both results and proofs are substantially different from previous works.

Another genre of work allows sellers to control the amount of commodities they would generate for sale [9, 30, 62, 15, 21, 3]. In these works, sellers (e.g. power generators) bid a supply function ¹ to a central auctioneer. Based on the demand and the bids submitted, the auctioneer solves an optimization problem to determine the number of units needed to be generated by the sellers and subsequently the price that should be paid to them. In

¹A supply function is a function that maps the price of the commodity under sale to the amount a producer will produce for sale.

[9, 30, 62], the setting is a uniform-price procurement auction in which the price is equal for different sellers, i.e. the clearing price. However, in [15, 21, 3], authors investigate a pay-as-bid (discriminatory) procurement auction, in which the central entity accepts the offers submitted by the sellers and pays the accepted offers based on the bid submitted. In [3], authors provide a characterization of mixed equilibria over increasing supply curves. In other words, in their characterization, the price per infinitesimal unit of the commodity is increasing, i.e., the higher the number of units produced, the higher the price per unit. We instead consider scenarios where sellers do not control the amount of commodities they produce². Thus, each seller quotes a price depending on the number of available units and her belief about other sellers. The distinctions in the setup lead to major differences in the formulation, analyses, and results. Our results reveal that the Nash equilibrium pricing of our model is in stark contrast with the optimal curves found in [3]. Specifically, we show that sellers with high availability quote a lower price.

Note that the setting considered in this chapter is an asymmetric discriminatory multi-unit auction in which sellers are the bidders. As stated in [22]: “*Unfortunately, computing equilibrium strategies in (asymmetric) discriminatory multi-unit auctions is still an open question*”. In this chapter, we provide an algorithm to compute the equilibrium strategies for a duopoly case. Using the results for duopoly, we provide an algorithm to compute the equilibrium strategies for a symmetric duopoly.

²This implies that in [3] the market will be cleared (firms produce up to the point that satisfies the demand), while in our case, there is no guarantee that all the available units would be sold.

1.5 Non-Neutrality Framework II- Quality Sponsored Data

1.5.1 The Research Challenges and Goals

In this work, we consider the cases in which ISPs can increase their profit by charging CPs for the service they provide, and the CPs can provide their users a better reception quality in exchange of monetary incentives for ISPs. In contrary to the previous case, we consider the CPs to have an active role in the market, and decide on the number of resources they want to reserve/buy from ISPs based on the price that ISPs quote. We refer to this model as the *quality-sponsored data* (QSD) model, wherein spectral resources at ISPs are sponsored to ensure quality for the data bytes being delivered to the end users.

Hence, the over-arching goal of this work is to analyze and understand the implications of the QSD model on the market dynamics. Using game-theoretic [45] tools, we study the market equilibria and dynamics under various scenarios and assumptions involving the three key players of the market, namely the CPs, ISPs and end users. We investigate the scenarios under which the QSD model is plausible, and one can expect a stable outcome for the market that involves sponsoring the quality of the content by CPs. In addition, we discuss about the division of profit between ISPs and CPs in two cases (1) when the decision makers do not cooperate and at least one of them is myopic optimizer, and (2) when both cooperatively maximize the payoff in the long-run. In the process, we devise strategies for the CPs (respectively, ISPs) to determine if they should participate in QSD, what quality to sponsor, and how the ISPs should price their resources.

In our model, ISPs make a portion of their resources available for sponsorship, and

price it appropriately to maximize their payoff. Their payoff depends on monetary revenue and satisfaction of end-users both for the non-sponsored and sponsored content. Note that the QSD model couples market decisions to the scarce (wireless) resources. Thus, resources allocated to sponsored contents will affect those allocated to non-sponsored content and hence their quality. Thus, one should consider the impact of the quality of the two types of data (sponsored and non-sponsored) on satisfaction of end-users.

1.5.2 Contributions

We consider one CP and one ISP. We consider that the CP has an advertisement revenue model³, and characterize the myopic pricing strategies for the CP and the ISP given the quality of the content that needs to be guaranteed and the available resource using a non-cooperative sequential game framework. Assuming the demand for content to be dynamic, wherein the change in the demand is dependent on the quality end-users experience, we investigate the asymptotic behavior of the market when at most one of the decision makers (ISP or CP) is short-sighted, i.e. not involving the dynamics of demand in their decision making. We show that depending on certain key parameters, such as the importance of non-sponsored data for ISPs and the parameters of the dynamic demand, the market can be asymptotically (in long run) stable or unstable. Furthermore, four different stable outcomes are possible: 1. no-sponsoring, 2. maximum bit sponsoring: the CP sponsors all the available resources, 3. minimum quality sponsoring: the CP sponsors minimum resources to deliver a minimum desired rate to her users, and 4. Interior solution in which the CP sponsors more than the minimum but not all the available resources. We

³A CP that earns money through advertisements.

characterize the conditions under which each of these asymptotic outcomes is plausible. The effects of different market parameters on the asymptotic outcome of the market is investigated through numerical simulations.

Note that there may exist multiple equilibria, and a non-cooperative framework may lead to a Pareto-inefficient outcome. Thus, when both of the decision makers are long-sighted, it is natural to consider a cooperative scheme such as a bargaining game framework. Thus, we investigate the role of a CP and an ISP with long-sighted business models⁴ in stabilizing the market and equilibrium selection. We characterize the Nash Bargaining Solution (NBS) of the game to determine the profit sharing mechanism between the ISP and CP.

1.5.3 Related Works:

Works related to the emerging subject of sponsored content are scarce. In [24], [5], [4], and [68], authors investigate the economic aspects of content sponsoring in a framework similar to At&t sponsored data plans. Note that in At&t sponsored data plan, the CP pays for the quantity of the data carried to the end-users, while in our scheme the CP pays for the *quality* of the data, and end-user is responsible for paying for the quantity. We take into the account the quality of the content and the coupling it has with scarce resources. We consider more strategic CPs that decide on the portion of ISP's resources they want to sponsor, based on the price ISPs quote and the demand from end-users.

This work falls in the category of economic models for a non-neutral Internet. Most

⁴in which decision makers maximize their payoff in long-run considering the dynamics of the demand for the content.

of the works in this area study the social welfare of the market under neutrality and non-neutrality regimes. In these works the decision of CPs does not depend on the demand for the content, and simply is a take-it-or-leave-it choice, i.e. either the CP pays for the premium quality or uses the free quality. In addition, most of the works do not consider the coupling between limited resources available to ISPs and the strategies of the decision makers. Exceptions are [32] and [43]. We consider that CPs decide on the number of resources they want to sponsor based on the dynamics of their demand. Depending on the demand and number of resources available with the ISP, the number of sponsored resources by the CP determines the quality of experience for users of sponsored and non-sponsored contents. Thus, we consider the coupling between market decisions and the limited wireless resources. Moreover, we study problems like stability of the market and the effects of being short-sighted or long-sighted. Therefore, we focus on one-to-one interaction between CPs and ISPs, and its implications on the payoff of individual decision makers.

The closest work to ours is [12] in which authors study the interaction between an ISP and a CP when the CP can sponsor a quality higher than the minimum quality under a private contract with the ISP. Their main focus is to compare the social welfare of the sequential game when either the ISP or the CP is the leader, with the Pareto optimal outcome resulting from a bargaining game between the ISP and the CP. Authors assume that the number of subscribers to the ISP is an *increasing* function of the quality it provides for the CP. In other words, as the quality for the sponsored content enhances, end-users of the ISP become more satisfied. However, in our work, the main focus is the coupling between the limited resources and the quality. Thus, providing a better

quality for a sponsored content may degrade the quality of non-sponsored contents in peak congestion times. Therefore, in our model, the satisfaction of end-users which is a function of both sponsored and the non-sponsored content is not necessarily increasing with respect to the sponsored quality. This changes the nature of the problem.

1.6 Publications

- Chapter 2 is based on [39]. The shorter versions have been published in [35] and [40].
- Chapter 3 is based on [37]. The shorter version has been published in [36].
- Chapter 4 is based on [42]. The shorter version has been published in [41].
- Other published papers are [34] and [38].

Chapter 2

Migration to a Non-Neutral Internet¹

We consider early stages of migrating to non-neutrality in which *some ISPs would adopt a non-neutral regime before others*. In this setting, we assess the benefits of different entities in an emerging non-neutral network. Such an assessment is crucial in whether a non-neutral Internet would be adopted. Thus, we consider a system in which there exists two ISPs, one “big” CP, and a continuum of End-Users (EUs). One of the ISPs is neutral and the other is non-neutral. We consider that the CP can differentiate between ISPs by controlling the quality of the content she is offering on each one. We also consider that EUs have different levels of innate preferences for ISPs. We formulate a sequential game, and explicitly characterize all the possible Sub-game Perfect Nash Equilibria (SPNE) of the game. We prove that if an SPNE exists, it would be one of the five possible strategies

¹Presented in W-PIN+NetEcon 2014 (as a poster)[35], the Information Science and Systems conference (CISS) [40], and submitted to Operations Research [39].

each of which we explicitly characterize. We prove that when EUs have sufficiently low innate preferences for ISPs, a unique SPNE exists in which the neutral ISP would be driven out of the market. We also prove that when these preferences are sufficiently high, there exists a unique SPNE with a non-neutral outcome in which both ISPs are active. Numerical results reveal that the neutral ISP receives a lower payoff and the non-neutral ISP receives a higher payoff (most of the time) in a non-neutral scenario. However, we identify scenarios in which the non-neutral ISP loses payoff by adopting non-neutrality. We also show that a non-neutral regime yields a higher welfare for EUs in comparison to a neutral one if the market power of the non-neutral ISP is small, the sensitivity of EUs (respectively, the CP) to the quality is low (respectively, high), or a combinations of these factors.

The chapter is organized as follows: First, in Section 2.1, we present the model. Then, we find the SPNE(s) strategies in Section 2.2. In Section 2.3, we present the results for the benchmark case, i.e. both ISPs neutral. In Section 2.4, we summarize and discuss about the key results of the work. We provide numerical examples in Section 2.5. Finally, we comment on some of the assumptions of the model and their generalizations in Section 2.6. All proofs are presented in the Appendix (Section 2.7).

2.1 Model and Formulations

We consider two ISPs, a CP, and a continuum of EUs.

ISPs:

We consider one of the ISPs to be neutral (ISP N) and the other to be non-neutral (ISP NoN), i.e. the latter can offer a premium quality for CPs in exchange of a side-payment. The strategies of the neutral and non-neutral ISPs are to determine Internet access fees for EUs, i.e. p_N and p_{NoN} , respectively. We show that most of the results will depend on the difference in the Internet access fees, i.e. $\Delta p := p_{NoN} - p_N$.

In addition, the non-neutral ISP determines \tilde{p} , i.e. the side-payment per quality. Note that \tilde{p} can be positive or negative, in which a negative side-payment implies a reverse flow of money from the non-neutral ISP to the CP. The CP will pay premium quality fee, i.e. the side-payment, to the non-neutral ISP if she chooses to offer a quality higher than the free quality threshold (\tilde{q}_f), and can offer with up to the quality \tilde{q}_f for free on both ISPs. The side-payment paid to the non-neutral ISP is considered to be a linear function of the quality. Thus,

$$\text{Side Payment} = \begin{cases} \tilde{p}q & \text{if } q > \tilde{q}_f \\ 0 & \text{Otherwise} \end{cases}$$

We assume that the neutral ISP generates her profit from EUs, and the non-neutral generates her profit from EUs and potentially from the CP (if $\tilde{p} > 0$ and the CP is willing to pay for a premium quality). The payoff of the neutral and non-neutral ISPs are as follows:

$$\pi_N(p_N) = (p_N - c)n_N \quad \& \quad \pi_{NoN}(\tilde{p}, p_{NoN}) = (p_{NoN} - c)n_{NoN} + z\tilde{p}q_{NoN} \quad (2.1)$$

where n_N and n_{NoN} are the fraction of EUs that have access to Internet via the neutral and non-neutral ISPs, respectively. The parameter q_{NoN} is the quality of the content

on the non-neutral ISP, and c is the marginal cost of providing Internet for EUs. The parameter z indicates whether the CP chooses to offer her content with premium quality ($z=1$ when the CP offers with premium quality, and $z = 0$ otherwise). From (2.1), for a positive payoff, $p_N \geq c$, and $p_{NoN} \geq c$, if $z = 0$. However, if $z = 1$, there may exist cases that even with $p_{NoN} < c$, the payoff of ISP NoN would be positive.

The CP:

The CP can potentially offer different quality levels on different ISPs. The strategy of the CP is to choose a quality of $q_N \in \{0, \tilde{q}_f\}$ on the neutral ISP, and a quality of $q_{NoN} \in \{0, \tilde{q}_f, \tilde{q}_p\}$ on the non-neutral ISP, with $\Delta q := q_{NoN} - q_N$. In our model, the CP generates revenue through advertisement. We also assume that the advertising profit that the CP receives is a function of the number of EUs and the content quality she delivers to these EUs ². Thus, the advertising profit is proportional to q_N and q_{NoN} (As seen in the first two terms of (2.2)). In addition, the CP pays (or receive if $\tilde{p} < 0$) a side-payment to the non-neutral ISP based on the side-payment per quality fee determined by the non-neutral ISP and the quality. Thus, the profit of the CP is,

$$\pi_{CP}(q_N, q_{NoN}, z) = n_N \kappa_{ad} q_N + n_{NoN} \kappa_{ad} q_{NoN} - z \tilde{p} q_{NoN} \quad (2.2)$$

where κ_{ad} is a constant ³, $z = 0$ if $q_{NoN} = \{0, \tilde{q}_f\}$ (using free quality) and $z = 1$ if $q_{NoN} = \tilde{q}_p$ (using premium quality).

²Note that we are assuming that advertisements are quality-dependent. For example they are video or sound. Some examples of the CPs that provide these types of ads are YouTube and Spotify.

³We assume a linear dependency between the quality and the advertising revenue and the cost. Thus, κ_{ad} can be considered to be $\kappa_{ad} = \kappa_{ad,rev} - \kappa_{ad,cost}$.

It may appear from (2.2) that the CP would lose nothing by choosing at least a free quality on both ISPs. However, this is not the case. As we explain later, n_N and n_{NoN} are dependent on q_N and q_{NoN} , and there is a negative correlation between them. In other words, increasing one of them (e.g. n_N), decreases the other one (e.g. n_{NoN}). Therefore, the CP may stop offering her content on the neutral ISP to move EUs to the non-neutral ISP on which they can receive a better quality. This may lead to higher advertisement revenues for the CP.

End-Users:

The strategy of an EU is to choose one of the ISPs to buy Internet access from. We assume that the neutral ISP is located at 0, the non-neutral one is located at 1, and EUs are distributed uniformly along the unit interval $[0, 1]$. The closer an EU to an ISP, the more this EU prefers this ISP to the other. Note that the notion of closeness and distance is used to model the preference of EUs and market power of ISPs, and may not be the same as the physical distance.

More formally, the EU located at $x \in [0, 1]$ incurs a *transport cost* of $t_N x$ (respectively, $t_{NoN}(1 - x)$) when joining the neutral ISP (respectively, non-neutral ISP), where t_N (respectively, t_{NoN}) is the marginal transport cost for the neutral (respectively, non-neutral) ISP. Two possible interpretations of the transport costs are reluctance of EUs to change their ISP and initial set-up costs of a new service upon switching the ISP. In sum, we consider t_N and t_{NoN} as the reluctance of EUs for connecting to the neutral and non-neutral ISPs, respectively.

We consider a common valuation for connecting to the Internet for EUs regardless of

the content of the CP. This common valuation also models the valuation of EUs for CPs other than the CP considered in this work, i.e. the valuation for connecting to the Internet regardless of the status of the CP considered. Let v^* denote this common valuation. The overall valuation of an EU located at $x \in [0, 1]$ for connecting to the Internet via the neutral ISP (respectively, non-neutral ISP) is considered to be $v^* + \kappa_u q_N - t_N x$ (respectively, $v^* + \kappa_u q_{NoN} - t_{NoN}(1 - x)$). Thus, the utility of an EU who connects to the ISP $j \in \{N, NoN\}$ located at distance x_j of the ISP, and is receiving the content with quality q_j , is:

$$u_{EU,j}(x_j) = v^* + \kappa_u q_j - t_j x_j - p_j, \quad j \in \{N, NoN\} \quad (2.3)$$

This model is generally known as the hotelling model. A symmetric version ($t_N = t_{NoN}$) of this model is used in the context of the Internet market in [6].

Note that the lower t_N and t_{NoN} , the easier EUs can switch between ISPs, and thus the lower would be the *inertia* of EUs. Therefore, high transport cost for an ISP is associated with EUs that are locked in with the other ISP. We consider the ratio of t_N and t_{NoN} as the relative bias of EUs for ISPs. More specifically, the higher $\frac{t_N}{t_N + t_{NoN}}$ (respectively, $\frac{t_{NoN}}{t_N + t_{NoN}}$), the higher the bias of EUs for connecting to the Internet via ISP NoN (respectively, ISP N). We define the market power of an ISP to be the relative biases, i.e. the market power of the neutral and non-neutral ISPs are $\frac{t_{NoN}}{t_N + t_{NoN}}$ and $\frac{t_N}{t_N + t_{NoN}}$, respectively.

A schematic of the market is presented in Figure 2.1.

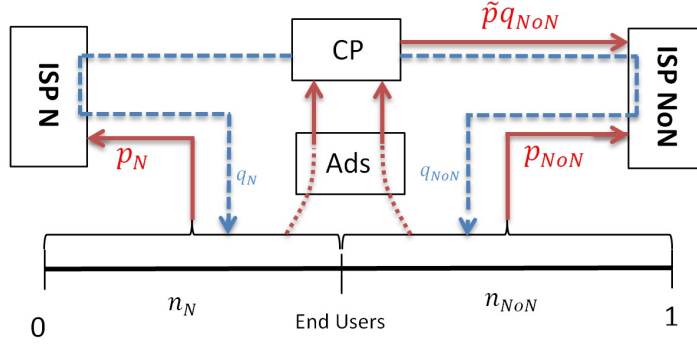


Figure 2.1: A schematic of the market - red (solid) lines are the flow of the money and blue (dashed) lines are the flow of the content.

Formulations:

We assume that ISPs are the leaders of the game, and the CP and EUs are followers.

Thus, the sequence of the game is as follows:

1. The neutral and non-neutral ISPs determine Internet access fees for EUs (p_N and p_{NoN}).
2. The non-neutral ISP announces the premium quality fee side-payment (\tilde{p}).
3. The CP decides on the quality of the content (q_N and q_{NoN}) for EUs of each ISP.
4. EUs decide which ISP to join.

We assumed the selection of Internet access fees to happen before the selection of the side-payment because of the rate of change in these selections. Note that the Internet access fees are expected to be kept constant for a longer time horizons in comparison to the side-payment that is expected to change more frequently depending on the demand and the network specifications.

In the sequential game framework, we seek a *Subgame Perfect Nash Equilibrium* (SPNE) using *backward induction*.

Definition 1. Subgame Perfect Nash Equilibrium (SPNE): *A strategy is an SPNE if and only if it constitutes a Nash Equilibrium (NE) of every subgame of the game.*

Definition 2. Backward Induction: *Characterizing the equilibrium strategies starting from the last stage of the game and proceeding backward.*

We also assume that each EU chooses exactly one ISP to buy Internet access. This is known as the full market coverage of EUs by ISPs. This assumption is common in hotelling models and is necessary to ensure competition between ISPs. An equivalent assumption is to consider the common valuation v^* to be sufficiently large so that the utility of EUs for connecting to the Internet is positive regardless of the choice of ISP.

2.2 The Sub-Game Perfect Nash Equilibrium

In this section, we seek a sub-game perfect equilibrium using backward induction. In Sections 2.2.1 to 2.2.4, we characterize the equilibrium strategies of each stage in a reverse order starting from Stage 4. *For each stage, we assume that each decision maker is aware of the strategies chosen by other decision makers in previous stages.*

2.2.1 Stage 4: Customers decide which ISP to join

In this subsection, we characterize the division of EUs between ISPs in the equilibrium, i.e. n_N and $n_{N \circ N}$, using the knowledge of the strategies chosen by the ISPs and the CP in Stages 1, 2, and 3. To do so, we characterize the location of the EU that is

indifferent between joining either of the ISPs, x_n . Thus, EUs located at $[0, x_n)$ join the neutral ISP, and those located at $(x_n, 1]$ joins the non-neutral ISP. The EU located at $x_n \in [0, 1]$ is indifferent between connecting to the neutral and non-neutral ISP (Recall that we assumed full market coverage by ISPs) if:

$$\begin{aligned} v^* + \kappa_u q_{NoN} - t_{NoN}(1 - x_n) - p_{NoN} &= v^* + \kappa_u q_N - t_N x_n - p_N \\ \Rightarrow x_n &= \frac{t_{NoN} + \kappa_u(q_N - q_{NoN}) + p_{NoN} - p_N}{t_{NoN} + t_N} \end{aligned} \quad (2.4)$$

Thus, the fraction of EUs with each ISP (n_N and n_{NoN}) is:

$$n_N = \begin{cases} 0 & \text{if } x_n < 0 \\ \frac{t_{NoN} + \kappa_u(q_N - q_{NoN}) + p_{NoN} - p_N}{t_{NoN} + t_N} & \text{if } 0 \leq x_n \leq 1 \\ 1 & \text{if } x_n > 1 \end{cases} \quad \& \quad n_{NoN} = 1 - n_N \quad (2.5)$$

2.2.2 Stage 3: The CP decides the qualities to offer over each ISP (q_N and q_{NoN})

In this section, we characterize q_N , q_{NoN} in the equilibrium using the knowledge of the vector of access fees $\vec{p} = (p_N, p_{NoN})$ and \tilde{p} from stages 1 and 2. Recall that $z = 1$ if $q_{NoN} > \tilde{q}_f$, and $z = 0$ otherwise. First, we find the strategies that maximize $\pi_{CP}(q_N, q_{NoN}, z)$ (2.2). Then, using appropriate tie-breaking assumptions, we characterize the equilibrium strategies in Theorems 1 and 2.

Note that the CP maximizes (2.2) by choosing the optimum strategies, (q_N^*, q_{NoN}^*) , from the sets F_0 or F_1 :

$$\begin{aligned} F_0 &= \{(0, 0), (0, \tilde{q}_f), (\tilde{q}_f, 0), (\tilde{q}_f, \tilde{q}_f)\} \\ F_1 &= \{(0, \tilde{q}_p), (\tilde{q}_f, \tilde{q}_p)\} \end{aligned} \quad (2.6)$$

Conditions				
	$x_N \leq 0$ $\left(q_{NoN} - q_N \geq \frac{\Delta p + t_{NoN}}{\kappa_u}\right)$ All EUs join Non-neutral	$0 < x_N < 1$ $\left(\frac{\Delta p - t_N}{\kappa_u} < q_{NoN} - q_N < \frac{\Delta p + t_{NoN}}{\kappa_u}\right)$ EUs divide between both ISPs	$x_N \geq 1$ $\left(q_{NoN} - q_N \leq \frac{\Delta p - t_N}{\kappa_u}\right)$ All EUs join Neutral	Union (\cup)
$z = 0$	F_0^L	F_0^I	F_0^U	F_0
$z = 1$	F_1^L	F_1^I	F_1^U	F_1
Union (\cup)	F^L	F^I	F^U	\mathcal{F}

Table 2.1: Notations for different subsets of the feasible set. Expressions in parenthesis are equivalent form of the conditions, e.g. $x_N \leq 0 \iff q_{NoN} - q_N \geq \frac{\Delta p + t_{NoN}}{\kappa_u}$.

Note that F_0 and F_1 are the set of strategies by which $z = 0$ and $z = 1$, respectively. Each of the sets F_0 and F_1 is further divided into three subsets, F_i^L , F_i^I , and F_i^U , for $i \in \{0, 1\}$, depending on whether $x_N \leq 0$, $0 < x_N < 1$, or $x_N \geq 1$ (using (2.4)). Since x_N is a function of q_N and q_{NoN} , these conditions on x_N lead to constraints on q_N and q_{NoN} . In Table 2.1, we present the division of the feasible set into the above-mentioned subsets and the constraints on q_N and q_{NoN} for each subset. Note that $F_0^L \cup F_1^L = F^L$, $F_0^I \cup F_1^I = F^I$, and $F_0^U \cup F_1^U = F^U$.

Next, we present the tie-breaking assumptions used to prove these results (Section 2.2.2). Then, we present the statement of the main results in Section 2.2.2. We prove the results in Appendix 2.7.1.

Tie- Breaking Assumptions

We assume that for choosing the equilibrium strategy, the CP uses the following tie-breaking assumptions that one can expect to arise in practice.

First note that $(q_N^*, q_{NoN}^*) \in F^L$ (respectively, $(q_N^*, q_{NoN}^*) \in F^U$) yields that $n_N^* = 0$

(respectively, $n_{NoN}^* = 0$). Thus, in this case, the quality that the CP offers on the neutral ISP (respectively, non-neutral ISP) is of no importance. Therefore:

Assumption 1. *If $(q_N^*, q_{NoN}^*) \in F^L$ (respectively, $(q_N^*, q_{NoN}^*) \in F^U$), then without loss of generality, $q_N^* = 0$ (respectively, $q_{NoN}^* = 0$).*

In addition, in practice, it is natural to expect that the CP prefer higher qualities to lower ones, e.g. $z = 1$ over $z = 0$, if this selection does not affect the payoff.

Assumption 2. *If the optimum solutions exist in F_0 and F_1 , then the CP chooses the ones in F_1 . In other words, if $z = 1$ and $z = 0$ yield equal maximum payoffs for the CP, then the CP will pick $z = 1$, i.e. will use the premium quality.*

The following tie-breaking assumptions are based on the natural assumption that the CP would prefer to diversify her content over different ISPs if she is indifferent:

Assumption 3. *If there exists global optimum solutions in F^I , then they are preferred by the CP over global optimum solutions in F^L and F^U . In other words, if the outcome in which only one ISP is operating and the outcome by which both ISPs are operating yield the global maximum payoff for the CP, then the CP chooses the strategies by which the latter outcome occurs.*

Assumption 4. *Consider two strategies: (i) (q'_N, q'_{NoN}) such that at least one of q'_N or q'_{NoN} is zero, and (ii) (q''_N, q''_{NoN}) such that $q''_N > 0$ and $q''_{NoN} > 0$. If these two strategies yield the same payoff for the CP, then the CP chooses (ii), i.e. the one with positive quality on both ISPs.*

In the following tie-breaking assumption, we assume that the CP takes into the account the welfare of EUs for tie-breaking between strategies.

Assumption 5. *If the payoff of the CP when only the neutral ISP is operating is equal to the payoff when only the non-neutral is operating, then the CP prefers the strategy by which the ISP that offers the lower Internet access fee, i.e. p_i , $i \in \{N, NoN\}$, is operating. In other words, the CP chooses the strategy that yields a higher social welfare for EUs.*

The above-mentioned assumptions over-ride each other in the order specified. For example, if two strategies one in F_1^L and the other in F_0^I are both global maximum, then Assumption 2 suggests that the CP chooses the strategy in F_1^L , and Assumption 3 suggests that the CP chooses the strategy in F_0^I . Since Assumption 2 comes before Assumption 3, the CP chooses the strategy in F_0^I . Next, using these tie-breaking assumptions, we characterize the equilibrium strategies:

Main Results

First, we define certain thresholds that appear in the results:

Definition 3. • $\tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$

- $\tilde{p}_{t,2} = \kappa_{ad}(n_{NoN} - \frac{\tilde{q}_f}{\tilde{q}_p})$, where $n_{NoN} = \frac{t_N + \kappa_u \tilde{q}_p - \Delta p}{t_N + t_{NoN}}$.
- $\tilde{p}_{t,3} = \kappa_{ad} n_{NoN} (1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, where $n_{NoN} = \frac{t_N + \kappa_u (\tilde{q}_p - \tilde{q}_f) - \Delta p}{t_N + t_{NoN}}$.
- $\Delta p_t = \kappa_u (2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$

We would observe that, when characterizing the optimum strategies, $\tilde{p}_{t,1}$, $\tilde{p}_{t,2}$, and $\tilde{p}_{t,3}$ would be thresholds on side-payment, and Δp_t would be a threshold on the difference in the access fees.

In Theorem 1, we characterize the equilibrium strategies of the CP by which $z^{eq} = 0$ for different values of Δp . Then, using these results, in Theorem 2, we characterize the equilibrium strategies of the CP in general case for different regions of Δp .

Theorem 1. *If $(q_N^{eq}, q_{NoN}^{eq}) \in F_0$, then:*

1. *if $-t_{NoN} < \Delta p < t_N$, then $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_f) \in F_0^I$.*
2. *if $\Delta p \geq t_N$, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, 0) \in F_0^U$.*
3. *if $\Delta p \leq -t_{NoN}$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_f) \in F_0^L$.*

In addition, the utility of the CP by each candidate equilibrium strategy is $\kappa_{ad}\tilde{q}_f$.

For proving this theorem, we characterize optimum strategies among all $(q_N, q_{NoN}) \in F_0$. Then, using these optimum strategies and tie-breaking assumptions, we characterize (q_N^{eq}, q_{NoN}^{eq}) . Later, we will see that this theorem also characterizes the optimum strategies of the CP in the benchmark case in which both ISPs are forced to be neutral.

Intuitively, as Δp increases, the number of EUs with ISP NoN decreases. Thus, as the results of Theorem 1 confirms, as Δp increases, the outcome of the market moves from F_0^L , i.e. all EUs join ISP NoN, to F_0^I , i.e. both ISPs have positive share of EUs, and to F_0^U , i.e. all EUs join the ISP N.

In Theorem 2, we characterize the equilibrium strategy of the CP in general case. We prove that results are threshold-type: when the side-payment, i.e. \tilde{p} , is less than a threshold, the CP chooses the premium quality, i.e. $z^{eq} = 1$, and when \tilde{p} is higher than the threshold, $z^{eq} = 0$ and the CP chooses the strategies according to Theorem 1. We also characterize the value of this thresholds for different regions of Δp . Note that as Δp increases, the number of EUs with ISP NoN decreases. This affects the payoff of the CP,

and subsequently the value of the side-payment that ISP NoN charges to the CP. Thus, the value of the threshold on the side-payment depends on Δp .

Theorem 2. *Let the thresholds Δp_t , $\tilde{p}_{t,1}$, $\tilde{p}_{t,2}$, and $\tilde{p}_{t,3}$ as characterized in Definition 3, then:*

1. *If $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$:*

- *if $\tilde{p} \leq \tilde{p}_{t,1}$, then $z^{eq} = 1$, and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$.*
- *if $\tilde{p} > \tilde{p}_{t,1}$, then $z^{eq} = 0$, and q_N^{eq} and q_{NoN}^{eq} are determined by Theorem 1.*

2. *If $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u \tilde{q}_p$, and $\tilde{q}_f \leq \frac{t_N + t_{NoN}}{\kappa_u}$:*

(a) *if $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, and:*

i. *if $\Delta p \geq \Delta p_t$:*

- *if $\tilde{p} \leq \tilde{p}_{t,3}$, then $z^{eq} = 1$ and $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$.*
- *if $\tilde{p} > \tilde{p}_{t,3}$, then $z^{eq} = 0$, and q_N^{eq} and q_{NoN}^{eq} are determined by Theorem 1.*

ii. *if $\Delta p < \Delta p_t$:*

- *if $\tilde{p} \leq \tilde{p}_{t,2}$, then $z^{eq} = 1$ and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$.*
- *if $\tilde{p} > \tilde{p}_{t,2}$, then $z^{eq} = 0$, and q_N^{eq} and q_{NoN}^{eq} are determined by Theorem 1.*

(b) *if $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u \tilde{q}_p$:*

i. *if $\tilde{p} \leq \tilde{p}_{t,2}$, then $z^{eq} = 1$, and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$.*

ii. *if $\tilde{p} > \tilde{p}_{t,2}$, then $z^{eq} = 0$, and q_N^{eq} and q_{NoN}^{eq} are determined by Theorem 1.*

3. If $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u \tilde{q}_p$, and $\tilde{q}_f > \frac{t_N + t_{NoN}}{\kappa_u}$:

(a) if $\tilde{p} \leq \tilde{p}_{t,2}$, then $z^{eq} = 1$, and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$.

(b) if $\tilde{p} > \tilde{p}_{t,2}$, then $z^{eq} = 0$, and q_N^{eq} and q_{NoN}^{eq} are determined by Theorem 1.

4. If $\Delta p \geq t_N + \kappa_u \tilde{q}_p$, then $z^{eq} = 0$, and q_N^{eq} and q_{NoN}^{eq} are determined by Theorem 1.

Note that the thresholds $\tilde{p}_{t,1}$, $\tilde{p}_{t,2}$, and $\tilde{p}_{t,3}$ are decreasing with respect to $\frac{\tilde{q}_f}{\tilde{q}_p}$. Thus, as theorem implies, the higher $\frac{\tilde{q}_p}{\tilde{q}_f}$, the higher would be the threshold on \tilde{p} after which the CP chooses the free quality over the premium one. In addition, with high \tilde{q}_p and low t_{NoN} , the CP prefers to choose the strategy by which the neutral ISP is driven out of the market.

2.2.3 Stage 2: ISP NoN determines the side-payment, \tilde{p} :

In this stage, ISP NoN chooses the equilibrium strategy $\tilde{p} = \tilde{p}^{eq}$, with the knowledge of p_{NoN} and p_N , to maximize her payoff:

$$\pi_{NoN}(p_{NoN}, \tilde{p}) = (p_{NoN} - c)n_{NoN} + z\tilde{p}q_{NoN} \quad (2.7)$$

First, we introduce a tie-breaking assumption (Assumption 6) for ISP NoN. In Theorem 3, we characterize the necessary and sufficient condition on \tilde{p}^{eq} by which $z^{eq} = 1$, i.e. the CP chooses the premium quality. Subsequently, in Theorem 4, we characterize \tilde{p}^{eq} by which $z^{eq} = 1$. Note that if $z^{eq} = 0$, (2.7) would be independent of \tilde{p} . Thus, we only need to characterize \tilde{p}^{eq} by which $z^{eq} = 1$. The proofs of theorems are presented in Appendix 2.7.2.

The following tie-breaking assumption for ISP NoN is used to determine the optimum strategy in this stage. In this tie-breaking assumption, we assume that because of legal

complexities of a non-neutral regime, whenever ISP NoN is indifferent between $z^{eq} = 0$ and $z^{eq} = 1$, she chooses \tilde{p} such that $z^{eq} = 0$, i.e. choosing neutrality over non-neutrality⁴.

Assumption 6. *If \tilde{p}_1 by which $(q_N^{eq}, q_{NoN}^{eq}) \in F_1$, i.e. $z^{eq} = 1$, and \tilde{p}_2 by which $(q_N^{eq}, q_{NoN}^{eq}) \in F_0$ yield the same payoff for ISP NoN, this ISP chooses \tilde{p}_2 , i.e. the one that yields $z^{eq} = 0$.*

Recall that in Definition 3, we characterized some threshold values for the side payment. For each value of Δp , the actual threshold on the side payment is equal to one of the thresholds characterized. We define and characterize the actual threshold, i.e. \tilde{p}_t , based on the results in Theorem 2:

Definition 4. *We define $\tilde{p}_t = \tilde{p}_{t,1}$ if conditions of item 1 of Theorem 2 is met, $\tilde{p}_t = \tilde{p}_{t,2}$ if the conditions of items 2-a-ii, 2-b, and 3 of Theorem 2 is met, and $\tilde{p}_t = \tilde{p}_{t,3}$ if the conditions of the item 2-a-i of Theorem 2 is met. Note that $\tilde{p}_{t,1}$, $\tilde{p}_{t,2}$, and $\tilde{p}_{t,3}$ are characterized in Definition 3, respectively.*

The following Theorem characterizes a necessary and sufficient condition on \tilde{p}_t by which $z^{eq} = 1$.

Theorem 3. *$z^{eq} = 1$ if and only if $\pi_{NoN}(p_{NoN}, \tilde{p}_t) > \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ and $\Delta p < t_N + \kappa_u \tilde{q}_p$, where $\pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ is the payoff of ISP NoN when $z^{eq} = 0$.*

The theorem implies that Δp being less than a threshold and the existence of \tilde{p} by which the payoff of ISP NoN is greater than the payoff of this ISP when $z = 0$ are necessary and sufficient conditions for $z^{eq} = 1$. The reason for the former is explained after Theorem 4. The latter follows from the fact that, if the payoff of ISP NoN is not

⁴Although the new rules are not final yet, it is expected that non-neutrality would be accepted by the FCC only under extensive traffic monitoring by the FCC. This introduces an implicit cost for the ISPs.

greater than the payoff of this ISP when $z = 0$, in an NE strategy, ISP would not choose \tilde{p} such that $z^{eq} = 1$, since the strategy of choosing an extremely large \tilde{p} by which $z = 0$ yields a better payoff.

In the following theorem, we characterize \tilde{p} chosen by ISP NoN by which $(q_N^{eq}, q_{NoN}^{eq}) \in F_1$, and also necessary conditions for \tilde{p} by which $z^{eq} = 1$.

Theorem 4. *If $z^{eq} = 1$, then $\tilde{p}^{eq} = \tilde{p}_t$, $\pi_{NoN}(p_{NoN}, \tilde{p}_t) > \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$, and $\Delta p < t_N + \kappa_u \tilde{q}_p$, where $\pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ is the payoff of ISP NoN when $z^{eq} = 0$.*

Thus, the necessary conditions are: (i) in each region, \tilde{p}^{eq} is the maximum side payment by which the CP chooses $z^{eq} = 1$, i.e. the threshold defined in Definition 4, (ii) the payoff of ISP NoN with \tilde{p}^{eq} should be strictly larger than the payoff when $z^{eq} = 0$, and (iii) Δp should be smaller than a threshold (if not the number of EUs on ISP NoN would be zero, and trivially the CP does not offer her content on this ISP).

Remark 1. *Note that, if $z^{eq} = 0$, then the payoff of ISP NoN (2.1) is independent of \tilde{p} . Thus, $\pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ is independent of \tilde{p} .*

2.2.4 Stage 1: ISPs determine p_N^{eq} and p_{NoN}^{eq} :

First, in Theorem 5, we prove that if inertias are small, then there is no NE by which $z^{eq} = 0$. Then, in Theorem 6, we characterize the NE strategies by which $z^{eq} = 1$ for the case that the inertias are small. In Theorem 6, we prove that if inertias are sufficiently small, then a unique NE exists. If not, only under certain conditions a unique NE exists. Numerical simulations under a wide range of parameters (presented in Section 2.5.1) reveal that these conditions are always satisfied.

Then, we focus on the case that inertias are not small. In Theorem 7, we characterize all possible NE strategies by which $z^{eq} = 1$. In Theorem 8, we prove that when one of the inertias is large, the only NE strategy by which $z^{eq} = 1$ is the third candidate strategy of Theorem 7. Then, in Theorem 9, we characterize the only candidate NE strategy by which $z^{eq} = 0$.

By (2.1) and without loss of generality, in the equilibrium, $p_N^{eq} \geq c$. In addition, if $z = 0$, $p_{N_oN}^{eq} \geq c$. If $0 \leq x_n \leq 1$, i.e. $(q_N^{eq}, q_{N_oN}^{eq}) \in F^I$, from (2.5), the payoff of neutral and non-neutral ISPs are:

$$\pi_N(p_N) = (p_N - c) \frac{t_{N_oN} + \kappa_u(q_N - q_{N_oN}) + p_{N_oN} - p_N}{t_N + t_{N_oN}} \quad (2.8)$$

$$\pi_{N_oN}(p_{N_oN}, \tilde{p}) = (p_{N_oN} - c) \frac{t_N + \kappa_u(q_{N_oN} - q_N) + p_N - p_{N_oN}}{t_N + t_{N_oN}} + z q_{N_oN} \tilde{p} \quad (2.9)$$

First, given the strategies of the CP and EUs described in previous sections, we prove that if inertias are small, then there is no NE by which $z^{eq} = 0$:

Theorem 5. *If $t_N + t_{N_oN} \leq \kappa_u \tilde{q}_p$, there is no NE by which $(q_N^{eq}, q_{N_oN}^{eq}) \in F_0$, i.e. $z^{eq} = 0$.*

Next, we characterize the NE strategies by which $z^{eq} = 1$ when inertias are small:

Theorem 6. *If $t_N + t_{N_oN} \leq \kappa_u \tilde{q}_p$, the NE strategies, p_N^{eq} and $p_{N_oN}^{eq}$ by which $(q_N^{eq}, q_{N_oN}^{eq}) \in F_1$, i.e. $z^{eq} = 1$, are:*

1. $p_{N_oN}^{eq} = c + \kappa_u \tilde{q}_p - t_{N_oN}$ and $p_N^{eq} = c$ if and only if $\tilde{q}_p \geq \frac{t_N + 2t_{N_oN}}{\kappa_u + \kappa_{ad}}$.
2. $p_{N_oN}^{eq} = c + \frac{t_{N_oN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{N_oN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$ if and only if $\tilde{q}_p < \frac{t_N + 2t_{N_oN}}{\kappa_u + \kappa_{ad}}$, and $\pi_N(p_N^{eq}) \geq p_t^d - c$ where $p_t^d = \frac{\kappa_{ad} \tilde{q}_p (t_N + t_{N_oN})}{p_{N_oN}^{eq} - c + \kappa_{ad} \tilde{q}_p} + p_{N_oN}^{eq} - t_{N_oN} - \kappa_u \tilde{q}_p$.

We show in Corollary 2 that both sets of strategies are associated with the case that the CP offers with premium quality on ISP NoN and with zero quality on ISP N. In the first set, ISP N would be driven out of the market, while with the second set, ISP N would be active.

Now, we focus on the case that inertias are not small. In the following theorem, we characterize the NE strategies by which $z^{eq} = 1$:

Theorem 7. *If $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$, then the only possible NE strategies by which $(q_N^{eq}, q_{NoN}^{eq}) \in F_1$, i.e. $z^{eq} = 1$, are:*

1. *If $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$, then $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$.*
2. *If (i) $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p^{eq} < \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ or $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) < \Delta p^{eq} < t_N + \kappa_u \tilde{q}_p$, then $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$, and $\pi_{NoN}^{eq} = \pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,2})$. The necessary conditions: (ii) $\tilde{q}_p \leq \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$, and (iii) $\pi_{NoN}^{eq} > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$.*
3. *If (i) $\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p^{eq} < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, then $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{NoN} + t_N - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3}$, and $\pi_{NoN}^{eq} = \pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,3})$. The necessary conditions: (ii) $\tilde{q}_p - \tilde{q}_f \leq \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$, and (iii) $\pi_{NoN}^{eq} > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$.*
4. *$p_{NoN}^{eq} = c$ and $p_N^{eq} = c - \kappa_u(2\tilde{q}_p - \tilde{q}_f) + t_{NoN}$, and $\pi_{NoN}^{eq} = \pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,3})$. The necessary conditions: (i) $2\tilde{q}_p - \tilde{q}_f \leq \frac{t_{NoN}}{\kappa_u}$ and (ii) $\pi_{NoN}^{eq} > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$.*

We show in Corollary 3 that the first two sets of strategies are associated with the case that the CP offers with premium quality on ISP NoN and with zero quality on ISP

N. With the first set, ISP N would be driven out of the market, while with the second, ISP N would be active. The third and fourth sets of strategies are associated with the case that both ISPs are active and the CP offers her content with premium quality on ISP NoN and with free quality on ISP N.

Next, we prove that when either of the transport costs is large enough, then the only NE strategy by which $z^{eq} = 1$ is the third candidate strategy of the previous theorem:

Theorem 8. *When either t_N or t_{NoN} is large enough, for the case that $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$, the only NE strategy by which $z^{eq} = 1$ is $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{NoN} + t_N - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3}$.*

Remark 2. *Note that when at least one of t_N and t_{NoN} is large, then the effect of \tilde{q}_p can be ignored. Thus, this scenario can be considered to be similar to the case that both ISPs are neutral, i.e. the benchmark case. Later, in Theorem 10, we prove that a unique SPNE exists in this case, and it is similar to the NE strategies characterized in Theorem 8 with $\tilde{q}_p = \tilde{q}_f$.*

Now, we characterize the equilibrium strategy by which $z^{eq} = 0$ when inertias are not small:

Theorem 9. *If $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$, the only possible NE strategy by which $(q_N^{eq}, q_{NoN}^{eq}) \in F_0$, i.e. $z^{eq} = 0$ is $p_N^{eq} = c + \frac{1}{3}(2t_{NoN} + t_N)$ and $p_{NoN}^{eq} = c + \frac{1}{3}(2t_N + t_{NoN})$. The necessary condition for this strategy to be a candidate NE strategy is $\pi_{NoN}(p_{NoN}^{eq}, z = 0) \geq \pi_{NoN}(p_{NoN}^{eq}, \tilde{p}t)$.*

Remark 3. *Note that the candidate strategies listed in Theorems 7 and 9 are NE if and only if the conditions listed in the theorem hold and no unilateral deviation is profitable for each ISPs.*

2.3 Benchmark Case: A Neutral Regime

In this section, we consider a benchmark case in which both ISPs are forced to be neutral. Our goal is to find the SPNE when both ISPs are neutral. We compare the results of the benchmark case with the results we found in the previous section. Note that we do not restrict the analysis of this section to any particular range of transport costs, and the analysis is done for a general case.

The main result of this section is Theorem 10. In order to characterize the equilibrium in this case, we can consider a simple change to our previous model and use some of the previous results. We assume that in this case, the CP chooses $z^{eq} = 0$, regardless of the strategy of ISPs. Thus, $(q_N^{eq}, q_{NoN}^{eq}) \in F_0$, and as a result both ISPs are neutral.

Note that in this case, the equilibrium strategy of Stage 4 is similar as before, and the equilibrium strategy of Stage 3 is characterized in Theorem 1. Recall that in Theorem 1, we characterize the equilibrium strategies within F_0 without considering the strategies in F_1 . In addition, note that the strategy of Stage 2 is of no importance since with $z^{eq} = 0$, the effect of \tilde{p} would be eliminated in all analyses. Thus, we only need to find the new equilibrium strategies in Stage 1 of the game:

Theorem 10. *The unique NE strategies chosen by the ISP are $p_N^{eq} = c + \frac{1}{3}(2t_{NoN} + t_N)$ and $p_{NoN}^{eq} = c + \frac{1}{3}(2t_N + t_{NoN})$.*

2.4 The Outcome of the Game and Discussions

First, in Section 2.4.1, we summarize, discuss, and interpret the possible outcomes of the model characterized in the previous section. Then, in Section 2.4.2, we summarize and

discuss the results for a benchmark case in which both ISPs are neutral.

2.4.1 Possible Outcomes of the Market

In Sections 2.2 and 2.3, we have characterized all the possible SPNE strategies. Using these strategies, we have characterized the SPNE outcomes in Appendix 2.7.5. In this section, we summarize, discuss, and interpret these possible outcomes.

Candidate Outcome (a): $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$, $p_N^{eq} = c$, $z^{eq} = 1$, i.e. the CP pays for the premium quality, $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$, $n_N^{eq} = 0$, and $n_{NoN}^{eq} = 1$ (outcome associated with Theorem 6-1 and Theorem 7-1).

Note that from Theorems 5 and 6, this outcome is the unique SPNE of the game if t_N and t_{NoN} are sufficiently small, i.e. EUs are not locked-in with ISPs. This outcome represents the case in which the CP offers the content with premium quality and pays the side-payment to the non-neutral ISP. Note that EUs can receive a better quality of content on the non-neutral ISP, and that yields a better advertisement revenue for the CP. Thus, in the equilibrium, the CP offers her content only on the non-neutral ISP to increase the number of EUs connecting to the Internet via the non-neutral ISP. By doing so, given the conditions of this candidate outcome, all EUs would join the non-neutral ISP and the neutral ISP would be driven out of the market.

In addition, note that the Internet access fee chosen by ISP NoN (p_{NoN}^{eq}) increases with (i) increasing the sensitivity of end-users to the quality (κ_u), (ii) increasing the value of the premium quality (\tilde{q}_p), and (iii) decreasing the transport cost of ISP NoN (t_{NoN}). Recall that t_{NoN} has an inverse relationship with the market power of ISP NoN if t_N is fixed.

Moreover, note that the side-payment charged for the premium quality ($\tilde{p}^{eq}\tilde{q}_p$) is positive, and is dependent on (i) the sensitivity of the payoff of the CP to the quality of the advertisement, i.e. κ_{ad} , and (ii) the difference between the premium and free quality, i.e. $\tilde{q}_p - \tilde{q}_f$. The latter implies that ISP NoN chooses the side-payment in proportion to the additional value created for the CP.

Candidate Outcome (b): $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$, $p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$, $z^{eq} = 1$, $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN}^{eq} - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{t_N + 2t_{NoN} - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$, and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN} + \tilde{q}_p(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$ (outcome associated with Theorem 6-2 and Theorem 7-2).

Candidate outcome (b) represents the case in which both ISPs are active. However, similar to (a), with this outcome, the CP does not offer her content over the neutral ISP, and offers her content only over the non-neutral ISP with premium quality. Thus, although the CP stops offering her content on the neutral ISP, she cannot move all EUs to ISP NoN. The loss in the number of EUs would be compensated by receiving higher advertisement revenue (due to the premium quality) and paying a lower side payment (will be explained in the associated paragraph).

It is noteworthy to observe that if t_N (respectively, t_{NoN}) increases, p_{NoN}^{eq} (respectively, p_N^{eq}) increases with a rate $\frac{2}{3}$ rd the rate of the growth of this transport cost. This is intuitive. The higher t_N (while t_{NoN} fixed), the higher would be the market power of ISP NoN, and subsequently the higher would be p_{NoN}^{eq} . In addition, counter-intuitively, p_N (respectively, p_{NoN}) also increases with a rate $\frac{1}{3}$ rd of the rate of growth of t_N (respectively, t_{NoN}). This counter-intuitive result (Internet access fee of an ISP being an increasing function of the transport cost of this ISP) is because of competition between ISPs. For example, with

the increase of t_{NoN} , EUs have more incentive to join the neutral ISP and less incentive to switch to the non-neutral ISP. Thus the neutral ISP can set a higher price for EUs. This allows her competitor, i.e. ISP NoN, to increase her price, but with a rate lower than the rate by which the price of ISP N increases.

In addition, note that p_N^{eq} is a decreasing function of \tilde{q}_p , κ_u , and κ_{ad} : The higher the premium quality or the sensitivity of EUs and the CP to the quality, the lower would be the Internet access fee of ISP N. On the other hand, the relationship between these parameters and p_{NoN}^{eq} is more complicated. The Internet access fee of ISP NoN is increasing with respect to the sensitivity of EUs to the quality, and is decreasing with respect to the sensitivity of the CP to the quality. Thus, the more the CP is sensitive to the quality, the more the ISP NoN provides subsidies for EUs (cheaper Internet access fees), and compensates the payoff through charging the CP. In addition, note that p_{NoN}^{eq} is decreasing or increasing with respect to the amount of premium quality (\tilde{q}_p) depending on the sensitivity of EUs and the CP to the quality: If the sensitivity of EUs to the quality dominates the sensitivity of the CP ($\kappa_u > 2\kappa_{ad}$), then p_{NoN}^{eq} is increasing with respect to \tilde{q}_p . If not, then ISP NoN generates more revenue from the CP, and thus provide a cheaper Internet access fee for EUs. The higher this sensitivity, the higher would be the side payment from the CP (can be seen from the expression of \tilde{p}^{eq}), and the higher would be the discount on the Internet access fees for EUs.

Moreover, note that the side-payment charged for the premium quality ($\tilde{p}^{eq}\tilde{q}_p$) is increasing with respect to (i) κ_{ad} (the sensitivity of the CP to the quality), (ii) the premium quality (\tilde{q}_p), and (iii) number of EUs with the non-neutral ISP (n_{NoN}^{eq}), and is decreasing with respect to the free quality (\tilde{q}_f). Note that since in this case $n_{NoN} < 1$,

the side payment would be lower than the side payment in candidate outcome (a). This side-payment can be positive or negative. However, as we explain later, the numerical results reveal that the side-payment is positive whenever this candidate outcome is an SPNE.

In addition, note that n_{NoN} is increasing with respect to the premium quality, i.e. \tilde{q}_p , and the sensitivity of the CP and EUs to the quality, i.e. κ_u and κ_{ad} . The relationship between n_{NoN} (and thus n_N) and the transport costs, i.e. t_N and t_{NoN} is more complex and is discussed in Section 2.5.2.

Candidate Outcome (c): $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad})}{3}$, $p_N^{eq} = c + \frac{2t_{NoN} + t_N - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3}$, $z^{eq} = 1$, $\tilde{p}^{eq} = \tilde{p}_{t,3} = \kappa_{ad} n_{NoN}^{eq} (1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{t_N + 2t_{NoN} - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$, and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$ (outcome associated with Theorem 7-3). Recall that in Theorem 8, we proved that when either of t_N or t_{NoN} is large, then the only candidate outcome by which $z^{eq} = 1$ is (c).

Candidate outcome (c) represents the case that both ISPs are active, and the CP offers her content with free quality on the neutral ISP and with premium quality on the non-neutral one. The dependencies of the access fees (p_N^{eq} and p_{NoN}^{eq}) to t_N , t_{NoN} , κ_u , and κ_{ad} are the same as what described for candidate outcome (b). In addition, note that p_N^{eq} is decreasing with the difference between the premium and free qualities, i.e. $\tilde{q}_p - \tilde{q}_f$, and p_{NoN}^{eq} is decreasing or increasing with respect to the difference in the qualities depending on the sensitivity of EUs and the CP to the quality (similar to the description for the candidate outcome (b)).

Moreover, note that the side-payment charged for the premium quality ($\tilde{p}^{eq} \tilde{q}_p$) is increasing with respect to (i) κ_{ad} (the sensitivity of the CP to the quality), (ii) the

difference between the premium and free qualities ($\tilde{q}_p - \tilde{q}_f$), (iii) number of EUs with the non-neutral ISP (n_{NoN}^{eq}). This side-payment is always positive. The dependencies of n_{NoN} to the parameters are similar to what described for candidate outcome (b), with the difference that n_{NoN} depends on the difference in the qualities, i.e. $\tilde{q}_p - \tilde{q}_f$, instead of only \tilde{q}_p .

Note that when either of t_N or t_{NoN} is large, then (c) is the only candidate outcome by which $z^{eq} = 1$. First, recall that the payoff that an ISP receives depends on both the number of EUs and the Internet connection fee of that ISP. In addition, we discussed that when either of t_N or t_{NoN} is large, then both of the Internet connection fees would be large in candidate outcomes (b) and (c). It turns out that when t_N or t_{NoN} is large, ISPs prefer candidate outcomes (b) and (c) to the outcomes by which they discount the price heavily to attract EUs ((a) and (d)).

Moreover, when both ISPs are active, large t_{NoN} or t_N decreases the effect of quality of the content on the decision of EUs (both through high transport costs and increase in the Internet access fees). Thus, the CP cannot control the number of EUs with each ISP by strategically controlling her quality. Therefore the CP simply chooses to provide with maximum possible quality on both ISPs instead of choosing strategic qualities to control the equilibrium. Thus, (c) is expected to occur.

Candidate Outcome (d): $p_{NoN}^{eq} = c$, $p_N^{eq} = c - \kappa_u(2\tilde{q}_p - \tilde{q}_f) + t_{NoN}$, $z^{eq} = 1$, $\tilde{p}^{eq} = \tilde{p}_{t,3} = \kappa_{ad}n_{NoN}^{eq}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{\kappa_u\tilde{q}_p}{t_N+t_{NoN}}$, and $n_{NoN}^{eq} = \frac{t_N+t_{NoN}-\kappa_u\tilde{q}_p}{t_N+t_{NoN}}$ (outcome associated with Theorem 7-4).

Candidate outcome (d) represents the scenario in which the non-neutral ISP is forced to provide a low Internet access fee for EUs. This strategy can only be valid when t_{NoN} is

large (so that $p_N^{eq} \geq c$). In other words, the only scenario that this strategy is possible is when EUs are reluctant joining the non-neutral ISP. Thus, this ISP should provide large discounts for EUs. Note that in this case, both ISPs are active, and the CP offers her content over both ISPs, with free quality on the neutral ISP and with premium quality on the non-neutral one.

In this case, p_N^{eq} is decreasing with respect to κ_u and \tilde{q}_p , and increasing with respect to \tilde{q}_f and t_{NoN} . In addition, the side payment is similar to the one in candidate outcome (c).

In this candidate outcome, p_{NoN}^{eq} is fixed, while p_N^{eq} is decreasing with respect to \tilde{q}_p and κ_u . In addition, the rate of decrease of p_N^{eq} is twice of the rate of increase of utility of EUs from κ_u and \tilde{q}_p when connecting to ISP NoN. Thus, The rate of increase in the utility of EUs for ISP N is higher than that of ISP NoN, and as result confirms, n_N^{eq} would be increasing with respect to the premium quality and the sensitivity of EUs to the quality. In addition, p_N^{eq} is increasing with t_{NoN} . Thus, as results confirm, n_N^{eq} would be decreasing with respect to the transport cost of ISP NoN⁵. Finally, note that the Internet access fees are independent of t_N , but the utility of EUs connecting to neutral ISP is decreasing with t_N (2.3). Thus, as result confirms, the number of EUs with the neutral ISP, i.e. n_N^{eq} , is decreasing with respect to both t_N .

Candidate Outcome (e): $p_{NoN}^{eq} = c + \frac{1}{3}(2t_N + t_{NoN})$, $p_N^{eq} = c + \frac{1}{3}(2t_{NoN} + t_N)$,

⁵Note that the utility of EUs connecting to ISP NoN is also decreasing with t_{NoN} (2.3). However, the rate of decrease in the utility of EUs connecting to ISP NoN (t_{NoN} is multiplies to a coefficient smaller than one) is lower than the rate of increase of the price of the neutral ISP (multiplied by one). Thus, overall, the number of EUs with the neutral (respectively, non-neutral) ISP is decreasing (respectively, increasing).

$z^{eq} = 0$, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_f) \in F_0^L$, $n_N^{eq} = \frac{2t_{NoN} + t_N}{3(t_{NoN} + t_N)}$, $n_{NoN}^{eq} = \frac{2t_N + t_{NoN}}{3(t_N + t_{NoN})}$, and since $z^{eq} = 0$, \tilde{p}^{eq} is of no importance (outcome associated with Theorem 9).

This case characterizes the only possible SPNE outcome by which $z^{eq} = 0$. This outcome is similar to the benchmark one (Section 2.4.2). Outcome (c) would be reduced to (e), if $\tilde{q}_p = \tilde{q}_f$.

Remark: Note that candidate strategies in different theorems (defined for different regions of t_N and t_{NoN}) can be similar and yield similar outcomes, e.g. Theorem 6-1 and Theorem 7-1. In addition, there is no outcome in which the CP offers her content only on the neutral ISP. From the expression of payoff of the CP (2.2), the CP can get at most $\kappa_{ad}\tilde{q}_f$ by offering only on the neutral ISP. On the other hand, the CP can guarantee a payoff of this amount by offering on both ISPs and $z = 0$. Assumption 4, i.e. the CP prefers to offer on both ISP whenever she is indifferent, yields that the CP never choose the strategy in which she offers only on the neutral ISP.

Interplay Between Sensitivities to Quality and the Outcome: Intuitively, we expect that high sensitivity of EUs and the CP to the quality, i.e. large κ_u and κ_{ad} , respectively, yields high payoff for the non-neutral ISP, since this ISP can provide a premium quality and charge the EUs accordingly to increase her payoff. Thus, the payoff can be collected from EUs or the CP, or both. Results reveal that in all candidate outcomes ISP NoN charges the CP in proportion to her sensitivity to the quality of the content. In addition, in candidate outcomes (a) to (c), the payoff collected from EUs through the Internet connection fees is always increasing with respect to the sensitivity of the EUs to the quality. In candidate outcomes (b) and (c), the Internet connection fees are decreasing with respect to the sensitivity of the CP to the quality. Thus, in these

candidate outcomes, EUs receive a discount in proportion to the sensitivity of the CP to the quality. In candidate outcome (d), the Internet connection fee of ISP NoN does not depend on the qualities, but it is as low as the marginal cost.

Existence of NE: An SPNE may not always exist. For example, for parameters $\tilde{q}_f = 1$, $\tilde{q}_p = 1.5$, $c = 1$, $\kappa_u = 1$, $\kappa_{ad} = 0.5$, $t_N = 3$, and $t_{NoN} = 2$, none of the candidate outcomes listed above would be an SPNE. The reason is that there exists a profitable deviation for at least one of the ISPs for those candidate strategies that their conditions are satisfied given these parameters. Later in Section 2.5.1, we identify the regions with no SPNE.

2.4.2 Benchmark: A Neutral Scenario

In the benchmark case, i.e. when both ISPs are neutral, we proved that there exists a unique SPNE, and the unique equilibrium outcome of the game is (the subscript B refers to “Benchmark”):

- Stage 1 - Internet access Fees chosen by ISPs: $p_{N,B}^{eq} = c + \frac{1}{3}(2t_{NoN} + t_N)$ and $p_{NoN,B}^{eq} = c + \frac{1}{3}(2t_N + t_{NoN})$.
- Stage 2 - Side Payment chosen by ISP NoN is of no importance.
- Stage 3 - Qualities chosen by the CP: $q_{NoN,B}^{eq} = \tilde{q}_f$ and $q_{N,B}^{eq} = \tilde{q}_f$.
- Stage 4 - Fractions of EUs with ISPs: $n_{N,B}^{eq} = \frac{2t_{NoN} + t_N}{3(t_{NoN} + t_N)}$ and $n_{NoN,B}^{eq} = \frac{2t_N + t_{NoN}}{3(t_{NoN} + t_N)}$.

Note that this case is similar to candidate outcome (e), i.e. the only candidate outcome of our model by which $z^{eq} = 0$. In this case, both ISPs are active and the CP offers the free quality on both ISPs. Note that in this case, the asymmetries of the model only arise

from the asymmetry in t_N and t_{NoN} . Thus, EUs are divided between ISPs depending on t_N and t_{NoN} , and the Internet access fees (p_N and p_{NoN}) are a function of transport costs (t_N and t_{NoN}). Also, similar to the candidate outcome (b) of the previous section, if t_N (respectively, t_{NoN}) increases, p_{NoN} (respectively, p_N) increases with a rate $\frac{2}{3}$ rd the rate of the growth of this transport cost. Also, counter-intuitively, p_N (respectively, p_{NoN}) increases with a rate $\frac{1}{3}$ rd of the rate of growth of t_N (respectively, t_{NoN}).

In this case, Internet access fees are independent of the quality provided for EUs, i.e. \tilde{q}_f . Recall that in contrast, in a non-neutral regime, the Internet access fee quoted by ISP NoN is dependent on the quality she provides (\tilde{q}_p). The reason lies in the product differentiation in the latter. The non-neutral ISP can charge for the quality she provides for EUs through differentiating her product from the neutral ISP. While in a neutral regime, no ISP can charge for the quality they provide because of competition. It is noteworthy that if $t_{NoN} \& t_N \rightarrow 0$, $p_{NoN}^{eq} \& p_N^{eq} \rightarrow c$. In other words, in the absence of inertias, since there is no differentiation between the quality offered by the ISPs in the neutral regime, price competition drives the access fees to the marginal cost. This implies that by removing the inertias (t_N and t_{NoN}), the model would be similar to a Bertrand competition [45]. Thus, considering the inertias brings a realistic dimension to the model.

The relationship between n_N^{eq} and n_{NoN}^{eq} and the transport costs are similar to that of candidate outcomes (b) and (c) of the previous section, and is investigated in Section 2.5.1.

2.5 Numerical Results

First, in Section 2.5.1, using numerical analysis, we find the NE strategies of Stage 1 for various parameters. Recall that strategies of Stage 1 yield one of the outcomes (a)-(e). In Section 2.5.2, we complement the discussions in Section 2.4.1, by providing more intuitions about n_{NoN}^{eq} , \tilde{p}^{eq} , and the payoff of ISPs, based on the numerical results. We assess the benefits of non-neutrality by comparing the results of the model with the benchmark case in Section 2.5.3. In Section 2.5.4, we provide regulatory comments based on the results.

2.5.1 NE Strategies

Recall that if SPNE exists, it would be of the form of outcomes (a)-(e) (Section 2.4.1). Now, we check whether these outcomes are indeed SPNE. We only need to check whether the candidate strategies of Stage 1 are NE. For doing so, we check for any profitable deviation for each ISPs. To check for unilateral deviations, we consider different regions of Δp (regions characterized in Theorem 2). In each region, we can identify potential profitable deviations (using first order condition for some regions, and the fact that payoff of ISPs are monotonic in other regions). Thus, the search for the best deviations is equivalent to comparing the payoff of finite number of candidate deviations with the payoff of the candidate equilibrium. We also check conditions listed in Theorems 6, 7, and 9.

We now present two illustrative examples. In Figure 2.2a, we identify the NE strategies of stage 1 for different regions of t_N and t_{NoN} when $\kappa_u = 1$ and $\kappa_{ad} = 0.5$. In Figure 2.2b, we identify the NE strategies when $\kappa_u = 0.5$ and $\kappa_{ad} = 1$. For the figures, we consider $\tilde{q}_f = 1$, $\tilde{q}_p = 1.5$, and $c = 1$. Numerical results for a large set of parameters reveal that

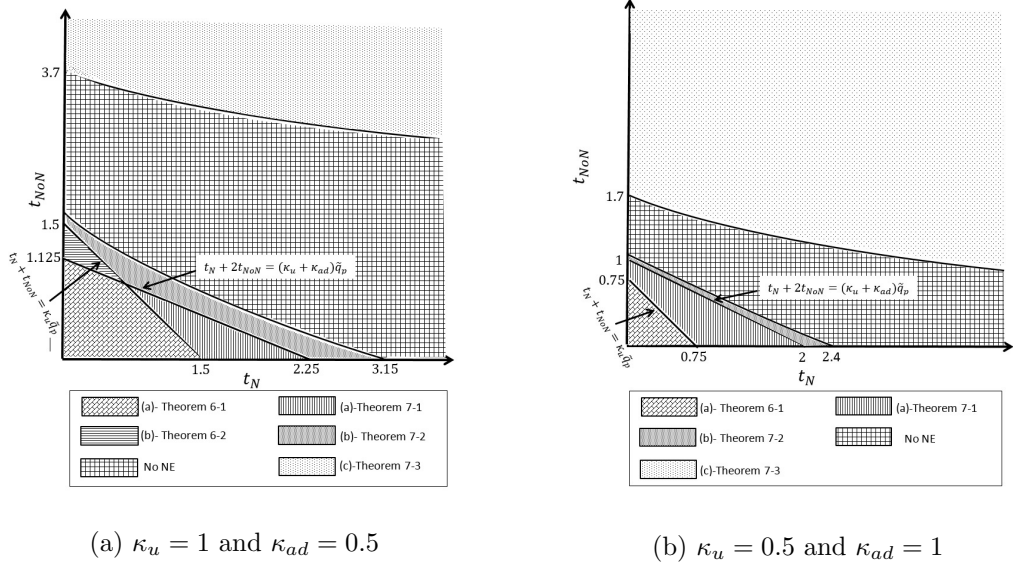


Figure 2.2: NE strategies of Stage 1 for various t_N and t_{NoN}

the pattern of NE strategies for different values of parameters is similar to one of the two pattern presented in Figures 2.2a and 2.2b. Overall, the outcome in which the neutral ISP is driven out of the market occurs when t_N and t_{NoN} are small. As t_N and t_{NoN} increases, we expect to have equilibrium outcomes in which both ISPs are active. Next, we discuss about the trends we observe in the results.

Note that in Theorem 6, we proved that, for $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$ and $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$, candidate strategy (Theorem 6-1) is an NE. Numerical results for a large set of parameters also reveal that for $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$ and $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$, candidate strategy (Theorem 7-1) is also an NE strategy. Note that these two strategies are the same and are listed under candidate outcome (a). Therefore, when $\frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}} \leq \tilde{q}_p$, (a) is an SPNE outcome. In this case, since the transport costs are low, EUs can easily switch ISPs. Thus, ISP NoN is able to attract all EUs by discounting the Internet access fee for EUs using some of the side payment received from the CP. Therefore, the neutral ISP would be driven out of

the market.

With increase in t_N or t_{NoN} , EUs have more inertia. Thus, one of the ISPs should provide a low Internet access fee for EUs to attract them all. However, in this case, ISPs prefer to maintain a high Internet access fee for EUs⁶, and split the EUs. Thus, as t_N and t_{NoN} increases, we expect to have equilibrium outcomes in which both ISPs are active. Numerical results reveal that if $\tilde{q}_p < \frac{t_N+2t_{NoN}}{\kappa_u+\kappa_{ad}}$ and $\tilde{q}_p \geq \frac{t_N+t_{NoN}}{\kappa_u}$, candidate strategy (Theorem 6-2) is an NE. In addition, consider the lines $t_N + 2t_{NoN} = \tilde{q}_p(\kappa_u + \kappa_{ad})$ and $t_N + t_{NoN} = \kappa_u \tilde{q}_p$. Results reveal that when the point (t_N, t_{NoN}) is just above these lines, the candidate strategy (Theorem 7-2) is an NE strategy. When (t_N, t_{NoN}) is substantially above these lines, then candidate strategy (Theorem 7-3) is an NE strategy. This result have been proved in Theorem 8. In addition, when (t_N, t_{NoN}) is above these lines, but is in an intermediate range, then no NE exists.

Numerical results for large set of parameters also reveal that the NE is unique, if it were to exist (in Figures there exists only one NE in each region). In addition, extensive numerical results reveal that candidate outcomes (d) and (e) are never SPNE. Thus, henceforth we do not include (d) and (e) in our discussions about the results.

2.5.2 Dependencies of n_{NoN}^{eq} , \tilde{p}^{eq} , and Payoffs of ISPs to t_N and t_{NoN}

Note that in Section 2.4.1, we explained that the relationship between n_{NoN}^{eq} and the transport costs is not obvious from the expressions. Thus, in this section, we provide intuitions for the behavior of n_{NoN}^{eq} , and subsequently \tilde{p}^{eq} and the payoffs of ISPs with

⁶As we discussed, when both ISPs are active, the Internet connection fees are increasing with the transport costs. In other words, each ISP lock in some EUs and charge high Internet access fees to them.

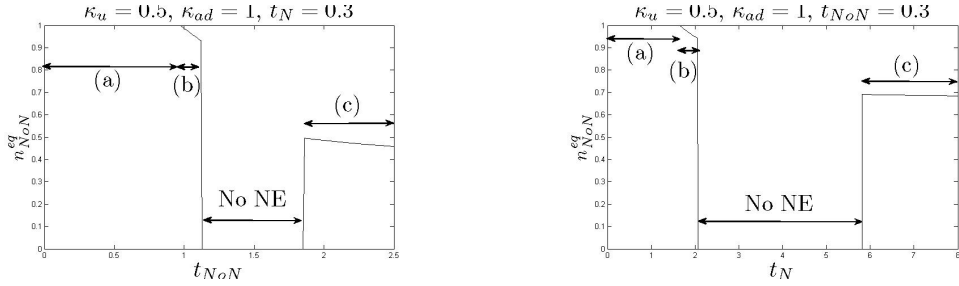


Figure 2.3: n_{NoN}^{eq} with respect to t_N and t_{NoN}

respect to t_N and t_{NoN} .

Numerical Results on n_{NoN}^{eq} : Numerical results reveal that n_{NoN}^{eq} is non-increasing with respect to both transport costs. For instance, in Figure 2.3, we plot the value of n_{NoN}^{eq} with respect to t_{NoN} and t_N , when $\tilde{q}_f = 1$ and $\tilde{q}_p = 1.5$. Recall that $n_N^{eq} = 1 - n_{NoN}^{eq}$. Thus, we only plot n_{NoN}^{eq} .

Note that for candidate outcome (a), as we know from the results, $n_{NoN}^{eq} = 1$. To understand the results for candidate outcomes (b) and (c), note that from (2.5) the number of EUs with each ISP has a decreasing relation with (i) the transport costs of the ISP, and (ii) the Internet access fee of the ISP which itself is increasing with both transport costs. In addition, the number of EUs with the ISP has an increasing relation with respect to the same parameters for the other ISP. Thus, different factors, some decreasing and some increasing with respect to the transport cost of an ISP, play a role in determining the number of EUs with each ISP. Overall, it turns out that the effect of increasing either of the transport costs decreases the incentive of EUs to join ISP NoN. Thus, in candidate outcomes (b) and (c), n_{NoN}^{eq} is decreasing with respect to both transport costs.

Numerical Results on \tilde{p}^{eq} : Note that the higher the number of EUs with ISP NoN, the higher would be the benefit of the CP from the premium quality. Thus, we expect

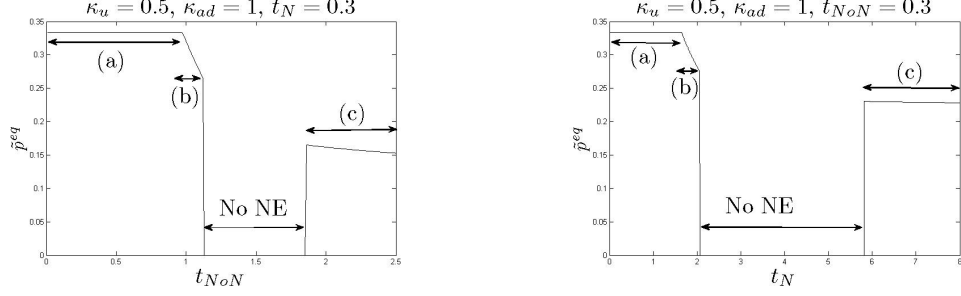


Figure 2.4: \tilde{p}^{eq} with respect to t_N and t_{NoN}

the side-payment, i.e. \tilde{p}^{eq} to be increasing with respect to number of EUs with ISP NoN. Results in Section 2.4.1 also confirms this fact. Thus, the relationship between \tilde{p}^{eq} and the transport costs is similar to the relationship between n_{NoN}^{eq} and the transport costs. Therefore, in candidate outcome (b) and (c), the higher one of the transport costs, the lower would be the side payments. For instance, in Figures 2.4, we plot the value of \tilde{p}^{eq} with respect to t_{NoN} and t_N , respectively, when $\tilde{q}_f = 1$ and $\tilde{q}_p = 1.5$.

Note that as we discussed in Section 2.4.1, in candidate outcome (b), \tilde{p}^{eq} can be positive or negative. However, numerical results for a large set of parameters reveal that \tilde{p}^{eq} is positive, whenever this candidate outcome is SPNE.

Numerical Results on the Payoffs of ISPs: Numerical results for the case $\tilde{q}_f = 1$ and $\tilde{q}_p = 1.5$ are plotted in Figure 2.5. If there is no NE strategy, we plot the payoff of ISPs in the benchmark case, i.e. when both ISPs are neutral.

Note that when the market of power ISP NoN is small, i.e. the fraction $\frac{t_N}{t_N + t_{NoN}}$ is small, then the payoff of ISP NoN would be lower than the payoff of ISP N (Figure 2.5-left).

For candidate outcome (a), the payoff of ISP N is zero (since the number of EUs with this ISP is zero), and the payoff of ISP NoN is independent of t_N (since ISP N is

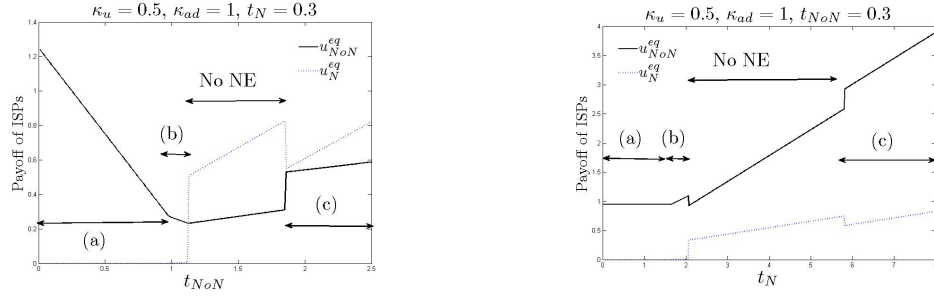


Figure 2.5: Payoff of ISPs with respect to t_N and t_{NoN}

out of the market), but decreasing with respect to t_{NoN} (since p_{NoN}^{eq} is decreasing with t_{NoN}). Intuitively, we expect the utility of an ISP to be decreasing with respect to the transport cost of that ISP, and increasing with respect to the transport cost of the other ISP. However, for some parameters and some of the candidate outcomes, results reveal that the payoff of an ISP is increasing with the transport cost of the ISP. Next, we explain the underlying reasons for this counter-intuitive behavior.

Note that the payoff of an ISP is increasing with (i) the number of EUs with the ISP and also (ii) the Internet access fee charged to the EUs. Recall that for the neutral ISP, in candidate outcomes (b), (c), and the benchmark case, both of (i) and (ii) are increasing with respect to both transport costs. Thus, the payoff of ISP N is increasing with respect to both transport costs. On the other hand, for ISP NoN, the number of EUs is decreasing and the Internet access fee is increasing with the transport costs. Thus, depending on which of these factors overweighs the other one, the payoff of ISP NoN can be decreasing or increasing with respect to the transport costs.

2.5.3 Profits of Entities Due to Non-neutrality

We compare the results of the model and the benchmark case in which both ISPs are neutral. We compare Internet access fees, payoff of ISPs, the welfare of EUs, and the payoff of the CP:

Internet Access Fees

In a non-neutral case, the neutral ISP would always decrease her Internet access fee, while that of the non-neutral ISP could be higher or lower depending on the parameters of the market. We now provide insights on when each of these scenarios happens.

First, note that the discount that ISP N provides for EUs in a non-neutral case, i.e. $p_{N,B}^{eq} - p_N^{eq}$, is always positive for candidate outcomes (a), (b), and (c) (using previous results). Thus, the neutral ISP would always decrease her Internet access fee in a non-neutral scenario in order to compete with the non-neutral ISP which is now offering a better quality.

In a non-neutral regime, if (a) occurs, then the discount that ISP NoN provides for EUs in a non-neutral case is $p_{NoN,B}^{eq} - p_{NoN}^{eq} = \frac{1}{3}(5t_{NoN} + t_N) - \kappa_u \tilde{q}_p$ (using the previous results). This discount can be negative or positive, and is decreasing with κ_u and \tilde{q}_p , and increasing with t_{NoN} and t_N . Thus, if (i) EUs are not sensitive to the quality, i.e. small κ_u , (ii) ISP NoN does not provide a high premium quality, i.e. small \tilde{q}_p , (iii) end-users cannot switch between ISPs easily, i.e. t_N and t_{NoN} large enough, or a combination of these factors, then ISP NoN provides a cheaper Internet access fee for EUs in comparison to the neutral scenario.

For candidate outcome (b) (respectively, (c)), using the results in Sections 2.4.1 and

2.4.2, the amount of discount is $p_{NoN,B}^{eq} - p_{NoN}^{eq} = \frac{1}{3}\tilde{q}_p(2\kappa_{ad} - \kappa_u)$ (respectively, $p_{NoN,B}^{eq} - p_{NoN}^{eq} = \frac{1}{3}(\tilde{q}_p - \tilde{q}_f)(2\kappa_{ad} - \kappa_u)$). Thus, if $2\kappa_{ad} > \kappa_u$, i.e. the sensitivity of the CP is high enough, then the discount is positive and is increasing with the premium quality (respectively, the difference between the premium and free quality). On the other hand, if the sensitivity of the CP is low, then the discount is negative, i.e. ISP NoN charges higher access fees to the EUs. The reason is that if the CP is sensitive to the quality, ISP NoN can charge higher side-payments to the CP. Thus, she can provide some of these new revenue to EUs as a discount even though they receive a premium quality. This is not possible when the CP is not sensitive to the quality of her content. In this case, ISP NoN charges the premium quality to the EUs directly, i.e. higher Internet access fees for EUs.

Payoff of ISPs

Consider the payoffs of ISPs N and NoN under both neutral and non-neutral scenarios. The difference in the payoffs for the case $\kappa_u = 0.5$, $\kappa_{ad} = 1$, $\tilde{q}_f = 1$, $\tilde{q}_p = 1.5$, and $t_N = 0.3$ are plotted in Figure 2.6 (using different parameters values yields same intuitions).

Results reveal that the neutral ISP will lose payoff in all of the non-neutral NE strategies, i.e. those that yield $z^{eq} = 1$ (Figure 2.6-right). Note that in case (a), ISP N would be driven out of the market. Thus, $\pi_N^{eq} = 0$, while $\pi_{N,B}^{eq} > 0$. In cases (b) and (c), although ISP N is active, she has to subsidize the Internet connection fee for EUs to be able to compete with ISP NoN, while possibly can attract lower number of EUs. This yields a loss in the payoff under a non-neutral scenario.

Results also reveal that for a wide range of parameters, ISP NoN receives a better payoff

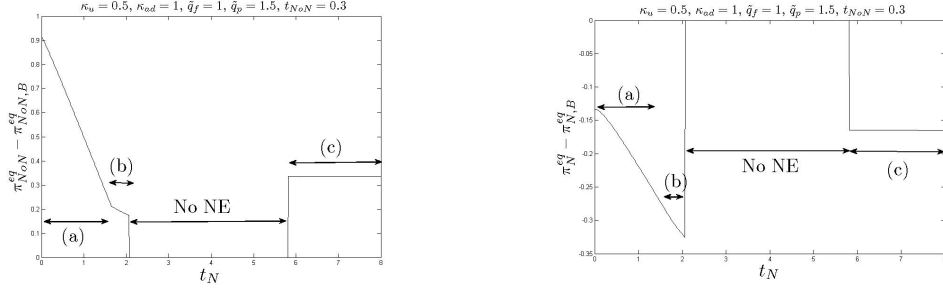


Figure 2.6: The difference between the payoff of ISPs for two scenarios with respect to t_N and t_{NoN}

under a non-neutral scenario (Figure 2.6-left). We discussed that ISP NoN extracts the additional profit of the CP (from the premium quality her EUs receive) in a non-neutral scenario. In addition, we also explained that for some parameters ($\kappa_u > 2\kappa_{ad}$), ISP NoN charges higher prices to EUs. Even when ISP NoN subsidizes the Internet access fee for EUs ($2\kappa_{ad} > \kappa_{ad}$), she would compensate through the side payment charged to the CP (high κ_{ad} yields a high side payment). Moreover, ISP NoN can potentially attract more EUs by providing a cheaper fee or a premium quality (or both). Thus, overall we expect the non-neutral ISP to receive a better payoff under a non-neutral regime.

However, we can find scenarios in which ISP NoN loses payoff by switching to non-neutrality. For example, with $\kappa_u = \kappa_{ad} = 0.85$, $\tilde{q}_f = 1$, $\tilde{q}_p = 1.03$, $t_N = 0.05$, and $t_{NoN} = 0.8$, then $\pi_{NoN}^{eq} < \pi_{NoN,B}^{eq}$. In particular, the payoff of ISP NoN decreases in a non-neutral regime if the outcome of the market is (a), and κ_u , κ_{ad} , $\tilde{q}_p - \tilde{q}_f$, and $\frac{t_N}{t_N + t_{NoN}}$ (the market power of ISP NoN) are small.

We now explain the underlying reason for this counter-intuitive result. Note that knowing that the other ISP has switched to non-neutrality, the neutral ISP would decrease

her Internet access fee for EUs to compensate for the superior quality that her competitor offers. On the other hand, the non-neutral ISP also has to significantly decrease her Internet access fee for EUs (because of her low market power, competition, and low sensitivity of EUs to the quality), while not generating enough revenue from the side-payments received from the CP (because of low sensitivity of the CP to quality or a premium quality that is not significantly better than a free quality). This makes both ISPs, lose revenue in a non-neutral setting under the specified conditions. Note that the non-neutral ISP still extracts the additional profit she creates for EUs.

EU's Welfare

Recall that from (2.3), the utility of an EU who connects to the ISP $j \in \{N, \text{NoN}\}$ located at distance x_j of the ISP, and is receiving the content with quality q_j , is $u_{EU,j} = v^* + \kappa_u q_j - t_j x_j - p_j$. Now, let us define the Welfare of EUs (EUW) for an EU connected to ISP j located at distance x_j from this ISP to be $u_{EU,j}(x_j) - v^* = \kappa_u q_j - p_j - t_j x_j$. Note that we dropped the common valuation v^* since it is equal for all EUs in all scenarios, and is only used to guarantee the full coverage of the market, i.e. to prevent negative utility for EUs. Thus, the total welfare of EUs is:

$$\begin{aligned} EUW &= \int_0^{n_N} (\kappa_u q_N - p_N - t_N x) dx + \int_{n_N}^1 (\kappa_u q_{NoN} - p_{NoN} - t_{NoN}(1-x)) dx \\ &= (\kappa_u q_N - p_N) n_N - \frac{t_N}{2} n_N^2 + (\kappa_u q_{NoN} - p_{NoN}) n_{NoN} - \frac{t_{NoN}}{2} n_{NoN}^2 \end{aligned} \quad (2.10)$$

Note that since we dropped v^* , EUW could be negative. In Figures 2.7 and 2.8, we plot the difference in the EUW of the non-neutral case with the benchmark case for various parameters of the market, when $\tilde{q}_f = 1$ and $\tilde{q}_p = 1.5$.

Results reveal that in general, EUW would be higher in a non-neutral setting if (i) the market power of ISP NoN is low, (ii) the sensitivity of the CP to the quality is high, or (iii)

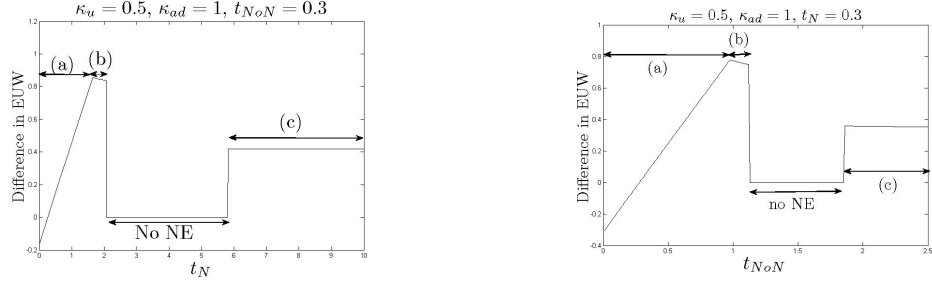


Figure 2.7: Difference in EUW with respect to t_N and t_{NoN}

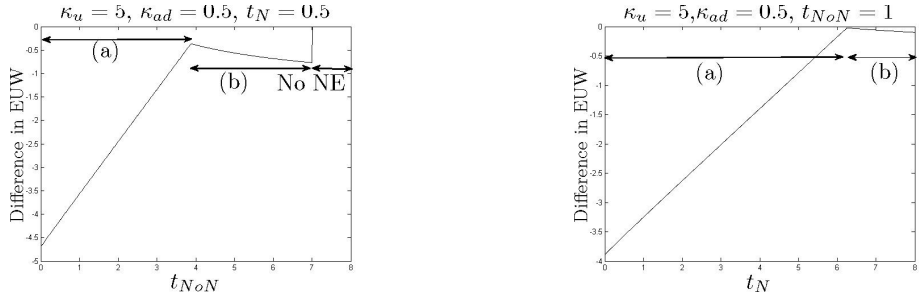


Figure 2.8: Difference in EUW with respect to t_N and t_{NoN}

EUs are not very sensitive to the quality, or a combination of these conditions. However, when both transport costs are sufficiently small, or the sensitivity of EUs (respectively, the CP) to the quality is high (respectively, low), then the benchmark case yields a better EUW in comparison to the non-neutral case. We next explain the reasons behind these results.

Consider the benchmark case. In this case, the welfare of EUs is dependent on the transport costs and the Internet access fees determined by ISPs N and NoN. Recall that both access fees are increasing with t_N and t_{NoN} . Thus, intuitively, EUW of the benchmark case is decreasing with t_N and t_{NoN} .⁷

⁷Note that n_N and n_{NoN} are sum up to one. Thus, the effect of access fees on EUW is more than the effect of number of EUs with each ISP.

In case (a), in which only the non-neutral ISP is active, EUW is dependent on the Internet access fee of ISP NoN, i.e. $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$. Thus, EUW of the non-neutral scenario with outcome (a) is increasing with respect to t_{NoN} . In other words, if t_{NoN} is large, ISP NoN should provide a cheaper Internet access fee (subsidizing the access fee), to attract EUs and keep the neutral ISP out of the market. Thus, EUW would be high. In addition, the EUW is independent of t_N . Thus, as Figures 2.7 and 2.8 confirm, the difference between the EUW of the non-neutral scenario in case (a) and the EUW of the benchmark case is increasing with respect to t_N and t_{NoN} .

We observe that when both transport costs are sufficiently small, the benchmark case yields a higher EUW than the non-neutral scenario. Note that if t_{NoN} is small, i.e. EUs can join (switch to) ISP NoN without incurring high transport costs, ISP NoN attracts all EUs even when quoting a high Internet access fee for EUs (since it offers a premium quality). Thus, ISP NoN charges a high Internet access fee, and the EUW would be small. On the other hand, if t_N is also small, the EUW of the benchmark case would be high (as discussed previously). Thus, when both transport costs are sufficiently small, we expect the benchmark case to yield a better EUW in comparison to the non-neutral case. Negative differences in Figures 2.7 and 2.8 confirm this intuition. Note that in Figure 2.8, because of high sensitivity of EUs to the quality, EUW of the neutral scenario is higher than the non-neutral scenario even when t_N or t_{NoN} are not small. Finally, observe that the maximum difference in the EUWs is achieved for the highest t_N and t_{NoN} by which the outcome of the game is (a), i.e. when only the non-neutral is active.

For candidate outcomes (b) and (c), similar to the benchmark case, the Internet access fees are increasing with respect to t_N and t_{NoN} . Thus, EUW is expected to be decreasing

with respect to these transport costs. Results in the figures reveal that the difference in EUWs is decreasing with respect to t_{NoN} and t_N . This means that EUW of the non-neutral case decreases more than EUW of the benchmark case. This difference is positive when the sensitivity of the EUs to the quality is low, i.e. small κ_u (Figure 2.7), and negative when κ_u is large (Figure 2.8). Recall that the non-neutral ISP provides discount to EUs when the sensitivity of the CP to the quality is high enough. If not, ISP NoN charges higher prices to EUs in comparison to the benchmark case. This is the reason that EUW of the non-neutral case is lower than the benchmark case when EUs are highly sensitive to the quality they receive.

Thus, the transport costs and the sensitivity of EUs and the CP to the quality are the important factors in comparing the EUW of the neutral and non-neutral scenario. As explained, the higher the sensitivity of the CP (respectively, EUs) to the quality, the higher (respectively, lower) would be EUW in the non-neutral case.

Payoff of the CP

Using (2.2) and the candidate outcomes listed in Sections 2.4.1 and 2.4.2, we can calculate the payoff of the CP in different outcomes. Results yield that the equilibrium payoff of the CP in all the possible outcomes of the non-neutral scenario and also in the benchmark scenario are equal and are $\pi_{CP}^{eq} = \pi_{CP,B}^{eq} = \kappa_{ad}\tilde{q}_f$. The reason is that the non-neutral ISP is the leader in the this leader-follower game. Thus, knowing the parameters of the game and the tie-breaking assumption 2 of the CP, it can extract all the profits of the CP and make it indifferent between taking the non-neutral option and not taking it.

2.5.4 Does the Market Need to be Regulated?

We showed that in the presence of a “big” monopolistic CP and when EUs can switch between ISPs, if a non-neutral regime emerges, then neutral ISPs are likely to lose their market share, and are expected to be forced out of the market. In addition, in any NE outcome, the neutral ISP would lose payoff. Thus, if the regulator is interested in keeping some of the neutral ISPs in the market⁸, she should provide incentives for them. These incentives could be in the form of monetary subsidies or tax deductions.

Although for many parameters, the payoff of the non-neutral ISP would be higher by adopting a non-neutral regime, as explained before, with certain conditions on the parameters, an ISP is likely to receive a lower payoff by switching to non-neutral regime. These conditions are when (i) EUs are not sensitive to the quality, i.e. small κ_u , (ii) the CP is not sensitive to the quality her EUs receive, i.e. small κ_{ad} , (iii) ISP NoN does not offer enough differentiation in the quality, i.e. small $\tilde{q}_p - \tilde{q}_f$, (iv) the market power of the non-neutral ISP is low, or a combination of these factors. Thus, with these conditions a non-neutral regime is unlikely to emerge, and there is no need for a government intervention.

2.6 Discussions on Generalization of the Model

Note that we assumed $q_N \in \{0, \tilde{q}_f\}$ and $q_{NoN} \in \{0, \tilde{q}_f, \tilde{q}_p\}$. This assumption can be generalized to selecting quality strategies from continuous sets, i.e. $q_N \in [0, \tilde{q}_f]$ and

⁸For example, the reason could be to prevent non-neutral ISPs from becoming monopoly or it could be the social pressure to preserve some neutrality in the market.

$q_{NoN} \in [0, \tilde{q}_p]$. In this case, the CP pays a side payment of $\tilde{p}q_{NoN}$ if she chooses $q_{NoN} \in (\tilde{q}_f, \tilde{q}_p]$. In Appendix 2.7.6, we prove that our results herein would continue to hold under this generalization.

Recall that in our model, being neutral or non-neutral is fixed and is not a decision variable for ISPs. This means that the non-neutral ISP has already have the infrastructure for offering a premium quality to the CP. For this reason the fixed cost of investment on the infrastructure for offering a non-neutral service is not considered in the utility of ISP NoN (2.1). Even when considering this fixed cost, analyses yield that the results would be the same as before. Even if we consider both the investment cost and the decision of ISP NoN on being neutral or non-neutral, then the fixed cost of investment would affect the comparison between the payoff of ISP NoN in neutral and non-neutral scenarios only by a constant. This increases the regions of parameters in which an ISP would lose payoff by switching to a non-neutral regime. The overall intuitions are expected to be the same as before.

The result that over some parameters, an ISP can lose payoff by switching to a non-neutral regime is dependent on the assumption that the neutral and non-neutral ISPs first decide on the Internet access fees, and then the non-neutral ISP decides on the side-payment in the second stage. If we swap the order of these two stages, then the non-neutral ISP would not lose payoff by switching to non-neutrality since in this case, she would be the sole leader of the game. Thus, ISP NoN, in the worst case, obtains the payoff of the neutral scenario. Recall that the reason for our choice of the orders of the stages of the game is that Internet access fees are expected to be kept constant for a longer time horizons in comparison to the side-payment.

Recall that in the hotelling model, we considered EUs to be distributed uniformly between zero and one. In the case of considering a non-uniform distribution, depending on the skewness of the probability measure, results would be similar to small t_N or t_{NoN} . For example, if the probability measure is skewed toward zero, i.e. EUs are distributed close to the neutral ISP, results would be similar to uniform distribution and t_N small.

2.7 Appendix

2.7.1 Proofs of Section 2.2.2- Stage 3

First, note that by (2.2), $(q_N, q_{NoN}) = (0, 0)$ yields a payoff of zero, while $(\tilde{q}_f, \tilde{q}_f)$ yields a payoff of $\kappa_{ad}\tilde{q}_f > 0$. Thus, we can discard strategy $(q_N, q_{NoN}) = (0, 0)$ from the candidate solutions in (2.6). In addition, we use tie-breaking Assumption 1 to discard $(0, \tilde{q}_f) \in F_0^U$, $(\tilde{q}_f, 0) \in F_0^L$, $(\tilde{q}_f, \tilde{q}_f) \in F_0^U \cup F_0^L$, $(0, \tilde{q}_p) \in F_0^U$, and $(\tilde{q}_f, \tilde{q}_f) \in F_1^U \cup F_1^L$. Thus, the candidate solutions in (2.6) can be divided into the sub-sets characterized in Table 2.1 as follows:

$$(0, \tilde{q}_f) \in F_0^I \cup F_0^L, (\tilde{q}_f, 0) \in F_0^I \cup F_0^U, (\tilde{q}_f, \tilde{q}_f) \in F_0^I, (0, \tilde{q}_p) \in F_1^I \cup F_1^L, (\tilde{q}_f, \tilde{q}_p) \in F_1^I \quad (2.11)$$

Moreover, if $0 < n_{NoN} < 1$, then the expression for the payoff in (2.2), would be (using (2.5)):

$$\begin{aligned} \pi_{CP}(q_N, q_{NoN}, z) = & \frac{t_{NoN} + \kappa_u(q_N - q_{NoN}) + p_{NoN} - p_N}{t_N + t_{NoN}} \kappa_{ad} q_N \\ & + \frac{t_N + \kappa_u(q_{NoN} - q_N) + p_N - p_{NoN}}{t_N + t_{NoN}} \kappa_{ad} q_{NoN} - z \tilde{p} q_{NoN} \end{aligned} \quad (2.12)$$

The following lemmas are used in proving the main results of this section:

Lemma 1. Let $(\tilde{q}_f, \tilde{q}_p)$ and $(0, \tilde{q}_p)$ belong to the set F^I , i.e. for them $0 < x_N < 1$. Then $\pi_G(\tilde{q}_f, \tilde{q}_p, z = 1) \geq \pi_G(0, \tilde{q}_p, z = 1)$ if and only if $\Delta p \geq \Delta p_t$, where $\Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$.

Proof. The proof is done by comparing the payoffs (note that in both cases $0 < x_N < 1$).

We use (2.12) to write the expression of $\pi_G(q_N, q_{NoN}, z)$:

$$\begin{aligned} \pi_G(\tilde{q}_f, \tilde{q}_p, z = 1) &\geq \pi_G(0, \tilde{q}_p, z = 1) \iff \\ &(t_{NoN} - \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_{NoN} - p_N)\kappa_{ad}\tilde{q}_f + (t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_N - p_{NoN})\kappa_{ad}\tilde{q}_p \\ &\geq (t_N + \kappa_u\tilde{q}_p + p_N - p_{NoN})\kappa_{ad}\tilde{q}_p \\ &\iff t_{NoN} - \kappa_u(2\tilde{q}_p - \tilde{q}_f) + p_{NoN} - p_N \geq 0 \iff \Delta p \geq \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} = \Delta p_t \end{aligned}$$

The result follows. \square

Lemma 2. Let $(0, \tilde{q}_p) \in F_1^L$, i.e. by which $n_{NoN} = 1$. Then, $\pi_{CP}(0, \tilde{q}_p, z = 1) \geq \kappa_{ad}\tilde{q}_f$ if and only if $\tilde{p} \leq \tilde{p}_{t,1}$, where $\tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$.

Proof. We use (2.2) to write the expression of the payoff of the CP:

$$\pi_{CP}(0, \tilde{q}_p, z = 1) \geq \kappa_{ad}\tilde{q}_f \iff \kappa_{ad}\tilde{q}_p - \tilde{p}\tilde{q}_p \geq \kappa_{ad}\tilde{q}_f \iff \tilde{p} \leq \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p}) = \tilde{p}_{t,1}$$

\square

Lemma 3. Let $(0, \tilde{q}_p) \in F_1^I$, i.e. by which $0 < n_{NoN} < 1$. Then, $\pi_{CP}(0, \tilde{q}_p, z = 1) \geq \kappa_{ad}\tilde{q}_f$ if and only if $\tilde{p} \leq \tilde{p}_{t,2}$, where $\tilde{p}_{t,2} = \kappa_{ad}(n_{NoN} - \frac{\tilde{q}_f}{\tilde{q}_p})$ and $n_{NoN} = \frac{t_N + \kappa_u\tilde{q}_p - \Delta p}{t_N + t_{NoN}}$.

Proof. We compare the payoff with $\kappa_{ad}\tilde{q}_f$. We use (2.2) to write the expression of the payoff of the CP:

$$\pi_{CP}(0, \tilde{q}_p, z = 1) \geq \kappa_{ad}\tilde{q}_f \iff n_{NoN}\kappa_{ad}\tilde{q}_p - \tilde{p}\tilde{q}_p \geq \kappa_{ad}\tilde{q}_f \iff \tilde{p} \leq \kappa_{ad}(n_{NoN} - \frac{\tilde{q}_f}{\tilde{q}_p}) = \tilde{p}_{t,2}$$

where, by (2.5), $n_{NoN} = \frac{t_N + \kappa_u\tilde{q}_p - \Delta p}{t_N + t_{NoN}}$. The result follows. \square

Lemma 4. Let $(\tilde{q}_f, \tilde{q}_p) \in F_1^I$, i.e. by which $0 < n_{NoN} < 1$. Then, $\pi_G(\tilde{q}_f, \tilde{q}_p, z = 1) \geq \kappa_{ad}\tilde{q}_f$ if and only if $\tilde{p} \leq \tilde{p}_{t,3}$, where $\tilde{p}_{t,3} = \kappa_{ad}n_{NoN}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$ and $n_{NoN} = \frac{t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) - \Delta p}{t_N + t_{NoN}}$.

Proof. We compare the payoff with $\kappa_{ad}\tilde{q}_f$. We use (2.2) to write the expression of the payoff of the CP:

$$\begin{aligned} \pi_G(\tilde{q}_f, \tilde{q}_p, z = 1) \geq \kappa_{ad}\tilde{q}_f &\iff (1 - n_{NoN})\kappa_{ad}\tilde{q}_f + n_{NoN}\kappa_{ad}\tilde{q}_p - \tilde{p}\tilde{q}_p \geq \kappa_{ad}\tilde{q}_f \\ &\iff \tilde{p} \leq \kappa_{ad}n_{NoN}(1 - \frac{\tilde{q}_f}{\tilde{q}_p}) = \tilde{p}_{t,3} \end{aligned}$$

where, by (2.5), $n_{NoN} = \frac{t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) - \Delta p}{t_N + t_{NoN}}$. The result follows. \square

Remark 4. The values of Δp_t , $\tilde{p}_{t,1}$, $\tilde{p}_{t,2}$, and $\tilde{p}_{t,3}$ characterized in the above lemmas are used in Definition 3.

We should distinguish between the solutions that maximize (2.2), i.e. (q_N^*, q_{NoN}^*) which is not unique, and the strategy that is chosen by the CP in the equilibrium, which is a unique choice among the optimum solutions. Thus, we denote the equilibrium strategy of the CP by (q_N^{eq}, q_{NoN}^{eq}) , which subsequently yields the equilibrium fraction of EUs with each ISP, i.e. x_N^{eq} , N_N^{eq} , and N_{NoN}^{eq} .

Now, by comparing the payoffs of the candidate solutions and using tie-breaking assumptions, we prove one of the main results of this section, Theorem 1:

Proof. Proof of Theorem 1: Note that an equilibrium strategy, i.e. (q_N^{eq}, q_{NoN}^{eq}) , should be a global maxima of (2.2). Suppose $(q_N^*, q_{NoN}^*) \in F_0$. First, in Part A, we separate the cases that (q_N^*, q_{NoN}^*) is in F_0^L , F_0^I , or F_0^U , characterize the candidate optimum strategy, i.e. (q_N^*, q_{NoN}^*) , chosen by the CP in each of these subsets (Note that $F_0^L \cup F_0^I \cup F_0^U = F_0$), and identify the necessary condition on Δp for each of these candidate optimums to be in

a particular subset. In Part B, we summarize the candidate optimum strategies. Finally, in Part C, by comparing the payoffs of the candidate strategies in different regions of Δp and using the tie-breaking assumptions, we characterize the equilibrium strategies.

Part A: First, consider F_0^I . If $(q_N^*, q_{NoN}^*) \in F_0^I$, i.e. $z^* = 0$, then (q_N^*, q_{NoN}^*) , by (2.11), is (a) $(0, \tilde{q}_f)$, or (b) $(\tilde{q}_f, 0)$, or (c) $(\tilde{q}_f, \tilde{q}_f)$. Note that the necessary and sufficient condition for each of these candidate outcomes to be in F_0^I is $\frac{\Delta p - t_N}{\kappa_u} < \Delta q^* < \frac{\Delta p + t_{NoN}}{\kappa_u}$ (Table 2.1). First consider case (a). Note that $\Delta q^* = \tilde{q}_f$. Thus, the necessary and sufficient condition for (a) to be in F_0^I becomes $\frac{\Delta p - t_N}{\kappa_u} < \tilde{q}_f < \frac{\Delta p + t_{NoN}}{\kappa_u}$, which yields $\kappa_u \tilde{q}_f - t_{NoN} < \Delta p < \kappa_u \tilde{q}_f + t_N$. Similarly, For cases (b), the necessary and sufficient condition is $-\kappa_u \tilde{q}_f - t_{NoN} < \Delta p < -\kappa_u \tilde{q}_f + t_N$, and for (c) is $-t_{NoN} < \Delta p < t_N$.

Now, consider F_0^L . If $(q_N^*, q_{NoN}^*) \in F_0^L$, then (q_N^*, q_{NoN}^*) , by (2.11), is (d) $(0, \tilde{q}_f) \in F_0^L$. Note that, using the condition in Table 2.1, the necessary and sufficient condition for $(0, \tilde{q}_f) \in F_0^L$ is $\Delta p \leq \kappa_u \tilde{q}_f - t_{NoN}$.

Finally, consider F_0^U . If $(q_N^*, q_{NoN}^*) \in F_0^U$, then (q_N^*, q_{NoN}^*) , by (2.11), is (e) $(\tilde{q}_f, 0) \in F_0^U$. Using the condition in Table 2.1, the necessary and sufficient condition for $(\tilde{q}_f, 0) \in F_0^U$ is $\Delta p \geq t_N - \kappa_u \tilde{q}_f$.

Part B: Note that, as mentioned before, the strategy that is chosen by the CP in the equilibrium is a unique choice among the possible optimum solutions. Thus, if $(q_N^{eq}, q_{NoN}^{eq}) \in F_0$, then (q_N^{eq}, q_{NoN}^{eq}) is of the form of *one* of the followings (the necessary condition for each to be optimum is also listed):

(a) $(0, \tilde{q}_f) \in F_0^I$, if this is overall optimum then $\kappa_u \tilde{q}_f - t_{NoN} < \Delta p < \kappa_u \tilde{q}_f + t_N$ (the necessary condition).

(b) $(\tilde{q}_f, 0) \in F_0^I$, the necessary condition: $-\kappa_u \tilde{q}_f - t_{NoN} < \Delta p < -\kappa_u \tilde{q}_f + t_N$.

(c) $(\tilde{q}_f, \tilde{q}_f) \in F_0^I$, the necessary condition: $-t_{NoN} < \Delta p < t_N$.

(d) $(0, \tilde{q}_f) \in F_0^I$, the necessary condition: $\Delta p \leq \kappa_u \tilde{q}_f - t_{NoN}$.

(e) $(\tilde{q}_f, 0) \in F_0^U$, the necessary condition: $\Delta p \geq -\kappa_u \tilde{q}_f + t_N$.

Part C: Now, we compare the payoffs of the CP at each candidate solutions, and use tie-breaking assumptions whenever needed to get the equilibrium strategies of the CP.

The payoff of the CP, for each candidate solution, is as follows (by (2.2)):

$$\begin{aligned}
\pi_{CP,(a)} &= n_{NoN} \kappa_{ad} \tilde{q}_f && \& \ 0 < n_{NoN} < 1 \\
\pi_{CP,(b)} &= n_N \kappa_{ad} \tilde{q}_f && \& \ 0 < n_N < 1 \\
\pi_{CP,(c)} &= \kappa_{ad} \tilde{q}_f \\
\pi_{CP,(d)} &= \kappa_{ad} \tilde{q}_f \\
\pi_{CP,(e)} &= \kappa_{ad} \tilde{q}_f
\end{aligned} \tag{2.13}$$

Next, we characterize the equilibrium strategies in different intervals of Δp . First consider $-t_{NoN} < \Delta p < t_N$. Note that in this case, Δp satisfies the necessary condition of (c) being a candidate strategy, and also the necessary and sufficient condition of (c) being in F_0^I . In addition, $\pi_{CP,(c)} > \pi_{CP,(a)}$ and $\pi_{CP,(c)} > \pi_{CP,(b)}$. Thus, (a) and (b) cannot be overall optimum solutions. Moreover, $\pi_{CP,(c)} = \pi_{CP,(d)}$ and $\pi_{CP,(c)} = \pi_{CP,(e)}$. Using tie-breaking assumption 3 yields that the CP prefers (c) to (d) and (e). Thus, $(\tilde{q}_f, \tilde{q}_f) \in F_0^I$ is chosen as the equilibrium strategy in this interval, and case 1 of the lemma follows.

Now, consider $\Delta p \geq t_N$. Note that in this case, Δp satisfies the necessary condition

of (e) being a candidate strategy, and also the necessary and sufficient condition of (e) to be in F_0^U . In addition, this condition rules out (b) and (c). However, for certain intervals of $\Delta p \geq t_N$, the necessary condition of candidate strategies (a) and (d) can be satisfied. We now compare the payoff of (e) to (a) and (d). First note that $\pi_{CP,(e)} > \pi_{CP,(a)}$. Thus candidate strategy (a) can be discarded. Also, $\pi_{CP,(e)} = \pi_{CP,(d)}$. Since $\Delta p = p_{NoN} - p_N \geq t_N > 0$, and by using tie-breaking assumption 5, candidate strategy (e), i.e. $(\tilde{q}_f, 0) \in F_0^U$ is chosen as the equilibrium strategy in this interval by the CP. Thus, case 2 of the lemma follows.

Finally, consider $\Delta p \leq -t_{NoN}$. Note that in this case, Δp satisfies the necessary condition of (d) to be a candidate strategy, and also the necessary and sufficient condition of (d) to be in F_0^L . In addition, this condition rules out (a) and (c). However, for certain intervals of $\Delta p \leq -t_{NoN}$, the necessary condition of candidate strategies (b) and (e) can be satisfied. We now compare the payoff of (d) to (b) and (e). First note that $\pi_{CP,(d)} > \pi_{CP,(b)}$. Thus candidate strategy (b) can be discarded. Also, $\pi_{CP,(d)} = \pi_{CP,(e)}$. Since $\Delta p = p_{NoN} - p_N \leq -t_{NoN} < 0$, and by using tie-breaking assumption 5, candidate strategy (d), i.e. $(0, \tilde{q}_f) \in F_0^L$ is chosen as the equilibrium strategy in this interval by the CP. Thus, case 3 of the lemma follows.

Note that by (2.13), $\pi_{CP,(a)} = \pi_{CP,(b)} = \pi_{CP,(c)} = \kappa_{ad}\tilde{q}_f$ and these are all the candidate solutions. Thus, the utility of the CP by each candidate equilibrium strategy would be $\kappa_{ad}\tilde{q}_f$. The result follows. \square

Now, we focus on characterizing the candidate strategies and the necessary condition for each of them when $z^{eq} = 1$.

Theorem 11. *If $(q_N^{eq}, q_{N_oN}^{eq}) \in F_1$, then $(q_N^{eq}, q_{N_oN}^{eq})$ is of the form of one of the followings:*

(a) $(0, \tilde{q}_p)$, *the necessary condition: $\kappa_u \tilde{q}_p - t_{N_oN} < \Delta p < \kappa_u \tilde{q}_p + t_N$. In addition, $(0, \tilde{q}_p) \in$*

F_1^I if and only if $\kappa_u \tilde{q}_p - t_{N_oN} < \Delta p < \kappa_u \tilde{q}_p + t_N$.

(b) $(\tilde{q}_f, \tilde{q}_p)$, *the necessary condition: $\kappa_u(\tilde{q}_p - \tilde{q}_f) - t_{N_oN} < \Delta p < \kappa_u(\tilde{q}_p - \tilde{q}_f) + t_N$. In*

addition, $(\tilde{q}_f, \tilde{q}_p) \in F_1^I$ iff $\kappa_u(\tilde{q}_p - \tilde{q}_f) - t_{N_oN} < \Delta p < \kappa_u(\tilde{q}_p - \tilde{q}_f) + t_N$.

(c) $(0, \tilde{q}_p)$, *the necessary condition: $\Delta p \leq \kappa_u \tilde{q}_p - t_{N_oN}$. In addition, $(0, \tilde{q}_p) \in F_1^L$ iff*

$\Delta p \leq \kappa_u \tilde{q}_p - t_{N_oN}$.

Proof. Suppose $(q_N^*, q_{N_oN}^*) \in F_1$. We separate the cases that $(q_N^*, q_{N_oN}^*)$ is in F_1^L , F_1^I , or F_1^U , characterize the candidate optimum solutions chosen by the CP in each of these subsets, and identify the necessary condition on Δp for each of these candidate optimum strategies to be in a particular subset.

Note that by (2.11), no optimum strategy is chosen in the set F_1^U . Thus, we characterize the optimum strategies chosen in F_1^I and F_1^L by the CP.

Now, consider F_1^I . By (2.11), if $(q_N^*, q_{N_oN}^*) \in F_1^I$, then $(q_N^*, q_{N_oN}^*)$ is (a) $(0, \tilde{q}_p)$ or (b) $(\tilde{q}_f, \tilde{q}_p)$. The necessary condition for each of them to be optimum is to be in F_1^I . In addition, the necessary and sufficient condition for each of these candidate outcomes to be in F_1^I is $\frac{\Delta p - t_N}{\kappa_u} < \Delta q^* < \frac{\Delta p + t_{N_oN}}{\kappa_u}$ (by Table 2.1). Thus, for case (a), the necessary and sufficient condition is $\kappa_u \tilde{q}_p - t_{N_oN} < \Delta p < \kappa_u \tilde{q}_p + t_N$ (note that $\Delta q^* = \tilde{q}_p$), and for case (b) is $\kappa_u(\tilde{q}_p - \tilde{q}_f) - t_{N_oN} < \Delta p < \kappa_u(\tilde{q}_p - \tilde{q}_f) + t_N$. These yields candidate strategies (a) and (b) and their conditions of the Theorem.

Consider F_1^L . By (2.11), if $(q_N^*, q_{N_oN}^*) \in F_1^L$, then $(q_N^*, q_{N_oN}^*)$ is (c) $(0, \tilde{q}_p)$. Note that the necessary and sufficient condition for $(0, \tilde{q}_p) \in F_1^L$ is $\Delta p \leq \kappa_u \tilde{q}_p - t_{N_oN}$ (by the

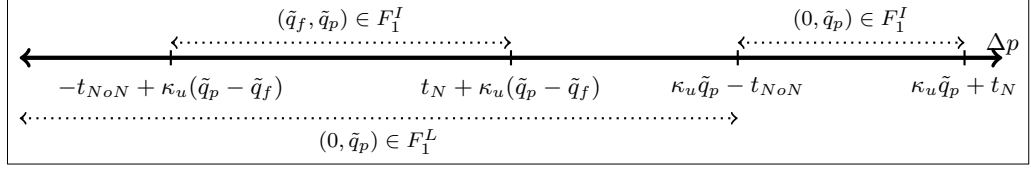


Figure 2.9: A schematic view of the ordering of different candidate equilibrium strategies characterized in Theorem 11 with respect to Δp when $\tilde{q}_f > \frac{t_N + t_{NoN}}{\kappa_u}$ and $z = 1$.

condition in Table 2.1 and $\Delta q = \tilde{q}_p$). The theorem follows. \square

The payoff of the CP in each candidate solution of Theorem 11 is as follows (using (2.2)):

$$\begin{aligned}
 \pi_{CP,(a)} &= n_{NoN} \kappa_{ad} \tilde{q}_p - \tilde{p} \tilde{q}_p \quad \& 0 < n_{NoN} < 1 \\
 \pi_{CP,(b)} &= (1 - n'_{NoN}) \kappa_{ad} \tilde{q}_f + n'_{NoN} \kappa_{ad} \tilde{q}_p - \tilde{p} \tilde{q}_p \quad \& 0 < n'_N < 1 \\
 \pi_{CP,(c)} &= \kappa_{ad} \tilde{q}_p - \tilde{p} \tilde{q}_p
 \end{aligned} \tag{2.14}$$

Thus, the payoffs are a function of \tilde{p} and Δp . Now, to get the second main result of this section, we compare the payoff of the candidate answers with the payoff of the candidate strategies when $z = 0$ considering different values of \tilde{p} and Δp , and pick the maximum as the equilibrium strategy of the CP. Thus Theorem 2 is proved as follows:

Proof of Theorem 2:

Proof. Now, for different regions of Δp , we compare the payoffs of the candidate equilibrium strategies characterized in Theorem 11 to each other and also to the equilibrium strategies in Theorem 1, and use tie-breaking assumptions (whenever needed) to characterize the equilibrium strategies of the CP.

First consider $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. In this case, Δp satisfies the necessary condition of candidate strategy (c) in Theorem 11. In addition, note that by (2.14), $\pi_{CP,(c)} > \pi_{CP,(a)}$

and $\pi_{CP,(c)} > \pi_{CP,(b)}$ (by $\tilde{q}_p > \tilde{q}_f$). Thus, for this region, (c) is chosen if and only if this strategy yields a higher or equal (by tie-breaking assumption 2) payoff than the payoff when $z^{eq} = 0$, that is $\kappa_u \tilde{q}_f$ (by Theorem 1). Thus, using Lemma 2, $z^{eq} = 1$, and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$ if $\tilde{p} \leq \tilde{p}_{t,1}$. Otherwise $z^{eq} = 0$, since the payoff of (c) and subsequently (a) and (b) are smaller than the payoff when $z^{eq} = 0$. Thus, in this case, the equilibrium strategy can be found using Theorem 1. This is item 1 of the theorem.

For $\Delta p \geq t_N + \kappa_u \tilde{q}_p$, the necessary condition of none of the candidate strategies in Theorem 11 can be satisfied. Therefore, $z^{eq} = 0$. This is item 4 of the theorem.

Now, for the rest of the proof, we consider $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u \tilde{q}_p$. In this case, the necessary condition of candidate strategy (c) of Theorem 11 cannot be satisfied. Therefore, we can eliminate (c). On the other hand, the necessary and sufficient condition of (a) of Theorem 11 can be met. Now, consider two different cases, $\tilde{q}_f \leq \frac{t_N + t_{NoN}}{\kappa_u}$ and $\tilde{q}_f > \frac{t_N + t_{NoN}}{\kappa_u}$:

- $\tilde{q}_f \leq \frac{t_N + t_{NoN}}{\kappa_u}$. This yields that $\kappa_u \tilde{q}_p - t_{NoN} \leq t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. For this case, consider two sub-cases:

- $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. In this case, Δp satisfies the necessary and sufficient condition of (b) in Theorem 11. Now, we should compare $\pi_{G,(a)}$ and $\pi_{G,(b)}$. In Lemma 1, we compare the payoff of the two solutions. In addition, by tie breaking assumption 4, when the payoffs are equal the CP chooses (b) over (a). Thus, if $\Delta p \geq \Delta p_t$, (b), i.e. $(\tilde{q}_f, \tilde{q}_p)$ would be chosen versus (a). Otherwise (a), i.e. $(0, \tilde{q}_p)$ would be chosen. Now, we compare the payoff of the one chosen with the payoff of the case $z = 0$, i.e. $\kappa_{ad} \tilde{q}_f$:

- * If $\Delta p \geq \Delta p_t$, then by Lemma 4 and tie-breaking assumption 2, $z^{eq} = 1$ and $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$ if $\tilde{p} \leq \tilde{p}_{t,3}$ (by comparing the payoff of strategy (b) by the payoff when $z = 0$, i.e. $\kappa_{ad}\tilde{q}_f$). Otherwise $z^{eq} = 0$, and the equilibrium strategy can be found using Theorem 1. This is item 2-a-i of the theorem.
 - * If $\Delta p < \Delta p_t$, then by Lemma 3 and tie-breaking assumption 2, $z^{eq} = 1$ and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$ if $\tilde{p} \leq \tilde{p}_{t,2}$ (by comparing the payoff of strategy (a) by the payoff when $z = 0$, i.e. $\kappa_{ad}\tilde{q}_f$). Otherwise $z^{eq} = 0$, and the equilibrium strategy can be found using Theorem 1. This is item 2-a-ii of the theorem.
- $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u\tilde{q}_p$: In this range, the necessary condition of (b) of Theorem 11 cannot be satisfied. Thus, the only candidate solution by which $z = 1$, whose necessary and sufficient conditions can be satisfied, is (a) (as stated before). Therefore, we should compare the payoff of (a) with that of when $z^{eq} = 0$, i.e. $\kappa_{ad}\tilde{q}_f$. Using Lemma 3 and Assumption 2, if $\tilde{p} \leq \tilde{p}_{t,2}$ then $z^{eq} = 1$ and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$. Otherwise $z^{eq} = 0$, and the equilibrium strategy can be found using Theorem 1. This is item 2-b of the theorem.
- $\tilde{q}_f > \frac{t_N + t_{NoN}}{\kappa_u}$: In this case, $\kappa_u\tilde{q}_p - t_{NoN} > t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Thus, the necessary condition of (b) cannot be satisfied. Therefore, we can eliminate (c) (eliminated before) and (b). On the other hand, the necessary and sufficient condition of (a) of Theorem 11 can be met. Therefore, we should compare the payoff of (a) with that of when $z^{eq} = 0$, i.e. $\kappa_{ad}\tilde{q}_f$. Using Lemma 3 and Assumption 2, if $\tilde{p} \leq \tilde{p}_{t,2}$ then

$z^{eq} = 1$ and $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$. Otherwise $z^{eq} = 0$, since the payoff of (a) is smaller than the payoff when $z^{eq} = 0$. Thus, in this case, the equilibrium strategy can be found using Theorem 1. This is item 3 of the theorem. The result follows. □

The following lemma simplify item 2-a of Theorem 2, and is useful in the next stages:

Lemma 5. *Consider $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. If $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$, then $\Delta p < \Delta p_t$. If $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$, then $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p_t < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, where $\Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ characterized in Lemma 1.*

Proof. Proof: First, consider $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$. Note that:

$$\Delta p_t - (t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)) = \kappa_u \tilde{q}_p - t_N - t_{NoN} \geq 0$$

Thus for every Δp such that $\Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, $\Delta p_t > \Delta p$. This establish the first part of the lemma.

Now, consider $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$. In this case:

$$\Delta p_t - (t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)) = \kappa_u \tilde{q}_p - t_N - t_{NoN} < 0$$

and

$$\Delta p_t - (\kappa_u \tilde{q}_p - t_{NoN}) = \kappa_u(\tilde{q}_p - \tilde{q}_f) > 0 \quad (\text{since } \tilde{q}_p > \tilde{q}_f)$$

Thus, $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p_t < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. The result follows. □

Theorem 2 and Lemma 5 yields the following corollary:

Corollary 1. Let $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$. Then the structure of the optimum answers of the CP (results in Theorem 2) for the case that $\tilde{q}_f \leq \frac{t_N + t_{NoN}}{\kappa_u}$ is the same as the results when $\tilde{q}_f > \frac{t_N + t_{NoN}}{\kappa_u}$.

Proof. Proof: Note that items 1 and 4 of Theorem 3 are the same for both cases, regardless of \tilde{q}_f . In addition by Lemma 5, when $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$, then $\Delta p < \Delta p_t$. Thus, 2-a-i in Theorem 2 would not happen. Note that 2-a-ii and 2-b yields is similar to 3. Thus, the two structures are similar, and the corollary follows. \square

2.7.2 Proofs of Section 2.2.3 - Stage 2

First, we prove Theorem 4. Then using the results of this theorem, we prove Theorem 3.

Proof of Theorem 4

It is sufficient to prove that if any one of the conditions (1) $\Delta p < t_N + \kappa_u \tilde{q}_p$, (2) $\tilde{p}^{eq} = \tilde{p}_t$, or (3) $\pi_{NoN}(p_{NoN}, \tilde{p}_t) > \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ is not true, then $z^{eq} = 0$. Thus, in each of the following cases, we consider one of these conditions to be not true, and prove that $z^{eq} = 0$.

- Case 1- $\Delta p \geq t_N + \kappa_u \tilde{q}_p$: By Theorem 2, when $\Delta p \geq t_N + \kappa_u \tilde{q}_p$, $z^{eq} = 0$. This case follows.
- Case 2- $\tilde{p}^{eq} \neq \tilde{p}_t$: if $\Delta p \geq t_N + \kappa_u \tilde{q}_p$, using case 1, $z^{eq} = 0$. Now, consider $\Delta p < t_N + \kappa_u \tilde{q}_p$. In this case, either $\tilde{p}^{eq} > \tilde{p}_t$ or $\tilde{p}^{eq} < \tilde{p}_t$. We claim that no \tilde{p} such that $\tilde{p} < \tilde{p}_t$ can be an optimum solution (the claim is proved in the next paragraph). Thus, $\tilde{p}^{eq} > \tilde{p}_t$. Note that $\tilde{p}^{eq} > \tilde{p}_t$ yields $z^{eq} = 0$ (by Theorem 2). Thus, the case

follows.

Now, we prove that no \tilde{p} such that $\tilde{p} < \tilde{p}_t$ can be an optimum solution. Note that by Theorem 2, when $\Delta p < t_N + \kappa_u \tilde{q}_p$, for $\tilde{p} \leq \tilde{p}_t$, the CP chooses $z = 1$. Thus, the payoff of ISP NoN (2.7) is equal to $(p_{NoN} - c)n_{NoN} + \tilde{p}\tilde{q}_f$, and is a strictly increasing function of \tilde{p} (note that p_{NoN} is fixed and by (2.5), n_{NoN} is independent of \tilde{p}). Thus, every \tilde{p} such that $\tilde{p} < \tilde{p}_t$, yields a strictly smaller payoff for ISP NoN in comparison to the the payoff when $\tilde{p} = \tilde{p}_t$. Thus, no \tilde{p} such that $\tilde{p} < \tilde{p}_t$ can be an optimum solution. The result follows.

- Case 3- $\pi_{NoN}(p_{NoN}, \tilde{p}_t) \leq \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$: In this case, either $\tilde{p}^{eq} \neq \tilde{p}_t$ or $\tilde{p}^{eq} = \tilde{p}_t$. Note that by Case 2, $\tilde{p}^{eq} \neq \tilde{p}_t$ yields $z^{eq} = 0$, which yields the result.

Now, consider $\tilde{p}^{eq} = \tilde{p}_t$. Note that by Theorem 2, the non-neutral ISP can ensure $z^{eq} = 0$, by choosing \tilde{p} greater than $\max\{\tilde{p}_{t,1}, \tilde{p}_{t,2}, \tilde{p}_{t,3}\}$. Thus, since $\tilde{p}^{eq} = \tilde{p}_t$, $\pi_{NoN}(p_{NoN}, \tilde{p}_t) = \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$.⁹ By tie-breaking assumption 6, $z^{eq} = 0$. The theorem follows.

Proof of Theorem 3

First, note that by Theorem 4, if $z^{eq} = 1$ then $\pi_{NoN}(p_{NoN}, \tilde{p}_t) > \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ and $\Delta p < t_N + \kappa_u \tilde{q}_p$. To prove the reverse, note that if $\pi_{NoN}(p_{NoN}, \tilde{p}_t) > \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ and $\Delta p < t_N + \kappa_u \tilde{q}_p$, \tilde{p} that yields $z^{eq} = 0$ cannot be an optimum answer. Note that by Theorem 2, when $\tilde{p} = \tilde{p}_t$, the ISP NoN can make sure that $z^{eq} = 1$. Thus, in the equilibrium, $z^{eq} = 1$. The result follows.

⁹if not, then $\tilde{p}^{eq} \neq \tilde{p}_t$, since \tilde{p}_t is not optimum.

2.7.3 Proofs of Section 2.2.4 - Stage 1

Proof of Theorem 5

We consider different regions of Δp in Theorem 1 and Theorem 2. For each region, we prove that there is no NE.

First, consider $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Note that in this region, the payoff of non-neutral ISP if $z^{eq} = 0$ is at most $p_{NoN}^{eq} - c$ (by (2.1)). On the other hand, by Theorem 2, by choosing $\tilde{p}' = \tilde{p}_{t,1}$, ISP NoN can ensure that the CP chooses $z^{eq} = 1$. In this case, the payoff of non-neutral ISP (by (2.1)) is $p'_{NoN} - c + \tilde{p}_{t,1} \tilde{q}_{NoN} = p_{NoN}^{eq} - c + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) > p_{NoN}^{eq} - c$. Thus, $\pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,1}) > \pi_{NoN, z=0}(p_{NoN}^{eq}, \tilde{p})$, and by Theorem 3, $z^{eq} = 1$. Thus, in this case, there is no NE by which $z^{eq} = 0$.

Now, Consider p_N^{eq} and p_{NoN}^{eq} to be NE strategies by which $z^{eq} = 0$ and $\Delta p^{eq} > \kappa_u \tilde{q}_p - t_{NoN}$. Note that $t_N + t_{NoN} \leq \kappa_u \tilde{q}_p$ (assumption of the theorem) yields $\kappa_u \tilde{q}_p - t_{NoN} \geq t_N$, and $\Delta p^{eq} > t_N$. Thus, by item 2 of Theorem 1, $n_N^{eq} = 1$. Consider a unilateral deviation by neutral ISP such that $p'_N = p_N^{eq} + \epsilon$ in which $\epsilon > 0$ such that $p_{NoN}^{eq} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$. Note that the values of z^{eq} , q_N^{eq} , and q_{NoN}^{eq} is the same as before, since still $\Delta p' = p_{NoN}^{eq} - p'_N > t_N$. Thus, again $n_N^{eq} = 1$, and by (2.1), the payoff of neutral ISP is an increasing function of p_N . Thus, p'_N is a profitable unilateral deviation. This contradicts the assumption that p_N^{eq} and p_{NoN}^{eq} is NE. Thus, the result of the theorem follows.

Proof of Theorem 6

Before proving the theorem, we state two lemmas with their proof which are used in the proof of the theorem:

Lemma 6. *If $p_{NoN} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N = c$, then $z^{eq} = 1$.*

Proof. Proof: Note that in this case, $\Delta p = \kappa_u \tilde{q}_p - t_{NoN}$. Thus, $\tilde{p}_t = \tilde{p}_{t,1}$. Therefore, using Theorem 3, it is sufficient to prove that $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,1}) > \pi_{NoN,z=0}(p_{NoN}, \tilde{p})$, where $\pi_{NoN,z=0}(\tilde{p}_{NoN}, \tilde{p})$ is the payoff of ISP NoN when $z^{eq} = 0$. Note that $\pi_{NoN,z=0}(p_{NoN}, \tilde{p}) \leq p_{NoN} - c = \kappa_u \tilde{q}_p - t_{NoN}$ and $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,1}) = \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f)$ (since by Theorem 2, $n_{NoN} = 1$, and by (2.1)). In addition, note that, $\tilde{q}_p > \tilde{q}_f$. Thus, this condition holds, and the result follows. \square

Lemma 7. *If $p_{NoN} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$, $p_N = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$, $\tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$, and $\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$, then $z^{eq} = 1$.*

Proof. Proof: Note that if $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u \tilde{q}_p$, by definition of \tilde{p}_t (Definition 4), $\tilde{p}_t = \tilde{p}_{t,2}$. Thus, by Theorem 3, it is enough to prove that $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,2}) > \pi_{NoN,z=0}(\tilde{p}_{NoN}, \tilde{p})$, where $\pi_{NoN,z=0}(\tilde{p}_{NoN}, \tilde{p})$ is the payoff of ISP NoN when $z^{eq} = 0$.

First, we prove that $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,2}) > p_N - c + \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f)$:

$$\begin{aligned} \pi_{NoN}(p_{NoN}, \tilde{p}_{t,2}) &\geq p_N - c + \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) \\ &\iff \frac{(t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})} \geq \frac{t_N - t_{NoN} + 2\tilde{q}_p(\kappa_u + \kappa_{ad})}{3} \\ &\iff (\tilde{q}_p(\kappa_u + \kappa_{ad}) - t_N - 2t_{NoN})^2 \geq 0 \end{aligned}$$

In addition, note that $p_N - c + \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) > 0$, since $p_N \geq c$ (under the condition $\tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$), $\kappa_u \tilde{q}_p - t_{NoN} \geq t_N > 0$ (by the assumption of the lemma), and $\tilde{q}_p > \tilde{q}_f$. Thus, $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,2}) > 0$.

Now, consider $\pi_{NoN,z=0}(\tilde{p}_{NoN}, \tilde{p})$. Note that by the assumption of the lemma $\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$. Thus, $\Delta p > t_N$, and by item 2 of Theorem 1, if $z^{eq} = 0$, $n_{NoN} = 0$. Thus, by (2.1), $\pi_{NoN,z=0}(\tilde{p}_{NoN}, \tilde{p}) = 0$. Therefore, $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,2}) > \pi_{NoN,z=0}(\tilde{p}_{NoN}, \tilde{p})$, and the result follows. \square

Now, we prove Theorem 6:

Proof. Proof of Theorem 6: We use the optimum strategies of the CP characterized in Theorem 2 to characterize Nash equilibria. Note that for the case that $\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$, by Corollary 1, the structure of the equilibrium strategies chosen by the CP is similar to the case that $\kappa_u \tilde{q}_f > t_N + t_{NoN}$. Thus, in this case, items 1, 3, and 4 of Theorem 2 characterizes the NE strategies chosen by the CP. Thus, henceforth we assume $\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$, and use these items to prove the theorem.

We denote $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$ by region A, $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u \tilde{q}_p$ by region B, and $\Delta p \geq t_N + \kappa_u \tilde{q}_p$ by region C. Using Theorem 2, if $z^{eq} = 1$, then $\Delta p < t_N + \kappa_u \tilde{q}_p$. Thus, to characterize NE strategies by which $z^{eq} = 1$, we should characterize any possible NE strategies in regions A and B. In Case A, we prove that the only possible NE in region A is $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$. In addition, we prove that these strategies are NE if $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$. If not, then there is no NE in region A. In Case B, we prove that the only possible NE in region B is $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$. In addition, we prove that these strategies can be NE strategies if $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$. If not, then there is no NE in region B.

Case A: We characterize the NE strategies p_N^{eq} and p_{NoN}^{eq} such that $\Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq} \leq \kappa_u \tilde{q}_p - t_{NoN}$. First, in Case A-1, we prove that if $z^{eq} = 1$ the only possible NE in this region is $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$, and with these strategies, z^{eq} is indeed equal to 1. In Case A-2, we characterize the necessary and sufficient conditions by which there is no unilateral profitable deviation for ISPs. This provides the necessary and sufficient condition for these strategies to be NE.

Case A-1: Note that by Theorem 2, for region A, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$ if and only if $\tilde{p} \leq \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$. In addition, by Theorem 4, if $z^{eq} = 1$ then $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$. Thus, in this region, if $z^{eq} = 1$, the payoff of ISP NoN is equal to $p_{NoN} - c + \tilde{q}_p \tilde{p}_{t,1}$ (by (2.1)) since $n_{NoN} = 1$. Therefore, the payoff is an increasing function of p_{NoN} . In addition, note that in region A, $n_N = 0$ and regardless of p_N , the neutral ISP receives a payoff of zero (by (2.1)). Thus, p_{NoN}^{eq} , i.e. the equilibrium Internet access fee, should be such that the neutral ISP cannot get a positive payoff by increasing or decreasing p_N , and changing the region of Δp to B or C . Using this condition, we find the equilibrium strategy.

Note that increasing p_N decreases Δp , and cannot change the region of Δp . We claim that by decreasing p_N to p'_N such that $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, the ISP N can fetch a positive payoff as long as $p'_N > c$ (the claim is proved in the next paragraph). Therefore, in the equilibrium, p_{NoN}^{eq} is such that even with $p'_N = c$ (the minimum plausible price), $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Thus, $p_{NoN}^{eq} \leq c + \kappa_u \tilde{q}_p - t_{NoN}$. Given that the payoff of ISP NoN is an increasing function of p_{NoN} , we get $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$. In addition, we claim that $p_N^{eq} = c$. If not, then $p_N^{eq} > c$. In this case, $\Delta p = p_N^{eq} - p_{NoN}^{eq} < \kappa_u \tilde{q}_p - t_{NoN}$. We argued that the payoff of ISP NoN is an increasing function of p_{NoN} . Thus, by increasing p_{NoN} such that $\Delta p = \kappa_u \tilde{q}_p - t_{NoN}$, ISP NoN can increase her payoff, which is a contradiction with p_N^{eq} and p_{NoN}^{eq} being NE strategies.

To prove the claim, note that if $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, then either (i) $z^{eq} = 0$ or (ii) $z^{eq} = 1$. Note that $\Delta p > \kappa_u \tilde{q}_p - t_{NoN} \geq t_N$, since $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$. Thus, for case (i), (q_N^{eq}, q_{NoN}^{eq}) is of the form of part 2 of Theorem 1. Thus, $n_N = 1$. Therefore ISP N can fetch a positive payoff as long as $p_N > c$ (by (2.1)). Now consider case (ii), i.e. $z^{eq} = 1$.

Note that when $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, Δp is either in region B or C. By Theorem 2, the only deviation that yields $z^{eq} = 1$ is p'_N such that Δp in region B. Note that in this region, by item 3 of Theorem 2, $n_N > 0$. Thus, ISP N can fetch a positive payoff as long as $p_N > c$ (by (2.1)). This completes the proof of the claim that by decreasing p_N to p'_N such that $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, the ISP N can fetch a positive payoff as long as $p'_N > c$.

Therefore, the NE strategies are $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$, and the payoff of the ISP NoN at this price by (2.1) and $\tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$ is equal to (note that $n_{NoN} = 1$), and

$$\pi_{NoN}^{eq} = \kappa_u \tilde{q}_p - t_{NoN} + \tilde{q}_p \tilde{p}_{t,1} = \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) \quad (2.15)$$

which is strictly positive since $\tilde{q}_p \geq \frac{t_N + t_{NoN}}{\kappa_u}$ and $\tilde{q}_p > \tilde{q}_f$.

Note that Lemma 6 yields that with p_N^{eq} and p_{NoN}^{eq} $z^{eq} = 1$.

Case A-2: Now, in order to prove that p_N^{eq} and p_{NoN}^{eq} are indeed NE strategies, we show that there is no unilateral profitable deviation for ISPs. First, in Case (A-2-i) we rule out the possibility of a unilateral profitable deviation for ISP N. Then, in Case (A-2-ii) we rule out a possibility of a downward unilateral profitable deviation, i.e. $p_{NoN} < p_{NoN}^{eq}$, for ISP NoN. Finally, in Case (A-3-iii), we consider a deviation of the form $p_{NoN} > p_{NoN}^{eq}$ for ISP NoN, and prove that the necessary and sufficient condition for this deviation to be not profitable is $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

Case A-2-i: The construction of strategies p_N^{eq} and p_{NoN}^{eq} yields that there is no profitable deviation for ISP N. To prove this formally, note that the only deviation for ISP N that might be profitable is $p_N > c$. With this deviation, Δp would be still in region A, in which $n_N = 0$, and the payoff of ISP N is zero. Thus, such a deviation is not

profitable.

Case A-2-ii: Now, consider a deviation by ISP NoN such that $p_{NoN} < p_{NoN}^{eq}$. In this case, Δp is in region A, and the payoff of ISP NoN is equal to $p_{NoN} - c + \tilde{q}_p \tilde{p} t_{,1}$ (by (2.1) and $n_{NoN} = 1$). Thus, the payoff of ISP NoN is strictly increasing in region A. Therefore, p_{NoN}^{eq} dominates all prices $p_{NoN} < p_{NoN}^{eq}$. Thus, this kind of deviation is not profitable for ISP NoN.

Case A-2-iii: In this case, we consider a deviation such that $p_{NoN} > p_{NoN}^{eq}$. Thus, $\Delta p > \kappa_u \tilde{q}_p - t_{NoN}$. Therefore, Δp is either in Region B or C. First, in Case A-2-iii-a we rule out the possibility of a profitable unilateral deviation in region C. Then, in Case A-2-iii-b, we rule out the possibility of a profitable unilateral deviation in region B if $z^{eq} = 0$. Finally, in Case A-2-iii-c, we prove that a deviation to region B if $z^{eq} = 1$ is not profitable if and only if $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

Case A-2-iii-a: Using item 4 of Theorem 2, if Δp in region C, i.e. $\Delta p \geq t_N + \kappa_u \tilde{q}_p$, then $z^{eq} = 0$. In this case, (q_N^{eq}, q_{NoN}^{eq}) is of the form of part 2 of Theorem 1 (note that $\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$). Thus, $n_{NoN} = 0$. Therefore, the ISP NoN receives a payoff of zero, and a deviation of this kind is not profitable for this ISP (since the equilibrium payoff is positive.).

Case A-2-iii-b: Consider a deviation to Region B by ISP NoN by which $z^{eq} = 0$. then by item 2 of Theorem 1, $n_{NoN} = 0$. Therefore, the ISP NoN receives a payoff of zero, and a deviation of this kind is not profitable for this ISP.

Case A-2-iii-c: Now, consider Consider a deviation to Region B by ISP NoN by which $z^{eq} = 1$. In this case, by item 3 of Theorem 2, $(0, \tilde{q}_p) \in F_1^I$, and by Theorem 4 and

Lemma 3, $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN} - \frac{\tilde{q}_f}{\tilde{q}_p})$ and $n_{NoN} = \frac{t_N + \kappa_u \tilde{q}_p - \Delta p}{t_N + t_{NoN}}$. Therefore, using (2.1):

$$\begin{aligned} \pi_{NoN}(\tilde{p}'_{NoN}, \tilde{p}_{t,2}) &= (p'_{NoN} - c)n_{NoN} + \kappa_{ad}(n_{NoN}\tilde{q}_p - \tilde{q}_f) \\ &= (p'_{NoN} - c + \kappa_{ad}\tilde{q}_p)n_{NoN} - \kappa_{ad}\tilde{q}_f \end{aligned} \quad (2.16)$$

in which $n_{NoN} = \frac{t_N + \kappa_u \tilde{q}_p - p'_{NoN} + c}{t_N + t_{NoN}}$. The maximum $\pi_{NoN}(\tilde{p}'_{NoN}, \tilde{p}_{t,2})$ can be found by applying the first order condition on the payoff, which gives us:

$$p'_{NoN} = c + \frac{1}{2}(t_N + \tilde{q}_p(\kappa_u - \kappa_{ad})) \quad (2.17)$$

This deviation is a profitable deviation in region B if (i) $\pi_{NoN}(\tilde{p}'_{NoN}, \tilde{p}_{t,2}) > \pi_{NoN}^{eq}$ and (ii) $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$. We also claim (claim is proved in the next two paragraphs) that if any deviation to region B is profitable, then (i) $\pi_{NoN}(\tilde{p}'_{NoN}, \tilde{p}_{t,2}) > \pi_{NoN}^{eq}$ and (ii) $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$. Thus, a deviation to this region is profitable if and only if (i) $\pi_{NoN}(\tilde{p}'_{NoN}, \tilde{p}_{t,2}) > \pi_{NoN}^{eq}$ and (ii) $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$.

Now, we prove the claim that (i) $\pi_{NoN}(\tilde{p}'_{NoN}, \tilde{p}_{t,2}) > \pi_{NoN}^{eq}$ and (ii) $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$ are necessary condition for a profitable deviation. First, we prove that (ii) is a necessary condition. Suppose (ii) is not true. We claim that no p'_{NoN} such that $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$ can be a profitable deviation. To prove this, note that by concavity of (2.16), if p'_{NoN} is not such that $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$, then all p'_{NoN} such that $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$ yields a strictly lower payoff than the maximum of payoffs at the boundary points. Note that with the upper boundary point, $\Delta p = p'_{NoN} - c = t_N + \kappa_u \tilde{q}_p$. In this case, by item 4 of Theorem 2, $z^{eq} = 0$, and by item 2 of Theorem 1, $n_{NoN} = 0$. Thus, the payoff of ISP NoN is zero (by (2.1)). On the other hand, in the lower boundary point, i.e. $p'_{NoN} = \kappa_u \tilde{q}_p - t_{NoN} + c$ is equal to p_{NoN}^{eq} .

Thus, the maximum payoff at the boundary points is equal to the equilibrium payoff. Therefore, if p_{NoN}^* is not such that $\kappa_u \tilde{q}_p - t_{NoN} < p_{NoN}^* - c < t_N + \kappa_u \tilde{q}_p$, then all p'_{NoN} such that $\kappa_u \tilde{q}_p - t_{NoN} < p'_{NoN} - c < t_N + \kappa_u \tilde{q}_p$, yields a payoff which is strictly less than the equilibrium payoff. The proof of (ii) being a necessary condition is complete.

Now, we prove that (i) is a necessary condition. Suppose (i) is not true and:

$$\pi_{NoN}(\tilde{p}_{NoN}^*, \tilde{p}_{t,2}) \leq \pi_{NoN}^{eq}$$

Then, either (ii) is true or not. If (ii) is not true, in the previous paragraph, we prove that no p'_{NoN} if Region B can be a profitable deviation, which yields the result. Now, consider the case that (ii) holds. In this case, by concavity of the payoff, p_{NoN}^* yields the highest payoff among p_{NoN} 's in Region B. Thus, $\pi_{NoN}(\tilde{p}_{NoN}^*, \tilde{p}_{t,2}) \leq \pi_{NoN}^{eq}$ yields that a deviation to Region B cannot be profitable. This completes the proof of the claim.

Thus, a deviation to region B is profitable if and only if (i) $\pi_{NoN}(\tilde{p}_{NoN}^*, \tilde{p}_{t,2}) > \pi_{NoN}^{eq}$ and (ii) $\kappa_u \tilde{q}_p - t_{NoN} < p_{NoN}^* - c < t_N + \kappa_u \tilde{q}_p$. First we check (i) and then (ii). Using (2.16), (2.17), and the expressions of n_{NoN} , we find the payoff of ISP NoN after deviation and compare it to the value of (2.25). We claim that (i) is always true unless $\tilde{q}_p = \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

Note that:

$$\begin{aligned} \pi_{NoN}(\tilde{p}_{NoN}^*, \tilde{p}_{t,2}) \geq \pi_{NoN}^{eq} &\iff \frac{(t_N + \tilde{q}_p(\kappa_{ad} + \kappa_u))^2}{4(t_N + t_{NoN})} \geq \tilde{q}_p(\kappa_u + \kappa_{ad}) - t_{NoN} \\ &\iff ((\kappa_u + \kappa_{ad})\tilde{q}_p - t_N - 2t_{NoN})^2 \geq 0 \end{aligned}$$

Thus, (i) is true if and only if $\tilde{q}_p \neq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

Now, we check (ii). Note that $p_{NoN}^* - c < t_N + \kappa_u \tilde{q}_p$ since:

$$p_{NoN}^* - c < t_N + \kappa_u \tilde{q}_p \iff \tilde{q}_p(\kappa_u + \kappa_{ad}) > -t_N$$

is always true. Now, we should check the lowerbound, i.e. $\kappa_u \tilde{q}_p - t_{NoN} < p_{NoN}^* - c$:

$$\kappa_u \tilde{q}_p - t_{NoN} < p_{NoN}^* - c \iff \tilde{q}_p(\kappa_u + \kappa_{ad}) < t_N + 2t_{NoN}$$

which is true if and only if $\tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

Now, using the conditions for (i) and (ii) to be true, we can say that (i) and (ii) are true if and only if $\kappa_u \tilde{q}_p - t_{NoN} < p_{NoN}^* - c$. Thus, there is no profitable deviation to region B if and only if $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

This completes the proof of item 1 of theorem that $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$ are NE strategies if and only if $\tilde{q}_p \geq \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

Case B: Now, we characterize any possible NE strategies in region B, i.e. $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < t_N + \kappa_u \tilde{q}_p$, by which $z^{eq} = 1$. First, in case B-1 we prove that if $z^{eq} = 1$, the only possible NE in this region is $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$. We also prove that the necessary condition for these strategies to be a NE is $\tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$, and verify that these strategies yield $z^{eq} = 1$. In case B-2, we characterize the necessary and sufficient condition by which there is no unilateral profitable deviation for ISPs.

Case B-1: Note that in this region, by item 3 of Theorem 2, if $z^{eq} = 1$, then $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN} - \frac{\tilde{q}_f}{\tilde{q}_p})$ and $n_{NoN} = \frac{t_N + \kappa_u \tilde{q}_p - \Delta p}{t_N + t_{NoN}}$ (by (2.5)). Thus, by (2.1), the payoff of ISP NoN in this region is $\pi_{NoN,B}(p_{NoN}, \tilde{p}_{t,2}) = (p_{NoN} - c)n_{NoN} + \tilde{p}_{t,2}\tilde{q}_p$, and the payoff of ISP N is $\pi_{N,B} = (p_N - c)(1 - n_{NoN})$. Note that $\tilde{p}_{t,2}\tilde{q}_p = \kappa_{ad}(\tilde{q}_p n_{NoN} - \tilde{q}_f)$. Thus, using the expression of n_{NoN} , the payoffs are:

$$\pi_{NoN,B} = (p_{NoN} - c + \kappa_{ad}\tilde{q}_p) \left(\frac{t_N + \kappa_u \tilde{q}_p + p_N - p_{NoN}}{t_N + t_{NoN}} \right) - \kappa_{ad}\tilde{q}_f \quad (2.18)$$

$$\pi_{N,B} = (p_N - c) \left(\frac{t_{NoN} - \kappa_u \tilde{q}_p + p_{NoN} - p_N}{t_N + t_{NoN}} \right) \quad (2.19)$$

Note that any NE inside this region should satisfy the first order optimality condition (note that the payoffs are concave). Thus,

$$\begin{aligned} \frac{d\pi_N}{dp_N} = 0 &\Rightarrow t_{NoN} - \kappa_u \tilde{q}_p + p_{NoN} - 2p_N + c = 0 \\ \frac{d\pi_{NoN,B}}{dp_{NoN}} = 0 &\Rightarrow t_N + \tilde{q}_p(\kappa_u - \kappa_{ad}) + p_N - 2p_{NoN} + c = 0 \end{aligned} \quad (2.20)$$

Solving the equation yields:

$$p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3} \quad (2.21)$$

$$p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3} \quad (2.22)$$

The equilibrium payoffs for ISP are:

$$\pi_{NoN}^{eq} = \frac{(t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})} - \kappa_{ad} \tilde{q}_f \quad (2.23)$$

$$\pi_N^{eq} = \frac{(2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})} \quad (2.24)$$

Now, we check the necessary conditions for these strategies to be NE. First, note that if $\tilde{q}_p > \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$, then $p_N^{eq} < c$, and p_N^{eq} cannot be an NE. Thus, the first necessary condition for these strategies to be NE is $\tilde{q}_p \leq \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$. The next necessary condition is that $\Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq}$ to be in region B, i.e. $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p^{eq} < t_N + \kappa_u \tilde{q}_p$. We claim that the upperbound always holds. To prove this consider:

$$\Delta p^{eq} < t_N + \kappa_u \tilde{q}_p \iff 2t_N + t_{NoN} + \tilde{q}_p(\kappa_u + \kappa_{ad}) > 0$$

which is always true. Now, we check the lower bound:

$$\begin{aligned}\kappa_u \tilde{q}_p - t_{NoN} < \Delta p^{eq} &\iff \kappa_u \tilde{q}_p - t_{NoN} < \frac{1}{3}(t_N - t_{NoN} + \tilde{q}_p(2\kappa_u - \kappa_{ad})) \\ &\iff \tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}\end{aligned}$$

Thus, this necessary condition together with the previous necessary condition yields that if p_N^{eq} and p_{NoN}^{eq} , NE strategies, then $\tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$.

In addition, note that by Lemma 7, p_N^{eq} and p_{NoN}^{eq} indeed yields $z^{eq} = 1$.

Thus, if $\tilde{q}_p < \frac{t_N + 2t_{NoN}}{\kappa_u + \kappa_{ad}}$, then the NE strategies can be p_N^{eq} and p_{NoN}^{eq} . To prove that these strategies are NE, we should rule out the possibility of a unilateral profitable deviation by both ISPs which we proceed to do in the next case.

Case B-2: In this case, we consider the possibility of a unilateral deviation by ISPs. First, in Case B-2-i, we rule out the possibility of a profitable deviation by the non-neutral ISP, and then in Case B-2-ii, we provide necessary and sufficient condition for a unilateral deviation to be not profitable for the neural ISP.

Case B-2-i: A deviation by the non-neutral ISP can be to regions A, C, and other prices in region B. In the following cases, we prove that a deviation by ISP NoN to each of these regions is not profitable:

Case B-2-i-A: Consider p_N^{eq} fixed and decreasing p_{NoN} such that Δp in regions A, i.e. $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Note that in A the payoff of the ISP NoN is an increasing function of p_{NoN} (as discussed in Case A). Thus, all other prices are dominated by $p'_{NoN} = p_N^{eq} + \kappa_u \tilde{q}_p - t_{NoN}$. The payoff in this case is $\pi'_{NoN} = p_N^{eq} + \kappa_u \tilde{q}_p - t_{NoN} - c + z \tilde{q}_p \tilde{p}_{t,1}$ (by (2.1)), and $\tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$ (by definition 3). We claim that this deviation is not

profitable for ISP NoN, since:

$$\begin{aligned}
\pi_{NoN}(p_{NoN}, \tilde{p}_{t,2}) &\geq p_N^{eq} - c + \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) \\
&\iff \frac{(t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})} \geq \frac{t_N - t_{NoN} + 2\tilde{q}_p(\kappa_u + \kappa_{ad})}{3} \\
&\iff (\tilde{q}_p(\kappa_u + \kappa_{ad}) - t_N - 2t_{NoN})^2 \geq 0
\end{aligned}$$

which is true always. Thus, no deviation is profitable for ISP NoN.

Case B-2-i-B: Now, consider a deviation by ISP NoN inside region B. By optimality of the solution inside B, if $z^{eq} = 1$, since $p_N = p_N^{eq}$ is fixed, all other p_{NoN} such that Δp in region B is dominated in payoff by $p_{NoN} = p_{NoN}^{eq}$. If p_{NoN} is such that $z^{eq} = 0$, then $n_{NoN} = 0$ (by item 2 of Theorem 1 and $\kappa_u \tilde{q}_p - t_{NoN} \geq t_N$). Thus, the payoff of ISP NoN is zero and this deviation is also not profitable.

Case B-2-i-C: In this case, consider a deviation to region C, i.e. $\Delta p \geq t_N + \kappa_u \tilde{q}_p$. Fixing p_N^{eq} and increasing p_{NoN} such that Δp in regions C yields a payoff of zero to ISP NoN (since by item 4 of Theorem 2, $z^{eq} = 0$ in this region, and by Theorem 1, $n_{NoN}^{eq} = 0$). Thus, this deviation is also not profitable.

Case B-2-ii: Now, consider a unilateral deviation by the non-neutral ISP. Similar to the case B-2-i, this deviation can be to regions A, C, and other prices in region B:

Case B-2-ii-A: In this case, we consider the possibility of a deviation by ISP N to region A, i.e. $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Note that in region A, $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,1}) > \pi_{NoN,z=0}(p_{NoN}, \tilde{p})$, where $\pi_{NoN,z=0}(p_{NoN}, \tilde{p})$ is the payoff of ISP NoN when $z^{eq} = 0$. To prove this note that by $\tilde{q}_p \tilde{p}_{t,1} = \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) > 0$, we can write:

$$\pi_{NoN}(p_{NoN}, \tilde{p}_{t,1}) = p_{NoN} - c + \tilde{q}_p \tilde{p}_{t,1} > p_{NoN} - c > \pi_{NoN,z=0}(p_{NoN}, \tilde{p})$$

Thus, in region A, $\pi_{NoN}(p_{NoN}, \tilde{p}_{t,1}) > \pi_{NoN,z=0}(p_{NoN}, \tilde{p})$, and by Theorem 3, $z^{eq} = 1$.

Thus, using Theorem 2, in this region $n_{NoN} = 1$. Therefore, $n_N = 0$, and by (2.1), the

payoff of ISP N is zero. Thus, a deviation to this region is not profitable.

Case B-2-ii-B: Now, consider a deviation inside region B by ISP N. If $z^{eq} = 1$, by optimality of the solution inside B (since $p_N = p_N^{eq}$ is fixed) all other p_N such that Δp in region B is dominated in payoff by $p_N = p_N^{eq}$.

Now, consider the case that p_N is such that $z^{eq} = 0$. In this case, $n_{NoN} = 0$ (by item 2 of Theorem 1 and $\kappa_u \tilde{q}_p - t_{NoN} \geq t_N$), and such a deviation might be profitable.

In order to have $z^{eq} = 0$, by Theorem 3, $\pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,2}) \leq \pi_{NoN,z=0}(p_{NoN}^{eq}, \tilde{p})$, where $\pi_{NoN,z=0}(p_{NoN}^{eq}, \tilde{p})$ is the payoff when $z^{eq} = 0$. Note that by the assumption of the theorem ($\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$), and in this region $\Delta p > \kappa_u \tilde{q}_p - t_{NoN} \geq t_N$. Thus, by Theorem 1, if $z^{eq} = 0$, then $n_{NoN} = 0$. Therefore, by (2.1), $\pi_{NoN,z=0}(p_{NoN}^{eq}, \tilde{p}) = 0$. Using (2.18), we can find $\pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,2})$, and compare the payoffs:

$$\begin{aligned} \pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,2}) \leq \pi_{NoN,z=0}(p_{NoN}^{eq}, \tilde{p}) &\iff (p_{NoN}^{eq} - c + \kappa_{ad} \tilde{q}_p) \left(\frac{t_N + \kappa_u \tilde{q}_p + p'_N - p_{NoN}^{eq}}{t_N + t_{NoN}} \right) - \kappa_{ad} \tilde{q}_f \leq 0 \\ &\iff p'_N \leq \frac{\kappa_{ad} \tilde{q}_f (t_N + t_{NoN})}{p_{NoN}^{eq} - c + \kappa_{ad} \tilde{q}_p} + p_{NoN}^{eq} - t_{NoN} - \kappa_u \tilde{q}_p = p_t^d \end{aligned}$$

Therefore, a deviation is only profitable if $p'_N \leq p_t^d$. If this condition holds, we need to check whether this deviation is indeed profitable. Note that in region B, if $z^{eq} = 0$, (as explained before) by Theorem 1, $n_N = 1$. Thus, by (2.1), the payoff of ISP N is an increasing function of p_N , and is equal to $p'_N - c$. Thus, $p'_N = p_t^d$ yields the maximum payoff after deviation. Therefore, such a deviation is not profitable if and only if $p_t^d - c \leq \pi_N(p_N^{eq})$.

Case B-2-ii-C: Now, consider a deviation by ISP N to region C, i.e. $\Delta p \geq \kappa_u \tilde{q}_p + t_N$. Note that in region C, $z^{eq} = 0$, and by item 2 of Theorem 1, $n_N = 1$. Thus, the payoff of ISP N (2.1) is an increasing function of p_N . Thus, $p'_N = p_{NoN}^{eq} - \kappa_u \tilde{q}_p - t_N$ (by definition of region C) yields the highest payoff after deviation. Note that by (2.34), $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$. Therefore, $p'_N = c + \frac{t_{NoN} - t_N - 2\tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$. In addition,

note that by the assumption of the theorem, $\kappa_u \tilde{q}_p \geq t_N + t_{NoN}$. Thus, $p'_N < c$, and by (2.1), the payoff of neutral ISP is negative. Thus, this deviation is not profitable.

Therefore, we only need to check the condition in Case B-2-ii-B for ruling out profitable deviations. This is item 2 of the theorem. The theorem follows. \square

Proof of Theorem 7

In this case, note that $\tilde{q}_f < \tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$. Thus, we characterize the optimum strategies for the CP using items 1, 2, and 4 of Theorem 2.

First, note that by Lemma 5, since $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$, $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p_t < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, where $\Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ characterized in Lemma 1. Thus, using this result, we characterize the regions characterized in items 1, 2, and 4 of Theorem 2. We denote $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$ by region A, $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ by region B_1 , $\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} \leq \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$ by region C, $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u \tilde{q}_p$ by set B_2 , and $\Delta p \geq t_N + \kappa_u \tilde{q}_p$ by D. Using Theorem 2, if $z^{eq} = 1$, then $\Delta p < t_N + \kappa_u \tilde{q}_p$. Thus, we characterize any possible NE strategies by which $z^{eq} = 1$, in regions A and B_1 , C, and B_2 :

Case A: First, we consider $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. In this case, we show that the payoff of ISP NoN is an increasing function of Δp . Then, we characterize the NE as $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$, using the fact that when choosing an NE, no player can increase her payoff by unilaterally changing her strategy.

Note that by Theorem 2, for region A, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$ if and only if $\tilde{p} \leq \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$. In addition, by Theorem 4, if $z^{eq} = 1$ then $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$ (Definition 3). Thus, in this region, if $z^{eq} = 1$, the payoff of ISP NoN is equal

to $p_{NoN} - c + \tilde{q}_p \tilde{p}_{t,1}$ (by (2.1)) since $n_{NoN} = 1$. Therefore, the payoff is an increasing function of p_{NoN} . In addition, note that in region A, $n_N = 0$ and regardless of p_N , the neutral ISP receives a payoff of zero (by (2.1)). Thus, p_{NoN}^{eq} , i.e. the equilibrium Internet access fee, should be such that the neutral ISP cannot get a positive payoff by increasing or decreasing p_N , and changing the region of Δp to B_1 , B_2 , or C . Using this condition, we find the equilibrium strategy.

First consider a unilateral deviation by ISP N. Note that increasing p_N decreases Δp , and cannot change the region of Δp . Thus, a deviation of this kind would not be profitable. We claim that by decreasing p_N to p'_N such that $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, the ISP N can fetch a positive payoff as long as $p'_N > c$ (the claim is proved in the next paragraph). Therefore, in the equilibrium, p_{NoN}^{eq} is such that even with $p'_N = c$ (the minimum plausible price), $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Thus, $p_{NoN}^{eq} \leq c + \kappa_u \tilde{q}_p - t_{NoN}$ (Otherwise, there exists a $p'_N > c$ by which $\Delta p > \kappa_u \tilde{q}_p - t_{NoN}$). Given that the payoff of ISP NoN is an increasing function of p_{NoN} , we get $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$. In addition, we claim that $p_N^{eq} = c$. If not, then $p_N^{eq} > c$. In this case, $\Delta p = p_N^{eq} - p_{NoN}^{eq} < \kappa_u \tilde{q}_p - t_{NoN}$. We argued that the payoff of ISP NoN is an increasing function of p_{NoN} . Thus, by increasing p_{NoN} such that $\Delta p = \kappa_u \tilde{q}_p - t_{NoN}$, ISP NoN can increase her payoff, which is a contradiction with p_N^{eq} and p_{NoN}^{eq} being NE strategies.

To prove the claim, note that if $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, then either (i) $z^{eq} = 0$ or (ii) $z^{eq} = 1$. For case (i), since $\kappa_u \tilde{q}_p - t_{NoN} > -t_{NoN}$, when $\Delta p > \kappa_u \tilde{q}_p - t_{NoN}$, then (q_N^{eq}, q_{NoN}^{eq}) is of the form of items 1 or 2 of Theorem 1. Thus, $n_N > 0$. Therefore ISP N can fetch a positive payoff as long as $p_N > c$ (by (2.1)). Now consider case (ii), i.e. $z^{eq} = 1$. In this case, if $z^{eq} = 1$, then by using item 2 of Theorem 2, $n_N > 0$ (since

solutions that yield $z^{eq} = 1$ are in F^I). Thus, ISP N can fetch a positive payoff as long as $p_N > c$ (by (2.1)). This completes the proof of the claim that by decreasing p_N to p'_N such that $p_{NoN} - p'_N > \kappa_u \tilde{q}_p - t_{NoN}$, the ISP N can fetch a positive payoff as long as $p'_N > c$.

Therefore, the NE strategies are $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$, and the payoff of the ISP NoN at this price by (2.1) and $\tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$ is equal to (note that $n_{NoN} = 1$), and

$$\pi_{NoN}^{eq} = \kappa_u \tilde{q}_p - t_{NoN} + \tilde{q}_p \tilde{p}_{t,1} = \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) \quad (2.25)$$

which is strictly positive since $\tilde{q}_p > \frac{t_N + t_{NoN}}{\kappa_u}$ and $\tilde{q}_p > \tilde{q}_f$. Note that Lemma 6 yields that with p_N^{eq} and p_{NoN}^{eq} , $z^{eq} = 1$. The first item of the theorem follows.

Case B_1 and B_2 : Now, consider regions B_1 and B_2 , i.e. $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ and $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u \tilde{q}_p$, respectively.

Note that in these regions, by items 2-a-ii and 2-b of Theorem 2, if $z^{eq} = 1$, then $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN} - \frac{\tilde{q}_f}{\tilde{q}_p})$ and $n_{NoN} = \frac{t_N + \kappa_u \tilde{q}_p - \Delta p}{t_N + t_{NoN}}$ (Definition 3). Thus, by (2.1), the payoff of ISP NoN in this region is $\pi_{NoN,B}(p_{NoN}, \tilde{p}_{t,2}) = (p_{NoN} - c)n_{NoN} + \tilde{p}_{t,2}\tilde{q}_p$, and the payoff of ISP N is $\pi_{N,B} = (p_N - c)(1 - n_{NoN})$. Note that $\tilde{p}_{t,2}\tilde{q}_p = \kappa_{ad}(\tilde{q}_p n_{NoN} - \tilde{q}_f)$. Thus, using the expression of n_{NoN} , the payoffs are:

$$\pi_{NoN,B}(p_{NoN}, \tilde{p}_{t,2}) = (p_{NoN} - c + \kappa_{ad}\tilde{q}_p) \left(\frac{t_N + \kappa_u \tilde{q}_p + p_N - p_{NoN}}{t_N + t_{NoN}} \right) - \kappa_{ad}\tilde{q}_f \quad (2.26)$$

$$\pi_{N,B}(p_N) = (p_N - c) \left(\frac{t_{NoN} - \kappa_u \tilde{q}_p + p_{NoN} - p_N}{t_N + t_{NoN}} \right) \quad (2.27)$$

First, we rule out any NE such that $\Delta p^{eq} = t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Suppose that $\Delta p^{eq} =$

$p_{NoN}^{eq} - p_N^{eq} = t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Consider a deviation by ISP N such that $p'_N = p_N^{eq} + \epsilon > c$ for $\epsilon > 0$ such that $\Delta p' = p_{NoN}^{eq} - p'_N$ to be in region C. Note that by item 2-a-i of Theorem 2, in region C, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$. Thus, the payoff of this ISP with this deviation is (by (2.8)):

$$\pi_N(p'_N) = (p_N^{eq} + \epsilon - c) \left(\frac{t_{NoN} - \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_{NoN}^{eq} - p_N^{eq} - \epsilon}{t_N + t_{NoN}} \right)$$

Note that $\lim_{\epsilon \downarrow 0} \pi_N(p'_N) > \pi_{N,B}(p_N^{eq})$. Thus, for $\epsilon > 0$ small enough, this deviation is profitable. Thus, the strategies by which $\Delta p^{eq} = t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$ cannot be NE.

Now, we characterize any NE in $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ and $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) < \Delta p < t_N + \kappa_u \tilde{q}_p$. Note that any NE inside this region should satisfy the first order optimality condition (note that the payoffs are concave). Thus,

$$\frac{d\pi_N}{dp_N} = 0 \Rightarrow t_{NoN} - \kappa_u \tilde{q}_p + p_{NoN} - 2p_N + c = 0 \quad (2.28)$$

$$\frac{d\pi_{NoN,B}}{dp_{NoN}} = 0 \Rightarrow t_N + \tilde{q}_p(\kappa_u - \kappa_{ad}) + p_N - 2p_{NoN} + c = 0$$

Solving the equation yields:

$$p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3} \quad (2.29)$$

$$p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3} \quad (2.30)$$

First, note that if $\tilde{q}_p > \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$, then $p_N^{eq} < c$, and p_N^{eq} cannot be an NE. Thus, the first necessary condition for these strategies to be NE is $\tilde{q}_p \leq \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$. In addition, by Theorem 3, $\pi_{NoN}^{eq} > \pi_{NoN,z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$ (for these strategies to yield $z^{eq} = 1$). The second item of the theorem follows.

Case C: Now, consider region C, i.e. $\Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} \leq \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Note that in this regions, by items 2-a-i of Theorem 2, if $z^{eq} = 1$, then $(q_N^{eq}, q_{NoN}^{eq}) =$

$(\tilde{q}_f, \tilde{q}_p) \in F_1^I$. In addition, by Theorem 4 and Definition 4, $\tilde{p}^{eq} = \tilde{p}_{t,3} = \kappa_{ad}n_{NoN}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$ and $n_{NoN} = \frac{t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) - \Delta p}{t_N + t_{NoN}}$ (Definition 3). Thus, by (2.1), the payoff of ISP NoN in this region is $\pi_{NoN,C}(p_{NoN}, \tilde{p}_{t,3}) = (p_{NoN} - c)n_{NoN} + \tilde{p}_{t,3}\tilde{q}_p$, and the payoff of ISP N is $\pi_{N,B} = (p_N - c)(1 - n_{NoN})$. Note that $\tilde{p}_{t,3}\tilde{q}_p = \kappa_{ad}n_{NoN}(\tilde{q}_p - \tilde{q}_f)$. Thus, using the expression of n_{NoN} , the payoffs are:

$$\pi_{NoN,C}(p_{NoN}, \tilde{p}_{t,3}) = (p_{NoN} - c + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f))\left(\frac{t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_N - p_{NoN}}{t_N + t_{NoN}}\right) \quad (2.31)$$

$$\pi_{N,C}(p_N) = (p_N - c)\left(\frac{t_{NoN} - \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_{NoN} - p_N}{t_N + t_{NoN}}\right) \quad (2.32)$$

First, in Part C-1, we characterize any NE in region $\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Later, in Part C-2, we consider the case that $\Delta p^{eq} = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$.

Part C-1: Note that any NE in region $\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$ should satisfy the first order optimality condition (note that the payoffs are concave).

Thus,

$$\frac{d\pi_{N,C}}{dp_N} = 0 \Rightarrow t_{NoN} - \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_{NoN} - 2p_N + c = 0 \quad (2.33)$$

$$\frac{d\pi_{NoN,C}}{dp_{NoN}} = 0 \Rightarrow t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - \kappa_{ad}) + p_N - 2p_{NoN} + c = 0$$

Solving the equation yields:

$$p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad})}{3} \quad (2.34)$$

$$p_N^{eq} = c + \frac{2t_{NoN} + t_N - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3} \quad (2.35)$$

First, note that if $\tilde{q}_p - \tilde{q}_f > \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$, then $p_N^{eq} < c$, and p_N^{eq} cannot be an NE. Thus, the first necessary condition for these strategies to be NE is $\tilde{q}_p - \tilde{q}_f \leq \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$. In addition, by Theorem 3, $\pi_{NoN}^{eq} > \pi_{NoN,z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$ (in order for these strategies to yield $z^{eq} = 1$). The third item of the theorem follows.

Part C-2: Now, consider p_N^{eq} and p_{NoN}^{eq} such that $\Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq} = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$. These strategies are not NE if ISP NoN can strictly increase her payoff by decreasing her price such that Δp in region B_1 . Note that using (2.31) and the expression for Δp^{eq} , the payoff of ISP NoN in this case is:

$$\pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,3}) = (p_{NoN} - c + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f)) \left(\frac{t_N - \kappa_u \tilde{q}_p + t_{NoN}}{t_N + t_{NoN}} \right) \quad (2.36)$$

By choosing $p'_{NoN} = p_{NoN}^{eq} - \epsilon$ such that $\epsilon \downarrow 0$, ISP NoN can get a limit payoff of (since $\Delta p = \Delta p^{eq}$ when $\epsilon \rightarrow 0$, and it is in region B_1 , and using (2.26)):

$$\pi'_{NoN} = \lim_{\epsilon \downarrow 0} \pi_{NoN}^{eq}(p_{NoN} - \epsilon, \tilde{p}_{t,3}) = (p_{NoN}^{eq} - c + \kappa_{ad} \tilde{q}_p) \left(\frac{t_N - \kappa_u(\tilde{q}_p - \tilde{q}_f) + t_{NoN}}{t_N + t_{NoN}} \right) - \kappa_{ad} \tilde{q}_f$$

Thus, p_N^{eq} and p_{NoN}^{eq} such that $\Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq} = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ are not NE if:

$$\begin{aligned} \pi'_{NoN} > \pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,3}) &\iff (p_{NoN}^{eq} - c + \kappa_{ad} \tilde{q}_p) \frac{\kappa_u \tilde{q}_f}{t_N + t_{NoN}} - \frac{\kappa_{ad} \kappa_u \tilde{q}_f \tilde{q}_p}{t_N + t_{NoN}} > 0 \\ &\iff p_{NoN}^{eq} > c \end{aligned}$$

Thus, the necessary condition for these strategy to be NE is $p_{NoN}^{eq} \leq c$. Note that from (2.31) and (2.32), since Δp is fixed, the payoffs of ISP NoN and N are an increasing function of p_{NoN} and p_N , respectively. Thus, $p_{NoN}^{eq} = c$, and $p_N^{eq} = c - \kappa_u(2\tilde{q}_p - \tilde{q}_f) + t_{NoN}$. Note that a necessary condition for p_N^{eq} to be an NE is that $p_N^{eq} \geq c$. Thus, one necessary condition is that $2\tilde{q}_p - \tilde{q}_f \leq \frac{t_{NoN}}{\kappa_u}$. In addition, $\pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,3}) > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$ (using Theorem 3, in order for these strategies to yield $z^{eq} = 1$). The forth item of the theorem follows.

Proof of Theorem 8

We use Theorem 7 to prove the result. First, in Part 1, we prove that when one of t_N or t_{NoN} is large, then strategies 1), 2), and 4) listed in Theorem 7 are not NE. In Part 2, we

prove that when one of t_N or t_{NoN} is high, then strategy 3) of Theorem 7 is an NE. This completes the proof of the theorem.

Part 1: We prove that strategies 1), 2), and 4) listed in Theorem 7 are not NE in Parts 1-i, 1-ii, and 1-iii, respectively.

Part 1-i: In this part, we prove that, item 1 of Theorem 7, i.e. $p_{NoN}^{eq} = c + \kappa_u \tilde{q}_p - t_{NoN}$ and $p_N^{eq} = c$ is not an NE. We do so in Parts 1-i-a and 1-i-b, by introducing a unilateral profitable deviation for ISP NoN for the cases that t_{NoN} is large and t_N is large, respectively. Note that in this case, by item 1 of Theorem 2, $(q_N^{eq}, q_{NoN}^{eq}) \in (0, \tilde{q}_f) \in F_1^L$. Thus, $n_{NoN} = 1$, and the payoff of ISP NoN is (by (2.1), Theorem 4, and Definition 3):

$$\pi_{NoN} = \kappa_u \tilde{q}_p - t_{NoN} + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) \quad (2.37)$$

Part 1-i-a: If t_{NoN} is large, then (2.37) would be less than zero. A deviation to price $p'_{NoN} = c$ yields a payoff of at least zero for the ISP NoN (by (2.1)). Thus, this is a profitable deviation.

Part 1-i-b: Now, consider t_N to be large, and a deviation by ISP NoN such that $p'_{NoN} = \frac{1}{2}t_N$ (Thus, $\Delta p = p'_{NoN} - p_N^{eq} = \frac{1}{2}t_N - c$). Note that in this case, $\Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Thus, by item 2-a-i of Theorem 2, $(q_n^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$. Thus, by (2.1), the payoff of ISP NoN after deviation (by the definition of $\tilde{p}_{t,3}$ in Definition 3 and Theorem 4) is at least¹⁰:

$$\pi'_{NoN} = \frac{1}{2}t_N n_{NoN} + \kappa_{ad} n_{NoN}(\tilde{q}_p - \tilde{q}_f) \quad (2.38)$$

, where $n_{NoN} = \frac{\frac{1}{2}t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) + c}{t_N + t_{NoN}}$. Thus, for t_N large, $n_{NoN} \rightarrow \frac{1}{2}$. Thus, comparing (2.38)

¹⁰Note that the payoff of NoN is equal to the maximum of the payoff when $\tilde{p}^{eq} = \tilde{p}_t$ and when $\tilde{p}^{eq} > \tilde{p}_t$, i.e. when $z^{eq} = 0$

with (2.37) yields:

$$\pi'_{NoN} = \frac{1}{4}t_N + \frac{1}{2}\kappa_{ad}(\tilde{q}_p - \tilde{q}_f) > \pi_{NoN} \quad \text{since } t_N \text{ is large}$$

Thus, this deviation is profitable .

Part 1-ii: In this part, we prove that item 2 of Theorem 7, i.e. $p_{NoN}^{eq} = c + \frac{t_{NoN} + 2t_N + \tilde{q}_p(\kappa_u - 2\kappa_{ad})}{3}$ and $p_N^{eq} = c + \frac{2t_{NoN} + t_N - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3}$ is not an NE. We do so by proving that Δp^{eq} does not satisfy $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p^{eq} < \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ and $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) < \Delta p^{eq} < t_N + \kappa_u \tilde{q}_p$, in the cases that t_{NoN} or t_N is large.

First, note that:

$$\Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq} = \frac{1}{3}(t_N - t_{NoN} + \tilde{q}_p(2\kappa_u - \kappa_{ad})) \quad (2.39)$$

If $\Delta p^{eq} < \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$, then $t_N + 2t_{NoN} < 3\kappa_u(2\tilde{q}_p - \tilde{q}_f) - \tilde{q}_p(2\kappa_u - \kappa_{ad})$, which is not correct when t_{NoN} or t_N is large. Thus, (a) $\Delta p^{eq} \geq \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$. In addition, if $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) < \Delta p^{eq}$, then $2t_N + t_{NoN} < \tilde{q}_p(2\kappa_u - \kappa_{ad}) - 3\kappa_u(\tilde{q}_p - \tilde{q}_f)$, which is not correct when t_{NoN} or t_N is large. Thus, (b) $\Delta p^{eq} \leq t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Therefore, (a) and (b) yields that Δp^{eq} is not in the regions specified. Thus, item 2 cannot be an NE.

Part 1-iii: In this part, we prove that item 4 of Theorem 7, i.e. $p_{NoN}^{eq} = c$ and $p_N^{eq} = c - \kappa_u(2\tilde{q}_p - \tilde{q}_f) + t_{NoN}$ is not an NE. To do so, we prove that there exists a profitable unilateral deviation for ISP NoN. Note that, in this case, $\Delta p^{eq} = \Delta p_t$. By item 2-a-i of Theorem 2, when $\Delta p_t \leq \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, then $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$. Thus, the expression of the payoff of ISP NoN is (by $\tilde{p}_t = \tilde{p}_{t,3}$, Definition 3, Theorem 4, and (2.9)):

$$\pi_{NoN,C}(p_{NoN}, \tilde{p}_{t,3}) = (p_{NoN} - c + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f)) \left(\frac{t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) + p_N - p_{NoN}}{t_N + t_{NoN}} \right)$$

Note that:

$$\frac{d\pi_{NoN,C}}{dp_{NoN}} = \frac{t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - \kappa_{ad}) + p_N - 2p_{NoN} + c}{t_N + t_{NoN}}$$

Thus,

$$\frac{d\pi_{NoN,C}}{dp_{NoN}} \Big|_{p_N^{eq}, p_{NoN}^{eq}} = \frac{t_N + t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - \kappa_{ad}) - \kappa_u(2\tilde{q}_p - \tilde{q}_f)}{t_N + t_{NoN}}$$

Note that $\frac{d\pi_{NoN,C}}{dp_{NoN}} \Big|_{p_N^{eq}, p_{NoN}^{eq}} > 0$, when either t_N or t_{NoN} are large enough. Thus, in this case, the payoff is increasing with respect to p_{NoN} . Thus, $p'_{NoN} = p_{NoN}^{eq} + \epsilon$ for $\epsilon > 0$ small, is a unilateral profitable deviation.

Part 2: We now prove that when one of t_N or t_{NoN} is large, then strategy 3) of Theorem 7 is an NE. To do so, we check conditions (i), (ii), and (iii) of strategy 3) of Theorem 7, in Parts 2-i, 2-ii, and 2-iii, respectively. Later, in Part 2-iv, we prove that there is no unilateral profitable deviation for ISPs. This completes the proof.

Part 2-i: In this part, we check the condition, i.e. $\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p^{eq} < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$. Note that in this case:

$$\Delta p^{eq} = \frac{1}{3}(t_N - t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(2\kappa_u - \kappa_{ad})) \quad (2.40)$$

Comparing the lower boundary yields that:

$$\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p^{eq} \Rightarrow 2t_{NoN} + t_N + (\tilde{q}_p - \tilde{q}_f)(2\kappa_u - \kappa_{ad}) - 3\kappa_u(2\tilde{q}_p - \tilde{q}_f) > 0$$

which is true when one of t_N or t_{NoN} is large. Now, consider the upper boundary:

$$\Delta p^{eq} < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \Rightarrow 2t_N + t_{NoN} + \kappa_u(\tilde{q}_p - \tilde{q}_f) - (\tilde{q}_p - \tilde{q}_f)(2\kappa_u - \kappa_{ad}) > 0$$

which is true when one of t_N or t_{NoN} is large. Thus, condition (i) of strategy 3) of Theorem 7 is true.

Part 2-ii: Condition (ii) of this strategy is $\tilde{q}_p - \tilde{q}_f \leq \frac{2t_{NoN} + t_N}{\kappa_u + \kappa_{ad}}$. This condition holds when one of t_N or t_{NoN} is large.

Part 2-iii: Now, we check the third condition, i.e.

$$\pi_{NoN}^{eq} = \pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,3}) > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$$

We use (2.1) to find $\pi_{NoN}^{eq} = \pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,3})$. Note that by using item 2-a-i of Theorem 2 (since $z^{eq} = 1$), $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p)$. Thus, by the definition of p_{NoN}^{eq} , Δp^{eq} , $\tilde{p}_{t,3}$, using Definition 3, and Theorem 4:

$$\pi_{NoN}^{eq} = \frac{(t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})} \quad (2.41)$$

Now, we obtain $\pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$. Consider the case that \tilde{p} is such that $z^{eq} = 0$. Note that since $\kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} < \Delta p^{eq} < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$, then $-t_{NoN} < \Delta p^{eq} < t_N$ or $\Delta p^{eq} \geq t_N$. Using item 2 of Theorem 1, if $\Delta p^{eq} \geq t_N$, then $n_{NoN} = 0$, and by (2.1), $\pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p}) = 0$. Thus, $\pi_{NoN}^{eq} > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$, and this part follows. Now, consider the case that $-t_{NoN} < \Delta p^{eq} < t_N$. Using item 1 of Theorem 1, if $-t_{NoN} < \Delta p^{eq} < t_N$, then $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_0^I$. Since $(q_N^{eq}, q_{NoN}^{eq}) \in F_0^I$, we can use (2.9). Thus, by using p_{NoN}^{eq} , Δp^{eq} , and $\pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$ is:

$$\begin{aligned} \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p}) &= \frac{1}{9(t_N + t_{NoN})} (2t_N + t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad})) \\ &\quad \times (2t_N + t_{NoN} - (\tilde{q}_p - \tilde{q}_f)(2\kappa_u - \kappa_{ad})) \end{aligned} \quad (2.42)$$

Next, we prove that $t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}) > 2t_N + t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad})$ and $t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}) > 2t_N + t_{NoN} - (\tilde{q}_p - \tilde{q}_f)(2\kappa_u - \kappa_{ad})$. This yields $\pi_{NoN}^{eq} > \pi_{NoN, z=0}(\tilde{p}_{NoN}^{eq}, \tilde{p})$. To prove the inequalities, note that:

$$t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}) > 2t_N + t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(\kappa_u - 2\kappa_{ad}) \iff 3\kappa_{ad}(\tilde{q}_p - \tilde{q}_f) > 0$$

$$t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}) > 2t_N + t_{NoN} - (\tilde{q}_p - \tilde{q}_f)(2\kappa_u - \kappa_{ad}) \iff 3\kappa_u(\tilde{q}_p - \tilde{q}_f) > 0$$

Since $\tilde{q}_p > \tilde{q}_f$, both inequalities hold. This completes the proof of this part.

Part 2-iv: In this part, we prove that there is no profitable unilateral deviation by ISPs when one of t_N or t_{NoN} is large. To do so, first, in Part 2-iv-NoN, we rule out the possibility of a profitable deviation by the non-neutral ISP. Then, in Part 2-iv-N, we rule out profitable deviations by the neutral ISP.

Note that, by (2.41), the equilibrium payoff of ISP NoN, $\pi_{NoN}^{eq} = \pi_{NoN}(\tilde{p}_{NoN}^{eq}, \tilde{p}_{t,3})$ is:

$$\pi_{NoN}^{eq} = \frac{(t_{NoN} + 2t_N + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})}$$

In addition, using $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p)$, p_N^{eq} , Δp^{eq} , and (2.8), we can find $\pi_N^{eq} = \pi_N(\tilde{p}_N^{eq})$:

$$\pi_N^{eq} = \frac{(2t_{NoN} + t_N - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad}))^2}{9(t_N + t_{NoN})} \quad (2.43)$$

Note that when t_N and t_{NoN} are large, π_N^{eq} and π_{NoN}^{eq} would be large.

Consider different regions in Theorem 2. We denote $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$ by region A, $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < \Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$ by region B_1 , $\Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN} \leq \Delta p < t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f)$ by region C, $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u \tilde{q}_p$ by B_2 , and $\Delta p \geq t_N + \kappa_u \tilde{q}_p$ by D. Recall that $\Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq}$ is in region C. Note that the payoffs are concave in C, and we found the strategies by solving the first order condition. Thus, there is no unilateral profitable deviation in C.

Part 2-iv-NoN: Now, we consider unilateral deviations by ISP NoN. We prove that any deviation to regions A, B_1 , B_2 , and D is not profitable in Cases 2-iv-NoN-A, 2-iv-NoN- B_1 , 2-iv-NoN- B_2 , and 2-iv-NoN-D, respectively. This yields that no deviation is profitable for ISP NoN.

Case 2-iv-NoN-A: First, we prove that in Region A, $z^{eq} = 1$. Note that in this case, by

Definition 4, $\tilde{p}_t = \tilde{p}_{t,1}$. Thus, $\pi_{NoN}(p_{NoN}, \tilde{p}_t) = p_{NoN} - c + \tilde{q}_p \tilde{p}_{t,1} = p_{NoN} - c + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f)$ (by (2.1), Definition 3, and since $n_{NoN} = 1$ by item 1 of Theorem 2). On the other hand, $\pi_{NoN, z=0}(p_{NoN}, \tilde{p}) = (p_{NoN} - c)n_{NoN}$. Thus, $\pi_{NoN}(p_{NoN}, \tilde{p}_t) > \pi_{NoN, z=0}(p_{NoN}, \tilde{p})$ (since $\tilde{q}_p > \tilde{q}_f$ and $0 \leq n_{NoN} \leq 1$). Thus, by Theorem 3, in region A, $z^{eq} = 1$.

Now, consider p_N^{eq} fixed and decreasing p_{NoN} such that Δp in region A, i.e. $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Since in Region A, $z^{eq} = 1$, and by Theorem 4, the payoff after deviation is $\pi'_{NoN} = p_{NoN} - c + \tilde{q}_p \tilde{p}_{t,1}$ (by (2.1), Definition 3, and since $n_{NoN} = 1$ by item 1 of Theorem 2). Thus, the payoff of the ISP NoN is an increasing function of p_{NoN} . Therefore, all other prices are dominated by $p'_{NoN} = p_N^{eq} + \kappa_u \tilde{q}_p - t_{NoN}$. The payoff in this case is $\pi'_{NoN} = p_N^{eq} + \kappa_u \tilde{q}_p - t_{NoN} - c + \tilde{q}_p \tilde{p}_{t,1}$ (by (2.1)). Therefore:

$$\pi'_{NoN} = \frac{1}{3}(t_N - t_{NoN}) + \alpha \quad (2.44)$$

where α is a constant independent of t_N and t_{NoN} . Now, in Cases (i), (ii), and (iii), we prove that $\pi_{NoN}^{eq} > \pi'_{NoN}$ when (i) t_N is sufficiently larger than other parameters, (ii) t_{NoN} is sufficiently larger than other parameters, and (iii) t_N and t_{NoN} are of the same order of magnitude and both are sufficiently larger than other parameters, respectively.

Case (i): If t_N is sufficiently larger than other parameters, then:

$$\pi_{NoN}^{eq} \approx \frac{4t_N}{9} > \pi'_{NoN} \approx \frac{1}{3}t_N$$

Thus, this deviation is not profitable.

Case (ii): If t_{NoN} is sufficiently larger than other parameters, then:

$$\pi_{NoN}^{eq} \approx \frac{t_{NoN}}{9} > \pi'_{NoN} \approx -\frac{1}{3}t_{NoN}$$

Thus, this deviation is also not profitable.

Case (iii): If t_N and t_{NoN} are of the same order of magnitude ($t_N \approx t_{NoN}$) and both

are sufficiently larger than other parameters, then:

$$\pi_{NoN}^{eq} = \frac{t_N}{2} > \frac{t_N}{3} > \pi'_{NoN}$$

Thus, this deviation is not profitable.

Thus, any deviation to region A by ISP NoN is not profitable. This completes the proof of this case.

Case 2-iv-NoN-B₁: Now, consider a deviation by ISP NoN to region B_1 , i.e. $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < \Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$. Note that with this deviation, $p'_{NoN} = \frac{1}{3}(t_N - t_{NoN}) + \alpha$, where $\alpha_l < \alpha < \alpha_u$, in which α_l and α_u are bounded. In addition, by (2.5), after the deviation, $n'_{NoN} = \frac{t_N + t_{NoN} - \beta}{t_N + t_{NoN}}$, where $\beta > 0$ is bounded ($\beta_l < \beta < \beta_u$, and β_l and β_u bounded). Therefore, for large t_N and t_{NoN} , $n'_{NoN} \rightarrow 1$. Thus, by (2.9), the payoff of ISP NoN after deviation is:

$$\pi'_{NoN} = \frac{1}{3}(t_N - t_{NoN}) + \gamma$$

where γ is bounded (Note that \tilde{p} is independent of t_N and t_{NoN}). This expression is similar to (2.44). Thus, we can exactly repeat the arguments in Cases i, ii, and iii to prove that any deviation to region B_1 by ISP NoN is not profitable. This completes the proof of this case.

Case 2-iv-NoN-B₂: Now, consider a deviation by ISP NoN to region B_2 , i.e. $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u \tilde{q}_p$. Note that with this deviation, $\Delta p' = t_N + \alpha$, and $p'_{NoN} = \frac{2t_{NoN} + 4t_N}{3} + \gamma$ where $\kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \alpha \leq \kappa_u \tilde{q}_p$ and thus γ is bounded. Thus, by (2.5), after this deviation, $n'_{NoN} = \frac{\beta}{t_N + t_{NoN}}$, where $\beta > 0$ is a constant independent of t_N and t_{NoN} , and the payoff of ISP NoN after deviation is $\pi'_{NoN} = \frac{2t_{NoN} + 4t_N}{3(t_N + t_{NoN})}\beta + \eta$ (by (2.1) and considering that by Theorem 4, if $z^{eq} = 1$, then $\tilde{p} = \tilde{p}_{t,2}$, and independent of

t_N and t_{NoN}), where η is a constant independent of t_N and t_{NoN} . Thus, when one of t_N and t_{NoN} is large, $\pi'_{NoN} \rightarrow \text{constant}$. Therefore, $\pi_{NoN}^{eq} > \pi'_{NoN}$. Thus, any deviation to region B_2 by ISP NoN is not profitable.

Case 2-iv-NoN-D: By item 4 of Theorem 2, in region D, $n_{NoN} = 0$. Thus, a deviation to this region, yields a payoff of zero, by (2.9) and $z^{eq} = 0$. Thus, a deviation of this kind is not profitable for ISP NoN.

Part 2-iv-N: Now, consider unilateral deviations by the neutral ISP. Similar to Part 2-iv-N, we prove that any deviation to regions A , B_1 , B_2 , and D is not profitable. We do so in Cases 2-iv-N-A, 2-iv-N- B_1 , 2-iv-N- B_2 , and 2-iv-N-D, respectively.

Case 2-iv-N-A: Consider a deviation by ISP N to region A. In this case, by item 1 of Theorem 2, $n_N = 0$. Thus, the payoff of ISP N after deviation is zero (by (2.8)), and this deviation is not profitable.

Case 2-iv-N- B_1 : Now, consider a deviation by ISP N to region B_1 , i.e. $\kappa_u \tilde{q}_p - t_{NoN} < \Delta p < \Delta p_t = \kappa_u(2\tilde{q}_p - \tilde{q}_f) - t_{NoN}$. Note that with this deviation, $\Delta p = -t_{NoN} + \alpha$, and $p'_N = \frac{4t_{NoN} + 2t_N}{3} + \gamma$, where $\kappa_u \tilde{q}_p < \alpha < \kappa_u(2\tilde{q}_p - \tilde{q}_f)$ and thus γ is bounded. Thus, by (2.5), $n'_N = \frac{\beta}{t_N + t_{NoN}}$, where $\beta > 0$ is bounded. By (2.1). The payoff of ISP N after deviation is $\pi_N = \frac{4t_{NoN} + 2t_N}{3(t_N + t_{NoN})} \beta$ (by (2.1)). Thus, when one of t_N and t_{NoN} is large, $\pi'_N \rightarrow \text{constant}$. Thus, $\pi_N^{eq} > \pi'_N$. Therefore, any deviation to region B_1 by ISP N is not profitable.

case 2-iv-N- B_2 : Now, consider a deviation by ISP NoN to region B_2 , i.e. $t_N + \kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \Delta p < t_N + \kappa_u \tilde{q}_p$. Note that with this deviation, $\Delta p' = t_N + \alpha$, where $\kappa_u(\tilde{q}_p - \tilde{q}_f) \leq \alpha < \kappa_u \tilde{q}_p$. Thus, $p'_N = \frac{1}{3}(t_{NoN} - t_N) + \beta$, where β is bounded. In addition, by (2.5), after the deviation, $n'_N = \frac{t_N + t_{NoN} - \gamma}{t_N + t_{NoN}}$, where $\gamma > 0$ is bounded. Therefore, for

large t_N or t_{NoN} , $n'_N \rightarrow 1$. Thus, by (2.8), the payoff of ISP N after deviation is:

$$\pi'_N = \frac{1}{3}(t_{NoN} - t_N) + \eta \quad (2.45)$$

where η is bounded. Now, in Cases i, ii, and iii, we prove that $\pi_N^{eq} > \pi'_N$ when (i) t_N is sufficiently larger than other parameters, (ii) t_{NoN} is sufficiently larger than other parameters, and (iii) t_N and t_{NoN} are of the same order of magnitude and both are sufficiently larger than other parameters, respectively.

Case i: If t_N is sufficiently larger than other parameters, then:

$$\pi_N^{eq} \approx \frac{t_N}{9} > \pi'_N \approx -\frac{1}{3}t_N$$

Thus, this deviation is not profitable.

Case ii: If t_{NoN} is sufficiently larger than other parameters, then:

$$\pi_N^{eq} \approx \frac{4t_{NoN}}{9} > \pi'_N \approx \frac{1}{3}t_{NoN}$$

Thus, this deviation is also not profitable.

Case iii: If t_N and t_{NoN} are of the same order of magnitude ($t_N \approx t_{NoN}$) and both are sufficiently larger than other parameters, then:

$$\pi_N^{eq} = \frac{t_{NoN}}{2} > \frac{t_{NoN}}{3} > \pi'_N$$

Thus, this deviation is not profitable.

Thus, any deviation to Region B_2 by ISP N is not profitable. This completes the proof of this case.

Case 2-iv-N-D: Now, consider decreasing p_{NoN} such that Δp in region D, i.e. $\Delta p \geq \kappa_u \tilde{q}_p + t_N$. Note that by item 4 of Theorem 2, $z^{eq} = 0$, and $n_N = 1$. Thus, the payoff

of ISP N is equal to $p_N - c$ (by (2.1)). Thus, the payoff of the ISP N is an increasing function of p_N . Therefore, all other prices are dominated by $p'_N = p_{NoN}^{eq} - (\kappa_u \tilde{q}_p + t_N)$. Thus, the payoff in this case is $\pi'_N = \frac{1}{3}(t_{NoN} - t_N) + \alpha$, where α is a constant and is independent of t_N and t_{NoN} . This expression is similar to (2.45). Thus, we can exactly repeat the arguments in Cases 2-iv-N- B_2 -a, 2-iv-N- B_2 -b, and 2-iv-N- B_2 -c to prove that any deviation to region D by ISP NoN is not profitable. This completes the proof of this case. This completes the proof of this case, and the theorem.

Proof of Theorem 9

We consider different regions of Δp in Theorem 1 and Theorem 2. For each region, we characterize all possible NE strategies.

First, consider $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$. Note that in this region, the payoff of non-neutral ISP if $z^{eq} = 0$ is at most $p_{NoN}^{eq} - c$ (by (2.1)). On the other hand, by Theorem 2, by choosing $\tilde{p}' = \tilde{p}_{t,1}$, ISP NoN can ensure that the CP chooses $z^{eq} = 1$. In this case, the payoff of non-neutral ISP (by (2.1)) is $p'_{NoN} - c + \tilde{p}_{t,1} \tilde{q}_{NoN} = p_{NoN}^{eq} - c + \kappa_{ad}(\tilde{q}_p - \tilde{q}_f) > p_{NoN}^{eq} - c$. Thus, $\pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_{t,1}) > \pi_{NoN,z=0}(p_{NoN}^{eq}, \tilde{p})$. Therefore, in this case, there is no NE by which $z^{eq} = 0$.

Now, consider $\Delta p > \kappa_u \tilde{q}_p - t_{NoN}$. Note that $\tilde{q}_p < \frac{t_N + t_{NoN}}{\kappa_u}$. Thus, two possibility may arise: (i) $-t_{NoN} < \Delta p < t_N$, and (ii) $\Delta p \geq t_N$. We consider these two cases in Case 1 and 2, respectively.

Case 1: In this case, $-t_{NoN} < \Delta p < t_N$. By item 1 of Theorem 1, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_f) \in F_0^I$. Note that in this region, $0 < x_N < 1$, and an NE strategy for ISPs should

satisfy the first order optimality conditions. Thus, using (2.8) and (2.9):

$$\begin{aligned}\pi_N(p_N) &= (p_N - c) \frac{t_{NoN} + p_{NoN} - p_N}{t_N + t_{NoN}} \\ \pi_{NoN}(p_{NoN}, \tilde{p}) &= (p_{NoN} - c) \frac{t_N + p_N - p_{NoN}}{t_N + t_{NoN}}\end{aligned}$$

Solving the first order optimality condition yields:

$$\begin{aligned}p_N^{eq} &= c + \frac{1}{3}(2t_{NoN} + t_N) \\ p_{NoN}^{eq} &= c + \frac{1}{3}(2t_N + t_{NoN})\end{aligned}\tag{2.46}$$

which is unique. Note that $p_N^{eq} \geq c$ and $p_{NoN}^{eq} \geq c$. First, note that $-t_{NoN} < \Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq} = \frac{t_N - t_{NoN}}{3} < t_N$.

The necessary condition for this strategy to be an NE is $\pi_{NoN, z=0}(p_{NoN}^{eq}) \geq \pi_{NoN}(p_{NoN}^{eq}, \tilde{p}_t)$ (by Theorem 3). The candidate strategies and this necessary condition is listed in the statement of the theorem.

Case 2: Now, consider $\Delta p \geq t_N$. We consider two cases $\Delta p = t_N$ and $\Delta p > t_N$ in Cases 2-i and 2-ii, respectively.

Case 2-i: Now consider strategies p_{NoN} and p_N such that $\Delta p = t_N$. In this case, using case 2 of Theorem 1, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, 0) \in F_0^U$. Thus, $n_{NoN} = 0$ and $\pi_{NoN}(p_{NoN}, z = 0) = 0$, i.e. the payoff of the non-neutral ISP is zero. Consider $\epsilon > 0$ such that $p'_{NoN} = p_{NoN} - \epsilon > c$. In this case, $p'_{NoN} - p_N < t_N$. Thus, by Theorem 1, $(q_N^{eq}, q_{NoN}^{eq}) \in F_0^I$ or $(q_N^{eq}, q_{NoN}^{eq}) \in F_0^L$. Thus, $n_{NoN} > 0$, and $\pi_{NoN}(p'_{NoN}, z = 0) > 0$. Thus, p'_{NoN} is a profitable deviation for the non-neutral ISP. Therefore, as long as such a deviation exist p_{NoN} and p_N such that $\Delta p = t_N$ cannot be NE.

Case 2-ii: Now, consider $\Delta p > t_N$. Thus, by item 2 of Theorem 1, $n_N^{eq} = 1$. Consider a unilateral deviation by neutral ISP such that $p'_N = p_N^{eq} + \epsilon$ in which $\epsilon > 0$ such that

$p_{N_oN}^{eq} - p'_N > t_N$. Note that the values of q_N^{eq} and $q_{N_oN}^{eq}$ is the same as before, since still $\Delta p' = p_{N_oN}^{eq} - p'_N > t_N$. Thus, again $n_N^{eq} = 1$, and by (2.1), the payoff of neutral ISP is an increasing function of p_N . Thus, p'_N is a profitable unilateral deviation. This contradicts the assumption that p_N^{eq} and $p_{N_oN}^{eq}$ is NE. Thus, the result of the theorem follows.

2.7.4 Proof of Theorem 10

The following lemmas allow us to characterize the NE when $(q_N^{eq}, q_{N_oN}^{eq}) \in F_0$, i.e. $z^{eq} = 0$.

Lemmas 8 and 9 are useful in proving Theorem 10.

Lemma 8. *No p_{N_oN} and p_N such that $\Delta p = p_{N_oN} - p_N \leq -t_{N_oN}$ can be equilibrium strategies.*

Proof. Proof: First, we rule out the existence of an NE when $\Delta p < -t_{N_oN}$, and then when $\Delta p = -t_{N_oN}$.

First, consider p_{N_oN} and p_N such that $\Delta p < -t_{N_oN}$. In this case, $p_{N_oN} < p_N - t_{N_oN}$. Note that the payoff of the non-neutral ISP when $\Delta p \leq -t_{N_oN}$ is $p_{N_oN} - c$ (by (2.1) and $n_{N_oN} = 1$, using case 3 of Theorem 1), and is strictly increasing with respect to p_{N_oN} . Thus, every price $p_{N_oN} < p_N - t_{N_oN}$ yields a strictly lower payoff for the non-neutral ISP in comparison with the payoff of the this ISP when $p_{N_oN} = p_N - t_{N_oN}$. Thus, there exist a profitable deviation for the non-neutral ISP for strategies such that $p_{N_oN} - p_N < -t_{N_oN}$. Therefore, no p_{N_oN} and p_N such that $p_{N_oN} - p_N < -t_{N_oN}$ can be NE strategies.

Now consider strategies p_{N_oN} and p_N such that $\Delta p = -t_{N_oN}$. In this case, using case 3 of Theorem 1, $(q_N^{eq}, q_{N_oN}^{eq}) = (0, \tilde{q}_f) \in F_0^L$. Thus, $n_N = 0$ and $\pi_N(p_N) = 0$, i.e. the payoff of the neutral ISP is zero. Consider $\epsilon > 0$ such that $p'_N = p_N - \epsilon > c$. In this case, $p_{N_oN} - p'_N > -t_{N_oN}$. Thus, by Theorem 1, $(q_N^{eq}, q_{N_oN}^{eq}) \in F_0^I$ or $(q_N^{eq}, q_{N_oN}^{eq}) \in F_0^U$.

Thus, $n_N > 0$, and $\pi_N(p'_N) > 0$. Thus, p'_N is a profitable deviation for the neutral ISP. Therefore, as long as such a deviation exist p_{N_oN} and p_N such that $\Delta p = -t_{N_oN}$ cannot be NE. Now, we prove that such deviation always exist. This complete the proof. Note that this deviation does not exist if and only if $p_N - \epsilon \leq c$ for all $\epsilon > 0$. Therefore, this deviation does not exist if only if $p_N \leq c$. Thus, $p_{N_oN} \leq c - t_{N_oN} < c$, which contradicts the fact that if $z = 0$, $p_{N_oN}^{eq} \geq c$ (as mentioned in the beginning of the section). The lemma follows. \square

Lemma 9. *No p_{N_oN} and p_N such that $\Delta p \geq t_N$ can be equilibrium strategies.*

Proof. Proof: First, we rule out the existence of an NE when $\Delta p > t_N$, and then when $\Delta p = t_N$. Consider p_{N_oN} and p_N such that $\Delta p > t_N$. In this case, $p_N < p_{N_oN} - t_N$. Note that the payoff of the neutral ISP when $\Delta p \geq t_N$ is $p_N - c$ (by (2.1) and $n_N = 1$, using case 2 of Theorem 1), and is strictly increasing with respect to p_N . Thus, every price $p_N < p_{N_oN} - t_N$ yields a strictly lower payoff for the neutral ISP in comparison with the payoff of the this ISP when $p_N = p_{N_oN} - t_N$. Thus, no p_{N_oN} and p_N such that $p_{N_oN} - p_N > t_N$ can be Ne strategies.

Now consider strategies p_{N_oN} and p_N such that $\Delta p = t_N$. In this case, using case 2 of Theorem 1, $(q_N^{eq}, q_{N_oN}^{eq}) = (\tilde{q}_f, 0) \in F_0^U$. Thus, $n_{N_oN} = 0$ and $\pi_{N_oN}(p_{N_oN}, z = 0) = 0$, i.e. the payoff of the non-neutral ISP is zero. Consider $\epsilon > 0$ such that $p'_{N_oN} = p_{N_oN} - \epsilon > c$. In this case, $p'_{N_oN} - p_N < t_N$. Thus, by Theorem 1, $(q_N^{eq}, q_{N_oN}^{eq}) \in F_0^I$ or $(q_N^{eq}, q_{N_oN}^{eq}) \in F_0^L$. Thus, $n_{N_oN} > 0$, and $\pi_{N_oN}(p'_{N_oN}, z = 0) > 0$. Thus, p'_{N_oN} is a profitable deviation for the non-neutral ISP. Therefore, as long as such a deviation exist p_{N_oN} and p_N such that $\Delta p = t_N$ cannot be NE. Now, we prove that such deviation always exist. This

complete the proof. Note that this deviation does not exist if and only if $p_{NoN} - \epsilon \leq c$ for all $\epsilon > 0$. Therefore, this deviation does not exist if only if $p_{NoN} \leq c$. Therefore, $p_N \leq c - t_N < c$, which contradicts the fact that $p_N^{eq} \geq c$ (as mentioned after at the beginning of the section.). The lemma follows. \square

Now, we proceed to prove Theorem 10:

Proof. Proof of Theorem 10: First, in Part 1, we characterize the candidate equilibrium strategies by applying the first order condition on the payoffs. Then, in Part 2, we prove that no unilateral deviation is profitable for ISPs. Thus, the strategies characterized in Part 1 are NE.

Part 1: Note that $z^{eq} = 0$. First note that by Lemmas 8 and 9, no p_N and p_{NoN} such that $\Delta p \leq -t_{NoN}$ or $\Delta p \geq t_N$ can be Nash equilibrium. Thus, we consider $-t_{NoN} < \Delta p < t_N$. Note that in this region, $0 < x_N < 1$, and an NE strategy for ISPs should satisfy the first order optimality conditions. Thus, using (2.8) and (2.9), and item 1 of Theorem 1:

$$\begin{aligned} p_N^{eq} &= c + \frac{1}{3}(2t_{NoN} + t_N) \\ p_{NoN}^{eq} &= c + \frac{1}{3}(2t_N + t_{NoN}) \end{aligned} \tag{2.47}$$

which is unique. Note that $p_N^{eq} \geq c$ and $p_{NoN}^{eq} \geq c$. In order to prove that this is an NE, it is enough to prove that (i) $-t_{NoN} < \Delta p^{eq} = p_{NoN}^{eq} - p_N^{eq} < t_N$, (ii) a deviation of one of the ISPs by which Δp is shifted to the region $\Delta p \leq -t_{NoN}$ or $\Delta p \geq t_N$ is not profitable for that ISP.

The condition (i) can be proved by (2.47). From this equation, $\Delta p^{eq} = \frac{t_N - t_{NoN}}{3}$. Thus, $\Delta p^{eq} > -t_{NoN}$ and $\Delta p^{eq} < t_N$. Therefore, (i) is true for this case.

Part 2: Now, we should prove that condition (ii) holds, i.e. no unilateral deviation is profitable. First, in Case 2-a, we rule out the possibility of a unilateral deviation when $-t_{NoN} < \Delta p < t_N$ for both neutral and non-neutral ISPs. Then, we consider $\Delta p \leq -t_{NoN}$ and $\Delta p \geq t_N$, and in Cases 2-NoN and 2-N, we rule out the possibility of a unilateral deviation in these regions for ISP N and NoN, respectively.

Case 2-a: First, note that by concavity of the payoffs (using (2.8) and (2.9)) as long as $-t_{NoN} < \Delta p < t_N$, i.e. $0 < x_N < 1$, a unilateral deviation by one of the ISPs from p_N^{eq} or p_{NoN}^{eq} decreases this ISP's payoff. Thus, we should consider the deviation by ISPs by which $\Delta p \leq -t_{NoN}$ or $\Delta p \geq t_N$.

Case 2-NoN: Now, consider the deviations by the non-neutral ISP. Fix $p_N = p_N^{eq}$, and consider two cases. In Case 2-NoN-i (respectively, Case 2-NoN-ii), we consider deviation by ISP NoN such that $\Delta p \geq t_N$ (respectively, $\Delta p \leq -t_{NoN}$).

Case 2-NoN-i: Suppose the non-neutral ISP increases her price from p_{NoN}^{eq} to make $\Delta p \geq t_N$. In this case, $n_{NoN} = 0$, and the payoff of the ISP is zero (by (2.1)). Since in the candidate equilibrium strategy this payoff is non-negative, this deviation is not profitable.

Case 2-NoN-ii: Now, consider the case in which the non-neutral ISP decreases her price to make $\Delta p \leq -t_{NoN}$. In this case, $n_{NoN} = 1$ and $\pi_{NoN}(p'_{NoN}, z = 0) = p'_{NoN} - c$ (by (2.1)). Thus, the payoff is a strictly increasing function of p'_{NoN} , and is maximized at $p'_{NoN} = p_N^{eq} - t_{NoN}$. We show that $\pi_{NoN}(p'_{NoN}, z = 0) < \pi_{NoN}(p_{NoN}^{eq}, z = 0)$. Note that $\pi_{NoN}(p'_{NoN}, z = 0) = \frac{1}{3}(t_N - t_{NoN})$. In addition, using (2.47), (2.1), $0 \leq x_N \leq 1$, (2.5), and the fact that with p_N^{eq} and p_{NoN}^{eq} , $q_{NoN}^{eq} - q_N^{eq} = 0$:

$$\pi_{NoN}(p_{NoN}^{eq}, z = 0) = \frac{1}{9} \frac{(2t_N + t_{NoN})^2}{t_{NoN} + t_N}$$

Thus:

$$\begin{aligned}\pi_{NoN}(p'_{NoN}, z = 0) < \pi_{NoN}(p_{NoN}^{eq}, z = 0) &\iff 3(t_N^2 - t_{NoN}^2) < 4t_N^2 + t_{NoN}^2 + 4t_N t_{NoN} \\ &\iff t_N^2 + 4t_{NoN}^2 + 4t_N t_{NoN} > 0\end{aligned}$$

where the last inequality is always true. Thus, this deviation is not profitable for ISP NoN.

These cases prove that no deviation form (2.47) is profitable for ISP NoN.

Case 2-N: Now, consider a deviation by the neutral ISP from (2.47). Similar argument can be done for the neutral ISP. Fix, $p_{NoN} = p_{NoN}^{eq}$, and consider two cases. In Case 2-N-i (respectively, Case 2-N-ii), we consider deviation by ISP N such that $\Delta p \leq -t_{NoN}$ (respectively, $\Delta p \geq t_N$).

Case 2-N-i: Suppose the neutral ISP increases her price from p_N^{eq} to get $\Delta p \leq -t_{NoN}$. In this case, $n_N = 0$, and the payoff of this ISP is zero. Since in the candidate equilibrium strategy the payoff is non-negative, this deviation is not profitable.

Case 2-N-ii: Now, consider the case in which the non-neutral ISP decreases her price such that $\Delta p \geq t_N$. In this case, $n_N = 1$ and $\pi_N(p'_N) = p'_N - c$. Thus, the payoff is a strictly increasing function of p'_N , and is maximized at $p'_N = p_{NoN}^{eq} - t_N$. We show that $\pi_N(p'_N) < \pi_N(p_N^{eq})$. Note that $\pi_N(p'_N) = \frac{1}{3}(t_{NoN} - t_N)$ (by (2.1)). In addition, using (2.47), (2.1), $0 \leq x_N \leq 1$, (2.5), and the fact that with p_N^{eq} and p_{NoN}^{eq} , $q_{NoN}^{eq} - q_N^{eq} = 0$:

$$\pi_N(p_N^{eq}) = \frac{1}{9} \frac{(2t_{NoN} + t_N)^2}{t_{NoN} + t_N}$$

Thus:

$$\begin{aligned}\pi_N(p'_N) < \pi_N(p_N^{eq}) &\iff 3(t_{NoN}^2 - t_N^2) < 4t_{NoN}^2 + t_N^2 + 4t_N t_{NoN} \\ &\iff t_{NoN}^2 + 4t_N^2 + 4t_N t_{NoN} > 0\end{aligned}$$

where the last inequality is always true. Thus, this deviation is not profitable for ISP N. Thus, there is no profitable deviation for ISP N. This completes the proof, and the lemma follows. \square

2.7.5 Proofs of Corollaries Characterizing the Outcome of the Market

Now, using the equilibrium strategies characterized in previous theorems, we characterize the equilibrium outcomes of the market for different parameters in the following corollaries:

Corollary 2. *If $t_N + t_{NoN} \leq \kappa_u \tilde{q}_p$, the equilibrium outcome of the market is:*

- *If $t_N + 2t_{NoN} \leq \tilde{q}_p(\kappa_u + \kappa_{ad})$, then $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$, $n_N^{eq} = 0$, and $n_{NoN}^{eq} = 1$.*
- *If $t_N + 2t_{NoN} > \tilde{q}_p(\kappa_u + \kappa_{ad})$ and conditions of item 2 of Theorem 6 is satisfied, then $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN}^{eq} - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{t_N + 2t_{NoN} - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$, and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN} + \tilde{q}_p(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$.*

Proof. Proof: First, consider Strategy 1 of Theorem 6. Item 1 of Theorem 2 yields that $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$. Thus, $n_N^{eq} = 0$, and $n_{NoN}^{eq} = 1$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$.

Now, consider Strategy 2 of Theorem 6. Note that we constructed this strategy such that Δp satisfies item 3 of Theorem 2. Thus, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN}^{eq} - \frac{\tilde{q}_f}{\tilde{q}_p})$. Using the expression for $\Delta p = p_{NoN}^{eq} - p_N^{eq}$, and (2.5), the expressions for n_N^{eq} and n_{NoN}^{eq} follow. \square

Corollary 3. *If Strategy 1 of Theorem 7 is an NE, it yields $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$, $n_N^{eq} = 0$, and $n_{NoN}^{eq} = 1$. If Strategy 2 of Theorem 7 is an NE, it yields $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN}^{eq} - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{t_N + 2t_{NoN} - \tilde{q}_p(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$, and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN} + \tilde{q}_p(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$. If Strategy 3 of Theorem 7 is an NE, it yields $\tilde{p}^{eq} = \tilde{p}_{t,3} = \kappa_{ad}n_{NoN}^{eq}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{t_N + 2t_{NoN} - (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$, and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN} + (\tilde{q}_p - \tilde{q}_f)(\kappa_u + \kappa_{ad})}{3(t_N + t_{NoN})}$. If Strategy 4 of Theorem 7 is an NE, it yields $\tilde{p}^{eq} = \tilde{p}_{t,3} = \kappa_{ad}n_{NoN}^{eq}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$, $n_N^{eq} = \frac{\kappa_u \tilde{q}_p}{t_N + t_{NoN}}$, and $n_{NoN}^{eq} = \frac{t_N + t_{NoN} - \kappa_u \tilde{q}_p}{t_N + t_{NoN}}$.*

Proof. Proof: First, consider Strategy 1 of Theorem 7. Item 1 of Theorem 2 yields that $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^L$. Thus, $n_N^{eq} = 0$, and $n_{NoN}^{eq} = 1$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,1} = \kappa_{ad}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$.

Now, consider Strategy 2 of Theorem 7. Note that we constructed this strategy such that Δp satisfies items 2-a-ii or 2-b of Theorem 2. Thus, $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p) \in F_1^I$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,2} = \kappa_{ad}(n_{NoN}^{eq} - \frac{\tilde{q}_f}{\tilde{q}_p})$. Using the expression for $\Delta p = p_{NoN}^{eq} - p_N^{eq}$, and (2.5), the expressions for n_N^{eq} and n_{NoN}^{eq} follow.

Consider Strategies 3 and 4 of Theorem 7. In this case, Δp satisfies item 2-a-i of Theorem 2 (by construction of these strategies). Thus, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_p) \in F_1^I$. In addition, by Theorem 4, $\tilde{p}^{eq} = \tilde{p}_{t,3} = \kappa_{ad}n_{NoN}^{eq}(1 - \frac{\tilde{q}_f}{\tilde{q}_p})$. Using the expression of Δp^{eq} for each of the strategies, n_N^{eq} and n_{NoN}^{eq} follow. \square

Corollary 4. *If the strategy of Theorem 9 is an NE, it yields $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_f) \in F_0^L$, $n_N^{eq} = \frac{2t_{NoN} + t_N}{3(t_{NoN} + t_N)}$, and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN}}{3(t_N + t_{NoN})}$. Since $z^{eq} = 0$, \tilde{p}^{eq} is of no importance.*

Proof. Proof: Note that we constructed this strategy such that Δp satisfies item 1 of Theorem 1. Thus, $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_f) \in F_0^I$. Using the expression for $\Delta p = p_{NoN}^{eq} - p_N^{eq}$, and (2.5), the expressions for n_N^{eq} and n_{NoN}^{eq} follow. \square

Corollary 5. *If both ISPs are neutral, then in the equilibrium, ISPs chooses $p_N^{eq} = c + \frac{1}{3}(2t_{NoN} + t_N)$ and $p_{NoN}^{eq} = c + \frac{1}{3}(2t_N + t_{NoN})$ as the Internet access fees. The CP chooses the vector of qualities $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, \tilde{q}_f)$. The fraction of EUs with each ISP is $n_N^{eq} = \frac{2t_{NoN} + t_N}{3(t_{NoN} + t_N)}$ and $n_{NoN}^{eq} = \frac{2t_N + t_{NoN}}{3(t_{NoN} + t_N)}$.*

Results follow from Theorem 1 (note that $-t_{NoN} < \Delta p^{eq} < t_N$), and (2.5).

2.7.6 Continuous Strategy Set for the CP

In this section, we consider $q_N \in [0, \tilde{q}_f]$ and $q_{NoN} \in [0, \tilde{q}_p]$. In this case, the CP pays a side payment of $\tilde{p}q_{NoN}$ if she chooses $q_{NoN} \in (\tilde{q}_f, \tilde{q}_p]$. The rest of the model is the same as before. Note that in this case, the optimum strategies in Stage 4 of the game, in which end-users decide on the ISP, is the same as before. We prove that the optimum decisions made by the CP is similar to the decisions of the CP when she has a discrete set of strategies. This yields that the results of the model would be the same as before when the CP chooses her strategy from a continuous set.

Therefore, we focus on characterizing the optimum strategies of the CP when she chooses her strategy from continuous sets, i.e. $q_N \in [0, \tilde{q}_f]$ and $q_{NoN} \in [0, \tilde{q}_p]$. The following lemma is useful in defining the maximization and to characterize the optimum answers.

Lemma 10. $\pi_{CP}(q_N, \tilde{q}_f, n_{NoN}, z = 0) \geq \pi_{CP}(q_N, \tilde{q}_f, n_{NoN}, z = 1)$.

Remark 5. Note that although we considered z to be a dummy variable, in this lemma and for the purpose of analysis, we treat it as an independent variable.

Proof. Proof: The lemma follows by (2.2), and comparing the expressions in these two cases:

$$\pi_{CP}(q_N, \tilde{q}_f, n_{oN}, z = 0) - \pi_{CP}(q_N, \tilde{q}_f, n_{oN}, z = 1) = \tilde{q}_f n_{oN} \tilde{p} \geq 0$$

Note that we used the fact that from (2.5), since the qualities are the same in both cases, n_N and n_{NoN} are equal for both cases. \square

Lemma 10 provides the ground to formally define the maximization for the CP as:

$$\max_{z, q_N, q_{NoN}} \pi_{CP}(q_N, q_{NoN}, z) = \max_{z, q_N, q_{NoN}} (n_N \kappa_{ad} q_N + n_{NoN} \kappa_{ad} q_{NoN} - z \tilde{p} q_{NoN})$$

s.t:

$$q_N \leq \tilde{q}_f \tag{2.48}$$

$$\text{if } z = 1 \quad \tilde{q}_f < q_{NoN} \leq \tilde{q}_p$$

$$\text{if } z = 0 \quad q_{NoN} \leq \tilde{q}_f$$

Existence of the maximum: Note that the mixed integer programming (2.48) can be written as two convex maximizations, one for $z = 0$ and one for $z = 1$. In addition, note that for the case $z = 1$, the feasible set is not closed (since $\tilde{q}_f < q_{NoN} \leq \tilde{q}_p$). Thus, in this case, we should use the ‘‘supremum’’ instead of ‘‘maximum’’. However, using Lemma 10, we prove that the maximum of (2.48) exists, and therefore the term maximum can be used safely. To prove this, consider the closure of the feasible set when $z = 1$ formed by adding \tilde{q}_f to the set, i.e. \tilde{F}_1 . Since the feasible set associated with $z = 0$ (F_0) and the closure of the feasible set associated to $z = 1$ (\tilde{F}_1) are closed and bounded (compact) and

the objective function is continuous for each $z \in \{0, 1\}$, using Weierstrass Extreme Value Theorem, we can say that a maximum exists in each of these two sets and for the overall optimization (2.48). If the maxima in \tilde{F}_1 is not \tilde{q}_f , then the maxima is in the original feasible set (F_1). Therefore the maximum of (2.48) exists. If not and \tilde{q}_f is the maxima in the set \tilde{F}_1 , then by Lemma 10, the maximum in the set F_0 dominates the maximum of the set \tilde{F}_1 . Thus, the maxima of (2.48) is in F_0 . Therefore, the maximum of (2.48) exists, and we can use the term maximum safely.

Henceforth, the solution $(\tilde{q}_N^*, \tilde{q}_{NoN}^*, z^*)$ of the maximization (2.48) would be called the optimum strategies of the CP. This solution yields x_N^* and subsequently n_N^* and n_{NoN}^* by (2.5). In addition, we denote the feasible set of (2.48) by \mathcal{F} .

Finding the optimum strategies of the CP: To characterize the optimum strategies, we use the partition the feasible set in Table 2.1, and characterize the candidate optimum strategies, i.e. the strategies that yield a higher payoff than the rest of the feasible solutions, in each subset. The overall optimum, which is chosen by the CP, is the one that yields the highest payoff among candidate strategies.

Note that although the maximum of the overall optimization exist, a maximum may not necessarily exist in each of the subsets. We will show in the next set of lemmas that the optimization in each subset of the feasible set can be reduced to a convex maximization over linear constraints. Thus, only the extreme points of the feasible sets may constitute the optimum solution. This means that the CP chooses her strategy among the discrete strategies, $q_N \in \{0, \tilde{q}_f\}$ and $q_{NoN} \in \{0, \tilde{q}_f, \tilde{q}_p\}$.

We now characterize optimum strategies of the CP, by considering each of the sub-feasible sets and characterizing the optimum solutions in each of them. In Lemma 13, we

prove that if $(q_N^*, q_{NoN}^*) \in F^I$, then $q_N^* \in \{0, \tilde{q}_f\}$, $q_{NoN}^* \in \{0, \tilde{q}_f, \tilde{q}_p\}$, and $(q_N^*, q_{NoN}^*) \neq (0, 0)$. In Lemma 15, we prove that if $(q_N^*, q_{NoN}^*) \in F^L$, then $q_{NoN}^* = \tilde{q}_f$, if $q_{NoN}^* \in F_0^L$, and $q_{NoN}^* = \tilde{q}_p$, if $q_{NoN}^* \in F_1^L$. Moreover, $0 \leq q_N^* \leq \frac{1}{\kappa_u}(\kappa_u q_{NoN}^* - t_{NoN} - \Delta p)$, and $\Delta p \leq \kappa_u q_{NoN}^* - t_{NoN}$. In Lemma 16, we prove that if $(q_N^*, q_{NoN}^*) \in F^U$, then $q_N^* = \tilde{q}_f$ and $0 \leq q_{NoN}^* \leq \frac{1}{\kappa_u}(\kappa_u \tilde{q}_f - t_N + \Delta p)$ and $\Delta p \geq t_N - \kappa_u \tilde{q}_f$. In addition, Lemmas 11 and 12 provide some results that are useful in proving Lemmas 13-16.

Lemma 11. *In an optimum solution of (2.48), $n_{NoN}\kappa_{ad} - z\tilde{p} \geq 0$.*

Proof. Proof: Suppose there exists an optimum answer such that $n_{NoN}\kappa_{ad} - z\tilde{p} < 0$. Note that $0 \leq n_N, n_{NoN} \leq 1$ and qualities are non-negative. Thus, in this case, $\pi_{CP} < \kappa_{ad}q_N$. However, choosing $z = 0$ and $q_{NoN} = q_N$, yields a profit equal to $\kappa_{ad}q_N$. This contradicts the solution with $n_{NoN}\kappa_{ad} - z\tilde{p} < 0$ to be optimum. Thus, the Lemma follows. \square

Lemma 12. *In an optimum solution, the CP offers the content quality equal to one of the threshold at least on one ISP, i.e. $q_N^* = \tilde{q}_f$ OR $(q_{NoN}^* = \tilde{q}_p$ XOR $q_{NoN}^* = \tilde{q}_f)$, where XOR means only one the qualities is chosen.*

Proof. Proof: Suppose not. Let the optimum qualities to be $\hat{q}_{NoN} < \tilde{q}_f$ if $z = 0$, or $\tilde{q}_f < \hat{q}_{NoN} < \tilde{q}_p$ if $z = 1$, and $\hat{q}_N < \tilde{q}_f$. The difference between the qualities offered in two platforms is $\Delta q = \hat{q}_{NoN} - \hat{q}_N$. Consider $q'_{NoN} = \hat{q}_{NoN} + \epsilon$ and $q'_N = \hat{q}_N + \epsilon$ in which $\epsilon > 0$ and is such that $q'_{NoN} \leq \tilde{q}_f$ if $z = 0$, or $\tilde{q}_f \leq q'_{NoN} \leq \tilde{q}_p$ if $z = 1$, and $q'_N \leq \tilde{q}_f$. Note that z remains fixed and $q'_{NoN} - q'_N = \hat{q}_{NoN} - \hat{q}_N = \Delta q$. Since Δq is the same for two cases, the number of subscriber to each ISP is the same for both cases by (2.5). Lemma 11, (2.2), and the fact that $n_N, n_{NoN} \geq 0$ yield that $\pi'_{CP} \geq \hat{\pi}_{CP}$, where $\hat{\pi}_{CP}$ (, respectively π'_{CP}) is the payoff of the CP when the vector of qualities is $(\hat{q}_N, \hat{q}_{NoN})$ (, respectively,

(q'_N, q'_{NoN}) .

We now prove if $(\hat{q}_N, \hat{q}_{NoN})$ is the optimum solution, then the inequality is strict, i.e. $\pi'_{CP} > \hat{\pi}_{CP}$. Suppose not, and $\pi'_{CP} = \hat{\pi}_{CP}$. This only happens if $n_{NoN}\kappa_{ad} - z\tilde{p} = 0$ and $n_N = 0$. Note that in this case, $\pi'_{CP} = \hat{\pi}_{CP} = 0$. However, in the previous paragraph, we argued that with $q_N = \tilde{q}_f$ and $q_{NoN} = \tilde{q}_f$, the CP can get a payoff of $\kappa_{ad}\tilde{q}_f > 0$. This contradicts the assumption that $(\hat{q}_N, \hat{q}_{NoN})$ is the optimum solution. Thus, $\pi'_{CP} > \hat{\pi}_{CP}$.

This inequality contradicts the assumption that $(\hat{q}_N, \hat{q}_{NoN})$ is the optimum solution. Thus, the result follows. \square

Clearly, the decision of the CP about the vector of qualities depends on the parameter x_N (2.4), and subsequently on n_N . First, we characterize the candidate strategies of the CP when $0 \leq x_N \leq 1$, i.e. $(q_N^*, q_{NoN}^*) \in F^I$ and therefore $n_N = x_N$. Then, we consider the case of $x_N < 0$ ($n_N = 0$ and $(q_N^*, q_{NoN}^*) \in F^L$) and $x_N > 1$ ($n_N = 1$ and $(q_N^*, q_{NoN}^*) \in F^U$). Finally, we combine both cases to determine the optimum strategies of the CP. In the following lemma, we characterize the candidate optimum qualities in F^I , i.e. the strategies by which $0 \leq x_N \leq 1$.

Lemma 13. *If $(q_N^*, q_{NoN}^*) \in F^I$, i.e. optimum strategies are such that $0 < x_N^* < 1$, then $q_N^* \in \{0, \tilde{q}_f\}$, $q_{NoN}^* \in \{0, \tilde{q}_f, \tilde{q}_p\}$, $(q_N^*, q_{NoN}^*) \neq (0, 0)$.*

Remark 6. *Note that to be in F^I and from (2.5), (q_N^*, q_{NoN}^*) should be such that $\frac{\Delta p - t_N}{\kappa_u} < \Delta q^* = q_{NoN}^* - q_N^* < \frac{\Delta p + t_{NoN}}{\kappa_u}$. In Lemma 12, we have proved that the quality on at least one of the ISPs is equal to a threshold. In this lemma, we prove that the qualities offered on both ISPs are equal to thresholds or one of them is zero.*

Proof. Proof: We would like to characterize the optimum qualities in $F^I = F_0^I \cup F_1^I$, i.e.

optimum strategies for which $0 < x_N < 1$. First note that by Lemma 12, either (a) $q_N^* = c$ and $q_{NoN}^* = c + \Delta q$ where $c = \tilde{q}_f$, or (b) $q_{NoN}^* = c$ and $q_N^* = c - \Delta q$ where $c \in \{\tilde{q}_f, \tilde{q}_p\}$. Note that the feasible sets for each case can be rewritten as a function of Δq . We characterize the candidate solutions for each case:

- Case (a): The feasible set for the case (a) is $\Delta q \in G_0 = [-c, \tilde{q}_f - c]$ (for $z = 0$) and $\Delta q \in G_1 = (\tilde{q}_f - c, \tilde{q}_p - c]$ (for $z = 1$), where $c = \tilde{q}_f$. Let $G = G_0 \cup G_1$. Note that if $0 \leq x_N \leq 1$, then $n_N = x_N$ and $n_{NoN} = 1 - x_N$. Thus, (2.48) can be written as,

$$\begin{aligned} & \max_{z, \Delta q \in G = G_0 \cup G_1} \pi_{CP}(c, c + \Delta q, z) = \\ & \max_{z, \Delta q \in G} (t_{NoN} - \kappa_u \Delta q + p_{NoN} - p_N) \kappa_{ad} c + \\ & + (t_N + \kappa_u \Delta q + p_N - p_{NoN}) \kappa_{ad} (c + \Delta q) - z \tilde{p}(c + \Delta q) \end{aligned} \quad (2.49)$$

Note that although the feasible set G_1 is not closed, we used maximum instead of supremum. We will show that the maximum of (2.49) exists. Thus, the term maximum can be used safely. Note that the objective functions of (2.49) is a strictly convex functions of Δq . Note that henceforth wherever we refer to maximum without further clarification, we refer to the solution of (2.48).

Let \tilde{G}_1 be the closure of G_1 , then $\tilde{G}_1 \setminus G_1 = \{\tilde{q}_f - c\}$. First, we prove that the maximum of (2.49) exists. Note that G_0 and \tilde{G}_1 are closed and bounded (compact) and the objective function of (2.49) is continuous with respect to Δq for each $z \in \{0, 1\}$. Using Weierstrass Extreme Value Theorem, we can say that a maxima for $\pi_{CP}(c, \Delta q + c, z = 0)$ and $\pi_{CP}(c, \Delta q + c, z = 1)$ exists in each of two sets G_0 and \tilde{G}_1 , respectively. Thus, the overall maximum for the objective function of (2.49)

over G_0 and \tilde{G}_1 exists. Now, consider two cases:

1. If the maxima of $\pi_{CP}(c, \Delta q + c, z = 1)$ in \tilde{G}_1 is not $\Delta q = \tilde{q}_f - c$, then the maxima is in the original feasible set (G_1). Therefore the maximum of (2.49) exists (since G_0 is closed).
2. If $\Delta q = \tilde{q}_f - c$ is the maxima of $\pi_{CP}(c, \Delta q + c, z = 1)$ in the set \tilde{G}_1 , then by Lemma 10, the maximum of $\pi_{CP}(c, \Delta q + c, z = 0)$ in the set G_0 greater than or equal to the maximum of $\pi_{CP}(c, \Delta q + c, z = 1)$ in \tilde{G}_1 . Thus, the maxima of (2.49) over G_0 and G_1 exists and is in G_0 .

Now, that we have proved the existence of the maximum for (2.49), we aim to find all the candidate optimum solutions. Note that the set G_0 is closed. Thus, by the strict convexity of the objective function of (2.49), the candidate optimums in G_0 are the extreme points of G_0 . Using the definition of this feasible set, the candidate answers are (i) $q_N^* = \tilde{q}_f$ and $q_{NoN}^* \in \{0, \tilde{q}_f\}$.

Now, consider the feasible set \tilde{G}_1 , and consider two cases:

1. If $\Delta q = \tilde{q}_f - c$ is not the unique maxima of (2.49) in \tilde{G}_1 , then the maxima is in G_1 or G_0 . The candidate answers in the set G_0 are already characterized. In addition, by strict convexity of the objective function, the maxima can only be an extreme point of \tilde{G}_1 . Since $\tilde{q}_f - c$ is not the unique maxima of (2.48) in \tilde{G}_1 , \tilde{q}_p is a maxima of (2.48) in G_1 . Thus, by strong convexity, for all $\Delta q \in G_1$ $\pi_{CP}(c, \tilde{q}_p, z = 1) > \pi_{CP}(c, \Delta q + c, z = 1)$, and the only candidate optimum solution over G_1 is at $\Delta q = \tilde{q}_p - c \in G_1$ which yields (ii) $q_N^* = \tilde{q}_f$ and $q_{NoN}^* = \tilde{q}_p$.

2. If $\tilde{q}_f - c$ is the unique maxima in \tilde{G}_1 , then $\pi_{CP}(c, \tilde{q}_f, z = 1) > \pi_{CP}(c, \Delta q + c, z = 1)$ for $\Delta q \in G_1$. By Lemma 10, $\pi_{CP}(c, \tilde{q}_f, z = 0) \geq \pi_{CP}(c, \tilde{q}_f, z = 1)$. Therefore, the overall maximum of (2.49) is in the set G_0 , and is as characterized previously.

- Case (b): The feasible set for the case (b) is $\Delta q \in \hat{G}_0 = [c - \tilde{q}_f, c]$ where $c = \tilde{q}_f$ (for $z = 0$), and $\Delta q \in \hat{G}_1 = [c - \tilde{q}_f, c]$ where $c = \tilde{q}_p$ (for $z = 1$). For this case, (2.48) can be written as:

$$\begin{aligned} \max_{z, \Delta q \in \hat{G} = \hat{G}_0 \cup \hat{G}_1} \pi_{CP}(c - \Delta q, c, z) = \\ \max_{z, \Delta q \in \hat{G}} \kappa_{ad}(t_{NoN} - \kappa_u \Delta q + p_{NoN} - p_N)(c - \Delta q) + \kappa_{ad}c(t_N + \kappa_u \Delta q + p_N - p_{NoN}) - z\tilde{p}c \end{aligned} \quad (2.50)$$

Note that the feasible set is closed. Thus the term maximum is fine. In addition, the objective functions of (2.50) are strictly convex functions of Δq . Thus, using the strict convexity and the definition of the feasible set, i.e. $c - \tilde{q}_f \leq \Delta q^* \leq c$ where c is \tilde{q}_f and \tilde{q}_p , respectively, we can get the other set of candidate answers, (iii) $q_{NoN}^* = \tilde{q}_f$ and $\tilde{q}_N^* \in \{0, \tilde{q}_f\}$, and (iv) $\tilde{q}_{NoN}^* = \tilde{q}_p$ and $\tilde{q}_N^* \in \{0, \tilde{q}_f\}$.

From, (i), (ii), (iii), and (iv), the result follows. □

The following corollary follows immediately from Lemma 13:

Corollary 6. *The possible candidate optimum strategies by which $0 < x_N^* < 1$, i.e. $(q_N^*, q_{NoN}^*) \in F^I$, are (1) $(0, \tilde{q}_f)$, (2) $(\tilde{q}_f, 0)$, and (3) $(\tilde{q}_f, \tilde{q}_f)$ when $z = 0$, i.e. $(q_N^*, q_{NoN}^*) \in F_0^I$, and (1) $(0, \tilde{q}_p)$ and (2) $(\tilde{q}_f, \tilde{q}_p)$ when $z = 1$, i.e. $(q_N^*, q_{NoN}^*) \in F_1^I$. Note that the*

necessary and sufficient condition for each of these candidate outcomes to be in F^I is

$$\frac{\Delta p - t_N}{\kappa_u} < \Delta q^* < \frac{\Delta p + t_{NoN}}{\kappa_u}.$$

Note that Corollary 6 lists all the candidate answers by which $0 < x_N < 1$. In the next three lemmas, we focus on the candidate answers when $x_N \geq 1$ or $x_N \leq 0$.

Lemma 14. *If $\Delta p > \kappa_u \tilde{q}_f - t_{NoN}$ then $x_N > 0$ for all choices of q_{NoN} and q_N in the feasible set F_0 (that is F_0^L is an empty set). Similarly, If $\Delta p > \kappa_u \tilde{q}_p - t_{NoN}$ then $x_N > 0$ for all choices of q_{NoN} and q_N in the feasible set F_1 (that is F_1^L is an empty set). In addition, if $\Delta p < t_N - \kappa_u \tilde{q}_f$ then $x_N < 1$ for all choices of q_{NoN} and q_N in the overall feasible set \mathcal{F} (that is F^U is an empty set).*

Proof. Proof: First note that from (2.5), $x_N > 0$ is equivalent to:

$$\Delta p > \kappa_u(q_{NoN} - q_N) - t_{NoN} \tag{2.51}$$

Consider $\Delta p > \kappa_u \tilde{q}_f - t_{NoN}$ (respectively, $\Delta p > \kappa_u \tilde{q}_p - t_{NoN}$), if $(q_N, q_{NoN}) \in F_0$ (respectively, $(q_N, q_{NoN}) \in F_1$) then $\Delta p > \kappa_u \tilde{q}_f - t_{NoN} \geq \kappa_u(q_{NoN} - q_N) - t_{NoN}$ (respectively, $\Delta p > \kappa_u \tilde{q}_p - t_{NoN} \geq \kappa_u(q_{NoN} - q_N) - t_{NoN}$) for every choice of $(q_N, q_{NoN}) \in F_0$ (respectively, $(q_N, q_{NoN}) \in F_1$). The inequality $\Delta p > \kappa_u(q_{NoN} - q_N) - t_{NoN}$ yields $x_N > 0$. The first result of the lemma follows.

Now, we prove the second statement. From (2.5), $x_N < 1$ is equivalent to:

$$\Delta p < t_N + \kappa_u(q_{NoN} - q_N) \tag{2.52}$$

Consider $\Delta p < t_N - \kappa_u \tilde{q}_f$. Note that:

$$\Delta p < t_N - \kappa_u \tilde{q}_f \leq t_N + \kappa_u(q_{NoN} - q_N)$$

for every choice of $0 \leq q_N \leq \tilde{q}_f$ and $0 \leq q_{NoN} \leq \tilde{q}_p$ which are all the possible choices in \mathcal{F} . The inequality $\Delta p < t_N + \kappa_u(q_{NoN} - q_N)$ yields that $x_N < 1$. The result follows. \square

The following lemma characterizes all the candidate answers when $x_N^* \leq 0$, and characterize the necessary condition on parameters for this solutions to be feasible.

Lemma 15. *Let $(q_N^*, q_{NoN}^*) \in F^L$. If $(q_N^*, q_{NoN}^*) \in F_0^L$ (respectively, if $(q_N^*, q_{NoN}^*) \in F_1^L$), then $q_{NoN}^* = \tilde{q}_f$ (respectively, $q_{NoN}^* = \tilde{q}_p$). Moreover, for every $x \in [0, \frac{1}{\kappa_u}(\kappa_u \tilde{q}_f - t_{NoN} - \Delta p)]$ (respectively, $x \in [0, \frac{1}{\kappa_u}(\kappa_u \tilde{q}_p - t_{NoN} - \Delta p)]$) and $\Delta p \leq \kappa_u \tilde{q}_f - t_{NoN}$ (respectively, $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$), (x, \tilde{q}_f) (respectively, (x, \tilde{q}_p)) constitutes an optimum solution in F_0^L (respectively, in F_1^L).*

Proof. Proof: From (2.5), $x_N \leq 0$ is equivalent to:

$$\Delta p \leq \kappa_u(q_{NoN} - q_N) - t_{NoN} \quad (2.53)$$

Note that from (2.5), if $x_N \leq 0$ then $n_N = 0$ and $n_{NoN} = 1$. In this case, the payoff of the CP is,

$$\pi_G = \kappa_{ad}q_{NoN} - z\tilde{p}q_{NoN} \quad (2.54)$$

Note that the value of the payoff is independent of q_N as long as $n_N = 0$, and from (2.5) n_N is a function of q_N and q_{NoN} . In addition, note that if there exist a q_{NoN} that satisfies the constraint $\Delta p \leq \kappa_u(q_{NoN} - q_N) - t_{NoN}$ (and therefore $n_N = 0$) then $q'_{NoN} \geq q_{NoN}$ also satisfies this constraint. Therefore for $q'_{NoN} \geq q_{NoN}$, $n_N = 0$ and (2.54) is true. Note that from Lemma 11, (2.54) is an increasing function of q_{NoN} . Thus, if $x_N \leq 0$, then $q_{NoN}^* = \tilde{q}_f$ if $(q_N^*, q_{NoN}^*) \in F_0^L$ or $q_{NoN}^* = \tilde{q}_p$ if $(q_N^*, q_{NoN}^*) \in F_1^L$ (using the feasible sets in Table 2.1 and their definitions).

Using (2.53), $(q_N^*, q_{NoN}^*) \in F_0^L$ (respectively, $(q_N^*, q_{NoN}^*) \in F_1^L$) if and only if,

$$q_N^* \leq \frac{1}{\kappa_u}(\kappa_u \tilde{q}_f - \Delta p - t_{NoN}) \quad \left(\text{respectively, } q_N^* \leq \frac{1}{\kappa_u}(\kappa_u \tilde{q}_p - \Delta p - t_{NoN}) \right) \quad (2.55)$$

Note that every q_N^* that satisfies (2.55) is an optimum answer since when $(q_N^*, q_{NoN}^*) \in F^L$, $n_N = 0$ and q_N^* is of no importance. Also, note that $q_N \geq 0$. Thus, (2.55) is true for at least one q_N^* if $\Delta p \leq \kappa_u \tilde{q}_f - t_{NoN}$ (respectively, $\Delta p \leq \kappa_u \tilde{q}_p - t_{NoN}$). The result follows. \square

The following lemma characterizes all the candidate answers when $x_N \geq 1$, and characterize the necessary condition on parameters for this solutions to be feasible.

Lemma 16. *If $(q_N^*, q_{NoN}^*) \in F^U$, i.e. optimum strategies are such that $x_N^* \geq 1$. Then $q_N^* = \tilde{q}_f$. Moreover, for all $x \in [0, \frac{1}{\kappa_u}(\kappa_u \tilde{q}_f - t_N + \Delta p)]$ and $\Delta p \geq t_N - \kappa_u \tilde{q}_f$, (q_N^*, x) constitutes an optimum solution in F^U .*

Proof. Proof: From (2.5), $x_N \geq 1$ is equivalent to:

$$\Delta p \geq t_N + \kappa_u(q_{NoN} - q_N) \quad (2.56)$$

Now, we prove the first result of the lemma. Note that from (2.5), if $x_N \geq 1$ then $n_N = 1$ and $n_{NoN} = 0$. In this case, the payoff of the CP is,

$$\pi_G = \kappa_{ad} q_N \quad (2.57)$$

Note that the value of the payoff is independent of q_{NoN} as long as $n_N = 1$, and from (2.5), n_N is a function of q_N and q_{NoN} . In addition, note that if there exist a q_N that satisfies $\Delta p \geq t_N + \kappa_u(q_{NoN} - q_N)$, then $q_N' \geq q_N$ also satisfies this constraint. Therefore,

for $q'_N \geq q_N$, $n_N = 1$ and (2.57) is true. Note that (2.57) is an increasing function of q_N .

Thus, $q_N^* = \tilde{q}_f$ (using the feasible sets in Table 2.1 and their definitions).

Using (2.56), $(q_N^*, q_{NoN}^*) \in F^U$ if and only if:

$$q_{NoN}^* \leq \frac{1}{\kappa_u} (\kappa_u \tilde{q}_f - t_N + \Delta p) \quad (2.58)$$

Note that every q_{NoN}^* that satisfies (2.58) is an optimum answer since when $(q_N^*, q_{NoN}^*) \in F^U$, $n_{NoN}^* = 0$ and q_{NoN}^* is of no importance. Also, note that $q_{NoN} \geq 0$. Thus, the condition (2.58) is true for at least one q_{NoN}^* if $\kappa_u \tilde{q}_f - t_N + \Delta p \geq 0$. The result follows. \square

Corollary 7. *If $(q_N^{eq}, q_{NoN}^{eq}) \in F_0^L$, then $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_f)$. If $(q_N^{eq}, q_{NoN}^{eq}) \in F_1^L$, then $(q_N^{eq}, q_{NoN}^{eq}) = (0, \tilde{q}_p)$. If $(q_N^{eq}, q_{NoN}^{eq}) \in F^U$, then $(q_N^{eq}, q_{NoN}^{eq}) = (\tilde{q}_f, 0)$.*

Proof. Proof: Note that when $(q_N^*, q_{NoN}^*) \in F^L$ (, respectively $(q_N^*, q_{NoN}^*) \in F^U$), then the payoff of the CP is independent of q_N^* and q_{NoN}^* . Thus, result of the corollary follows from Tie-Breaking Assumption 1. \square

Theorem 12. *All possible equilibrium strategies are:*

$$\begin{aligned} (0, \tilde{q}_f) \in F_0^I \cup F_0^L, (\tilde{q}_f, 0) \in F_0^I \cup F_0^U, (\tilde{q}_f, \tilde{q}_f) \in F_0^I, \\ (0, \tilde{q}_p) \in F_1^I \cup F_1^L, (\tilde{q}_f, \tilde{q}_p) \in F_1^I \end{aligned} \quad (2.59)$$

Results follow directly from Corollaries 6 and 7.

Note that (2.59) and (2.11) are exactly similar. This implies that the strategies chosen by the CP when she chooses from continuous sets is exactly similar to the strategies when she chooses from the discrete set characterized in our model. This completes our proof.

Chapter 3

Non-Neutrality Framework I - Uncertain Price Competition in an Internet Market¹

In Chapter 2, we assessed the benefits of migrating to non-neutrality for different entities of the market. Given the incentives of different entities of the market to migrate to a non-neutral regime under appropriate conditions, in this chapter, we formulate and analyze the strategic choices of decision makers in a non-neutral Internet market. More specifically, we analyze the interactions, pricings, and the consequences of different non-neutral frameworks. In this chapter, we study the price competition in a duopoly with an arbitrary number of buyers. In this case, ISPs can be considered to be sellers selling/leasing a number of their resources to buyers, i.e. CPs. Each seller can offer multiple

¹Presented in Allerton 2012 [36] and published in IEEE Transaction on Automatic Control [37].

units of resources depending on the availability of the resources which is random and may be different for different sellers. Sellers seek to select a price that will be attractive to the buyers and also fetch adequate profits. The selection will in general depend on the number of units available with the seller and also that of its competitor - the seller may only know the statistics of the latter. We analyze this price competition as a game, and identify a set of necessary and sufficient properties for the Nash Equilibrium (NE). The properties reveal that sellers randomize their price using probability distributions whose support sets are mutually disjoint and in decreasing order of the number of availability. We prove the existence and uniqueness of a symmetric NE in a symmetric market, and explicitly compute the price distribution in the symmetric NE. In addition, we propose a heuristic pricing strategy for sellers in a symmetric oligopoly market which satisfies the necessary and sufficient properties identified for a NE in a symmetric duopoly. Numerical evaluations reveal that our proposed strategy constitutes a good approximation for the NE of the symmetric oligopoly market.

The chapter is organized as follows: We model the price selection problem as a one-shot non-cooperative game in Section 3.1. In Section 3.2, we identify key properties that every NE pricing strategy should satisfy when demand is greater than the maximum possible availability level. In Section 3.3, we prove that any strategy profile that satisfies the properties listed in Section 3.2 constitutes an NE regardless of the relation between the demand and the number of available units. This sufficiency result naturally leads to an algorithm for computing the strategies that satisfy the properties in Section 3.2 (presented in Appendix, Section 3.9.3). In Section 3.4, we present the results for a symmetric market. Results are generalized to the case of random demand in Section 3.5. In Section 3.6,

numerical evaluations are presented. In Section 3.7, we outline the connection between the decision problem we considered and two other different emerging application domains: primary/secondary and microgrid markets. In addition, we discuss about some aspects of the model in connection to the applications considered. Finally, in Section 3.8 we conclude the chapter. Additional details and some of the proofs are presented in the Appendix of the chapter (Section 3.9).

3.1 Market Model and Problem Formulation

3.1.1 Market Model

First, we define some preliminary notations. Then sellers' decision and information are described.

Preliminary notations

We consider a market with two sellers in which each seller owns multiple number of the same commodity and quotes a price per unit. The total demand of the market is d units. For simplicity, the demand is assumed to be deterministic. The generalization to random d is straightforward, and is presented in Section 3.5.

Buyers prefer the seller who quotes a lower price per unit, and they are equally likely to buy a unit from sellers who select equal prices. Thus, if sellers have a, b units to sell respectively and quote prices of x, y per unit, where $x < y$, then they respectively sell $\min\{a, d\}$, $\min\{b, (d - a)^+\}$ units, where z^+ denotes $\max\{z, 0\}$. The cost of each transaction is c . Therefore, a seller earns a profit of $i(x - c)$ when she sells i units with

price x per unit. Because of regulatory restrictions or because of valuations that buyers associate with purchase of each unit, the price selected by each seller should be bounded by some constant $v > c$, i.e. $x \leq v$. The availability of each seller is random:

Terminology 1. We denote m_k as the maximum possible number of available units of seller k . Let $q_{kj} \in [0, 1]$ be the probability that seller k has $j \in \{0, \dots, m_k\}$ units available, and $\vec{q}_k = (q_{k0}, \dots, q_{km_k})$.

The availability of sellers may for example follow binomial distributions $\mathcal{B}(m_1, p_1)$ and $\mathcal{B}(m_2, p_2)$. Specifically, if $p_1 = 0.5$, $p_2 = 0.3$, $m_1 = 3$, and $m_2 = 2$, then $\vec{q}_1 = (\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$ and $\vec{q}_2 = (\frac{49}{100}, \frac{42}{100}, \frac{9}{100})$.

We assume that sellers have zero unit available for sale with positive probability, i.e., $q_{k0} > 0$ for $k \in \{1, 2\}$, and the competition is uncertain, i.e., $q_{\bar{k}i} < 1$ for $i \in \{0, 1, \dots, m_k\}$ for at least one seller k^2 . Note that if competition is deterministic for both sellers, then the problem is trivial.

Terminology 2. For each seller k , let \bar{k} denote the other seller, i.e., if $k = 1$ (respectively, $k = 2$), then $\bar{k} = 2$ (respectively $\bar{k} = 1$).

Sellers' decisions and information

Sellers select their price based on the number of units they offer in the market. Before choosing her price, a seller does not know the number of units of the commodity that her competitor has available for sale and the price per unit her competitor selects. She

²Note that if there exists $i, j \in \{0, \dots, m_k\}$ such that $q_{1i} = 1$ and $q_{2j} = 1$, then both sellers know the exact number of available units with the other seller. Thus the competition is deterministic.

is however aware of the demand and the distributions for the above quantities. A seller may select her price randomly.

Terminology 3. Let $\Phi_{kj}(\cdot)$ be the probability distribution that the seller $k \in \{1, 2\}$ uses for selecting price per unit when she offers j units. Let \tilde{p}_{kj} and \tilde{v}_{kj} be the infimum and the supremum of the support set³ of $\Phi_{kj}(\cdot)$. The strategy profile of seller k is $\Theta_k(\cdot) = (\Phi_{k1}(\cdot), \dots, \Phi_{km_k}(\cdot))$.

An example of probability distributions, support sets, and their infimums and supremums is presented in Figure 3.1. In this figure, the infimums (\tilde{p}_{kj} 's) are illustrated explicitly, and $\tilde{v}_{kj} = \tilde{p}_{k,j-1}$ (For instance, $\tilde{v}_{13} = \tilde{p}_{12}$). Note that, Figure 3.1 presents the distributions which are strictly increasing between the infimum and the supremum of their support sets. However, the probability distributions in general may consist of strictly increasing and flat parts. For example, a probability distribution that is strictly increasing over intervals $[a, b]$ and $[c, d]$, and flat over interval $[b, c]$. Unlike the previous example, the support set of this probability distribution ($[a, d] \cup [c, d]$) is not connected.

3.1.2 Problem Formulation

Clearly, the number of units a seller sells and her profit are random.

Terminology 4. Let $u_k(\Theta_k(\cdot), \Theta_{\bar{k}}(\cdot))$ denotes the expected profit of seller k when she adopts strategy profile $\Theta_k(\cdot)$ and her competitor adopts $\Theta_{\bar{k}}(\cdot)$.

³The support set of a probability distribution is the smallest closed set such that its complement has probability zero under the distribution function. In other words, if there is another set such that its complement has probability zero, it should be a super set of the support set.

Definition 5. A Nash equilibrium (NE)⁴ is a strategy profile such that no seller can improve her expected profit by unilaterally deviating from her strategy. Therefore, $(\Theta_1^*(\cdot), \Theta_2^*(\cdot))$ is a NE if for each seller k :

$$u_k(\Theta_k^*(\cdot), \Theta_{\bar{k}}^*(\cdot)) \geq u_k(\tilde{\Theta}_k(\cdot), \Theta_{\bar{k}}^*(\cdot)), \quad \forall \tilde{\Theta}_k(\cdot).$$

Terminology 5. With slight abuse of notation, we denote $u_{kl}(x)$ as the expected profit that seller k earns, and $B_{kl}(x)$ as the expected number of units that seller k sells, when she offers l units for sale with price x per unit, respectively (the dependence on the competitor's strategy is implicit in this simplified notation).

$$\text{Clearly, } u_{kl}(x) = B_{kl}(x)(x - c). \quad (3.1)$$

Note that $\frac{u_{kl}}{l}$ is the expected utility per unit of availability. Thus, $A_{k,l,j}(x) = \frac{1}{l}u_{kl}(x) - \frac{1}{j}u_{kj}(x)$ is the difference between the utility per availability for availability levels l and j . We will see that $A_{k,l,j}(x)$ plays an important role throughout in the proofs, which motivates the following terminology:

Terminology 6. Let $A_{k,l,j}(x) = \frac{1}{l}u_{kl}(x) - \frac{1}{j}u_{kj}(x) = (x - c)B_{k,l,j}(x)$, where $B_{k,l,j}(x) = \frac{1}{l}B_{kl}(x) - \frac{1}{j}B_{kj}(x)$.

Terminology 7. Let $e_k = (d - m_{\bar{k}})^+$.

Note that for all $x \leq v$,

$$B_{kl}(x) = l \quad l = 1, \dots, e_k \quad (3.2)$$

⁴Clearly, our game is a Bayesian game with the number of available units for sale being the type of a player. For the sake of notational convenience, we use Nash equilibrium as an alternative for Bayesian Nash equilibrium.

as k will sell all she offers in this case given that the total offering is less than the demand. We would later obtain the expression for $B_{kl}(x)$ under the NE strategy profiles when $l > e_k$.

Definition 6. *A price x is said to be the best-response price for seller k when she offers j units if $u_{kj}(x) \geq u_{kj}(a)$ for all $a \in [0, v]$.*

Note that a NE-strategy profile selects with positive probability only amongst the best-response prices. Thus, all the elements of support sets are best responses except potentially those on the boundaries (elements of boundaries may not be best responses) if there is a discontinuity in the utility at those points.

We seek to determine the Nash equilibrium strategy profile of sellers. If $m_1 + m_2 \leq d$, since there is no competition between sellers, both sellers offer their units with the monopoly price, v at the NE. We therefore assume that $m_1 + m_2 > d$.

3.2 Properties of a NE when $d > \max\{m_1, m_2\}$

We investigate the necessary conditions for a strategy to be an NE when $d > \max\{m_1, m_2\}$ (Theorem 13). We will explicitly point out whenever we use the assumption that $d > \max\{m_1, m_2\}$.

Theorem 13. *A NE must satisfy the following properties when $d > \max\{m_1, m_2\}$,*

1. *For each k , there exists a threshold such that seller k offers price v with probability one if she has the availability level less than or equal to this threshold. This threshold, denoted as l_k henceforth, is such that:*

(a) $l_k \in \{e_k, \dots, m_k - 1\}$

(b) $l_1 + l_2 = d - 1$ or $l_1 + l_2 = d$

2. When seller k has $l_k + 1$ ⁵ units, she uses distribution $\Phi_{k, l_k + 1}(\cdot)$

(a) whose support set is $[\tilde{p}_{k, l_k + 1}, v]$,

(b) which is continuous throughout except possibly at v , and

(c) has a jump at v for at most one value of $k \in \{1, 2\}$, and size of such a jump is less than 1

3. When the availability level is $i \in \{l_k + 2, \dots, m_k\}$ ⁶, seller k uses distribution $\Phi_{ki}(\cdot)$

(a) whose support set is $[\tilde{p}_{k, i}, \tilde{p}_{k, i-1}]$,

(b) which is continuous throughout

(c) $\tilde{p}_{1, m_k} = \tilde{p}_{2, m_k}$

4. The utility of seller k when she offers i units is equal for all prices in the support set of $\Phi_{ki}(\cdot)$, except possibly at price v (if v belongs to her support set).

In Appendix 3.9.3, we will present an algorithm to explicitly compute the NE strategies satisfying properties in Theorem 13. Using this algorithm, in Figure 3.1, we plot an NE probability distribution of price when the vector of availability distributions are $\vec{q}_1 = [0.3, 0.2, 0.2, 0.3]$ and $\vec{q}_2 = [0.4, 0.2, 0.2, 0.2]$, the demand, i.e. d , is 3, $v = 10$, and $c = 6$. Note that in this case $l_1 = l_2 = 1$, and $l_1 + l_2 = d - 1$ (part 1 at Theorem 13). This means that both sellers offer price v with probability one if they have one unit of commodity

⁵The same l_k as the one in part 1.

⁶The same l_k as the one in part 1

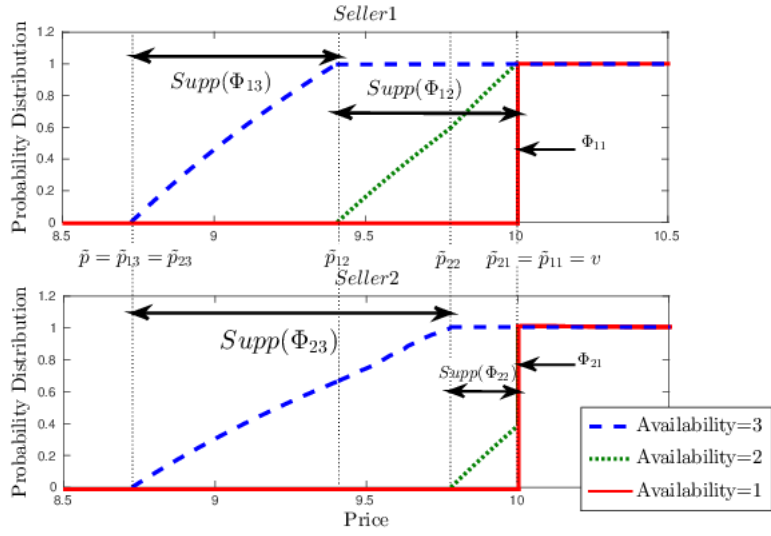


Figure 3.1: An example of an NE pricing strategy, $Supp = Support Set$. Note that Φ_{11} and Φ_{21} have a jump of magnitude one, and Φ_{22} has a jump of size 0.6 at v .

available. When sellers have availability $l_1 + 1 = 2$ and $l_2 + 1 = 2$ units available for sale, they use probability distributions $\Phi_{12}(\cdot)$ and $\Phi_{22}(\cdot)$, respectively, whose support sets are $[\tilde{p}_{12}, v]$ and $[\tilde{p}_{22}, v]$, respectively (part 2a of the Theorem). In addition, these distributions are continuous throughout except possibly at v (part 2b). Furthermore, only the probability distribution $\Phi_{22}(\cdot)$ has a jump at price v and the size of this jump is less than one (part 2c of Theorem 13). When sellers have availability level $l_1 + 2 = l_2 + 2 = 3$, they use probability distributions $\Phi_{13}(\cdot)$ and $\Phi_{23}(\cdot)$, respectively, whose support sets are $[\tilde{p}_{13}, \tilde{p}_{12}]$ and $[\tilde{p}_{23}, \tilde{p}_{22}]$, respectively (part 3a of Theorem 13). In addition, these probability distributions are continuous throughout (part 3b). Note that $\tilde{p}_{13} = \tilde{p}_{23} = \tilde{p}$ (part 3c of the Theorem). More numerical examples are presented in Appendix 3.9.3.

We prove Theorem 13 using the following results which we first state and prove later.

1. The probability distribution of price, $\Phi_{ki}(x)$ for $i \in \{1, \dots, m_k\}$, is continuous for $x < v$ (Section 3.2.2, Property 3).
2. The lower bound of prices are equal for both sellers (Section 3.2.3, Property 4).
3. There is no gap between support sets (Section 3.2.4, Property 5).
4. Support sets are disjoint barring common boundary points, and are in decreasing order of the number of available units for sale (Section 3.2.4, Property 6).
5. The structure of NE at price v : A seller selects v with probability one, if and only if the number of available units with her is less than or equal to a threshold $l_k \in \{0, 1, \dots, m_k - 1\}$, where $l_1 + l_2 = d$ or $l_1 + l_2 = d - 1$ (Section 3.2.6, Property 7).

Note that in Figure 3.1, the distributions are continuous and the lower bound of prices are equal. In addition, every element of the set $[\tilde{p}, v]$ belongs to a support set, i.e. there is no gap between support sets. The support sets of seller one when she offers 3, 2, and 1 unit are $[\tilde{p}, \tilde{p}_{12}]$, $[\tilde{p}_{12}, v]$, and $\{v\}$, respectively. This illustrates the result 4. The result 5 is the same as part 1 in Theorem 13, and is previously connected to Figure 3.1.

Henceforth in this section, we focus on proving the necessary results and properties needed to prove Theorem 13.

3.2.1 Results that we use throughout

Property 1. *For each i and k , $\Phi_{ki}(c) = 0$.*

This result follows directly since prices less than cost c are not chosen by sellers. Property 1 therefore rules out jumps at prices $x \leq c$.

Proof. Note that for each i , $u_{ki}(x) \leq 0$ for $x \leq c$. But, since $B_{ki}(x) \geq iq_{\bar{k}0} > 0$ for all $x \in [0, v]$, $u_{ki}(x) > 0$ for all $x \in (c, v]$. Thus, no price in $[0, c]$ is a best response for a seller. \square

Lemma 17 rules out jumps at prices higher than c :

Lemma 17. *Let the strategy profile of player k be $\Theta_k(\cdot) = (\Phi_{k1}(\cdot), \dots, \Phi_{km_k}(\cdot))$, and $\Phi_{ki}(\cdot)$ have a jump at $x > c$. Then for l such that $l + i > d$, $u_{\bar{k}l}(x - \epsilon') > u_{\bar{k}l}(a)$, $\forall a \in [x, \min\{x + \epsilon, v\}]$, and for all sufficiently small but positive ϵ and ϵ' .*

We provide the intuition behind the result and defer the proof to Appendix 3.9.1. Note that offering a lower price increases the expected number of units sold by a seller, but decreases the revenue per unit sold. Suppose that a seller k offers i units with price x with a positive probability. Let her competitor \bar{k} have l units available where $l + i > d$; \bar{k} can sell a strictly larger number of units in an expected sense by choosing a price in the left neighborhood of x (eg, $x - \epsilon$) rather than x or in its right neighborhood. In addition the difference is bounded away from zero even as the size of the left neighborhood approaches zero. On the other hand, the difference in the revenue per unit approaches zero as the size of the left neighborhood approaches zero. Therefore, prices in the left neighborhood of x constitute better responses for the seller than x or those in its right neighborhood.

The following property fully characterizes the NE when seller k offers $i \in \{1, \dots, e_k\}$ units.

Property 2. $\Phi_{ki}(x)$ selects v with probability 1 and any other prices with probability 0 when $i = 1, \dots, e_k$ for each k .

The proof relies on the fact that if a seller offers less than or equal to e_k units of commodity, she can sell all units regardless of the price she quotes. Therefore v strictly dominates all other prices.

Proof. This statement holds by vacuity if $\max\{m_1, m_2\} \geq d$. Now consider $d > \max\{m_1, m_2\}$. If the seller k offers $i \leq e_k$ units, the total offerings from both sellers are at most d , since the other seller offers at most $m_{\bar{k}}$ units. Thus, the seller k can sell everything it offers with any price x in interval $[0, v]$. Therefore for all $x \in [0, v)$, $u_{ki}(x) = i(x - c) < i(v - c) = u_{ki}(v)$. Thus, no price in $[0, v)$ is a best response. The result follows. \square

3.2.2 Continuity of Price Distribution for Price $x < v$

Utilizing Lemma 17, we can prove that the distribution of price is continuous for prices less than v ,

Property 3. $\Phi_{ki}(x)$ is continuous for $x < v$.

Note that in Fig 3.1, there is no jump in the distributions for prices less than v .

Proof. If $i \leq e_k$, the property follows from Property 2. Now let $i > e_k$. If $x \leq c$, the property follows from Property 1. Now consider $x \in (c, v)$. We use contradiction argument. Suppose $\Phi_{ki}(\cdot)$ has a jump at price $x < v$. Since $i > e_k$, there exists $l \leq m_{\bar{k}}$ such that $l + i > d$. Using lemma 17, we can say that if $\Phi_{ki}(\cdot)$ has a jump at x , for each l such that $l + i > d$, $u_{\bar{k}l}(x - \epsilon') > u_{\bar{k}l}(a)$, where $a \in [x, \min\{x + \epsilon, v\}]$, and for all sufficiently small but positive ϵ and ϵ' . Therefore no price in this interval is a best response for the seller \bar{k} when she offers l units. Therefore $\Phi_{\bar{k}l}(x + \epsilon) = \Phi_{\bar{k}l}(x)$ for all sufficiently small but positive ϵ and all l such that $l > d - i$, i.e. the other seller does not choose any price in

$[x, x + \epsilon)$ whenever she offers l units. Knowing this we can say that $B_{ki}(a) = B_{ki}(x)$ for all $a \in [x, x + \epsilon)$ for some $\epsilon > 0$ such that $x + \epsilon \leq v$. Therefore,

$$u_{ki}(x) = (x - c)B_{ki}(x) < (x + \frac{\epsilon}{2} - c)B_{ki}(x + \frac{\epsilon}{2}) = u_{ki}(x + \frac{\epsilon}{2}) \quad (3.3)$$

Thus, x is not a best response for a seller who offers i units. Hence x is chosen with probability zero, which rules out a jump at x for $\Phi_{ki}(\cdot)$. The property follows. \square

Based on this property, the distribution of price is continuous for $x < v$. We will later show that the price distribution has a jump at v for some availabilities.

Based on the above continuity result, the expression for the expected number of units sold for all $x \in [0, v)$ and $l = e_k + 1, \dots, m_k$ is,

$$B_{kl}(x) = l \sum_{i=0}^{d-l} q_{\bar{k}i} + l \sum_{i=d-l+1}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x))q_{\bar{k}i} + \sum_{i=d-l+1}^{m_{\bar{k}}} \Phi_{\bar{k}i}(x)q_{\bar{k}i}(d - i) \quad (3.4)$$

Note that we assumed $d \geq \max\{m_1, m_2\}$ in (3.4). The first term in the left hand side corresponds to the situation in which the other seller offers at most $d - l$ units. In this case, seller k will sell all l units she offered in the market. The second and the third terms are corresponding to the situation in which the other seller offers more than $d - l$ units with a price higher than and less than x , respectively. If the other seller offers with price higher than x , seller k is able to sell the entire l units. On the other hand, if \bar{k} offers with a price less than x , k will sell $d - l$ units of commodity.

We can now obtain an expression for $u_{kl}(x)$ for $x < v$ from (3.1), (3.2), and (3.4).

3.2.3 Sellers Have Equal Lowerbound of Prices

Note that the example NE distributions presented in Figure 3.1 have equal lower bounds ($\tilde{p} = \tilde{p}_{13} = \tilde{p}_{23}$). We now prove that all NE distributions must satisfy this property:

Property 4. *The minimum of lower end points of support sets are equal for both sellers.*

Mathematically,

$$\tilde{p}_1 = \tilde{p}_2$$

where, $\tilde{p}_k = \min\{\tilde{p}_{ki} : i = 1, \dots, m_k\}$. Furthermore, $\tilde{p}_1 = \tilde{p}_2 < v$ if $d < m_1 + m_2$.

If the lower bound of prices for seller k , i.e. \tilde{p}_k , is lower than that for the other seller, $\tilde{p}_{\bar{k}}$, then k sells equal number of units in an expected sense by choosing \tilde{p}_k as any other price in $(\tilde{p}_k, \tilde{p}_{\bar{k}})$. Using continuity of distributions for prices less than v , we can say that $\tilde{p}_{\bar{k}}$ is a better response than \tilde{p}_k for k , which is a contradiction. The formal proof follows:

Proof. Suppose not. Without loss of generality suppose $\tilde{p}_1 < \tilde{p}_2 \leq v$. Therefore there exists j such that \tilde{p}_1 belongs to the support set of $\Phi_{1j}(\cdot)$. Since player 2 does not offer with any price in the interval $[\tilde{p}_1, \tilde{p}_2)$, $B_{1j}(\tilde{p}_1) = B_{1j}(\tilde{p}_2^-)$ ⁷. Thus $u_{1j}(\tilde{p}_1) < u_{1j}(\tilde{p}_2^-)$ which contradicts the assumption that \tilde{p}_1 is a best response for the first player when she offers i units of commodity. Therefore, the first part of the property follows.

Suppose $\tilde{p}_1 = \tilde{p}_2 = v$. Thus, both sellers choose the price v with probability 1 regardless of the number of units they have available. Consider seller k . Let $l = m_{\bar{k}}$. Since $m_1 + m_2 > d$, Lemma 17 implies that $u_{\bar{k}m_{\bar{k}}}(v - \epsilon) > u_{\bar{k}m_{\bar{k}}}(v)$. This contradicts the assumption that v is the best response for seller k . The result follows. \square

⁷ $f(x^-) = \lim_{y \uparrow x} f(y)$

Terminology 8. Let \tilde{p} denote the minimum of lower end points of prices in the NE, i.e. $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}$.

3.2.4 The union of support sets cover $[\tilde{p}, v]$

We show that there does not exist an interval of prices in $[\tilde{p}, v]$ which is eschewed with probability 1 by both sellers. If such an interval existed, the cumulative distribution functions of both sellers would be flat in it, which we rule out below. Note that in Fig 3.1, the NE distributions are strictly increasing throughout their support sets, and there is no flat region.

Property 5. *There does not exist a, b such that $\tilde{p} \leq a < b \leq v$ and $\Phi_{ki}(b) = \Phi_{ki}(a)$ for all $i \in \{e_k + 1, \dots, m_k\}$ and $k = 1, 2$.*

If such a and b exist for seller k , this means that regardless of the number of available units, k does not select any price in the interval (y, z) where $y \leq a$, $z \geq b$, and y is a best response when k has an availability level l . This implies that for the competitor, \bar{k} , the expected utility is strictly increasing in interval $[y, b]$. Thus \bar{k} does not select any price in the interval $[y, b)$. This again implies that for seller k , when she offers l units, price b yields a strictly higher payoff than y , which is in contradiction with y being a best response for k when offering l units. The formal proof is as follows:

Proof. Let there be a, b , and k such that $\tilde{p} \leq a < b \leq v$ and $\Phi_{ki}(b) = \Phi_{ki}(a)$ for all i . Thus for ζ such that $a < b - \zeta < b \leq v$, $\Phi_{ki}(b - \zeta) = \Phi_{ki}(a)$. Consider y such that,

$$y = \sup\{x | x < a, x \in \text{support set of } \Phi_{kl}(\cdot) \text{ for an } l\}$$

Since support sets are closed, y belongs in the support set of $\Phi_{kl}(\cdot)$ for some l . Thus, y is a best response when the availability of player k is l (note that $y < v$).

In addition, note that $\Phi_{ki}(y) = \Phi_{ki}(b - \zeta)$ for all i . Since $a < b - \zeta < v$, from Property 3 and equation (3.4), the expected number of units sold for the second seller remains constant for prices in $[y, b - \zeta]$, regardless of the number of units she offers, i.e. $B_{\bar{k},\cdot}(y) = B_{\bar{k},\cdot}(b - \zeta)$. Thus, $u_{\bar{k},\cdot}(b - \zeta) > u_{\bar{k},\cdot}(y)$, and player \bar{k} does not offer any price in the interval $[y, b - \zeta)$. Therefore $\Phi_{\bar{k},\cdot}(y) = \Phi_{\bar{k},\cdot}(b - \zeta)$. Since $a < b - \zeta < v$, from Property 3 and equation (3.4), $B_{kl}(y) = B_{kl}(b - \zeta)$. Thus, $u_{kl}(b - \zeta) > u_{kl}(y)$. This is in contradiction with y being a best response when the availability of player k is l . Therefore, there does not exist a, b such that $\tilde{p} \leq a < b \leq v$ and $\Phi_{ki}(b) = \Phi_{ki}(a)$ for all $i \in \{1, \dots, m_k\}$ and $k = 1, 2$. Also, note that for $i \in \{1, \dots, e_k\}$, $\Phi_{ki}(b) = \Phi_{ki}(a)$ for $\tilde{p} \leq a < b \leq v$, since support sets for these distributions only contain v . The result follows. \square

Remark: In all the previous results, we considered $d \geq \max\{m_1, m_2\}$. In the next section, we need to consider that $d > \max\{m_1, m_2\}$.

3.2.5 Support Sets Are Mutually Disjoint and in Decreasing Order of the Number of Availabilities

We start with proving a result, Lemma 18, on $A_{k,l,j}(x)$ (defined in Section 3.1, Terminology 6). Note that we use Lemma 18 in subsequent sections as well. We next prove Property 6 using this result, which leads to the main results of this section: Corollaries 8 and 9.

First, using (3.2) and (3.4),

$$B_{k,l,j}(x) = -\frac{1}{l} \sum_{i=d-l+1}^{d-j} \Phi_{\bar{k}i}(x) q_{\bar{k}i}(i-d+l) + \sum_{i=d-j+1}^{m_{\bar{k}}} \Phi_{\bar{k}i}(x) q_{\bar{k}i}(d-i) \left(\frac{1}{l} - \frac{1}{j}\right) \quad (3.5)$$

Thus, $B_{k,l,j}(\cdot)$ is non increasing and non positive with respect to the price x when $l > j$. Therefore if $l > j$ then $A_{k,l,j}(x)$ is non increasing and non positive with respect to x . Based on the following lemma, $A_{k,l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}$ and $l > j$ if $d > \max\{m_1, m_2\}$.

Lemma 18. *For each seller $k \in \{1, 2\}$ and every l and j , $j < l \leq m_k$, $A_{k,l,j}(x)$ is (strictly) decreasing for $\tilde{p} \leq x < v$ when $d > \max\{m_1, m_2\}$.*

Since $A_{k,l,j}(\cdot) = (x - c)B_{k,l,j}(x)$, knowing that $B_{k,l,j}(x)$ is non-increasing, lemma follows if we prove that $B_{k,l,j}(\cdot)$ is negative. We will prove that $\Phi_{km_k}(x)$, which is included in the summation of $B_{k,l,j}(\cdot)$, is positive for $x > \tilde{p}$ and $k \in \{1, 2\}$. In addition, the coefficient of $\Phi_{km_k}(x)$ is negative since $d > \max\{m_1, m_2\}$. Thus, the result follows.

Proof. It is enough to prove that $B_{k,l,j}(x)$ is non-increasing for $x \geq \tilde{p}$ and negative for $x > \tilde{p}$. This yields that $A_{k,l,j}(x) = (x - c)B_{k,l,j}(x)$ is strictly decreasing with respect to x .

Note that in (3.5), $\Phi_{\bar{k}j}(\cdot)$'s are non-negative and non-increasing since they are probability distributions. In addition, they have negative weights: $-(i - d - l) \leq -1 < 0$, $\frac{1}{l} - \frac{1}{j} < 0$, and since $d > \max\{m_1, m_2\}$, $d - i \geq d - m_{\bar{k}} > 0$. Thus $B_{k,l,j}(x)$ is non increasing and non positive with respect to the price x when $l \geq j$. To prove that $B_{k,l,j}(x)$ is negative for $x > \tilde{p}$, since the distributions in (3.5) have (strictly) negative weights, it is enough to prove that at least one of the $\Phi_{\bar{k}j}(\cdot)$'s is included in the summation of $B_{k,l,j}(\cdot)$ is positive, i.e. not all of them are zero. We will prove that $\Phi_{km_k}(x) > 0$ for $x > \tilde{p}$ and $k \in \{1, 2\}$.

Suppose not and there exists $x > \tilde{p}$ such that $x \leq \tilde{p}_{km_k}$. By Property 5, there exists an $\epsilon > 0$ and an availability level $j \neq \{1, \dots, e_k, m_k\}$ such that $[\tilde{p}_{km_k} - \epsilon, \tilde{p}_{km_k}]$ belongs to the support set of $\Phi_{kj}(\cdot)$ and $\tilde{p}_{kj} < \tilde{p}_{km_k}$. Thus $u_{kj}(\tilde{p}_{km_k}) = u_{kj}(\tilde{p}_{km_k} - \epsilon)$. In addition, $B_{k,m_k,j}(x)$ is the weighted summation of $\Phi_{\bar{k}i}(\cdot)$ for $i \in \{e_{\bar{k}}+1, \dots, m_{\bar{k}}\}$. Property 5 implies that \tilde{p}_{kj} belongs to at least one of the support sets of $\Phi_{\bar{k}i}(\cdot)$ for $i \in \{e_{\bar{k}}+1, \dots, m_{\bar{k}}\}$. The distribution $\Phi_{\bar{k}i}(\cdot)$ is included in the summation of $B_{k,m_k,j}(x)$, and its coefficient is negative. Thus, $A_{k,m_k,j}(x)$ is strictly decreasing with respect to x for $x > \tilde{p}_{kj}$. Thus $A_{k,m_k,j}(\tilde{p}_{km_k} - \epsilon) > A_{k,m_k,j}(\tilde{p}_{km_k})$. Using $u_{kj}(\tilde{p}_{km_k}) = u_{kj}(\tilde{p}_{km_k} - \epsilon)$, we can conclude that $u_{km_k}(\tilde{p}_{km_k}) = u_{km_k.max} < u_{km_k}(\tilde{p}_{km_k} - \epsilon)$. This contradicts with \tilde{p}_{k,m_k} belonging to the support set of $\Phi_{km_k}(\cdot)$. The result follows. \square

Note that in the previous lemma, we used $d > \max\{m_1, m_2\}$ to prove that $A_{k,l,j}(x)$ is decreasing for $\tilde{p} \leq x < v$. The following properties characterize the NE for price less than v .

Property 6. For $k \in \{1, 2\}$, the support set of $\Phi_{kl}(\cdot)$ is a subset of $[\tilde{p}, \tilde{p}_{kj}] \cup [v]$ for all integers $j \in [1, l]$.

For example, in Figure 3.1, the support set for seller 1 and availability 3 is $[\tilde{p}, \tilde{p}_{12}]$, which is a subset of the mentioned set.

Proof. First note that for $j \in \{1, \dots, e_k\}$ property follows, since $\tilde{p}_{kj} = v$ by Property 2. Now consider $j > e_k$. Consider support sets of $\Phi_{kj}(\cdot)$, $\Phi_{kl}(\cdot)$, and $j < l$. We will show that $u_{kl}(a) < u_{kl}(\tilde{p}_{kj})$ for all $a \in (\tilde{p}_{kj}, v)$. Thus, no $a \in (\tilde{p}_{kj}, v)$ is a best response for the seller k with availability of l units. Therefore, the support set of $\Phi_{kl}(\cdot)$ is a subset of $[c, \tilde{p}_{kj}] \cup [v]$.

We now complete the proof, by showing that $u_{kl}(a) < u_{kl}(\tilde{p}_{kj})$ for all $a \in (\tilde{p}_{kj}, v)$:

$$\frac{1}{l}u_{kl}(a) - \frac{1}{j}u_{kj}(a) = A_{k,l,j}(a)$$

Since $l > j$ and $\tilde{p} \leq \tilde{p}_{kj} < a < v$, by Lemma 18, $A_{k,l,j}(a)$ is decreasing function of a for $a \in [\tilde{p}_{kj}, v)$. Thus, $A_{k,l,j}(a) < A_{k,l,j}(\tilde{p}_{kj})$ for $a \in (\tilde{p}_{kj}, v)$. On the other hand $u_{kj}(a) \leq u_{kj}(\tilde{p}_{kj})$ for all $a > \tilde{p}_{kj}$, since \tilde{p}_{kj} is a best response of a seller with availability j , therefore $u_{kl}(\tilde{p}_{kj}) > u_{kl}(a)$. \square

Note that, in this stage, since $\Phi_{kl}(\cdot)$ can have a jump at v , we cannot rule out v as a member of the support set of $\Phi_{kl}(\cdot)$.

Corollary 8. *The support sets of $\Phi_{kl}(\cdot)$ and $\Phi_{kj}(\cdot)$ overlap at most at one point in $[\tilde{p}, v)$.*

For instance, note that in Figure 3.1, the support sets of Φ_{13} and Φ_{12} overlap only at \tilde{p}_{12} , the support sets of Φ_{12} and Φ_{11} overlap only at v , and there is no overlap between support sets of Φ_{13} and Φ_{11} .

Proof. Suppose two points x_1 and x_2 , where $x_1 < x_2 < v$, and both points belong to the intersection of the support sets of $\Phi_{kj}(\cdot)$ and $\Phi_{kl}(\cdot)$. Without loss of generality, consider $j < l$. The price $x_2 > \tilde{p}_j$ belongs to the support set of $\Phi_{kl}(\cdot)$, which is a contradiction with Property 6. \square

Corollary 9. *For prices less than v support sets are contiguous (Property 5), disjoint (except possibly at one point) (Corollary 8), and in decreasing order of the number of available units for sale (Property 6). Thus, there exists an increasing sequence $a_{km_k}, a_{k,m_k-1}, \dots$ of positive real numbers in $(c, v]$ such that the seller k will randomize her price in the interval $[a_{ki}, a_{k,i+1}]$ and possibly $\{v\}$ when she has i units of commodity available for sale.*

For instance, note that in Figure 3.1, the support sets of seller one are in decreasing order of the number of available units for sale, and the aforementioned increasing sequence is $\tilde{p}, \tilde{p}_{12}$, and v .

3.2.6 The Structure of Nash Equilibrium at Price v

We will investigate the possibility of having a jump at v . First, we prove Lemma 19 which complements previous results by identifying the nature of overlap between $\Phi_{kj}(\cdot)$ and $\Phi_{\bar{k}l}(\cdot)$ for $j \in \{1, \dots, m_k\}$ and $l \in \{1, \dots, m_{\bar{k}}\}$ for prices less than v . Using this lemma, we prove Property 7 which is the main result of this section.

Lemma 19. *For every price $\tilde{p} \leq x < v$, x should belong to the support sets $\Phi_{kl}(\cdot)$ and $\Phi_{\bar{k}j}(\cdot)$ such that $l + j > d$.*

A contradiction argument is used to prove the lemma. Assume that there exist x , l , and j such that x belongs to say $\Phi_{kl}(\cdot)$ and $\Phi_{\bar{k}j}(\cdot)$, and $l + j \leq d$. We show that in this case, the expected number of units sold at x and $x + \epsilon$ are equal for seller k when offering l units, i.e. $B_{kl}(x) = B_{kl}(x + \epsilon)$, and subsequently that $u_{kl}(x + \epsilon) > u_{kl}(x)$. Thus x is not a best response for seller k who offers l units, which is a contradiction.

Proof. Suppose not. There exist x , l , and j such that x belongs to say $\Phi_{kl}(\cdot)$ and $\Phi_{\bar{k}j}(\cdot)$, and $l + j \leq d$. We show that there exist $\tilde{j}, \epsilon > 0$ such that $x + \epsilon$ belongs in the support set of $\Phi_{\bar{k}\tilde{j}}(\cdot)$, and subsequently that $u_{kl}(x + \epsilon) > u_{kl}(x)$. Thus x is not a best response for seller k who offers l units which is a contradiction. Consider two cases:

- $x = \tilde{v}_{\bar{k}j}$. Using Corollary 9, x and $x + \epsilon$ belongs to the support set of $\Phi_{\bar{k},j-1}(\cdot)$ when ϵ is small enough. Take $\tilde{j} = j - 1$.

- $x < \tilde{v}_{\bar{k}j}$. If ϵ is small enough, x and $x + \epsilon$ belongs to the support set of $\Phi_{\bar{k}j}(\cdot)$. Take $\tilde{j} = j$.

Note that since $l + j \leq d$, $l + \tilde{j} \leq d$. We are going to argue that the expected number of units sold at x and $x + \epsilon$ are equal for seller k , i.e. $B_{kl}(x) = B_{kl}(x + \epsilon)$. To show this, we condition on the number of available units with the seller \bar{k} . If \bar{k} has more than \tilde{j} number of available units, say f , then she will offer with price less than x with probability one. Thus $\tilde{B}_{kl}(x|f) = \tilde{B}_{kl}(x + \epsilon|f) = d - f$ in which $\tilde{B}(\cdot|\cdot)$ is the conditional expected number of units sold. If \bar{k} offers less than \tilde{j} number of units, she will offer with price higher than $x + \epsilon$ with probability one. Thus $\tilde{B}_{kl}(x|f) = \tilde{B}_{kl}(x + \epsilon|f) = l$. If \bar{k} offers \tilde{j} units, since $l + \tilde{j} \leq d$, $\tilde{B}_{kl}(x|\tilde{j}) = \tilde{B}_{kl}(x + \epsilon|\tilde{j}) = l$. Therefore the expected number of units sold at x and $x + \epsilon$ are equal for seller k , and $u_{kl}(x + \epsilon) > u_{kl}(x)$. The proof is complete. □

Finally, the following property characterizes the behavior of NE at v .

Property 7. *For each k , there exists a threshold such that seller k offers price v with probability one if she has the availability level less than or equal to this threshold. We denote this threshold with l_k . This threshold is such that:*

- $l_k \in \{e_k, \dots, m_k - 1\}$
- $l_1 + l_2 = d - 1$ or $l_1 + l_2 = d$

The price distribution $\Phi_{kj}(\cdot)$ does not have a jump at v if $j > l_k + 1$, at most one of the distributions $\Phi_{1,l_1+1}(\cdot)$ and $\Phi_{2,l_2+1}(\cdot)$ can have a jump at v , and size of such a jump is less than 1.

Note that in Fig 3.1, $l_1 = l_2 = 1$, and $l_1 + l_2 = d - 1$. In addition, both sellers have a jump of magnitude one at price v when they have one unit available, only seller two has a jump at price v when the availability level is two, and there is no jump in the distribution functions when sellers have three units available.

Proof. Take z_k such that k offers price v with probability one if she has $i \in \{1, \dots, z_k\}$ units. Property 2 shows that $z_k \geq e_k$. We will prove that the z_k should be less than m_k . Note that if seller k has m_k units of availability and she offers her units with a single price v , then $\tilde{p}_k = v$. By Properties 4 and 6, the other seller, \bar{k} , offers her units with a single price v regardless of the number of available units. This is a contradiction. The reason is because of Lemma 17. Since $m_1 + m_2 > d$, if $\Phi_{1,m_1}(\cdot)$ has a jump at v , then $u_{2m_2}(v - \epsilon) > u_{2m_2l}(v)$, for all sufficiently small but positive ϵ . Thus v is not a best response for the second player when she offers m_2 units, which is a contradiction. Thus $z_k < m_k$. Therefore $z_k \in \{e_k, \dots, m_k - 1\}$.

First, suppose $z_1 + z_2 \geq d + 1$. By lemma 17, v is not a best response for the player k when she offers z_k units, which is a contradiction. Therefore $z_1 + z_2 \leq d$. Next, we will prove that either $z_1 + z_2 = d - 1$ or $z_1 + z_2 = d$. Note that by the definition of z_k , seller k with availability $z_k + 1$ cannot choose the price v with probability 1. Thus using this fact and Corollary 9, the price $x = v - \epsilon$ for $\epsilon > 0$ small enough is in the support sets of $\Phi_{1,z_1+1}(\cdot)$ and $\Phi_{2,z_2+1}(\cdot)$. Thus, by Lemma 19, $z_1 + z_2 \geq d - 1$. Knowing that $z_1 + z_2 \leq d$. Take $l_k = z_k$, and the first part of the property follows.

Now we should consider the possibility of having a jump at v for $\Phi_{kj}(\cdot)$ for $j \geq l_k + 1$. We will prove that the price distribution does not have a jump at v when seller k offers

more than $l_k + 1$ units. Suppose $\Phi_{kj}(\cdot)$ has a jump for $j > l_k + 1$. Note that $j + l_{\bar{k}} > l_k + l_{\bar{k}} + 1 \geq d$. By Lemma 17, v is not a best response for the seller \bar{k} under availability $l_{\bar{k}}$ which contradicts the definition of $l_{\bar{k}}$.

Now consider $l_k + 1$. By definition of l_k such a jump must have a size less than 1, should it exist. We will prove that at most one of the distributions $\Phi_{1,l_1+1}(\cdot)$ and $\Phi_{k,l_2+1}(\cdot)$ can have a jump at v . Suppose not and both have a jump at v . By Lemma 17, since $(l_1 + 1) + (l_2 + 1) > d$, v is not a best response for the player k when she offers $l_k + 1$ units. This is a contradiction. The result follows. □

Revisiting Equation (3.4) implies that utility, $u_{ki}(\cdot)$, is continuous not only in interval $[c, v)$, but also at price v , if $i \leq d - l_{\bar{k}} - 1$. The reason is that for $i \leq d - l_{\bar{k}} - 1$, equation (3.4) depends only on $\Phi_{\bar{k}j}(\cdot)$ where $j \geq l_{\bar{k}} + 2$, which is continuous at price v based on Property 7. If $\Phi_{\bar{k}l_{\bar{k}}+1}(\cdot)$ is continuous at v then $u_{ki}(\cdot)$ is continuous in $[c, v]$ for $i \leq d - l_{\bar{k}}$.

3.2.7 Proof of Theorem 13

Proof. Part 1 of Theorem 13 follows from Property 7. We now prove part 2. The support set of $\Phi_{k,l_k+1}(\cdot)$ includes at least one $x < v$ from Property 7. Thus, Properties 6 and 5 imply part 2a of this part. Parts 2b and 2c follow from Properties 3 and 7, respectively.

We now prove part 3. We start with 3a. Consider $i > l_{k+1}$. From Property 7, $\Phi_{k,i}(\cdot)$ does not have a jump at v . From part 2a and Property 6, v is not in the supports set of $\Phi_{k,i}(\cdot)$ and $\tilde{v}_{k,i} \leq \tilde{p}_{k,i-1}$. The result can now be proved by induction starting with $i = l_{k+2}$ using the fact that there is no gap between the support sets (Property 5). Since v is not

in the support set of $\Phi_{k,i}(\cdot)$, part 3b follows from Property 3. Part 3c follows from part 3a and Property 4.

Part 4 follows from the fact that every price in the support set of a NE, except those on the boundaries, should be a best response for a seller. Thus they yield the same utility value. The result follows for the boundary points of the support sets other than v from Property 3. □

3.3 Arbitrary Demand

In this section, first we present the sufficiency theorem for $d \geq \max\{m_1, m_2\}$ (Theorem 14). Theorem 14 establishes that a strategy profile which satisfies the mentioned properties in Theorem 13 constitutes an NE when $d \geq \max\{m_1, m_2\}$. Note that unlike Theorem 13, the sufficiency theorem holds even when $d = \max\{m_1, m_2\}$. Thus, the properties in Theorem 13 are both necessary and sufficient conditions for an NE when $d > \max\{m_1, m_2\}$, and only sufficient conditions when $d = \max\{m_1, m_2\}$. The sufficiency theorem naturally leads to an algorithm for computing NE strategy profiles that satisfy the properties in Theorem 13 (Appendix 3.9.3). Any strategy profile obtained by the algorithm constitutes an NE by Theorem 14. In Section 3.3.2, we argue that the computation of the NE strategies for $d < \max\{m_1, m_2\}$ can be reduced to $d = \max\{m_1, m_2\}$. This completes the entire framework.

3.3.1 The Sufficiency Theorem when $d \geq \max\{m_1, m_2\}$

Theorem 14. *Consider a strategy profile that satisfies the properties enumerated in Theorem 13. This strategy profile is a Nash equilibrium when $d \geq \max\{m_1, m_2\}$.*

The proof is presented in Appendix 3.9.2. In the proof, we use the fact that $A_{k,l,j}(\cdot)$ is non increasing and non positive when $d \geq \max\{m_1, m_2\}$.

3.3.2 Allowing $d \leq \max\{m_1, m_2\}$

Note that all results before equation (3.4) also hold when $d \leq \max\{m_1, m_2\}$. Thus (3.4) can be restated by replacing $e_k = d - m_{\bar{k}}$ with $e_k = (d - m_{\bar{k}})^+$:

$$B_{kj}(x) = j \sum_{i=0}^{(d-j)^+} q_{\bar{k}i} + \min\{j, d\} \sum_{i=(d-j)^++1}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x))q_{\bar{k}i} + \sum_{i=(d-j)^++1}^{m_{\bar{k}}} \Phi_{\bar{k}i}(x)q_{\bar{k}i}(d-i)^+ \quad (3.6)$$

Note that if $m_k > d$, the utilities of all number of availability levels $j \geq d$ for player k are equal:

$$u_{kd} = u_{k,d+1} = \dots = u_{km_k} = d \sum_{i=1}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x))q_{\bar{k}i} \quad (3.7)$$

Let $\tilde{q}_{\bar{k}d} = \sum_{i=d}^{m_{\bar{k}}} q_{\bar{k}i}$ and $\tilde{\Phi}_{\bar{k}d}(x) = \sum_{i=d}^{m_{\bar{k}}} \frac{q_{\bar{k}i}}{\tilde{q}_{\bar{k}d}} \Phi_{\bar{k}i}(x)$. Thus, $\tilde{q}_{\bar{k}d}$ is the probability that the availability level of seller \bar{k} is greater than or equal to d and $\tilde{\Phi}_{\bar{k}d}(x)$ is the average probability distribution associated with selecting the price if seller \bar{k} availability is d or higher. Now, the term $\sum_{i=d}^{m_{\bar{k}}} (1 - \Phi_{\bar{k}i}(x))q_{\bar{k}i}$ in the expression for $u_{ki}(\cdot)$ in (3.6) can be replaced by $\tilde{q}_{\bar{k}d}(1 - \tilde{\Phi}_{\bar{k}d}(x))$. Thus the problem is reduced to finding the structure when $d = \max\{m_1, m_2\}$. It was proved previously that a strategy profile that satisfies properties in Theorem 13 is a NE when $d = \max\{m_1, m_2\}$. Thus, a set of equilibria of the game when $d < \max\{m_1, m_2\}$ can be found by defining $\tilde{\Phi}_{kd}(\cdot)$ and using the properties in Theorem 13. The distribution of each individual $\Phi_{kj}(\cdot)$ for $j \geq d$ cannot be determined uniquely and is not of significant interest.

3.4 The Symmetric Setting

We now consider the symmetric setting in which $\vec{q}_1 = \vec{q}_2 = \vec{q}$ (clearly $m_1 = m_2 = m$). In this case, it is natural to consider a symmetric NE, defined as follows,

Definition 7. *An NE $(\Theta_1(\cdot), \Theta_2(\cdot))$ is said to be symmetric if $\Theta_1(\cdot) = \Theta_2(\cdot)$.*

Thus, when considering symmetric NE, in terminologies like $\Phi(\cdot), \Theta(\cdot), u(\cdot), \tilde{p}$, we drop the index that represents the seller and only retain the index that represents the number of units available for sale. As a special case of the general setting (Sections 3.2 and 3.3), every symmetric NE should satisfy the properties in Theorem 13 when $d > m$, and every strategy profile that satisfies these properties is a NE when $d \geq m$ (Theorem 14). In Section 3.4.1, we extend Theorem 13 to the case of $d = m$. In Section 3.4.2, we will present an algorithm to find symmetric Nash equilibria of the game when $d \geq m$. Using the results in Section 3.3.2, the algorithm can be extended to $d < m$.

Note that the algorithm reveals that there is only one symmetric strategy profile that satisfies the properties. It follows from Theorems 13 and 14 that a symmetric NE strategy profile exists uniquely when $d \geq m$. In contrast, in Appendix 3.9.3, we show that there may exist multiple Nash equilibria for an asymmetric market.

3.4.1 Properties of a Symmetric Nash Equilibrium

Theorem 15. *Let $d = m$. A symmetric NE in a symmetric market satisfies the properties in Theorem 13.*

The proof is technical and is relegated to the Appendix. It implies that properties in Theorem 13 are necessary and sufficient conditions for a symmetric NE when $d \geq m$.

Since NE is symmetric, $l^* = l_1 = l_2$. Thus, $l^* = \frac{d-1}{2}$ or $l^* = \frac{d}{2}$, whichever is an integer. Since at most one seller can have a jump at v at $l^* + 1$, in a symmetric NE, none of them do. Thus, the properties in Theorem 13 transform to the following in the symmetric context.

1. Sellers offer price v with probability 1, if they have $i \in \{1, \dots, l^*\}$ available units.
2. There exists an increasing sequence $a_m, a_{m-1}, \dots, a_{l^*+1}, a_{l^*}$ of positive real numbers in $(c, v]$ with $a_{l^*} = v$ such that each seller randomizes her price in the interval $[a_i, a_{i-1}]$ when she has i units of commodity available for sale for $i \in \{l^* + 1, \dots, m\}$.

Thus,

- (a) Support sets are contiguous.
 - (b) Support sets are disjoint (except possibly at one point).
 - (c) Support sets are in decreasing order of the number of available units for sale.
3. Price distribution is continuous for $i \geq l^*$.
 4. The utility of a seller when she offers i units is equal for all prices in the support set of $\Phi_i(\cdot)$, except possibly at price v (if it belongs to her support set).

3.4.2 Algorithm for computing a symmetric NE for the symmetric setting

We will now identify an algorithm to compute strategies that exhibit the properties in the previous subsection. The algorithm reveals that there is only one symmetric strategy profile that satisfies the same. It follows from Theorem 13 and 14 that a symmetric NE

strategy profile exists uniquely when $d \geq m$. Note that the algorithm is developed for $d \geq m$. However, with the method presented in Section 3.3.2, the algorithm can be used to find the equilibrium for $d \leq m$.

Since $\Phi_j(\cdot)$ is completely characterized for $j < \frac{d+1}{2}$, we should characterize $\Phi_j(\cdot)$ for $j \geq \frac{d+1}{2}$, and outline a framework for computing the same. We proceed in an increasing order of j starting with $j = \lceil \frac{d+1}{2} \rceil$. Then moving to $j = \lceil \frac{d+1}{2} \rceil + 1$, etc.

Now, let $\lceil \frac{d+1}{2} \rceil$. Note that $\tilde{v}_{\lceil \frac{d+1}{2} \rceil} = v$ and $\tilde{p}_k = v$ for $k < \lceil \frac{d+1}{2} \rceil$, and $\tilde{v}_k \leq \tilde{p}_{\lceil \frac{d+1}{2} \rceil}$ for $k > \lceil \frac{d+1}{2} \rceil$ (Properties 1 and 2c). Since support sets are ordered (Property 2c) and disjoint (Property 2b), the expression for $u_{\lceil \frac{d+1}{2} \rceil}(x)$ for $x \in [\tilde{p}_{\lceil \frac{d+1}{2} \rceil}, v)$ only depends on $\Phi_{\lceil \frac{d+1}{2} \rceil}(x)$ (Equation (3.4)). In particular, $u_{\lceil \frac{d+1}{2} \rceil}(v^-)$ can be obtained using the fact that $\Phi_{\lceil \frac{d+1}{2} \rceil}(v^-) = 1$ which follows from the continuity of $\Phi_{\lceil \frac{d+1}{2} \rceil}(\cdot)$ (Properties 3). Next, $u_{\lceil \frac{d+1}{2} \rceil}(x) = u_{\lceil \frac{d+1}{2} \rceil}(v^-)$ for every $x \in [\tilde{p}_{\lceil \frac{d+1}{2} \rceil}, v)$. Thus having $u_{\lceil \frac{d+1}{2} \rceil}(v^-)$, and using continuity, we can find a unique expression for $\Phi_{\lceil \frac{d+1}{2} \rceil}(x)$. Using $\Phi_{\lceil \frac{d+1}{2} \rceil}(\tilde{p}_{\lceil \frac{d+1}{2} \rceil}) = 0$, $\tilde{p}_{\lceil \frac{d+1}{2} \rceil}$ can be found uniquely.

We now compute the structure of $\Phi_i(\cdot)$, $\forall i > \lceil \frac{d+1}{2} \rceil$ using $\Phi_{i-1}(\cdot), \Phi_{i-2}(\cdot), \dots, \Phi_{\lceil \frac{d+1}{2} \rceil}(\cdot)$ that are computed before $\Phi_i(\cdot)$. We utilize the facts that,

1. $\Phi_j(x) = 1$ for $j > i, x \in [\tilde{p}_i, \tilde{v}_i]$
2. $\Phi_j(x) = 0$ for $j < i, x \in [\tilde{p}_i, \tilde{v}_i]$
3. $\tilde{v}_i < v$

Thus, from (3.4),

$$u_i(\tilde{v}_i) = (\tilde{v}_i - c) \left(i \sum_{g=0}^{i-1} q_g + \sum_i^m q_g (d - g) \right) \quad (3.8)$$

Since $\tilde{v}_i = \tilde{p}_{i-1}$, and \tilde{p}_{i-1} is computed during the computation of $\Phi_{i-1}(\cdot)$, which precedes that of $\Phi_i(\cdot)$, (3.8) fully specifies $u_i(\tilde{v}_i)$. Furthermore, for $x \in [\tilde{p}_i, \tilde{v}_i]$ the only unknown variable in the expression of $u_i(x)$ is $\Phi_i(x)$. Since $u_i(x) = u_i(v_i)$ for $x \in [\tilde{p}_i, \tilde{v}_i]$,

$$\Phi_i(x) = \frac{i \sum_{g=0}^{i-1} q_g + i q_i + \sum_{g=i+1}^m q_g (d-g) - \frac{u_i(\tilde{v}_i)}{x-c}}{q_i(2i-d)} \quad (3.9)$$

From (3.9), $\Phi_i(\tilde{v}_i) = 1$. Thus, for $x \geq \tilde{v}_i$, $\Phi_i(x) = 1$. Now, \tilde{p}_i can be uniquely identified using the fact that $\Phi_i(\tilde{p}_i) = 0$,

$$\tilde{p}_i = c + \frac{(\tilde{v}_i - c) \left(i \sum_{g=0}^{i-1} q_g + \sum_{g=i}^m q_g (d-g) \right)}{i \sum_{g=0}^{i-1} q_g + i q_i + \sum_{g=i+1}^m q_g (d-g)} \quad (3.10)$$

Therefore $\Phi_i(x) = 0$ for $x \leq \tilde{p}_i$. Clearly, $\Phi_i(\cdot)$ has been characterized uniquely. Note that the denominator of (3.10) is positive since $d \geq m$ and $q_m < 1$ (uncertainty assumption in Section 3.1). In addition, $\tilde{p}_i > c$. This is because of the fact that the second term of RHS of (3.10) is positive.

We now prove that $\Phi_i(\cdot)$ is a valid probability distribution. Clearly, $\Phi_i(\cdot)$ is continuous. Note that in (3.9) for $x \in [\tilde{p}_i, \tilde{v}_i]$, by increasing x , the term $\frac{u_i(\tilde{v}_i)}{x-c}$ will strictly decrease (since $u_i(\tilde{v}_i) > 0$), and we can say that $\Phi_i(x)$ is strictly increasing. Also, $\Phi_i(\tilde{p}_i) = 0$ and $\Phi_i(\tilde{v}_i) = 1$. Thus, $0 \leq \Phi_i(x) \leq 1$ for $x \in [\tilde{p}_i, \tilde{v}_i]$. Therefore, $\Phi_i(\cdot)$ is non-decreasing and assumes values in $[0, 1]$ for all x . The claim follows. Thus we have uniquely identified a symmetric strategy that satisfies the properties required by a Nash equilibrium.

3.5 Random Demand

We have so far assumed that the demand d is deterministic. In this section, we will generalize the results to a random demand, D . Let r_d denote the probability that the

demand is d , $B_{kld}(x)$ be the expected number of units that seller k sells if she offers l units for sale and quotes x as the price per unit when the total demand is d , and $u_{kld}(x)$ be the expected utility in this case. Clearly,

$$u_{kl}(x) = \sum_d r_d u_{kld}(x) = \sum_d r_d B_{kld}(x)(x - c)$$

We introduce $\underline{d} = \min\{d : d > 0 \text{ and } r_d > 0\}$. Utilizing similar proofs, we can show that all the previous results about the structure of NE are valid for the random demand, once d is replaced with \underline{d} . This is but expected as each seller now chooses her price knowing that she is assured of an overall demand of at least \underline{d} (instead of d in the deterministic demand case). Algorithms similar to those in the deterministic case can be developed for computation of the NE in both symmetric and general cases.

3.6 Numerical Evaluations

In this section, we present numerical results for a symmetric market. In Section 3.6.1, using the results we proved for a duopoly market, we propose a heuristic pricing strategy for sellers in an oligopoly market, i.e. a market with multiple number of sellers. Numerical results reveal that our proposed strategy constitutes a good approximation for the NE of the oligopoly market. In Section 3.6.2, we investigate the asymptotic behavior of the symmetric NE of a symmetric duopoly market when the number of available units with a seller increases to infinity.

3.6.1 Oligopoly Market

Suppose that the setting is symmetric and there exist n sellers in the market. We consider a strategy that satisfies the properties identified for a symmetric NE in Section 3.4 with the threshold $l^* = \lfloor \frac{d}{n} \rfloor$. Note that the algorithm for finding such a strategy is similar to what is presented in Section 3.4.2. We now investigate how well this strategy approximates an NE strategy in an oligopoly market.

We numerically compute the maximum expected utility for a particular seller, when all other sellers choose the proposed strategy (best response utility, $U_{\text{Best Response}}$). We observe that over a large set of parameters for all possible availability levels, the best response utility is either the same as the expected utility obtained by following the proposed strategy ($U_{\text{Proposed Strategy}}$), or is fairly close to this value ⁸.

For instance, consider a market in which the availability of each seller follows a binomial distribution, $\mathcal{B}(m, p)$, with binomial probability $p = 0.4$ and $m = 3$ (m is the maximum possible available units with each seller). In addition, in this market the demand is $d = \max\{n, m\}$, $v = 10$, and $c = 1$. We plot the relative difference, described as follows, between the best response utility and the expected utility of the proposed strategy versus different number of sellers, i.e. n , for different availability levels in Figure 3.2:

$$\text{Relative Difference} = \frac{U_{\text{Best Response}} - U_{\text{Proposed Strategy}}}{U_{\text{Proposed Strategy}}}$$

Note that the relative difference is zero for all availability levels when there exist 2, 3, and 6 sellers in the market. Thus, the proposed strategy is a NE of the market in these

⁸For large sets of parameters, the difference is at most 5 percent of the value of the expected utility resulted by the proposed strategy.

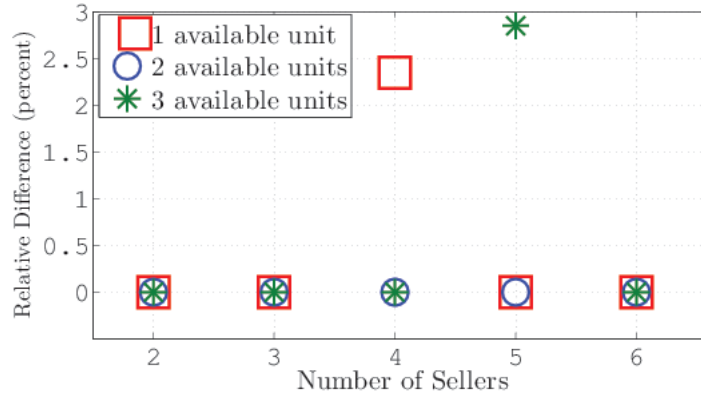


Figure 3.2: The relative difference of the best response expected utility and the expected utility of the proposed strategy versus different number of sellers

cases. Although, in the case of 4 and 5 sellers the proposed strategy is not an NE when a seller has 1 and 2 units of commodity available, respectively, the relative difference in these cases is less than 3 percent. Thus, overall, we can say that the proposed strategy is a good approximation for the oligopoly market in this scenario.

3.6.2 The Asymptotic Behavior

The focus of this section is on the asymptotic behavior of the symmetric NE of a symmetric duopoly market when the number of available units with a seller increases to infinity. In asymptotic scenario, many of availability probability distributions that arise naturally concentrate around the mean. Thus, $q_k \rightarrow 0$, when k is far from the mean. First, we show that the length of the support set for availability of k units approaches zero as $q_k \rightarrow 0$: From equation (3.10),

$$\tilde{p}_i = c + \frac{(\tilde{p}_{i-1} - c)(i \sum_{g=0}^{i-1} q_g + \sum_{g=i}^m q_g(d-g))}{i \sum_{g=0}^i q_g + \sum_{g=i+1}^m q_g(d-g)} = \tilde{p}_{i-1} + (\tilde{p}_{i-1} - c) \frac{q_i(d-2i)}{i \sum_{g=0}^i q_g + \sum_{g=i+1}^m q_g(d-g)}$$

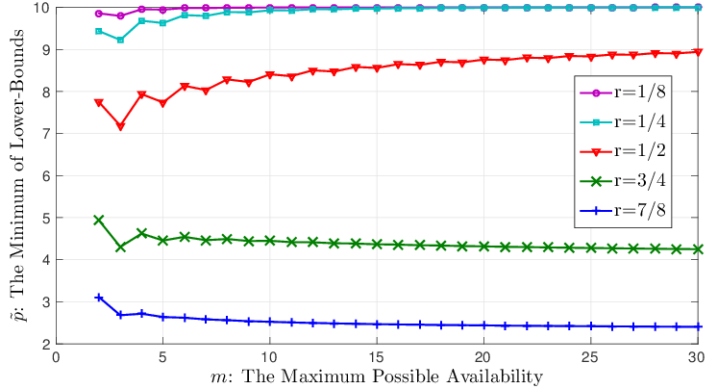


Figure 3.3: \tilde{p} versus m for when availability level is binomial with probability p and demand is m

It is immediate that if $q_i \rightarrow 0$, then $\tilde{p}_i \rightarrow \tilde{p}_{i-1}$ ⁹. This implies that the length of the support set for the availability level i units approaches zero.

We investigate the asymptotic behavior using numerical simulations when the availability of each seller follows a binomial distribution ($m, r < 1$). With this distribution, as $m \rightarrow \infty$, the binomial distribution can be approximated by a normal distribution with mean mr and variance $mr(1 - r)$. Thus $m \rightarrow \infty$ yields that $\tilde{p}_i \rightarrow \tilde{p}_{i-1}$ when $|i - mr|$ is large enough. In other words, the length of the support set for the availability level i units approaches zero if i is far from the mean. Other parameters are considered to be $v = 10$, $c = 1$, and $d = m$.

In Figure 3.3, the value of \tilde{p} , i.e. the lowest lower-bound is plotted versus m , i.e. the highest possible level of availability. As you can see, the larger the probability r , the smaller \tilde{p} . Note that when r is large, the seller is more likely to offer with higher levels of

⁹Note that the denominator is positive since $d \geq m$, and we assume uncertainty in competition, i.e. $q_m < 1$.

availability. Therefore the competition is more intense. In addition, when m is increased, the distribution \vec{q} of the availability levels concentrates around the mean, mr . If $r > \frac{1}{2}$, when a seller offers $k = mr$, knowing that the other seller offers $mr > \frac{m}{2}$ with positive probability, she will offer price less than v (note that $d = m$). Furthermore, the higher m , the more intense the competition, and consequently \tilde{p} is decreasing. On the other hand, when $r \leq \frac{1}{2}$, if a seller offers around mr units, there is no competition between sellers knowing that $2mr \leq d = m$. Furthermore, the availability probability q_k , when k is far from mr , tends to zero when m is large. Thus the associated support sets shrink to zero. This explains the increasing behavior of \tilde{p} . We notice oscillation in the figure, since m alternates between odd and even.

3.7 Applications and Discussion

The framework we described in this chapter can also be used to model two other applications in which uncertainty in competition naturally emerges: secondary spectrum access and micro grid networks.

Pricing in secondary spectrum access networks [1] is one of the applications of our model. Recent developments in wireless devices have resulted in a significant growth in demand for the radio spectrum. This leads to spectrum congestion. On the other hand, the available radio spectrum is greatly under-utilized [57]. Spectrum congestion and under-utilization have directed researchers to adopt new techniques in order to use the available spectrum more efficiently and to decrease congestion. Secondary spectrum access is an example of these techniques. In these networks, there are two types of users:

(i) Primary/licensed users, who lease a number of frequency bands (channels) directly from the regulator, and (ii) Secondary/unlicensed users, who lease frequency bands from primary users for a certain amount of time in exchange for money or other types of credit. Note that primary and secondary users correspond to sellers and buyers in our model, respectively. Each primary user may have multiple vacant frequency bands available for sale, and a secondary user can lease a channel only if it is not in use by the primary user who owns it. The usage of subscribers of primary users is random and different for different primaries. Thus primaries are uncertain about the competition, and they need to select prices for the frequency bands they offer for sale, without knowing the number of frequency bands available for sale with their competitors.

The next example scenario pertains to pricing in micro grids[14]. A micro grid network is a network of distributed power generating systems connected to local subscribers, and also to the central macro power grid. The distributed generation of power at small on-site stations is a promising alternative to the traditional generation at large stations. Decreasing the loss of transmission by reducing the distance to consumption units¹⁰, utilizing renewable energy sources, decreasing the risk of blackout, and increasing security are some of the advantages of distributed power generating scheme [33]. In these networks, a microgrid equipped with a distributed power generating system can sell its excess power to other microgrids as well as the macro grid. Since micro grids are emerging technologies¹¹, their market structure has not been finalized yet. Thus, different market structures needs to be investigated. One possible scenario is a centralized market in which micro

¹⁰In microgrid networks, the power can be sold to or bought from other local micro grids. This reduces the distance the power should be transmitted via the macro grid from a generation to a consumption site.

¹¹Microgrids are emerging in different countries such as United States [63] and India[46].

grids sell their excess power to the macro grid or a local utility at a feed-in tariff¹²[56]. Another scenario, which is investigated in this chapter, is a distributed market in which micro grids trade the power among themselves as also with macro grid at a price quoted by them in a competitive market. Our model captures the second scenario in which each micro grid with excess power (seller) sells its excess power to micro grids with deficient power or the macro grid (buyers)¹³. The amount of power generated by a power generating system is not apriori known and is different for different sellers. Thus, the sellers need to select prices for the excess power they offer for sale, without knowing the number of power units available for sale with their competitors (uncertainty in competition).

We now discuss about some details of the applications that arise in practice. Note that one unit of commodity might be valued differently by different buyers in the above mentioned applications. For instance, different secondary users receive different rates for the same frequency band, depending on their location. Similarly, different microgrids receive different amounts of power owing to differences in power loss. Hence, different buyers have different utilities even when they buy the same amount of commodity. However, in our formulations, we assumed that the pricing structure is the same for all buyers, regardless of the differences in the utilities. We justify this assumption as follow.

First note that in microgrid networks, the transmission loss is typically negligible, due to the proximity of generators and consumers. Thus, all consumers receive approximately

¹²A feed-in tariff is an offer by the macro grid to purchase some or all of the output of a micro grid at a fixed or formula rate.

¹³Note that each microgrid can be a seller or a buyer depending on the number of power units generated and the demand of its subscribers. However, at a fixed time, the identity of a micro grid as a seller or a buyer is fixed.

the same utility for a unit of power they purchase. For Primary/Secondary markets and a Non-Neutral Internet market, the utility of secondary users and CPs (as buyers) depends on the utility of their end-users, and subsequently is different for different secondaries and CPs, depending on the characteristics of their end-users. Sellers would not in general know the characteristics and identities of the subscribers of potential buyers. Hence, prices quoted by the sellers cannot depend on the utility of buyers. In addition, note that introducing a *differential pricing* for customers complicates the pricing structure for them, and prevents an easy cost prediction and management. For instance, in wireless settings, the channel quality of end-users and the rate perceived by them are time and location dependent[34]. Thus, in a differential pricing scheme, customers know the current pricing only when they use the service. But, customers are usually reluctant to adopt differential pricing schemes, owing to the rapid variability of prices which is not usually well-received by them [58]. In addition, sellers are also reluctant using a differential pricing scheme for their end-users, as they are usually computationally complex. Therefore, we did not consider different valuations for different customers in determining the pricing strategy of sellers. However, differential pricing for users with different valuation might arise for other applications; this constitutes a topic of future research.

3.8 Conclusion

We investigated price competition in a duopoly market with uncertain competition when different sellers may have different number of units available for sale. We modelled the interactions among sellers as a non-cooperative game and listed a set of properties that are

sufficient conditions for a strategy profile to be an NE. We proved that these properties are also necessary conditions for an NE in a symmetric market, or for some values of demand values in an asymmetric market. We showed that there exists a unique symmetric NE and presented an algorithm for computing the same. Using the results proved for a duopoly, we proposed a heuristic pricing strategy for sellers in a symmetric oligopoly market which approximate the NE. Directions for future work is to consider different pricing for different types of demand.

3.9 Appendix

3.9.1 Proof of Lemma 17

Proof. First consider the tuple $\langle l, y \rangle$ associated with the seller \bar{k} in which the first element is the number of units she offers and the second one is the price she chooses. We introduce $D_{kl}^{(1)}(y, i, x)$ as the expected number of units sold by the seller k who wants to offer l units with price y when her competitor's tuple $\langle g, z \rangle \neq \langle i, x \rangle$, and $D_{kl}^{(2)}(y, i, x)$ as the expected number of units sold by the seller who wants to offer l units with price y when her competitor's tuple $\langle g, z \rangle = \langle i, x \rangle$. The expected number of units sold by a seller can be written as,

$$B_{kl}(y) = D_{kl}^{(1)}(y, i, x)Pr\{\langle g, z \rangle \neq \langle i, x \rangle\} + D_{kl}^{(2)}(y, i, x)Pr\{\langle g, z \rangle = \langle i, x \rangle\}$$

Note that $D_{kl}^{(1)}(a, i, x) \leq D_{kl}^{(1)}(x, i, x)$ and $D_{kl}^{(2)}(a, i, x) \leq D_{kl}^{(2)}(x, i, x)$ for $a \geq x$ because the number of units a seller sells is a non-increasing function of her price for any given amounts offered by both sellers and any given price chosen by the competitor. Thus

$B_{kl}(a) \leq B_{kl}(x)$. In addition,

$$\begin{aligned} B_{kl}(x - \epsilon') - B_{kl}(x) &= (D_{kl}^{(1)}(x - \epsilon', i, x) - D_{kl}^{(1)}(x, i, x))Pr\{\langle g, z \rangle \neq \langle i, x \rangle\} \\ &\quad + (D_{kl}^{(2)}(x - \epsilon', i, x) - D_{kl}^{(2)}(x, i, x))Pr\{\langle g, z \rangle = \langle i, x \rangle\} \end{aligned} \quad (3.11)$$

As we discussed $D_{kl}^{(1)}(x, i, x) \leq D_{kl}^{(1)}(x - \epsilon', i, x)$. For $D_{kl}^{(2)}(x, i, x)$, we should consider ties. Since each buyer is equally likely to buy a unit from both sellers if both select equal prices, we can say that $D_{kl}^{(2)}(x, i, x) = l \frac{d}{i+l} < l$ (since $i + l > d$) and $D_{kl}^{(2)}(x - \epsilon, i, x) = l$. Note that $Pr\{\text{other seller's tuple } \langle g, z \rangle = \langle i, x \rangle\} = q_i \times \text{Jump Size of } \Phi_{ki}(\cdot) \text{ at } x$. Thus, for all positive ϵ' , RHS of (3.11) is greater than or equal to $\theta(x)$, where $\theta(x)$ is a positive number that does not depend on ϵ . Therefore since $B_{kl}(a) \leq B_{kl}(x)$, $\forall a \geq x$, $B_{kl}(x - \epsilon') \geq B_{kl}(a) + \theta(x)$, for all $a \geq x$. Thus,

$$u_{kl}(x - \epsilon') - u_{kl}(a) \geq (x - \epsilon' - a)B_{kl}(a) + \theta(x)(x - \epsilon' - c)$$

Since $x > c$, for all sufficiently small ϵ' , $x - \epsilon' - c > 0$. In addition, since $a \leq x + \epsilon$ by the statement of the lemma, the lowest value for $x - \epsilon' - a$ is $-\epsilon - \epsilon'$, and $B_{kl}(a) \leq l$. Therefore $(x - \epsilon' - a)B_{kl}(a) + \theta(x)(x - \epsilon' - c) \geq (-\epsilon - \epsilon')l + \theta(x)$. Therefore, for all sufficiently small but positive ϵ and ϵ' ,

$$u_{kl}(x - \epsilon') > u_{kl}(a) \quad a \in [x, \min\{x + \epsilon, v\}]$$

□

3.9.2 Proof of Theorem 14

Proof. The goal is to show that for each i and k all $x \in [\tilde{p}_{ki}, \tilde{v}_{ki})$ constitutes a best response for the seller k who offers i units. That is, for each $x \in [\tilde{p}_{ki}, \tilde{v}_{ki})$ and for all

y , $u_{ki}(x) \geq u_{ki}(y)$. In addition, if $\Phi_{ki}(\cdot)$ associates positive probability with \tilde{v}_{ki} , then $u_{ki}(\tilde{v}_{ki}) \geq u_{ki}(y)$ for all y , i.e., v_{ki} is a best response when the seller k offers i units. Note that the distributions, $\Phi_{ki}(\cdot)$'s, should satisfy Property 3. Thus, equations (3.4) and (3.5) holds for $x < v$, and $A_{k,l,j}(x)$ is non increasing and non positive with respect to x for $l > j > e_{\bar{k}}$.

We consider the case $j \leq e_{\bar{k}}$ here. Thus, $B_{k,j}(x) = j$ and $B_{k,l,j}(x) = \frac{1}{l}B_{k,l}(x) - 1$. Note that the expected number of units $B_{k,l}(x)$ sold at price x when l units are offered is a non-increasing function of x and $B_{k,l}(x) \leq l$. Thus, $B_{k,l,j}(x)$ and therefore $A_{k,l,j}(x)$ is non increasing and non positive with respect to x for $l > j$ regardless of how j compares with $e_{\bar{k}}$.

Consider $x < \tilde{p}$. $u_{ki}(x) \leq i(x - c) < i(\tilde{p} - c) = u_{ki}(\tilde{p})$. The last equality follows from (3.4), since $\Phi_{\bar{k}j}(\tilde{p}) = 0$ for all j . Therefore we consider $x \geq \tilde{p}$ throughout the proof.

Suppose $l_k \in \{0, 1, \dots, m_k - 1\}$ in Property 7 is fixed. We first start with $i \geq l_k + 1$. From the assumption in Theorem 14, we know that $u_{ki}(x) = u_{ki}(y)$ for any x, y in the interior of the support set of $\Phi_{ki}(\cdot)$, the support set of $\Phi_{ki}(\cdot)$ is $[\tilde{p}_{ki}, \tilde{v}_{ki}]$, $\Phi_{ki}(\cdot)$ is continuous at all $x < v$, $\tilde{v}_{ki} < v$ for $i > l_k + 1$, and $\tilde{v}_{ki} = v$ for $i = l_k + 1$. Thus, if $i > l_k + 1$ $u_{ki}(x) = u_{ki}(y)$ for all $x, y \in [\tilde{p}_{ki}, \tilde{v}_{ki}]$, and for $i = l_k + 1$, $u_{ki}(x) = u_{ki}(y)$ for all $x, y \in [\tilde{p}_{ki}, \tilde{v}_{ki})$. We consider the last case in detail. Here, $\tilde{v}_{ki} = v$. If \bar{k} has a jump at v when she offers $l_{\bar{k}} + 1$ units, by Lemma 17, $u_{ki}(v) < u_{ki}(v - \epsilon)$ for arbitrary small but positive ϵ .¹⁴ If not, using equation (3.4) and continuity of the price distributions included in that equation, it follows that $u_{ki}(v) = u_{ki}(\tilde{p}_{ki})$. Thus, we only need to prove that for all x , $u_{ki}(\tilde{p}_{ki}) \geq u_{ki}(x)$. We do so by separately considering three cases: 1. $i \geq l_k + 1$ and

¹⁴Note that Lemma 17 holds for any arbitrary price distributions and not only those that are NE.

$x \in [\tilde{p}, \tilde{p}_{ki}]$ 2. $i \geq l_k + 1$ and $x \in (\tilde{v}_{ki}, v]$ 3. $i \leq l_k$.

1) $i \geq l_k + 1$ and $x \in [\tilde{p}, \tilde{p}_{ki}]$: The claim follows by vacuity for $i = m_k$. We therefore consider $i < m_k$. Since $\tilde{v}_{kj} = \tilde{p}_{k,j-1}$ for $j \geq l_k + 1$, any such x is in $[\tilde{p}_{kg}, \tilde{p}_{k,g-1}]$ for some $g > i$. We prove this claim by induction on g , starting with the base case of $g = i + 1$.

For $x \in [\tilde{p}_{k,i+1}, \tilde{p}_{ki}]$,

$$\begin{aligned} \frac{1}{i+1}u_{k,i+1}(x) - \frac{1}{i}u_{ki}(x) &= A_{k,i+1,i}(x) \\ \frac{1}{i+1}u_{k,i+1}(\tilde{p}_{ki}) - \frac{1}{i}u_{ki}(\tilde{p}_{ki}) &= A_{k,i+1,i}(\tilde{p}_{ki}) \\ u_{k,i+1}(x) &= u_{k,i+1}(\tilde{p}_{ki}) \end{aligned}$$

Note that $\tilde{p}_{ki} = \tilde{v}_{k,i+1}$. Subtracting the first and the second equation, we get,

$$\frac{1}{i}(u_{ki}(x) - u_{ki}(\tilde{p}_{ki})) = A_{k,i+1,i}(\tilde{p}_{ki}) - A_{k,i+1,i}(x) \leq 0$$

Since $A_{k,l,j}(x)$ is non increasing and non positive with respect to x for $l > j$. Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{k,i+1}, \tilde{p}_{ki}]$. We want to prove that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{k,g+1}, \tilde{p}_{kg}]$, knowing that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{kg}, \tilde{p}_{k,g-1}]$ and $m_k - 1 \geq g \geq i + 1$ (at the base we had $g = i + 1$).

$$\begin{aligned} \frac{1}{g+1}u_{k,g+1}(x) - \frac{1}{i}u_{ki}(x) &= A_{k,g+1,i}(x) \\ \frac{1}{g+1}u_{k,g+1}(\tilde{p}_{kg}) - \frac{1}{i}u_{ki}(\tilde{p}_{kg}) &= A_{k,g+1,i}(\tilde{p}_{kg}) \\ u_{k,g+1}(x) &= u_{k,g+1}(\tilde{p}_{kg}) \end{aligned}$$

Note that $\tilde{p}_{kg} = \tilde{v}_{k,g+1}$. Subtracting the first and the second equation, we get,

$$\frac{1}{i}(u_{ki}(x) - u_{ki}(\tilde{p}_{kg})) = A_{k,g+1,i}(\tilde{p}_{kg}) - A_{k,g+1,i}(x) \leq 0$$

Thus, $u_{ki}(x) \leq u_{ki}(\tilde{p}_{kg})$ for $x \in [\tilde{p}_{k,g+1}, \tilde{p}_{kg}]$. The induction hypothesis yields $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for $x \in [\tilde{p}_{k,g+1}, \tilde{p}_{kg}]$.

2) $i \geq l_k + 1$ and $x \in (\tilde{v}_{ki}, v]$: We have just shown that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{ki})$ for all $x \in [\tilde{p}, \tilde{p}_{ki})$. We now show the same for all $x \in (\tilde{v}_{ki}, v]$. The claim follows by vacuity for $i = l_k + 1$, since $\tilde{v}_{ki} = v$. We therefore consider $i > l_k + 1$. Since $\tilde{v}_{kj} = \tilde{p}_{k,j-1}$ for $l_k + 1 \leq j \leq m_k$, and $\tilde{v}_{k,l_k+1} = v$, any such x is in $(\tilde{p}_{kg}, \tilde{p}_{k,g-1}]$ for some $l_k + 1 < g < i$. We prove this claim by induction on g , starting with the base case of $g = i - 1$. Let $x < v$.

$$\begin{aligned} \frac{1}{i}u_{ki}(x) - \frac{1}{i-1}u_{k,i-1}(x) &= A_{k,i,i-1}(x) \\ \frac{1}{i}u_{ki}(\tilde{p}_{k,i-1}) - \frac{1}{i-1}u_{k,i-1}(\tilde{p}_{k,i-1}) &= A_{k,i,i-1}(\tilde{p}_{k,i-1}) \\ u_{k,i-1}(x) &= u_{k,i-1}(\tilde{p}_{k,i-1}) \end{aligned}$$

Subtracting the first and the second equation, we get,

$$\frac{1}{i}(u_{ki}(x) - u_{ki}(\tilde{p}_{k,i-1})) = A_{k,i,i-1}(x) - A_{k,i,i-1}(\tilde{p}_{k,i-1}) \leq 0$$

Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{k,i-1}, \tilde{p}_{k,i-2}] \setminus v$. The claim is established in the base case if $\tilde{p}_{k,i-2} < v$. Else, if $\tilde{p}_{k,i-2} = v$, the claim has been shown only for $x \in (\tilde{p}_{k,i-1}, v)$ and we still need to show that $u_{ki}(v) \leq u_{ki}(\tilde{p}_{k,i-1})$, which we proceed to do. Now, let $x = v$. if the seller \bar{k} has a jump when it offers $l_{\bar{k}} + 1$ units, since $i > l_k + 1$, for all sufficiently small but positive ϵ , $u_{ki}(v) < u_{ki}(v - \epsilon)$, and for sufficiently small but positive ϵ , $v - \epsilon \in (\tilde{p}_{k,i-1}, v)$. Since $u_{ki}(v - \epsilon) \leq u_{ki}(\tilde{p}_{k,i-1})$, the base case follows. If not, that is seller \bar{k} does not have a jump when it offers $l_{\bar{k}} + 1$ units, using equation (3.4) and continuity, we can deduce that $u_{ki}(v) \leq u_{ki}(\tilde{p}_{k,i-1})$. The base case follows.

Now we want to prove that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{k,g-1}, \tilde{p}_{k,g-2}]$, knowing that $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{kg}, \tilde{p}_{k,g-1}]$ and $g \leq i - 1$ and $g - 1 \geq l_k + 1$. First, let $x < v$.

$$\begin{aligned}\frac{1}{i}u_{ki}(x) - \frac{1}{g-1}u_{k,g-1}(x) &= A_{k,i,g-1}(x) \\ \frac{1}{i}u_{ki}(\tilde{p}_{k,g-1}) - \frac{1}{g-1}u_{k,g-1}(\tilde{p}_{k,g-1}) &= A_{k,i,g-1}(\tilde{p}_{k,g-1}) \\ u_{k,g-1}(x) &= u_{k,g-1}(\tilde{p}_{k,g-1})\end{aligned}$$

Subtracting the first and the second equation, we get,

$$\frac{1}{i}(u_{ki}(x) - u_{ki}(\tilde{p}_{k,g-1})) = A_{k,i,g-1}(x) - A_{k,i,g-1}(\tilde{p}_{k,g-1}) \leq 0$$

The inequality is because of the fact that $A_{k,l,j}(x)$ is non increasing and non positive with respect to x if $l > j$. Therefore $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,g-1})$. Furthermore we know from the assumption of induction that $u_{ki}(\tilde{p}_{k,g-1}) \leq u_{ki}(\tilde{p}_{k,i-1})$, thus $u_{ki}(x) \leq u_{ki}(\tilde{p}_{k,i-1})$ for $x \in (\tilde{p}_{k,g-1}, \tilde{p}_{k,g-2}] \setminus v$. We can show that $u_{ki}(v) \leq u_{ki}(\tilde{p}_{k,i-1})$ if $v \in (\tilde{p}_{k,g-1}, \tilde{p}_{k,g-2}]$ exactly as in the base case. The proof that for each $i \geq l_k + 1$ each $x \in [\tilde{p}_{ki}, \tilde{v}_{ki})$ is a best response when a seller offers i units is therefore complete.

3) $i \leq l_k$: Now let $i \leq l_k$. Thus, $l_k > 0$. Consider two cases:

- $l_k + l_{\bar{k}} = d - 1$. Therefore $i \leq l_k = d - l_{\bar{k}} - 1$. As we previously mentioned, utility $u_{ki}(\cdot)$, is continuous not only in interval $[c, v)$, but also at price v , if $i \leq d - l_{\bar{k}} - 1$. Using (3.5), and the fact that $A_{k,l,j}(x)$ is non increasing and non positive with respect to x , for $l > j$ and a similar argument to case 1, we can get $u_{ki}(x) \leq u_{ki}(v)$ for all $x \in [\tilde{p}, v)$. The result follows.
- $l_k + l_{\bar{k}} = d$. Therefore $i \leq l_k = d - l_{\bar{k}}$. Since $l_k + l_{\bar{k}} + 1 > d$, neither $\Phi_{kl_k+1}(\cdot)$ nor $\Phi_{\bar{k}l_{\bar{k}}+1}(\cdot)$ have a jump at v , and $u_{ki}(\cdot)$ is continuous in $[c, v]$. The result follows by a similar argument to that of in the previous case.

□

3.9.3 Computation of NE Strategies in an Asymmetric Setting

In this section, we consider the general case in which the setting may not be symmetric. First we develop a framework to obtain the strategy profiles that satisfy the properties listed in Theorem 13. Then, we compute these strategies for a simple case of an asymmetric market in which $m_1 = m_2 = d = 3$. We then show that the system may have multiple Nash equilibria.

Framework for computation

In Theorem 14, it has been proved that the properties listed in Theorem 13 are sufficient properties for a NE whether $d > \{m_1, m_2\}$ or $d = \max\{m_1, m_2\}$. In this section, we use Theorem 13 to obtain a framework to identify a set of Nash equilibria for the game.

First, fix l_1 and l_2 (refer to Property 7). In addition, note that Theorem 13 specifies the ordering of support sets for a seller and not the relative ordering of support sets of the two sellers. Thus, we fix an ordering of \tilde{p}_{ki} 's and $\tilde{p}_{\bar{k}j}$'s for $i \in \{l_k + 1, \dots, m_k\}$ and $j \in \{l_{\bar{k}} + 1, \dots, m_{\bar{k}}\}$ such that for seller k and \bar{k} the lower bounds are ordered with a decreasing relation with i and j respectively, and $\tilde{p}_{km_k} = \tilde{p}_{\bar{k}m_{\bar{k}}} = \tilde{p}$. The unknowns that we should determine for a NE are \tilde{p} , $m_k - l_k - 1$ and $m_{\bar{k}} - l_{\bar{k}} - 1$ number of lower bounds other than \tilde{p} for seller k and \bar{k} respectively, and the distribution of price over each support set.

For these particular l_1 , l_2 , and relative ordering of support sets, the NE is the solution of:

$$\begin{aligned}
u_{ki}(\tilde{p}_{ki}) &= u_{ki}(\tilde{p}_{k,i-1}^-) & i \in A \\
u_{\bar{k}j}(\tilde{p}_{\bar{k}j}) &= u_{\bar{k}j}(\tilde{p}_{\bar{k},j-1}^-) & j \in A \\
u_{ki}(\tilde{p}_{ki}) &= u_{ki}(\tilde{p}_{\bar{k}j}^-) & i \in A, j : \tilde{p}_{\bar{k}j} \in (\tilde{p}_{ki}, \tilde{p}_{k,i-1}) \\
u_{\bar{k}j}(\tilde{p}_{\bar{k}j}) &= u_{\bar{k}j}(\tilde{p}_{ki}^-) & j \in A, i : \tilde{p}_{ki} \in (\tilde{p}_{\bar{k}j}, \tilde{p}_{\bar{k},j-1}) \\
f_1 f_2 &= 0
\end{aligned} \tag{3.12}$$

where $A = \{l_k + 1, \dots, m_k\}$. In addition, f_1 and f_2 are the magnitude of jump at v for the first and second seller when they offer $l_k + 1$ and $l_{\bar{k}} + 1$ units, respectively. Note that the first four sets of equations are derived using the fact that the utility of a seller should be equal over the entire support set. The fifth equation ensures that only one seller can have a positive jump at v .

In equation (3.12), the unknowns are \tilde{p} , $m_1 + m_2 - l_1 - l_2 - 2$ number of lower-bounds other than \tilde{p} , p_1 , p_2 , and $m_1 + m_2 - l_1 - l_2 - 2$ number of probability distributions at some specific points. That is $\Phi_{ki}(\tilde{p}_{\bar{k}j})$ for $i \in \{l_k + 1, \dots, m_k\}$ and j such that $\tilde{p}_{\bar{k}j} \in (\tilde{p}_{ki}, \tilde{p}_{k,i-1})$. By solving the system of equations (3.12), we can get a candidate NE.

Using the solution, $\Phi_{ki}(\cdot)$ for $k \in \{1, 2\}$ and $i \in \{1, \dots, m_k\}$ can be found. To find the distributions of price for prices less than v , first note that each price $x \in [\tilde{p}, v)$ which is not a lower bound for the support set belongs to exactly one of the support sets of each seller. Therefore, by (3.4), the expression of utility of player k when it offers i units depends only on x and $\Phi_{\bar{k}j}(x)$, i.e. $u_{ki}(x) = (x - c)G(\Phi_{\bar{k}j}(x))$, where $G(\Phi(\cdot))$ is a decreasing function of $\Phi(\cdot)$, and therefore its inverse exists. On the other hand, the utilities at the lower bounds are obtained from (3.12) for both sellers. Using Property 4, $\Phi_{\bar{k}j}(x) = G^{-1}(\frac{u_{ki}(\tilde{p}_{kj})}{x-c})$. If the resulting $\Phi_{\bar{k}j}(\cdot)$ are valid probability distribution functions, using Theorem 14 we can

conclude that they constitute a NE for the given l_1, l_2 , and the fixed ordering of lower bounds.

We have shown how to obtain a Nash equilibrium given one exists for a particular choice of l_1, l_2 , and a relative ordering between the support sets of the two sellers. Note that by changing the choices of the above we can possibly obtain multiple Nash equilibria. In the next sections, we present an example in which there exist at least two equilibria. It is not clear that there always exists an NE; our extensive numerical evaluations have not however lead to an instance where there does not exist one.

Example illustration of computation of Nash Equilibria

Consider the case in which each seller offers up to three units and the total demand is exactly three units, i.e. $d = 3$. Without loss of generality we assume that $l_1 \geq l_2$; the strategy profiles in the other case $l_1 < l_2$ can be obtained by swapping the indices of the sellers.

1) First we focus on the case in which $l_1 + l_2 = d - 1 = 2$. In this case, $l_1 = l_2 = 1$ or $l_1 = 2, l_2 = 0$. If $l_1 = l_2 = 1$, then sellers choose v with probability 1, if they offer 1 unit of commodity. In order to specify the NE, we should find the lower bounds $\tilde{p}_{13} = \tilde{p}_{23} = \tilde{p}$, $\tilde{p}_{12}, \tilde{p}_{22}$, jumps at price v (f_1 and f_2), and each distribution $\Phi_{kj}(\cdot)$ for all $k = 1, 2$, and $j = 2, 3$.

First consider the ordering of lower bounds in which $\tilde{p}_{22} \geq \tilde{p}_{12}$ (Figure 3.4). The system of equations is:

$$u_{13}(\tilde{p}) = u_{13}(\tilde{p}_{12}) \Rightarrow 3(\tilde{p} - c) = (3 - 3q_{23}\Phi_{23}(\tilde{p}_{12}))(\tilde{p}_{12} - c) \quad (3.13)$$

$$u_{23}(\tilde{p}) = u_{23}(\tilde{p}_{12}) \Rightarrow 3(\tilde{p} - c) = (3 - 3q_{13})(\tilde{p}_{12} - c) \quad (3.14)$$

$$u_{23}(\tilde{p}) = u_{23}(\tilde{p}_{22}) \Rightarrow 3(\tilde{p} - c) = (3 - 3q_{13} - 2q_{12}\Phi_{12}(\tilde{p}_{22}))(\tilde{p}_{22} - c) \quad (3.15)$$

$$u_{12}(v^-) = u_{12}(\tilde{p}_{22}) \Rightarrow (v - c)(2q_{20} + 2q_{21} + 2q_{22}f_2 + q_{22}(1 - f_2)) = (\tilde{p}_{22} - c)(2 - 2q_{23}) \quad (3.16)$$

$$u_{12}(v^-) = u_{12}(\tilde{p}_{12}) \Rightarrow (v - c)(2q_{20} + 2q_{21} + 2q_{22}f_2 + q_{22}(1 - f_2)) = (\tilde{p}_{12} - c)(2 - 2q_{23}\Phi_{23}(\tilde{p}_{12})) \quad (3.17)$$

$$u_{22}(v^-) = u_{22}(\tilde{p}_{22}) \Rightarrow (v - c)(2q_{10} + 2q_{11} + 2q_{12}f_1 + q_{12}(1 - f_1)) = (\tilde{p}_{22} - c)(2 - 2q_{13} - q_{12}\Phi_{12}(\tilde{p}_{22})) \quad (3.18)$$

$$f_1 f_2 = 0 \quad (\text{At most one seller can have a jump at } v) \quad (3.19)$$

Using equations (3.13), (3.15), (3.17), and (3.18), we can find \tilde{p}_{22} as,

$$\tilde{p}_{22} = \frac{(v - c)A}{\frac{1}{2} - \frac{1}{2}q_{13}} + c \quad (3.20)$$

$$A = \left(2q_{10} + 2q_{11} + q_{12}(1 + f_1) - \frac{3}{2}q_{20} - \frac{3}{2}q_{21} - \frac{3}{4}q_{22}(1 + f_2) \right)$$

On the other hand, from (3.16),

$$\tilde{p}_{22} = \frac{(v - c)(2q_{20} + 2q_{21} + q_{22}(1 + f_2))}{2 - 2q_{23}} + c \quad (3.21)$$

The values of \tilde{p}_{22} in (3.20) and (3.21) should be equal. Utilizing this and (3.19),

$$\frac{2f_1 q_{12}}{1 - q_{13}} - \frac{1}{2}q_{22}f_2 A = (q_{20} + q_{21} + \frac{1}{2}q_{22})A - \frac{4q_{10} + 4q_{11} + 2q_{12}}{1 - q_{13}} = B \quad (3.22)$$

where $A = \frac{1}{1-q_{23}} + \frac{3}{1-q_{13}}$. Therefore,

$$\left\{ \begin{array}{ll} f_1 = f_2 = 0 & \text{if } B = 0 \\ f_1 > 0 \& f_2 = 0 & \text{if } B > 0 \\ f_2 > 0 \& f_1 = 0 & \text{if } B < 0 \end{array} \right. \quad (3.23)$$

Therefore f_1 , f_2 , and \tilde{p}_{22} are uniquely determined. Using (3.18), $\Phi_{12}(\tilde{p}_{22})$ can be derived uniquely,

$$\Phi_{12}(\tilde{p}_{22}) = \frac{1}{q_{12}} \left(2 - 2q_{13} - \frac{v-c}{(\tilde{p}_{22}-c)} (2q_{10} + 2q_{11} + q_{12}(1+f_1)) \right) \quad (3.24)$$

By (3.15), \tilde{p} can be derived uniquely, (3.14) determines \tilde{p}_{12} uniquely, and (3.13) provides us $\Phi_{23}(\tilde{p}_{12})$ uniquely. However, we should check whether $\Phi_{23}(\tilde{p}_{12})$ and $\Phi_{12}(\tilde{p}_{22})$ are between zero and one or not. If not, then this NE candidate is not valid. The distributions can be found by the process explained previously.

Another possible ordering of lower bounds is when $\tilde{p}_{22} \leq \tilde{p}_{21}$. The system of equations corresponding to this case can be obtained by swapping the index of sellers.

In the case of $l_1 = 2$ and $l_2 = 0$, Figure 3.5 illustrates a schematic view of the support sets for the unique relative ordering of support sets. Equations can be obtained with a similar approach to the previous case.

2) $l_1 + l_2 = 3 = d$. Note that $l_k = 3$ and $l_{\bar{k}} = 0$ can be ruled out since l_k should be less than $m_k = 3$. Thus, $l_1 = 2$ and $l_2 = 1$ (Figure 3.6). The approach to find the equilibria is similar to the previous cases.

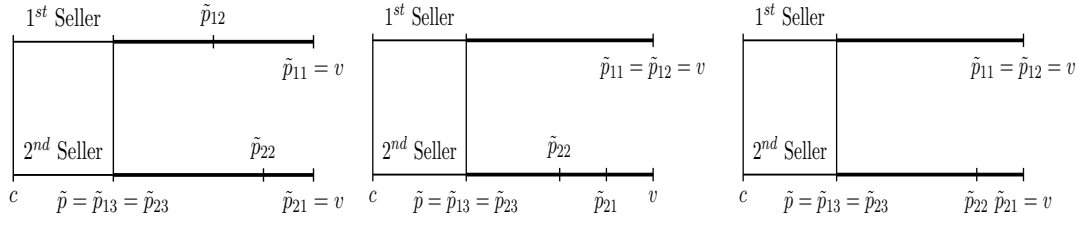


Figure 3.4: $l_1 = 1$ and $l_2 = 1$ Figure 3.5: $l_1 = 2$ and $l_2 = 0$ Figure 3.6: $l_1 = 2$ and $l_2 = 1$

Multiple Nash Equilibria

In Section 3.4, we proved that the symmetric NE exists uniquely. In this section, we show that an asymmetric market allows for multiple Nash equilibria. Nash equilibria are computed using the above framework with $v = 10$ and $c = 1$ and for different values of \vec{q}_1 and \vec{q}_2 . Some lead to a unique NE and some others to multiple Nash equilibria. For instance, the NE is unique, if

$$\vec{q}_1 = [0.45, 0.1, 0.4, 0.05] \quad \vec{q}_2 = [0.2, 0.2, 0.45, 0.15]$$

In this case, in the NE strategy, $l_1 = 1$, $l_2 = 2$, $\tilde{p}_{12} = 9.0526$, $\tilde{p} = 8.65$, and $\Phi_{23}(\tilde{p}_{12}) = 0.3333$, and the second seller has a jump of size 0.625 at price $v = 10$. However, there are two Nash equilibria if:

$$\vec{q}_1 = [0.05, 0.1, 0.4, 0.45] \quad \vec{q}_2 = [0.2, 0.2, 0.4, 0.2]$$

In both NE, $l_1 = 2$, $l_2 = 1$, and $\Phi_{13}(\tilde{p}_{22}) = 0.4444$. In the first NE, $f_2 = 0.06525$, $f_1 = 0$, $\tilde{p} = 5.95$, and $\tilde{p}_{22} = 7.1875$. In the second NE, $f_2 = 0$, $f_1 = 0.7778$, $\tilde{p} = 5.8$, and $\tilde{p}_{22} = 7$.

3.9.4 Proof of Theorem 15

Before going to the proof of Theorem 15, we need to prove some lemmas and theorems. First we prove that $A_{l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}_{m-1}$ when $d = m$ (Lemma 20). Then, in Lemma 21, we prove that the minimum of the lower end points is the lower end point of $\Phi_m(x)$, i.e., $\tilde{p} = \tilde{p}_m$. Next, using Lemmas 20 and 21, we prove that $\tilde{p}_i \notin [\tilde{p}_m, \tilde{p}_{m-1})$ for $i \in \{1, \dots, m-2\}$. This establishes the ordering for $\Phi_m(\cdot)$ and $\Phi_{m-1}(\cdot)$. After that we proceed to establish the ordering for the remaining support sets $\Phi_j(\cdot)$ for $j \in \{1, \dots, m-2\}$, knowing that for them $\tilde{p}_j \geq \tilde{p}_{m-1}$. A similar result to the Property 6 is proved in Property 8. Finally, we prove Theorem 15.

Note that a symmetric NE in a symmetric market is considered in this section. Lets define $A_{l,j}(x) = \frac{1}{l}u_l(x) - \frac{1}{j}u_j(x)$. $B_{l,j}(x)$ is defined such that,

$$A_{l,j}(x) = (x - c)B_{l,j}(x)$$

where,

$$B_{l,j}(x) = -\frac{1}{l} \sum_{i=d-l+1}^{d-j} \Phi_i(x)q_i(i-d+l) + \sum_{i=d-j+1}^m \Phi_i(x)q_i(d-i)\left(\frac{1}{l} - \frac{1}{j}\right) \quad (3.25)$$

Based on the following lemma, $A_{l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}_{m-1}$ and $l > j$, when $d = m$.

Lemma 20. *For every l and j , $l > j \geq 1$, $A_{l,j}(x)$ is (strictly) decreasing for $v > x \geq \tilde{p}_{m-1}$ when $d = m$.*

We argued that $B_{l,j}(\cdot)$ is non increasing and non positive with respect to the price x . To prove that $A_{l,j}(\cdot) = (x - c)B_{l,j}(x)$ is strictly decreasing, it is enough to prove

that $B_{l,j}(\cdot)$ is negative. We will prove that $\Phi_{m-1}(x)$ is included in the summation of $B_{l,j}(\cdot)$ and obviously positive for $x > \tilde{p}_{m-1}$. In addition, its coefficient is negative since $d = m > m - 1$. Thus, the result follows.

Proof. It is enough to prove that $B_{l,j}(x)$ is non-increasing for $x \geq \tilde{p}_{m-1}$ and negative for $x > \tilde{p}_{m-1}$ when demand is m . This yields that $A_{l,j}(x) = (x - c)B_{l,j}(x)$ is strictly decreasing with respect to x .

Note that in (3.25), $\Phi_i(\cdot)$'s are non-negative and non-increasing since they are probability distributions. In addition, they have non-positive weights: $-(i - d - l) \leq -1 < 0$, $\frac{1}{l} - \frac{1}{j} < 0$, and $d - i \geq d - m = 0$ (note that $d = m$). Thus $B_{l,j}(x)$ is non increasing and non positive with respect to the price x when $l \geq j$. To prove that $B_{l,j}(x)$ is negative for $x > \tilde{p}_{m-1}$, since $d - (m - 1) = 1 > 0$ and $-(i - d - l) \leq -1 < 0$ (possible coefficients of $\Phi_{m-1}(x)$), it is enough to prove that $\Phi_{m-1}(\cdot)$ is included in the summation of $B_{l,j}(\cdot)$ and it is positive, i.e. $\Phi_{m-1}(x) > 0$ for $x > \tilde{p}_{m-1}$. The later follows from the definition of \tilde{p}_{m-1} .

Now we prove that $\Phi_{m-1}(\cdot)$ is included in the summation of $B_{l,j}(\cdot)$. Note that $l > j \geq 1$. Thus $l \geq 2$, and the lowest index of the (3.25) is $d - l + 1 \leq m - 2 + 1 = m - 1$. The result follows. \square

To prove the ordering and disjoint properties in the symmetric setting we should alter the proofs. First we will prove that $\tilde{p} = \tilde{p}_m$, i.e. the minimum of lower bounds is the lower bound of $\Phi_m(x)$. Then we will prove that $\tilde{p}_j \notin [\tilde{p}_m, \tilde{p}_{m-1})$ for $j \in \{1, \dots, m - 2\}$. This proves that the next lowest support set is the support set of $\Phi_{m-1}(\cdot)$. After that using Lemma 18 will prove that the support set of $\Phi_l(\cdot)$ for $l < m$ is a subset of $[\tilde{p}_{m-1}, p_j]$

for all integers $j \in [1, l)$. These three all together establishes the ordering.

Lemma 21. $\tilde{p} = \tilde{p}_m$, i.e. the minimum of lower end points is the lower end point of $\Phi_m(x)$.

Proof. Suppose not and there exists $x > \tilde{p}$ such that $x \leq \tilde{p}_m$. By Property 5, there exists an $\epsilon > 0$ and an availability level $j \neq m$ such that $[\tilde{p}_m - \epsilon, \tilde{p}_m]$ belongs to the support set of $\Phi_j(\cdot)$ and $\tilde{p}_j < \tilde{p}_m$. Thus $u_j(\tilde{p}_m) = u_j(\tilde{p}_m - \epsilon)$. In addition, $B_{m,j}(x)$ is the weighted summation of $\Phi_i(\cdot)$ for $i \in \{1, \dots, m\}$. Thus, the distribution $\Phi_j(\cdot)$ is included in the summation of $B_{m,j}(x)$, and its coefficient is negative. In addition, $\Phi_j(x) > 0$ for $x > \tilde{p}_j$. Thus, $A_{m,j}(x)$ is strictly decreasing with respect to x for $x > \tilde{p}_j$. Thus $A_{m,j}(\tilde{p}_m - \epsilon) > A_{m,j}(\tilde{p}_m)$. Note that $u_j(\tilde{p}_m) = u_j(\tilde{p}_m - \epsilon)$. Thus, $u_m(\tilde{p}_m) = u_{m,max} < u_m(\tilde{p}_m - \epsilon)$. This contradicts with \tilde{p}_m belonging to the support set of $\Phi_m(\cdot)$. The result follows. □

Lemma 22. $\tilde{p}_i \notin [\tilde{p}_m, \tilde{p}_{m-1})$ for $i \in \{1, \dots, m-2\}$.

To prove this, we use a contradiction argument. Suppose that there exists $\tilde{p}_j \in [\tilde{p}_m, \tilde{p}_{m-1})$ such that $j \in \{1, \dots, m-2\}$. We will prove that no $x \in (\tilde{p}_j, \tilde{p}_{m-1}]$ is in the support of $\Phi_m(\cdot)$. Thus there exists $u \in \{1, \dots, m-2\}$ such that \tilde{p}_{m-1} is in the support set of $\Phi_u(\cdot)$. We prove that the payoff of the seller when she offers u units with price $\tilde{p}_{m-1} + \epsilon$ is strictly greater than the payoff when offering with price \tilde{p}_{m-1} . This is in contradiction with \tilde{p}_{m-1} being the best response for player with availability u .

Proof. The lemma follows by vacuity if $m \leq 2$. Take $m > 2$. Note that $\tilde{p}_{m-1} < v$. If not there is a jump of size 1 at price v when the seller offers $m-1$ units. Since

$2m - 2 > d = m$ for $m > 2$, using Lemma 17, $u_{m-1}(v - \epsilon) > u_{m-1}(v)$ for ϵ small enough. This is in contradiction with assigning a positive probability to price v in the equilibrium when seller offers $m - 1$ units. Thus $\tilde{p}_{m-1} < v$.

Suppose there exists $\tilde{p}_j \in [\tilde{p}_m, \tilde{p}_{m-1})$ such that $j \in \{1, \dots, m - 2\}$. We will prove that no $x \in (\tilde{p}_j, \tilde{p}_{m-1}]$ is in the support of $\Phi_m(\cdot)$. Thus (using this and Property 5), there exists $u \in \{1, \dots, m - 2\}$ such that \tilde{p}_{m-1} is in the support set of $\Phi_u(\cdot)$. Consider $B_{m-1,u}(x)$ which is the summation of weighted distributions $\Phi_i(x)$ when $i \in \{2, \dots, m - 1\}$. Thus, the distribution $\Phi_{m-1}(\cdot)$ is included in the summation of $B_{m-1,u}(x)$ (note that $m > 2$), and its coefficient is negative (Note that $d - (m - 1) = 1 > 0$). Thus, $A_{m-1,u}(x)$ is strictly decreasing with respect to x for $x > \tilde{p}_{m-1}$. Thus $A_{m-1,u}(\tilde{p}_{m-1} + \epsilon) < A_{m-1,u}(\tilde{p}_{m-1})$. Using $u_{m-1}(\tilde{p}_{m-1}) = u_{m-1}(\tilde{p}_{m-1} + \epsilon)$, we can conclude that $u_u(\tilde{p}_{m-1}) = u_{u,max} < u_u(\tilde{p}_{m-1} + \epsilon)$. This is in contradiction with \tilde{p}_{m-1} being the best response for player with availability u . Note that $\tilde{p}_{m-1} < v$, and every price less than v which belongs to the support set of a distribution $\Phi_i(\cdot)$ should be a best response for players when offering i units. The lemma follows.

Now we complete the proof by proving that no $x \in (\tilde{p}_j, \tilde{p}_{m-1}]$ is in the support of $\Phi_m(\cdot)$. Suppose not. We will show that there exist an availability level f and two prices y_1 and y_2 , such that $\tilde{p}_j < y_1 < \tilde{p}_{m-1}$, belongs to the support set of $\Phi_m(\cdot)$, and both y_1 and y_2 belong to the support set of $\Phi_f(\cdot)$. Then we will show that $u_m(y_1) < u_m(y_2)$, which contradicts with y_1 being in the support set of $\Phi_m(\cdot)$.

Using the contradiction assumption, w is defined as,

$$w = \inf_{x \in (\tilde{p}_j, \tilde{p}_{m-1}] \text{ \& } x \in \text{Supp}(\Phi_m(\cdot))} x$$

Note that w is in the support set of $\Phi_m(\cdot)$. Now consider two cases:

1. $w > \tilde{p}_j$: Using continuity, the definition of support sets, and Property 5, there exist ϵ and $f \in \{1, \dots, m-2\}$ such that w and $w - \epsilon$ is in the support set of $\Phi_f(\cdot)$. Take $y_1 = w$ and $y_2 = w - \epsilon$.
2. $w = \tilde{p}_j$: Using continuity and the definition of infimum, there exists ϵ such that every $w + \epsilon$ belong to the support set of $\Phi_m(\cdot)$ and $\Phi_j(\cdot)$. Take $f = j$, $y_1 = w + \epsilon$, and $y_2 = w$.

Next, we will prove that $u_m(y_1) < u_m(y_2)$, which contradicts with y_1 being in the support set of $\Phi_m(\cdot)$. Note that $y_1 < v$, and every price less than v which belongs to the support set of a distribution $\Phi_i(\cdot)$ should be a best response for players when offering i units. This completes the proof.

Consider $B_{m,f}(x)$ which is the summation of weighted distributions $\Phi_i(x)$ when $i \in \{1, \dots, m-1\}$. Thus, the distribution $\Phi_f(\cdot)$ is included in the summation of $B_{m,f}(x)$, and its coefficient is negative. Thus, $A_{m,f}(x)$ is strictly decreasing with respect to x for $x \geq \tilde{p}_f$. Thus $A_{m,f}(y_2) > A_{m,f}(y_1)$. Using $u_f(y_1) = u_f(y_2)$, we can conclude that $u_m(y_1) < u_m(y_2)$. The contradiction argument is complete.

□

Therefore we established the ordering for $\Phi_m(\cdot)$ and $\Phi_{m-1}(\cdot)$. Now we are set to establish the ordering for the remaining support sets $\Phi_j(\cdot)$ for $j \in \{1, \dots, m-2\}$, knowing that for them $\tilde{p}_j \geq \tilde{p}_{m-1}$. The next is the counterpart of the Property 8 in symmetric setting.

Property 8. *The support set of $\Phi_l(\cdot)$ is a subset of $[\tilde{p}, \tilde{p}_j] \cup [v]$ for all integers $j \in [1, l]$.*

Proof. Consider support sets of $\Phi_j(\cdot)$, $\Phi_l(\cdot)$, and $j < l$. We will show that $u_l(a) < u_l(\tilde{p}_j)$ for all $a \in (\tilde{p}_j, v)$. Thus, no $a \in (\tilde{p}_j, v)$ is a best response for the seller with availability of l units. Therefore, the support set of $\Phi_l(\cdot)$ is a subset of $[\tilde{p}, \tilde{p}_j] \cup [v]$.

We now complete the proof, by showing that $u_l(a) < u_l(\tilde{p}_j)$ for all $a \in (\tilde{p}_j, v)$:

$$\frac{1}{l}u_l(a) - \frac{1}{j}u_j(a) = A_{l,j}(a)$$

Note that if $\tilde{p}_j \geq v$, property follows by vacuity. Now we consider $\tilde{p}_j < v$. Since $j < l \leq m$, $j \leq m - 1$. By Lemma 22, $\tilde{p}_{m-1} \leq \tilde{p}_j < a < v$, by Lemma 20, $A_{l,j}(a)$ is decreasing function of a for $a \in [\tilde{p}_{m-1}, v)$. Thus, $A_{l,j}(a) < A_{l,j}(\tilde{p}_j)$ for $a \in (\tilde{p}_j, v)$. On the other hand $u_j(a) \leq u_j(\tilde{p}_j)$ for all $a > \tilde{p}_j$, since \tilde{p}_j is a best response of a seller with availability j , therefore $u_l(\tilde{p}_j) > u_l(a)$. \square

Now we will prove Theorem 15:

Proof. Note that the first place that we used the condition $d > \max\{m_1, m_2\}$ (in symmetric setting $d > m$) instead of $d = \max\{m_1, m_2\}$ ($d = m$) was in Section 3.2.5. Thus all of the results before that apply also to the case that $d = m$. Property 8 provides exactly the same property in the Property 6 for the symmetric scenario. Thus the corollaries after the property follows. In addition, results in the Section 3.2.6 follows, since they are based on results before the Section 3.2.5 and Property 6 and its corollaries. Thus Theorem 13 goes through in the case of a symmetric NE and $d = m$. The result follows. \square

Chapter 4

Non-Neutrality Framework II - Quality Sponsored Data¹

In Chapter 2, we assessed the benefits of different entities for migrating to a non-neutral Internet. In Chapter 3, we considered a non-neutral framework in which CPs are passive in the equilibrium selection and are only price-takers. In this chapter, we consider a non-neutral framework in which, in contrast to the previous work, CPs have an active role in the market by selecting appropriate strategies. Note that while SPs currently only provide best-effort services to their CPs, it is plausible to envision a model in near future, where CPs are willing to sponsor quality of service for their content in exchange of sharing a portion of their profit with SPs. This quality sponsoring becomes invaluable especially when the available resources are scarce such as in wireless networks, and can be accommodated in a non-neutral network. In this work, we consider the problem of

¹Presented in WiOpt 2015 [41] and accepted for publication in IEEE/ACM Transaction on Networking [42].

Quality-Sponsored Data (QSD) in a non-neutral network. In our model, SPs allow CPs to sponsor a portion of their resources, and price it appropriately to maximize their payoff. The payoff of the SP depends on the monetary revenue and the satisfaction of end-users both for the non-sponsored and sponsored content, while CPs generate revenue through advertisement. Note that in this setting, end-users still pay for the data they use. We analyze the market dynamics and equilibria in two different frameworks, i.e. sequential and bargaining game frameworks, and provide strategies for (i) SPs: to determine if and how to price resources, and (ii) CPs: to determine if and what quality to sponsor. The frameworks characterize different sets of equilibrium strategies and market outcomes depending on the parameters of the market.

The chapter is organized as follows: In Section 4.1, we model the market. Subgame Perfect Nash Equilibrium (SPNE) of the sequential game is characterized in Section 4.2, and the Nash Bargaining Solution (NBS) of the game is characterized in Section 4.3. In Section 4.4, we summarize the key results of the chapter and comment on some of the assumptions and their generalizations. Finally, in Section 4.5, we conclude the chapter. Additional details and some of the proofs are presented in the appendix of the chapter in Section 4.6.

4.1 Model

4.1.1 Problem Formulation:

We model the ecosystem as a market consisting of three players: CPs, SPs, and end-users. We focus on the interaction between SPs and CPs, and not on the competition among SPs

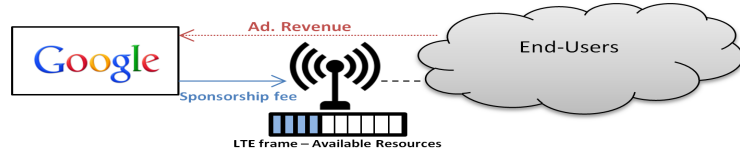


Figure 4.1: Market when CP has an advertisement revenue model

and CPs. Thus the interaction between one CP and one SP is considered. The strategy for the CP is to determine how much resources to sponsor (i.e. quality), and the strategy of the SP is to determine how to price her resources. Decisions are made by the players based on an estimated demand update function (explained later) at the beginning of every time-epoch, which captures the typical time granularity of sponsorship decisions².

The CP has an advertisement revenue model, and sponsors b_t resources (e.g. bits in an LTE frame) out of a total of N resources at t^{th} time-epoch to sponsor the average quality of at least ζ ($\frac{\text{bit}}{\text{frame}}$) for her content, and pays a price of p_t per resource sponsored to the SP. Thus, on average a quality of ζ should be satisfied for the users. If not, the CP exits the sponsorship program. Note that this does not guarantee the quality of an individual user to be higher than ζ . An example Schematic picture of the market in this case is presented in Figure 4.1.

The CP and the SP choose their strategy at time-epoch t after observing the previous demand, i.e. the number of end-users desiring content from the previous epoch. Obviously,

²Using the estimate of the demand, players decide on their strategy to maximize an “estimated” payoff (and not the actual one). Note that the shorter the interval of epochs, the more accurate the estimates, and the more inconvenient the implementation would be. We will observe that, in our framework, the algorithm of decision making in NE would be simple. Thus, the decision making can be done in shorter time intervals, e.g. minutes.

the demand is non-negative. Note that the demand for the content of the CP changes over time depending on the satisfaction of users, which in turn depends on the resources that the CP decides to sponsor and hence the quality. We suppose that the satisfaction of users depends on the average quality, i.e. $\frac{b_t}{d_t}$ ³, and the demand for content updates as follows⁴,

$$d_{t+1} = \begin{cases} d_t(1 + \gamma \log(\kappa_u \frac{b_t}{d_t}))^+ & \text{if } d_t > 0 \\ 0 & \text{if } d_t = 0 \end{cases} \quad (4.1)$$

where $z^+ = \max\{z, 0\}$, d_t is the demand between epoch t and $t + 1$, $\frac{b_t}{d_t}$ is the rate a single user receives, and $\log(\kappa_u \frac{b_t}{d_t})$ models the satisfaction of end users: the higher the rate received by users, the higher their satisfaction. The parameter $\gamma > 0$ represents the *sensitivity* of end-users to their satisfaction. A higher γ is associated with a higher rate of change with respect to the satisfaction of users (higher fluctuation in demand). An instance of this type of users is customers of a streaming website like Netflix that are sensitive to the quality they receive. Parameter $\kappa_u > 0$ is a constant.

Note that the total available wireless resources (for sponsored and non-sponsored contents) is limited (N). This limits the number of sponsored resources (b_t) which in turn determines the upperbound of resources that can be allocated to non-sponsored contents.

³Note that we are analyzing the model from the perspective of the CP and the SP. Thus, we are assuming that the CP and the SP see the demand for the content as a whole and want to sponsor an average sponsored quality. The demand of the individual end-users could be potentially different from each other.

⁴Note that receiving a satisfactory quality, increases the chance of user repeating the visit to the website and increases the number of new users that are going to use the service. Therefore, a satisfactory QoS will likely increase the demand for the data in the next session. In addition, we assume that the increase in the demand would be slower with high rates (a diminishing return behavior).

This is a key distinction of our work from previous works as the limited resources available couples the utility of end users for both sponsored and non-sponsored content with the decisions of the market players. We assume that the number of resources (bits) a CP can sponsor is bounded above by \hat{N} ($\hat{N} \leq N$).

The utility of the CP if she chooses to enter the sponsorship program consists of the utility she receives by sponsoring the content minus the price she pays for sponsoring the sponsored bits. The latter is $p_t b_t$. The former, i.e. the utility of the CP for sponsoring the content, depends on the advertisement revenue which in turns depends on the demand for the content as well as the quality received by the users (throughput is $\frac{b_t}{d_t}$)⁵. We consider the utility from advertisement for the CP to be:

$$u_{ad}(b_t) = \begin{cases} \alpha d_t \log\left(\frac{\kappa_{CP} b_t}{d_t}\right) & \text{if } d_t > 0 \\ 0 & \text{if } d_t = 0 \end{cases} \quad (4.2)$$

Note that the better the quality of advertisement, more successful the advertisement would be, and therefore the higher the utility that the CP receives from advertisement. Thus, the utility of advertisement is dependent on the satisfaction of users. This is the reason that we use a similar function to (4.1) for the utility of advertisement⁶. The

⁵Note that $\frac{b_t}{d_t}$ can be the quality of the content, ads, or both. One example of CPs whose revenue depends on the quality of the ad is a CP that support video ads, e.g. YouTube. Another example is websites loaded with several “flash ads” for which users may have difficulty loading the ads which can lead to the decrease of number of clicks on the ads. In addition to these CPs, we can think about scenarios in which increasing the quality of the content of a website (not only the ads) increases the revenue of this website. An example of such contents are shopping websites (e.g. Amazon). Improving the quality of the experience of users, increases the chance of spending more time on these website. This would increase the chance of a transaction which increases the revenue of the CP.

⁶Note that we expect a diminishing return of ad utility based on quality, i.e. after a certain point

constant κ_{CP} in general can be different from κ_u . The parameter α is a constant that models the the unit income of the CP for each end-user based on the quality that the end-user receives: The higher α , the higher the profit of the CP per rates sponsored. An example of CPs with high α is shopping websites (e.g. Amazon) that in contrast with streaming websites (e.g. Netflix) have a high profit per user rate.

Thus, the utility of the CP at time t if she chooses to join the sponsorship program is:

$$u_{CP,t}(b_t) = u_{ad}(b_t) - p_t b_t \quad (4.3)$$

To have a non-trivial problem, we assume $\kappa_{CP}\zeta > e = 2.72$.

The utility of the SP at time t if she chooses to offer the sponsorship program is the revenue she makes by sponsoring the bits plus the users' satisfaction function:

$$u_{SP,t}(p_t) = p_t b_t + u_s(b_t(p_t)) \quad (4.4)$$

where the users' satisfaction function, i.e. $u_s(\cdot)$, is a function of the number of sponsored bits which subsequently depends on the price p_t . This function consists of two parts: (i) the satisfaction of users for access to the sponsored content and its quality, and (ii) the satisfaction of users when using non-sponsored content. This function could be decreasing or increasing depending on the users, the cell condition, etc. We define the satisfaction function as follows⁷

increasing the quality would not significantly increase the utility of advertisement. Thus, we used a log function to model the ad utility from an end-user ($\log(\frac{b_t}{d_t})$). Thus, we assume the utility of advertisement to be $\sum_{d_t} \text{constant} \times \log(\frac{b_t}{d_t}) = \text{constant} \times d_t \log(\frac{b_t}{d_t})$. If we consider a linear dependency between quality and ad revenue, then the utility would be $\sum_{d_t} \text{constant} \times \frac{b_t}{d_t} = \text{constant} \times b_t$. However, we believe that a function with diminishing return would model the ad utility more closely.

⁷Note that in the case of no sponsoring, the demand of the CP should be added to the demand of in the

$$u_s(b_t) = \begin{cases} \nu_1 d_t \log\left(\frac{\kappa_{SP} b_t}{d_t}\right) + \nu_2 D \log\left(\kappa_{SP} \frac{N - b_t}{D}\right) & \text{if } d_t > 0 \& b_t > 0 \\ \nu_2 D \log\left(\kappa_{SP} \frac{N - b_t}{D}\right) & \text{Otherwise} \end{cases} \quad (4.5a)$$

where D is the total demand for all CPs other than the strategic CP⁸ and κ_{SP} is a positive constant. In addition, ν_1 and ν_2 are constants corresponding to the weights that end-users assign to the sponsored and non-sponsored data, respectively. We considered the users' satisfaction function to be a part of the SP's utility since it is natural to think that SPs not only care about the money they receive for the sponsored content, but also about the satisfaction (or the overall quality of experience) of end-users for both sponsored and non-sponsored content⁹. Another reason for considering the satisfaction of users is best effort categor, i.e. D in (4.5a) should be substituted by $d_t + D$. However, we assume that $d_t \ll D$, i.e. the demand for one content is much smaller than the aggregate demand for all other contents. This often arises in practice. In Appendix 4.6.8, we show that this modification does not alter any results in essence, and the insights of the model would be the same as before.

⁸We now argue why D is considered to be constant. We consider the content of the CP that is willing to sponsor her content to be different from the content of other CPs, i.e. no competition over the content. An example of such CP is Youtube (for personal video streaming). This yields that the demands for the strategic CP (that can be sponsored) and other CPs are independent of each other. Thus, no demand will be switched from the content of the sponsored CP to other CPs. In addition, since we assume that other CPs are not sensitive to the quality they deliver, their demand is independent of the quality their end-users receive. Thus, D can be considered as a constant, and independent of the demand for the sponsored content.

⁹Note that in reality, end-users can switch between SPs if they are not satisfied, and this incurs loss to the SP they leave. To capture this, we need to consider the competition between SPs which makes the analysis much more complicated. Instead, we consider only one SP and the user satisfaction function to be an element in the utility of the SP. This captures some aspects of competition over end-users between SPs, without complicating the analysis unnecessarily.

the regulatory policies on the quality of the experience of users who use non-sponsored contents. Thus, v_1 and v_2 can be determined by the SP or the regulator¹⁰. The higher $\frac{v_2}{v_1}$, the higher would be the importance of the non-sponsored content.

Note that, despite the dependencies between κ_u , κ_{CP} , and κ_{SP} , these parameters could be potentially different. For example, the ad revenue is paid by an advertiser. This advertiser may value the quality of the content delivered to the end-users different from the end-users. Thus, for this reason, κ_{CP} might be different from κ_u . We do not mandate the parameters to be different from each other and they can be potentially equal.

Recall that we assume if $d_t = 0$ or one of the decision makers exits the sponsoring program, then the game ends, and we have a stable outcome of no-sponsoring.

A summary of important symbols is presented in Table 4.1.

¹⁰ Although in the current set up, the regulator does not provide quality constraints for the SP, one can envision that in a non-neutral framework, the regulator imposes explicit or implicit constraints on the behavior of SP toward the sponsored and non-sponsored data. In other words, in a non-neutral regime, it is natural to suppose that the regulator forces the SPs to take into the account the satisfaction of their users regardless of the fact that they are using sponsored or non-sponsored data. Thus, the SP wants to maximize her utility (which depends on the money collected from the CPs) given some constraints. In this sense, including the satisfaction of users with parameters v_1 and v_2 is similar to the *Lagrangian penalty (reward) function* by which we solve the mentioned maximization. Note that eventually v_1 and v_2 is set by the SP and not the regulator. However, their value depends on the restriction determined by the regulator. Therefore, a strict net-neutrality rule, mandates the SP to assign high weights to the quality of the content of non-sponsored data, i.e. high v_2 .

Symbol	Description
p_t	the price per unit of resources sponsored at time t
b_t	the number of sponsored bits in an LTE frame at time t
d_t	the demand between epoch t and $t + 1$
ζ	the minimum average quality desired by end-users
γ	sensitivity of end users to the quality they receive.
α	constant, the unit income
$\kappa_u, \kappa_{CP}, \kappa_{SP}$	constants
\hat{N}	the number of available bits for sponsoring
N	the total number of bits (resources) in an LTE frame
$u_s(\cdot)$	end-users' satisfaction function
$u_{ad}(\cdot)$	CP's advertisement profit
ν_1	the weight end-users assign to the sponsored data
ν_2	the weight end-users assign to the non-sponsored data
D	the total demand of end-users for non-sponsored data
$\frac{1}{\kappa_u}$	the stable quality, the rate that stabilizes the demand
$z \ \& \ y$	the participation factor for the CP and SP, 1 =join, 0 =exit

Table 4.1: Important Symbols

4.1.2 Preliminary Notations and Definitions:

In this section, we define some notations that we use throughout the chapter. Section-specific notations and definitions are presented in the corresponding sections.

Note that, we model the problem of QSD as a sequential game if at least one of the decision makers is short-sighted, and as a bargaining game when both CPs and SPs are long-sighted.

Definition 8. Short-Sighted (Myopic) Decision Maker: *A decision maker is short-sighted if she maximizes the myopic payoff knowing the present demand (d_t),¹¹ i.e. does not involve the evolution of demand (4.1) in her decision making.*

Definition 9. Long-Sighted Decision Maker: *A decision maker is long-sighted if she maximizes her payoff in long-run considering the current demand and the evolution of the demand in (4.1).¹²*

Since we consider an evolving demand of end-users based on their satisfaction, one of the contributions of this work is to characterize the stability conditions of the market.

Definition 10. Stable Market: *We say that the market is stable if and only if the demand of end-users is asymptotically stable, i.e. if and only if:*

$$\lim_{t \rightarrow \infty} |d_{t+1} - d_t| = 0$$

Note that it is not apriori clear that the demand would be stable. In fact, we see that in the short-sighted scenario, under certain parameters, the demand is unstable. The definition

¹¹Mathematical definitions for the optimization solved by the short-sighted SP and CP are presented in Equations (4.6) and (4.7), respectively.

¹²Mathematical definitions for the payoff of the long-sighted SP and CP are presented in Equations (4.11) and (4.12), respectively.

of the stable market and (4.1) yield the following lemma that is useful in determining the stable outcome of the market.

Lemma 23. *The market is stable if and only if the quality $\frac{b_t}{d_t} \xrightarrow{t \rightarrow \infty} \frac{1}{\kappa_u}$ is sponsored for end-users.*

Proof. The result follows immediately from (4.1). □

Definition 11. *Stable Quality and Stable Demand: We refer to $\frac{b}{d} = \frac{1}{\kappa_u}$ as the stable quality, and $d = \kappa_u b$ as the stable demand.*

4.2 Sequential Framework: SPNE Analysis

In the sequential game framework, we seek a *Subgame Perfect Nash Equilibrium* (SPNE) using *backward induction*.

Definition 12. *Subgame Perfect Nash Equilibrium (SPNE): A strategy is an SPNE if and only if it constitutes a Nash Equilibrium (NE) of every subgame of the game.*

Definition 13. *Backward Induction: Characterizing the equilibrium strategies starting from the last stage of the game and proceeding backward.*

In this section, we first present the stages of the game (Section 4.2.1). Then, in Section 4.2.2, we consider the case in which both the CP and the SP have a short-sighted (myopic) business model and play the one-shot game infinitely. We characterize the equilibrium strategies and asymptotic outcome of the game. When parameters of the market are such that a stable sponsoring outcome is not plausible, considering decision makers with long-sighted vision about the market may ensure a stable sponsoring outcome

for the market. Thus, in Sections 4.2.3 and 4.2.4, using the sequential framework, we investigate the cases in which either one of the SP and the CP is long-sighted and the other is short-sighted. In Section 4.2.5, we present numerical results and discuss about them.

4.2.1 Stages of the Game:

We suppose a complete information setting for the game. The timing of the game at time epoch t is as follow:

1. The SP decides on (1) offering the sponsorship program, $y_t \in \{0, 1\}$ (with $y_t = 1$ implying offering) and (2) if $y_t = 1$, on the price per sponsored bit in an LTE frame, p_t , by solving the following optimization:

$$\max_{p_t} u_{SP,t}(p_t), \tag{4.6}$$

where $u_{SP,t}(p_t)$ is defined in (4.24). The SP sets $y_t = 0$ if $u_{SP,t}^* < v_2 D \log(\kappa_{SP} \frac{N}{D})$ (the payoff is less than no-sponsoring payoff) or $d_t = 0$, and $y_t = 1$ otherwise, where $u_{SP,t}^*$ is the optimum outcome of the optimization¹³.

2. The CP decides on (1) whether to participate in the sponsorship program, $z_t \in \{0, 1\}$ (with $z_t = 1$ implying participation) and (2) if $z_t = 1$ on the number of bits in an LTE frame (i.e. quality) she wants to sponsor, b_t , by solving the following optimization

¹³Note that we consider that in the case of indifference $u_{SP,t}^* = 0$, $y_t^* = 1$

problem:

$$\begin{aligned}
& \max_{b_t > 0} u_{CP,t}(b_t) \\
s.t. \quad & \frac{b_t}{d_t} \geq \zeta \\
& b_t \leq \hat{N}
\end{aligned} \tag{4.7}$$

where $u_{CP,t}$ is defined in (4.3). The first constraint is associated with the minimum quality that the CP wants to deliver to her end-users. The second constraint puts an upperbound to the number of bits that a CP can sponsor in an LTE frame. The CP sets $z_t = 0$ if $u_{CP,t}^* < 0$ or $d_t = 0$, and $z_t = 1$ otherwise, where $u_{CP,t}^*$ is the optimum outcome of the optimization¹⁴. In addition, the CP exits the sponsorship program, i.e $z_t = 0$, if there is no feasible solution for (4.7). Note that, $d_t = (1 + \gamma \log(\kappa_u \frac{b_{t-1}}{d_{t-1}}))^+$, and is known as the history of the game is known.

We use the Backward Induction method to find the Sub game Perfect Nash Equilibrium (SPNE) of the game. Thus, first, we find the best response strategy of the CP in the second stage given the strategy of the SP in the first stage and the history of the game. This allows the CP to decide on (1) joining the sponsorship program and also on (2) the number of bits to sponsor. Then, using this best response strategy and the history, the SP chooses (1) whether to launch the sponsorship program or not, and (2) the optimum per-bit price, p_t , in the first stage.

4.2.2 Short-Sighted CP, Short-Sighted SP

CP's Strategy: In the second stage, knowing the decision of the SP at stage one, the CP solves (4.7) at each time-epoch t .

¹⁴Note that we consider that in the case of indifference $u_{CP,t}^* = 0$, $z_t^* = 1$

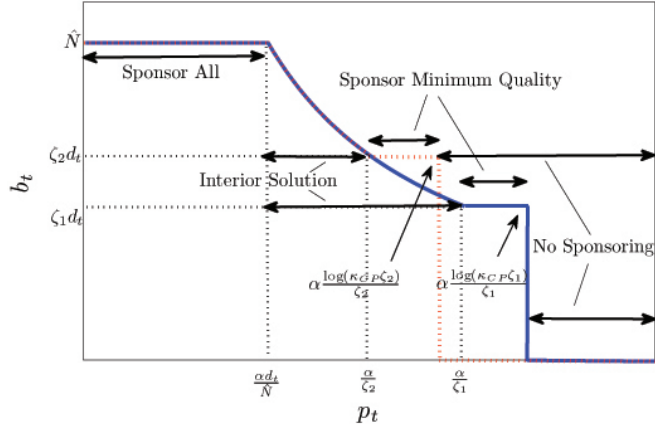


Figure 4.2: The optimum strategy of the CP presented in theorem 16 for ζ_1 (blue) and ζ_2 (red) when $0 < d_t \leq \frac{\hat{N}}{\zeta}$ and $\zeta_2 > \zeta_1$.

Theorem 16. *Equilibrium Strategy of Stage 2:* *The strategy of the CP in the SPNE is as follows:*

$$\text{if } 0 < d_t \leq \frac{\hat{N}}{\zeta}, \quad (b_t^*, z_t^*) = \begin{cases} (\hat{N}, 1) & \text{if } p_t \leq \frac{\alpha d_t}{\hat{N}} \\ (\frac{\alpha d_t}{p_t}, 1) & \text{if } \frac{\alpha d_t}{\hat{N}} \leq p_t \leq \frac{\alpha}{\zeta} \\ (\zeta d_t, 1) & \text{if } \frac{\alpha}{\zeta} \leq p_t \leq \frac{\alpha \log(\kappa_{CP} \zeta)}{\zeta} \\ (-, 0) & \text{if } p_t > \frac{\alpha \log(\kappa_{CP} \zeta)}{\zeta} \end{cases} \quad (4.8)$$

$$\text{if } d_t > \frac{\hat{N}}{\zeta} \text{ or } d_t = 0, \quad (b_t^*, z_t^*) = (-, 0) \quad (4.9)$$

Remark 7. *It is intuitive that the number of sponsored bits is a decreasing function of the price per sponsored bit. In addition, one can expect that if the price per sponsored bit is lower (respectively, higher) than a threshold, the CP sponsors all the available bits (respectively, the amount to satisfy only the minimum quality requested by the end-users). Moreover, if the price is so high that in case of sponsoring the CP receives a negative*

payoff, the CP would exit the sponsoring program. This also characterizes another threshold for the price per sponsored bit. In theorem 16, we confirm the intuitions, and go beyond it by characterizing the thresholds on the price per sponsored bit and optimum number of sponsored bits in different regions characterized by the thresholds. Figure 4.2 illustrates the optimum strategy of the CP and the regions described in the theorem for two different values of ζ . Note that the higher the minimum quality requested by end-users, the lower the thresholds on p_t after which the CP sponsors only the minimum quality or exits the sponsorship program.

In order to prove the theorem, we apply the first order optimality condition since the utility of the CP is concave. The proof is presented in the Appendix.

SP's Strategy: Now, having the optimum strategy of the CP in stage 2, we can find the optimum strategy for the SP:

Theorem 17. Equilibrium Strategy of Stage 1: The optimum strategies of the SP are:

$$\begin{aligned} \text{if } 0 < d_t \leq \frac{\hat{N}}{\zeta}, \quad (p_t^*, y_t^*) &= \begin{cases} (\operatorname{argmax}\{u_{SP,t}(p_t) : p_t \in P^*\}, 1) & \text{if } u_{SP,t}(p_t^*) \geq u_{SP,0} \\ (-, 0) & \text{if } u_{SP,t}(p_t^*) < u_{SP,0} \end{cases} \\ \text{if } d_t > \frac{\hat{N}}{\zeta} \text{ or } d_t = 0, \quad (p_t^*, y_t^*) &= (-, 0) \end{aligned} \tag{4.10}$$

where $P^* = \{\frac{\alpha d_t}{N}, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, \alpha \frac{\nu_1 d_t + \nu_2 D}{\nu_1 N}\}$ is the set of candidate optimum pricing strategies, and $u_{SP,0}$ is considered to be the utility of the SP in case of no-sponsoring, i.e. $\nu_2 D \log(\kappa_{SP} \frac{N}{D})$. In addition, the necessary condition for the candidate stable point $\alpha \frac{\nu_1 d_t + \nu_2 D}{\nu_1 N}$ to be an optimum is $\frac{\alpha d_t}{N} \leq \alpha \frac{\nu_1 d_t + \nu_2 D}{\nu_1 N} \leq \frac{\alpha}{\zeta}$. Note that the variable y_t determines whether the SP offers the sponsorship program or not, with $y_t = 1$ implying the offering.

Remark 8. *The immediate plausible range for the price per sponsored bit that one can think of is the interval between the lowest price that makes the CP to sponsor all the available bits and the highest price that makes the CP to sponsor only to satisfy the minimum desired quality. In Theorem 17, first, we narrow down this interval to prices between the highest price that makes the CP to sponsor all the available bits and the highest price that makes the CP to sponsor only to satisfy the minimum desired quality. Then, we characterize the interior optimum price. This choice is conditional on getting a payoff greater than or equal to the utility of the SP in case of no-sponsoring. Otherwise, the SP exits the sponsorship program.*

In order to prove the theorem, we use the monotonic behavior of the utility of the SP in some regions, and apply the first order optimality condition for the remaining regions. The proof is presented in the Appendix.

Corollary 10. *Choosing the price $\frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$ by the SP, i.e. the highest price by which the CP sponsors only to guarantee the minimum quality, renders the utility of the CP to be zero, and the CP to be indifferent between joining or not joining the sponsorship program.*

Proof. Results follow from (4.3), and that from Theorem 17, when $p_t = \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$, $d > 0$ and $b_t = \zeta d_t$ (from Theorem 16). □

Outcome of the Game: Now that we have characterized the SPNE at each time-epoch for a short-sighted CP and SP, the next step is to analyze the asymptotic behaviour of the market given the demand update function (4.1) and considering the one-shot game to be repeated infinitely. The goal is to characterize the asymptotically stable 5-tuple

equilibrium outcome of the game, i.e. (d, y, p, z, b) (table 4.1), if it were to exist. In the next Theorem, all possible asymptotically stable outcomes are listed. However, the existence of such a stable outcome is not guaranteed, and the market can be unstable in some cases.

Theorem 18. *The possible asymptotically stable outcomes of the game are:*

1. $(-, 0, -, 0, -)$: no sponsoring is offered, none taken.
2. $(\kappa_u \hat{N}, 1, \alpha \kappa_u, 1, \hat{N})$: the maximum bit sponsoring; if this is the stable outcome then $\kappa_u \leq \frac{1}{\zeta}$.
3. $(d, 1, \frac{\alpha \log(\kappa_{CP} \zeta)}{\zeta}, 1, \zeta d)$: the minimum quality sponsoring; if this is the stable outcome then $\kappa_u = \frac{1}{\zeta}$ and $0 < d \leq \frac{\hat{N}}{\zeta}$.
4. $(N \kappa_u - \frac{\nu_2}{\nu_1} D, 1, \alpha \kappa_u, 1, N - \frac{\nu_2}{\kappa_u \nu_1} D)$: the interior stable points; if this is the stable outcome, then $\kappa_u \leq \frac{1}{\zeta}$ and $0 < b = N - \frac{\nu_2}{\kappa_u \nu_1} D \leq \hat{N}$.

Remark 9. *Since the CP is shortsighted, in every stable outcome of the game, the strategy of the CP would be a myopic optimum strategy. Thus, using Theorem 16, one can expect the strategy of the CP to take one of the four possibilities in a stable outcome: (1) no sponsoring (2) sponsoring the maximum amount available (3) sponsoring only to satisfy the minimum required quality, or (4) sponsoring an optimum interior amount of bits. Subsequently, depending on the strategy of the CP, Theorem 17 characterizes the stable strategy of the SP. In order to prove the theorem we use Lemma 23. The proof is presented in the Appendix.*

Corollary 11. *There is no stable outcome involving sponsoring for the game if the stable quality is smaller than the minimum quality set by the CP, i.e. $\frac{1}{\kappa_u} < \zeta$.*

Remark 10. *Unlike other plausible stable outcomes, the third possible stable point, i.e. $(d, 1, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, 1, \zeta d)$ when $\zeta = \frac{1}{\kappa_u}$, can assume a range of different values. Whenever the SP sets $p = \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$, the CP sets $\frac{b_t}{d_t} = \zeta$, and the market will be stable. By choosing that price, the SP ensures that she will extract all the profit of the CP and makes her indifferent between joining the sponsorship program and opting out, i.e. $u_{CP}(b) = 0$ (using Corollary 10).*

In the next theorem, we find the stable demand that maximizes the payoff of the SP when she chooses the third stable point, i.e. the minimum quality.

Theorem 19. *Let $d^* = \frac{N}{\zeta} - \frac{1}{(\alpha + \nu_1) \log(\kappa_{SP}\zeta)}$. The payoff of the SP when the 5-tuple $(d, 1, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, 1, \zeta d)$ (the minimum quality stable point) is the stable outcome of the market is maximized when either (1) $d = \min\{d^*, \frac{N}{\zeta}\}$ and $d^* \geq 0$, or (2) $d = 0$ and $d^* < 0$.*

The proof of the theorem is presented in the appendix.

In the next sections, we investigate the case in the SP is long-sighted and the CP is short-sighted.

4.2.3 Long-Sighted SP, Short-Sighted CP

A long-sighted SP sets the per-bit sponsorship fee in order to achieve a stable market, i.e. a stable demand for the content, and also to maximize the payoff in the long-run:

$$U_{SP, Long Run}(\vec{p}) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T u_{SP,t}(p_t) \quad (4.11)$$

In this scenario, the SP is the leader of the game and therefore can set the equilibrium of the game individually by knowing that the CP is a myopic optimizer unit and follows the results in Theorem 16. Note that the long-sighted SP wants to asymptotically set a strategy that given the strategy of the CP, yields the highest profit. Thus, even with a long-sighted SP, the optimum strategy follows Theorem 17, and we can use the result in Theorem 18.

The difference between this case and the previous case is the ability of the long-sighted SP to choose between the candidate stable points in Theorem 18. Thus, the SP sets appropriate sponsoring fees at the beginning of the sponsoring program in order to asymptotically lock the stable outcome of the market in the chosen equilibrium.

Note that from Theorem 18, when $\kappa_u > \frac{1}{\zeta}$, there is no stable sponsoring outcome, and if $\kappa_u < \frac{1}{\zeta}$, depending on the parameters of the market, the stable point 2, i.e. maximum bit sponsoring, or 4, i.e. interior stable point, is chosen by the SP. In this case, if ν_2 , i.e. the importance of non-sponsored data for end-users and SP, is high enough, the stable point 4 is chosen and set by the SP. In addition, increasing the number of resources available with the SP, i.e. N , makes the stable point 2, i.e. maximum bit sponsoring, to yield the highest payoff, and thus is chosen by the SP. In the next theorem, we prove that when $\kappa_u = \frac{1}{\zeta}$, the stable point 3 yields the highest payoff.

Theorem 20. *If $\kappa_u = \frac{1}{\zeta}$, the minimum quality stable point, i.e. $(d, 1, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, 1, \zeta d)$, with the demand characterized in Theorem 19 yields the highest payoff for the SP.*

Remark 11. *Note that in a minimum quality stable point, the CP is indifferent, i.e. all profit of the CP is extracted by the SP. Therefore, we can expect this stable outcome to*

be the most favorable for the SP. Thus, a long-sighted SP sets this stable point as the asymptotic outcome of the market when $\kappa_u = \frac{1}{\zeta}$.

The proof of the theorem is presented in the appendix.

4.2.4 Short-Sighted SP, Long-Sighted CP

Consider a CP that chooses b_t in order to achieve a stable demand and to maximize the payoff in the long-run:

$$U_{CP, Long\ Run}(\vec{b}) = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T u_{CP,t}(b_t) \quad (4.12)$$

In the next theorem, we prove that for a long-sighted CP, the maximum bit sponsorship yields the highest payoff amongst the stable outcomes characterized in Theorem 18. Note that if the CP sponsors all the available units at the start of the sponsoring program, the sudden increase in the demand may push the market to the stable outcome of no sponsoring. Thus, given that the SP is short-sighted, the CP sets the number of bits for sponsoring appropriately over time, in order to achieve the demand of $\kappa_u \hat{N}$ eventually. With this demand and $b = \hat{N}$, the market would be stable. However, note that not the SP is the leader of the game and may chooses a price other than $\alpha \kappa_u$, i.e. the price in the maximum bit sponsoring. In this case, the CP cannot set the stable outcome she prefers. Thus, the CP is forced to set a stable outcome that is also preferable for a short-sighted SP.

Theorem 21. *The 5-tuple plausible stable sponsoring points, characterized in Theorem 18, in a decreasing order of the utility they yield to the CP are: maximum bit sponsorship, interior stable point, and minimum quality.*

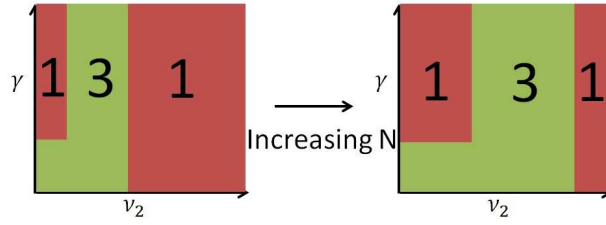


Figure 4.3: Market Asymptotic Outcomes with Short-Sighted Decision Makers when $\kappa_u = \frac{1}{\zeta}$

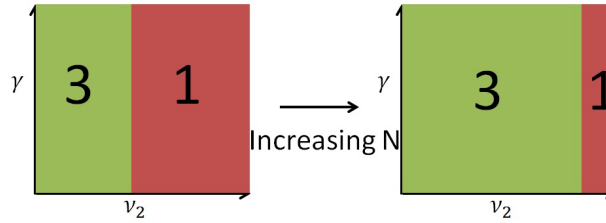


Figure 4.4: Market Asymptotic Outcomes when One of the Decision Makers is Long-Sighted, when $\kappa_u = \frac{1}{\zeta}$

Remark 12. *In order to establish the results, note that the stable point of no sponsoring is not considered a sponsoring stable point. Thus not listed in the theorem. In addition, since the CP is indifferent in the minimum quality stable point, this point should be the least favorite one for the CP. The ordering of the maximum bit and the interior stable points follows from the fact that the payoff of the CP is strictly increasing in those outcomes. The proof is presented in the Appendix.*

4.2.5 Numerical Results

In this section, we consider at least one of the SP and the CP to have a short-sighted model, and investigate the effects of ζ , κ_u , v_2 , N , and γ on the asymptotic outcome of the market. The fixed parameters considered are $\nu_1 = 1$, $\hat{N} = 25$, $D = 50$, $\kappa_{SP} = \kappa_{CP} = 10$,

and $\zeta = 0.3$. We observe the effect of important parameters such as κ_u , the sensitivity of the demand to the quality (γ), the weight an SP assigns to non-sponsored data (ν_2), and the total number of available bits in an LTE frame (N) on the asymptotic outcome of the market.

Market asymptotic outcomes for the case of $\kappa_u = \frac{1}{\zeta}$ when both decision makers are short-sighted and when one of them is long-sighted are presented in Figures 4.3 and 4.4, respectively. Similar plots for the case of $\kappa_u = \frac{1}{2\zeta} < \frac{1}{\zeta}$ are presented in Figures 4.5 and 4.6, respectively. Recall from Theorem 18 that the asymptotic stable outcome of the game is one of the four candidates: 1. No-Sponsoring, 2. Maximum bit sponsoring, 3. Minimum quality, and 4. Interior stable point. In the figures, each of the numbers are corresponding to one of the candidates. We also denote the unstable outcome by 0.

Next, we discuss about the effect of the framework and parameters on the asymptotic outcome of the game:

Impact of a decision-maker with long-sighted model: Note that in both cases $\kappa_u = \frac{1}{\zeta}$ and $\kappa_u < \frac{1}{\zeta}$, the outcome of the market is independent of the CP or the CP being long-sighted or the SP being long-sighted. The reason is that in the sequential game the SP is the leader of the game. Thus, although a long-sighted SP can set the stable outcome she prefers in the long-run, a long-sighted CP cannot enforce the most preferred stable outcome, and should choose the stable outcome that is also preferable for the SP. This yields the same asymptotic outcome for the market as the case that the SP is long-sighted.

Impact of the minimum quality (ζ) and the stable quality ($\frac{1}{\kappa_u}$): Theorem 18 implies that depending on the relation between the minimum quality set by the CP (ζ) and the stable quality ($\frac{1}{\kappa_u}$), the market has different stability outcomes in the short-

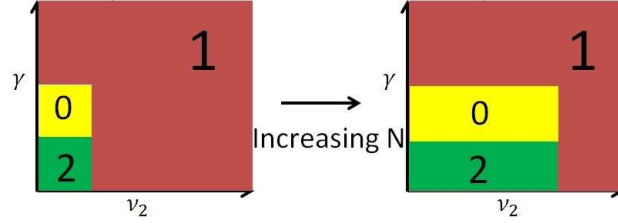


Figure 4.5: Market Asymptotic Outcomes with Short-Sighted Decision Makers, when $\kappa_u < \frac{1}{\zeta}$

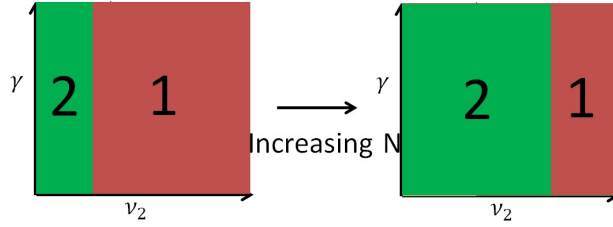


Figure 4.6: Market Asymptotic Outcomes when One of the Decision Makers is Long-Sighted, when $\kappa_u < \frac{1}{\zeta}$

sighted scenario. We seek to identify the stable outcomes that arise in each different parameter ranges:

- Using Corollary 11, If the CP *over-provisions* the minimum quality for the satisfaction of users ($\zeta > \frac{1}{\kappa_u}$), there is no stable sponsoring outcome since the demand of users grows drastically forcing the SP and CP to exit the sponsoring program. Thus, we do not study this case through simulations.

- If the CP *under-provisions* the minimum quality ($\zeta < \frac{1}{\kappa_u}$), simulation results in Figure 4.5 reveal that the market is set on the maximum bit sponsoring stable outcome for a particular range of parameters. However, the market is unstable or has the stable outcome of no-sponsoring for the rest of parameters.

- If the CP sets the minimum quality equal to the stable quality ($\zeta = \frac{1}{\kappa_u}$), the market

is set either on the no-sponsoring or the minimum quality stable outcomes. Comparing Figures 4.3, 4.4, 4.5, and 4.6 reveals that in the case of $\kappa_u = \frac{1}{\zeta}$, the market is more likely to have a stable outcome that involves sponsoring.

Impact of the sensitivity of the demand to the quality, γ : Note that, in Figures 4.3 and 4.5, and in general, increasing the value of γ shifts the stable outcome of the market from sponsoring to no-sponsoring. The exception is a range of v_2 for the case $\kappa_u = \frac{1}{\zeta}$, which we explain about when we discuss about the impact of v_2 later.

Therefore, in a market with short-sighted entities, the sensitivity of the demand to the quality, i.e. γ , greatly influences the stability of the market. When γ is high, the satisfaction and subsequently the demand of end-users increases/decreases drastically with small changes in the rate perceived by them. Thus, players would exit the sponsorship program since the demand may go down to zero or the demand may exceed $d_{max} = \frac{\hat{N}}{\zeta}$ ¹⁵, i.e. the jump in the demand decreases the quality received by the users below the requested minimum quality (ζ) which leads the CP to stick to the best-effort scenario. On the other hand, if γ is small, the market is more likely to be set on a sponsoring stable outcome.

Thus, in order to have a stable outcome of sponsoring, γ should be sufficiently small. Note that this parameter is small for a CP whose users are less sensitive to the quality they receive, such as shopping websites. This is in contrast with streaming websites whose users are sensitive to the quality (high γ). The parameter γ is also small for a CP which has a well-established end-user side, i.e. a more stable demand, such as Google, in contrast with the emerging CPs and start ups whose demand usually fluctuate more. Thus, in a short-sighted setting, the QSD may not be a viable option for streaming websites and

¹⁵the highest number of end-users that can be satisfied with the minimum quality.

emerging CPs.

In addition, note that the effect of γ would be canceled if one of the decision makers is long-sighted. This implies that in QSD framework, CPs with volatile demand (high γ) should be long-sighted to have a market with a stable outcome that involves sponsoring the content.

Impact of the importance of non-sponsored contents, v_2 : The parameter v_2 being large, when v_1 is normalized to one, represents the fact that the SP assigns more weight to the satisfaction of users for using non-sponsored content. Thus, the SP wants to restrict the number of bits she offers for sponsoring, and the best strategy for the SP is to set her per-bit sponsorship fee high enough so that the CP sponsors a smaller number of bits. Thus, we expect the market to have a stable outcome of no-sponsoring when v_2 is high.

Results in Figures 4.3, 4.4, 4.5, and 4.6 confirm that the market has the stable outcome of no-sponsoring when v_2 is large. One of the differences between the cases $\kappa_u = \frac{1}{\zeta}$ and $\kappa_u < \frac{1}{\zeta}$ is that when $\kappa_u = \frac{1}{\zeta}$, for a certain range of v_2 , the stable sponsoring outcome is 3, i.e. the minimum quality stable outcome, regardless of γ , i.e. the sensitivity of the demand to the quality. Next, we explain the reason for this behavior. Note that, as we mentioned, v_2 being high is associated with lower bits sponsored. Thus, for a certain range of v_2 , we expect the CP to start the sponsoring program with a quality near the minimum quality (since the CP wants to sponsor at least the minimum quality). In the case that $\kappa_u = \frac{1}{\zeta}$, (4.1) implies that the demand increases more slowly (the logarithm in the expression is smaller). Thus, the effect of γ is not significant, and the market can be stabilized on the minimum quality stable point regardless of γ . However, this does not

happen for the case of $\kappa_u < \frac{1}{\zeta}$, since in this case, from (4.1), the demand of end-users diminishes to zero¹⁶, and the market is set on the stable outcome of no-sponsoring.

Impact of total available resources, N : Figures 4.3, 4.4, 4.5, and 4.6 reveal that increasing the number of available resources (N) stretches the regions. In other words, as N increases, results would be similar to that of smaller v_2 's.

For example, in Figure 4.3, increasing the number of available bits (resources) increases the area of no-sponsoring region for small v_2 . This is *counter-intuitive* since one can expect that increasing the amount of available resources should facilitate sponsoring the content. This counter-intuitive result is due to the fact that by increasing N , the value of the SP for each bit decreases and the SP sets a lower sponsoring fee. This leads to sponsoring more bits by the CP which leads to a significant increase in the demand for the content when γ is large. This derives the market to the point of no-sponsoring. Therefore, the outcome is the same as the case in which v_2 is very small: the minimum quality stable point when γ is small and no-sponsoring when γ is large.

Impact of v_1 and v_2 on the social welfare: If we define the social welfare of the QSD regime as the sum of the payoffs of the CP and the SP¹⁷, then important parameters for determining the social welfare of the system are v_1 and v_2 (can be imposed by the regulator), and D . In this case, if $\frac{v_2}{v_1}$ or D are high, i.e. when the weight on the content

¹⁶Note that in this case the logarithm in (4.1) is negative for a quality near the minimum quality.

¹⁷Note that the payoff of the SP includes a term for users' satisfaction function that captures the welfare of EUs for sponsored and non-sponsored contents (possibly with constants different from v_1 and v_2). In addition, the effect of the model on other CPs is also hidden in the users' satisfaction function (the term $v_2 D \log(\kappa_{SP} \frac{N-b\epsilon}{D})$). Thus, sum of the utility of the CP and the SP (with possibly different v_1 and v_2) is a good indicator of the social welfare.

of a non-sponsored content is high, then the SP restricts the number of bits she offers for sponsoring by quoting a high sponsorship fee (as explained before). Thus, either the CP reserves a smaller number of bits or exits the sponsorship program. In either cases, the outcome would be aligned with maximizing the social welfare. Thus, v_1 and v_2 can be imposed by the regulator to control the social welfare.

Remark 13. *Figure 4.5 and 4.6 illustrate that the stable point 4, i.e. interior stable point, does not emerge, and only stable points 1 and 2 occur. In other words, in a stable outcome, when stakeholders of the market are short-sighted, either the CP sponsors all the available resources or no sponsoring occurs. Note that in the stable point 4, the number of bits sponsored by the CP in the equilibrium is $N - \frac{\nu_2}{\kappa_u \nu_1} D$. In addition, a stable sponsoring 5-tuple occurs only when ν_2 is small which makes $N - \frac{\nu_2}{\kappa_u \nu_1} D > \hat{N}$ for a wide range of parameters. Thus, the stable point 4 does not emerge in many scenarios. One can argue that by decreasing N or increasing D , we may have a scenario in which $N - \frac{\nu_2}{\kappa_u \nu_1} D < \hat{N}$. However, note these changes, is similar to having a large v_2 . Thus, similar to previous arguments, in this case, the SP is willing to set a price so high that leads the CP and market to a no-sponsoring outcome. Therefore, again in the regions that support an interior sponsoring solution, i.e. when $N - \frac{\nu_2}{\kappa_u \nu_1} D < \hat{N}$, the stable outcome 4 would not occur. None of the parameters we considered results in such a stable outcome.*

In the next section, using a bargaining framework, we investigate the scenario in which the decision makers have a *long-sighted model*, i.e. consider the effect of their decisions on the evolution of the demand and subsequently their payoff.

4.3 Bargaining Framework: NBS Analysis

In the previous section, we proved that a long-sighted CP and SP can prefer different stable outcomes, i.e. the stable outcomes that yield the highest payoff for them. If both decision makers are long-sighted, since multiple asymptotic outcomes are plausible, playing a sequential game may lead to a Pareto-inefficient outcome in the long-run¹⁸. Therefore, when both of the decision makers are long-sighted, it is natural to consider a bargaining game framework¹⁹. A bargaining game provides the framework to model the scenario in which two selfish agents can cooperatively select an equilibrium outcome (possibly among multiple equilibria) when non-cooperation, i.e. *disagreement*, yields Pareto-inefficient results. Note that both cases (multiple Nash equilibria and Pareto-inefficient outcome) occur in our modeling in Section 4.2 when at least one of the decision makers is short-sighted. After selecting the equilibrium, the division of profits can be characterized using the

¹⁸ The strategies of the SP and the CP are Pareto-inefficient in the long-run if at least one of the CP or the SP can increase her payoff, by changing her strategy, without decreasing the other player's payoff. An example of an inefficient outcome that occurs in our model is when v_2 is small and γ is large and both players are short-sighted. According to Figure 4.3 the asymptotic outcome of the game would be the no-sponsoring outcome. On the other hand, in Figure 4.4, with the same parameters, when one of the decision makers is long-sighted (which means that she chooses different strategies to maximize her long-run payoff), the asymptotic outcome of the market would be the minimum quality sponsoring outcome in which the SP receives a strictly higher payoff than the previous case. Thus, the outcome of the sequential game, can be Pareto inefficient in the long-run.

¹⁹We can also consider a bargaining game when decision makers are short-sighted. However, in this chapter, we consider two extreme scenarios: (1) non-cooperation/at least one decision maker short-sighted and (2) cooperation/both long-sighted, and compare the outcome of the market in these two extreme cases.

bargaining game frameworks considering the *bargaining power* of each decision maker.

In Section 4.3.1, we formulate and analyze the bargaining game. In Section 4.3.2, we present numerical results for this framework and discuss about the results.

4.3.1 Nash Bargaining Solution (NBS)

Thus, we formulate the interaction between the CP and the SP as a bargaining game, and use the *Nash Bargaining Solution* (NBS) to characterize the bargaining solution to the problem when both the SP and the CP are long-sighted.

Definition 14. Nash Bargaining Solution (NBS): *is the unique solution (in our case the tuple of the payoffs of the CP and the SP) that satisfies the four “reasonable” axioms (Invariant to affine transformations, Pareto optimality, Independence of irrelevant alternatives, and Symmetry) characterized in [53].*

Let $0 \leq w \leq 1$ be the relative bargaining power of the CP over SP: the higher w , more powerful is the bargaining power of a CP. In addition, u_{CP} and u_{SP} denote the payoff of the CP and SP respectively, and d_{CP} and d_{SP} denote the payoff each decision maker receives in case of disagreement, i.e. *disagreement payoff*. In order to characterize the disagreement payoffs, we assume that in case of disagreement, the SP and the CP will interact as short-sighted entities playing the sequential game previously described²⁰. Thus, the disagreement payoffs can be found by determining the asymptotic status of the market: the asymptotic payoff of the CP and the SP if the market is asymptotically stable,

²⁰The reason is that if in the case of disagreement, the CP and the SP continue their selfish non-cooperative behavior, they can obtain a payoff higher than or equal to the payoff of no-sponsoring. The inequality is strict for the cases that a sequential game yields a sponsoring outcome.

or the average payoffs if the market is unstable. Note that the value of the disagreement payoff for the CP and the SP can have an effect similar to the bargaining power (w for the CP and $1 - w$ for the SP).

Using standard game theoretic results in [53], the pair of u_{CP}^* and u_{SP}^* can be identified as the Nash bargaining solution of the problem if and only if it solves the following optimization problem:

$$\begin{aligned} & \max_{u_{CP}, u_{SP}} (u_{CP} - d_{CP})^w (u_{SP} - d_{SP})^{1-w} \\ & \text{s.t.} \end{aligned} \tag{4.13}$$

$$(u_{CP}, u_{SP}) \in U$$

$$(u_{CP}, u_{SP}) \geq (d_{CP}, d_{SP})$$

where U is the set of feasible payoff pairs. Note that the long-sighted SP and CP want to set a stable market in the long-run²¹, and based on Lemma 23 in a stable outcome $\frac{b}{d} = \frac{1}{\kappa_u}$. Thus, the expressions for u_{CP} and u_{SP} in a stable outcome (using (4.3) and (4.24)) are functions of the demand (d) and as follows:

$$u_{CP}(d) = u_{ad}(d) - p \frac{d}{\kappa_u} \tag{4.14}$$

$$u_{SP}(d) = p \frac{d}{\kappa_u} + u_s(d) \tag{4.15}$$

where $u_{ad}(d) = \alpha d \log(\frac{\kappa_{CP}}{\kappa_u})$ is the advertisement profit for the CP, and $u_s(d) = \nu_1 d \log(\frac{\kappa_{CP}}{\kappa_u}) + \nu_2 D \log(\kappa_{SP} \frac{N - \frac{d}{\kappa_u}}{D})$ is the satisfaction of the end-users of the SP. In addition, $p \frac{d}{\kappa_u}$ is the

²¹An unstable market makes it difficult for the CP and the SP to make decisions, predict the demand, or manage the network. Thus, an unstable market has its implicit costs for the CP and the SP. This is the reason that we assumed that the CP and the SP want to set an stable market.

side-payment transferred from the CP to the SP in exchange of securing a quality of $\frac{1}{\kappa_u}$ for the demand (d). Note that in Section 4.1, we introduced $u_{ad}(\cdot)$ and $u_s(\cdot)$ as functions of the number of sponsored bits (b). Here, we redefine them to be functions of demand (d) since in a stable outcome $d = \kappa_u b$.

Thus, the maximization (4.13) is over $d > 0$ and p . In addition, note that the maximum demand that can be satisfied with maximum available resources of \hat{N} to provide the quality of $\frac{1}{\kappa_u}$ is $\kappa_u \hat{N}$ ($d = \kappa_u b \leq \kappa_u \hat{N}$), which constitutes the feasible set. Thus the maximization is,

$$\begin{aligned} & \max_{d,p} (u_{CP} - d_{CP})^w (u_{SP} - d_{SP})^{1-w} \\ & \text{s.t.} \\ & 0 \leq d \leq \hat{N} \kappa_u \\ & u_{CP} \geq d_{CP} \\ & u_{SP} \geq d_{SP} \end{aligned} \tag{4.16}$$

We define p^* and d^* to be the optimum solution of (4.16). Note that p^* and d^* characterize the optimum division of profit (u_{CP}^* and u_{SP}^*) and thus the NBS. In addition, we define the *aggregate excess profit* to be the additional profit yielded from the cooperation in the bargaining framework:

Definition 15. Aggregate Excess Profit (u_{excess}): *The aggregate excess profit is defined as follows:*

$$u_{excess} = u_{CP} - d_{CP} + u_{SP} - d_{SP} = u_{ad} - d_{CP} + u_s - d_{SP} \tag{4.17}$$

Note that u_{excess} is independent of p and is only a function of d . We define $u_{excess}^* = u_{excess}|_{d=d^*}$. Note that the bargaining would only occur if $u_{excess}^* > 0$, i.e. the framework

creates additional joint profit that can be divided between the SP and the CP. Thus, henceforth, we characterize the NBS for the case that $u_{excess}^* > 0$. We use u_{excess} in the following theorem:

Theorem 22. *If $u_{excess}^* > 0$, the optimum solution of the optimization (4.16) is (d^*, p^*) in which $d^* = \arg \max_{0 \leq d \leq \hat{N}\kappa_u} u_{excess}$, and p^* is:*

$$p^* = \frac{\kappa_u}{d} \left((1-w)(u_{ad}^* - d_{CP}) - w(u_s^* - d_{SP}) \right) = \frac{\kappa_u}{d} \left((u_{ad}^* - d_{CP}) - w u_{excess}^* \right) \quad (4.18)$$

where $u_{ad}^* = u_{ad}|_{d=d^*}$, $u_s^* = u_s|_{d=d^*}$.

Remark 14. *The theorem characterizes p^* and d^* which directly lead to the NBS (using (4.14) and (4.15)), i.e. (u_{CP}^*, u_{SP}^*) . Based on the theorem, before splitting the profit, the SP and the CP cooperatively set a stable quality and subsequently a stable demand to maximize the aggregate excess profit, u_{excess} by solving the concave maximization problem $(\max_d u_{excess})$ with the single parameter d . Subsequently, they decide the split of the additional profit, i.e. the side payment paid to the SP by the CP $(p^* \frac{d^*}{\kappa_u})$, based on (4.18) which depends on the bargaining power each has (w and $1-w$). The proof of Theorem is presented in the Appendix.*

Remark 15. *As we mentioned before, the value of the disagreement payoffs can also play a similar role as the bargaining power (w). From (4.18): $d_{CP} \uparrow \Rightarrow p \downarrow$, and $d_{SP} \uparrow \Rightarrow p \uparrow$.*

Remark 16. Price vs. Bargaining Power: *The price per sponsored bit (4.18) is a decreasing function of w , i.e. the bargaining power of the CP: the higher the bargaining power of the CP the lower the side payment paid to the SP. It follows from (4.18) that there exists a threshold on w , $w_t = \frac{u_{ad}^* - d_{CP}}{u_{excess}^*}$, such that when $w > w_t$, $p^* < 0$, when $w < w_t$,*

$p^* > 0$, and when $w = w_t$, $p^* = 0$. In other words, for the CP with a bargaining power higher than the threshold w_t , the flow of money is reversed, and the SP pays the CP. This counter-intuitive case occurs either due to a high bargaining power of the CP (high w), or in the scenario that the SP gain significantly more than the CP from the cooperative scenario ($u_{excess}^* \gg u_{ad}^* - d_{CP}$, i.e. low w_t). For example, a powerful CP, e.g. Google, which already has a large established demand for the content might be reluctant to cooperate with the SP unless the SP pays some of the additional profit to it.

4.3.2 Numerical Results

Now, consider the SP and the CP with long-sighted business model that play a bargaining game as described in this Section. We investigate the effects of bargaining and cooperation between the CP and the SP in increasing the utility of each of them and stabilizing the market. In addition, we discuss about the relation between the number of available resources (N) and the Nash bargaining price (p^*).

We consider $w = 0.5$, i.e. the CP and the SP have the same bargaining power, and the values of v_1 , \hat{N} , D , κ_{SP} , and κ_{CP} to be the same as those considered in Section 4.2.5.

First, we plot the percentage of increase in the payoff of the CP and the SP after bargaining versus v_2 for different values of ζ when $\kappa_u = \frac{1}{\zeta}$ in Figures 4.7 and 4.8. The percentage of increase in the utility after bargaining is defined as follows:

$$\text{increase (percentage)} = \frac{\text{utility after bargaining} - \text{utility before bargaining}}{\text{utility after bargaining}} \times 100 \quad (4.19)$$

Note that when the utility before bargaining is zero and the utility after bargaining

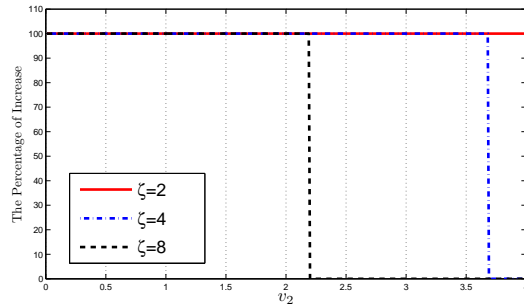


Figure 4.7: The percentage of increase in the utility of the CP when $\kappa_u = \frac{1}{\zeta}$ with respect to v_2 for different values of ζ .

is positive, then the increase in the utility, using (4.19), is 100 percent, and this value is zero if the utility after bargaining is equal to the utility before bargaining.

Results in Figure 4.7 reveal that the percentage of increase in the payoff of the CP is either zero or 100. Note that when $\kappa_u = \frac{1}{\zeta}$, the market either has a stable outcome of no sponsoring or a stable outcome of minimum quality sponsoring. In both cases, the utility of the CP is zero. Thus, if bargaining occurs, the CP would get a positive payoff, and the percentage of the increase in the utility of the CP would be 100. In Figure 4.7, we can see a threshold on v_2 after which the bargaining does not occur. This threshold is decreasing with respect to ζ . The reason is intuitive: even in a bargaining framework, due to limited resources, sponsoring does not occur if the CP needs a high quality to be sponsored, and/or the quality of non-sponsored data is important for the SP.

Results in Figure 4.8 reveal that the percentage of increase in the utility of the SP after bargaining is decreasing with respect to v_2 . In other words, the higher the importance of non-sponsored data for the end-users and subsequently the SP, the lower the incentive of the SP for participating in a bargaining game. Note that the case $\zeta = 2$ is corresponding

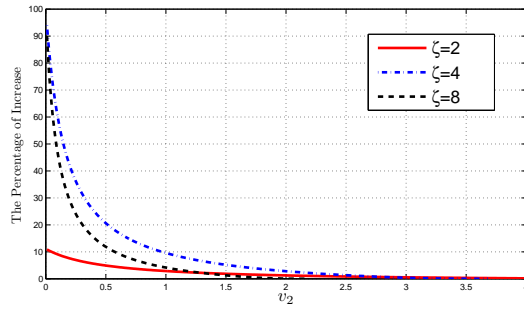


Figure 4.8: The percentage of increase in the utility of the SP when $\kappa_u = \frac{1}{\zeta}$ with respect to v_2 for different values of ζ .

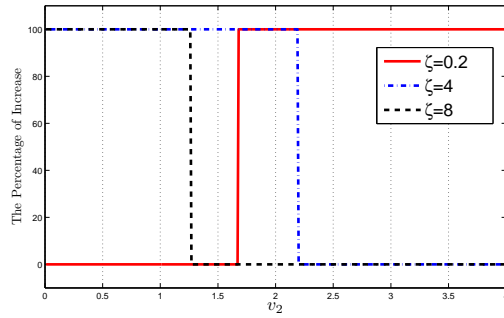


Figure 4.9: The percentage of increase in the utility of the CP when $\kappa_u = \frac{1}{2\zeta}$ with respect to v_2 for different values of ζ .

to a minimum quality stable outcome in the short-sighted framework. Thus, bargaining does not add greatly to the utility of the SP. On the other hand, $\zeta = 4$ and $\zeta = 8$ are corresponding to a stable point of no sponsoring in the short sighted framework. Therefore, the increase in the utility of the SP from bargaining is higher in these two cases than $\zeta = 2$. In addition, the percentage of increase is decreasing with respect to ζ . In other words, the higher the minimum quality needed to be sponsored, the lower the incentive of the SP for a bargaining framework.

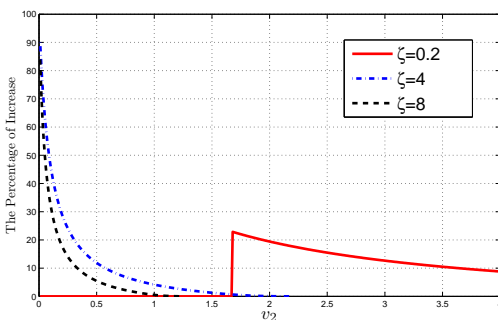


Figure 4.10: The percentage of increase in the utility of the SP when $\kappa_u = \frac{1}{2\zeta}$ with respect to v_2 for different values of ζ .

In Figures 4.9 and 4.10, the percentage of the increase in the payoff of the CP and the SP is plotted when $\kappa_u = \frac{1}{2\zeta}$. Note that for the case of $\zeta = 0.2$, when v_2 is small, the stable outcome of a short-sighted market would be the maximum bit sponsoring. Since, in this case, this stable outcome, yields the highest payoff for the SP and the CP, bargaining cannot create additional profit. Thus, the percentage of increase in the utility of the SP and the CP is zero up a threshold. For v_2 higher than this threshold, and the cases $\zeta = 4$ and $\zeta = 8$, the corresponding short-sighted outcome of the market is no sponsoring stable outcome. Thus, the results is similar to the previous figures (Figures 4.7 and 4.8).

Note that bargaining can enforce sponsoring for the set of parameters that have no stable sponsoring outcome in a sequential game. However, the bargaining framework cannot always enforce sponsoring. In particular, if the CP needs to sponsor a high quality (high ζ) for the content, or the quality of non-sponsored content is important for the end-users and subsequently the SP (high v_2), then sponsoring does not occur regardless of the framework used.

The next set of numerical results investigate the relation between the number of avail-

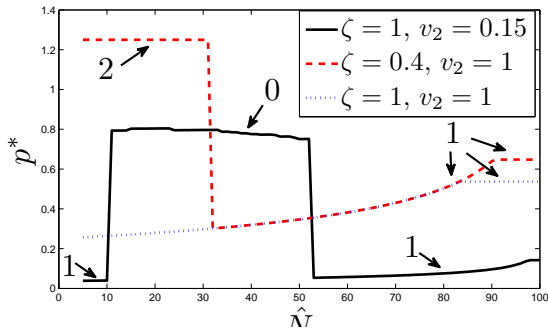


Figure 4.11: The price per bit in a Nash bargaining solution versus the available number of bits when $\kappa_u = \frac{1}{2\zeta}$.

able resources (\hat{N}) and the Nash bargaining price (p^*). Intuitively, one may expect that higher number of available resources yields a lower valuation of the SP for each unit of resources, and subsequently a lower price for each bit. While this line of thought seems to be true in the sequential framework, numerical results reveal a more complex relationship between p^* and \hat{N} in the bargaining framework: the negotiated price can be increasing, decreasing, or a combination of both (Figures 4.11 and 4.12).

The reason for this counter-intuitive behavior is the different disagreement payoffs resulting from different asymptotic outcomes of the game when decision makers are short-sighted. The disagreement payoffs can be considered as a form of bargaining power for each decision maker, and can affect the excess profit resulted by bargaining. Thus, different disagreement payoffs lead to different amounts of excess profit and its division between the CP and the SP, and subsequently different behavior of price per sponsored bit.

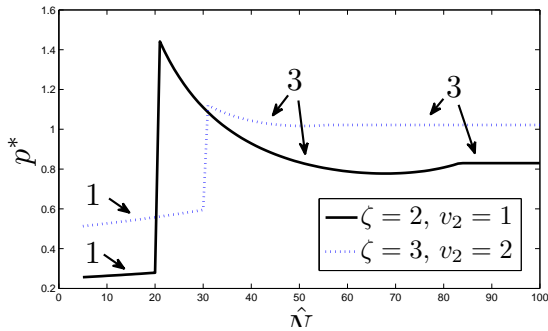


Figure 4.12: The price per bit in a Nash bargaining solution versus the available number of bits when $\kappa_u = \frac{1}{\zeta}$.

4.4 Discussions

First in Section 4.4.1, we present high-level perspective of the results. Then in Section 4.4.2, we discuss about the modeling and assumptions of this the work in this chapter, their implications, and their generalizations.

4.4.1 Summary of Key Results

We discussed that relation between the minimum quality the CP requests, i.e. ζ , and the stable quality, i.e. $\frac{1}{\kappa_u}$,²² is an important factor in determining the asymptotic outcome of the market (Section 4.2.5). In particular if the CP over-provisions the minimum quality, i.e. $\zeta > \frac{1}{\kappa_u}$, then there is no stable sponsoring outcome. The stability can be achieved when $\zeta < \frac{1}{\kappa_u}$ (under-provision). However, the set of parameters for which the market is stable is larger when $\zeta = \frac{1}{\kappa_u}$. Thus, a QSD framework is more likely to emerge for CPs that know the dynamic of their demand ($\frac{1}{\kappa_u}$) and are willing to disclose it (by requesting $\zeta = \frac{1}{\kappa_u}$). However, note that in a sequential framework and if $\zeta = \frac{1}{\kappa_u}$, the utility of the

²²Recall that the stable quality is defined in Definition 11.

CP would be zero, i.e. the additional profit of the CP by sponsoring the content would be fully extracted if the CP reveals the true value of the stable quality. Thus, the CP would be indifferent between this scheme and neutrality, while the SP receives a higher payoff in QSD scheme. While in a bargaining framework (if it happens), both the CP and the SP receive a higher payoff in comparison to a neutral framework. Thus, a bargaining framework is preferable especially for the CP.

We showed that a CP with a volatile demand, i.e. a CP whose users are sensitive to the quality (high γ), leads to a no-sponsoring outcome in a sequential framework (non-cooperative scenario) if both the SP and CP are short-sighted. Examples of such CPs are streaming websites and emerging CPs (start ups). Thus, a QSD framework is not a viable scenario in long run for these CPs, if the decision-makers are short-sighted. For these CPs, we can expect a stable QSD framework only if one of the CP or the SP is long-sighted, or in a bargaining game framework. In addition, we showed that even in a bargaining framework, an SP who assigns high weights to the satisfaction of users that use the non-sponsored data (high $\frac{v_2}{v_1}$), chooses to not sponsor the content of a CP who needs high quality (high ζ).²³

Moreover, results reveal that investment by the SP is not always in favor of having a stable QSD. Increasing the number of available resources for sponsoring (investment by the SP), when at least one of the decision makers is long-sighted, increases the range of the parameters by which a stable QSD framework occurs. However, when both the SP

²³As explained in the model, the parameter v_2 can also represent the regulatory policies for the quality of the experience of the users that use the non-sponsored content. In this case, a high v_2 is corresponding to stricter (net-neutrality) rules.

and CP are short-sighted, increasing the number of resources may lead to a scenario in which the CP sponsors a large number of resources. If the demand is volatile, this yields a sudden jump in the demand, and drives the market to a no-sponsoring outcome (the jump in demand decreases the quality below the minimum quality), and leads the CP to use the best-effort scenario.

4.4.2 Comments on the Assumptions of the Model

We assume logarithmic functions for the demand update function and utilities owing to its concavity. However, our analysis and insights are expected to be applicable to other concave functions with diminishing returns.

Note that the focus of this work is on the interaction between an SP and a CP, and not on the competition among SPs and CPs. In particular, we consider that only one CP wants to sponsor a quality for her users, and the rests stick to the best effort scenario. The effects of other CPs are considered by the SP as part of her utility. Introducing competition among CPs and SPs would introduce another level of strategic decisions by them. It does not necessarily alter the high-level intuitions for the interaction of the CP and the SP provided in this work. For example, we can expect that even under competition, a CP with a volatile demand would not be a good options for a QSD framework in a non-cooperative scenario. However, considering the competition among CPs provides intuitions on the possible structure of the Internet market in future under a QSD framework. For example, a possible outcome would be the case that competitive CPs divide SPs (and subsequently end-users) among themselves and each sponsors the quality of the content on only one of the SPs. Using this, each CP can secure a monopoly over users. This would be a mild

version of the “Internet fragmentation” which might be an undesirable outcome for users and from the perspective of the FCC. A possible direction for future work is to consider the competition over end-users among ISPs and CPs.

In addition, we assume that quality is sponsored by reserving a number of resources, e.g. LTE time-frames. In general, SPs can sponsor high quality for users of CPs using various methods, e.g. by prioritization of the content of a CP. Analyzing different methods of sponsoring the quality of a content is beyond the scope of this work.

4.5 Conclusion

We introduced the problem of quality-sponsored data (QSD) in cellular networks and studied its implications on market entities in sequential and bargaining game frameworks in various scenarios. The direct coupling between the scarce (wireless) resources and the market decisions resulting from QSD has been taken into account, Subgame Perfect Nash Equilibrium and Nash Bargaining Solution of the problem is characterized, and the market dynamics and equilibria have been investigated. We provided strategies for (i) SPs: to determine if and how to price resources, and (ii) CPs: to determine if and how many resources to sponsor (what quality). In this work, we focused on the interaction between ISPs and CPs. A possible direction for future work is to consider the competition over end-users among ISPs and CPs. Another direction is to consider the effects of QSD on the payments of user to SPs, and its implications on the results.

4.6 Appendix

4.6.1 Proof of Theorem 16

First we consider the case in which $d_t > \frac{\hat{N}}{\zeta}$. In this case, $\zeta d_t > \hat{N}$. Therefore, there is no feasible solution for b_t . Thus, as we mentioned previously after (4.7), in this case of infeasibility, the CP exits the sponsorship program, i.e. $z_t^* = 0$. In addition, from (4.3), $d_t = 0$ yields $u_{CP,t}(b_t) < 0$ for every $b_t > 0$, and subsequently $z_t^* = 0$. This completes the proof of (4.9).

Thus, henceforth, we consider $0 < d_t \leq \frac{\hat{N}}{\zeta}$. Clearly, the utility of the CP (4.3) is concave. Thus, the first order optimality condition provides us with the candidate optimum answer for (4.7). The first order condition yields that $\hat{b}_t = \frac{\alpha d_t}{p_t}$. In order to be an optimum answer, \hat{b}_t should be feasible, i.e. $\zeta d_t \leq \hat{b}_t \leq \hat{N}$. This characterizes a region for p_t , $\frac{\alpha d_t}{\hat{N}} \leq p_t \leq \frac{\alpha}{\zeta}$. In order to determine z^* , we should check non-negativity of $u_{CP,t}^*$. The utility of the CP with $\hat{b}_t = \frac{\alpha d_t}{p_t}$ is non-negative if $p_t \leq \frac{\alpha \kappa_{CP}}{e}$. Since $\zeta \kappa_{CP} > e$,²⁴ $\frac{\alpha}{\zeta} < \frac{\alpha \kappa_{CP}}{e}$. Therefore, a feasible solution for (4.8) yields a non-negative payoff. Thus, $z_t^* = 1$. This is the second region from top in (4.8).

If $p_t \leq \frac{\alpha d_t}{\hat{N}}$, then the top boundary condition $b_t^* = \hat{N}$ is the optimum answer of (4.8). In addition, since in this region $u_{CP,t}(\hat{N})$ is positive, $z_t^* = 1$. This is the first optimality region of (4.8). On the other hand, if $p_t \geq \frac{\alpha}{\zeta}$, then the lower boundary condition, i.e. $\bar{b}_t = \zeta d_t$, is the optimum answer of the optimization. The condition for $u_{CP,t}(\bar{b}_t) \geq 0$ and therefore $z_t^* = 1$ is $p_t \leq \frac{\alpha \log(\kappa_{CP} \zeta)}{\zeta}$ which yields the third optimality region in (4.8). If $p_t > \frac{\alpha \log(\kappa_{CP} \zeta)}{\zeta}$, $u_{CP,t}(b_t) < 0$. Thus, $z_t^* = 0$. This concludes the proof.

²⁴The condition to have a non-trivial problem stated in Section 4.1.

4.6.2 Proof of Theorem 17

Theorem 16 implies that if $d_t > \frac{\hat{N}}{\zeta}$, or $d_t = 0$, or $p_t > \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$, the CP does not participate in the sponsoring program. Thus, the value of y_t^* does not affect the outcome of the market in these cases. Without loss of generality, we assume that in these cases the SP does not offer the program, i.e. $y_t^* = 0$.

Thus, henceforth, we consider $0 < d_t \leq \frac{\hat{N}}{\zeta}$ and $p_t \leq \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$. Note that in this region, by Theorem 16, $b_t > 0$. Thus, the SP maximization problem is,

$$\max_{p_t} u_{SP,t}(p_t) = \max_{p_t} \left(p_t b_t^* + \nu_1 d_t \log \left(\frac{\kappa_{SP} b_t^*}{d_t} \right) + \nu_2 D \log \left(\kappa_{SP} \frac{N - b_t^*}{D} \right) \right), \quad (4.20)$$

where b_t^* is the equilibrium outcome of the second stage. Let $p_t \leq \frac{\alpha d_t}{N}$. Then from Theorem 16, $b_t^* = \hat{N}$. Thus, $u_{SP,t}(p_t)$ is a strictly increasing function of p_t . Therefore, all prices less than $\frac{\alpha d_t}{N}$ yields a strictly lower payoff than $p_{1,t}^* = \frac{\alpha d_t}{N}$, which is the first candidate pricing strategy. Next, let $\frac{\alpha}{\zeta} \leq p_t \leq \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$. Thus, from Theorem 16, $b_t^* = \zeta d_t$. Again, in this region, $u_{SP,t}(p_t)$ is a strictly increasing function of p_t . Thus, $p_{2,t}^* = \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$ strictly dominates all other prices in this interval, which yields the second candidate pricing strategy²⁵. For the case that $\frac{\alpha d_t}{N} \leq p_t \leq \frac{\alpha}{\zeta}$, from Theorem 16, $b_t^* = \frac{\alpha d_t}{p_t}$. In this region, the first order condition on $u_{SP,t}(p_t)$ provides us with the local extremum,

$$p_{3,t}^* = \alpha \frac{\nu_1 d_t + \nu_2 D}{\nu_1 N} \quad (4.21)$$

Since the second order derivative can be negative or positive, the first order condition provides us with only a candidate optimum answer, which is the third candidate pricing

²⁵ Note that $p_{2,t}^* = \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$ yields a payoff of zero for the CP. However, since we have assumed that the indifferent CP chooses to join the sponsorship program, $z_t^* = 1$ and subsequently $y_t^* = 1$.

strategy. This candidate strategy should satisfy the condition $\frac{\alpha d_t}{N} \leq p_{3,t}^* \leq \frac{\alpha}{\zeta}$. If not, it would not be an optimum answer since, as we discussed earlier in the proof, every price less than (respectively, higher than) $\frac{\alpha d_t}{N}$ (respectively, $\frac{\alpha}{\zeta}$) is dominated by $\frac{\alpha d_t}{N}$ (respectively, $\frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$)²⁶. Note that these candidate strategies are optimum only if they yield a payoff higher than the payoff of the SP in the case of no-sponsoring, i.e. $v_2 D \log(\kappa_{SP} \frac{N}{D})$. The result follows.

4.6.3 Proof of Theorem 18

We characterize the possible stable outcomes of the game when at each time t , the SP and the CP choose their strategy to be the SPNE of the game characterized in Theorems 16 and 17.

The first candidate stable outcome is trivial: as soon as one of the CP or SP exits the sponsorship program, or $d_t > \frac{\hat{N}}{\zeta}$, or $d_t = 0$, the program will not be resumed.

Now consider the case that sponsoring occurs. In this case, $y = 1$ $z = 1$, and from Theorem 17, the SP chooses one of the candidate optimum pricing strategies from the set $P^* = \{ \frac{\alpha d_t}{N}, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, \alpha \frac{\nu_1 d_t + \nu_2 D}{\nu_1 N} \}$. We show that the first, the second, and the third candidate pricing strategies are corresponding to the second, the third, and the fourth stable outcome, respectively. Note that when choosing these prices, by Theorem 17, the demand should be feasible, i.e. $0 < d_t \leq \frac{\hat{N}}{\zeta}$. In addition, recall that by Lemma 23, the demand is stable when $d = \kappa_u b$.

Now, we obtain the second stable outcome by considering that $p = \frac{\alpha d}{N}$ and $0 < d_t \leq \frac{\hat{N}}{\zeta}$. In this case, from Theorem 16, $b = \hat{N}$. Thus $p = \alpha \kappa_u$ since $d = \kappa_u \hat{N}$. The feasibility

²⁶Note that from Theorem 16, prices higher than $\frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$ leads to no sponsoring on the CP side.

condition yields that $d = \kappa_u \hat{N} \leq \frac{\hat{N}}{\zeta} \Rightarrow \kappa_u \leq \frac{1}{\zeta}$ ²⁷.

Next, we obtain the third stable outcome by considering $p = \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$ and $0 < d_t \leq \frac{\hat{N}}{\zeta}$. In this case, from Theorem 16, $b = \zeta d$, and subsequently from the stability condition, $d = \kappa_u b = \kappa_u \zeta d$. Therefore, this case occurs if $\kappa_u \zeta = 1$. Note that the demand could be any positive value less than or equal to $\frac{\hat{N}}{\zeta}$ (feasibility condition), and with this demand, $0 < b = \zeta d \leq \hat{N}$.

Finally, the fourth possible stable outcome happens when $p = \alpha \frac{\nu_1 d + \nu_2 D}{\nu_1 N}$, $p \in [\frac{\alpha d}{\hat{N}}, \frac{\alpha}{\zeta}]$ (from Theorem 17), and $0 < d_t \leq \frac{\hat{N}}{\zeta}$. In this case, from Theorem 16, $b = \frac{\alpha d}{p}$. In order to have a stable outcome, $d = \kappa_u b \Rightarrow p = \alpha \kappa_u$. Thus, from $p = \alpha \frac{\nu_1 d + \nu_2 D}{\nu_1 N}$, $d = N \kappa_u - \frac{\nu_2}{\nu_1} D$ and $b = N - \frac{\nu_2}{\nu_1 \kappa_u} D$. Note that b should satisfy $0 < b \leq \hat{N}$, and from Theorem 17, we know that $p = \alpha \frac{\nu_1 d + \nu_2 D}{\nu_1 N}$ is optimum if it is in the interval $[\frac{\alpha d}{\hat{N}}, \frac{\alpha}{\zeta}]$. The latter yields that $\frac{\alpha d}{\hat{N}} \leq p = \alpha \kappa_u \leq \frac{\alpha}{\zeta}$, which yields that $\kappa_u \leq \frac{1}{\zeta}$ and $b = \frac{\alpha d}{p} \leq \hat{N}$. Note that these conditions automatically lead to a feasible demand: from $b = \frac{\alpha d}{p} \leq \hat{N}$, then $d \leq \frac{\hat{N} p}{\alpha} = \hat{N} \kappa_u \leq \frac{\hat{N}}{\zeta}$. Thus, in this stable outcome, $\kappa_u \leq \frac{1}{\zeta}$ and $0 < b \leq \hat{N}$. The result follows.

4.6.4 Proof of Theorem 19

By (4.24), the utility of the SP when choosing the tuple $(d, 1, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, 1, \zeta d)$ is:

$$u_{SP} = \alpha d \log(\kappa_{SP}\zeta) + \nu_1 d \log(\kappa_{SP}\zeta) + \nu_2 D \log\left(\kappa_{SP} \frac{N - \zeta d}{D}\right)$$

First, note that the expression of the utility is concave in d . Thus, the first order condition gives the optimum answer. The solution of the first order condition is:

$$d^* = \frac{N}{\zeta} - \frac{1}{(\alpha + \nu_1) \log(\kappa_{SP}\zeta)}$$

²⁷Note that $d = \kappa_u \hat{N} > 0$.

Based on Theorem 18, for d^* to be the demand corresponding to the minimum quality stable outcome, it should satisfy the constraint $0 < d^* \leq \frac{\hat{N}}{\zeta}$. If $d^* > \frac{\hat{N}}{\zeta}$ or $d^* < 0$, the concavity implies that the optimum is $d = \frac{\hat{N}}{\zeta}$ or $d = 0$, respectively.

4.6.5 Proof of Theorem 20

First, note that in Theorem 18, when $\kappa_u = \frac{1}{\zeta}$, the stable points 2, 3, and 4 can occur. In addition, the demand is fixed in the stable point 2 and 4, while it can take a range of values for the stable point 3, including the fixed demands in the other two stable points. On the other hand, the price is fixed in all these three stable points. In these stable points, the stable quality is $\frac{b}{d} = \zeta$. Thus, by (4.24), the payoff of the SP is:

$$u_{SP} = p\zeta d + \nu_1 d \log(\kappa_{SP}\zeta) + \nu_2 D \log\left(\kappa_{SP} \frac{N - \zeta d}{D}\right)$$

Therefore, for a fixed demand, the payoff of the SP in this case is an increasing function of the price p . Note that the third stable point, i.e. minimum quality stable point, has the highest price among the possible stable points since $\log(\kappa_{CP}\zeta) > 1$.²⁸ In addition, it can take a range of demand including the fixed demands of the stable point 2 and 4. Thus, the third stable outcome of the market yields the highest payoff for the SP. The optimum demand is chosen by Theorem 19 as discussed before. The result follows.

4.6.6 Proof of Theorem 21

First note that from Corollary 10, the minimum quality stable point, i.e. $\left(d, 1, \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}, 1, \zeta d\right)$, yields a payoff of zero for the CP. From Lemma 23, in both the maximum bit sponsorship, i.e. $\left(\kappa_u \hat{N}, 1, \alpha \kappa_u, 1, \hat{N}\right)$, and interior stable point, i.e. $\left(N \kappa_u - \frac{\nu_2}{\nu_1} D, 1, \alpha \kappa_u, 1, N - \frac{\nu_2}{\kappa_u \nu_1} D\right)$,

²⁸In Section 4.1, we assumed that in order to have a non-trivial problem $\kappa_{CP}\zeta > e$

the stable quality ($\frac{b}{d}$) is $\frac{1}{\kappa_u}$. Thus, using (4.3), the payoff of the CP in these plausible stable outcomes is:

$$u_{CP} = \alpha d \left(\log \left(\frac{\kappa_{CP}}{\kappa_u} \right) - 1 \right)$$

Note that from the condition for plausibility of these stable points ($\kappa_u \leq \frac{1}{\zeta}$), and our previous assumption that $\kappa_{CP}\zeta > e$,²⁹ $\frac{\kappa_{CP}}{\kappa_u} > e$. Thus, the payoff of the CP in the maximum bit sponsorship and the interior stable point is strictly greater than zero, and is strictly increasing with respect to the demand. Given that the quality $\frac{b}{d} = \frac{1}{\kappa_u}$, and is a constant, the higher the number of sponsored bits, the higher the demand, and therefore the higher the payoff of the CP would be. In addition, note that the number of sponsored bits in the maximum bit sponsorship point is greater than or equal to the number of sponsored bits in the interior stable point. Thus, the utility of the CP in the maximum bit sponsoring point is greater than or equal to the utility in the interior stable point. The result follows.

4.6.7 Proof of Theorem 22

d_{CP} and d_{SP} are independent of d and p . In addition, $u_{excess} = (u_{CP} - d_{CP}) + (u_{SP} - d_{SP})$ is independent of p , and is only a function of d . Thus, for a given d , using equation (2) of [49], the optimum value of p is such that:

$$\frac{u_{CP} - d_{CP}}{w} = \frac{u_{SP} - d_{SP}}{1 - w}$$

²⁹The condition to have a non-trivial problem.

if the solution for p satisfies other constraints. Thus, by plugging the expressions for the CP and the SP ((4.14) and (4.15)), the candidate optimum p as a function of d is:

$$p^* = \frac{\kappa_u}{d} \left((1-w)(u_{ad} - d_{CP}) - w(u_s - d_{SP}) \right) = \frac{\kappa_u}{d} \left((u_{ad} - d_{CP}) - w u_{excess} \right) \quad (4.22)$$

Substituting (4.22) in the objective function of (4.16) and using (4.14) and (4.15) yield the new objective function:

$$w^w (1-w)^w (u_{ad} - d_{CP} + u_s - d_{SP}) = w^w (1-w)^w u_{excess}$$

Substituting (4.22), (4.14), and (4.15) in the constraint $u_{CP} \geq d_{CP}$, yields the new constraint $u_{ad} - d_{CP} + u_s - d_{SP} \geq 0$. Similar substitutions for $u_{SP} \geq d_{SP}$ yields the same constraint. Thus, the optimization can be written as,

$$\begin{aligned} & \max_d u_{excess} \\ & \text{s.t.} \end{aligned} \quad (4.23)$$

$$0 \leq d \leq \hat{N} \kappa_u$$

$$u_{ad} - d_{CP} + u_s - d_{SP} = u_{excess} \geq 0$$

The theorem follows from above and (4.22).

4.6.8 Comments on the Approximations in the Model

Note that in our model, we have assumed that either the CP sponsor a quality for her end-users or she uses the best effort scenario (both cannot happen together). This means that in the second case (no sponsoring) the demand of the CP would be added to the pool of the demand for the best effort scenario, i.e. would be added to D . In our model, we do not considered the augmentation since we naturally expect the demand for a CP

to be much smaller than the total demand for all CPs. In this section, we discuss if and how the results change if we consider this augmentation.

Change in the Model

The augmentation in the demand can be accommodated as follows:

The SP:

$$u_{SP,t}(p_t) = p_t b_t + u_s(b_t(p_t)) \quad (4.24)$$

where now the users' satisfaction function, i.e. $u_s(\cdot)$, becomes:

$$u_s(b_t) = \begin{cases} \nu_1 d_t \log\left(\frac{\kappa_{SP} b_t}{d_t}\right) + \nu_2 D \log\left(\kappa_{SP} \frac{N - b_t}{D}\right) & \text{if } d_t > 0 \& b_t > 0 \\ \nu_2 (D + d_t) \log\left(\kappa_{SP} \frac{N - b_t}{D + d_t}\right) & \text{Otherwise} \end{cases} \quad (4.25a)$$

Note that (4.25a) is the same as (4.5a). Thus, the only change is for the case of no sponsoring ($d_t = 0$ or $b_t = 0$) (4.25a) in which d_t is added to the total demand of the best effort scenario, i.e. D . Note that (4.25a) becomes similar to (4.5a) when $d_t \ll D$.

The CP:

Note that we have considered that the CP receives a payoff of zero in the case of no sponsoring. This is justified as in many cases in which if the CP does not sponsor the data, then she will only transmit the content with a best effort scenario and because of limited bandwidth do not transmit advertisements. An example of this can be seen in Youtube: If the quality of the content is low, then Youtube automatically skips the ad. Thus, in this case, when the CP transmits with best effort, it receives zero ad revenue.

Change in the Analytical Results

This change may only affect the results when (i) the exact expression of $u_s(b_t)$ in the case of no sponsoring, i.e. (4.25a), or (ii) the expressions for the optimum strategies of the SP, i.e. p_t^* is used. Note that in Theorem 1 we do not use any of (i) and (ii). Thus, the Theorem would be similar to before. In the next paragraph, we will argue that the expressions for p_t^* in Theorem 2 would be the same as before. In Theorem 3, we only use the expression for p_t^* . Thus, the results of Theorem 3 would be the same as before. In Theorems 4 and 5, we use (4.25a) (which is similar to (4.5a)) and the expression of p_t^* (which are the same as before). Thus, the the results for these theorems also would be as before. For the long-sighted case, we do not use the exact expression of $u_s(b)$. Thus, all the results of long-sighted would be as before.

Now, we argue that the expressions for the optimum strategies of the SP in Theorem 2 would be the same as before. The first paragraph of the proof would be the same as before since we do not use (4.25a). In addition, in the next paragraph of the proof and when characterizing the optimum strategies of the SP, we focus on $0 < d_t \leq \frac{\hat{N}}{\zeta}$ and $p_t \leq \frac{\alpha \log(\kappa_{CP}\zeta)}{\zeta}$. With these conditions, $b_t > 0$ and sponsoring occurs. Thus, we use the expression of $u_{SP}(p_t)$ for the case of sponsoring (4.25a) which is the same as before, i.e. (4.5a). Thus, the expressions for the optimum strategies of the SP would be the same as before.

Note that the only change that should be applied to Theorem 17 is to the expression of $u_{SP,0}$, i.e. the utility of the SP in the case of no sponsoring. This utility should be changed from (4.5a) to (4.25a), i.e. $u_{SP,0} = v_2(D + d_t) \log(\kappa_{SP} \frac{N}{D+d_t})$.

Change in the Simulation Results

Since the SP now receives a greater utility in the case of no sponsoring (compare (4.5a) with (4.25a)), the option of no-sponsoring becomes more attractive for the SP. We have redone all the simulations with the new model. We comment on all the changes in the numerical results, and present the results for one sample scenario (Figure 4.13).

Changes to Figures 3 to 6: In the numerical results, we observe that these figures will remain similar in general. The only change is that the region of no sponsoring for large v_2 slightly increases (since the no-sponsoring is now more attractive for the SP). Thus, the insights associated with these figures would be the same as before.

Changes to Figures 7 and 9: Now, consider the numerical results for the long-sighted scenario. In this case, for Figures 7 and 9, we observe the thresholds for the jump to no-sponsoring region slightly decreases (as we expect because of the explanations in the first paragraph of Appendix 4.6.8). Otherwise, the figures would be the same as before. This is because of the fact that the utility of the CP is the same as before.

Changes to Figures 8 and 10: We plot the counterpart of Figure 8, in Figure 4.13. Note that the results are similar. The only difference is that the percentage of increase in the utility of the SP decreases in some regions (regions in which short-sighted yields no sponsoring). This is because of the increase in the utility of the SP in the case of no sponsoring. The same happens to Figure 9. Thus, the insights associated with these figures remain the same.

Changes to Figures 11 and 12: Recall that p^* is the price of sponsored bits in the bargaining framework, and is distinct from p_t^* which is the price of sponsored bits in the

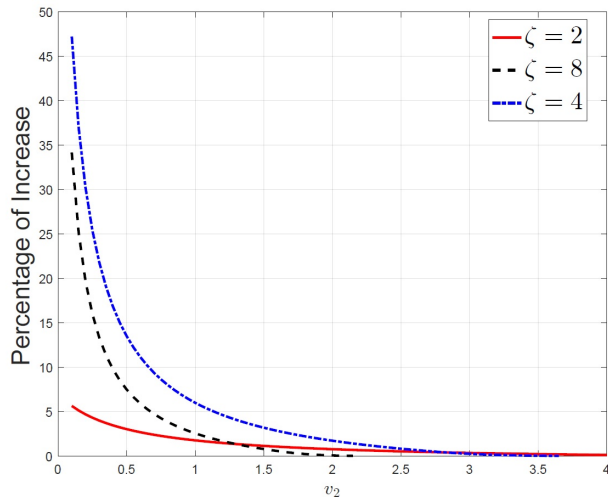


Figure 4.13: The percentage of increase in the utility of the SP when $\kappa_u = \frac{1}{\zeta}$ with respect to v_2 for different values of ζ (new model).

short-sighted framework. Results reveal that the insights associated with these figures follow the same trend as before. Note that p^* depends on the disagreement payoff which is the payoff of short-sighted framework. Thus, the only change to the value of p^* happens when the disagreement yields no sponsoring. In this case, since the payoff of disagreement increases slightly (4.25a), p^* increases slightly.

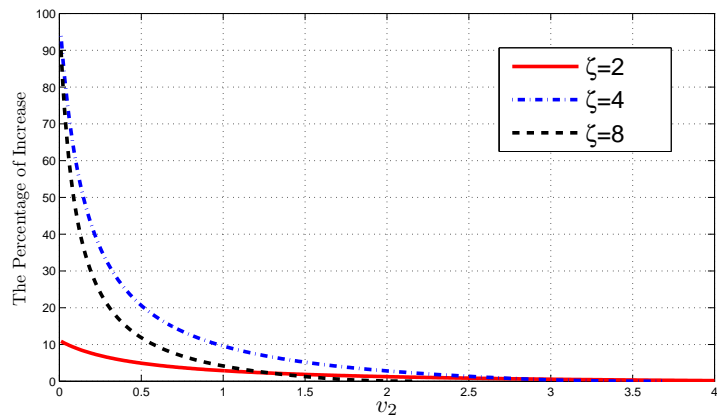


Figure 4.14: The percentage of increase in the utility of the SP when $\kappa_u = \frac{1}{\zeta}$ with respect to v_2 for different values of ζ (old model).

Chapter 5

Conclusion

5.1 Summary

In this thesis, we considered economic frameworks to investigate different questions about the departure toward a non-neutral regime and its possible consequences. We assessed whether different entities of the market have the incentive to adopt a non-neutral pricing scheme; what are the pricing strategies they choose; and how these changes affect the Internet market. To answer these questions, we modeled, analyzed, and characterized a non-neutral Internet market under different scenarios and parameters.

First, in Chapter 2, we investigated the incentives of different entities of the Internet market for migrating to a non-neutral regime. We considered a diverse set of parameters for the market, e.g. market powers of Internet Service Providers (ISPs), sensitivity of End-Users (EUs), and Content Providers (CPs) to the quality of the content. The goal was to obtain founded insights on whether there exists a market equilibrium, the structure of the equilibria, and how they depend on different parameters of the market when the

current equilibrium (neutral regime) is disrupted and some ISPs have switched to a non-neutral regime. We considered a system in which there exists two ISPs, one “big” CP, and a continuum of EUs. One of the ISPs is neutral and the other is non-neutral. We considered that the CP can differentiate between ISPs by controlling the quality of the content she is offering on each one. We also considered that EUs have different levels of innate preferences for ISPs. We formulated a sequential game, and explicitly characterized all the possible Sub-game Perfect Nash Equilibria (SPNE) of the game. We proved that if an SPNE exists, it would be one of the five possible strategies each of which we explicitly characterized. We proved that when EUs have sufficiently low innate preferences for ISPs, a unique SPNE exists in which the neutral ISP would be driven out of the market. We also proved that when these preferences are sufficiently high, there exists a unique SPNE with a non-neutral outcome in which both ISPs are active. Through numerical analysis, we observed that the neutral ISP receives a lower payoff and the non-neutral ISP receives a higher payoff (most of the time) in a non-neutral scenario. However, we identified scenarios in which the non-neutral ISP loses payoff by adopting non-neutrality. We also showed that a non-neutral regime yields a higher welfare for EUs in comparison to a neutral one if the market power of the non-neutral ISP is small, the sensitivity of EUs (respectively, the CP) to the quality is low (respectively, high), or a combinations of these factors.

Then, we investigated frameworks using which ISPs and CPs select appropriate incentives for each other, and investigated the implications of these new schemes on the entities of the Internet market. Thus, we considered a market consisting of ISPs, CPs, and EUs in which ISPs sell the bandwidth to CPs in exchange of financial incentives. We

analyzed two non-neutral frameworks:

In Chapter 3, we studied the price competition in a duopoly with an arbitrary number of buyers. In this case, ISPs was considered to be sellers selling/leasing a number of their resources to buyers, i.e. CPs. Each seller can offer multiple units of resources depending on the availability of the resources which is random and may be different for different sellers. Sellers seek to select a price that will be attractive to the buyers and also fetch adequate profits. A seller may only know the statistics of the number of available resources to her competitor. We analyzed this price competition as a game, and identified a set of necessary and sufficient properties for the Nash Equilibrium (NE). The properties reveal that sellers randomize their price using probability distributions whose support sets are mutually disjoint and in decreasing order of the number of availability. We proved the existence and uniqueness of a symmetric NE in a symmetric market, and explicitly computed the price distribution in the symmetric NE. In addition, we proposed a heuristic pricing strategy for sellers in a symmetric oligopoly market which satisfies the necessary and sufficient properties identified for a NE in a symmetric duopoly. Numerical evaluations reveal that our proposed strategy constitutes a good approximation for the NE of the symmetric oligopoly market. Note that in this case, CPs have a passive role, in the sense that they cannot alter their demand in accordance with the price set by ISPs. However, CPs have the ability to choose amongst the ISPs based on their price.

In Chapter 4, we considered a non-neutral framework in which CPs have an active role in the market, and decide on the number of resources they want to reserve/buy from the ISPs based on the price ISPs quote. In our model, ISPs allow CPs to sponsor a portion of their resources, and price it appropriately to maximize their payoff. The payoff

of an ISP depends on the monetary revenue and the satisfaction of end-users both for the non-sponsored and sponsored content, while CPs generate revenue through advertisement. Moreover, in this work, we considered the coupling between limited resources and the strategies of the decision makers. We analyzed the market dynamics and equilibria in two different frameworks, i.e. sequential and bargaining game frameworks, and provide strategies for (i) SPs: to determine if and how to price resources, and (ii) CPs: to determine if and what quality to sponsor. The frameworks characterize different sets of equilibrium strategies and market outcomes depending on the parameters of the market.

5.2 Future Works

Although different models and problems were presented and analyzed in this thesis, several additional questions remain open. In this section, we present some of the extensions and new questions that can be addressed in the future:

CPs:

Note that in Chapter 3, we considered a random number of identical CPs that are price-takers. On the other hand, in Chapters 2 and 4, we considered a single strategic CP while the rest of the CPs are passive and their effects can be captured by constant parameters. A possible direction for future work is to consider multiple number of “strategic” CPs that are of different types. These CPs not only interact with ISPs but also compete with each other to attract EUs and to reserve/buy the resources of ISPs. The effects of this competition on the pricing of the Internet market and resources are not apriori clear. One might think about scenarios in which a competition between CPs leads to higher

prices quoted by ISPs for resources and subsequently heavier discounts for EUs. Thus, a better welfare for EUs might be expected if this outcome occurs. On the other hand, a “fragmented” Internet might be another outcome of the market. In this case, each CP picks an ISP and provides her content with premium quality exclusively on that ISP to avoid competition and also to secure a monopoly over EUs. This is not desirable from the perspective of a policy maker since it may have negative effects on the social welfare of the market.

The first step in extending our work toward this direction is to consider multiple CPs that belong to two broad types, e.g. those with high market power such as Google and Netflix, and those with low market power such as start-ups. It is interesting to investigate under what conditions migration to a non-neutral Internet would drive the CPs with low market power out of the market and how a regulator can prevent from this. Using this model, we can also compare between inter-type cooperations/competitions with cross-type cooperations/competitions. In other word, we can investigate whether CPs with low market power prefer to cooperate among themselves to be able to compete with those with high market power or they prefer to team up with powerful CPs in their competition with other small CPs.

ISPs:

Another possible direction, is to consider the effects of investment decisions by ISPs on the outcome of the market. In this case, ISPs invest the additional profit on their infrastructure and can compensate the cost through CPs or EUs. For example, in the model of Chapter 2, investment decisions can be accommodated by considering different levels of premium quality, each corresponding to different costs. The non-neutral ISP

would select a level of quality based on the cost and other parameters. It is of interest to investigate how the quality-cost relationship can affect the results of that chapter.

In addition, note that in the models presented in this thesis, we focused on either one (Chapter 4) or two ISPs (Chapters 2 and 3). A possible extension is to consider more than two ISPs and the interaction among them. In Chapter 3, we provided a heuristic algorithm to compute NE strategies for the case of a symmetric oligopoly, i.e. more than two identical ISPs. We provided numerical results on the performance of this algorithm. However, providing analytic results on the structure of equilibria in the case of an asymmetric oligopoly is a topic of future work. As previously mentioned, computing Nash equilibria in this case is considered to be an open problem. Thus, finding possible structures on the equilibria may at least provide approximate solutions if not solve this open problem.

Another possible direction to expand this work is to enable ISPs to share resources among themselves. The interplay between cooperation and competition among ISPs can change the pricing strategies for the CPs and EUs, and subsequently affect the social welfare of the market. In [38], we investigate this interplay in the context of the interaction between Mobile Network Operators (MNOs) and Mobile Virtual Network Operators (MVNOs).

EUs:

Note that in Chapter 2, we considered EUs to have different reluctance for ISPs. The lower the reluctance for ISPs, the easier an EU can switch between ISPs. Thus, high reluctance is associated with EUs that are locked-in with ISPs. A possible future work is to take into the account the duration of the contract that EUs have with ISPs. A possible

extension that can be applied to the work in Chapter 4 is to consider end-users that can switch between ISPs. Note that in Chapter 4, we assumed that end-users are locked with ISPs due to contracts.

The Regulator:

In the models presented in this thesis, we do not explicitly consider the strategic decisions a regulator can make. Instead, whenever possible, using the results of the models, we commented on the role of the regulator in affecting the equilibrium outcome. A future work is to consider the regulator as one of the decision makers of the market alongside the ISPs, CPs, and EUs. The decision of the regulator could be to select the amount of incentive (for example through tax breaks) or penalty (for example through increasing tax) for the decision makers in order to control the equilibrium of the market.

Game Frameworks:

In this thesis, we mainly considered a myopic interaction among entities (except for the model in Chapter 4). Investigating a long-run interaction among decision makers using repeated game is a topic of future work. Through analyzing a repeated framework, we can find out whether CPs or ISPs with high market power are able to secure monopoly power over the market by driving their competitors out of the market through adoption of sub-optimal strategies for a limited time.

In addition, in most of the models analyzed in this thesis, we considered non-cooperative sequential game frameworks (except for one part of Chapter 4). It is of interest to investigate the equilibrium outcome of the models using cooperative or bargaining game frameworks, and to compare the results with the results of this thesis.

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