2016

Resource-Aware Design Of Wireless Control Systems

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Abstract

This work is motivated by modern monitoring and control infrastructures appearing in smart homes, urban environments, and industrial plants. These systems are characterized by multiple sensor and actuator devices at different physical locations, communicating wirelessly with each other. Desired monitoring and control performance requires efficient wireless communication, as the more information the sensors convey the more precise actuation becomes. However wireless communication is constrained by the inherent uncertainty of the wireless medium as well as resource limitations at the devices, e.g., limited power resources. The increased number of wireless devices in such environments further necessitates the management of the shared wireless spectrum with direct account of control performance. To address these challenges, the goal of this work is to provide control-aware and resource-aware communication policies. This is first examined in the fundamental problem of allocating transmit power resources for wireless closed loop control. Opportunistic online adaptation of power to plant and wireless channel conditions is shown to be essential in achieving the optimal tradeoff between control performance and power utilization. Optimal structural properties of channel access mechanisms are also considered for the problem of guaranteeing multiple control performance requirements over a shared wireless medium. This includes scheduling mechanisms implemented by central authorities, as well as decentralized mechanisms implemented independently by the wireless devices with emerging wireless interferences. Again the mechanisms exhibit an opportunistic adaptation to varying wireless channel conditions, especially designed to explore the tradeoffs between different communication links and meet control performance requirements. The structural characterization is augmented with tractable optimization algorithms to compute these channel access mechanisms. Finally, as control is naturally a dynamic task that requires a long term planning, appropriate dynamic algorithms adapting to the varying control system states are examined. Besides adapting dynamically, the proposed algorithms provide guarantees about long term control performance and resource utilization by construction.

Degree Type

Dissertation

Degree Name

Doctor of Philosophy (PhD)

Graduate Group

Electrical & Systems Engineering

First Advisor

George J. Pappas

Keywords

Control systems, Optimal control, Resource allocation, Wireless communications

Subject Categories

Electrical and Electronics

RESOURCE-AWARE DESIGN OF WIRELESS CONTROL SYSTEMS

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A DISSERTATION

in

Electrical and Systems Engineering

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2016

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ACKNOWLEDGEMENT

My supervisor Prof. George J. Pappas has supported me by creating the environment for fruitful research during my graduate studies. His help has been indispensable and I am grateful. Prof. Alejandro Ribeiro's guidance and advice, technical and academic, has been influential during all my years at Penn and I would like to express my gratitude. I would also like to thank my committee members for their time invested and constructive comments on my work. I would like to thank all my lab and department colleagues for making studying at Penn such a great experience.

I would like to acknowledge the support of "Andreas Mentzelopoulos Scholarships for the University of Patras".

Finally, I would like to thank my family for all the support all these years, and I would like to especially thank my wife Anita for making our life in Philadelphia so happy and memorable.

ABSTRACT

RESOURCE-AWARE DESIGN OF WIRELESS CONTROL SYSTEMS

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George J. Pappas

This work is motivated by modern monitoring and control infrastructures appearing in smart homes, urban environments, and industrial plants. These systems are characterized by multiple sensor and actuator devices at different physical locations, communicating wirelessly with each other. Desired monitoring and control performance requires efficient wireless communication, as the more information the sensors convey the more precise actuation becomes. However wireless communication is constrained by the inherent uncertainty of the wireless medium as well as resource limitations at the devices, e.g., limited power resources. The increased number of wireless devices in such environments further necessitates the management of the shared wireless spectrum with direct account of control performance. To address these challenges, the goal of this work is to provide control-aware and resource-aware communication policies. This is first examined in the fundamental problem of allocating transmit power resources for wireless closed loop control. Opportunistic online adaptation of power to plant and wireless channel conditions is shown to be essential in achieving the optimal tradeoff between control performance and power utilization. Optimal structural properties of channel access mechanisms are also considered for the problem of guaranteeing multiple control performance requirements over a shared wireless medium. This includes scheduling mechanisms implemented by central authorities, as well as decentralized mechanisms implemented independently by the wireless devices with emerging wireless interferences. Again the mechanisms exhibit an opportunistic adaptation to varying wireless channel conditions, especially designed to explore the tradeoffs between different communication links and meet control performance requirements. The structural characterization is augmented with tractable optimization algorithms to compute these channel access mechanisms. Finally, as control is naturally a dynamic task that requires a long term planning, appropriate dynamic algorithms adapting to the varying control system states are examined. Besides adapting dynamically, the proposed algorithms provide guarantees about long term control performance and resource utilization by construction.

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Chapter 1: Introduction

Wireless sensors are abundant in modern smart infrastructures, where they are deployed to monitor and control physical processes in our homes, urban environments, and industrial plants. Essential to the development of such systems is wireless communication, allowing the separation between the physical locations where sensing and data collection is performed, and different physical locations where control operations are decided and actuated to the system. To achieve desired monitoring and control performance, efficient and reliable communication of information from sensors to actuators is required. However the latter is constrained by the uncertain stochastic nature of the wireless medium as well as by resource limitations.

Wireless sensors for example are typically battery-operated. The more information a sensor conveys the more precise decision making becomes, but the resulting increase in power consumption at the sensor leads to rapid depletion of its energy resources. Hence tradeoffs between device lifetime and overall system performance emerge. Besides power limitations, as the number of wireless sensors and actuators in these environments increases, a need to share the available wireless medium between these devices arises. Efficient communication requires frequent channel access for each sensor, but if this access is not coordinated wireless interferences arise between transmissions and performance degrades. As a result, the shared wireless medium necessitates mechanisms for maintaining desired levels of performance among the various devices/systems in the environment.

Examples of such systems are ubiquitous. Smart buildings (Nghiem and Pappas (2011); Ma et al. (2012); Oldewurtel et al. (2012)) are instrumented with sensors collecting and transmitting local information to locations where operation decisions are being made. Monitoring physical processes with a distributed or large-scale nature also requires efficient sensor communication, as illustrated in monitoring of, e.g., agricultural applications (Yiming et al. (2007); Ruiz-Garcia et al. (2009); Cardell-Oliver et al. (2004)), contagious processes (Sadilek et al. (2012); Nowzari et al. (2016)), air quality and concentration of chemical substances (Kim et al. (2009); De Vito et al. (2011)), traffic (Horvitz et al. (2012)), or water networks (Whittle et al. (2013)).

Designing wireless sensor-actuator systems with desirable characteristics involves the following challenges:

- **Dynamic Physical Processes:** The wireless sensors and actuators of the systems considered are deployed to collect measurements and control physical processes that are dynamically and stochastically changing over time. To characterize the performance of such systems it is indispensable to consider metrics that account for operation over a long time horizon.

Hence, appropriate metrics of dynamic monitoring or control performance are required.

- **Wireless Communication Uncertainties:** Due to the intrinsic stochastic behavior of the wireless medium, communication between devices is not always reliable but is subject to randomness and non-determinism. The varying wireless conditions create opportunities or adversities, and their effects on the monitoring and control performance needs to be accounted for. As a result, appropriate wireless communication models which capture this stochastic behavior need to be employed.

- **Adaptive Resource Allocation:** Static mechanisms for allocating the available communication resources are conservative, as in order to maintain desired performance at all times they have to consider worst operation conditions. In view of the dynamic nature of the considered physical processes, as well as the variability of the wireless medium, the use of dynamic resource allocation mechanisms becomes apparent. By adapting to the system conditions during operation, wireless communication resources can be appropriately allocated only when necessary or only when opportunities arise. The development of such mechanisms that can adapt online but can as well provide long time performance guarantees emerges as a principal research challenge.

1.1. Related Work

The topics considered in this work relate to the area of networked control systems, that is, control systems implemented over communication networks. The related work can be classified with respect to the employed communication and channel models. A widely adopted paradigm, which will also be followed in this work, is that of *packet-based communication* (Hespanha et al. (2007)). In this paradigm communication is organized in packets, i.e., when a sensor collects a plant sample or measurement it transmits the acquired information to the intended receiver in a packet. Packets are assumed to be long enough so that any quantization effects from converting the collected information to, e.g., a sequence of bits, can be ignored.

Early works on packet-based networked control systems ignore the cost of conveying information, i.e., the cost of transmitting packets, and focus their analysis on the performance of control loops when various communication uncertainties are taken into account. A fundamental type of uncertainty arises when transmitting packets over erasure channels which randomly drop some of the packets. If packets are dropped in an independent and identically distributed fashion there is a maximum packet drop rate above which controlling or estimating a plant becomes impossible, i.e., the system becomes unstable. Characterizations of this maximum packet drop rate are examined by Seiler and Sengupta (2001); Sinopoli et al. (2004); Imer et al. (2006); Hespanha et al. (2007); Gupta et al. (2007). A different type of communication uncertainty arises when transmitted packets are received with unknown or random delay. Related characterizations of stability under delays are considered by Walsh et al. (2002); Hespanha et al. (2007); Heemels et al. (2010).

Besides fundamental stability characterizations, considerable attention has been given on applying appropriate control operations to counteract these communication uncertainties, that is, random packet drops or delays. This entails the design of appropriate state estimators and controllers, e.g., linear feedback and observer gains, as discussed by Seiler and Sengupta (2001); Smith and Seiler (2003); Xu and Hespanha (2005); Imer et al. (2006); Hespanha et al. (2007); Schenato et al. (2007); Gupta et al. (2007). Overall the fact that packets/measurements are sometimes available at the receiver while other times they are not requires an appropriate analysis of the system as a switched or hybrid system.

While in the aforementioned works communication is treated as a constraint or disturbance to the overall closed loop system, more recent efforts consider communication as an active part of the design. The resulting setup typically departs from the classic periodic control paradigm, where the control loop is closed periodically after a fixed amount of time. In particular frameworks such as event-triggered sampling (Astrom and Bernhardsson (2002); Rabi et al. (2012)), event-triggered control (Tabuada (2007); Rabi et al. (2008); Heemels et al. (2012, 2013)), or self-triggered control (Anta and Tabuada (2010); Nowzari and Cortés (2012)) have been proposed to limit the amount of required communication. The underlying concept in these contributions is to prolong the time elapsed between successive sampling or input updates as long as some plant performance criterion is satisfied. Such schemes exhibit in general an average communication rate lower than periodic schemes that attain similar plant performance. However, communication costs are not explicitly accounted for in the triggering design.

Communication costs are explicitly modeled in the context of remote state estimation by Xu and Hespanha (2004); Cogill et al. (2007); Li and Lemmon (2011); Lipsa and Martins (2011); Mesquita et al. (2012). In this framework a sensor measuring the plant state decides at each time step whether to transmit its measurement to an estimator or not, and every transmission incurs a fixed cost. The overall goal is to minimize jointly a penalty on the estimation error and the communication costs aggregated over time. Alternatively a hard constraint on the number of transmissions between the sensor and the estimator is considered by Imer and Başar (2005, 2010); Rabi et al. (2012). The optimal communication is shown to be event-triggered by Xu and Hespanha (2004), similar to, e.g., Tabuada (2007); Rabi et al. (2008), meaning that transmissions are triggered when a measure of the plant state exceeds a threshold. Computing the optimal transmission-triggering sets is in general computationally hard, motivating the development of suboptimal schemes Cogill et al. (2007); Cogill (2009); Li et al. (2010); Antunes and Heemels (2014).

Related contributions consider the joint design of communication policies as well as control input policies, assuming again a fixed cost per transmission (Molin and Hirche (2009); Ramesh et al. (2013)). The problem becomes complex as the policies of the two entities, i.e., the sensor and the controller, have an information structure that is coupled over time. More specifically, the controller requires a local state estimator that is in general dependent on the communication policy. Some structural results for the joint policy design have been recently considered by Nayyar et al. (2013). Alternatively, one can impose special policy structures leading a separation principle (Ramesh et al. (2013)) so that optimal inputs and communication schedules can be found by dynamic programming for a finite time horizon. Other structural properties of the communication policies to simplify the state estimation are considered by Han et al. (2015).

The above approaches within the packet-based communication model are related to the approach followed in this work, however it is worth noting that different lines of research emerge by other communication or channel models as follows:

- *Information-theoretic approaches and Coding:* An alternative communication model arises by considering a digital channel with a limited data-rate. In this case to examine the transfer of information from sources to destinations, i.e., from sensors to controllers, information-theoretic approaches are usually followed. A fundamental characterization that arises is the minimimum bit rate under which an unstable system can be stabilized; see, e.g., Nair et al. (2007); Franceschetti and Minero (2014). To achieve the communication of plant states, typically taking values in real vector spaces, an efficient coding and decoding scheme is required and, in contrast to the packet-based paradigm, quantization effects need to be taken into account (Wong and Brockett (1997); Brockett and Liberzon (2000); Tatikonda and Mitter (2004); Yuksel and Başar (2006); Ostrovsky et al. (2009); Sukhavasi and Hassibi (2016)). As the encoder and decoder policies depend on each other but are implemented at different physical locations, the joint design is complex and structural properties are explored by Walrand and Varaiya (1983); Varaiya and Walrand (1983); Borkar and Mitter (1997); Mahajan (2008). Further connections between information theoretic measures and control performance are explored by Martins and Dahleh (2008); Tanaka et al. (2015).

- *Input-ouput models of communication:* Apart from the packet-based communication paradigm and the limited data-rate channel model, communication between a sensor and a controller can also be modeled as an input-output system (Elia (2005)). The uncertainty introduced in the closed loop due to channel randomness is then treated as stochastic model uncertainty. This facilitates controller synthesis using robust control techniques (Braslavsky et al. (2007)). Communication is again not treated as a design variable in this context but a constraint to closed loop control.

Control over Shared Wireless Channels

The problem of sharing a communication medium between different sensor and actuator devices has also received considerable attention in the context of networked control systems. The prevalent approach to this problem is to design a centralized scheduling mechanism with respect to control performance objectives. The scheduling mechanisms usually examined are either static or dynamic. Typical examples of the first type are periodic protocols where the wireless devices transmit in a predefined repeating order, e.g., round-robin. Stability under such protocols can be analyzed by a switched system approach – see, e.g., Zhang et al. (2001); Hespanha et al. (2007); Schenato et al. (2007); Donkers et al. (2011). The problem of designing static schedules suitable for control applications has also been addressed. Periodic sequences leading to stability

(Hristu-Varsakelis (2001)), controllability and observability (Zhang and Hristu-Varsakelis (2006)), or minimizing linear quadratic objectives (Le Ny et al. (2011)) have been proposed. Deriving otherwise optimal scheduling sequences is recognized as a hard combinatorial problem (Meier et al. (1967); Rehbinder and Sanfridson (2004); Gupta et al. (2006)).

The second type of schedulers, the dynamic ones, do not rely on a predefined sequence but access to the communication medium is decided dynamically at each step. The decision typically depends on the current plant/control system states, i.e., informally the subsystem with the largest state discrepancy is scheduled to communicate. Examples can be found in, e.g., Walsh et al. (2002); Hristu-Varsakelis and Kumar (2002); Egerstedt and Wardi (2002); Donkers et al. (2011). Recent efforts have also focused on scheduling for event-based controllers (Cervin and Henningsson (2008); Molin and Hirche (2014)). Another approach, motivated by the problem of scheduling control tasks sharing a computation resource (CPU) rather than a communication medium, is to abstract control performance requirements in the time/frequency domain. Knowing, e.g., how often a task needs resource access to communicate and close the loop, static and dynamic schedules meeting the desired requirements can be obtained using algorithms from real-time scheduling theory (Liu (2000); Branicky et al. (2002)).

In contrast to centralized scheduling, decentralized mechanisms where sensors independently decide access to the shared wireless medium are easier to implement. They do not require predesigned sequences of how sensors access the medium, or a central authority to take scheduling decisions. The drawback however is that *packet collisions* can occur from simultaneously transmitting sensors, resulting in lost packets and control performance degradation. Control under decentralized channel access mechanisms has drawn limited attention in the literature. Comparisons between different medium access mechanisms for networked control systems and the impact of packet collisions have been considered either in numerical simulations (Liu and Goldsmith (2004); Ramesh et al. (2013)) or analytically in simple cases (Rabi et al. (2010); Blind and Allgöwer (2011)). These include random access mechanisms, where each sensor independently and randomly decides whether to transmit, and related Aloha-like schemes, where after a packet collision the involved sensors wait for a random time interval and retransmit. Stability conditions under packet collisions were examined in Zhang (2003); Tabbara and Nesic (2008). Besides closed loop control, optimal remote estimation over collision channels is considered recently in Vasconcelos and Martins (2014).

1.2. Outline and Contributions

The goal of this work is to develop mechanisms for efficient communication and resource allocation in sensor-actuator systems implemented over wireless channels. The topics considered in this context and the related contributions are presented next.

Resource-aware Wireless Control Systems

The packet-based paradigm is adopted in modeling the communication between sensors and actuators. In order to explicitly account for the allocation of transmit power resources however this model is generalized. In particular, the probability of successfully decoding the transmitted packet at the receiver is modeled as a function of the channel conditions (fading) as well as the selected transmit power. This constitutes a generalization of the independent packet success model, as here a transmitting sensor has the ability to directly control the packet success by adjusting the transmit power. This communication model allows us to mathematically formulate the problem of power allocation in wireless control systems. Moreover, it allows us to capture the uncertainties introduced by the inherent stochastic nature of the wireless medium via channel fading. Finally, this communication model is extended to the case where multiple sensors need to communicate over a shared wireless medium (multiple-access channels) and allows for allocation of resources across multiple systems. These contributions are detailed in Chapter 2.

Power Management for Wireless Sensor-Actuator Systems

Under the introduced power-aware channel model we examine a single loop control system, where a sensor transmits information over a wireless fading channel to a controller. We formulate the problem of minimizing jointly the average control performance of the plant as well as the average power consumption at the sensor. We identify a special information structure that allows the decoupling of the two designs, so that the control input policy and the power allocation policy can be designed separately. Then we derive structural properties of the optimal power allocation. We show that in order to effectively balance the tradeoff between control performance and resource utilization the sensor needs to opportunistically exploit online the channel conditions as well as the control system state. Our approach leads to novel designs as it is shown that the sensor needs to allocate a wide range of transmit powers, in contrast to the binary event-triggered approaches where a sensor decides just whether to transmit or not. In fact event-triggered policies arise as a special case in our context. These contributions are detailed in Chapter 3.

Scheduling Sensor-Actuator Systems over Shared Wireless Channels

For the case of multiple wireless control systems sharing a wireless channel we develop centralized scheduling and power allocation policies. The goal of the policies is to maintain a level of control performance for each control system, modeled via Lyapunov functions, while minimizing the total power consumption. This formulation leads to a constrained stochastic optimization problem and the structure of the optimal solution is characterized. The scheduler opportunistically exploits the channel conditions experienced by the control systems to decide which system should get access to the channel. The optimal transmit powers are shown to be decoupled among systems, allowing for an easy implementation. We further describe algorithms to find the optimal policies either offline, when the channel distributions are available, or online based on samples from the channel states observed during operation. In the latter case we show that the algorithm converges almost surely to the optimal scheduling and power allocation rule. These contributions are detailed in Chapter 4.

Random Access Design for Wireless Sensor-Actuator Systems

When alternatively a central authority is not available to implement scheduling decisions, sensors randomly and independently decide whether to access the channel. The goal is again to guarantee closed loop control performance, however in this case a mitigation of packet collisions arising from simultaneously transmitting sensors is required. The proposed decentralized sensor access policies are derived by a constrained optimization formulation. Despite the fact that sensor policies are coupled, as each sensor is causing collisions to all other links and is subject to collisions from all other sensors, a decentralized structure emerges. Each sensor should access the channel when its local channel conditions exceed a threshold, i.e., are favorable enough. The threshold is selected in a way that balances the gains from transmitting with the losses due to packet collisions on all other systems. Decentralized algorithms to obtain the optimal mechanism are also explored. These contributions are detailed in Chapter 5.

Online Control-aware Resource Allocation

While the optimal power allocation problem examined in Chapter 3 indicates that the sensor needs to allocate its transmit power online based on plant and channel states, computing the optimal policy is computationally hard. In particular, the dynamic nature of the control system state makes it hard to account for the optimal future behavior of the system (or technically the optimal cost-to-go function). To alleviate this computational complexity we provide suboptimal yet tractable state-aware policies based on the approximate dynamic programming technique of rollout algorithms. This approach guarantees by design that the suboptimal state-aware policies have a control performance and resource utilization that is superior to other non-state-aware policies, which are hereby used as reference policies.

We further develop state-aware policies for the decentralized channel access problem. In this case the rollout policies for the sensors are explicitly constrained to prescribed average channel access rates that do not violate control performance of other systems due to collisions. The design of such constrained policies becomes tractable based on Lagrange duality arguments. These contributions are detailed in Chapter 6.

Chapter 2: Resource-aware Wireless Control Systems

The control systems considered in this work are characterized by physical separation between the sensors and controllers/actuators. The sensors are deployed to measure the state of a dynamic physical system (plant) and transmit their measurements over wireless channels to controllers/actuators. The latter are responsible for providing appropriate control inputs in order to regulate the plant given the received information. In this context the most basic architecture is presented in Fig. 1 and consists of a single sensor-actuator pair over a wireless channel. The goal of this work is to provide resource-aware and control-aware communication designs. In this chapter we discuss the employed plant and controller models, as well as the wireless communication model which explicitly takes into account the employed resources.

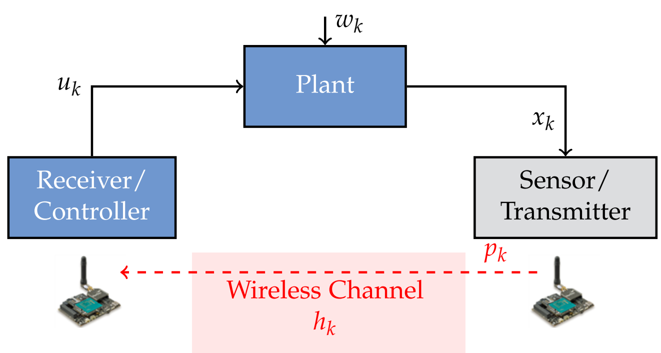

Figure 1: Wireless control system architecture. At each time step k the sensor measures and transmits the plant state x_k over a wireless channel to the controller. The selected transmit power p_k and the wireless fading state h_k determine whether the message is successfully received. The goal is to achieve a good trade-off between power consumption and closed loop performance.

2.1. Control System Model

The goal of the architecture in Fig. 1 is to regulate a plant. The evolution of the plants considered in this work can be described by a discrete-time linear time-invariant system given by

$$x_{k+1} = Ax_k + Bu_k + w_k,\ k \geq 0. \tag{2.1}$$

Here $k \geq 0$ represents discrete time steps, $x_k \in \mathbb{R}^n$ is the plant's state with x_0 being an initial given plant state, $u_k \in \mathbb{R}^m$ is the driving input, and $\{w_k, k \geq 0\}$ is the process noise which unless otherwise noted is an independent identically distributed (i.i.d) n-dimensional Gaussian process

$\mathcal{N}_{0,W}$ with zero mean and covariance W. The process noise repeatedly perturbs the system from the desirable equilibrium operating point 0, taken without loss of generality. If the system is unstable, i.e., its spectral radius is larger than unity $\rho(A) := \max_i |\lambda_i(A)| > 1$, the plant state will grow unbounded unless appropriate feedback control inputs are provided.

Given a wireless channel model and a communication policy, which is the main part of our design and is detailed in the following section, plant state measurements from the sensor/transmitter become available to the receiver/controller. Throughout this work we assume that communication is organized in packets containing each sensor measurement x_k at each time slot (Hespanha et al. (2007)). A packet transmitted might get received at the receiver during that slot or it might get dropped. We model the packet success for each time step k as an indicator γ_k taking value $\gamma_k = 1$ when information is successfully decoded and $\gamma_k = 0$ otherwise. Variables γ_k are random due to uncertainty introduced in the wireless communication as we will see in the following section. In general they are binary random variables with a time-varying success probability $\mathbb{P}(\gamma_k = 1) \in [0, 1], k \geq 0$.

When a packet is successfully received, the plant state x_k becomes completely known to the controller – the effects of measurement quantization and transmission delays are considered negligible and are thus ignored henceforth. When a packet is dropped, the controller does not know the current state of the plant. More formally, given the packet success model $\gamma_k \in \{0, 1\}$ the controller receives an output $\gamma_k x_k$ which is the actual system state when a packet is received and 0 otherwise. We can further assume that the controller also gets the value of γ_k so that it can distinguish between the cases $x_k = 0$ and $\gamma_k = 0$. Overall the information available at the controller at time k consists of all previously received messages given by $\{\gamma_0, \ldots, \gamma_k, \gamma_0 x_0, \ldots, \gamma_k x_k\}$. A standing assumption throughout this work is that the receiver also sends lossless packet acknowledgments to the sensor/transmitter.

Given the available information, the controller is responsible for deciding what control input u_k to be applied as a feedback to the plant at each time step. The general problem of designing appropriate control input policies in this context is examined in detail in Chapter 3. In particular the goal of the control policy is to provide good closed loop control performance. The latter can be measured as a typical quadratic cost on the plant state and the control input, i.e.,

$$\limsup_{N \to +\infty} \frac{1}{N} \mathbb{E} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k. \tag{2.2}$$

The limit in this expression accounts for long term performance while the expectation accounts for the randomness introduced by the process noise and the wireless communication.

Alternatively some control input policy may be fixed, i.e., the controller is pre-designed, and the remaining goal is to design the communication policy. To mathematically formulate this case we consider some examples of frequently used controller policies in networked control systems.

Example 2.1. *Consider the control of a plant given by (2.1). A very simple controller is described by*

$$u_k = \begin{cases} K\, x_k, & \text{if } \gamma_k = 1 \\ 0, & \text{if } \gamma_k = 0 \end{cases} . \tag{2.3}$$

That is, the controller applies a linear feedback with gain $K \in \mathbb{R}^{m \times n}$ when a measurement is received, otherwise does not apply any input when no measurement is received (Hadjicostis and Touri (2002)). The resulting closed loop system has two modes of operation depending on whether the measurement is received or not. In particular it can be written as a linear switched system of the form

$$x_{k+1} = \begin{cases} (A + BK)\, x_k + w_k, & \text{if } \gamma_k = 1 \\ A\, x_k + w_k, & \text{if } \gamma_k = 0 \end{cases} . \tag{2.4}$$

Intuitively the closed loop mode $A + BK$ is chosen to be a stable system.

Example 2.2. *Alternatively, as a second example consider a controller with the ability to hold the last applied input until a new measurement is received (Zhang et al. (2001); Seiler and Sengupta (2001)), that is,*

$$u_k = \begin{cases} K\, x_k, & \text{if } \gamma_k = 1 \\ u_{k-1}, & \text{if } \gamma_k = 0 \end{cases} . \tag{2.5}$$

where K is a state feedback gain. The overall evolution of the system can be expressed as a switched system again where the plant state x_k is augmented with the last applied input u_{k-1} to rewrite the system dynamics in the form

$$\begin{bmatrix} x_{k+1} \\ u_k \end{bmatrix} = \begin{cases} \begin{bmatrix} A + BK & 0 \\ BK & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} w_k \end{bmatrix}, & \text{if } \gamma_k = 1 \\ \begin{bmatrix} A & I \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} w_k \end{bmatrix}, & \text{if } \gamma_k = 0 \end{cases} . \tag{2.6}$$

This is again a switched linear time-invariant system with two modes of operation, open and closed.

Example 2.3. *As a third example, consider a more sophisticated controller which may compute and apply a different control input at each time step. Since the controller does not always have access to the plant state, it is reasonable to keep a local state estimate $\hat{x}_k$ and then apply a control input as a linear feedback with respect to the estimate, that is,*

$$u_k = K\hat{x}_k. \tag{2.7}$$

An intuitive rule for updating the state estimate is as follows (Xu and Hespanha (2004); Ramesh et al. (2013))

$$\hat{x}_k = \begin{cases} x_k & \text{if } \gamma_k = 1, \\ A\hat{x}_{k-1} + Bu_{k-1} & \text{if } \gamma_k = 0 \end{cases} , \tag{2.8}$$

In other words, when no measurement is received the plant dynamics are used in order to propagate the last estimate. We can combine the plant state with the state estimate in an augmented state vector to express the *overall dynamics as*

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_k \end{bmatrix} = \begin{cases} \begin{bmatrix} A+BK & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_{k-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} w_k \end{bmatrix}, & \text{if } \gamma_k = 1 \\ \begin{bmatrix} A & BK\,(A+BK) \\ 0 & A+BK \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_{k-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} w_k \end{bmatrix}, & \text{if } \gamma_k = 0 \end{cases} \tag{2.9}$$

This is again a switched linear time-invariant system with two modes of operation, open and closed. In Chapter 3 it will be shown that this controller becomes optimal for the wireless control system under appropriate assumptions.

As can be seen from the preceding examples a general model for the evolution of the plant and pre-designed controller is a switched linear time-invariant system of the form

$$x_{k+1} = \begin{cases} A_c\, x_k + w_k, & \text{if } \gamma_k = 1 \\ A_o\, x_k + w_k, & \text{if } \gamma_k = 0 \end{cases} . \tag{2.10}$$

Here $x_k \in \mathbb{R}^n$ is the overall state of the control system, with the understanding that this may include controller states as well as in the examples above. The evolution of the state has two possible linear modes of operation depending on transmission success. At a successful transmission the system dynamics are described by the matrix $A_c \in \mathbb{R}^{n\times n}$, where 'c' stands for closed-loop, and otherwise by $A_o \in \mathbb{R}^{n\times n}$, where 'o' stands for open-loop. We assume that A_c is asymptotically stable, implying that if system were to transmit at each slot its respective state evolution is stable. The open loop matrix A_o may be unstable. The terms $\{w_k, k \geq 0\}$ are again the i.i.d. Gaussian process noise with zero mean and covariance W.

For a fixed wireless control system of the form (2.10) our goal will be to design appropriate communication policies that yield good closed loop control performance, i.e., ability to regulate plant states to the desirable operating point 0 despite external disturbances or communication uncertainties. In particular we consider two different measures of control performance that will facilitate the communication design. These are presented next.

2.1.1. Lyapunov-based Control Performance Abstraction

First we capture the ability of the wireless control system to drive fast the system state to the desirable operating point 0. Convergence to 0 is typically analyzed in control systems via non-negative functions $V(x)$ of the plant state, i.e., Lyapunov functions. The Lyapunov function of the state decreases if $V(x_{k+1}) \leq \rho V(x_k)$ for all $x_k \in \mathbb{R}^n$ for some scalar $\rho < 1$ which denotes some *desired rate of decrease*. Due to the randomness introduced by the plant disturbance and the wireless communication it is not possible to guarantee that the Lyapunov function decreases at each time step. In fact when a plant state measurement is not received the system runs in open loop and it typically increases $V(x_{k+1}) > V(x_k)$. As an alternative then we can require that the

Lyapunov function decreases *in expectation* with respect to the system uncertainties, that is,

$$\mathbb{E}[V(x_{k+1}) \,|\, x_k] \leq \rho V(x_k) + \varepsilon \;\text{ for all } x_k \in \mathbb{R}^n \text{ and all } k \geq 0 \tag{2.11}$$

for a desired decrease rate $\rho < 1$ and some constant ε, independent of the system state. The expectation in this expression is with respect to the random disturbance $w_k, k \geq 0$ and the random wireless communication process $\gamma_k, k \geq 0$.

For the special case of switched linear systems in (2.10) we suppose that a quadratic Lyapunov function of the form $V(x) = x^T P x$ for some positive definite matrix $P \in \mathbb{S}^n_+$ is given. In that case we can write the control performance specification in the form

$$\mathbb{E}[V(x_{k+1}) \,|\, x_k] \leq \rho V(x_k) + \mathrm{Tr}(PW) \;\text{ for all } x_k \in \mathbb{R}^n \text{ and all } k \geq 0. \tag{2.12}$$

Here $\rho < 1$ is a given desired decrease rate. The term $\mathrm{Tr}(PW)$ is a constant appearing due to the linearity of the system and the additive process noise with covariance W (cf. (2.10)) as well as the quadratic form of the Lyapunov function.

The intuition behind condition (2.12) is as follows. If (2.12) holds for each time step $k = 0, \ldots, N$, then by taking the expectation at both sides and by iterating backwards in time we find that

$$\mathbb{E}V(x_N) \leq \rho\, \mathbb{E}V(x_{N-1}) + Tr(PW) \leq \ldots \leq \rho^N\, \mathbb{E}V(x_0) + \sum_{k=0}^{N-1} \rho^k\; Tr(PW). \tag{2.13}$$

Hence, system states have second moments that decay exponentially with rate ρ with respect to initial states, and in the limit remain bounded by $Tr(PW)/(1-\rho)$, since the sum in (2.13) converges due to $\rho < 1$.

The above Lyapunov control performance abstraction facilitates communication design. In Chapter 4 we will convert the above control performance requirements to equivalent explicit wireless communication requirements, in particular required packet success rates.

2.1.2. *Quadratic Control Performance Abstraction*

An alternative approach is to consider quadratic regulation costs as usually associated with linear systems. In particular at each time step we can consider a cost of the form

$$\begin{cases} x_k^T\, Q_c\, x_k, & \text{if } \gamma_k = 1 \\ x_k^T\, Q_o\, x_k, & \text{if } \gamma_k = 0 \end{cases}. \tag{2.14}$$

This costs depends in general on whether transmission occurs ($\gamma_k = 1$) or not ($\gamma_k = 0$) at time k. Both $n \times n$ matrices Q_c, Q_o are assumed to be positive semidefinite and are design choices. The case where both matrices are equal is a special case. In view of the system evolution (2.10) the state x_k is a random variable affected by plant disturbance and wireless communication uncertainties. As it is of interest to examine the long term behavior of the system we consider the average expected quadratic cost

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}[x_k^T \left(\gamma_k Q_c + (1-\gamma_k) Q_o\right) x_k]. \tag{2.15}$$

Average penalties on actuation, i.e., the actual control inputs, can also be captured in (2.15) as seen in the following example.

Example 2.4. *Consider the control system described in Example 2.2 which is expressed with respect to an augmented state x_k, u_{k-1}. Suppose that we want to account for a joint cost on the plant state and control inputs $x_k^T Q x_k + u_k^T R u_k$ for some $n \times n$ and $m \times m$ symmetric positive definite matrices Q, R. It can be seen from (2.5) that whenever a packet is received this joint cost equals $x_k^T(Q + K^T R K)x_k$. Otherwise, when a packet is dropped, the joint cost becomes $x_k^T Q x_k + u_{k-1}^T R u_{k-1}$. In either case this is quadratic cost on the augmented state that depends on whether the current transmission is successful or not. This is of the general form (2.14) with respect to the augmented system state.*

In the following section we will describe in detail the wireless communication model which determines the packet indicator variables $\gamma_k, k \geq 0$. Before doing so we briefly describe a difference between the two control performance abstractions presented above.

Remark 2.1. *We point out that overall in this work we are interested in communication design for control performance, in contrast to determining what communication designs guarantee stability, as is commonly examined in the literature, e.g. Zhang et al. (2001); Hristu-Varsakelis (2001); Hespanha et al. (2007); Donkers et al. (2011). The two proposed control performance abstractions provide for different communication design approaches. We will show (Prop. 4.1 in Chapter 4) that the Lyapunov control performance abstraction leads to an explicit wireless communication requirement. In particular it corresponds to a convex requirement (lower bound) on the packet success rate at each time step, and leads to a design that is decoupled from the plant states over time. In contrast, the average quadratic control performance in (2.15) is in general a non-convex function of the packet success rates. The communication design in this case (Chapter 6) can exploit dynamic programming approaches and leads to solutions that are adapted online to plant states.*

2.2. Wireless Channel Model

Consider again the wireless control architecture of Fig. 1. At each time step k the wireless sensor transmits the current plant state measurement with some power $p_k \in [0, p_{\max}]$ which is a design variable over a wireless fading channel with coefficient h_k. Due to propagation effects the channel gain h_k changes unpredictably over time (Goldsmith, 2005, Ch. 3). We adopt the standard block fading model of wireless communications whereby channels $\{h_k, k \geq 0\}$ are modeled as i.i.d. random variables taking values in the positive reals $\mathbb{R}_+$ according to some distribution ϕ_h (Goldsmith, 2005, Ch. 4). The channel states are also independent of the plant process noise $\{w_k, k \geq 0\}$ (cf.(2.1) or (2.10)). To allow for transmissions adapted to the current channel conditions the transmitter has access to the channel state information h_k before transmitting at time k –

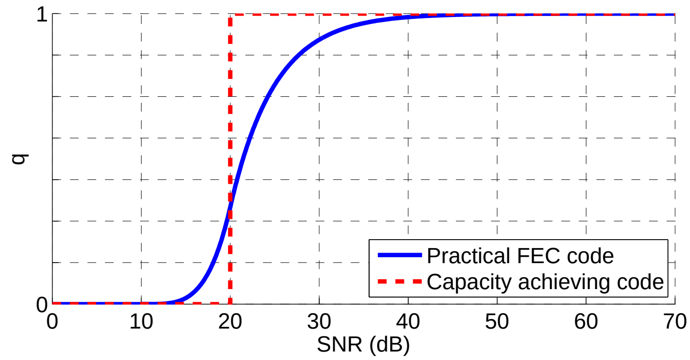

Figure 2: Complementary error function for FEC and capacity achieving codes. The probability of successful decoding q for a practical FEC code is a sigmoid function of the received SNR $= h\,p/N_0$, while for a capacity achieving code a threshold value SNR_0 determines whether a packet is successfully received.

see Remark 2.3.

At the controller side the received signal includes the information bearing signal and additive white Gaussian noise (AWGN). The noise power is denoted by N_0 and the power of the information bearing signal is the product $h_k\, p_k$. Assuming the receiver also has channel state information, successful decoding of the transmitted packet is determined by the signal to noise ratio (SNR) at the receiver defined as

$$\text{SNR}_k := \frac{h_k\, p_k}{N_0}. \tag{2.16}$$

More precisely, given the particular type of modulation and forward error correcting (FEC) code used, the SNR determines the probability of successful decoding $\mathbb{P}(\gamma_k = 1)$. To keep the analysis general we define a generic complementary error function

$$\mathbb{P}(\gamma_k = 1) = q\,(h_k, p_k)\,, \tag{2.17}$$

mapping $\text{SNR}_k := h_k p_k / N_0$ to the packet success probability. We assume that $q\,(h, p)$ is a known increasing function of the product $h\,p$ - see Remark 2.2.

Considering packet decoding as a part of the communication process, we can model communication as a sequence of indicator variables γ_k taking value $\gamma_k = 1$ when information is successfully decoded and $\gamma_k = 0$ otherwise. Variables γ_k are Bernoulli distributed with time-varying success probabilities $\mathbb{P}(\gamma_k = 1)$ given by (2.17). Packet receipt acknowledgment γ_k is also sent from the controller to the sensor as provided by 802.11 and TCP protocols. We assume lossless acknowledgments, so that the sensor knows what information is received at the controller.

A fundamental problem addressed in this work is the design of the transmit powers p_k of the sensor. The design should lead to desired control performance of the system as expressed, e.g., in the preceding section. Moreover the sensor should make an efficient use of its power resources. As a result the communication policy needs to take into account the average power utilization as measured by

$$\limsup_{N\to+\infty} \frac{1}{N}\,\mathbb{E} \sum_{k=0}^{N-1} p_k. \tag{2.18}$$

The expectation here accounts for the uncertainty introduced by the random channel fading process, the random packet success, as well as the random control system evolution. A formal problem specification for the optimal power allocation will be presented in Chapter 3.

The sensor may select transmit power per each packet as a function of the information available to the sensor, i.e., the plant state measurements, the observed channel realizations, and the receiver acknowledgments. Informally, to conserve power at the sensor side we want to transmit information only when the state x_k deviates from its desired value or when the channel realization h_k is favorable. In the first case transmission is necessary to keep the plant under control. In the latter case the transmission cost is minimal.

The present wireless channel model can be naturally extended to the case where multiple control systems are

Remark 2.2. *The error profiles $1 - q\,(h_k, p_k)$ of particular FEC codes are difficult to determine analytically but can be measured in actual or simulated experiments (Zhang et al. (2008); Ploplys et al. (2004)). Typically $q\,(h_k, p_k)$ is a sigmoid function of $h_k\, p_k$ with exponential tails as depicted in Fig. 2. In the theoretical limit, correct decoding depends on the channel capacity $C_k = \mathcal{W} \log_2(1 + SNR_k)$, where $\mathcal{W}$ is the channel bandwidth. If the packet is transmitted at a rate smaller than C_k bits per second it is almost surely successfully decoded, and it is almost surely incorrectly decoded otherwise. Thus, we can write the successful decoding probability as the indicator function*

$$q\,(h_k, p_k) = \mathbb{I}\left(\frac{h_k p_k}{N_0} \geq SNR_0\right), \tag{2.19}$$

for some constant SNR_0. Determining the threshold SNR_0 requires specification of the sampling rate and quantization resolution of the state x_k. With α samples per second and β bits per sample we require a transmission rate of $\alpha\beta$ bits per second. The SNR threshold is then given by $SNR_0 = 2^{\alpha\beta/\mathcal{W}} - 1$. Our interest in (2.19) is conceptual as it will allow us to recover results in event-triggered communication (Xu and Hespanha (2004)) as arising from the use of capacity achieving codes – see Section 3.4.1. The form of (2.19) is shown in Fig. 2.

Remark 2.3. *The assumption that channel state information (CSI) is available at the transmitter is typical in modern wireless communication setups (Goldsmith, 2005, Ch. 9). To measure the wireless channel conditions a short pilot signal of fixed power can be sent from the transmitter and then the fading characteristics can be estimated at the receiver and sent back to the transmitter by utilizing the reverse channel. Although accurate CSI is difficult to acquire at the transmitter side, our development can be modified if channel estimates are available in lieu of the actual channel value h_k with a reinterpretation of the function $q(.)$. Further discussion is provided in Appendix A. We additionally point out that even though the pilot signals for the channel estimation incur some power consumption, we assume that practically this is much lower than the*

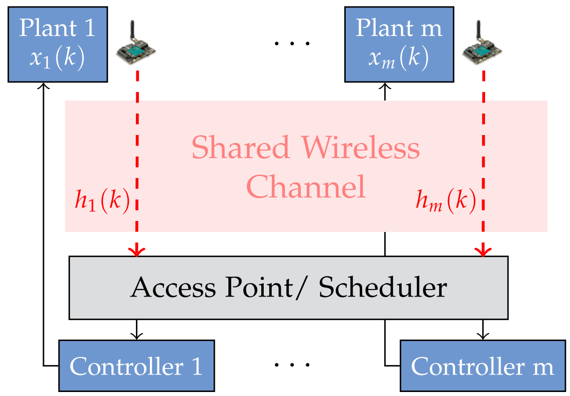

Figure 3: Architecture for control over multiple access channels. Independent control systems close their loops by transmitting over the shared wireless medium to a common receiver/access point. Each control system i experiences random channel conditions h_i. A centralized scheduler at the access point observes all channel states and opportunistically decides which system is scheduled to transmit and close the loop.

power necessary for transmitting the packets of the control system, especially for large packet lengths (e.g. long headers). Hence the power for channel estimation is not included in our objective in (2.18).

Remark 2.4. *There is a distinction to be made between errors that are detected by the receiver and errors that are undetected and may confuse the controller. The model here handles the former and ignores the latter. This is justified because practical communication schemes include the use of cyclic redundancy checks (CRC) for error detection that can drive the probability of undetected errors to very small values (Moon, 2005, Ch. 4). The use of simple CRCs reduces the probability of undetected errors to* 10^{-3}*, while longer codes can reduce this probability to* 10^{-7}.

2.2.1. Multiple Access Channels

Consider now the case where multiple sensors need to communicate over a shared wireless medium in order to close their loops, as depicted in Fig. 3. Each sensor is measuring the state of an independent plant and transmits its local state information to a common access point. For example this can be a common central controller computing the control inputs to the plants. Each control system has their own independent dynamics, expressed, e.g., in the form (2.10). The access point is responsible for deciding at each time step which sensor is scheduled, i.e., gets access to the channel. The goal is to achieve control performance for all systems.

Each control loop i ($i = 1, 2, ..., m$) experiences different wireless fading conditions over the shared channel. In particular by $h_{i,k}$ we refer to the channel fading coefficient that system i experiences if it transmits at time slot k. Following a block fading model as in the single control loop case, channel states $\{h_{i,k}, 1 \leq i \leq m\}$ are modeled as constant during each transmission slot k, but independent and identically distributed across different time slots k according to some joint distribution ϕ_h on $\mathcal{H}^m$. We assume the channel states are available to the access point before transmission – see

Remark 2.5 for a practical implementation.

Following our power allocation model, if system i is scheduled to access the channel at a given time step, the corresponding sensor selects a transmit power level $p_{i,k} \in [0, p_{\max}]$. As described in (2.17) the probability that the transmitted packet is successfully decoded is a function of the power level and the link's channel conditions given by a relationship of the form $q(h_{i,k},\ p_{i,k})$.

The goal of the access point/scheduler is to select one system to access the channel at each time step. We denote with $\alpha_{i,k} = 1$ the decision to schedule system i at time k, and $\alpha_{i,k} = 0$ otherwise. Since at most one system is scheduled we have $\sum_{i=1}^{m} \alpha_{i,k} \leq 1$. Let us indicate with $\gamma_{i,k} \in \{0, 1\}$ the event that a successful transmission occurs at time k for the subsystem i. This is a Bernoulli random variable with success probability

$$\mathbb{P}[\gamma_{i,k} = 1 \,|\, h_k, \alpha_k, p_k] = \alpha_{i,k}\, q(h_{i,k}, p_{i,k}) \tag{2.20}$$

This expression states that the probability of a message for system i being successfully received equals the probability that system i is scheduled to transmit and the message is correctly decoded.

The scheduler to be designed can take into account current channel information when making the scheduling decisions online. Intuitively the scheduler might decide to give access to the system currently experiencing the most favorable channel conditions, as that will result in a most reliable communication. An important restriction however is posed on the scheduler design due to the physical systems. To avoid loss of stability or deterioration of control performance, the scheduler needs to guarantee that each system gets access to the shared channel at a sufficient rate. The detailed scheduler design in presented in Chapter 4.

Remark 2.5. *The centralized scheduler of the multiple access channel architecture in Fig. 3 requires channel state information. The channel conditions for each system can be measured at the access point at the beginning of each time slot by short pilot signals sent from the wireless transmitters of all systems to the access point. Depending on the measured channel states the access point decides which plant is scheduled to close the loop during the time slot.*

2.2.2. Decentralized (Random) Channel Access

An alternative mechanism for sharing the wireless medium is random access. In contrast to centralized scheduling, this is a decentralized mechanism where sensors independently and randomly decide whether to access the shared wireless medium and transmit plant state measurements to the access point/controller (Fig. 4). Such a mechanism is easier to implement in practical scenarios, as it does not require a central authority to take scheduling decisions. The drawback of this decentralized approach however is that packet collisions can occur from simultaneously transmitting sensors, resulting in lost packets and control performance degradation. Hence sensor access policies need to be appropriately designed to mitigate these effects.

At every slot k each sensor i transmits over the shared channel with some probability $\alpha_{i,k} \in [0, 1]$

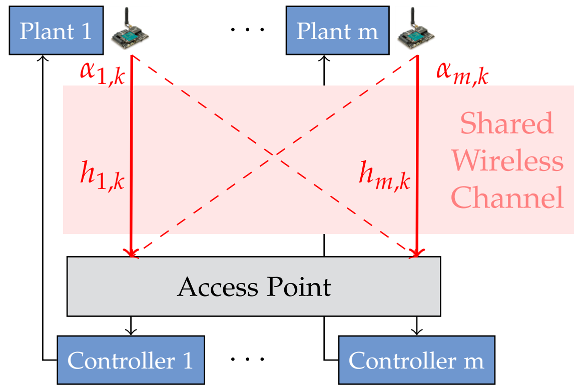

Figure 4: Random access architecture for m control loops over a shared wireless medium. Each sensor i randomly transmits with probability $\alpha_{i,k}$ at time k to a common access point computing the plant control inputs. If only sensor i transmits, the successful decoding probability depends on local channel conditions $h_{i,k}$. If other sensors transmit at the same time a collision might occur at sensor i's transmission, rendering i's packet lost.

which is a decision variable. A sensor's transmission might fail due to two reasons, packet decoding errors and packet collisions. A collision might be experienced on link i, thereby rendering packet i lost, if some other sensor $j \neq i$ transmits in the same time slot. We assume that such a collision event occurs with constant probability $q_{ji} \in [0, 1]$, *given* that both sensors i, j transmit in the slot. Thus, the probability that sensor i's transmission is free of collisions, i.e., that no other sensor transmits and causes collisions on link i, equals $\prod_{j \neq i}[1 - \alpha_{j,k}\, q_{ji}]$. See Remark 2.6 for details of this collision model.

We adopt once again a block fading channel model whereby $h_{i,k}$ is the channel fading coefficient that system i experiences if it transmits at time slot k. The channel fading coefficients are randomly varying over time. If sensor i transmits and has a collision free time slot, the success of decoding the message at the access point/receiver depends on the randomly varying channel conditions on link i and transmit power. Assuming for simplicity that transmit power is fixed to some value p_i in this setup, we denote by $q(h_{i,k}, p_i)$ the probability of successful transmission (cf.(2.17)).

Combining the effects of collisions and packet losses due to fading, the probability that a packet is successfully decoded at the access point can be written as

$$\mathbb{P}(\gamma_{i,k} = 1) = \alpha_{i,k}\, q(h_{i,k}, p_i) \prod_{j \neq i} \Big[1 - \alpha_{j,k}\, q_{ji}\Big]. \tag{2.21}$$

This expression states that the probability of system i in (2.10) closing the loop at time k equals the probability that transmission i is successfully decoded at the receiver, multiplied by the probability that no other sensor $j \neq i$ is causing collisions on ith transmission.

The goal of the random access design problem in this case it to design the sensor policies so that each involved control system has desirable control performance. One the one hand, this requires management of the arising packet collisions between sensors. On the other hand, sensors can opportunistically exploit their local plant or channel information to decide whether to access the channel. Due to the shared wireless medium the sensor policies are coupled (cf. (2.21)). A formal problem specification and an efficient design procedure is presented in Chapter 5.

Remark 2.6. *Our collision model captured by the probabilities q_{ji} subsumes: i) the conservative case where simultaneous transmissions certainly lead to collisions ($q_{ji} = 1$) usually considered in control literature, e.g., by Zhang (2003); Tabbara and Nesic (2008), ii) the case where simultaneously transmitted packets are not always lost ($q_{ji} < 1$), e.g., due to the capture phenomenon (Luo and Ephremides (2002)), and iii) the asymmetric case where different sensors j, ℓ interfere differently on link i, e.g., due to their spatial configuration.*

Chapter 3: Optimal Power Management for Wireless Sensor-Actuator Systems

3.1. Problem Description

We consider the wireless control architecture shown in Fig. 5 deployed to control a discrete-time linear time-invariant plant described by the difference equation

$$x_{k+1} = Ax_k + Bu_k + w_k,\ k \geq 0, \tag{3.1}$$

where $x_k \in \mathbb{R}^n$ is the plant's state with x_0 given, $u_k \in \mathbb{R}^m$ the driving input, and $\{w_k, k \geq 0\}$ is the process noise composed of independent identically distributed (i.i.d) n-dimensional Gaussian random variables $w_k \sim \mathcal{N}_{0,W}$ with zero mean and covariance W. We assume the plant is unstable ($\lambda_{\max}(A) > 1$) but that (A, B) is stabilizable.

The wireless sensor collects state measurements x_k that it communicates with power $p_k \in [0, p_{\max}]$ over a wireless fading channel with coefficient h_k. At the other side of the channel the receiver/controller uses the received information to determine a control input u_k that it feedbacks into the plant.

The channel fading state $h_k, k \geq 0$ is modeled as an i.i.d. process (block fading cf. 2.2) with some known probability distribution ϕ_h on the positive reals $\mathbb{R}_+$, independent of the plant process noise $\{w_k, k \geq 0\}$. We make the technical assumption that the distribution of the channel state is absolutely continuous, i.e., has a probability density function on $\mathbb{R}_+$. To allow for transmissions adapted to the current channel conditions the transmitter has access to the channel state information h_k before transmitting at time k.

Following the wireless channel modeled introduced in 2.2, the probability of successfully decoding the message at the receiver/controller at time k, denoted hereby by q_k, is a function of the current channel state h_k as well as the current transmit power p_k,

$$q_k = q\left(h_k, p_k\right). \tag{3.2}$$

The typical form of this function is depicted in Fig. 2. We assume that $q\left(h, p\right)$ is a known increasing function of the product $h\,p$. We denote the success of transmission at each time step k by an indicator variable $\gamma_k \in \{0, 1\}$. Variables $\gamma_k \sim \text{Bern}(q_k)$ are Bernoulli distributed with time-varying success probabilities q_k given by (3.2). With this communication model the controller receives the output of the decoding process which we model by the signal $y_k = \gamma_k x_k$. We further

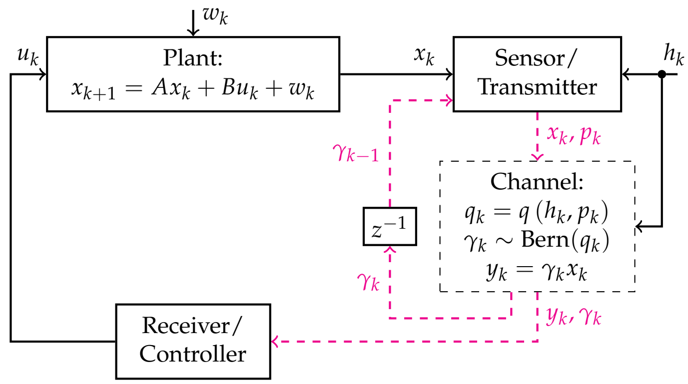

Figure 5: Wireless control system architecture. A sensor measures the plant and wireless fading channel states x_k, h_k respectively and transmits with power p_k. Messages are successfully decoded at the controller with probability q_k that depends on the channel state h_k and the power p_k. The sensor receives acknowledgments with a one-step delay.

assume that the controller also gets γ_k so that it can distinguish between the cases $x_k = 0$ and $\gamma_k = 0$. Packet receipt acknowledgment γ_k is also sent to the sensor as provided by 802.11 and TCP protocols. We assume lossless acknowledgments, so that the sensor knows what information is received at the controller.

The problem addressed in this chapter is the joint design of the control inputs u_k and the transmit powers p_k. The control input u_k is determined by the received information $y_{0:k}$, $\gamma_{0:k}$. The power p_k is determined as a function of the plant state measurements $x_{0:k}$, the observed channel realizations $h_{0:k}$, and the controller acknowledgments $\gamma_{0:k}$. Informally, to conserve power at the sensor side we want to transmit information only when the state x_k deviates from its desired value or when the channel realization h_k is favorable. In the first case transmission is necessary to keep the plant under control. In the latter case the transmission cost is minimal. A formal problem specification is presented in the next section after the following remarks.

To formulate the joint design of plant controller and power management we introduce an equivalent architecture. In view of (3.2), choosing p_k is equivalent to choosing the desired probability of successful decoding q_k at time k and transmitting with the minimum required power to achieve this q_k, namely

$$p_k = p(h_k, q_k) := \inf \{0 \leq p \leq p_{\max} : q(h_k, p) \geq q_k\}. \tag{3.3}$$

We can therefore interpret q_k as our decision variable with $p(h_k, q_k)$ denoting the cost of selecting transmission success probability q_k. This leads to the equivalent control system architecture shown in Fig. 6 where a scheduler block responsible for deciding q_k replaces the sensor/transmitter block of Fig 5. Our formulation generalizes the simple transmit-or-not decision as considered in, e.g., Xu and Hespanha (2004).

We note for future reference that the assumed monotonicity of the function $q(h, p)$ on the product $h\,p$ implies that the power function $p(h, q)$ is increasing in q and decreasing in h. Using maximum

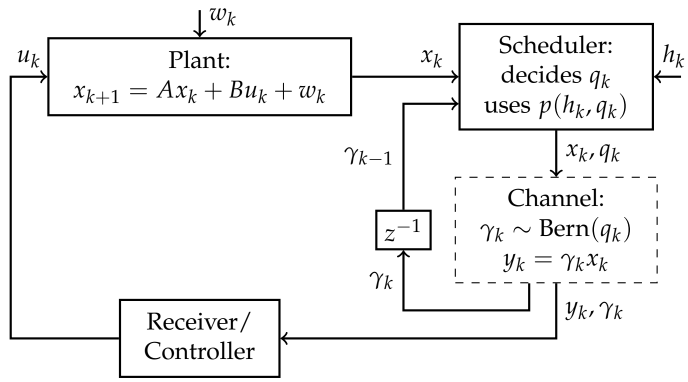

Figure 6: Equivalent wireless control system architecture. A scheduler decides the successful decoding probability q_k and transmits the state measurement x_k with the required power $p_k = p(h_k, q_k)$. The controller receives the message with probability q_k.

power $p_{\max}$, the transmitter can achieve a maximum successful decoding probability $q_{\max}(h) := q(h, p_{\max})$ for a given channel state h. Therefore, the decision variables q_k belong in the interval $[0, q_{\max}(h_k)]$. We also make the following assumptions.

Assumption 3.1. *The maximum achievable successful decoding probability $q_{\max}(h)$ satisfies*

$$\mathbb{E}_h q_{\max}(h) > q_{crit} := 1 - 1/\lambda_{\max}(A)^2, \tag{3.4}$$

where expectation is taken over the channel distribution ϕ_h.

Assumption 3.2. *For any channel realization h, the function $p(h, q)$ in (3.3) is continuous in the successful decoding probability variable q.*

Assumption 3.1 is essentially a stability condition, which as we will see in the following section states that transmitter has enough power to keep the plant state bounded in second moment, and it will be used to establish our main Theorems 3.1 and 3.2. Assumption 3.2 is of a technical nature and will be used in Theorem 3.2.

In the architecture of Fig. 6 the communication decision q_k is chosen as a function of the information available at the sensor, while the plant control signal u_k is a function of the information available at the controller. These choices are in general allowed to be randomized. The sequence $\pi := \{q_0, q_1, \ldots\}$, or equivalently the power allocation $\{p_0, p_1, \ldots\}$, is termed the communication policy, whereas the sequence $\theta := \{u_0, u_1, \ldots\}$ denotes the control policy. With fixed policies π, θ, all random variables are defined on an appropriate product space and have a measure that we denote as $\mathbb{P}^{\pi,\theta}$. We use $\mathbb{E}^{\pi,\theta}$ to denote integration with respect to $\mathbb{P}^{\pi,\theta}$, which we simplify to $\mathbb{E}$ when not leading to confusion. We remark that sensor and controller know each other's policy.

The policy pair (π, θ) incurs a control cost and a communication cost. As a control cost we adopt the standard linear quadratic regulator cost

$$J_{\text{LQR}}^N(\pi, \theta) := \mathbb{E}^{\pi,\theta} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k, \tag{3.5}$$

for some pair of matrices $R > 0$ and $Q \geq 0$, with $(A, Q^{1/2})$ detectable. The communication cost is given by the expected power consumption

$$J_{\text{PWR}}^N(\pi, \theta) := \mathbb{E}^{\pi,\theta} \sum_{k=0}^{N-1} p(h_k, q_k). \tag{3.6}$$

To quantify the tradeoff between plant performance and power consumption we combine the LQR cost in (3.5) and the power cost in (3.6) into the limit aggregate cost

$$J(\pi, \theta) := \limsup_{N \to \infty} \frac{1}{N} \left[J_{\text{LQR}}^N(\pi, \theta) + \lambda J_{\text{PWR}}^N(\pi, \theta) \right], \tag{3.7}$$

for some positive constant $\lambda > 0$. Our goal is to design plant and power control policies θ and π respectively that minimize the joint cost (3.7). These policies depend on what information is available to the sensor and controller. The specific information structure considered is introduced next.

3.2. Information structure

Denote as O_k the information known at the controller side at time k just before deciding the input u_k. This information includes the given initial plant state x_0, the history of decoding success variables $\gamma_{0:k}$ and the decoded signals $y_{0:k}$, as well as the previously chosen control inputs $u_{0:k-1}$, i.e.,

$$O_k := \{x_0, \gamma_{0:k}, y_{0:k}, u_{0:k-1}\}. \tag{3.8}$$

Then the control input u_k is chosen as a function of O_k, or more formally, measurable with respect to the σ-field generated by O_k.

Given the possibility of lost packets as indicated by $\gamma_k = 0$, the controller has partial information on the plant state x_k. It is then of importance to study the MMSE estimate $\mathbb{E}^{\pi,\theta}(x_k|O_k)$. This estimation is complicated by the fact that the event $\gamma_k = 0$ possibly contains information about the state x_k through the dependence of the probability q_k on the value of x_k – see Remark 3.1. To avoid this complication we discard the information given by events of the form $\gamma_k = 0$. Formally, define $\tau_k := \max\{0 \leq l \leq k : \gamma_l = 1\}$ as the time of the last successful transmission by time k and define the sequence

$$G_k := \{x_0, \gamma_{0:\tau_k}, y_{0:\tau_k}, u_{0:k-1}\}. \tag{3.9}$$

with $G_0 = \{x_0\}$. When $\gamma_k = 1$, G_k coincides with O_k. When $\gamma_k = 0$, G_k only contains information received till the last successful transmission which occurred at time $\tau_k < k$.

We restrict attention to control policies θ selecting inputs u_k as functions of G_k, possibly randomized, and denote the set of all such policies by Θ. Unlike $\mathbb{E}^{\pi,\theta}(x_k|O_k)$, the state MMSE estimate $\hat{x}_k := \mathbb{E}^{\pi,\theta}(x_k|G_k)$ with respect to G_k is easy to compute. When $\gamma_k = 1$ the state $x_k = y_k$ becomes known at the receiver side. When $\gamma_k = 0$ no new information becomes available and $\hat{x}_k$ is obtained by propagating $\hat{x}_{k-1}$ through the plant's dynamics in (3.1). Put together, we get

$$\hat{x}_k := \mathbb{E}^{\pi,\theta}(x_k|G_k) = \begin{cases} y_k & \text{if } \gamma_k = 1, \\ A\hat{x}_{k-1} + Bu_{k-1} & \text{if } \gamma_k = 0 \end{cases}, \tag{3.10}$$

with $\hat{x}_0 = x_0$ since the initial state is given.

At the other side of the link at time k the sensor/transmitter has access to the channel realization h_k and the plant state x_k which allows selection of the successful transmission probability q_k to depend on the values of both of h_k, x_k. This affects the controller design however, because when the controller decides u_{k-1} to control x_k, it should consider the indirect effect on q_k. This information structure renders the joint communication and control co-design problem hard to analyze. To overcome this, we restrict transmission policies to depend on the channel state h_k and the information about plant state x_k that the controller does not know. More precisely consider the difference between the sensor measurement x_k and the controller's estimate $\hat{x}_k$ by (3.10) if the kth packet is *not* successfully decoded, that is

$$\varepsilon_k := x_k - (A\hat{x}_{k-1} + Bu_{k-1}), \tag{3.11}$$

with $\varepsilon_0 := 0$. Observe that the term in the parenthesis is known to the sensor since by the acknowledgment mechanism the controller's previous estimate $\hat{x}_{k-1}$ and input u_{k-1} can be replicated at the sensor. Alternatively the terms ε_k can be viewed as the innovations of the controller's estimate (3.10) when a new message is received.

We restrict then information at the sensor side to the set F_k defined as a collection of the channel history $h_{0:k}$, the history of innovations $\varepsilon_{0:k}$, and past decisions $q_{0:k-1}$, i.e.,

$$F_k := \{\varepsilon_{0:k}, h_{0:k}, q_{0:k-1}\}. \tag{3.12}$$

We also add a technical requirement that the sensor selects maximum transmit power $p_{\max}$ when the innovation ε_k gets too large, $\|\varepsilon_k\| \geq L$ for some positive constant $L > 0$, and the channel gain h_k is favorable, $h_k \geq h_t$ for some threshold $h_t > 0$ on channel values. We assume that a positive success probability $q_{\max}(h_t) > 0$ is achieved at this threshold and also that

$$\int_{h \geq h_t} q_{\max}(h)\, d\phi_h(h) > q_{\text{crit}}, \tag{3.13}$$

where q_{crit} is given in (3.4) and the integration is over the channel distribution ϕ_h. Such a channel threshold exists by Assumption 3.1. We consider then communication policies π selecting decoding success q_k as functions of F_k for each k, possibly randomized, satisfying $q_k \in Q(\varepsilon_k, h_k)$

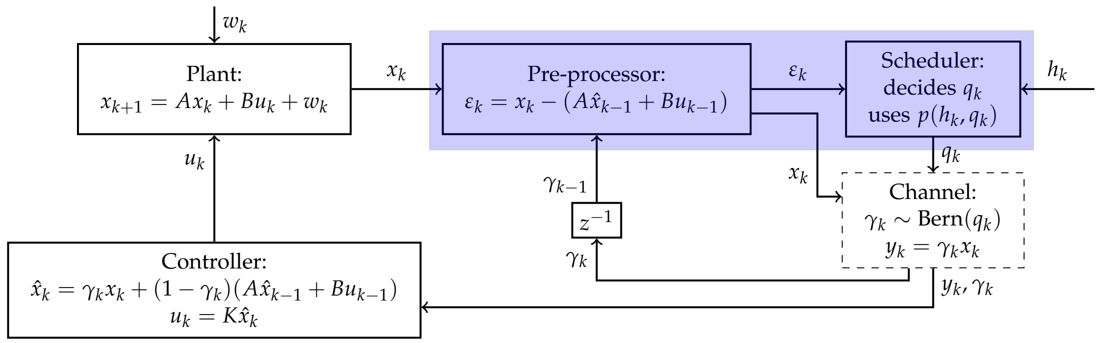

Figure 7: Wireless control system with a restricted information structure. The sensor consists of two blocks. A pre-processor computes the error ε_k given the measurement x_k and the acknowledgment γ_{k-1}. A scheduler decides q_k based on ε_k and the channel state h_k, and transmits x_k with the required power $p_k = p(h_k, q_k)$. The controller receives the message with probability q_k, computes the state estimate $\hat{x}_k$ and provides input u_k to the plant.

where

$$Q(\varepsilon, h) := \begin{cases} q_{\max}(h) & \text{if } \|\varepsilon\| \geq L \text{ and } h \geq h_t \\ [0, q_{\max}(h)] & \text{otherwise} \end{cases} . \tag{3.14}$$

We denote the set of all such policies with Π. The technical power saturation requirement is inconsequential as we may pick L arbitrarily large, and will be used to prove Proposition 3.2 and Theorem 3.2 in the sequel. Similar requirements have been introduced in Xu and Hespanha (2004); Mesquita et al. (2012), however our setup is further complicated by the availability of the random channel states.

The proposed information structure is depicted in Fig. 7. The sensor block is split into a pre-processor and a scheduler. The pre-processor computes ε_k based on the sample x_k and the acknowledgment γ_{k-1} and feeds it to the scheduler who, upon measuring the channel h_k decides the transmission success probability q_k while incurring power cost $p(h_k, q_k)$. Our goal in this paper is to study policies $\pi \in \Pi$ and $\theta \in \Theta$ that are optimal with respect to the joint objective (3.7), that is

$$\underset{\pi \in \Pi, \theta \in \Theta}{\text{minimize}} \quad J(\pi, \theta). \tag{3.15}$$

In particular, the next section shows that the information structure we introduced allows optimal communication and control policies to be designed separately. The standard LQR controller is shown to be optimal and we then leverage this result to study optimal communication policies in Section 3.4.

Remark 3.1. *If the controller uses the complete information O_k to estimate x_k, the optimal plant estimate is not $\hat{x}_k$ as given by (3.10). When a packet drop $\gamma_k = 0$ is observed, and since the communication policy is known, the controller should consider the possibility that the sensor did not transmit at all, which could in general give indirect information about the value of x_k – see also Molin and Hirche (2009); Imer and Başar (2010); Lipsa and Martins (2011); Rabi et al. (2012); Ramesh et al. (2013); Nayyar et al. (2013) for further discussion on this issue. The restriction to G_k in (3.9) allows to overcome this issue and obtain* *linear dynamics of the estimation error e_k and the related ε_k as described next in (3.16),(3.17), but it is not needed for the separation result of Prop. 3.1 in the following section to hold. An alternative approach to overcome this issue, followed by other works, is to restrict the communication policy design to transmission policies with suitable symmetries facilitating estimation (Imer and Başar, 2010; Rabi et al., 2012; Ramesh et al., 2013; Han et al., 2015).*

3.3. Separation of designs

In this section we show that with the imposed restrictions on the information available at sensor and controller the control law $\theta \in \Theta$ and the communication policy $\pi \in \Pi$ can be designed separately. In particular the control policy has no effect on the estimation process at the receiver and by utilizing a separation principle the optimal controller becomes the standard linear quadratic one.

Let us denote the difference between the plant state and the estimate kept at the controller by $e_k := x_k - \hat{x}_k$ and its covariance as seen at the controller by $\Sigma_k := \mathbb{E}^{\pi}\left[e_k e_k^T \,\middle|\, G_k\right]$. The estimation error dynamics can be found by subtracting (3.10) from the system dynamics (3.1) to get

$$e_k = (1 - \gamma_k)(Ae_{k-1} + w_{k-1}), \tag{3.16}$$

with $e_0 = 0$ since x_0 is given. Stabilizability of estimation error is guaranteed by Assumption 3.1. Indeed if transmitter were to use maximum power all the time the dynamics in (3.16) become a jump linear system since γ_k are Bernoulli with constant probability equal to the left hand side of (3.4). Then condition (3.4) is sufficient for bounded second moment as, e.g., in (Hespanha et al., 2007, Theorem 2). It is also tight in the sense that estimation error becomes unstable if $\mathbb{E}_h q_{\max}(h) < q_{\text{crit}}$.

Turning our attention to the innovation substituting x_k by (3.1) in the definition of ε_k in (3.11) gives $\varepsilon_k = Ae_{k-1} + w_{k-1}$. The term e_{k-1} equals $(1 - \gamma_{k-1})\varepsilon_{k-1}$ as seen by (3.16), therefore ε_k evolves according to

$$\varepsilon_k = (1 - \gamma_{k-1})A\varepsilon_{k-1} + w_{k-1}, \tag{3.17}$$

with initial value $\varepsilon_0 = 0$. The following proposition establishes a separation principle in our restricted information structure setup, stating that the control action has no effect on the quality of the future estimates at the controller.

Proposition 3.1. *Consider any communication policy π selecting successful decoding probabilities q_k as functions of F_k given in (3.12), possible randomized, with ε_k defined in (3.11) and channel states h_k independently drawn from a distribution ϕ_h. Then at any step k the distributions of the future processes $\{\varepsilon_\ell, q_\ell, \gamma_\ell, e_\ell, \ell > k\}$ given G_k do not depend on the chosen control policy $\theta \in \Theta$.*

Proof. First note that the processes $\{w_k, h_k, k \geq 0\}$ are by assumption independent of any other process. Then we follow an induction argument to prove the claim. At $k = 0$, ε_0 is equal to 0,

q_0 depends only on h_0 and ε_0, γ_0 is an independent Bernoulli with success q_0, and e_0 is also 0 since x_0 is initially known. Consider then a time k with a given G_k, the corresponding estimation error e_k given G_k having zero mean and covariance Σ_k, and a control input u_k that is a function of G_k as described by the control policy θ. The term ε_{k+1} equals $Ae_k + w_k$, as indicated by the arguments preceding (3.17), which given G_k has mean 0 and covariance $A\Sigma_k A^T + W$. The choice $q_{k+1} \in F_{k+1}$ by construction depends on past variables in F_k which by causality do not depend on the action u_k, as well as the new variables $\varepsilon_{k+1}, h_{k+1}$ which are also independent of u_k. Also the distribution of $\gamma_{k+1} \sim \text{Bern}(q_{k+1})$ only depends on the distribution of q_{k+1}, and the same holds for e_{k+1} which equals $(1-\gamma_{k+1})\varepsilon_{k+1}$ again by the arguments preceding (3.17). To sum up all variables $\varepsilon_{k+1}, q_{k+1}, \gamma_{k+1}, e_{k+1}$ given G_k do not depend on u_k. ■

The intuition behind this proposition is that the effect of control inputs is subtracted from x_k when forming the innovation terms ε_k in (3.11) that are fed to the communication policy π. Similar separation results based on innovation terms have been utilized in other communication/control design problems (Molin and Hirche (2009); Ramesh et al. (2013); Nair et al. (2007)). The above proposition restates the separation principle for our power allocation problem under channel state information.

Since the power cost $J^N_{\text{PWR}}(\pi, \theta)$ in (3.6) only depends on pairs (q_k, h_k), the above proposition shows that the control policy θ has no effect on the power cost. Thus we can rewrite the objective in (3.7) as

$$J(\pi, \theta) = \limsup_{N\to\infty} \frac{1}{N} J^N_{\text{LQR}}(\pi, \theta) + \lambda \limsup_{N\to\infty} \frac{1}{N} J^N_{\text{PWR}}(\pi). \tag{3.18}$$

This means that the optimal control policy $\theta \in \Theta$ for a given communication policy $\pi \in \Pi$ is the one minimizing the limit LQR cost. It turns out that the form of the optimal controller does not depend on the communication policy, leading to a stronger separability than what follows from (3.18).

Indeed by the above separation principle standard dynamic programming arguments show that the optimal control law for a finite horizon is given by the standard LQR one, as in, e.g., Molin and Hirche (2009); Ramesh et al. (2013). We are interested however in the infinite horizon problem. Our setup differs from the standard LQG/Kalman filtering problem with state observations y_k containing Gaussian noise, where the estimation error covariance Σ_k converges to some limit and the system is assumed to start at time $k = 0$ with this limit estimation error. In our setup whenever a packet is received the estimation error is reset to zero otherwise it grows (cf. (3.16)), so for the general communication policies $\pi \in \Pi$ under consideration it is not clear whether some limit covariance exists. Alternatively the following proposition shows that estimation errors admit a uniform bound in second moment.

Proposition 3.2. *Suppose Assumption 3.1 holds. Then there exists a finite positive constant M such that for any communication policy $\pi \in \Pi$ selecting successful decoding probabilities q_k with respect to F_k given* *in (3.12), possibly randomized, satisfying the additional restriction $q_k \in Q(\varepsilon_k, h_k)$ given by (3.14), and for every $k = 0, 1, \ldots$, it holds that*

$$\mathbb{E}^{\pi} e_k^T e_k \leq M. \tag{3.19}$$

With this bound on expected magnitude of estimation error established, uniform over k and over any policy $\pi \in \Pi$, the following theorem determines the optimal control law for the average infinite horizon problem (3.15).

Theorem 3.1 (Optimal control policy)**.** *Consider the wireless control system of Fig. 7 with any communication policy $\pi := \{q_0, q_1, \ldots\} \in \Pi$ selecting successful decoding probabilities q_k as functions of F_k given in (3.12), possibly randomized, with innovation terms ε_k as defined in (3.11) and channel states h_k independently drawn from a distribution ϕ_h, satisfying the additional restriction $q_k \in Q(\varepsilon_k, h_k)$ given by (3.14). Suppose Assumption 3.1 holds. Then for any control policy $\theta := \{u_0, u_1, \ldots\} \in \Theta$ composed of inputs u_k as possibly randomized functions of G_k in (3.9) such that*

$$\lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}^{\pi,\theta} x_N^T x_N = 0, \tag{3.20}$$

the joint objective $J(\pi, \theta)$ described by (3.5) - (3.7) satisfies

$$J(\pi,\theta) \geq Tr(PW) + \limsup_{N\to\infty} \frac{1}{N} \mathbb{E}^{\pi} \sum_{k=0}^{N-1} e_k^T \tilde{P} e_k + \lambda p(h_k, q_k) \tag{3.21}$$

where P is the solution to the standard algebraic Riccati equation $P = A^T PA + Q - A^T PB(R + B^T PB)^{-1} B^T PA$ for the system in (3.1) and the linear quadratic regulator cost (LQR) in (3.5), and the matrix $\tilde{P}$ is defined as

$$\tilde{P} := A^T PA + Q - P. \tag{3.22}$$

Moreover, the minimum value in (3.21) is achieved for the control policy

$$u_k = K\hat{x}_k, \tag{3.23}$$

with $\hat{x}_k$ the state estimate described in (3.10) and the steady state LQR gain $K := -(R + B^T PB)^{-1} B^T PA$.

The theorem determines the optimal control policy θ of problem (3.15) as the conventional LQR controller in (3.23), shown in Fig. 7. The optimal cost given in (3.21) equals a constant $Tr(PW)$ and a limit average sum term that only depends on the communication policy $\pi \in \Pi$. This term shows that the optimal communication policy needs to balance the power expenditures with a weighted version of the estimation error at the controller.

Observe that as per (3.10) and (3.16) it holds that $e_k = (1 - \gamma_k)\varepsilon_k$. Also $\mathbb{E}^{\pi}[\gamma_k | F_k] = \mathbb{P}^{\pi}[\gamma_k = 1 | F_k] = q_k$ and $\varepsilon_k \in F_k$. So we can write

$$\mathbb{E}^{\pi}[e_k^T \tilde{P} e_k | F_k] = \mathbb{E}^{\pi}[(1 - \gamma_k)\varepsilon_k^T \tilde{P} \varepsilon_k | F_k] = (1 - q_k)\varepsilon_k^T \tilde{P} \varepsilon_k, \tag{3.24}$$

and taking the expectation in both sides gives

$$\mathbb{E}^{\pi}[e_k^T \tilde{P} e_k] = \mathbb{E}^{\pi}[(1-q_k)\varepsilon_k^T \tilde{P} \varepsilon_k]. \tag{3.25}$$

Substituting the expression (3.25) into the second summand of (3.21) it follows that the optimal communication policy $\pi \in \Pi$ of problem (3.15) is the one achieving the infimum cost

$$J^*_{\text{COMM}} := \inf_{\pi \in \Pi} \lim_{N \to \infty} \frac{1}{N} \mathbb{E}^{\pi} \sum_{k=0}^{N-1} c(\varepsilon_k, h_k, q_k), \tag{3.26}$$

where we define

$$c(\varepsilon, h, q) := (1-q)\varepsilon^T \tilde{P} \varepsilon + \lambda p(h, q). \tag{3.27}$$

The difference between the sum in (3.21) and the objective in (3.26) is that in the former e_k is not known at the sensor at time k, while ε_k in the latter is. This way (3.26) takes the form of a Markov decision process (MDP) problem with an infinite horizon average cost criterion. The state of the problem at time k is the pair $(\varepsilon_k, h_k) \in \mathbb{R}^n \times \mathbb{R}_+$, the available action is $q_k \in Q(\varepsilon_k, h_k)$ by (3.14), and the cost-per-stage is $c(\varepsilon_k, h_k, q_k)$. The state transition probabilities can be obtained from (3.17) and are given by

$$\begin{aligned} &\mathbb{P}(\varepsilon_+, h_+ | \varepsilon, h, q) \\ &= [q\, \mathcal{N}_{0,W}(\varepsilon_+) + (1-q)\, \mathcal{N}_{A\varepsilon,W}(\varepsilon_+)]\ \phi_h(h_+). \end{aligned} \tag{3.28}$$

Here ε, h and ε_+, h_+ denote the current and next states respectively, and q the current action. When q is chosen at state (ε, h), a variable $\gamma \sim \text{Bern}(q)$ is drawn. By (3.17) on the event $\gamma = 1$, $\varepsilon_+ = w \sim \mathcal{N}_{0,W}$, while on the event $\gamma = 0$, $\varepsilon_+ = A\varepsilon + w$ with $w \sim \mathcal{N}_{0,W}$, which is equivalent to $\varepsilon_+ \sim \mathcal{N}_{A\varepsilon,W}$. Since h_+ is independent of $\varepsilon, h, \varepsilon_+$, its distribution ϕ_h appears as a product in (3.28). We denote $\mathbb{E}\left[\varepsilon_+, h_+ \,\middle|\, \varepsilon, h, q\right]$ the integration with respect to the above transition probability measure.

To sum up, we have exploited the proposed decoupling information structure to determine the optimal control policy as the standard LQR control input. We proceed in the following section to show that an optimal communication policy exists and we characterize its main features in the case of general FEC codes and in the special case of capacity achieving codes.

Remark 3.2. *The technical condition (3.20) for the controller in Theorem 3.1 can be viewed as an additional stability condition requiring that the norm of the plant state grows at a sub-linear rate. Such conditions appear in general average cost optimal control problems, see e.g. (Bertsekas, 2005, Vol.II, p.254-5), and have also been used in average LQG problems (Bertsekas, 2005, Vol.II, p.272-3). This technical condition may potentially be relaxed under a different proof technique.*

3.4. Optimal Communication Policy

Exploiting the MDP formulation of (3.26) we can show that optimal communication policies for the co-design problem in (3.15) exist. This existence result provides a characterization of these policies from which we infer the general features of optimal transmit powers p_k and corresponding successful decoding probabilities q_k as a function of innovation terms ε_k and channel realizations h_k.

The existence of optimal policies for average infinite-horizon MDPs on general state spaces requires some technical conditions (Hernández-Lerma and Lasserre (1996)). In our case restriction to communication policies $\pi \in \Pi$ that uniformly satisfy (3.14) guarantee existence, as the following theorem shows.

Theorem 3.2 (Optimal communication policy)**.** *Consider the Markov decision process with optimal cost as in (3.26), state transition probabilities as in (3.28), and actions restricted to $q_k \in Q(\varepsilon_k, h_k)$ with $Q(\varepsilon, h)$ abiding to (3.14). If Assumptions 3.1 and 3.2 hold true there exists a function $V : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ such that for all $\varepsilon \in \mathbb{R}^n$ and $h \in \mathbb{R}_+$ it satisfies*

$$V(\varepsilon, h) = \min_{q \in Q(\varepsilon,h)} \{c(\varepsilon, h, q) - J^*_{COMM} \\ + \mathbb{E}\left[V(\varepsilon_+, h_+) \,\middle|\, \varepsilon, h, q\right]\}. \tag{3.29}$$

The optimal communication cost can be written as $J^*_{COMM} = \mathbb{E}_{w,h} V(w, h)$, where $\mathbb{E}_{w,h}$ denotes integration with respect to the product measure $\mathcal{N}_{0,W} \times \phi_h$. The optimal communication policy π^* achieving the minimum cost can be written as a function of the error and channel states at time k, $q^*_k = q^*(\varepsilon_k, h_k)$, and is the one achieving the minimum in the right hand side of (3.29), i.e.

$$q^*(\varepsilon, h) := \operatorname*{argmin}_{q \in Q(\varepsilon,h)} \{c(\varepsilon, h, q) - J^*_{COMM} \\ + \mathbb{E}\left[V(\varepsilon_+, h_+) \,\middle|\, \varepsilon, h, q\right]\}. \tag{3.30}$$

The theorem states that the optimal communication policy exists, is deterministic, and also stationary in the sense that q^*_k adapts only to the current state (ε_k, h_k) and not the complete history F_k in (3.12). The optimal policy is described by (3.30) in terms of a function $V(\varepsilon, h)$ that solves (3.29). Note that this function is unique up to a constant. Related characterizations of optimal communication policies when the decision is whether to transmit or not appear in Xu and Hespanha (2004); Mesquita et al. (2012). Our setup however differs since the decision is on the transmit power and this depends on the random wireless channel state. The proof of the theorem relies on constructing a Lyapunov-like function that is common for all policies $\pi \in \Pi$, and applying the MDP results of Vega-Amaya (2003). This methodology has been used in Mesquita et al. (2012), however a refined construction is required here to account for the random channel states.

An informal interpretation of condition (3.29) based on finite state spaces (Bertsekas, 2005) is the following. Suppose the constant J^*_{COMM}, corresponding to the optimal cost of (3.26), was known. Subtracting this constant from the cost-per-stage in problem (3.26) does not change the optimal policy, and gives a relative cost per stage $c(\varepsilon, h, q) - J^*_{\mathrm{COMM}}$ indicating how far we are from the optimal average cost. Then equation (3.29) has exactly the form of a standard Bellman equation for a non-averaged infinite horizon problem with this relative cost per stage (Bertsekas, 2005, Vol.I, Ch.7). The function $V(\varepsilon, h)$ captures the expected future relative cost of following the optimal policy when starting from state (ε, h), and is termed the relative value function. Bellman's equation (3.29) states that the optimal choice q at every step minimizes the sum of the current-stage relative cost $c(\varepsilon, h, q) - J^*_{\mathrm{COMM}}$ and the expected future relative cost $\mathbb{E}\left[V(\varepsilon_+, h_+) \,\middle|\, \varepsilon, h, q\right]$. The minimization over the current action q gives again the value $V(\varepsilon, h)$ of the current state at the left hand side of (3.29).

In principle one can find $V(\varepsilon, h)$ using value iteration or policy iteration algorithms which involve iterative application of (3.29) (Hernández-Lerma and Lasserre, 1996). This procedure is, however, computationally onerous as each iteration requires minimizing the right hand side of (3.29) for all possible state pairs $(\varepsilon, h) \in \mathbb{R}^n \times \mathbb{R}_+$. Nevertheless, (3.29) still gives qualitative information on the optimal policy.

Let us ignore the case $\|\varepsilon\| \geq L, h \geq h_t$ in (3.14) as it is irrelevant for the following discussion. Integrating $V(\varepsilon, h)$ with respect to the transition (3.28) gives

$$\begin{aligned}&\mathbb{E}\left[V(\varepsilon_+, h_+)|\varepsilon, h, q\right] \\ &= q\mathbb{E}_{w,h_+}V(w, h_+) + (1-q)\mathbb{E}_{w,h_+}V(A\varepsilon + w, h_+).\end{aligned} \tag{3.31}$$

We substitute this, the cost-per-stage $c(\varepsilon, h, q)$ defined by (3.27), and the expression $J^*_{\mathrm{COMM}} = \mathbb{E}_{w,h}V(w, h)$ provided by the theorem in the minimization of (3.30), and upon reordering terms, the optimal communication policy can be written as

$$q^*(\varepsilon, h) = \operatorname*{argmin}_{q \in [0, q_{\max}(h)]} \lambda p(h, q) + (1-q)R(\varepsilon), \tag{3.32}$$

where for convenience we defined the function

$$R(\varepsilon) := \mathbb{E}_{w,h}\left[V(A\varepsilon + w, h) - V(w, h)\right] + \varepsilon^T \tilde{P}\varepsilon, \tag{3.33}$$

which can be thought as a penalty function on the error ε. The optimal policy $q^*(\varepsilon, h)$ depends on the shape of the function $p(h, q)$ and takes values anywhere in the interval $[0, q_{\max}(h)]$. The optimal power allocation can be found by converting (3.32) to power by (3.2), (3.3), and is described by

$$p^*(\varepsilon, h) := \operatorname*{argmin}_{p \in [0, p_{\max}]} \lambda p + (1 - q(h, p))R(\varepsilon). \tag{3.34}$$

Despite the fact that $V(\varepsilon, h)$ and $R(\varepsilon)$ are hard to compute, the above expression is an important characterization of the optimal power allocation. It provides a tool for qualitative analysis of

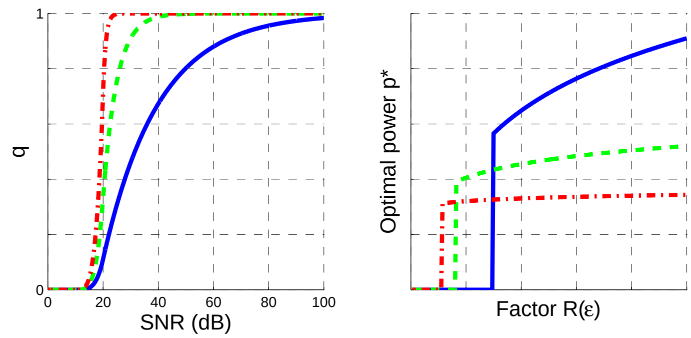

Figure 8: Optimal power allocation for FEC codes with different complementary error functions. The optimal transmit power p^* is plotted as a function of the factor $R(\varepsilon)$ for a fixed channel state h using FEC codes with different q-SNR characteristics. When the q-SNR curve becomes steeper, the optimal power allocation resembles a step function.

different FEC codes in wireless control systems. We illustrate this in Fig. 8 where we examine how the q-SNR relationship of a FEC code affects the optimal power allocation. For simplicity we fix the channel state h and plot p^* in Fig. 8 as a function of $R(\varepsilon)$. In all cases, when the error penalty $R(\varepsilon)$ is below some threshold, the best option is to not transmit. Above the threshold, the optimal transmit power increases as the error penalty $R(\varepsilon)$ gets larger. For powerful FEC codes characterized by a steep q-SNR relationship, close to the theoretical limit in (2.19), the optimal power allocation resembles a step function, since the probability of successful decoding becomes practically one for large powers. For fat q-SNR tails, this behavior deteriorates as the sensor needs to transmit with higher power to achieve a larger q.

Then in Fig. 9 we present qualitative plots of the optimal decoding probability q^* and optimal transmit power p^* as functions of both the factor $R(\varepsilon)$ and the channel state h for a given q-SNR characteristic. The blue region indicates the event where no transmission occurs. This happens if channel gain h is low, where transmission is costly, or if error ε has a low penalty, meaning that there is no need to update the receiver's estimate. This no-transmission region becomes larger for a higher penalty λ on power in (3.7). Outside this region a transmission occurs and transmit power adapts to both channel and error states. In principle when channel gain h is high, a small amount of power suffices. For intermediate values of channel h power takes a wide range of values depending on the error as well.

Overall this optimal power management displays different features than the standard "0-1" event-triggered transmission paradigm of, e.g., Xu and Hespanha (2004) or Tabuada (2007). It can be though as a 'soft" version of these policies since the power decision ranges between $[0, p_{\max}]$, or equivalently the decoding q between $[0, q_{\max}(h)]$. Finally we note that the transmit power/remote estimation problem has also been studied in the very recent works of Quevedo et al. (2012); Leong and Dey (2012), which however consider only power adaptation to the channel and the packet drop processes, not the plant state/error observed online. Hence the qualitative characterization we discuss here and the connections with the event-triggered paradigm were not apparent.

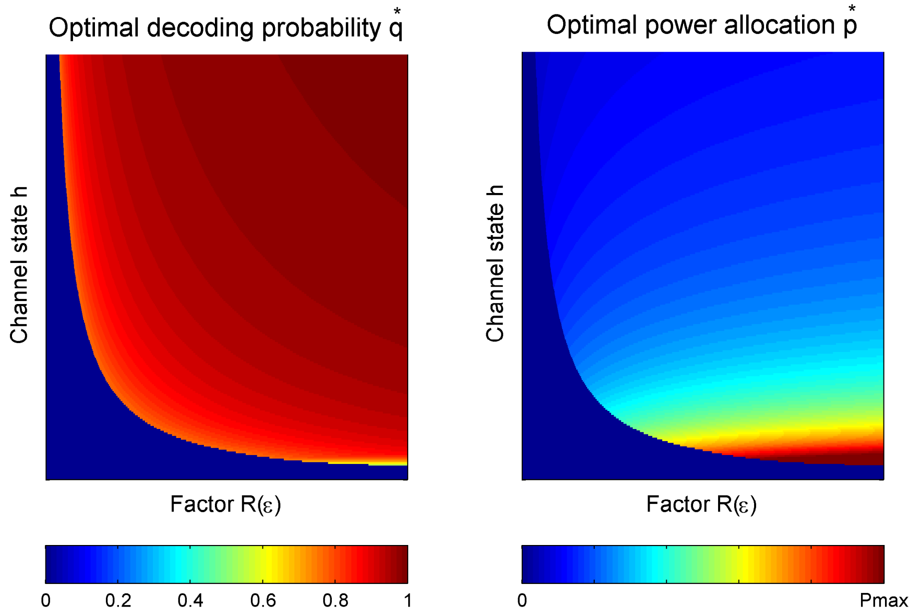

Figure 9: Optimal decoding probability and power allocation for a FEC code. Color intensity indicates the magnitude of optimal decoding probability q^* and optimal transmit power p^* as functions of the factor $R(\varepsilon)$ and the channel state h.

Since computing the optimal power allocation becomes a computationally hard problem, tractable suboptimal power allocation policies are developed in Chapter 6. In the following chapter we consider the problem of multiple wireless control systems sharing a wireless medium and design appropriate resource-aware and control-aware scheduling policies.

3.4.1. Optimal solution for capacity achieving codes

Consider now the case of capacity achieving codes. By (2.19), at time k the transmitter needs to use either $p_k = 0$, i.e. not transmitting, or $p_k = p_0/h_k$ with $p_0 := N_0\mathrm{SNR}_0$, which certainly guarantees correct packet delivery. Any other power allocation is unfavorable. However the instantaneous power is bounded by $p_k \leq p_{\max}$, so the sensor can transmit only when $p_0/h_k \leq p_{\max}$, or equivalently when the channel state exceeds $h_k \geq p_0/p_{\max}$.

In this case we are looking again for a randomized policy, i.e. a distribution on the two power options $\{0, p_0/h_k\}$ when $h_k \geq p_0/p_{\max}$. With a slight abuse of notation we denote $q_k \in [0,1]$ the probability of choosing power p_0/h_k. Then when $h_k \geq p_0/p_{\max}$ the transmitter draws independent $\gamma_k \sim \mathrm{Bern}(q_k)$ and transmits with power $p_k = \gamma_k\, p_0/h_k$. The decoding success at the receiver is given by the same γ_k. The expected power consumption becomes

$$\mathbb{E} \sum_{k=0}^{N-1} p_k = \mathbb{E} \sum_{k=0}^{N-1} q_k \frac{p_0}{h_k} \mathbb{I}\left(h_k \geq \frac{p_0}{p_{\max}}\right). \tag{3.35}$$

Observe that this is of the same form as the expected power consumption of the original problem given in (3.6) with the function $p(h,q)$ substituted with $q\, p_0/h\, \mathbb{I}\,(h \geq p_0/p_{\max})$. Then the statements of the results so far hold for the capacity achieving codes as well. For this special case of $p(h,q)$ however the minimization in (3.32) becomes linear in q, and the optimal communication policy is deterministic,

$$q^{CA}(\varepsilon,h) := \begin{cases} 0 & \text{if } \; h\, R(\varepsilon) \leq \lambda\, p_0 \text{ or } h \leq p_0/p_{\max} \\ 1 & \text{otherwise} \end{cases}, \tag{3.36}$$

or in terms of power

$$p^{CA}(\varepsilon,h) := \begin{cases} 0 & \text{if } \; h\, R(\varepsilon) \leq \lambda\, p_0 \text{ or } h \leq p_0/p_{\max} \\ p_0/h & \text{otherwise} \end{cases}. \tag{3.37}$$

This is an event-triggered transmission scheme along the lines of, e.g., Xu and Hespanha (2004), except that now the decision is also affected by the current channel state h apart from the error ε. This deterministic policy was expected as the limit behavior of powerful FEC codes in Fig. 8. The region of the plant/channel state space $\mathbb{R}^n \times \mathbb{R}_+$ where it is optimal to transmit is described in (3.36) as $h\, R(\varepsilon) > \lambda p_0$ and $h \geq p_0/p_{\max}$. Intuitively condition $h\, R(\varepsilon) > \lambda p_0$ states that when channel gain is large, transmitting is worthy as it does not cost much, while when an error penalty $R(\varepsilon)$ is large, it is necessary to transmit in order to reset it to zero. This region gets smaller when the constant $p_0 = N_0 \mathrm{SNR}_0$ (cf. (2.19)) increases, since transmission then requires more power, or when λ increases, since power then is penalized more in the objective (3.7).

3.5. Proofs

3.5.1. Proof of Proposition 3.2

First note that, by the same arguments we use to derive (3.25) later, conditioned on F_k we can rewrite

$$\mathbb{E}^{\pi}\, e_k^T e_k = \mathbb{E}^{\pi}(1-q_k)\varepsilon_k^T \varepsilon_k. \tag{3.38}$$

The bound in (3.19) will be shown by an equivalent bound on the innovation process $\{\varepsilon_k, k \geq 0\}$. By Proposition 3.1 for any communication policy $\pi \in \Pi$ this process is independent of the control policy $\theta \in \Theta$ and its evolution is given by (3.17). This evolution can be described more formally along with the i.i.d. channel process $h_k \sim \phi_h$ by a stochastic transition kernel given the values of ε, h and decision q at each step as

$$\begin{aligned} &\mathbb{P}(\varepsilon_+, h_+ | \varepsilon, h, q) \\ &= [q\, \mathcal{N}_{0,W}(\varepsilon_+) + (1-q)\, \mathcal{N}_{A\varepsilon,W}(\varepsilon_+)]\;\; \phi_h(h_+). \end{aligned} \tag{3.39}$$

This expression is included again in (3.28), where its derivation is explained in detail.

The following technical lemma shows that under Assumption 3.1 one can construct a Lyapunov-like function common for all communication policies, satisfying explicitly the technical requirements of (Vega-Amaya, 2003, Assumptions 3.1, 3.2). The uniform bound (3.19) will be a direct consequence of these requirements. We note also that the lemma will be subsequently used to prove Theorem 3.2 based on the results of Vega-Amaya (2003).

Lemma 3.1. *Suppose Assumption 3.1 holds and consider the innovation and channel processes* $\{\varepsilon_k, h_k, k \geq 0\}$ *described by the transition (3.39), with communication decisions satisfying* $q_k \in Q(\varepsilon_k, h_k)$ *given in (3.14). Then there exists a measurable function* W *on* $\mathbb{R}^n \times \mathbb{R}_+$ *bounded below by a constant* $\gamma > 0$ *such that*

$$(1-q)\varepsilon^T\varepsilon + c \leq KW(\varepsilon, h), \tag{3.40}$$

where $c \geq 0$ *is some constant, for all* $\varepsilon, h \in \mathbb{R}^n \times \mathbb{R}_+$*,* $q \in Q(\varepsilon, h)$*, for some positive* K*. Moreover there exists a non-trivial measure* ν *on* $\mathbb{R}^n \times \mathbb{R}_+$*, a non-negative measurable function* $r(\varepsilon, h, q) \geq 0$ *for* $\varepsilon, h \in \mathbb{R}^n \times \mathbb{R}_+$*,* $q \in Q(\varepsilon, h)$*, and a positive constant* $\mu < 1$ *such that*

$$\begin{aligned}
&\textit{(i) } \nu(W) := \int W(\varepsilon, h) d\nu(\varepsilon, h) < \infty,\\
&\textit{(ii) } \mathbb{P}(\varepsilon_+ \in B_1, h_+ \in B_2 | \varepsilon, h, q) \geq \nu(B_1, B_2) r(\varepsilon, h, q)\\
&\textit{for all measurable subsets } (B_1, B_2) \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}_+),\\
&\textit{(iii) } \mathbb{E}\left[W(\varepsilon_+, h_+) | \varepsilon, h, q\right] \leq \mu W(\varepsilon, h) + r(\varepsilon, h, q)\nu(W)\\
&\textit{(iv) } \int r(\varepsilon, h, q) d\nu(\varepsilon, q) > 0 \textit{ for all } q \in Q(\varepsilon, h).
\end{aligned}$$

Proof. The proof is constructive. Let

$$\nu := \mathcal{N}_{0,W} \times \phi_h \text{ and } r(\varepsilon, h, q) := q. \tag{3.41}$$

Let us denote the set where the choice of q is free as

$$S := \{(\varepsilon, h) \in \mathbb{R}^n \times \mathbb{R}_+ : \|\varepsilon\| < L \text{ or } h < h_t\}. \tag{3.42}$$

We choose $\mu < 1$ such that

$$\mu > 1 - q_{\max}(h_t) + q_{\max}(h_t)\nu(S), \tag{3.43}$$

$$\mu > (1-\bar{q})\,\lambda_{\max}(A)^2, \tag{3.44}$$

where $\bar{q}$ denotes the integral introduced in (3.13),

$$\bar{q} := \int_{h_t}^{+\infty} q_{\max}(h) d\phi_h(h). \tag{3.45}$$

The right hand side of (3.43) is less than 1 because the event S under the measure ν happens with probability less than 1 and we have assumed $q_{\max}(h_t) > 0$. The right hand side of (3.44) is also less than 1 because of Assumption 3.1 and by the choice for h_t that satisfies (3.13).

For future reference note that by construction of the set $Q(\varepsilon, h)$, for any $L > 0$, when $\|\varepsilon\| \geq L$ we can upper bound

$$1 - q \leq 1 - q_{\max}(h)\mathbb{I}\left(h \geq h_t\right) =: \psi(h), \tag{3.46}$$

where we named the quantity on the right $\psi(h)$ to be used within this proof. This inequality holds because when $h < h_t$, we have $q \geq 0$, and when $h \geq h_t$, we choose $q = q_{\max}(h)$.

Finally we pick

$$W(\varepsilon, h) := \psi(h)\varepsilon^T H\varepsilon + \beta\,\mathbb{I}\left(\varepsilon, h \in S\right) + \gamma, \tag{3.47}$$

where $\beta, \gamma > 0$ are appropriate positive constants that will be designed next, and $H \succ 0$ is a positive definite matrix satisfying

$$(1 - \bar{q})A^T HA - \mu H = -\Theta, \tag{3.48}$$

for some positive definite matrix $\Theta > 0$. This Lyapunov equation is feasible by our choice of μ that satisfies (3.44).

Next we show that the conditions of the lemma are satisfied for the constructed quantities. First observe that $W(\varepsilon, h) \geq \gamma > 0$ by construction. Then we check (3.40). When $\|\varepsilon\| < L$,

$$(1 - q)\varepsilon^T\varepsilon + c \leq L^2 + c \leq K(\beta + \gamma) \leq KW(\varepsilon, h), \tag{3.49}$$

for a sufficiently large K, where the last inequality follows from the form of $W(\varepsilon, h)$ on $\|\varepsilon\| < L$. On the other hand if $\|\varepsilon\| \geq L$, we may use (3.46) to upper bound

$$\begin{aligned}(1 - q)\varepsilon^T\varepsilon + c &\leq \psi(h)\varepsilon^T\varepsilon + c \\ &\leq K(\psi(h)\varepsilon^T H\varepsilon + \gamma) \leq KW(\varepsilon, h),\end{aligned} \tag{3.50}$$

for a sufficiently large K, by our choice for the function $W(\varepsilon, h)$ when $\|\varepsilon\| \geq L$.

We proceed to show that parts (i)-(iv) in the statement of the lemma also hold. Part (i) holds because the integral of $W(\varepsilon, h)$ with our chosen measure ν equals

$$\nu(W) = (1 - \bar{q})Tr(HW) + \nu(S)\beta + \gamma < \infty. \tag{3.51}$$

Part (ii) holds because the transition probability in (3.39) gives

$$\begin{aligned}&\mathbb{P}(\varepsilon_+ \in B_1, h_+ \in B_2 | \varepsilon, h, q) \\ &= \left[q\;\mathcal{N}_{0,W}(B_1) + (1 - q)\;\mathcal{N}_{A\varepsilon,W}(B_1)\right]\;\phi_h(B_2) \\ &\geq q\;\mathcal{N}_{0,W}(B_1)\;\phi_h(B_2) = r(\varepsilon, h, q)\nu(B_1, B_2).\end{aligned} \tag{3.52}$$

Part (iv) follows by our choice $r(\varepsilon, h, q) = q$ and the construction of the set $Q(\varepsilon, h)$ in (3.14) because

$$\begin{aligned}\int r(\varepsilon, h, q) d\nu(\varepsilon, h) &\geq \int_{\varepsilon, h \in S^c} q_{\max}(h) d\nu(\varepsilon, h) \\ &= \bar{q} \int_{\|\varepsilon\| \geq L} d\mathcal{N}_{0,W}(\varepsilon) > 0. \end{aligned} \tag{3.53}$$

The remainder of the proof shows that (iii) also holds. First observe that by the transition defined in (3.39) and our choices for the measure ν and the function $r(.)$ we have

$$\begin{aligned}&\mathbb{E}\left[W(\varepsilon_+, h_+) | \varepsilon, h, q\right] = r(\varepsilon, h, q) \nu(W) \\ &+ (1-q) \int W(\varepsilon_+, h_+) d\mathcal{N}_{A\varepsilon, W}(\varepsilon_+)\, d\,\phi_h(h_+). \end{aligned} \tag{3.54}$$

Substituting (3.54) in (iii), we only need to show that

$$(1-q) \int W(\varepsilon_+, h_+) d\mathcal{N}_{A\varepsilon, W}(\varepsilon_+)\, d\,\phi_h(h_+) \leq \mu W(\varepsilon, h). \tag{3.55}$$

Plugging the expression of $W(\varepsilon, h)$ given by (3.47) in the integral of the left hand side, condition (3.55) becomes

$$\begin{aligned}&(1-q)\,\{\,(1-\bar{q}) \left[\varepsilon^T A^T H A \varepsilon + Tr(HW)\right] \\ &\qquad + \beta \mathcal{N}_{A\varepsilon, W} \times \phi_h(S) + \gamma\} \leq \mu W(\varepsilon, h). \end{aligned} \tag{3.56}$$

We can bound $\mathcal{N}_{A\varepsilon, W} \times \phi_h(S) \leq \nu(S)$ for any $\varepsilon \in \mathbb{R}^n$, and also $(1-q)(1-\bar{q}) Tr(HW) \leq Tr(HW)$. So a sufficient condition for (3.56) is to show that

$$\begin{aligned}&(1-q) \left\{(1-\bar{q}) \varepsilon^T A^T H A \varepsilon + \beta\, \nu(S) + \gamma\right\} + Tr(HW) \\ &\leq \mu W(\varepsilon, h) \end{aligned} \tag{3.57}$$

holds for every choice of $q \in Q(\varepsilon, h)$. To show this we examine cases.

Case $\|\varepsilon\| \geq L$. Using (3.46) to upper bound $1 - q \leq \psi(h)$, and upon substituting $W(\varepsilon, h)$ in (3.57) and rearranging terms, we need to show equivalently that

$$\begin{aligned}&\psi(h) \left\{\varepsilon^T \left[(1-\bar{q}) A^T H A - \mu H\right] \varepsilon \right. \\ &+ \beta \nu(S) + \gamma\} + Tr(HW) \leq \mu \left\{\beta \mathbb{I}\left(h < h_t\right) + \gamma\right\} \end{aligned} \tag{3.58}$$

By the choice of H in (3.48) the quadratic on the left hand side is negative definite equal to to $-\varepsilon^T \Theta \varepsilon$. And since $\|\varepsilon\| \geq L$ we can upper bound $-\varepsilon^T \Theta \varepsilon \leq -\lambda_{\min}(\Theta) L^2 \leq 0$. After these, a sufficient condition for (3.58) is

$$Tr(HW) + \psi(h) \left\{\beta \nu(S) + \gamma\right\} \leq \mu \left\{\beta \mathbb{I}\left(h < h_t\right) + \gamma\right\} \tag{3.59}$$

Now consider two cases for h. If $h < h_t$ condition (3.59) becomes

$$Tr(HW) + \beta\nu(S) + \gamma \leq \mu(\beta + \gamma). \tag{3.60}$$

On the other hand if $h \geq h_t$ we have that $q_{\max}(h) \geq q_{\max}(h_t)$ by monotonicity of $q_{\max}$, so we may bound $\psi(h) = 1 - q_{\max}(h) \leq 1 - q_{\max}(h_t)$. Condition (3.59) becomes

$$Tr(HW) + (1 - q_{\max}(h_t))\{\beta\nu(S) + \gamma\} \leq \mu\gamma. \tag{3.61}$$

We pick a $\gamma >$ to satisfy (3.61) with equality, that is

$$\gamma = \frac{(1 - q_{\max}(h_t))\nu(S)\beta + Tr(HW)}{\mu - (1 - q_{\max}(h_t))} \tag{3.62}$$

where the denominator is positive by the choice of μ in (3.43). We will show that condition (3.60) also holds by an appropriate choice for $\beta > 0$.

Case $\|\varepsilon\| < L$. In this case $q \geq 0 \Rightarrow 1 - q \leq 1$ and it is sufficient for (3.57) to show that

$$\begin{aligned} &\sup_{\|\varepsilon\|<L} \varepsilon^T(1 - \bar{q})A^T HA\varepsilon + Tr(HW) + \beta\nu(S) + \gamma \\ &\leq \mu(\beta + \gamma) \end{aligned} \tag{3.63}$$

holds, where on the right hand side we lower bounded the quadratic term of W by 0. This is of the general form

$$C_2 + \beta\nu(S) + \gamma \leq \mu(\beta + \gamma) \tag{3.64}$$

for some constant C_2, and recall that the left over condition (3.60) is of the same form. Plugging the chosen γ by (3.62) in (3.64) leads to a condition of the form

$$C_3 \leq \left(1 - \frac{q_{\max}(h_t)\,\nu(S)}{\mu - (1 - q_{\max}(h_t))}\right)\mu\beta, \tag{3.65}$$

for some constant C_3. We want to be able select $\beta > 0$ that satisfies (3.65) for any value of the constant C_3. Hence we require the coefficient of β to be *strictly* positive. This turns out to be equivalent to $\mu > 1 - q_{\max}(h_t) + q_{\max}(h_t)\nu(S)$, which corresponds to our choice of μ in (3.43). Therefore we conclude that part (iii) of the lemma holds as well and this completed the proof. ■

Returning to the proof of Proposition 3.2, combining (3.38) with condition (3.40) of the above Lemma we have that $\mathbb{E}^\pi e_k^T e_k \leq K\,\mathbb{E}^\pi W(\varepsilon_k, h_k)$, so it suffices for (3.19) to show that a uniform bound on the expected value of $W(\varepsilon_k, h_k)$ exists.

By condition (ii) of the above lemma for $(B_1, B_2) = (\mathbb{R}^n, \mathbb{R}_+)$ we have that $r(\varepsilon, h, q) \leq 1/\nu(\mathbb{R}^n, \mathbb{R}_+)$.

Plugging this in (iii) leads to

$$\mathbb{E}\left[W(\varepsilon_+, h_+)|\varepsilon, h, q\right] \leq \mu W(\varepsilon, h) + \nu(W)/\nu(\mathbb{R}^n, \mathbb{R}_+) \tag{3.66}$$

Iterated applications of this inequality across some policy $\pi \in \Pi$ yields

$$\mathbb{E}^\pi W(\varepsilon_k, h_k) \leq \mu^k \, \mathbb{E}\, W(\varepsilon_0, h_0) + \frac{\nu(W)}{(1-\mu)\nu(\mathbb{R}^n, \mathbb{R}_+)} \tag{3.67}$$

Thus since $\mu < 1$ a uniform bound on $\mathbb{E}^\pi W(\varepsilon_k, h_k)$ exists and this completes the proof.

3.5.2. Proof of Theorem 3.1

First note that since $\Sigma_k := \mathbb{E}^\pi \left[e_k e_k^T \mid G_k\right]$ we have that

$$\mathbb{E}^\pi[Tr(\Sigma_k)] = \mathbb{E}^\pi e_k^T e_k. \tag{3.68}$$

Then under Assumption 3.1, Proposition 3.2 states that for any $\pi \in \Pi$ condition (3.19) holds and guarantees that both quantities in (3.68) are bounded uniformly over k.

To establish the optimality of the proposed control law we use the fact that the Bellman-like equation

$$\begin{aligned} &V(G_k) + Tr(PW) + Tr(\tilde{P}\Sigma_k) = \\ &\min_{u_k} \mathbb{E}^\pi \left[x_k^T Q x_k + u_k^T R u_k + V(G_{k+1})|G_k, u_k\right], \end{aligned} \tag{3.69}$$

is satisfied for the function

$$V(G_k) = \mathbb{E}^\pi \left[x_k^T P x_k \mid G_k\right], \tag{3.70}$$

with $V(G_0) = x_0^T P x_0$, where P is the solution to the standard algebraic Riccati equation and $\tilde{P}$ is given by (3.22). The existence of P is guaranteed by the stabilizability of (A, B) and detectability of $(A, Q^{1/2})$.

Indeed observe that we can use the tower property to rewrite the term on the right hand side of (3.69) as

$$\begin{aligned} \mathbb{E}^\pi\left[V(G_{k+1})|G_k, u_k\right] &= \mathbb{E}^\pi\left[x_{k+1}^T P x_{k+1}|G_k, u_k\right] \\ &= \mathbb{E}^\pi\left[(Ax_k + Bu_k)^T P (Ax_k + Bu_k)|G_k, u_k\right] + Tr(PW), \end{aligned} \tag{3.71}$$

where the last equality follows by substituting x_{k+1} from the system equation (3.1). The quadratic minimization over u_k at the right hand side of (3.69) takes the standard form appearing in LQR problems with partial state information - see e.g. (Bertsekas, 2005, Vol. II, Section 5.2). The argument of the minimization in (3.69) is given by the control law (3.23). Straightforward substitutions show that the optimal value of the minimization equals the left hand side of (3.69).

The equation (3.69) can be used to show that the optimal control policy is (3.23). First iterate (3.69) for $k = 0, \dots, N-1$ across some control policy $\theta \in \Theta$ to get

$$
\begin{aligned}
& V(G_0) + N\,Tr(PW) + \mathbb{E}^{\pi} \sum_{k=0}^{N-1} Tr(\tilde{P}\Sigma_k) \\
& \leq J_{\mathrm{LQR}}^{N}(\pi,\theta) + \mathbb{E}^{\pi,\theta} V(G_N)
\end{aligned} \tag{3.72}
$$

Dividing (3.72) by N and taking the limit as $N \to \infty$, the term on the left hand side tends to

$$
\begin{aligned}
& \limsup_{N\to\infty} 1/N \left[x_0^T P x_0 + N\,Tr(PW) + \mathbb{E}^{\pi} \sum_{k=0}^{N-1} Tr(\tilde{P}\Sigma_k) \right] \\
& = Tr(PW) + \limsup_{N\to\infty} 1/N\,\mathbb{E}^{\pi} \sum_{k=0}^{N-1} e_k^T \tilde{P} e_k
\end{aligned} \tag{3.73}
$$

where we used (3.68) to convert Σ_k to e_k.

Then consider the term on the right hand side of (3.72). Any control policy $\theta \in \Theta$ satisfying (3.20) also satisfies

$$
\lim_{N\to\infty} \frac{1}{N} \mathbb{E}^{\pi,\theta} V(G_N) = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}^{\pi,\theta} x_N^T P x_N = 0 \tag{3.74}
$$

by the form of V given in (3.70). Thus taking the limit as $N \to \infty$, by (3.74) the term on the right hand side of (3.72) tends to the average LQR cost. The inequality in (3.72) then shows that the average LQR cost of θ is larger or equal to the limit of the left hand side which was given in (3.73). The result (3.21) of the theorem follows by including the power cost that depends only on the communication policy π as suggested by (3.18).

The final step of the proof is to show that the control policy θ^* defined by (3.23) gives exactly the LQR cost given in (3.73). This policy satisfies (3.69) with equality, so (3.72) also holds with equality for θ^*. Dividing by N and taking the limit as before would prove the desired result if condition (3.74) also holds for θ^*. We next verify that this is the case.

Indeed use $u_k^* = K\hat{x}_k$ and $x_k = \hat{x}_k + e_k$ to rewrite the closed loop system equation (3.1) under θ^* as

$$
x_{k+1} = (A + BK)\hat{x}_k + Ae_k + w_k. \tag{3.75}
$$

Then denoting $\rho := \lambda_{\max}(A + BK)$ which is stable, $\rho < 1$, we can upper bound (3.71) under θ^* by

$$
\begin{aligned}
& \mathbb{E}^{\pi,\theta^*} \left[V(G_{k+1}) | G_k \right] \\
& \leq \rho^2 \hat{x}_k^T P \hat{x}_k + Tr(A^T P A \Sigma_k) + Tr(PW) \\
& = \rho^2 V(G_k) + Tr((A^T P A - \rho^2 P)\Sigma_k) + Tr(PW)
\end{aligned} \tag{3.76}
$$

where in the last equality we used the fact that $\hat{x}_k \hat{x}_k^T = \mathbb{E}^{\pi}\left[x_k x_k^T \mid G_k\right] - \Sigma_k$. Taking expectation on both sides of (3.76) we have that

$$\begin{aligned}\mathbb{E}^{\pi,\theta^*} V(G_{k+1}) \;&\le \rho^2 \mathbb{E}^{\pi,\theta^*} V(G_k) \\ &\quad + Tr((A^T P A - \rho^2 P)\, \mathbb{E}^{\pi,\theta^*} \Sigma_k) + Tr(PW)\end{aligned} \tag{3.77}$$

But (3.68) and (3.19) imply that $\mathbb{E}^{\pi,\theta^*}\Sigma_k$ is uniformly bounded over k so the term on the second line of (3.77) is bounded by some constant $\delta < \infty$. Iterating the above inequality (3.77) across θ^* up to $k = N$ yields

$$\mathbb{E}^{\pi,\theta^*}\left[x_N^T P x_N\right] \le \rho^{2N} x_0^T P x_0 + \delta/(1-\rho^2) \tag{3.78}$$

which guarantees the limit (3.74) since $\rho < 1$.

3.5.3. Proof of Theorem 3.2

The proof of the theorem is a direct application of the theorems contained in Vega-Amaya (2003). For these we need to show that (Vega-Amaya, 2003, Assumptions 3.1, 3.2, 3.4) hold in our case. In particular (Vega-Amaya, 2003, Assumption 3.1) requires that the cost per stage is bounded $|c(\varepsilon, h, q)| \le KW(\varepsilon, h)$ by a positive measurable function W. This is a consequence of (3.40) of Lemma 3.1, since

$$|c(\varepsilon, h, q)| \le (1-q)\lambda_{\max}(\tilde{P})\varepsilon^T\varepsilon + \lambda p_{\max} \tag{3.79}$$

which is of the same form as (3.40). Also (Vega-Amaya, 2003, Assumption 3.2) requires exactly the conditions given in (i)-(iv) of Lemma 3.1. Finally (Vega-Amaya, 2003, Assumption 3.4) requires the following conditions for the functions W and r satisfying Lemma 3.1.

Assumption 3.3. *For every $\varepsilon \in \mathbb{R}^n$, $h \in \mathbb{R}_+$*

(i) *$Q(\varepsilon, h)$ is compact,*

(ii) *$c(\varepsilon, h, q)$ is lower semi-continuous in $q \in Q(\varepsilon, h)$,*

(iii) *$\mathbb{P}(\varepsilon_+, h_+ | \varepsilon, h, q)$ is strongly continuous [1] in $q \in Q(\varepsilon, h)$,*

(iv) *the mapping $q \to \mathbb{E}\left[W(\varepsilon_+, h_+) | \varepsilon, h, q\right]$ is continuous,*

(v) *$r(\varepsilon, h, q)$ is continuous in $q \in Q(\varepsilon, h)$.*

Part (i) is trivial, and (ii) is a consequence of the continuity of $p(h, q)$ by Assumption 3.2. Strong continuity in (iii) is guaranteed by the fact that the transition kernel given in (3.28) has a probability density function. Part (iv) holds because the transition (3.28) is linear in q, and (v) is trivial.

[1] i.e. for every bounded measurable function Ψ on $\mathbb{R}^n \times \mathbb{R}_+$, the mapping $q \mapsto \mathbb{E}[\Psi(\varepsilon_+, h_+)|\varepsilon, h, q]$ is continuous

Having established (Vega-Amaya, 2003, Assumptions 3.1, 3.2, 3.4), then (Vega-Amaya, 2003, Theorems 3.5, 3.6) state that in our case the infimum J^*_{COMM} in (3.26) exists, there exists a function $V(\varepsilon, h)$ that satisfies (3.29), and the optimal policy is the minimizer of the right hand side of (3.29) as given in (3.30).

Finally note that (3.29) holds if we add any constant to $V(\varepsilon, h)$, so without loss of generality we may take $V(0, \hat{h}) = 0$ for some $\hat{h}$. Then for $\varepsilon = 0, h = \hat{h}$ (3.29) gives

$$\begin{aligned} V(0, \hat{h}) = 0 = \min_{q \in Q(0,\hat{h})} \Big\{ & c(0, \hat{h}, q) - J^*_{\mathrm{COMM}} \\ & + \mathbb{E}\left[V(\varepsilon_+, h_+) \,\middle|\, 0, \hat{h}, q \right] \Big\}. \end{aligned} \tag{3.80}$$

Note that by (3.28), $\mathbb{P}(\varepsilon_+, h_+ | 0, \hat{h}, q) = \mathcal{N}_{0,W}(\varepsilon_+)\, \phi_h(h_+)$, and also $c(0, \hat{h}, q) = \lambda p(\hat{h}, q)$ by (3.27), so (3.80) becomes

$$0 = \min_{q \in Q(0,\hat{h})} \Big\{ \lambda p(\hat{h}, q) - J^*_{\mathrm{COMM}} + \mathbb{E}_{w,h} V(w, h) \Big\}. \tag{3.81}$$

The minimizer is $q = 0$, giving the optimal value $J^*_{\mathrm{COMM}} = \mathbb{E}_{w,h} V(w, h)$ provided in the statement of the theorem.

Chapter 4: Opportunistic Control over Multiple Access Channels

4.1. Problem Description

In this chapter we consider a wireless control architecture with multiple control systems over a shared wireless medium. The architecture, introduced in Section 2.2.1, is repeated here as well in Fig. 10. The setup consists of m independent networked control systems. Each control loop i ($i = 1, 2, ..., m$) includes a wireless transmitter communicating to a common receiver/access point. For example this can be a wireless sensor transmitting plant measurements to a common controller computing the control inputs to the plants. A centralized scheduler, implemented at the access point, decides which control system is given access to the shared wireless channel.

We denote the state of system i at each time k by $x_{i,k} \in \mathbb{R}^{n_i}$. The evolution of each system at time k depends on whether a transmission occurs at time k or not for link/system i, hereby denoted as $\gamma_{i,k} \in \{0, 1\}$. We assume then that each system is described by a known switched linear time-invariant system of the form

$$x_{i,k+1} = \begin{cases} A_{c,i}\, x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1 \\ A_{o,i}\, x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0 \end{cases} . \tag{4.1}$$

This model was introduced in Section 2.1 where examples of such control systems were presented. We assume that the closed-loop matrix $A_{c,i}$ is asymptotically stable for each system i, implying that if system i were to transmit at each slot its respective state evolution is stable. The open loop matrices $A_{o,i}$ may be unstable. The additive terms $w_{i,k}$ model an independent (both across time k for each plant i, and across plants) identically distributed (i.i.d.) noise process with mean zero and covariance $W_i \succeq 0$.

Let us now describe the wireless communication system and model how it determines the packet transmission success $\gamma_{i,k}$. Suppose there are f different frequencies that each system may use to communicate to the access point and let the wireless channel fading conditions for a system i and frequency j at time slot k be denoted as $h_{ij,k}$. Channel conditions take values in a subset $\mathcal{H} \subseteq \mathbb{R}_+$ of the positive reals. We adopt a block fading model whereby channel states $\{h_{ij,k}, 1 \leq i \leq m, 1 \leq j \leq f\}$ are modeled as constant during each transmission slot k, but independent and identically distributed across different time slots k according to some joint distribution ϕ on $\mathcal{H}^{m \times f}$. They are also independent of the plant process noise $w_{i,k}$. We assume the channel states are available to the access point before transmission – see Remark 2.5 for a practical implementation. We also make

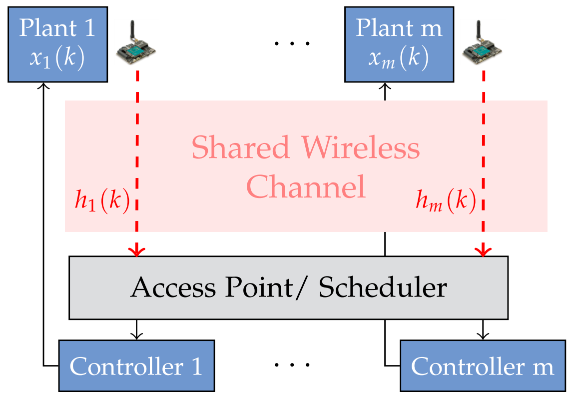

Figure 10: Architecture for control over multiple access channels. Independent control systems close their loops by transmitting over the shared wireless medium to a common receiver/access point. Each control system i experiences random channel conditions h_i. A centralized scheduler at the access point observes all channel states and opportunistically decides which system is scheduled to transmit and close the loop.

the following technical assumption on their joint distribution to exclude the possibility of channel states becoming degenerate random variables.

Assumption 4.1. *The joint distribution ϕ of channel states $\{h_{ij,k}, 1 \leq i \leq m, 1 \leq j \leq f\}$ is absolutely continuous, i.e., has a probability density function on $\mathcal{H}^{m\times f}$.*

If system i transmits at time k over frequency j it selects a transmit power level $p_{ij,k}$ taking values in $[0, p_{\max}]$. Following the wireless channel model introduced in Section 2.2, the probability of successfully decoding the message at the receiver depends on the received power level expressed by the product $h \cdot p$ of channel fading and the allocated transmit power. The probability of success is given by a known relationship of the form $q(h_{ij,k} \cdot p_{ij,k})$ – see also Fig. 2 where this relationship is depicted. The following technical assumption on the form of the function $q(hp)$ will be helpful in the subsequent sections.

Assumption 4.2. *The function $q(.)$ as a function of the product $r = h\,p$ for $r \geq 0$ satisfies:*

(a) $q(0) = 0$*,*

(b) $q(r)$ *is continuous, and strictly increasing when $q(r) > 0$, i.e., for $r' > r$ it holds that $q(r') > q(r) > 0$,*

(c) for any $\mu \in \mathbb{R}_+$ and for almost all values $h \in \mathcal{H}$ the set $\operatorname{argmin}_{0\leq p\leq p_{\max}} p - \mu\, q(h\,p)$ *is a singleton.*

Parts (a) and (b) of this assumption state that the probability of successful decoding $q(h\,p)$ is zero when the received power level $h\,p$ is small, and it becomes positive $q(h\,p) > 0$ and strictly increasing for larger values of $h\,p$. These properties are verified for cases of practical interest as shown in Fig. 2. Part (c) is more stringent but not restrictive in practice. The function $q(hp)$ typically has a sigmoid form with exponential tails as shown in Fig. 2. This verifies that the power minimizer in (c) is unique for almost all channel gains h, and is either equal to zero or belongs to the strictly concave exponential tail. We also note that the minimizer set in (c) exists by the continuity assumption in (b). These properties are assumed for technical reasons that will become clear later (see the discussion before Theorem 4.1).

We are interested in a centralized mechanism that selects which system accesses each of the available frequencies at the channel, i.e., which system is *scheduled* to transmit. We denote with $\alpha_{ij,k} = 1$ the decision to schedule system i on frequency j at time k, and $\alpha_{ij,k} = 0$ otherwise. To avoid packet collisions we let at most one system transmit on each frequency j, that is $\sum_{i=1}^{m} \alpha_{ij,k} \leq 1$. We allow each system i to transmit on at most one frequency, that is $\sum_{j=1}^{f} \alpha_{ij,k} \leq 1$. Mathematically we denote the set $\Delta_{m,f}$ of all feasible scheduling decisions $\alpha_{ij,k}$ at each time k as

$$\Delta_{m,f} = \left\{ \alpha \in \{0,1\}^{m \times f} : \begin{array}{l} \sum_{i=1}^{m} \alpha_{ij} \leq 1, \ 1 \leq j \leq f, \\ \sum_{j=1}^{f} \alpha_{ij} \leq 1, \ 1 \leq i \leq m \end{array} \right\}. \tag{4.2}$$

For compactness we group channel states, scheduling decisions, and power allocations of the communication model at time k into matrices $h_k \in \mathcal{H}^{m \times f}$, $\alpha_k \in \Delta_{m,f}$, and $p_k \in [0, p_{\max}]^{m \times f}$ respectively. We can then model the transmission event $\gamma_{i,k}$ of system i at time k given scheduling variables, power allocation, and channel state, as a Bernoulli random variable with success probability

$$\mathbb{P}[\gamma_{i,k} = 1 \,|\, h_k, \alpha_k, p_k] = \sum_{j=1}^{f} \alpha_{ij,k} \, q(h_{ij,k}, p_{ij,k}) \tag{4.3}$$

This expression states that the probability of a message for system i being successfully received equals the probability that the message is correctly decoded if system i is scheduled to transmit on any of the f available frequencies. Note that, by design of the scheduling variables, system i uses at most one frequency, and we make the implicit assumption that no interferences arise from transmissions on different frequencies.

Our goal is to design the communication variables of the shared wireless control system, i.e., the scheduling and power allocation. Since the randomly varying channel affects the communication process, we are interested in selecting scheduling and power variables that adapt to channel states h_k in order to counteract or exploit these effects. Overall we express the scheduling and power decisions at time k by mappings $\alpha_k = \alpha(h_k)$ and $p_k = p(h_k)$ selected from the sets

$$\begin{aligned} \mathcal{A} &= \{\alpha : \mathcal{H}^{m \times f} \to \Delta_{m,f}\}, \\ \mathcal{P} &= \{p : \mathcal{H}^{m \times f} \to [0, p_{\max}]^{m \times f}\}. \end{aligned} \tag{4.4}$$

Since channel states h_k are i.i.d. over time k these mappings do not need to change over time. Substituting the scheduling and power allocation mappings $\alpha(.), p(.)$ in our communication model described by (4.3) the probability of successful transmission for each system i at any given slot k

becomes

$$\begin{aligned}\mathbb{P}(\gamma_{i,k}=1) &= \mathbb{E}_{h_k}\left\{\mathbb{P}[\gamma_{i,k}=1 \,|\, h_k, \alpha(h_k), p(h_k)]\right\} \\ &= \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}(h)\, q(h_{ij}, p_{ij}(h)). \end{aligned} \tag{4.5}$$

Here the expectation is with respect to the joint distribution ϕ of the channel realization h_k which we assumed to be identical for any time k, hence we drop the index k. Note also that the communication process modeled by the sequence $\{\gamma_{i,k}, 1 \leq i \leq m,\ k \geq 0\}$ depends only on variables related to the wireless communication counterpart of the overall system, and is in particular independent of the system evolutions $\{x_{i,k}, 1 \leq i \leq m,\ k \geq 0\}$.

Our primary goal in designing the communication variables of the system is to guarantee a level of closed loop control performance for each subsystem. To formalize the problem description we consider Lyapunov-like performance requirements for each control system, as introduced in Section 2.1.1. In particular suppose that for each system i a quadratic Lyapunov function of the form

$$V_i(x_i) = x_i^T P_i x_i,\ x_i \in \mathbb{R}^{n_i}, \tag{4.6}$$

with positive definite matrix $P_i \in S_{++}^{n_i}$ is given. A Lyapunov-like requirement then states that *for all systems i at time k* the Lyapunov functions at the next time step decrease at the desired rates $\rho_i < 1$ *in expectation*, that is

$$\mathbb{E}\left[V_i(x_{i,k+1}) \,\middle|\, x_{i,k}\right] \leq \rho_i\, V_i(x_{i,k}) + Tr(P_i W_i) \tag{4.7}$$

for any possible value of the current plant states $x_{i,k} \in \mathbb{R}^{n_i}$. The expectation over the next system state $x_{i,k+1}$ on the left hand side accounts via (4.1) for the randomness introduced by the process noise $w_{i,k}$ as well as the transmission success $\gamma_{i,k}$. The latter is expressed in (4.5) and depends on the observed channel state h_k as well as the communication decisions α_k, p_k.

On the other hand, apart from control performance requirements an efficient communication design should make an efficient use of the available power resources at the devices. The induced overall expected power consumption on each slot k is given by

$$\mathbb{E}_{h_k} \sum_{i=1}^{m} \sum_{j=1}^{f} \alpha_{ij,k}(h_k) p_{ij,k}(h_k), \tag{4.8}$$

summing up the transmit power of each system i and frequency j if the system is scheduled to transmit. The expectation here is with respect to the joint distribution ϕ of channels h_k. We design scheduling and power allocation (cf. (4.4)) that are control-performance aware (cf. (4.11)) and also energy-efficient (cf. (4.8)) through a stochastic optimization framework that we present next.

We formulate the problem of designing scheduling and power allocation in an optimization framework as follows.

Problem 4.1 (Optimal Scheduling and Power Allocation Design)**.** *Consider a shared wireless control architecture with f frequencies and m systems of the form (4.1), quadratic Lyapunov performance requirements by (4.7), channel states $h_k \in \mathcal{H}^{m\times f}$ i.i.d. with distribution ϕ, and communication modeled by (4.3). The design of optimal scheduling and power allocation as functions of the current channel states $\alpha_k = \alpha(h_k)$ and $p_k = p(h_k)$ respectively is posed as*

$$\underset{\alpha,p\in(\mathcal{A},\mathcal{P})}{\text{minimize}} \quad \mathbb{E}_{h_k} \sum_{i=1}^{m}\sum_{j=1}^{f} \alpha_{ij,k}(h_k) p_{ij,k}(h_k) \tag{4.9}$$

$$\text{subject to} \quad \mathbb{E}\left[V_i(x_{i,k+1}) \,\middle|\, x_{i,k}\right] \le \rho_i V_i(x_{i,k}) + Tr(P_i W_i)$$

$$\text{for all } x_{i,k} \in \mathbb{R}^{n_i},\ i = 1,\dots,m. \tag{4.10}$$

In other words, at each time step we seek to minimize the total expected power consumption (4.8) of the design while satisfying the Lyapunov requirements (4.7) for all systems i and for any value of the current plant states $x_{i,k} \in \mathbb{R}^{n_i}$, since scheduling and power allocation adapt to channel states but are independent of the plant states.

The Lyapunov control performance abstraction can be equivalently translates to desired packet success rates for each link. This important observation will permit the design of control-aware scheduling policies. It is formally stated in the following proposition.

Proposition 4.1 (Lyapunov Control Performance Abstraction)**.** *Consider a switched linear system described by* (4.1) *where $\gamma_{i,k}, k \ge 0$ is a sequence of random binary variables, and a quadratic function $V_i(x_i) = x_i^T P_i x_i,\ x_i \in \mathbb{R}^{n_i}$ with a positive definite matrix $P_i \succ 0$. Further suppose that the function $V_i(x_i)$ is a Lyapunov function for the closed loop mode of the system, i.e., $A_{c,i}^T\, P_i\, A_{c,i} \prec \rho_i\, P_i$. Then the function $V_i(x_i)$ decreases with an expected rate $\rho_i < 1$ at each step, i.e., we have*

$$\mathbb{E}\left[V_i(x_{i,k+1}) \,\middle|\, x_{i,k}\right] \le \rho_i\, V_i(x_{i,k}) + Tr(P_i W_i) \quad \text{for all } x_{i,k} \in \mathbb{R}^{n_i},\ k \ge 0, \tag{4.11}$$

if and only if the packet success rate satisfies

$$\mathbb{P}(\gamma_{i,k} = 1) \ \ge\ c_i, \text{ for all } k \ge 0, \tag{4.12}$$

where c_i is computed by the semidefinite program

$$c_i = \min\{\theta \ge 0 :\ \theta\, A_{c,i}^T P_i A_{c,i} + (1-\theta)\, A_{o,i}^T P A_{o,i} \preceq \rho_i P_i\}. \tag{4.13}$$

This proposition is important as it transforms control performance requirements (4.11) to wireless communication requirements, i.e., desired packet success rates (4.12) for each link. In particular there exist easily computed non-negative constants $c_i \ge 0$ such that (4.7) is equivalent to requiring $\mathbb{P}(\gamma_{i,k} = 1) \ge c_i$. This makes explicit how the resource allocation functions $\alpha(.)$, $p(.)$ appear in the optimization problem via (4.5).

As a result the optimization problem (4.9) can be equivalently written as

$$\begin{aligned}
&\underset{\alpha,p\in(\mathcal{A},\mathcal{P})}{\text{minimize}} && \mathbb{E}_h \sum_{i=1}^{m}\sum_{j=1}^{f} \alpha_{ij}(h) p_{ij}(h) \\
&\text{subject to} && c_i \leq \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}(h)\, q(h_{ij}, p_{ij}(h)),\ i = 1, \dots, m
\end{aligned} \tag{4.14}$$

Here we have dropped the time indices k from the variables h_k since they are identically distributed over time. Finally we make a constraint qualification assumption that is typical in optimization theory, i.e., that a strictly feasible solution exists.

Assumption 4.3. *There exist variables $\alpha' \in \mathcal{A}$ and $p' \in \mathcal{P}$ that satisfy the constraints of the optimization problem (4.14) with strict inequality, i.e.,*

$$c_i < \mathbb{E}_h \sum_{j=1}^{f} \alpha'_{ij}(h)\, q(h_{ij}, p'_{ij}(h)), \quad i = 1, \dots, m \tag{4.15}$$

By the equivalence between problems (4.9) and (4.14), condition (4.15) can be interpreted as a feasibility/schedulability assumption for the shared wireless control system. It requires that there exist some channel-aware scheduling and power allocation such that the control performance requirements (4.7) of *all* control systems are met. This assumption however does not provide any information on how to find such a solution.

In the rest of the chapter we examine problem (4.14), which is equivalent to the optimal scheduling and power allocation for the shared wireless control architecture (Problem 4.1). Since this problem is feasible (Assumption 4.3) we denote the optimal value by P and an optimal solution pair by $\alpha^*(.), p^*(.)$. In the following section we characterize the form of the optimal solution and describe a methodology to obtain it.

4.2. Optimal scheduling and power allocation

In this section we examine how the optimal scheduling and power allocation for the wireless control system can be recovered by considering the optimization problem in the dual domain. This allows us to develop an offline algorithm to solve the problem and provides an explicit characterization of the form of the optimal solution.

First let us derive the Lagrange dual problem of (4.14). Consider non-negative dual variables $\mu \in \mathbb{R}^m_+$ corresponding to each one of the m constraints of (4.14). The Lagrangian then is defined as

$$L(\alpha, p, \mu) = \mathbb{E}_h \sum_{i=1}^{m} \sum_{j=1}^{f} \alpha_{ij}(h) p_{ij}(h)$$
$$+ \sum_{i=1}^{m} \mu_i \left[c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}(h)\, q(h_{ij}, p_{ij}(h)) \right], \tag{4.16}$$

while the dual function is defined as

$$g(\mu) = \inf_{\alpha, p \in (\mathcal{A}, \mathcal{P})} L(\alpha, p, \mu). \tag{4.17}$$

For future reference we also denote the set of functions $\alpha(.), p(.)$ that minimize the Lagrangian at μ by

$$(\mathcal{A}, \mathcal{P})(\mu) = \operatorname*{argmin}_{\alpha, p \in (\mathcal{A}, \mathcal{P})} L(\alpha, p, \mu), \tag{4.18}$$

whenever the minimizers exist. This set might contain in general multiple solutions and we denote with $\alpha(\mu), p(\mu)$ an arbitrary element pair of the set. Since the pair itself is a function on $\mathcal{H}^{m \times f}$ (cf. (4.4)), we denote the value of the pair at a point $h \in \mathcal{H}^{m \times f}$ by $\alpha(\mu; h), p(\mu; h)$.

The Lagrange dual problem is defined as follows.

$$D = \sup_{\mu \in \mathbb{R}_+^m} \quad g(\mu). \tag{4.19}$$

According to Lagrange duality theory the optimal dual value D is a lower bound on the optimal cost P of problem (4.14). The following proposition however establishes a strong duality result ($D = P$) for the problem under consideration and provides a relationship between the optimal primal and dual variables.

Proposition 4.2. *Let Assumptions 4.1 and 4.3 hold. Let P be the optimal value of the optimization problem (4.14) and (α^*, p^*) be an optimal solution, and let D be the optimal value of the dual problem (4.19) and μ^* be an optimal solution. Then*

(a) $P = D$ (strong duality)

(b) $\mu_i^* \left[c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}^*(h)\, q(h_{ij}, p_{ij}^*(h)) \right] = 0$ for all $i = 1, \ldots, m$ (complementary slackness)

(c) $(\alpha^*, p^*) \in (\mathcal{A}, \mathcal{P})(\mu^*)$

This proposition states that strong duality holds even though the original problem is not convex, regardless also of the form of the function $q(h, p)$ (Assumption 4.2 is not imposed). More importantly, part (c) suggests the possibility of finding the optimal primal variables α^*, p^* by solving first for the optimal point μ^* of the dual problem, and then searching for primal variables the minimize the Lagrangian function at μ^* (cf. (4.18)). As we present next, this direction provides a significant advantage. The design of infinite-dimensional scheduling and power allocation policies that meet the control performance specifications in Problem 4.1 is reduced to the problem of determining finite-dimensional optimal dual variables. A technical caveat of Proposition 4.2(c) is that the optimal policies are included in a set which could in general contain other irrelevant policies. As we show next, Assumption 4.2 helps overcome this issue.

4.2.1. Dual subgradient method

To solve the dual problem in (4.19), that is, to maximize the dual function $g(\mu)$, we employ a dual projected subgradient algorithm (Bertsekas et al., 2003, Ch. 8). We first note that function $g(\mu)$ is concave, as a pointwise infimum over functions linear in μ (cf. (4.17)). A subgradient direction for $g(\mu)$ at any point $\mu \in \mathbb{R}^m_+$ is a vector, denoted here as $s(\mu) \in \mathbb{R}^m$, that satisfies

$$g(\mu') - g(\mu) \leq (\mu' - \mu)^T s(\mu) \quad \text{for all} \mu' \in \mathbb{R}^m_+. \tag{4.20}$$

If we pick $\alpha(\mu), p(\mu) \in (\mathcal{A}, \mathcal{P})(\mu)$ by (4.18) then a subgradient $s(\mu)$ can be found as the constraint slack of the primal problem (4.14) evaluated at these points, i.e.,

$$s_i(\mu) = c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}(\mu; h)\, q(h_{ij}, p_{ij}(\mu; h)). \tag{4.21}$$

To show this observe that for any $\mu' \in \mathbb{R}^m_+$ in general we have $g(\mu') \leq L(\alpha(\mu), p(\mu), \mu')$ by the definition of the dual function in (4.17). Subtracting $g(\mu) = L(\alpha(\mu), p(\mu), \mu)$ from both sides of this inequality and expanding the terms of the Lagrangian as in (4.16) we get

$$\begin{aligned} &g(\mu') - g(\mu) \leq \\ &\sum_{i=1}^{m} (\mu'_i - \mu_i) \left[c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}(\mu; h)\, q(h_{ij}, p_{ij}(\mu; h)) \right]. \end{aligned} \tag{4.22}$$

Comparing this with the property of the subgradient in (4.20), we verify that (4.21) indeed gives a subgradient direction. We also note for future reference that for any μ the subgradients are bounded because at the right hand side of (4.21) the term c_i is bounded and the term in the expectation corresponds to a probability (cf.(4.5)).

A projected dual subgradient ascent method to maximize the concave dual function $g(\mu)$ consists of the following steps:

1. At iteration t given $\mu(t)$ find primal optimizers of the Lagrangian at $\mu(t)$ according to (4.18),

$$p(\mu(t)), \alpha(\mu(t)) \in (\mathcal{A}, \mathcal{P})(\mu(t)) \tag{4.23}$$

2. Evaluate the subgradient vector $s(\mu(t))$ by (4.21) and update the dual variables by an ascent

step

$$\mu(t+1) = [\mu(t) + \varepsilon(t)s(\mu(t))]_+ \tag{4.24}$$

where $[\quad]_+$ denotes the projection on the non-negative orthant and $\varepsilon(t) > 0$ is the stepsize.

The stepsizes are selected to be square summable but not summable, i.e.,

$$\sum_{t\geq 1} \varepsilon(t)^2 < \infty, \ \sum_{t\geq 1} \varepsilon(t) = \infty. \tag{4.25}$$

Before stating the convergence properties of the algorithm, we note that in order to implement it we need an efficient way to compute primal Lagrange optimizers in (4.23) that solve (4.18). This problem also relates to our capability of finding the optimal primal variables of interest α^*, p^* as we have argued by Proposition 4.2(c). Hence we turn our focus to problem (4.18). A more convenient expression for the Lagrangian defined in (4.16) can be obtained by rearranging terms to get

$$\begin{aligned} L(\alpha, p, \mu) = \ & \mu^T c + \\ & \mathbb{E}_h \sum_{i=1}^{m} \sum_{j=1}^{f} \alpha_{ij}(h) \left[p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h)) \right]. \end{aligned} \tag{4.26}$$

This form provides a useful separation structure for the primal Lagrangian optimizers that we exploit in the following proposition.

Proposition 4.3. *For any $\mu \in \mathbb{R}_+^m$ the following hold true:*

(a) Solutions $\alpha(\mu), p(\mu) \in (\mathcal{A}, \mathcal{P})(\mu)$ of problem (4.18) can be obtained at each $h \in \mathcal{H}^{m\times f}$ as

$$p_{ij}(\mu; h) = p_{ij}(\mu_i; h_{ij}) = \operatorname*{argmin}_{0\leq p\leq p_{\max}} \ p - \mu_i q(h_{ij}, p) \tag{4.27}$$

for any $i = 1, \ldots, m$ and $j = 1, \ldots, f$, and

$$\begin{aligned} \alpha(\mu; h) = & \operatorname*{argmin}_{\alpha \in \mathbb{R}_+^{m\times f}} && \sum_{i=1}^{m} \sum_{j=1}^{f} \alpha_{ij} \, \xi(h_{ij}, \mu_i) \\ & \text{subject to} && \sum_{i=1}^{m} \alpha_{ij} \leq 1, \ \sum_{j=1}^{f} \alpha_{ij} \leq 1 \end{aligned} \tag{4.28}$$

where

$$\xi(h_{ij}, \mu_i) = \min_{0\leq p\leq p_{\max}} \ p - \mu_i q(h_{ij}, p). \tag{4.29}$$

(b) If Assumptions 4.1 and 4.2 hold, then for any solution $\alpha(\mu), p(\mu) \in (\mathcal{A}, \mathcal{P})(\mu)$ the vector $s(\mu)$ defined in (4.21) has a unique value.

The first part of the proposition provides in (4.27) and (4.28) a method to obtain primal Lagrange optimizers that can be used in step (4.23) of the subgradient algorithm. Interestingly, the minimizing scheduling and power allocation decisions can be computed separately at each channel state value $h \in \mathcal{H}^{m \times f}$, hence significantly simplifying the computation. A further separability for the power allocation across systems and frequencies is revealed – see Remark 4.1.

The second part of the proposition relies on the properties of the function $q(h, p)$ by Assumption 4.2 in order to establish that the subgradient vector $s(\mu)$ takes a unique value. The proof relies on the fact that the expected value in $s_i(\mu)$ (cf. (4.21)) conditioned on the channel state h is almost surely either zero, because system i should not be scheduled, or unique, because the optimal Lagrange minimizers $\alpha(\mu; h), p(\mu; h)$ are unique. For example, Assumption 4.2(c) ensures that the power minimizer $p(\mu; h)$ in (4.27) is (almost surely) unique. Proposition 4.3(b) is important because it allows a stronger and more explicit characterization of the optimal scheduling and power allocation than the set-characterization of Proposition 4.2(c). This is established in the following theorem.

Theorem 4.1 (Optimal Scheduling and Power Allocation)**.** *Consider the design of channel-aware scheduling and power allocation variables in Problem 4.1 for the shared wireless control architecture of Fig. 10, and let Assumptions 4.1, 4.2, 4.3 hold. Then optimal scheduling α^* and power allocation p^* are obtained by (4.27)-(4.29) at any point $\mu^* \in \mathbb{R}^m_+$ that is an optimal solution of the dual problem (4.19). Moreover a point μ^* can be obtained by iterating (4.23)-(4.24), i.e., $\mu(t) \to \mu^*$, for stepsizes satisfying (4.25).*

The theorem characterizes the optimal scheduling and power allocation that meet the control performance specifications in our shared wireless control architecture – see Remarks 4.1, 4.2 for more details about the form of the optimal policy. It is worth noting that the optimal policy need not be unique. More precisely, there might be many optimal dual solutions μ^*, each corresponding to a different scheduling and power allocation policy according to the theorem. However all such policies will have the same objective value in (4.9).

The theorem also establishes a methodology to find the optimal communication policy by iterating (4.23)-(4.24). This can be viewed as an offline algorithm, and requires knowledge of the channel distribution. In the next section we develop an online algorithm that solves for the optimal communication policy based instead only on a random sequence of channel realizations observed during system execution.

Remark 4.1. *According to Theorem 4.1, the optimal power allocation can be obtained at each channel value h by solving (4.27) at the point μ^*. In particular $p^*_{ij}(h)$ depends on the variables μ^*_i and h_{ij} pertinent only to system i and frequency j and not on the whole vectors μ^* or h. This implies a decentralized power allocation among systems and frequencies, made explicit in (4.27) by the notation $p_{ij}(\mu_i; h_{ij})$. Similar separability results are also known in the context of resource allocation for wireless communication networks Liu et al. (2003); Georgiadis et al. (2006); Ribeiro (2012) even though in those works the goal is to maximize user utility (capacity, data rates, etc.) while the goal here is to meet closed loop control performance. The* *separability can be intuitively understood from the shared wireless control architecture of Fig. 10, since each transmitter experiences different channel conditions and is responsible for an independent control task. Moreover, this optimal power allocation can be easily implemented in practice. The transmitter of each control system i can store its value μ_i^* and adapt transmit power, whenever scheduled, based on the channel conditions it currently experiences. The optimal scheduling $\alpha^*(h)$ in (4.28), on the other hand, is centralized since it depends on the whole vector μ^* and all channel states h.*

Remark 4.2. *The optimal scheduling decision in (4.28) is posed as a linear program by relaxing the integer constraints of $\Delta_{m,f}$ in (4.2), hence the policy is computationally efficient. As mentioned in the proof of the theorem there is no loss in the relaxation, as the optimal solution to the linear program is integer. It is worth noting that (4.28) solves a standard assignment problem[2] which opportunistically tries to match control systems to frequencies at each time slot. A numerical example of this opportunistic behavior is shown in Fig. 11 of Section 4.4. Besides the linear program presented here, integer programming algorithms with complexity polynomial in the number of systems m and frequencies f exist (Bertsimas and Tsitsiklis, 1997, Ch. 7). In the special case of a single frequency ($f = 1$) the complexity of scheduling in (4.28) is linear in the number of systems ($O(m)$), since the scheduler looks for and schedules the system i with the minimum value $\xi(h_i, \mu_i)$.*

4.3. Online scheduling and power allocation

The algorithm presented in the previous section to obtain optimal scheduling and power allocation for the shared wireless control system of Problem 4.1 is hard to implement in practice. In the primal step (4.23) one needs to obtain a solution pair $\alpha(h), p(h)$ for a continuum of channel variables $h \in \mathcal{H}^{m \times f}$, while for the dual step in (4.24) one needs to compute the subgradient direction $s(\mu)$ in (4.21) by integrating over the channel distribution ϕ. A practical implementation would require drawing a large number of samples from ϕ and solving for primal variables at these samples to obtain an estimate of the actual subgradient direction. This is computationally intensive, does not scale for a large number of systems m and frequencies f, while also in most cases of practical interest the channel distribution is not available.

These drawbacks motivate us to develop an *online* algorithm to solve Problem 4.1. The algorithm is a stochastic version of the primal/dual steps (4.23), (4.24) of the offline subgradient method and does not rely on availability of the channel distribution. In particular, suppose that at time k a channel realization h_k is observed, and the current power and scheduling decision are selected by solving (4.27)-(4.28) at the current h_k, i.e.,

$$\begin{aligned} p_{ij,k} &= p_{ij}(\mu_{i,k}; h_{ij,k}), \quad i = 1, \dots, m, \ j = 1, \dots, f, \\ \alpha_k &= \alpha(\mu_k; h_k). \end{aligned} \tag{4.30}$$

Then in contrast to updating the dual variables μ_k by (4.24) after computing the vector (4.21), sup-

[2]Technically the standard assignment problem requires equal number of systems and frequencies. This can be accomplished by introducing dummy systems or frequencies with zero values $\xi(h_{ij}, \mu_i)$.

Algorithm 4.1 Online Scheduling and Power Allocation

Input: m, f, $c \in [0,1]^m$, $q : \mathcal{H} \times [0, p_{\max}] \mapsto [0,1]$, $\varepsilon_k \in \mathbb{R}_+, k \geq 0$

Initialize $\mu_0 \in \mathbb{R}_+^m$, $k \leftarrow 0$

loop

At time k observe channel state h_k

Compute power allocation for all systems i and frequencies j by

$$p_{ij,k} \leftarrow \underset{0 \leq p \leq p_{\max}}{\operatorname{argmin}} \; p - \mu_{i,k} q(h_{ij}, p) \tag{4.33}$$

$$\xi_{ij,k} \leftarrow \min_{0 \leq p \leq p_{\max}} \; p - \mu_{i,k} q(h_{ij}, p) \tag{4.34}$$

Decide scheduling by solving

$$\alpha_k \leftarrow \underset{\alpha \in \Delta_{m,f}}{\operatorname{argmin}} \sum_{i=1}^{m} \sum_{j=1}^{f} \alpha_{ij} \, \xi_{ij,k} \tag{4.35}$$

Compute for all $i = 1, \ldots, m$

$$s_{i,k} \leftarrow c_i - \sum_{j=1}^{f} \alpha_{ij,k} \, q(h_{ij,k}, p_{ij,k}) \tag{4.36}$$

Update dual variables by $\mu_{k+1} \leftarrow [\mu_k + \varepsilon_k s_k]_+$

end loop

pose only the current channel measurement and power/scheduling choices are used. In particular, suppose we compute

$$s_{i,k} = c_i - \sum_{j=1}^{f} \alpha_{ij,k} \, q(h_{ij,k}, p_{ij,k}), \quad i = 1, \ldots, m, \tag{4.31}$$

and update the variables μ_k by

$$\mu_{k+1} = [\mu_k + \varepsilon_k s_k]_+ \tag{4.32}$$

where $[\quad]_+$ is the projection on the non-negative orthant and $\varepsilon_k > 0$ is the stepsize.

To emphasize that this is an online algorithm we have explicitly indexed the variables with k corresponding to real time slots. This procedure, summarized in Algorithm 4.1, gives scheduling and power variables $\{\alpha_k, p_k, k \geq 0\}$ as well as dual variables $\{\mu_k, k \geq 0\}$ which are random because they depend on the random observed channel sequence $\{h_k, k \geq 0\}$. The main difference compared to the subgradient algorithm of the previous section is that it follows random directions s_k in (4.31) instead of the exact subgradient directions $s(\mu_k)$ by (4.21). Comparing these two expressions it is immediate that the expected value of s_k coincides with the subgradient $s(\mu_k)$, so it is reasonable to conjecture that the online algorithm is expected to move towards the maximum of the dual function, as the subgradient method does. The following proposition indeed establishes convergence in a strong sense.

Proposition 4.4. *Consider the optimization problem (4.14) and its dual derived in (4.19) and let Assumption 4.3 hold. Let a sequence $\mu_k, k \geq 0$ be obtained by steps (4.30)-(4.32) based on a sequence $\{h_k, k \geq 0\}$ of i.i.d. random variables with distribution ϕ, and stepsizes ϵ_k satisfying (4.25). Then almost surely we have that*

$$\lim_{k\to\infty} \mu_k = \mu^*, \quad \textit{and} \quad \lim_{k\to\infty} g(\mu_k) = D \tag{4.37}$$

where μ^* is an optimal solution of the dual problem and D is the optimal value of the dual problem.

The proposition states that the stochastic online algorithm yields a random sequence of dual variables μ_k that converges to the optimal point μ^* almost surely for any sequence of channel realizations that is observed. However the real problem of interest is the primal problem (4.14), or equivalently Problem 4.1. This is the problem of optimal design of scheduling and power allocation policies that satisfy the given Lyapunov performance requirements (4.11) for each control system i, while also minimizing the expected power expenditures of the communication process. Hence, in the following theorem, we characterize how the control systems would actually perform if the communication variables are selected according to the proposed online algorithm.

Theorem 4.2 (Online Scheduling and Power Allocation)**.** *Consider a shared wireless control architecture composed of m systems of the form (4.1), f frequencies, and communication modeled by (4.3) depending on channel states $h_k \in \mathcal{H}^{m\times f}$ which are i.i.d. with distribution ϕ, and scheduling and power allocation variables $\alpha_k \in \Delta_{m,f}$, $p_k \in [0, p_{\max}]^{m\times f}$. Also consider given quadratic Lyapunov performance requirements (4.11) for each system and let Assumptions 4.1, 4.2, 4.3 hold. If α_k, p_k are chosen according to (4.30)-(4.32), then almost surely with respect to the channel sequence $\{h_k, k \geq 0\}$ the control performances for all systems $i = 1, \dots, m$ satisfy*

$$\begin{aligned} \limsup_{k\to\infty} \; & \mathbb{E}[V_i(x_{i,k+1}) \,|\, x_{i,k} = x_i, h_0, \dots, h_{k-1}] \\ & \leq \rho_i V_i(x_i) + Tr(P_i W_i), \end{aligned} \tag{4.38}$$

for any state values $x_i \in \mathbb{R}^{n_i}$. In addition, the power consumption almost surely satisfies

$$\limsup_{k\to\infty} \; \mathbb{E}\left[\sum_{i=1}^{m} \sum_{j=1}^{f} \alpha_{ij,k} p_{ij,k} \middle| h_0, \dots, h_{k-1} \right] \leq P \tag{4.39}$$

where P is the optimal value of the optimization problem (4.9).

According to the theorem the scheduling and power allocation variables selected by the online algorithm lead in the limit to the desired Lyapunov requirements for all control systems and to the optimal power expenditure, for almost all channel sequences. We can also establish the following corollary.

Corollary 4.1. *Consider the setup of Theorem 4.2. Then for any positive constant $\delta > 0$ there exists a time* *step N such that for all times $k \geq N$ we have that*

$$\mathbb{E}\left[V_i(x_{i,k+1}) \,\middle|\, x_{i,k}\right] \leq (\rho_i + \delta)\; V_i(x_{i,k}) + Tr(P_i W_i) \tag{4.40}$$

for any possible value of plant states $x_{i,k} \in \mathbb{R}^{n_i}$ and for all systems $i = 1, \ldots, m$.

Recall that we initially asked for a communication design that guarantees expected control performance requirements at each time step k in (4.11). According to the above corollary our online algorithm approximately satisfy this. After a sufficiently long time horizon the expected decrease rates of all Lyapunov functions get arbitrarily close to the desired ones. Before proceeding to simulations of the stochastic online algorithm, we present an intuitive interpretation of the algorithm from an economic resource allocation point of view.

4.3.1. *Pricing interpretation of online scheduling and power allocation algorithm*

In this section we provide an interpretation of the problem variables as well at the online Algorithm 4.1 in economic terms. In particular we may view each transmitter in the wireless control architecture as an agent that utilizes some scarce resource, namely transmit power, to produce some 'good', namely the probability of successfully transmitting and closing the corresponding control loop. Our development shows that each closed loop has a Lyapunov control performance requirement (cf. (4.11)) that can be translated as requiring c_i units of good (cf. (4.14)). Under this view, the dual variables μ_i can be interpreted as the 'unit price' at which each agent can 'sell' the produced good. In this context the role of Algorithm 4.1 is to determine unit prices such that all demand levels c_i are met and in the most profitable manner from the agents' perspective.

More specifically, consider a time step k where prices are set to μ_k and the current channel conditions are described by h_k. If agent i gets access to the channel at frequency j, the agent can spend an amount $p_{ij,k}$ to produce $q(h_{ij,k}, p_{ij,k})$ units of good, which can be sold at a price of $\mu_{i,k}$ per unit. In this case the total profit for the agent can be expressed as

$$\mu_{i,k}\, q(h_{ij,k}, p_{ij,k}) - p_{ij,k}, \tag{4.41}$$

i.e., the difference between the total revenue $\mu_{i,k}\, q(h_{ij,k}, p_{ij,k})$ and the total cost $p_{ij,k}$. The optimal resource allocation $p_{ij,k}$ is the one maximizing the profit (4.41), matching exactly the optimization over power provided in (4.33). The optimal profit if agent i gets access to the channel at frequency j under conditions $h_{ij,k}$ equals $-\xi_{ij,k}$ given in (4.34).

Then the role of the scheduler is to opportunistically assign agents to the available frequencies in a way that maximizes the total aggregated profit. In particular the scheduler observes current conditions $h_{ij,k}$ for all agents i and frequencies j, computes the possible profit $-\xi ij, k$ of all agent/frequency pairs, and searches for the scheduling $\alpha \in \Delta_{m,f}$ defined by (4.2) that maximizes the total profit

$$\sum_{i,j} \alpha_{ij}(-\xi ij, k) \tag{4.42}$$

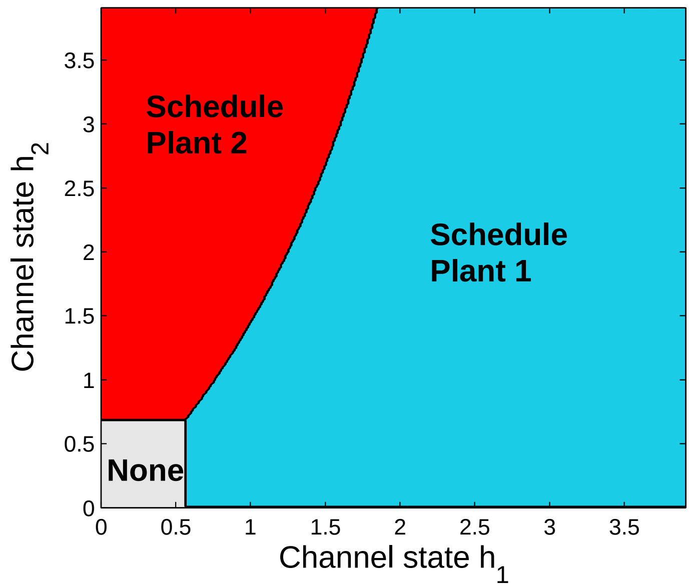

Figure 11: Optimal channel-aware scheduling for the example presented in Section 4.4. System 1 has a harder Lyapunov decrease rate requirement and is scheduled to transmit for most observed channel states h_1, h_2. System 2 is scheduled only if its channel conditions h_2 are much more favorable that those of system 1. When both channels are very adverse, systems select zero transmit powers so scheduling is irrelevant.

aggregated over all agents. This optimal scheduling matches the one implemented by Algorithm 4.1 (cf. line (4.35)).

After the current scheduling α_k and power p_k decisions have been made, the unit prices μ_{k+1} for the next step are adjusted depending on the current production levels. If the production for system i exceeds the required level c_i, i.e., $s_{i,k} < 0$ in (4.31), then the unit price for system i is reduced to $\mu_{i,k} + \epsilon_k s_{i,k}$ (cf. line 7 in Algorithm 4.1). If on the other hand the production for system i does not meet c_i, i.e., $s_{i,k} > 0$, then the unit price i increases to $\mu_{i,k} + \epsilon_k s_{i,k}$.

According to Theorem 4.2 the online algorithm converges almost surely to the optimal prices μ^*, under which the expected production meets demand. Moreover the expected total production cost (the objective of problem (4.14)) becomes optimal in the limit.

We note that Theorem 4.2 does not provide theoretical guarantees on how fast the solution converges to the optimal one. We discuss this issue along with other limitations of the algorithm in Section VI. In the following section we present simulations verifying our theoretical results, and also indicating that the convergence of the algorithm is relatively fast so that online control performance is not severely affected.

4.4. Numerical Simulations

We first illustrate through simulations the opportunistic nature of the communication mechanism for wireless control systems obtained in Section 4.2, in particular how scheduling and power decisions adapt appropriately to channel conditions to meet the control performance goals. Moreover we compare the resulting performance with other simple non channel-adaptive mechanisms. Recall that control systems with vector states are converted to scalar constraints in optimization problem (4.14) by introducing the constants c_i. Hence without loss of generality we consider scalar control systems.

Consider a heating system application controlling the temperature in two independent rooms of a building. Assuming the wireless control architecture of Fig. 10 with $m = 2$, wireless sensors transmit the temperatures of each room to a central location (the access point in Fig. 10) responsible for adjusting the heating in the rooms. For simplicity suppose both systems have identical dynamics of the form (4.1) with state $x_{i,k}$ denoting the difference between current and some desired temperature for room i. When system i transmits ($\gamma_{i,k} = 1$), heating is activated for system i and results in stable dynamics $A_{c,i} = 0.4$. Otherwise if $\gamma_{i,k} = 0$ heating is deactivated and the system is open loop unstable with $A_{o,i} = 1.1$.

For simplicity we assume there is one ($f = 1$) available frequency and for symmetry let channel states $h_{1,k}$ and $h_{2,k}$ be independent for each system, both having an exponential distribution with mean 1. The function $q(h, p)$ is shown in Fig. 2. For these scalar systems it suffices to consider Lyapunov functions $V_i(x) = x^2$. We require then that system 1 guarantees a high Lyapunov decrease $\rho_1 = 0.75$ rate according to (4.11), while system 2 only requires $\rho_2 = 0.90$. For these choices we get a higher required success of transmission $c_1 \approx 0.44$ for system 1 (solving (4.45) of Prop. 4.1), compared to a lower $c_2 \approx 0.30$ of system 2.

Using the offline subgradient method of Section 4.2 to solve problem (4.14), the optimal channel-aware scheduling and power allocation variables are depicted in Fig. 11 and Fig. 12 respectively. We observe in Fig. 11 that System 1, which requires higher transmission success c_1, is scheduled to transmit for most values of the channel states h_1, h_2. System 2, which has a lower requirement, is scheduled only if its channel h_2 is sufficiently favorable *and* system 1 experiences an adverse channel h_1. This illustrates how the scheduler exploits opportunistically the channel conditions to select which system will transmit to close the loop, in order to meet the Lyapunov constraints in a power efficient manner. Note also that when both systems experience very adverse channels scheduling is irrelevant because, as we will see in Fig. 12, the optimal transmit powers then are zero (no transmission).

The optimal power allocation is decentralized as we noted in Remark 4.1, i.e., the transmit power p_i for system i depends only on the channel h_i that system i experiences. Thus we plot in Fig. 12 the power allocation for both systems on same axes. For both systems, when the channel conditions are adverse it is not worth to spend transmit power. System 1, which has a more demanding control constraint, requires in general higher transmit power since, as we saw in Fig. 11, it is

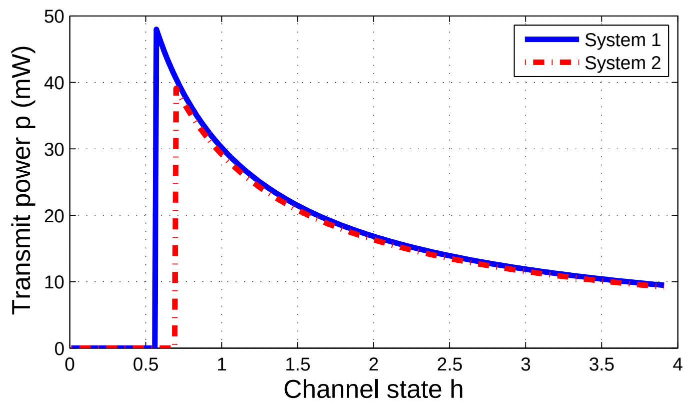

Figure 12: Optimal channel-aware power allocation for the example presented in Section 4.4. Under adverse channel conditions systems do not transmit. The channel threshold for transmission for system 1 is lower than that of system 2 because the former has a higher Lyapunov decrease rate requirement. System 1 also requires higher transmit power.

scheduled to transmit even under adverse channel conditions. This is also captured in the expected power consumption of each system computed numerically as $\mathbb{E}_h \alpha_1^*(h) p_1^*(h_1) \approx 11mW$ and $\mathbb{E}_h \alpha_2^*(h) p_2^*(h_2) \approx 6.5mW$. Hence the minimum total power budget required to meet the control objectives is $17.5mW$.

To demonstrate the power savings obtained by the opportunistic mechanism we compare with a simple non-channel-aware mechanism. In particular suppose that at each step a system is scheduled randomly to access the channel/frequency. With a slight abuse of notation systems 1 and 2 are chosen with probabilities α_1 and $\alpha_2 = 1 - \alpha_1$ respectively. When a system is selected, we suppose it transmits with a constant power level p_c. The control performance requirements (cf. (4.14)) in this case become $\alpha_i \, \mathbb{E}_{h_i} q(h_i \, p_c) \geq c_i$ for $i = 1, 2$ and the total power cost is $(\alpha_1 + \alpha_2) \, p_c = p_c$. We are interested then in selecting the minimum constant power p_c that would satisfy both control requirements. It can be easily argued that a necessary and sufficient condition for the requirements is $\sum_{i=1,2} \alpha_i \, \mathbb{E}_{h_i} q(h_i \, p_c) \geq c_1 + c_2$, which is equivalent to $\mathbb{E}_{h_i} q(h_i \, p_c) \geq c_1 + c_2$ since $\sum_{i=1,2} \alpha_i = 1$. We compute then numerically the minimum constant p_c that satisfies this equivalent requirement, which is $p_c \approx 73mW$. Note that this transmit power is higher than the optimal opportunistic power policy in Fig. 12. Moreover, in this example, the optimal opportunistic scheduling and power allocation achieves almost 80% decrease in power consumption compared to an optimal not channel-aware randomized schedule scheme.

4.4.1. Stochastic online scheduling and power allocation

Next we implement the stochastic online algorithm of Section 4.3 in a setup with three ($m = 3$) control loops sharing two ($f = 2$) frequencies. For example consider again the room heating system of the previous section including three rooms/systems with identical dynamics, $A_{o,i} = 1.1$ and $A_{c,i} = 0.4$ as before. The chosen desired Lyapunov decrease rates are shown in Table 1, implying that system 1 is the most demanding. We assume channel states h_{ij} are independent

	Control objective ρ	Mean Fading h_{i1}	Mean Fading h_{i2}	Transmit Rate at Freq. 1	Transmit Rate at Freq. 2
Plant 1	0.8	1	1	0.46	0.40
Plant 2	0.92	1	1	0.34	0.29
Plant 3	0.92	1	2	0.16	0.29

Table 1: System parameters & online transmission rates

across systems i and frequencies j, and have exponential distributions with means given in Table 1. In particular we model that system 3 experiences better channel quality (higher channel fading gain) in the second frequency.

The evolution of the dual variables μ_k during Algorithm 4.1 is shown in Fig. 13. After a number of iterations (time k in this example corresponds to seconds) they remain in a small neighborhood around the optimal μ^*, as anticipated by the theoretical a.s. convergence in Prop. 4.4. Consequently, the scheduling and power allocation decisions taken online are almost feasible for the constraints of problem (4.14) after a number of iterations. We observe that the dual variable corresponding to system 1 is the largest, consistent with the fact that it has a harder control requirement to meet. Using the economic interpretation of Section 4.3.1 about the dual variables, the price at which agent 1 can sell its produced good is higher, giving the incentive to schedule agent 1 to produce more often. On the other hand, systems 2 and 3 have the same control requirements but the dual variable for system 2 is larger. The reason is that system 2 experiences worse channel conditions than system 3 (cf. Table 1), which imply higher required transmit power, or in economic terms a higher production cost in (4.41). By setting a higher selling price μ_2, system 2 becomes profitable enough so that it is scheduled to produce at a sufficient rate to meet the requirement.

In Table 1 we show the average transmission rates selected by the online algorithm, i.e., the average number of time slots where each system i was selected to transmit (with a positive power level) at each frequency j as $1/N\sum_{k=1}^{N}\alpha_{ij,k}\mathbb{I}\left(p_{ij,k}>0\right)$. System 3 was scheduled mainly at frequency 2, exploiting its better channel quality. This forced systems 1 and 2 to use frequency 1 more often. Also system 1, which has higher control requirement, transmitted more often than the other systems. This behavior resulted from the online algorithm using only an observed channel sequence, not any prior knowledge on the channel quality distribution.

Finally, we examine the evolution of the three heating control systems when the online algorithm is employed for scheduling and power decisions. Suppose that for all systems i the states x_i, which measure deviations from reference room temperatures, are perturbed by disturbances $w_{i,k}$ as in (4.1), which we model as independent Gaussian with mean zero and variance $W_i = 1$. We plot in Fig. 14 the evolution of the empirical quadratic averages $1/N\sum_{k=1}^{N}x_{i,k}^2$. Recall that when the Lyapunov condition (4.11) is satisfied, we get from (2.13) of Section 2.1.1 that the expected limit quadratic costs are bounded by $W_i/(1-\rho_i)$. We observe from Fig. 14 that after some initial

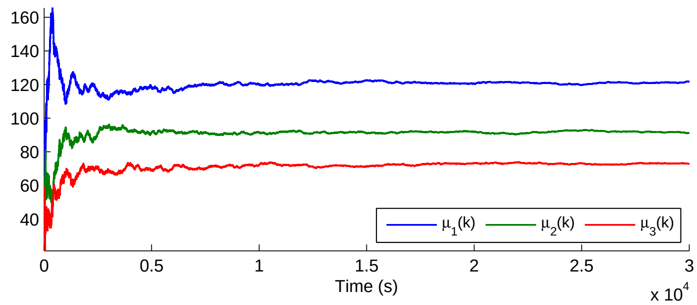

Figure 13: Evolution of dual variables μ_k over time k using the online algorithm. After a number of steps the dual variables μ_k remain in a neighborhood around the optimal values μ^*.

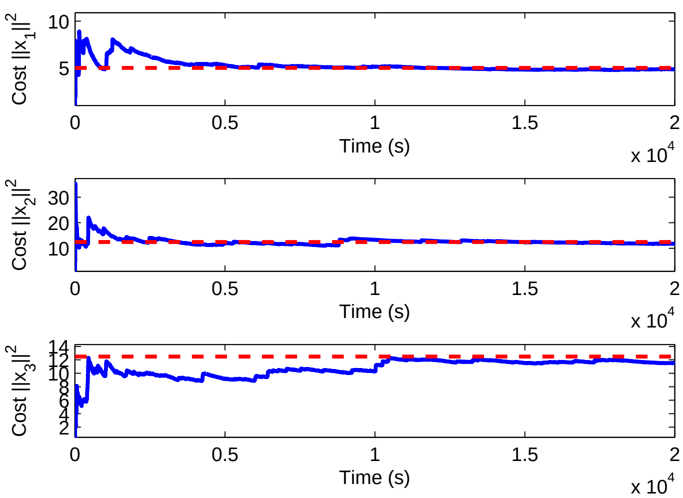

Figure 14: Average quadratic costs during the online scheduling and power allocation algorithm. The stochastic algorithm keeps the average quadratic cost of each control system close to the upper bound of the limit expected cost, shown with dashed lines, induced theoretically by the required Lyapunov decrease rates.

transient the online communication algorithm keeps the empirical average quadratic costs close to the theoretical upper bounds.

4.5. Proofs

4.5.1. Proof of Proposition 4.1

For simplicity we drop the indices i within the proof. The expectation over the next system state x_{k+1} on the left hand side of (4.11) accounts via (4.1) for the randomness introduced by the process noise w_k and the random packet success γ_k. In particular we have that

$$\mathbb{E}\left[V(x_{k+1}) \,\middle|\, x_k\right] = \mathbb{P}(\gamma_k = 1)\; x_k^T A_c^T P A_c x_k + \mathbb{P}(\gamma_k = 0)\; x_k^T A_o^T P A_o x_k + Tr(PW). \tag{4.43}$$

Here we used the fact that the random variable γ_k is independent of the system state x_k and depends just on the communication model. Plugging (4.43) at the left hand side of (4.11) we get

$$\mathbb{P}(\gamma_k = 1)\; x_k^T A_c^T P A_c x_k + \mathbb{P}(\gamma_k = 0)\; x_k^T A_o^T P A_o x_k \leq \rho x_k^T P x_k. \tag{4.44}$$

Since condition (4.11) needs to hold for all $x_k \in \mathbb{R}^n$ we can rewrite (4.44) as a linear matrix inequality (Boyd and Vandenberghe (2009))

$$\mathbb{P}(\gamma_k = 1) A_c^T P A_c + (1 - \mathbb{P}(\gamma_k = 1)) A_o^T P A_o \preceq \rho P, \tag{4.45}$$

The values $\mathbb{P}(\gamma_k = 1)$ that satisfy this linear matrix inequality belong in some closed convex set. We know by assumption that the case $\mathbb{P}(\gamma_k = 1) = 1$ belongs in this set by the assumption that $V(x)$ is a Lyapunov function for the closed loop mode of the system. As a result there is a minimum value c, given by the semidefinite program (4.13), such that condition (4.45) is equivalent to $\mathbb{P}(\gamma_k = 1) \geq c$.

4.5.2. Proof of Proposition 4.2

Statement (a) under assumptions 4.1 and 4.3 follows immediately from (Ribeiro, 2012, Theorem 1) where a similar optimization setup is examined. The proof is omitted due to space limitations.

To show (b) observe that, by definition of the dual function in (4.17), at the point μ^* we have that

$$g(\mu^*) \leq L(\alpha^*, p^*, \mu^*) \tag{4.46}$$

Since μ^* is optimal for (4.19) and using part (a) we have for the left hand side of (4.46) that $g(\mu^*) = D = P$. On the other hand, the right hand side of (4.46), by the definition of the Lagrangian at (4.16), equals

$$\begin{aligned} L(\alpha^*, p^*, \mu^*) &= P \\ &+ \sum_{i=1}^{m} \mu_i^* \Big[c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}^*(h)\, q(h_{ij}, p_{ij}^*(h)) \Big], \end{aligned} \tag{4.47}$$

because the objective of (4.14) at the optimal solution (α^*, p^*) equals the optimal value P. These expressions for the left and right hand sides of the inequality in (4.46) therefore give

$$P \leq P + \sum_{i=1}^{m} \mu_i^* \Big[c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha_{ij}^*(h)\, q(h_{ij}, p_{ij}^*(h)) \Big]. \tag{4.48}$$

This implies that the sum on the right hand side is non-negative. However all summands are non-positive, because $\mu^* \geq 0$ by dual feasibility (4.19), and the terms in the brackets in (4.48) are non-positive because (α^*, p^*) are feasible for the primal problem (4.14). The only possibility is that all summands in (4.48) are identically zero, which proves statement (b).

We have established that (4.48) holds with equality, so by tracing back our steps, we have that (4.46) holds with equality too, which, by the definition of the dual function on (4.17) translates to

$$\inf_{\alpha,p\in(\mathcal{A},\mathcal{P})} L(\alpha,p,\mu^*) = L(\alpha^*,p^*,\mu^*). \tag{4.49}$$

This verifies statement (c).

4.5.3. Proof of Proposition 4.3

We first show part (a) of the proposition. Consider the problem of minimizing the Lagrangian as given at the form (4.26) over variables $\alpha(.), p(.)$ for some $\mu \in \mathbb{R}^m_+$. Since $\mu^T c$ is constant the problem is equivalent to [3]

$$\inf_{\alpha,p\in(\mathcal{A},\mathcal{P})} \mathbb{E}_h \sum_{i,j} \alpha_{ij}(h) \left[p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h))\right]. \tag{4.50}$$

Without loss of generality we can exchange the expectation over h and the minimization over functions $\alpha(.), p(.)$ in (4.50) to equivalently solve for each $h \in \mathcal{H}^{m\times f}$

$$\inf_{\substack{\alpha(h)\in\Delta_{m,f} \\ p(h)\in[0,p_{\max}]^{m\times f}}} \sum_{i,j} \alpha_{ij}(h) \left[p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h))\right] \tag{4.51}$$

This step is valid because any pair of functions α, p that does not minimize the objective in (4.51) on a set of values of variables h with ϕ-positive measure must yield a strictly larger expected value in the objective of (4.50). In other words, the minimizers of (4.50) can only differ from the minimizers of (4.51) at a set of values for h with measure zero.

Then note that at any $h \in \mathcal{H}^{m\times f}$ and any choice for the variable $\alpha(h)$ we have that $\alpha_{ij}(h) \geq 0$. Hence the optimization over $p(h)$ in (4.51) can be rearranged to

$$\begin{aligned} &\inf_{\alpha(h)\in\Delta_{m,f}} \sum_{i,j} \alpha_{ij}(h) \\ &\qquad \inf_{p_{ij}(h)\in[0,p_{\max}]} p_{ij}(h) - \mu_i q(h_{ij}, p_{ij}(h)). \end{aligned} \tag{4.52}$$

The optimization over power variables $p_{i,j}(h)$ in this expression corresponds exactly to (4.27). Using the notation introduced in (4.29), the minimization over scheduling variables $\alpha(h)$ in (4.52) becomes

$$\inf_{\alpha(h)\in\Delta_{m,f}} \sum_{i,j} \alpha_{ij}(h)\, \xi(h_{ij}, \mu_i). \tag{4.53}$$

This is an integer programming problem over $\alpha_{ij} \in \{0,1\}$ according to the set $\Delta_{m,f}$ (cf.(4.2)). The expression given in (4.28) is a linear programming relaxation of (4.53) by assuming $\alpha_{ij} \geq 0$. The

[3] Within this proof we denote $\sum_{i=1}^m \sum_{j=1}^f$ as $\sum_{i,j}$ for compactness.

relaxation is tight (see, e.g., (Bertsimas and Tsitsiklis, 1997, Th. 7.5)), meaning that the optimal solution of (4.28) will be integer and feasible with respect to $\Delta_{m,f}$, hence optimal for (4.53) too.

Now let us prove part (b) of the proposition. We need to show that any pair $\alpha(\mu), p(\mu)$, which are functions of h, that solves (4.50) gives a unique evaluation of $s(\mu)$ given in (4.21). Since $s_i(\mu)$ involves integrating the term

$$\sum_{j=1}^{f} \alpha_{ij}(\mu;h)\, q(h_{ij}, p_{ij}(\mu;h)) \tag{4.54}$$

with respect to the distribution ϕ of $h \in \mathcal{H}^{m\times f}$, it suffices to show that (4.54) is unique ϕ-a.s.

By the argument presented already, minimizing (4.50) is a.s. equivalent to minimizing (4.51). The latter is again equivalent to the problem (4.52) since all $\alpha_{ij}(h) \geq 0$. Note that the only case where the optimizers in (4.51) can differ from the ones obtained in (4.52) is if $\alpha_{ij}(\mu;h) = 0$ for some i, j is optimal at some values $h \in \mathcal{H}^{m\times f}$ and the power minimizer $p_{ij}(\mu;h)$ in (4.51) can be chosen arbitrarily. But this does not affect the computation of $s_i(\mu)$ since (4.54) will equal zero. Hence we only need to show that the minimizers $\alpha(\mu;h), p(\mu;h)$ in (4.52) imply a.s. uniqueness of (4.54).

For values of h where the minimizers $\alpha(\mu;h), p(\mu;h)$ of problem (4.52) are unique it is immediate that (4.54) has a unique value, hence we only need to consider h where the minimizers are not unique. By Assumption 4.2(c) the minimizer $p(\mu;h)$, which is given in (4.27), is unique for almost all h, therefore we only need to focus on the set of values for h where the minimizer $\alpha(\mu;h)$, described by (4.28), is not unique.

Let us denote by E the set of interest, i.e., the set of $h \in \mathcal{H}^{m\times f}$ where $\alpha(\mu;h)$ in (4.28) is not unique. By considering all possible pairs of multiple solutions $\alpha' \neq \alpha''$ in the finite set $\Delta_{m,f}$, we can rewrite E as a union $E = \bigcup_{\alpha' \neq \alpha'' \in \Delta_{m,f}} E_{\alpha',\alpha''}$ where $E_{\alpha',\alpha''} \subseteq \mathcal{H}^{m\times f}$ such that

$$h \in E_{\alpha',\alpha''} \Leftrightarrow \alpha', \alpha'' \in \operatorname*{argmin}_{\alpha \in \Delta_{m,f}} \sum_{i,j} \alpha_{ij}\, \xi(h_{ij}, \mu_i). \tag{4.55}$$

In other words, the set $E_{\alpha',\alpha''}$ is the set of values h where both α', α'' are optimal for (4.28). The rest of the proof shows that on any $E_{\alpha',\alpha''}$ the value of (4.54) is almost surely unique.

The set $E_{\alpha',\alpha''}$ depends on the shape of the function ξ defined in (4.29), so next we point out two properties of $\xi(h_{ij}, \mu_i)$.

Fact 1: For almost all h_{ij} where the optimal value of problem (4.29) is $\xi(h_{ij}, \mu_i) = 0$, the optimal solution is unique and equals $p_{ij}(\mu;h) = 0$.

Proof of Fact 1: First we note that for any h_{ij}, the choice $p = 0$ is feasible for problem (4.29) and by Assumption 4.2(a) it gives an objective $p - \mu_i q(h_{ij}, p) = 0$. So whenever the optimal value of problem (4.29) is 0, then $p = 0$ is an optimal solution. This optimal solution is unique for almost all h_{ij} because of Assumption 4.2(c).

Fact 2: If at some h_{ij} the optimal value of problem (4.29) is $\xi(h_{ij}, \mu_i) < 0$, then for $h'_{ij} > h_{ij}$ we have that $\xi(h'_{ij}, \mu_i) < \xi(h_{ij}, \mu_i)$.

Proof of Fact 2: First note that at the given h_{ij} it must be that the optimal solution $p_{ij}(\mu; h)$ of problem (4.29) satisfies $q(h_{ij}, p_{ij}(\mu; h)) > 0$. This is true because otherwise $q(h_{ij}, p_{ij}(\mu; h)) = 0$ implies $\xi(h_{ij}, \mu_i) = p_{ij}(\mu; h) \geq 0$. Second by Assumption 4.2(b) when $q(.) > 0$, it is strictly increasing in its argument. Thus we have

$$\begin{aligned}\xi(h_{ij},\mu_i) &= p_{ij}(\mu; h) - \mu_i q(h_{ij}, p_{ij}(\mu; h)) \\ &> p_{ij}(\mu; h) - \mu_i q(h'_{ij}, p_{ij}(\mu; h)) \geq \xi(h'_{ij}, \mu_i)\end{aligned} \tag{4.56}$$

for $h'_{ij} > h_{ij}$.

Let us now fix some $\alpha' \neq \alpha'' \in \Delta_{m,f}$ and consider the set $E_{\alpha',\alpha''}$. Pick indices $\imath, \jmath$ where α', α'' differ, i.e., without loss of generality, $\alpha'_{\imath,\jmath} = 1, \alpha''_{\imath,\jmath} = 0$. Consider first the case of $h \in E_{\alpha',\alpha''}$ where $\xi(h_{\imath,\jmath}, \mu_\imath) = 0$. By Fact 1 above we know that this implies $p_{\imath,\jmath}(\mu; h) = 0$ is almost surely the unique optimizer of (4.27). But in that case $q(h_{\imath,\jmath}, p_{\imath,\jmath}(\mu; h)) = 0$, and the choice of $\alpha_{\imath,\jmath}(h)$ does not affect the value of (4.54), which is zero.

Second, we examine the set $h \in E_{\alpha',\alpha''}$ where $\xi(h_{\imath\jmath}, \mu_\imath) < 0$. We will show that this event happens with ϕ-probability zero. In particular by Assumption 4.1 ϕ has a probability density function on $\mathcal{H}^{m\times f}$, or more formally ϕ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{H}^{m\times f}$. Hence to show that the discussed event has ϕ-measure zero, it suffices to show that it has Lebesgue measure zero. Note that we can upper bound the set as follows

$$\begin{aligned}&E_{\alpha',\alpha''} \bigcap \{h : \xi(h_{\imath\jmath}, \mu_\imath) < 0\} \\ &\subseteq \{h : \sum_{i,j} (\alpha''_{ij} - \alpha'_{ij})\, \xi(h_{ij}, \mu_i) = 0, \quad \xi(h_{\imath\jmath}, \mu_\imath) < 0\} \\ &= \{h : \sum_{i\neq\imath, j\neq\jmath} (\alpha''_{ij} - \alpha'_{ij})\, \xi(h_{ij}, \mu_i) = \xi(h_{\imath\jmath}, \mu_\imath) < 0\}\end{aligned} \tag{4.57}$$

The subset in the first step is justified from the fact that, in contrary to the definition of $E_{\alpha',\alpha''}$ in (4.55), we do not take α', α'' to be optimal for problem (4.28). We only require that they yield the same objective in the problem. The second step follows by the appropriately selected indices $\imath, \jmath$.

We will now argue that the last set in (4.57) has Lebesgue measure zero. If we fix the values of all the variables/coordinates $h_{ij}, i \neq \imath, j \neq \jmath$, there is at most one value for the variable/coordinate $h_{\imath\jmath}$ that belongs in the set. The reason is that for values of the $h_{\imath\jmath}$ coordinate where $\xi(h_{\imath\jmath}, \mu_\imath) < 0$, Fact 2 above states that $\xi(h_{\imath\jmath}, \mu_\imath)$ is strictly monotonic in $h_{\imath\jmath}$. Hence there can be at most one value $h_{\imath\jmath}$ that equals the sum within the last set of (4.57). This means that the last set in (4.57) can be equivalently described by a mapping from an $m \cdot f - 1$ dimensional space to the space $\mathcal{H}^{m\times f}$, or in other words it is a lower-dimensional subset of $\mathcal{H}^{m\times f}$. Hence it has Lebesgue measure zero. This implies that the first set in (4.57) has Lebesgue (and ϕ) measure zero as well.

The above procedure can be iterated for any pair α', α'' to conclude that in their union set E the value of the subgradient vector is almost surely unique.

4.5.4. Proof of Theorem 4.1

Let μ^* be an optimal solution of the dual problem (4.19). First, we argue that every pair $\alpha(\mu^*), p(\mu^*)$ chosen from the set of Lagrangian minimizers $(\mathcal{A}, \mathcal{P})(\mu^*)$ at the point μ^* (cf. (4.18)) is an optimal solution to primal Problem 4.1 (equivalently (4.14)). Under Assumptions 4.1 and 4.2, Proposition 4.3(b) states that the vector $s(\mu^*)$ in (4.21) has the same value at any chosen pair $\alpha(\mu^*), p(\mu^*)$. Since $s(\mu^*)$ is also the constraint slack of the chosen pair in the primal problem (4.14), then any Lagrange optimizers $\alpha(\mu^*), p(\mu^*)$ have the same constraint slack. Moreover, under Assumptions 4.1 and 4.3, Proposition 4.2(c) states that the optimal primal variables α^*, p^* are one such pair of Lagrange optimizers at μ^*, and by definition they have a feasible constraint slack. Hence all Lagrange optimizers $\alpha(\mu^*), p(\mu^*)$ have the same feasible constraint slack as α^*, p^*. Additionally all optimizers $\alpha(\mu^*), p(\mu^*)$ yield the same minimum Lagrangian value $L(\alpha(\mu^*), p(\mu^*), \mu^*)$. By the form of the Lagrangian in (4.16) it follows that all optimizers $\alpha(\mu^*), p(\mu^*)$ also give the same primal objective in (4.14) as the point α^*, p^*, i.e., the minimum P. Hence any optimizer pair $\alpha(\mu^*), p(\mu^*)$ is primal optimal. The first statement of the theorem follows because the scheduling and power allocation obtained by (4.27)-(4.29) at μ^* describe one pair of Lagrange optimizers at μ^*, i.e., are optimal solutions to Problem 4.1.

The convergence of iterations (4.23)-(4.24) to the optimal dual variable μ^* for stepsizes in (4.25) relies on the boundedness of the subgradient vectors (as mentioned after (4.22)) and follows from a standard subgradient method argument – for a proof see, e.g., (Bertsekas et al., 2003, Prop. 8.2.6).

4.5.5. Proof of Proposition 4.4

We begin by noting that at every time k the vector s_k computed by (4.31) is a stochastic subgradient for the dual function $g(\mu)$ at the point μ_k, i.e.,

$$g(\mu') - g(\mu_k) \leq (\mu' - \mu_k)^T \mathbb{E}[s_k \mid \mu_k] \quad \text{for all} \mu' \in \mathbb{R}^m_+. \tag{4.58}$$

To show this fact compare equations (4.30)-(4.31) of the online algorithm with (4.21) to conclude that $\mathbb{E}[s_k \mid \mu_k] = s(\mu_k)$ because h_k is i.i.d for every k. Inequality (4.58) then follows directly from (4.20).

Then note that by Assumption 4.3 there exists a strictly feasible primal solution α', p'. Call P' the resulting objective value (4.14) at this point, and let a positive constant $\varepsilon' > 0$ denote the constraint slack of (4.15) at this point, i.e., $c_i + \varepsilon' \leq \mathbb{E}_h \sum_{j=1}^{f} \alpha'_{ij}(h)\, q(h_{ij}, p'_{ij}(h))$. Then we may bound the dual function (4.17) at the optimal μ^* by

$$D = g(\mu^*) \leq L(\alpha', p', \mu^*) = P' + \sum_{i=1}^{m} \mu_i^* \left[c_i - \mathbb{E}_h \sum_{j=1}^{f} \alpha'_{ij}(h)\, q(h_{ij}, p'_{ij}(h)) \right] \leq P' - \sum_{i=1}^{m} \mu_i^* \epsilon'$$

Rearranging the above inequality, and since $\mu^* \geq 0$, it follows that $\mu_\ell^* \leq \sum_{i=1}^m \mu_i^* \leq (P' - D)/\epsilon'$ for every ℓ, i.e., the optimal dual variables are finite.

Since the optimal dual variables are finite, the distance $\|\mu_k - \mu^*\|$ between any random μ_k obtained by Algorithm 4.1 and the set of optimal dual variables μ^* is well-defined and bounded. The following lemma gives an upper bound on this distance. Here recall that as we commented after (4.21) the subgradients $s(\mu)$ are always bounded in our problem.

Lemma 4.1. *Let D be the optimal value of the dual problem (4.19), μ^* be an optimal solution, and S be the bound on the subgradient $\|s(\mu)\| \leq S$ for any $\mu \in \mathbb{R}_+^m$. Then at each step k of Algorithm 4.1 the update of μ_{k+1} satisfies*

$$\mathbb{E}[\|\mu_{k+1} - \mu^*\|^2 \,|\mu_k] \leq \|\mu_k - \mu^*\|^2 + \epsilon_k^2 S^2 - 2\epsilon_k (D - g(\mu_k)) \tag{4.59}$$

Proof. First use the expression $\mu_{k+1} = [\mu_k + \epsilon_k s_k]_+$ in Algorithm 4.1 to write

$$\|\mu_{k+1} - \mu^*\| = \|[\mu_k + \epsilon_k s_k]_+ - \mu^*\| \leq \|\mu_k + \epsilon_k s_k - \mu^*\|, \tag{4.60}$$

where the last inequality holds because when projecting on the positive orthant the distance from a point μ^* in the orthant can only decrease. Taking expectation on both sides given μ_k and expanding the square norm of the right hand side, we get

$$\begin{aligned}\mathbb{E}[\|\mu_{k+1} - \mu^*\|^2 \,|\mu_k] \leq& \|\mu_k - \mu^*\|^2 + \epsilon_k^2 S^2 \\ &+ 2\epsilon_k (\mu_k - \mu^*)^T \mathbb{E}[s_k \mid \mu_k]\end{aligned} \tag{4.61}$$

where we bounded $\|\mathbb{E}[s_k \mid \mu_k]\|^2 < S^2$. Then (4.59) follows from (4.61) by applying inequality (4.58) with $\mu' = \mu^*$. ■

Based on (4.59), we will use a supermartingale convergence argument to show that $\|\mu_k - \mu^*\|^2 \to 0$ almost surely. Note first that at any $\mu_k \in \mathbb{R}_+^m$ the dual function is smaller than the optimal value (cf. (4.19)), so $D - g(\mu_k) \geq 0$. Hence (4.59) can be simplified to

$$\mathbb{E}[\|\mu_{k+1} - \mu^*\|^2 \,|\mu_k] \leq \|\mu_k - \mu^*\|^2 + \epsilon_k^2 S^2. \tag{4.62}$$

Then consider the sequence of random variables

$$a_k = \|\mu_k - \mu^*\|^2 + \sum_{\ell \geq k} \epsilon_l^2 S^2. \tag{4.63}$$

This stochastic process is: i) measurable with respect to the sequence (filtration) $\mathcal{F}_k = \{\mu_{0:k}\}$, ii) non-negative, iii) integrable because μ_k generated by Algorithm 4.1 is bounded at every k and stepsizes are square summable, iv) satisfies $\mathbb{E}[a_{k+1} \mid \mathcal{F}_k] \leq a_k$ as can be seen by definition (4.63) and (4.62).

Such a stochastic process is called a supermartingale (Durrett, 2010, Ch. 5), and a non-negative supermartingale converges almost surely to some limit random variable (Durrett, 2010, Th. 5.2.9). Observe that the second summand $\sum_{\ell=k}^{\infty} \varepsilon_l^2 S^2$ of a_k in (4.63) is deterministic and converges to 0 because of square summability of the stepsizes. Hence the random variable $\|\mu_k - \mu^*\|^2$ converges almost surely (to some limit random variable).

To arrive at a contradiction suppose the limit random random variable is not identically zero. Equivalently, with probability $\delta > 0$ we have $\|\mu_k - \mu^*\|^2 \geq \varepsilon$ for some $\varepsilon > 0$ for all sufficiently large k. This implies that μ_k and $g(\mu_k)$ are bounded away from the optimal μ^* and D respectively, hence the following expected value diverges,

$$\mathbb{E} \sum_{k \geq 0} 2\varepsilon_k (D - g(\mu_k)) = +\infty. \tag{4.64}$$

However taking expectation at both sides of (4.59) and iterating for $k = 0, \dots, N-1$ implies

$$\begin{aligned} \mathbb{E}\|\mu_N - \mu^*\|^2 \leq \|\mu_0 - \mu^*\|^2 \\ + \sum_{k=0}^{N-1} 2\varepsilon_k^2 S^2 - \mathbb{E} \sum_{k=0}^{N-1} 2\varepsilon_k (D - g(\mu_k)). \end{aligned} \tag{4.65}$$

The left hand side is non-negative, but (4.64) implies that in the limit as $N \to \infty$ the right hand side becomes negative. This is a contradiction. Therefore it must be that $\|\mu_k - \mu^*\|^2$ converges to zero with probability 1.

By continuity of the concave dual function $g(\mu)$ we also have that $g(\mu_k)$ converges to $g(\mu^*) = D$ a.s.

4.5.6. Proof of Theorem 4.2 and Corollary 4.1

The proofs of the two results are presented together. First we will show that the statements (4.38) of Theorem 4.2 and (4.40) of Corollary 4.1 hold, by converting them into equivalent ones involving variables relating to the dual problem (4.19). Imitating the steps leading from problem (4.9) to problem (4.14), the statement (4.38) is equivalent to

$$\limsup_{k \to \infty} c_i - \mathbb{E}_{h_k} \left[\sum_{j=1}^{f} \alpha_{ij,k}\, q(h_{ij,k}, p_{ij,k}) \,\middle|\, \mu_k \right] \leq 0 \tag{4.66}$$

holding a.s. with respect to the channel sequence $\{h_k, k \geq 0\}$. Here we exploited the fact that by the online algorithm the variables α_k, p_k conditioned on the value of μ_k are independent of the observed channel history (but μ_k does depend on the whole history). Similarly we see that the statement (4.40) is equivalent to

$$\limsup_{k\to\infty} c_i - \mathbb{E} \sum_{j=1}^{f} \alpha_{ij,k}\, q(h_{ij,k}, p_{ij,k}) \leq 0 \tag{4.67}$$

The term inside the limit is the expected (i.e., unconditioned) value of the term in (4.66) with respect to the random sequence $\{h_k, k \geq 0\}$.

Condition (4.66) is equivalent to $\limsup_{k\to\infty} \mathbb{E}_{h_k}[s_k \mid \mu_k] \leq 0$ using the expression of s_k given in (4.31). Also we already argued in the proof of Prop. 4.4 that $\mathbb{E}_{h_k}[s_k \mid \mu_k] = s(\mu_k)$ where $s(\mu_k)$ is given by (4.21) and is a subgradient of the dual function g at μ_k. To sum up, (4.38) is equivalent to

$$\limsup_{k\to\infty} s(\mu_k) \leq 0 \quad \text{a.s.} \tag{4.68}$$

Similarly, condition (4.40) via (4.67) is equivalent to $\limsup_{k\to\infty} \mathbb{E}s(\mu_k) \leq 0$. But the latter is a consequence of (4.68) because $\limsup_{k\to\infty} \mathbb{E}s(\mu_k) \leq \mathbb{E}\limsup_{k\to\infty} s(\mu_k) \leq 0$. The first inequality follows by applying Fatou's lemma (Durrett, 2010, Thm. 1.6.5) to the bounded below (as we comment after (4.21)) random variable $-s(\mu_k)$. The second inequality follows by monotonicity of expectation. Hence, to prove the statements (4.38) and (4.40) it suffices to show (4.68) which we do next.

Under Assumption 4.3 we have established in Proposition 4.4 that a.s. $\mu_k \to \mu^*$. Then we note a convex analysis fact by (Bertsekas et al., 2003, Prop. 4.2.3). If g is concave, and $\mu_k \to \mu^*$, and $s(\mu_k)$ is selected as a subgradient of g at μ_k, then every limit point of $s(\mu_k)$ is a subgradient of g at μ^*. Hence for the sequence of μ_k obtained by the online algorithm we have that $s(\mu_k)$ converges a.s. to a subgradient of g at μ^*.

Also, as follows from Danskin's theorem (Bertsekas et al., 2003, Prop. 4.5.1), the subgradients of the dual function g at any point μ belong in the convex hull of the vectors $s(\mu)$ obtained in (4.21). Hence the sequence $s(\mu_k)$ converges a.s. to the convex hull of the vectors $s(\mu^*)$. But under Assumptions 4.1, 4.2, and 4.3, as we argued in the proof of Theorem 4.1, the vectors $s(\mu^*)$ take a unique value that satisfies $s(\mu^*) \leq 0$. Hence for the sequence of μ_k obtained by the online algorithm we have that $\limsup_{k\to\infty} s(\mu_k) \leq 0$ a.s., which is exactly what we set out to prove in (4.68).

Finally let us prove (4.39). Recall that the dual function equals $g(\mu) = L(\alpha(\mu), p(\mu), \mu)$ where $\alpha(\mu), p(\mu)$ are chosen as Lagrange optimizers at μ according to (4.18). Using the definition of the Lagrangian at (4.16) and the interpretation of the subgradient $s(\mu)$ at (4.21) as the constraint slack, we have that for any μ_k

$$
\begin{aligned}
g(\mu_k) &= L(\alpha(\mu_k), p(\mu_k), \mu_k) \\
&= \mathbb{E}_h \sum_{i=1}^{m} \alpha_i(\mu_k; h) p_i(\mu_k; h) + \mu_k^T s(\mu_k)
\end{aligned} \tag{4.69}
$$

Now observe that the expectation in (4.39) equals the expectation given in (4.69) because by design of Algorithm 4.1 the primal variables α_k, p_k are selected as Lagrange optimizers at μ_k. Therefore to show that (4.39) holds a.s. it suffices to show that the expectation in (4.69) converges a.s. to P which equals D by strong duality.

Proposition 4.4 establishes that the left hand side of (4.69) converges to $g(\mu_k) \to D$, and also that $\mu_k \to \mu^*$ a.s. We have also already argued that $s(\mu_k) \to s(\mu^*)$ a.s. Therefore also $\mu_k^T s(\mu_k) \to \mu^{*T} s(\mu^*)$ a.s. But by Prop. 4.2(b) $\mu^{*T} s(\mu^*) = 0$. This shows that the expectation at the right hand side of (4.69) converges to D, which completes the proof.

Chapter 5: Random Access Design for Wireless Control Systems

5.1. Problem Formulation

In this chapter a decentralized channel access method is employed for multiple control systems sharing a wireless medium. The architecture was introduced in Section 2.2.2 and is depicted again in Fig. 15. Each sensor i ($i = 1, 2, ..., m$) in the architecture transmits measurements of plant i to an access point responsible for computing the plant control inputs. Packet collisions might arise on the shared medium between simultaneously transmitting sensors. We are interested in designing a mechanism for each sensor to independently decide whether to access the medium (random access) in a way that guarantees desirable control performance for all control systems.

We denote by $\gamma_{i,k} \in \{0, 1\}$ the success of the transmission at time k for link/system i, determined by the communication algorithm to be designed. As introduced in Section 2.1 the evolution of each system i depends on the success of transmission and is described by a switched linear time-invariant system of the form

$$x_{i,k+1} = \begin{cases} A_{c,i}\, x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 1, \\ A_{o,i}\, x_{i,k} + w_{i,k}, & \text{if } \gamma_{i,k} = 0. \end{cases} \tag{5.1}$$

Here $x_{i,k} \in \mathbb{R}^{n_i}$ denotes the state of control system i at each time k. At a successful transmission the system dynamics are described by the asymptotically stable closed-loop matrix $A_{c,i} \in \mathbb{R}^{n_i \times n_i}$, and otherwise by the unstable open-loop matrix $A_{o,i} \in \mathbb{R}^{n_i \times n_i}$. The additive terms $w_{i,k}$ model an independent (both across time k for each system i, and across systems) identically distributed (i.i.d.) noise process with mean zero and covariance $W_i \succeq 0$. Example of such control systems were presented in Section 2.1.

According to the random access communication mechanism, introduced in Section 2.2.2, at each time step k a sensor i transmits over the shared wireless medium with a probability $\alpha_{i,k} \in [0, 1]$ which is a design variable. A sensor's transmission might fail due to two reasons, packet decoding errors and packet collisions. A collision might be experienced on link i, thereby rendering packet i lost, if some other sensor $j \neq i$ transmits in the same time slot. We assume that such a collision event occurs with constant probability $q_{ji} \in [0, 1]$, *given* that both sensors i, j transmit in the slot. Thus, the probability that sensor i's transmission is free of collisions, i.e., that no other sensor transmits and causes collisions on link i, equals $\prod_{j \neq i}[1 - \alpha_{j,k}\, q_{ji}]$.

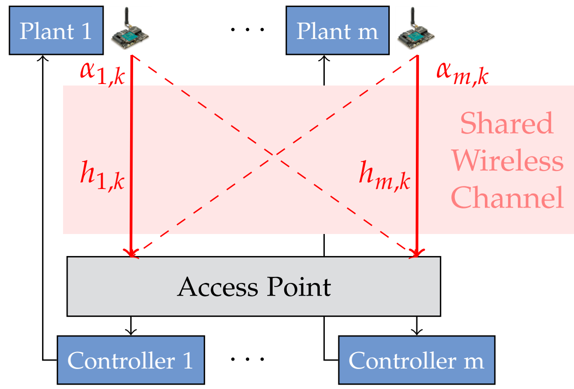

Figure 15: Random access architecture for m control loops over a shared wireless medium. Each sensor i randomly transmits with probability $\alpha_{i,k}$ at time k to a common access point computing the plant control inputs. If only sensor i transmits, the successful decoding probability depends on local channel conditions $h_{i,k}$. If other sensors transmit at the same time a collision might occur at sensor i's transmission, rendering i's packet lost.

If sensor i transmits and has a collision free time slot, the success of decoding the message at the access point/receiver depends on the randomly varying channel conditions on link i. Employing the block fading wireless channel model introduced in Section 2.2 we denote by $h_{i,k} \in \mathbb{R}_+$ the current fading state for link i at time k, which is assumed i.i.d. across time with distribution ϕ_i. We also assume channel states are independent among systems i, a common assumption in the literature Adireddy and Tong (2005); Qin and Berry (2006); Hu and Ribeiro (2011), as well as independent of the plant process noise $w_{i,k}$. The probability of successful decoding, in absence of collisions, depends on the current channel fading and transmit power of the sensor. For simplicity in this section we assume the transmit power for each sensor i is fixed to some constant value p_i. Then the decoding probability is given by a relationship of the form $q(h_{i,k}, p_i)$. An illustration of this function is depicted in Fig. 2. Since power p_i is fixed in this setup (not a design variable) it is omitted and the decoding probability is denoted as $q(h_{i,k})$ in the rest of this chapter. The function $q : \mathbb{R}_+ \to [0, 1]$ is assumed to be continuous and strictly increasing, i.e., higher channel fading states imply higher packet success probability.

Combining the effects of collisions and packet losses due to fading, the probability that a packet is successfully decoded at the access point can be written as

$$\mathbb{P}(\gamma_{i,k} = 1) = \alpha_{i,k}\, q(h_{i,k}) \prod_{j \neq i} \left[1 - \alpha_{j,k}\, q_{ji}\right]. \tag{5.2}$$

This expression states that the probability of system i in (5.1) closing the loop at time k equals the probability that transmission i is successfully decoded at the receiver, multiplied by the probability that no other sensor $j \neq i$ is causing collisions on ith transmission.

Channel states reveal information about how easy it is for each sensor to successfully communicate, assuming no other sensor transmits. We assume that before deciding whether to transmit each sensor has access to its respective channel state and may adapt accordingly. For example sensor i may transmit with higher or lower rate $\alpha_{i,k}$ under favorable or unfavorable channel states $h_{i,k}$ respectively. Hence we design policies that are measurable functions of the form $\alpha_{i,k} = \alpha_i(h_{i,k})$. Since channel states are i.i.d. over time we restrict attention to stationary policies, and drop the time index when not necessary. The set of all access policies for sensor i is then

$$\mathcal{A}_i = \{\alpha_i : \mathbb{R}_+ \to [0,1]\} \tag{5.3}$$

and the vector $\alpha(.)$ of access policies for all sensors belongs in the Cartesian product space $\mathcal{A} = \mathcal{A}_1 \times \ldots \mathcal{A}_m$. For fixed sensor access policies, the probability of successful transmission on link i can be expressed as

$$\mathbb{P}(\gamma_{i,k} = 1) = \mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)] \prod_{j \neq i} \Big[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)]\, q_{ji}\Big]. \tag{5.4}$$

This expression follows from (5.2) by taking expectation with respect to the channel states and using the independence of channels among systems. The expectation is well-defined as both functions $\alpha(.)$, $q(.)$ are measurable and bounded in $[0,1]$ hence integrable.

We make the following technical assumption on the probability distribution of channel states, which holds true for practically considered models (Goldsmith, 2005, Ch. 3).

Assumption 5.1. *The distributions ϕ_i of channel states $\{h_{i,k},\, k \geq 0\}$ for all $i = 1, \ldots, m$ are absolutely continuous, i.e., have a probability density function on $\mathbb{R}_+$.*

The random packet success on link i modeled by (5.4) causes each control system i in (5.1) to switch in a random fashion between the two modes of operation (open and closed loop). As a result, the sensor access policies $\alpha(.)$ to be designed affect the performance of all control systems. We account for control performance via a Lyapunov-function-based abstraction, as introduced in Section 2.1.1. In particular we assume that a quadratic function $V_i(x_i) = x_i^T P_i x_i,\ x_i \in \mathbb{R}^{n_i}$ with a positive definite matrix $P_i \succ 0$ is provided for each system. To achieve desired control performance *every function $V_i(x_i)$ is required to decrease with a desired rate $\rho_i < 1$ in expectation at each step*, i.e.,

$$\mathbb{E}\left[V_i(x_{i,k+1}) \,\middle|\, x_{i,k}\right] \leq \rho_i\, V_i(x_{i,k}) + Tr(P_i W_i) \tag{5.5}$$

for all $x_{i,k} \in \mathbb{R}^{n_i}$. The expectation in this expression accouns for the randomness introduced by the wireless communication as well as the plant process noise.

By Proposition 4.1 of Section 2.1.1 the above Lyapunov control performance requirements are equivalent to guaranteeing minimum packet success rates for each link i, i.e.,

$$\mathbb{P}(\gamma_{i,k} = 1) \;\geq\; c_i, \tag{5.6}$$

for some non-negative constants $c_i \geq 0$ that are easily computed (by the semidefinite program (4.45)). Hence to achieve desired control performance we need to ensure that (5.6) holds for all links i.

Besides control performance, it is desired that the sensors' channel access mechanism makes an efficient use of their power resources. Since sensor i transmits with a fixed power $p_i > 0$ when it decides to access the channel, the total expected power consumption at each slot is given by $\sum_{i=1}^{m} \mathbb{E}_{h_i} \alpha_i(h_i) p_i$ summing up the transmit power of each system i if the system decides to transmit. We pose then the design of the sensor access rates α that minimize the total expected power consumption subject to the desired control performance (5.5) (equivalently (5.6)) for all plants as

$$\underset{\alpha \in \mathcal{A}}{\text{minimize}} \quad \sum_{i=1}^{m} \mathbb{E}_{h_i} \alpha_i(h_i) p_i \tag{5.7}$$

$$\text{subject to} \quad c_i \leq \mathbb{E}_{h_i}[\alpha_i(h_i) q(h_i)] \prod_{j \neq i} \left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)] q_{ji}\right], \quad i = 1, \ldots, m. \tag{5.8}$$

Technically we assume that the problem is strictly feasible, as follows.

Assumption 5.2. *There exists $\alpha' \in \mathcal{A}$ that satisfies constraints* (5.8) *with strict inequality.*

In the following section we proceed to characterize the optimal access policies α^*. In particular we reveal a simple and intuitive decoupled structure. Each sensor i independently accesses the channel in a way that trades off the goal of closed loop i with the effect of collisions on all other closed loops $j \neq i$ collectively. Later we develop a procedure to find these optimal access policies.

5.2. Channel-aware Random Access Design

Our main result is the following characterization of the optimal access policies for the sensors.

Theorem 5.1 (Optimal sensor access policies)**.** *Consider a random access architecture with m control loops of the form* (5.1)*, communication modeled by (5.4), and control performance abstracted by* (5.5)-(5.6) *for each loop $i = 1, \ldots, m$. Consider the design of optimal sensor access policies* (5.7)-(5.8)*, and let Assumptions 5.1, 5.2 hold. Then there exists a matrix of non-negative elements $\nu^* \in \mathbb{R}_+^{m \times m}$ such that the optimal sensor access policy for each sensor $i = 1, \ldots, m$ is written as*

$$\alpha_i^*(h_i) = \begin{cases} 1 & \text{if } \nu_{ii}^* q(h_i) \geq p_i + \sum_{j \neq i} \nu_{ji}^* q_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

We observe the following interesting facts. First note that the optimal policies are deterministic, that is, given current channel conditions each sensor either transmits or not. Second we note that by the assumed strict monotonicity of the packet success function $q(.)$, the optimal sensor access policies in (5.9) are threshold policies. That is, a sensor transmits only when its corresponding channel quality is above some threshold. The intuitive interpretation is that a sensor should attempt to close its loop only when its channel is sufficiently favorable.

Third, and more importantly, the optimal policies are decoupled among the sensors. That is because the policy α_i^* (or equivalently the threshold for sensor i) in (5.9) only depends on parameters pertinent to system i, i.e., its transmit power p_i, and the values ν_{ii}^* and $\sum_{j\neq i}\nu_{ji}^*\, q_{ij}$ which belong in the ith column of matrix ν^*. Hence, as long as the matrix ν^* is available, each sensor can select its optimal channel access policy independently of what the other sensors are trying to achieve. We note that decentralized threshold-based policies have also been shown to be optimal for general wireless communication networks Adireddy and Tong (2005); Qin and Berry (2006); Hu and Ribeiro (2011). The context differs however, since in these works the objective is thoughput-based utility functions in contrast to the packet success rates used for control systems here.

As we explain in the proof, the matrix ν^* technically corresponds to the optimal Lagrange multiplier of an appropriately defined problem (cf. (5.25)-(5.28)). An intuitive alternative interpretation is as follows. We can think of each diagonal term ν_{ii}^* as the importance of control performance of system i, and of each offdiagonal term ν_{ji}^* as the collision effect that sensor i has on system j. The optimal access policy for sensor i in (5.9), or equivalently the optimal channel threshold, trades off the requirement on loop i and the collective negative effect ($\sum_{j\neq i}\nu_{ji}^* q_{ij}$) on all other control loops $j \neq i$. That is because a larger value ν_{ii}^* corresponds to a lower threshold (sensor transmits more often), while a larger value $\sum_{j\neq i}\nu_{ji}^* q_{ij}$ corresponds to a higher threshold (sensor transmits less often). Note also that the latter summands are normalized by the parameters q_{ij}, i.e., the probability that sensor i collides with link i when both sensors transmit. Morever, a high transmit power p_i in (5.9) also implies that sensor i should access the channel less often to limit expenditures.

The decoupled structure of the optimal sensor access policies in Theorem 5.1 relies on knowing the values ν^*. In the following section we develop a distributed iterative procedure to obtain the desired ν^*.

Remark 5.1. *In our work in Gatsis et al. (2015c) we consider simpler random access policies for the sensors, not taking into account channel state information. In particular we consider that at every time k each sensor i randomly and independently transmits with some constant probability* $\tilde{\alpha}_i \in [0,1]$ *to be designed. Similarly to* (5.4) *the probability of successfully closing each loop is given by* $\mathbb{P}(\gamma_{i,k}=1) = \tilde{\alpha}_i\, q_{ii}\, \prod_{j\neq i}\left[1-\tilde{\alpha}_j q_{ji}\right]$. *It turns out (Gatsis et al., 2015c, Theorem 2) that the optimal access rates* $\tilde{\alpha}^*$*, i.e., the solution to a problem equivalent to the channel-aware setup in* (5.7)-(5.8) *can be expressed as*

$$\tilde{\alpha}_i = \frac{\tilde{\nu}_{ii}}{p_i + \sum_{j\neq i}\tilde{\nu}_{ji}\, q_{ij}} \tag{5.10}$$

for each $i \in \{1,\dots,m\}$ *for some non-negative matrix* $\tilde{\nu} \in \mathbb{R}_+^{m\times m}$*. The matrix* $\tilde{\nu}$ *here has the same interpretation as the matrix* ν^* *of Theorem 5.1 but the two matrices are different as they correspond to different problems. Hence we see that for the non-channel-aware case the sensors need to randomize* ($0 < \tilde{\alpha}_i < 1$)*. In contrast, conditioned on channel state information being available the optimal policies for the* *sensors are deterministic, exploiting favorable channel conditions to transmit.*

We can now capitalize on the fact that the optimal sensor access policies that guarantee control performance of all closed loop systems and minimize power expenditures are characterized in terms of some appropriate matrix of values ν^* (Theorem 5.1). We develop an iterative procedure to determine the optimal sensor access policies by computing these values ν^*. The procedure is distributed and easily implementable in the architecture of Fig. 15. In particular the common access point/controller is responsible for finding ν^* and communicates them to the sensors via the reverse channel, so that the sensors do not need to directly coordinate or communicate among themselves.

Technically as we have argued in the proof of Theorem 5.1 the values ν^* are the optimal Lagrange dual variables of an appropriately defined problem (cf.(5.25)-(5.28)). The iterative procedure presented in Algorithm 5.1 corresponds mathematically to a dual subgradient algorithm (Bertsekas et al., 2003, Ch. 8) to find the optimal dual variables ν^*. Alternatively we can interpret the procedure as a distributed implementation in the wireless control architecture of Fig. 15 as follows.

At each period t the access point/controller of Fig. 15 maintains a tentative matrix of values $\nu(t)$. At the beginning of each period, the access point (AP) sends to each sensor i the values $\nu_{ii}(t)$ and $\sum_{j\neq i}\nu_{ji}(t)q_{ij}$ via the reverse channel (Step 3). For the rest of the period t each sensor uses a random access policy $\alpha(h_i,;t)$ as if the received values $\nu(t)$ corresponded to the optimal ν^* (Step 4). Here $\alpha(h_i;t)$ denotes the valuation of the policy during period t at any channel state $h_i \in \mathbb{R}_+$. Then the AP measures the gap between desired and current control performance of each system during this period and updates the values $\nu(t)$ to $\nu(t+1)$ to prepare for the next period (Step 7). To perform this update the AP needs to compute[4] the average transmission and packet success rates for each system during this period (Step 5) and keep track of some auxiliary variables (Step 6).

This algorithm is guaranteed to converge to the optimal sensor access policies as we state next.

Theorem 5.2 (Sensor access policy optimization)**.** *Consider the setup of Theorem 5.1. The iterations of Algorithm 5.1 with stepsizes in* (5.14)-(5.16) *satisfying* $\sum_{t\geq 0}\varepsilon(t)^2 < \infty$, $\sum_{t\geq 0}\varepsilon(t) = \infty$ *converge to the optimal sensor access policies, i.e.,*

$$c_i \leq \lim_{t\to\infty} \mathbb{E}_{h_i}[\alpha_i(h_i;t)q(h_i)] \prod_{j\neq i}\Big[1-\mathbb{E}_{h_j}[\alpha_j(h_j;t)]q_{ji}\Big], \tag{5.17}$$

for all $i = 1,\dots,m$, *and*

$$\lim_{t\to\infty}\sum_{i=1}^{m}\mathbb{E}_{h_i}[\alpha_i(h_i;t)]p_i = \sum_{i=1}^{m}\mathbb{E}_{h_i}[\alpha_i^*(h_i)]p_i. \tag{5.18}$$

[4]Here we assume that even when collisions arise the AP can identify which sensor transmits at each time slot. Hence it can measure the average rate $\mathbb{E}_{h_i}[\alpha_i(h_i;t)]$ at which each sensor i accesses the channel, as well as the term $\mathbb{E}_{h_i}[\alpha_i(h_i;t)\,q(h_i)]$ which is the packet success ratio when only sensor i transmits.

Algorithm 5.1 Distributed random access computation

Initialize $\lambda(0) \in \mathbb{R}_+^m$, $\nu(0) \in \mathbb{R}_+^{m \times m}$ at the AP
loop At period $t = 0, 1, \ldots$
  AP sends $\nu_{ii}(t)$, $\sum_{j \neq i} \nu_{ji}(t)\, q_{ij}$ to each sensor i.
  During the period each sensor i accesses the channel according to policy

$$\alpha_i(h_i; t) \leftarrow \begin{cases} 1 & \text{if } \nu_{ii}(t)\, q(h_i) \geq p_i + \sum_{j \neq i} \nu_{ji}(t)\, q_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{5.11}$$

  AP measures $\mathbb{E}_{h_i}[\alpha_i(h_i; t)\, q(h_i)]$, $\mathbb{E}_{h_i}[\alpha_i(h_i; t)]$ for all sensors $i = 1, \ldots, m$ during the period.
  AP computes the auxiliary variables

$$\beta_{ii}(t) \quad \leftarrow \quad \left[\frac{\lambda_i(t)}{\nu_{ii}(t)}\right]_{\mathcal{B}} \tag{5.12}$$

$$\beta_{ji}(t) \quad \leftarrow \quad \left[1 - \frac{\lambda_i(t)}{\nu_{ij}(t)}\right]_{\mathcal{B}} \tag{5.13}$$

  for all $i \neq j \in \{1, \ldots, m\}$, where $[\ ]_{\mathcal{B}}$ denotes the projection to the set defined in (5.24).
  AP updates the new dual variables

$$\nu_{ii}(t+1) \leftarrow \Big[\nu_{ii}(t) + \varepsilon(t)\, \big(\beta_{ii}(t) - \mathbb{E}_{h_i}[\alpha_i(h_i; t) q(h_i)]\big)\Big]_+ \tag{5.14}$$

$$\nu_{ij}(t+1) \leftarrow \Big[\nu_{ij}(t) + \varepsilon(t)\, \big(\, \mathbb{E}_{h_j}[\alpha_j(h_j; t)] q_{ji} - \beta_{ji}(t)\,\big)\Big]_+ \tag{5.15}$$

$$\lambda_i(t+1) \leftarrow \Big[\lambda_i(t) + \varepsilon(t)\, \big(\, \log(c_i) - \log(\beta_{ii}(t)) - \sum_{j \neq i} \log(1 - \beta_{ji}(t))\,\big)\Big]_+ \tag{5.16}$$

  or all $i \neq j \in \{1, \ldots, m\}$, where $[\ ]_+$ denotes the projection to the non-negatives $\mathbb{R}_+$.
end loop

The caveat of this distributed implementation is that it requires information exchange between sensors and the access point, hence it introduces some communication overhead. This overhead however burdens mainly the access point which is typically a base station with more capabilities compared to the simpler wireless sensors.

5.3. Numerical simulations

We present a numerical example of the random access design. We consider a case with $m = 2$ scalar control systems of the form (5.1). We assume the first system has open and closed loop dynamics given by $A_{o,1} = 1.1$, $A_{c,1} = 0.5$ respectively, i.e., it is open loop unstable. We assume the second system has integrator open loop dynamics $A_{o,i} = 1$ and stable closed loop dynamics $A_{c,2} = 0.4$. Both systems are perturbed by zero-mean unit-variance Gaussian noises, hence both

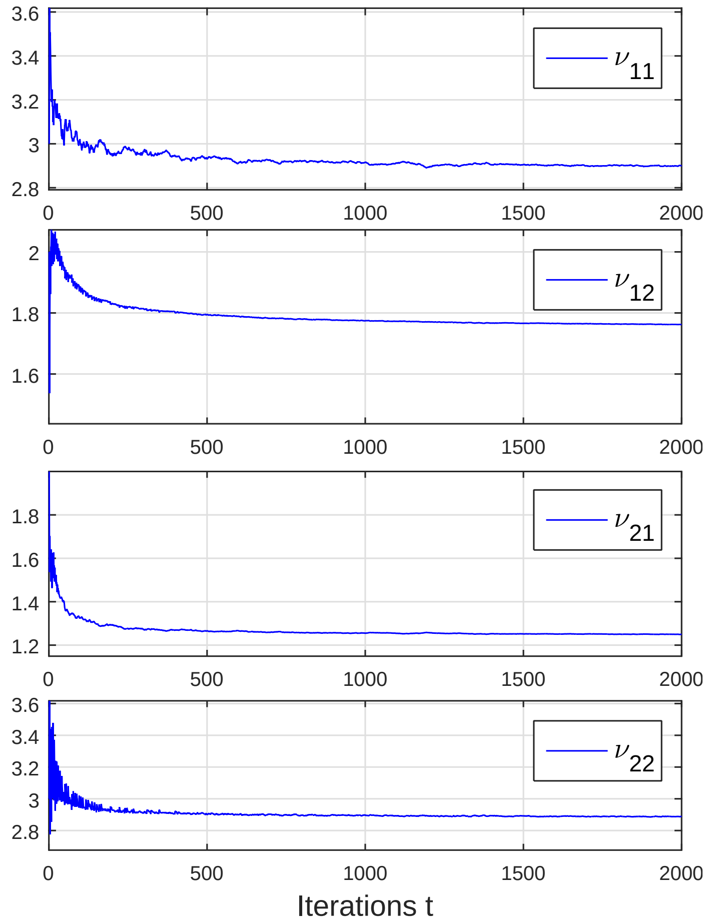

Figure 16: Evolution of dual variables during the optimization algorithm. The elements of the matrix $\nu(t)$ converge to the optimal values ν^* required to obtain the optimal sensor access policies.

system states will diverge unless the closed loops are applied appropriately. The systems are asymmetric, but we model a symmetric control performance requirement. The Lyapunov function $V_i(x_i) = x_i^2$ $(P_i = 1)$ for both plants $i = 1, 2$ is required to decrease with expected rate $\rho_1 = \rho_2 = 0.8$ (cf. (5.5)). By Proposition 4.1 these control performance requirements are equivalent to required packet success rates $c_1 \approx 0.43$, $c_2 \approx 0.27$ for the two sensors, computed by (4.45). Hence System 1, which is more unstable, requires a higher packet success rate.

We assume that both channel states $h_{1,k}, h_{2,k}$ are i.i.d. exponential with mean 1. In isolation each sensor faces a packet success probability modeled by the function $q(h_{i,k})$, $i = 1, 2$ shown in Fig. 2. Also when both sensors transmit at the same time, collisions occur with probability $q_{12} = q_{21} = 0.5$. The transmit powers are taken equal $p_1 = p_2 = 1$.

We solve the random access design problem (5.7)-(5.8) by implementing Algorithm 5.1, which as explained in the previous section solves the problem in the dual domain. We note that at each iteration of the algorithm some expectations with respect to the channel state distributions need

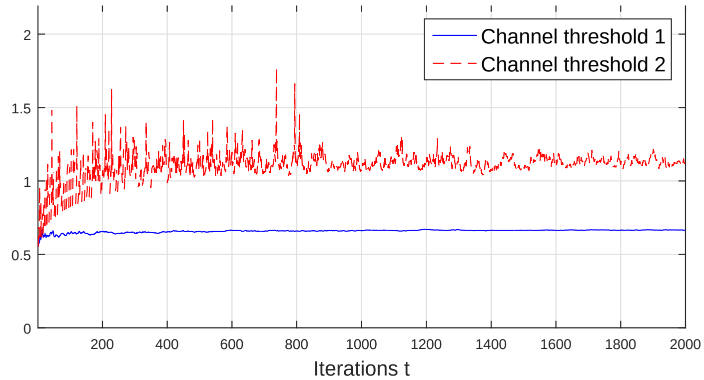

Figure 17: Channel thresholds corresponding to the access policies selected by the optimization algorithm. The channel thresholds for both sensors converge to their optimal values in the limit. Sensor 1 has a lower threshold, i.e., transmits more often, since it is required to guarantee control performance for a more demanding (unstable) plant.

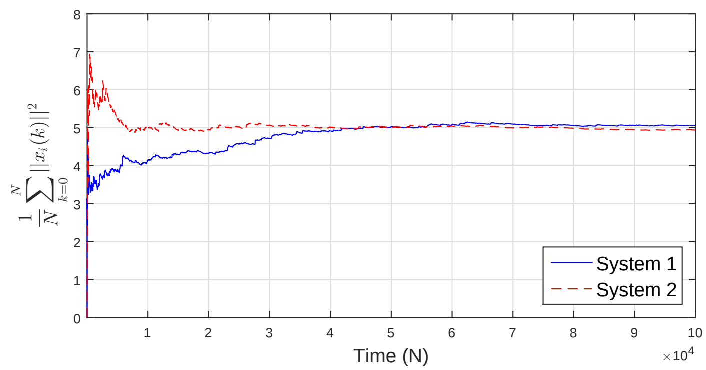

Figure 18: Evolution of control systems using the optimal random access policies. Both systems remain stable despite collisions and packet drops. Also their long run average quadratic cost converges to the same value, since be design both systems were required to have the same control performance.

to be computed, in particular in steps (5.14) and (5.15) of Algorithm 5.1. In our simulations we approximate these expectations with averages from a large number of samples, since samples from the exponential channel distributions can be readily simulated. The iterates of the matrix dual variables $\nu(t)$ during the simulation are shown in Fig. 16 where we observe that they converge to the optimal values ν^*, as was also shown in the proof of Theorem 5.2. We also plot the evolution of the sensor access policies $\alpha_i(h_i;t)$, or equivalently the thresholds of these policies during the simulation of the algorithm in Fig. 17. As also established in Theorem 5.2 the channel thresholds converge to their optimal values in the limit. We observe that the Sensor 1 has a lower threshold, meaning that it transmits more often, which is natural since it corresponds to the unstable plant.

After the optimal access policies (equivalently channel thresholds) have been found, we simulate the random access architecture with the obtained. In Fig. 18 we plot the empirical average long term quadratic cost of the systems $1/N\sum_{k=1}^{N} x_{i,k}^2$ for each system $i = 1, 2$. We first observe that both systems remain stable despite packet collisions over the shared channel. Moreover, even though the two systems are asymmetric, both long term average costs converge to the same value because we required the same control performance for both systems. More specifically this long term cost equals the value $\mathrm{Tr}(P_i W_i)/(1-\rho_i) = 1/(1-0.8) = 5$ for both systems $i = 1, 2$, as noted in Section 2.1.1. Hence even though the two plants have different dynamics, the obtained channel access policies provide symmetric performance by design. The empirical rates $1/N \sum_{k=1}^{N} \alpha_{i,k}$ at which each sensor transmits equal 0.51 and 0.32 for $i = 1, 2$ respectively. As expected, both sensors access the channel at a rate higher than the respective necessary packet success rate on each link, i.e., $\alpha_i^* > c_i$. This happens because the sensors need to counteract the effect of packet collisions, as well as packet drops due to decoding errors.

5.4. Proofs

5.4.1. Proof of Theorem 5.1

The first part of the proof involves converting problem (5.7)-(5.8) into an equivalent auxiliary optimization problem which has zero duality gap. Then in the second part we use Lagrange duality arguments to show that (5.9) describes an optimal solution for the auxiliary problem.

We begin by a modification to remove the product of the expectations appearing in the constraints (5.8). Taking the logarithm at each side of (5.8) preserves the feasible set of variables by monotonicity. Then the logarithm of the product at the right hand side of (5.8) becomes a sum of logarithms, and we can rewrite the optimal random access design problem equivalently as

$$\begin{aligned} \underset{\alpha \in \mathcal{A}}{\text{minimize}} \quad & \sum_{i=1}^{m} \mathbb{E}_{h_i} \alpha_i(h_i) p_i && (5.19) \\ \text{subject to} \quad & \log(c_i) \leq \log(\mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)]) \\ & \qquad + \sum_{j \neq i} \log(1 - \mathbb{E}_{h_j}[\alpha_j(h_j)] q_{ji}), && (5.20) \\ & i = 1, \dots, m. \end{aligned}$$

Here we make an implicit technical assumption that the terms $\mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)]$ and $\mathbb{E}_{h_i}[\alpha_i(h_i)]\, q_{ij}$ in (5.20), which in general take values in the unit interval $[0, 1]$ as all involved variables belong there too, are bounded away from 0 and 1. Then the logarithms in (5.20) are well-defined and finite. This does not restrict the feasible set of solutions, as intuitively each sensor i can neither choose $\alpha_i(h_i)$ too close to 0 otherwise it cannot meet its packet success requirement in (5.8), nor too close to 1 otherwise it causes significant packet collisions on other sensors.

Next, we replace the term $\mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)]$ in constraint (5.20) by an auxiliary variable β_{ii} for $i = 1, \dots, m$, and the terms $\mathbb{E}_{h_j}[\alpha_j(h_j)] q_{ji}$ in (5.20) by variables β_{ji} for $j \neq i$. Hence we rewrite (5.20) as

$$\log(c_i) \leq \log(\beta_{ii}) + \sum_{j \neq i} \log(1 - \beta_{ji}). \tag{5.21}$$

To force the auxiliary variables to behave like the expectations we introduce additional constraints of the form

$$\beta_{ii} \quad \leq \mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)] \tag{5.22}$$

$$\beta_{ji} \quad \geq \mathbb{E}_{h_j}[\alpha_j(h_j)]\, q_{ji} \tag{5.23}$$

for all $i, j \in \{1, \ldots, m\}$, $j \neq i$. Each of these variables are restricted to a subset

$$\beta_{ij} \in \mathcal{B} = [\beta_{\min}, \beta_{\max}] \tag{5.24}$$

of the unit interval $[0, 1]$. In a matrix form $\beta \in \mathcal{B}^{m\times m}$. These upper and lower bounds guarantee that all logarithms at constraints (5.21) are finite, as we also assumed for constraint (5.20). Overall we formulate the auxiliary optimization problem

$$\begin{aligned}
\underset{\alpha\in\mathcal{A},\, \beta\in\mathcal{B}^{m\times m}}{\text{minimize}} \quad & \sum_{i=1}^{m} \mathbb{E}_{h_i}\alpha_i(h_i) p_i && (5.25)\\
\text{subject to} \quad & \log(c_i) \leq \log(\beta_{ii}) + \sum_{j\neq i} \log(1-\beta_{ji}) && (5.26)\\
& \beta_{ii} \leq \mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)] && (5.27)\\
& \beta_{ji} \geq \mathbb{E}_{h_j}[\alpha_j(h_j)]\, q_{ji} && (5.28)\\
& i, j \in \{1, \ldots, m\},\ j \neq i
\end{aligned}$$

We argue that this auxiliary problem is equivalent to the original one in (5.7)-(5.8), in the sense that a feasible solution of one problem corresponds to a feasible solution with the same objective value for the other problem. Indeed let's start with a feasible solution α for (5.7)-(5.8). Let us define variables β that make (5.27), (5.28) hold with equality for all $i, j \in \{1, \ldots, m\}$, $j \neq i$. Then the pair α, β is also feasible for problem (5.25)-(5.28) and has the same objective. Reversely, consider a feasible pair α, β for problem (5.25)-(5.28). Without loss we can assume that all constraints (5.27)-(5.28) hold with equality. Otherwise if, say, an inequality i in (5.27) is strict, we can increase the value of variable β_{ii} till equality in (5.27) is reached without loss of feasibility in (5.26) and without changing the objective value in (5.25). A similar procedure can be performed if some inequality (5.28) is strict, leading finally to a new feasible point satisfying (5.27)-(5.28) with equalities. Then it is immediate that α is also feasible for (5.8) and has the same objective.

Based on the established equivalence, in the rest of the proof it suffices to show that (5.9) describes an optimal solution for the auxiliary problem (5.25)-(5.28). The advantage of formulating this auxiliary problem is that it has zero duality gap as can be shown by the results in Ribeiro (2012). To formally state this result, let us denote the optimal value of this problem by P^* (finite by feasibility Assumption 4.3) and let us define the Lagrange dual problem. We associate dual variables $\lambda_i \geq 0$ with inequalities (5.26), $\nu_{ii} \geq 0$ with (5.27), and $\nu_{ij} \geq 0$ with (5.28), for $i, j \in \{1, \ldots, m\}$. We write the Lagrangian function as

$$\begin{aligned}
L(\alpha,\beta,\lambda,\nu) = &\sum_{i=1}^{m} \mathbb{E}_{h_i}\alpha_i(h_i)p_i \\
&+ \sum_{i=1}^{m} \lambda_i \Big\{ \log(c_i) - \log(\beta_{ii}) - \sum_{j\neq i} \log(1-\beta_{ji}) \Big\} \\
&+ \sum_{i=1}^{m} \nu_{ii}(\beta_{ii} - \mathbb{E}_{h_i}[\alpha_i(h_i)\, q(h_i)]) \\
&+ \sum_{i=1}^{m} \sum_{j\neq i} \nu_{ij}(\mathbb{E}_{h_j}[\alpha_j(h_j)]\, q_{ji} - \beta_{ji})
\end{aligned} \tag{5.29}$$

Here the dual variables take values $\lambda \in \mathbb{R}_+^m$, $\nu \in \mathbb{R}_+^{m\times m}$. We can rearrange the terms of the Lagrangian in the form

$$\begin{aligned}
L(\alpha,\beta,\lambda,\nu) = \sum_{i=1}^{m} \Big\{ &\mathbb{E}_{h_i}\, \alpha_i(h_i) \Big[p_i + \sum_{j\neq i} \nu_{ji} q_{ij} - \nu_{ii} q(h_i) \Big] \\
&+ \sum_{j\neq i} \left[-\lambda_i \log(1-\beta_{ji}) - \nu_{ij}\beta_{ji} \right] \\
&+ \nu_{ii}\beta_{ii} - \lambda_i \log(\beta_{ii}) + \lambda_i \log(c_i) \quad \Big\}.
\end{aligned} \tag{5.30}$$

This form is useful because each primal variable ($\alpha_i(h_i)$ and β_{ji} for each i, j) is decoupled from the others, a fact we will exploit next. Then we can define the Lagrange dual function

$$g(\lambda,\nu) = \inf_{\alpha\in\mathcal{A},\, \beta\in\mathcal{B}^{m\times m}} L(\alpha,\beta,\lambda,\nu), \tag{5.31}$$

as well as the Lagrange dual problem whose optimal value we denote by D^* as

$$D^* = \inf_{\lambda\in\mathbb{R}_+^m,\, \nu\in\mathbb{R}_+^{m\times m}} g(\lambda,\nu). \tag{5.32}$$

Then we can establish the following zero duality property about the auxiliary problem (5.25)-(5.28).

Proposition 5.1 (Strong Duality)**.** *Let Assumptions 5.1 and 5.2 hold. Then the problem* (5.25)-(5.28) *has zero duality gap, i.e., $P^* = D^*$. Moreover if α^*, β^* are optimal solutions and λ^*, ν^* are optimal solutions for the dual problem* (5.32)*, then*

$$\alpha^*, \beta^* \in \operatorname*{argmin}_{\alpha\in\mathcal{A},\, \beta\in\mathcal{B}^{m\times m}} L(\alpha,\beta,\lambda^*,\nu^*). \tag{5.33}$$

The result follows from (Ribeiro, 2012, Theorems 1 and 4) where general stochastic optimization problems of the form (5.25)-(5.28) are examined under absolute continuity (Assumption 5.1) and strict feasibility (Assumption 5.2). The proof is omitted due to space limitations.

The above characterization suggests that we can recover the optimal variables α^*, β^* of our problem by just minimizing the unconstrained Lagrangian function. A technical caveat of (5.33) is that it describes an inclusion only, implying that in general there might be Lagrangian minimizers that are not optimal. The following lemma excludes such cases by establishing that the Lagrangian minimizers α, which are functions, i.e., infinite-dimensional variables, are unique up to a set of measure zero. Moreover the following lemma gives an explicit expression for these minimizers.

Lemma 5.1. *Consider any dual variables $\lambda \in \mathbb{R}_+^m$, $\nu \in \mathbb{R}_+^{m\times m}$. Then the functions $\alpha \in \mathcal{A}$ that minimize the Lagrangian $L(\alpha, \beta, \lambda, \nu)$ are uniquely defined except for a set of arguments $h \in \mathbb{R}_+^m$ of measure zero, and are given by*

$$\alpha_i(h_i; \lambda, \nu) = \begin{cases} 1 & \text{if } \nu_{ii} q(h_i) \geq p_i + \sum_{j\neq i} \nu_{ji} q_{ij} \\ 0 & \text{otherwise.} \end{cases} \tag{5.34}$$

for each $i = 1, \ldots, m$ and for every value $h_i \in \mathbb{R}_+$.

In (5.34) the term $\alpha_i(h_i; \lambda, \nu)$ denotes the function α_i that minimizes the Lagrangian $L(\alpha, \beta, \lambda, \nu)$ at given dual points λ, ν evaluated at an argument h_i.

To sum up we have shown in (5.33) that the optimal solution $\alpha(.)$ to problem (5.25)-(5.28) belongs in the set of Lagrange minimizers at λ^*, ν^*, and by Lemma 5.1 these minimizers are unique up to a set of measure zero. As a result, all these minimizers will have the same objective and constraint slack in problem (5.25)-(5.28), and they will all be optimal for this problem. In particular, the specific minimizer defined by $\alpha_i(h_i; \lambda^*, \nu^*)$ given in (5.34) will be optimal for the problem, and corresponds exactly to the one given in (5.9) at the statement of the theorem.

5.4.2. Proof of Lemma 5.1

Consider the problem of minimizing the Lagrangian $L(\alpha, \beta, \lambda, \nu)$ over variables $\alpha \in \mathcal{A}, \beta \in \mathcal{B}^{m\times m}$. Due to the separability of the Lagrangian given in the form (5.30) over variables α, β, we can separate the problem into subproblems

$$\underset{\beta_{ji}\in\mathcal{B}}{\operatorname{argmin}} \quad -\lambda_i \log(1-\beta_{ji}) - \nu_{ij}\beta_{ji} \tag{5.35}$$

$$\underset{\beta_{ii}\in\mathcal{B}}{\operatorname{argmin}} \quad \nu_{ii}\beta_{ii} - \lambda_i \log(\beta_{ii}) \tag{5.36}$$

$$\underset{\alpha_i\in\mathcal{A}_i}{\operatorname{argmin}} \quad \mathbb{E}_{h_i}\, \alpha_i(h_i) \left[p_i + \sum_{j\neq i} \nu_{ji} q_{ij} - \nu_{ii} q(h_i)\right] \tag{5.37}$$

for $i, j \in \{1, \ldots, m\}, i \neq j$. Next we need to verify that (5.34) is optimal for (5.37). Note that without loss of generality we can exchange the expectation operator $\mathbb{E}_{h_i}$ and the minimization over $\alpha_i \in \mathcal{A}_i$, which is a function $\alpha_i : \mathbb{R}_+ \to [0,1]$ defined for any channel value $h_i \in \mathbb{R}_+$, to equivalently solve

$$\underset{\alpha_i(h_i)\in[0,1]}{\text{argmin}} \ \alpha_i(h_i) \Big[p_i + \sum_{j\neq i} \nu_{ji}^* q_{ij} - \nu_{ii}^* q(h_i)\Big]. \tag{5.38}$$

pointwise at all values $h_i \in \mathbb{R}_+$. This is valid because any function α_i that minimizes (5.37) can differ form the minimizer in (5.38) at a set of values $h_i \in \mathbb{R}_+$ with measure at most zero.

Then we can verify that (5.34) is the minimizer in (5.38). That is because the right hand side in (5.38) is a linear expression of $\alpha_i(h_i) \in [0,1]$. Hence the minimizer $\alpha_i(h_i)$ is uniquely defined, and takes values either 0 or 1 except for the values h_i where $p_i + \sum_{j\neq i} \nu_{ji}^* q_{ij} - \nu_{ii}^* q(h_i) = 0$. In the latter case the minimizer is not uniquely defined. However due to the strict monotonicity assumption for $q(h_i)$ this case occurs for at most one value h_i, hence it is a measure zero event due to the absolute continuity of the measure ϕ_i by Assumption 5.1. This completes the proof.

We also note for future reference the terms β that minimize the Lagrangian. Since (5.35), (5.36) are strongly convex, their minimizers are unique and satisfy the first order conditions $\partial L/\partial\beta = 0$, that is

$$\nu_{ii} - \frac{\lambda_i}{\beta_{ii}} = 0 \tag{5.39}$$

$$\frac{\lambda_i}{1-\beta_{ji}} - \nu_{ij} = 0. \tag{5.40}$$

respectively subject to the box constraints $\beta_{ji} \in \mathcal{B}$ for all $i, j \in \{1, \ldots, m\}$. As a result the optimal solutions are given by

$$\beta_{ii}(\lambda,\nu) = \left[\frac{\lambda_i}{\nu_{ii}}\right]_{\mathcal{B}} \tag{5.41}$$

$$\beta_{ji}(\lambda,\nu) = \left[1 - \frac{\lambda_i}{\nu_{ij}}\right]_{\mathcal{B}} \tag{5.42}$$

for all $i \neq j \in \{1, \ldots, m\}$, where $[\ ]_{\mathcal{B}}$ denotes the projection to the set defined in (5.24).

5.4.3. Proof of Theorem 5.2

A sufficient condition for (5.17) and (5.18) is that

$$\lim_{t\to\infty} \quad \mathbb{E}_{h_i}[\alpha_i(h_i;t)] = \mathbb{E}_{h_i}[\alpha_i^*(h_i)] \tag{5.43}$$

$$\lim_{t\to\infty} \quad \mathbb{E}_{h_i}[\alpha_i(h_i;t)q(h_i)] = \mathbb{E}_{h_i}[\alpha_i^*(h_i)q(h_i)] \tag{5.44}$$

hold for all $i = 1, \ldots, m$. Indeed this immediately implies (5.18), while (5.17) is also implied since the optimal policy α^* for problem (5.7)-(5.8) needs to be feasible. In the proof of Theorem 5.1 we argued that problem (5.7)-(5.8) is equivalent to the auxiliary problem (5.25)-(5.28) with variables $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}^{m\times m}$. Hence it suffices to show that the algorithm converges to the optimal solution of this auxiliary problem in the sense of (5.43)-(5.43).

Recall that after introducing dual variables $\lambda \in \mathbb{R}^m_+$, $\nu \in \mathbb{R}^{m\times m}_+$, the Lagrange dual function $g(\lambda, \nu)$ of the auxiliary problem (5.25)-(5.28) is defined in (4.17). We begin by arguing that that at each iteration of the algorithm, the dual variables $\lambda(t)$, $\nu(t)$ according to (5.14)-(5.16) move towards a subgradient direction of the dual function. For convenience let us denote the direction of the steps at (5.14)-(5.15) by the matrix $s_\nu(t) \in \mathbb{R}^{m\times m}$ defined as

$$s_{\nu,ii}(t) \quad = \quad \beta_{ii}(t) - \mathbb{E}_{h_i}[\alpha_i(h_i;t)\, q(h_i)] \tag{5.45}$$

$$s_{\nu,ij}(t) \quad = \quad \mathbb{E}_{h_j}[\alpha_j(h_j;t)]q_{ji} - \beta_{ji}(t) \tag{5.46}$$

for all $i \neq j \in \{1, \dots, m\}$, and the steps at (5.16) by the vector $s_\lambda(t) \in \mathbb{R}^m$ defined as

$$s_{\lambda,i}(t) = \log(c_i) - \log(\beta_{ii}(t)) - \sum_{j\neq i} \log(1 - \beta_{ji}(t)) \tag{5.47}$$

for all $i \in \{1, \dots, m\}$. We argue that $s_\nu(t), s_\lambda(t)$ are subgradient directions for the dual function at the point $\lambda(t), \nu(t)$, i.e., that

$$g(\lambda', \nu') - g(\lambda(t), \nu(t)) \leq \qquad (\,\lambda' - \lambda(t)\,)^T s_\lambda(t) \\ + \mathrm{Tr}(\,(\nu' - \nu(t))\, s_\nu(t)\,) \tag{5.48}$$

for all $\lambda' \in \mathbb{R}^m_+$, $\nu' \in \mathbb{R}^{m\times m}_+$. This can be shown as follows.

Consider an iteration of Algorithm 5.1. The variable $\alpha(t)$ selected by the algorithm at step (5.11) is a variable that minimizes the Lagrangian $L(\alpha, \beta, \lambda(t), \nu(t))$ with respect to the variable $\alpha \in \mathcal{A}$. This follows directly from Lemma 5.1. Similarly the variables $\beta(t)$ at step (5.13) minimize the Lagrangian function $L(\alpha, \beta, \lambda(t), \nu(t)$ with respect to the variable $\beta \in \mathcal{B}^{m\times m}$. This fact is included in the proof of Lemma 5.1 at (5.41)-(5.42). As a result by the definition of the dual function in (4.17) it follows that $g(\lambda(t), \nu(t)) = L(\alpha(t), \beta(t), \lambda(t), \nu(t))$. Additionally we can substitute the Lagrangian at the right hand side with the form given at (4.16) to get

$$g(\lambda(t), \nu(t)) = \sum_{i=1}^{m}\, \mathbb{E}_{h_i}\alpha_i(h_i;t)p_i + \lambda(t)^T s_\lambda(t) \\ + \mathrm{Tr}(\nu(t)s_\nu(t)). \tag{5.49}$$

Here for convenience we replaced the lengthy parentheses of (4.16) by the equivalent terms $s_\lambda(t)$, $s_\nu(t)$ defined in (5.46), (5.47), (5.45).

Next note that at any point λ', ν' the dual function $g(\lambda', \nu')$ is by definition (4.17) the minimum of the Lagrangian $L(\alpha, \beta, \lambda', \nu')$, hence we must have that $g(\lambda', \nu') \leq L(\alpha(t), \beta(t), \lambda', \nu')$. Using again the notation $s_\lambda(t), s_\nu(t)$ at the right hand side we get that

$$g(\lambda', \nu') \leq \sum_{i=1}^{m} \mathbb{E}_{h_i} \alpha_i(h_i; t) p_i + \lambda'^T s_\lambda(t) + \mathrm{Tr}(\nu' s_\nu(t)), \tag{5.50}$$

Subtracting (5.49) from (5.50) by sides yields (5.48).

To sum up, at each iteration of the algorithm the dual variables $\lambda(t)$, $\nu(t)$ move towards a subgradient direction of the dual function. Additionally the subgradients are bounded. That is true for $s_\nu(t)$ because all the terms at the right hand side of (5.45)-(5.46) are between 0 and 1. It is also true for $s_\lambda(t)$ because the logarithms at the right hand side of (5.47) are finite by the restriction $\beta(t) \in \mathcal{B}^{m \times m}$ defined in (5.24). Under the bounded subgradient condition, convergence of $\lambda(t)$, $\nu(t)$ to the optimal dual variables λ^*, ν^* for stepsizes as in the statement of the theorem relies on standard subgradient method arguments – see, e.g., (Bertsekas et al., 2003, Prop. 8.2.6) for a proof.

In the rest of the proof, based on the established convergence of the dual variables to the optimal ones, we will show that the same holds for the primal variable $\alpha(.;t)$ in the sense of (5.43)-(5.43). Note that at any iteration t the function $\alpha_i(.;t)$ takes the value 1 when $\nu_{ii}(t)\, q(h_i) \geq p_i + \sum_{j \neq i} \nu_{ji}(t) q_{ij}$ and 0 otherwise. Due to the strict monotonicity of the function $q(.)$ this is a threshold-like function taking the value 1 when $h_i \geq \bar{h}_i(t) = q^{-1}(p_i + \sum_{j \neq i} \nu_{ji}(t) q_{ij}) / \nu_{ii}(t))$. Since we have established that $\nu(t) \to \nu^*$, and since the function $q(.)$ is continuous hence its inverse too, we conclude that the threshold $\bar{h}_i(t)$ converges to $\bar{h}_i^* = q^{-1}(p_i + \sum_{j \neq i} \nu_{ji}^* q_{ij}) / \nu_{ii}^*)$. By Theorem 5.1 this limit value equals the threshold of the optimal access policy, which is also of a threshold form.

Hence we conclude that $\alpha_i(.;t) \to \alpha_i^*(.)$ pointwise for all $h_i \in \mathbb{R}_+$ except perhaps for the point h_i^*, i.e., the optimal threshold point. By absolute continuity of the probability measure ϕ_i the point h_i^* has a probability measure zero. Hence $\alpha(.;t) \to \alpha^*(.)$ almost everywhere. Also both sequences of functions $\alpha_i(.;t)$ and $\alpha_i(.;t)q(.)$ are uniformly bounded in $[0,1]$. By the Bounded Convergence Theorem (Billingsley, 1995, Theorem 16.5) we conclude that convergence in expectation, i.e., (5.43) and (5.43), also holds. This completes the proof.

Chapter 6: Adaptation to Plant States

The allocation of communication resources in wireless control architectures needs to account for the dynamic evolution of the control system over time. For example, in the problem of power management examined in Chapter 3 it is shown that the optimal transmit power for a sensor needs to be dynamically and opportunistically adapted to the control system state and wireless channel state online. Formally it is shown that such communication design problems can be seen as Markov Decision Process (MDP) problems (Bertsekas (2005)), where the state is composed of the plant state x_k and the channel state h_k, while the decided actions are the resource allocation variables at each time step, e.g. the transmit power p_k. The difficulty in solving these MDP instances lies on the fact that the state space takes a continuum of values rendering standard dynamic programming algorithms, e.g., value or policy iteration (cf. Bertsekas (2005)), computationally hard. In Chapters 4, 5 this difficulty is bypassed using a different control performance abstraction (Lyapunov functions), however the developed solutions excluded the adaptation to physical plant states.

In this chapter we alleviate the technical difficulties of the optimal designs by proposing suboptimal yet tractable resource allocation algorithms that benefit from online adaptation to both control system states and channel conditions. The proposed policies have their basis in the approximate dynamic programming notion of rollout algorithms. More importantly, despite their suboptimality, the proposed algorithms provide control performance and resource utilization guarantees. These guarantees are enforced by design, as the proposed algorithms are designed in order to improve upon simple non-adaptive algorithms that are used as a reference. As we see in numerical simulations the proposed policies achieve in practice significant gains in performance.

We begin by introducing the proposed algorithms in the problem of transmit power allocation for control systems, and then in the scheduling problem. Finally, we propose suboptimal algorithms for the more complicated problem of random access design for control systems.

6.1. Rollout Power Allocation Policies

We consider again the problem of a sensor communicating over a wireless fading channel to a controller in order to control a plant. The control system is described by a switched linear time invariant model of the form (cf. Section 2.1)

$$x_{k+1} = \begin{cases} A_c\, x_k + w_k, & \text{if } \gamma_k = 1 \\ A_o\, x_k + w_k, & \text{if } \gamma_k = 0 \end{cases} \tag{6.1}$$

depending on whether a successful transmission occurs at time k or not, indicated with variables $\gamma_k \in \{0,1\}$. Here $x_k \in \mathbb{R}^n$ denotes the state of the overall control system at time k, and $A_c, A_o \in \mathbb{R}^{n\times n}$ are the stable closed loop and unstable open loop dynamics respectively. The plant disturbance $w_k, k \geq 0$ in (6.1) is an i.i.d. process with some given probability distribution ϕ_w with mean zero and positive definite covariance W.

The wireless channel has an i.i.d. fading state $h_k \in \mathbb{R}_+, k \geq 0$ with some probability distribution ϕ_h (cf. Section 2.2). The sensor selects a transmit power level $p_k \in [0, p_{\max}]$ at each time step, and the probability of successfully receiving the message at the controller and closing the loop is given by the relationship

$$\mathbb{P}(\gamma_k = 1) = q(h_k, p_k) \tag{6.2}$$

as described in Section 2.2.

We are interested in maintaining good control performance over a long time horizon. In particular we consider quadratic costs on the system state given by

$$\begin{cases} x_k^T\, Q_c\, x_k, & \text{if } \gamma_k = 1 \\ x_k^T\, Q_o\, x_k, & \text{if } \gamma_k = 0 \end{cases} . \tag{6.3}$$

These costs depend on the success of the transmission at each time step, as introduced in Section 2.1.2. Both matrices Q_c, Q_o are assumed to be positive semidefinite.

Our goal is to design transmit power policies, that is, transmission decisions $p_k, k \geq 0$ adapted to the available plant and channel information x_k, h_k at the sensor at each time step k. The objective is to minimize the long time average control performance described as

$$J_{\text{control}}(p_0, p_1, \ldots) = \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}[x_k^T\, (\gamma_k Q_c + (1-\gamma_k) Q_o)\, x_k]. \tag{6.4}$$

As the transmit power affects the mode of operation (open or closed) by (6.2), the evolution of the system in (6.1) and the cost in (6.3) lead to a different average expected cost in (6.4). Besides, the transmit power selection should make an efficient use of the available power resources accounted by

$$J_{\text{comm}}(p_0, p_1, \ldots) = \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E} p_k. \tag{6.5}$$

Our approach to designing control-aware and channel-aware power allocation policies begins by considering simpler policies that will be used as a reference. The proposed policies will then emerge as improvements upon the reference ones.

We consider a family of policies adapting transmit power only to the channel state h_k and not to the plant state x_k. Such a policy can be described as $p_k = \tilde{\mathbf{p}}(h_k)$ for some measurable function $\tilde{\mathbf{p}} : \mathbb{R}_+ \to [0, p_{\max}]$ mapping channel states to power values[5]. We will call such a policy a reference policy. Since channel states are independent of the plant states x_k, as well as independent and identically distributed over time, the policy $\tilde{\mathbf{p}}$ results in a successful packet decoding (cf. (6.2)) with a constant probability at each time step k denoted here by $\tilde{q}$ defined as

$$\tilde{q} := \mathbb{P}(\gamma_k = 1) = \mathbb{E}_h[q(h, \tilde{\mathbf{p}}(h))]. \tag{6.6}$$

The expectation in this expression is with respect to the channel distribution ϕ_h. Moreover this reference policy has an average power consumption given by

$$J_{\text{comm}}(\tilde{\mathbf{p}}) = \mathbb{E}_h[\tilde{\mathbf{p}}(h)] \tag{6.7}$$

where we used again the fact that the channel states are i.i.d. over time.

Under the considered reference policy for the system (6.1) the closed loop mode A_c is applied with constant probability $\tilde{q}$ in (6.6) while the open loop mode A_o is applied with the complement probability. The stability of the system under such a reference policy can be characterized using well established results in the context of random jump linear systems. In particular, a direct application of (Costa and Fragoso, 1993, Theorem 1) verifies the following stability result.

Proposition 6.1 (Reference policy stability)**.** *Consider the wireless control system (6.1) with communication modeled as (6.2). Consider a reference policy of the form* $p_k = \tilde{\mathbf{p}}(h_k), k \geq 0$*, leading to a packet success probability* $\tilde{q}$ *given in (6.6). Then the system is mean square stable, i.e.,* $\limsup_{k\to\infty} \mathbb{E} x_k x_k^T < \infty$*, if and only if*

$$\max_{i=1,\ldots,n^2} |\lambda_i\{\tilde{q}\, A_c^T \otimes A_c^T + (1-\tilde{q})\, A_o^T \otimes A_o^T\}| < 1 \tag{6.8}$$

where $\otimes$ *denotes the Kronecker product.*

After having established the stability properties of the reference policy, the following proposition characterizes the quadratic control performance of such a reference policy, denoted by $J_{\text{control}}(\tilde{\mathbf{p}})$.

Proposition 6.2 (Reference Policy Performance)**.** *Consider the wireless control system (6.1) with communication modeled as (6.2) and control costs described by (6.3). Consider a reference policy of the form* $p_k = \tilde{\mathbf{p}}(h_k), k \geq 0$*, leading to mean square stability with a packet success probability* $\tilde{q}$ *given in (6.6). Then there exists a positive semidefinite matrix* $P \in \mathbb{S}^n_+$ *satisfying the linear matrix equality*

$$P = \tilde{q}\,(Q_c + A_c^T P A_c) + (1-\tilde{q})(Q_o + A_o^T P A_o), \tag{6.9}$$

[5]Throughout the chapter we use the boldface notation to denote the functions/policies we are interested in designing.

and the average control cost (6.4) equals

$$J_{control}(\tilde{\mathbf{p}}) = Tr(P\,W). \tag{6.10}$$

The interpretation of the matrix P in this proposition is as follows. The quadratic function $x_k^T P x_k$ models the future expected control cost if the sensor follows the reference policy starting from system state x_k at time k, i.e., the cost-to-go function of this policy (Bertsekas (2005)). This characterization of the reference policy is important as it leads to an easily computable cost-to-go function by solving (6.9) – see Remark 6.2.

Given a reference policy and the associated cost-to-go function, we will develop policies that adapt to plant and channel states online while they improve upon the control and communication performance of the reference policies. To prepare for the proposed policies we define the matrix

$$M := Q_o + A_o^T P A_o - Q_c - A_c^T P A_c. \tag{6.11}$$

We interpret the quadratic form $x_k^T M x_k$ as the *relative value of transmission* at the current control system state x_k. Indeed if the sensor successfully transmits ($\gamma_k = 1$), the current control cost is $x_k Q_c x_k$ and the system evolves according to A_c meaning that the future expected costs becomes $x_k^T A_c^T P A_c x_k$ assuming the reference model for the future. Alternatively if no transmission occurs ($\gamma_k = 1$), the current and future control cost becomes $x_k(Q_o + A_o^T P A_o)x_k$. Hence the relative gains if the sensor transmits as compared to not transmitting involve the matrix difference in (6.11).

6.1.2. Rollout policy

We propose then the following power allocation policy $\mathbf{p}_\nu^{\text{roll}} : \mathbb{R}^n \times \mathbb{R}_+ \to [0, p_{\max}]$ mapping the current plant and channel conditions x_k, h_k to a transmit power p_k. In particular let the sensor choose the transmit power according to the following minimization

$$p_k = \mathbf{p}_\nu^{\text{roll}}(x_k, h_k) := \underset{p \in [0, p_{\max}]}{\operatorname{argmin}} \quad \nu p - q(h_k, p) x_k^T M x_k. \tag{6.12}$$

Here $\nu \geq 0$ is a non-negative constant. The expression at the right hand side of (6.12) represents a tradeoff between the allocated power (measured by νp) and the expected gains in control performance (given by $x_k^T M x_k$) if the transmission is successful (which happens with probability $q(h_k, p)$).

We call the proposed policy in (6.12) a *rollout* policy. We note that a rollout policy depends on the selected reference policy $\tilde{\mathbf{p}}$ via the matrix M appearing in (6.12). One may think of the reference policy as a knob or a tunable parameter that gives different rollout policies. We will not actively design or tune here the reference policy, but will assume one is given to us. However we note that the approaches in Chapters 4, 5 may be useful to come up with good channel-aware reference policies.

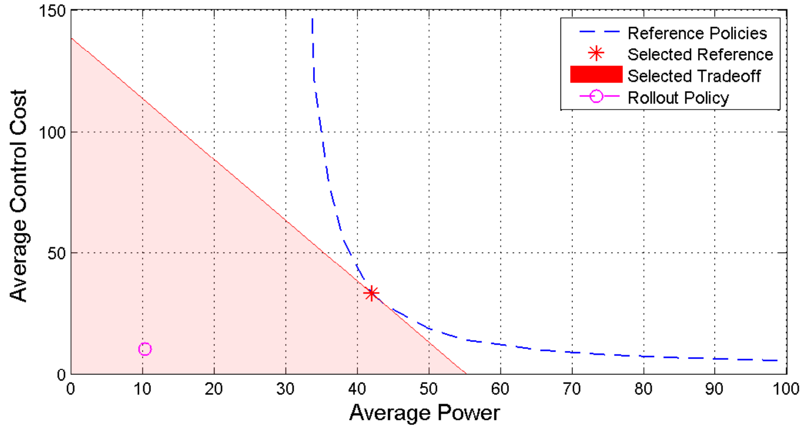

Figure 19: The control and communication performance of a selected reference policy is depicted, along with the selected tradeoff line $\nu \geq 0$ between the two objectives. By Theorem 6.1 the control and communication performance resulting from the rollout policy is guaranteed to lie left of the selected tradeoff (shaded area). The actual rollout performance for this example is also depicted. The set of reference policies with constant power allocation $\tilde{p}(.)$ are also depicted for comparison.

The following theorem is the main result of this section and characterizes the performance of the rollout policy.

Theorem 6.1 (Rollout Power Policy Performance)**.** *Consider the system described by (6.1) with communication modeled as (6.2) and control costs described by (6.3). Let $\tilde{\mathbf{p}}$ be a given reference policy leading to mean square stability, associated with a matrix M defined in (6.11). For any non-negative constant $\nu \geq 0$, suppose the sensor follows the rollout policy $p_k = \mathbf{p}_\nu^{roll}(x_k, h_k)$ given in (6.12). Then the average control and communication costs of this policy satisfy*

$$J_{control}(\mathbf{p}_\nu^{roll}) + \nu\, J_{comm}(\mathbf{p}_\nu^{roll}) \leq J_{control}(\tilde{\mathbf{p}}) + \nu\, J_{comm}(\tilde{\mathbf{p}}). \tag{6.13}$$

This theorem provides a guarantee on the joint control performance and power consumption of the proposed rollout policy. In particular it states that the rollout policy performs better that the reference policy with respect to a linear combination of the two objectives. It is worth noting that even though an optimal power policy might be hard to find, the proposed suboptimal policy is given explicitly in (6.12). By tuning the weight $\nu \geq 0$, i.e., by a simple modification of the proposed policy in (6.12), one can get a more improved control performance or power utilization. A geometric interpretation is given in Fig. 19. As we will also see in numerical simulations this results in significant performance improvements in practice.

An example is presented next.

Example 6.1. *Consider the setup of Chapter 3. In that case a specific controller is employed in order to control a linear plant (cf. Theorem 3.1). This leads to a problem where the sensors tries to regulate the* *estimation error ε_k evolving according to (3.17). This is special case of the general switched linear system (6.1). For this special case the dynamics are given by $A_o = A, A_c = 0$, and the estimation cost is given by $Q_c = 0, Q_o = \tilde{P}$ (cf.(3.27)). For a channel-adapted reference policy of the form $\tilde{\mathbf{p}}$, with an average packet success $\tilde{q}$ given by (6.6), one first solves for the reference matrix P in (6.9), which now becomes a Lyapunov equation of the form*

$$P = (1 - \tilde{q})\,(APA^T + \tilde{P}). \tag{6.14}$$

The resulting rollout power allocation policy (6.12) as a function of the system state, which here is the estimation error ε_k, and the channel state h_k becomes

$$\mathbf{p}_\nu^{roll}(\varepsilon_k, h_k) := \underset{p\in[0,p_{\max}]}{argmin} \quad \nu p - q(h_k, p)\varepsilon_k^T(APA^T + \tilde{P})\varepsilon_k. \tag{6.15}$$

Observe that (6.15) is of the same form as the optimal communication policy (3.32) except that the optimal unknown function $R(\varepsilon_k)$ in the latter is replaced by an easily computed quadratic form. Since the rollout policy is suboptimal the quadratic can be viewed as an approximation of the function $R(\varepsilon_k)$.

For the particular case of capacity achieving codes repeating the analysis of Section 3.4.1 we can modify (6.15) to obtain the suboptimal policy

$$\mathbf{p}_\nu^{roll,CA}(\varepsilon_k, h_k) := \begin{cases} 0 & \text{if } \; h_k\,\varepsilon_k^T(APA^T + \tilde{P})\varepsilon_k \le \nu\, p_0 \; \text{ or } \; h_k \le \frac{p_0}{p_{max}} \\ \frac{p_0}{h_k} & \text{otherwise} \end{cases}. \tag{6.16}$$

Again the unknown function $R(\varepsilon_k)$ in (3.36) is approximated by a quadratic that can be easily computed. This is an explicit event-triggered communication policy, where the events depend on the current channel state h_k and error ε_k. For more details see Gatsis et al. (2014c).

After the following remarks, we extend the rollout policy development to the case of state-aware scheduling of multiple control systems over a shared wireless medium.

Remark 6.1 (Rollout Algorithm Interpretation)**.** *The motivation behind rollout policies is as follows. In a Markov Decision Process there is a state space, e.g., composed of plant and channel states x, h in our case, and an action space, e.g., power p in our case, which affects the evolution of the states. The goal is to find an action policy that minimizes a long term cost, where at each time step a function $c(x, h, p)$ penalizes current states and actions. The optimal action at each step arises by solving a problem of the form*

$$\underset{p\in[0,p_{\max}]}{\text{argmin}} \quad c(x,h,q) + \mathbb{E}\left[V^*(x_+, h_+) \,\middle|\, x, h, q\right]. \tag{6.17}$$

for some function V^* called the optimal cost-to-go function, and where the expectation is over the state evolution. In cases where the optimal cost-to-go function V^* is not easily computed, rollout policies arise as suboptimal solutions (Bertsekas, 2005, Vol. I). One needs to find a reference policy (or a class of reference policies) for which the cost-to-go function, which is suboptimal in general, is easily computed. Say the policy *is π and the corresponding function is V^{π}. Then the rollout policy is given by solving*

$$\underset{p\in[0,p_{\max}]}{\operatorname{argmin}} \; c(x,h,q) + \mathbb{E}\left[V^{\pi}(x_+,h_+) \,\middle|\, x,h,q\right]. \tag{6.18}$$

in lieu of the optimal policy (6.17). In other words, the rollout policy finds the optimal current action assuming that the reference policy will be used from the next time step and onwards. At the next time step a rollout action is selected as well, and so on in a receding horizon fashion. The specialization of (6.18) in our setting yields (6.12).

The rollout policy (6.18) is a heuristic. Intuitively if the reference policy π is close to the optimal policy then the rollout is close to the optimal as well. Even though it is not easy to characterize how worse the rollout performs compared to the optimal policy, in practice it usually performs well as noted by (Bertsekas, 2005, Vol. I).

Rollout sensor transmit policies have also been used by Antunes and Heemels (2014) however without account of power resources but based on deterministic periodic transmission schedules as references. As a result the transmission decisions in that work are updated once every period, unlike our policies in (6.12) which allow the sensor to continuously exploit online at each time step the stochastic plant state and channel state processes.

Remark 6.2. *The matrix equation (6.9) is linear in the unknown variable P hence can be readily solved. A specific solution approach, often used in the context of jump linear systems (Costa and Fragoso (1993)) or of Lyapunov equations (Laub, 2005, Ch. 13), is to vectorize the matrix equation to convert it to a vector of linear equations. That is,*

$$vec(P) = (\tilde{q}\, A_c^T \otimes A_c^T + (1-\tilde{q})\, A_o^T \otimes A_o^T)\, vec(P) + vec(\tilde{q}\, Q_c + (1-\tilde{q})\, Q_o) \tag{6.19}$$

where $\otimes$ denotes the Kronecker product. Here we employed the identity $vec(ABC) = C^T \otimes A\, vec(B)$ (Laub, 2005, Ch. 13). The vectorized matrix P that solves (6.19) is written as

$$vec(P) = \left[I - (\tilde{q}\, A_c^T \otimes A_c^T - (1-\tilde{q})\, A_o^T \otimes A_o^T)\right]^{-1} vec(\tilde{q}\, Q_c + (1-\tilde{q})\, Q_o). \tag{6.20}$$

This way the cost-to-go matrix P is described explicitly as a function of the system dynamics, costs, and packet success of the reference policy. We also note that mean square stability of the reference policy implies the existence of such a solution (cf. Prop. 6.2). Indeed (6.8) implies that the matrix in (6.20) is invertible.

6.1.3. Numerical examples

Consider the control of a plant of the form (2.1) with dynamics

$$A = \begin{bmatrix} 1.05 & 0 \\ 0 & 0.6 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}. \tag{6.21}$$

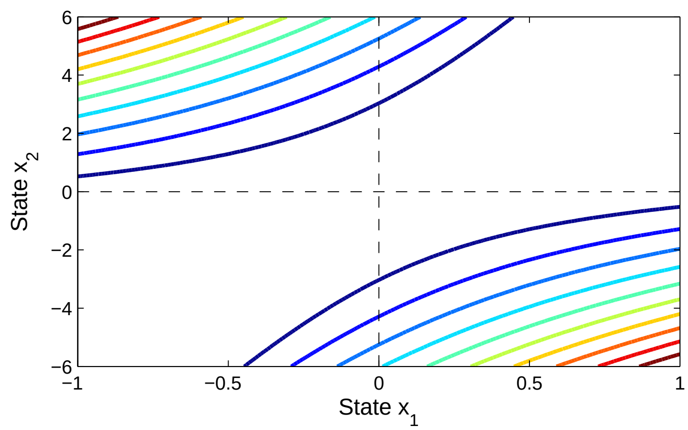

Figure 20: Contour plot of the indefinite value-of-transmission function example.

Suppose first we employ the controller introduced in Example 2.1 which applies a state feedback whenever a message is received and zero input otherwise, i.e., $A_c = A + BK, A_o = A$. Moreover let the feedback gain K be chosen as the optimal LQR controller with respect to quadratic state and control costs given by identity matrices $Q = I_2, R = I_1$. That would be the best feedback gain if communication was perfect.

Suppose we pick a reference power allocation policy $\tilde{\mathbf{p}}$ which is constant, resulting in an average packet success rate $\tilde{q} = 0.3$ (cf. (6.6)), meaning that the sensor would close the loop with the controller over the channel 30% of the time. To implement the rollout policy of the form (6.12) we solve for the reference matrix P by (6.9) and for the value-of-transmission matrix M in (6.11). We observe that the resulting matrix M is indefinite as it has one positive and one negative eigenvalue. In Fig. 20 we illustrate the contour plot of the quadratic $x^T M x$. The plot has a hyperbolic form, and sometimes the quadratic function takes negative values. Intuitively this negative value of transmission exists because the feedback gain of the controller is designed assuming a perfect communication link. The intermittent communication however degrades the value of sensor information to the controller. This numerical example serves to point out that communication and control policies are in general coupled leading to perhaps counter-intuitive results.

Alternatively for the same plant we consider the more sophisticated controller of Example 2.3 (or Chapter 3) which keeps a local plant state estimate $\hat{x}_k$ and applies the control input $u_k = K\hat{x}_k$ with K being the standard LQR feedback gain. For the same reference policy $\tilde{\mathbf{p}}$ and average packet success rate $\tilde{q} = 0.3$ we solve for the matrices P, M in (6.9) and (6.11) respectively. We observe then that the matrix M is positive semi-definite and exactly half of its eigenvalues are zero. Actually, we can prove theoretically that for any plant combined with a controller of this form the quadratic form is exactly equal to a positive definite quadratic function of just the terms $\varepsilon_k = x_k - A\hat{x}_{k-1}$. The latter terms can be thought of as the innovation process of the plant – see also Chapter 3. This contours of this value-of transmission function are plotted in Fig. 21 which in contrast to the previous case exhibits an ellipsoid form.

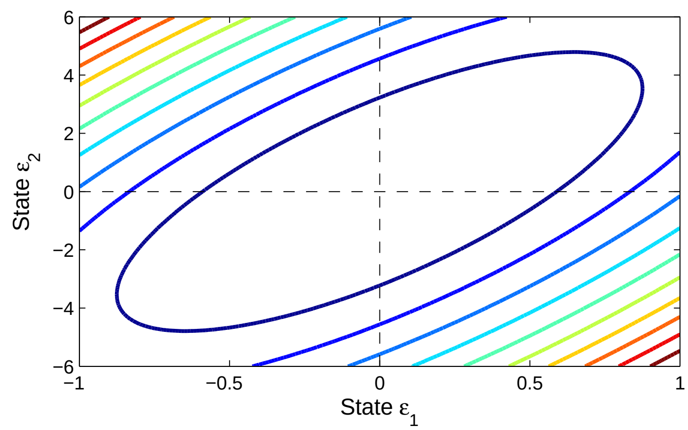

Figure 21: Contour plot of the positive definite value-of-transmission function example.

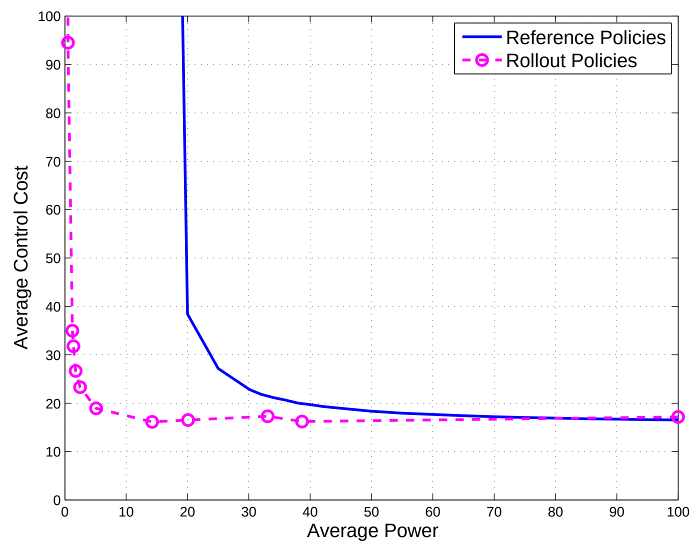

Figure 22: Control performance and average power consumption of proposed rollout policies, in comparison with the reference policies.

We assume now the plant is perturbed by zero mean Gaussian noise with covariance $W = 1$. We implement the rollout power allocation policy of the form (6.12) for the controller with the local estimator. The resulting average control performance and power consumption after varying the parameter ν are shown in Fig. 22. We observe significant performance improvements compared to the reference policies.

These performance improvements can be understood if we look at an example of the schedule of selected powers during operation Fig. 23. The sensor often times does not transmit at all, while when it does it adapts its transmit power. Typically higher system states require higher transmit power, and better channel conditions indicate opportunities for communication with low amounts of power.

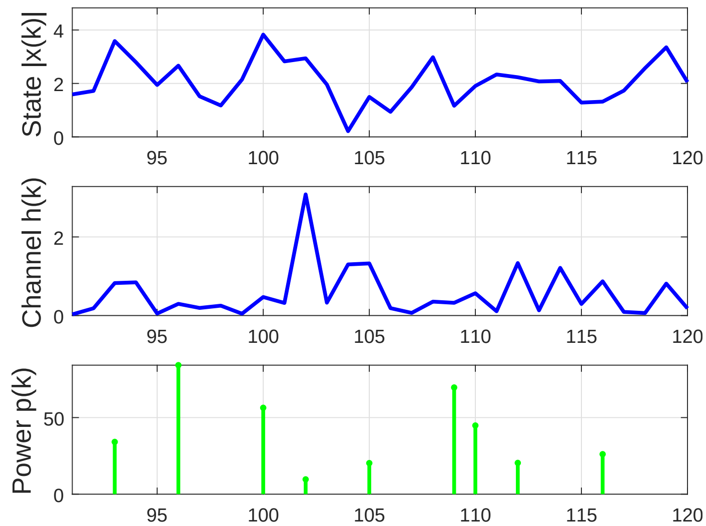

Figure 23: Schedule of selected transmit powers by the rollout algorithm.

6.2. Rollout Multiple Access Policies

In this section we discuss how the rollout power allocation procedure can be extended for the case where multiple independent plants need to communicate over the same wireless channel. In that case we design an online scheduler who decides at each time step which sensor gets access to the channel based on the plant and channel conditions experienced by all systems. We revisit the setup of Chapter 4. There are m plants of the form (6.1) indexed by $i = 1, \ldots, m$, i.e.,

$$x^i_{k+1} = \begin{cases} A^i_c\, x^i_k + w^i_k, & \text{if } \gamma^i_k = 1 \\ A^i_o\, x^i_k + w^i_k, & \text{if } \gamma^i_k = 0 \end{cases} \tag{6.22}$$

and corresponding quadratic costs of the form (6.3) given by

$$\begin{cases} {x^i_k}^T Q^i_c\, x_k, & \text{if } \gamma^i_k = 1 \\ {x^i_k}^T Q^i_o\, x_k, & \text{if } \gamma^i_k = 0 \end{cases}. \tag{6.23}$$

for positive semidefinite matrices Q^i_o, Q^i_c, $i = 1, \ldots, m$.

The channel fading state for each system i at time k is denoted by h^i_k. Channel conditions for all sensor-controller pairs are grouped in a vector h_k assumed to be i.i.d. over time with distribution ϕ_h. When system i is scheduled it decides upon a transmit power p^i_k and this results in a transmission success with probability given by a function $q(h^i_k, p^i_k)$ of the current channel and power. System i is scheduled with probability $\alpha^i_k \in [0, 1]$ at time k. Since at most one system can be scheduled at each time step, the scheduling vector α_k takes values in the probability simplex $\mathcal{A}_m = \{\alpha \in [0, 1]^m : \sum_{i=1}^m \alpha_i \leq 1\}$, where the inequality means that there is a probability that no plant is scheduled.

To follow the rollout procedure introduced in the previous section, we need to pick reference policies. Once again we consider reference scheduling and power allocation policies that are functions of the channel conditions but independent of the plant states. Let $\tilde{\boldsymbol{\alpha}}(h)$ and $\tilde{\mathbf{p}}(h)$ be the reference scheduling and power allocation policies. Under the reference policies, each system i closes its loop, i.e., successfully transmits, with probability

$$\tilde{q}^i := \mathbb{P}(\gamma_k^i = 1) = \mathbb{E}_h[\tilde{\boldsymbol{\alpha}}^i(h) q(h^i, \tilde{\mathbf{p}}^i(h^i))] \tag{6.24}$$

at each time step k, where the expectation is with respect to the joint channel distribution. The average power consumption for each system is given by

$$J^i_{\text{comm}}(\tilde{\boldsymbol{\alpha}}, \tilde{\mathbf{p}}) = \limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}\, \alpha_k^i \, p_k^i = \mathbb{E}_h[\tilde{\boldsymbol{\alpha}}^i(h)\, \tilde{\mathbf{p}}^i(h^i)]. \tag{6.25}$$

As in (6.9) in the previous section, we can compute the control cost-to-go function for each system $i = 1, \dots, m$ under this reference policy by solving for matrices P^i satisfying

$$P^i = \tilde{q}^i \,(Q_c^i + {A_c^i}^T P^i A_c^i) + \,(1 - \tilde{q}^i)(Q_o^i + {A_o^i}^T P^i A_o^i). \tag{6.26}$$

From these matrices we can deduce the relative value of transmission matrices

$$M^i := Q_o^i + {A_o^i}^T P^i A_o^i - Q_c^i - {A_c^i}^T P^i A_c^i. \tag{6.27}$$

The rollout scheduling and power allocation policy is then implemented as follows. Fix some vectors of non-negative weights $\mu \geq 0, \nu \geq 0$. Given the current channel conditions h_k^i and plant state x_k^i for all systems $i = 1, \dots, m$, the current scheduling and power allocation are selected as the solutions to the optimization

$$\min_{\alpha \in \mathcal{A}_m} \quad \left\{ \sum_{i=1}^m \alpha^i \min_{p_i \in [0, p_{\max}]} \left[\nu^i\, p^i - \mu^i\, q(h_k^i, p^i)\, {x_k^i}^T M^i x_k^i \right] \right\} \tag{6.28}$$

The structure of this rollout policy resembles the structure of the channel-only-aware scheduling in Chapter 4 but also involves plant states. In particular each sensor selects its transmit power in a decentralized fashion based only on its corresponding local plant and channel conditions, by solving the inner minimization problem in (6.28). We point out that this power optimization problem has the same structure as in the single loop case in (6.12). The scheduler decision then is as follows. Since (6.28) involves a sum over the scheduling probability vector $\alpha \in \mathcal{A}_m$, the scheduler will deterministically select to schedule the system currently resulting in the lowest power-control tradeoff. A similar opportunistic scheduling structure was also found in Chapter 4.

Moreover we can extend Theorem 6.1 in the current setup. By utilizing the rollout policy (6.28) the total average control and communication costs satisfy by design

$$\sum_{i=1}^{m} \mu^i J^i_{\text{control}} + \sum_{i=1}^{m} \nu^i J^i_{\text{comm}} \leq \sum_{i=1}^{m} \mu^i J^i_{\text{control}}(\tilde{\boldsymbol{\alpha}}, \tilde{\mathbf{p}}) + \sum_{i=1}^{m} \nu^i J^i_{\text{comm}}(\tilde{\boldsymbol{\alpha}}, \tilde{\mathbf{p}}). \tag{6.29}$$

The weights $\nu \geq 0, \mu \geq 0$ are parameters that can be tuned to penalize more or less the control and/or communication costs for different systems independently.

We note that in order to implement this state-aware policy, the scheduler needs to have access to the actual plant states. This might be unrealistic in the uplink case of, e.g., Fig. 10, where the access point receives sensor measurements and needs to schedule one without knowing the exact plant states. The setup becomes more practical in the downlink case where an access point has all the sensor data available and decides to communicate to one of many different receivers/controllers. We also note that rollout algorithms have been used for the problem of scheduling control systems elsewhere in the literature (Antunes et al. (2012)) however without consideration of power resources or wireless fading channels.

6.3. Rollout Random Access Policies

In this section we revisit the setup of Chapter 5 where multiple sensors decide in a decentralized manner whether to transmit or not over a shared wireless channel in order to close their loops. In this section we will design sensor policies adapted online to their local plant state as well as channel state conditions. That is in contrast to the policies designed in Chapter 5 which did not involve plant adaptation. As in the previous section there are m independent systems and each one has its own plant states x^i_k and dynamics of the form

$$x^i_{k+1} = \begin{cases} A^i_c\, x^i_k + w^i_k, & \text{if } \gamma^i_k = 1 \\ A^i_o\, x^i_k + w^i_k, & \text{if } \gamma^i_k = 0 \end{cases} \tag{6.30}$$

for $i = 1, \dots, m$. Here $w^i_k, k \geq 0$ denotes an i.i.d. noise process with some known distribution ϕ_{w^i} with zero mean and covariance W^i. We emphasize that knowledge of the distribution is important in the development that follows, however our results hold regardless of the type of distribution. As in the previous section the systems have their own control costs (cf. (6.23)) and i.i.d. channel states h^i_k with some distribution ϕ_{h^i}.

According to the decentralized mechanism each sensor $i = 1, \dots, m$ transmits at each time step k over the shared channel with probability $\alpha^i_k \in [0, 1]$. We suppose that a collision occurs if more than one sensors transmit at the same time slot, and a collision-free transmission is subject to packet decoding errors depending on channel fading conditions and transmit power. In this section we suppose each sensor's i transmit power is fixed to some value p^i, not a design variable. As a result, the probability that sensor i successfully closes its loop at time k equals

$$\mathbb{P}(\gamma^i_k = 1 \,|\, \alpha_k, h_k) = \alpha^i_k\, q(h^i_k, p^i) \prod_{j \neq i} \Big[1 - \alpha^i_k\Big]. \tag{6.31}$$

This expression was also introduced in Chapter 5 (cf. (5.2)) and states that the probability of system i closing its loop at time k equals the probability that transmission i is successfully decoded at the receiver, multiplied by the probability that no other sensor $j \neq i$ is causing collisions on ith transmission. Similar to Chapter 5 we will assume the success function $q(h_k^i, p^i)$ is strictly increasing in the channel conditions h_k^i.

Consider now reference channel access policies for the sensors independent of the plant states, but potentially adapting to local channel states. In particular we consider policies of the form $\alpha_k^i = \tilde{\boldsymbol{\alpha}}^i(h_k^i)$ for each $i = 1, \ldots, m$, where $\tilde{\boldsymbol{\alpha}}^i : \mathbb{R}_+ \to [0,1]$ is a mapping of current local channel states to the probability of accessing the shared channel. The vector of all sensor reference policies is denoted by $\tilde{\boldsymbol{\alpha}}$. Under these reference policies, a sensor j is accessing the channel with probability that we denote as

$$\tilde{\alpha}^j := \mathbb{E}[\tilde{\boldsymbol{\alpha}}^j(h_k^j)]. \tag{6.32}$$

Here the expectation is with respect to the jth channel distribution. Substituting these reference policies in (6.31), the probability of system i closing its loop becomes constant at each time step that we denote as

$$\tilde{q}^i := \mathbb{P}(\gamma_k^i = 1) = \mathbb{E}[\tilde{\boldsymbol{\alpha}}^i(h_k^i)\, q(h_k^i, p^i)] \prod_{j \neq i} [1 - \tilde{\alpha}^j]. \tag{6.33}$$

Here we used the fact that channel distributions are independent among systems.

Assuming all systems are mean square stable under the chosen reference policies $\tilde{\boldsymbol{\alpha}}$ (cf. Proposition 6.1), then according to Proposition 6.2 the control performance of each system i under this policy is given by $J^i_{\text{control}}(\tilde{\boldsymbol{\alpha}}) = \text{Tr}(P^i W^i)$ where the positive semidefinite matrix P^i solves

$$P^i = \tilde{q}^i\,(Q_c^i + {A_c^i}^T P^i A_c^i) + \,(1 - \tilde{q}^i)(Q_o^i + {A_o^i}^T P^i A_o^i) \tag{6.34}$$

for the term $\tilde{q}^i$ defined in (6.33).

Moreover we can define once again the relative value of transmitting for each system i under the current plant conditions as the quadratic term ${x_k^i}^T M^i x_k^i$, with the matrix M^i defined as in (6.27). In this section we also make the assumption that this value is always non-negative.

Assumption 6.1. *For each system i the matrix* $M^i = Q_o^i + {A_o^i}^T P^i A_o^i - Q_c^i - {A_c^i}^T P^i A_c^i$ *is positive semidefinite.*

Unlike the previous cases where the decision at each time step was centralized, here each sensor makes their own decision in a decentralized fashion based on their own local information. Unfortunately it is not obvious how to generalize the previous centralized rollout policies (Sections 6.1, 6.2) in this setting. We propose instead a different methodology for improving upon the reference policies.

According to the reference policy each sensor j accesses the channel on average with a constant rate $\tilde{\alpha}^j$ given in (6.32) and is mean square stable. Intuitively this rate is low enough so that the resulting collisions on all other communication links do not deteriorate the control performance of the rest of the systems. This rate can be thought of as a *safe* access rate. Our approach in the current section is to allow the sensors to adapt to their local plant and channel states, but explicitly require that the rate at which each sensor accesses the shared channel at every time step prescribes to the reference safe access rate. We term this approach *constrained rollout*.

6.3.1. Constrained Rollout Policies

To formalize our approach we first characterize the information available to each sensor at the beginning of a time step k. Before measuring the channel state h_k^i, the sensor knows that its distribution is given by ϕ_{h^i}. Additionally before measuring the plant state x_k^i, sensor i knows by (6.30) that x_k^i has a distribution ϕ_{w^i} centered at some current mean value denoted here by $\bar{x}_k^i$. In other words $x_k^i = \bar{x}_k^i + w_{k-1}^i$. Similarly to (6.30) the mean $\bar{x}_k^i$ evolves according to

$$\bar{x}_k^i = \begin{cases} A_c^i \, x_{k-1}^i, & \text{if } \gamma_{k-1}^i = 1 \\ A_o^i \, x_{k-1}^i, & \text{if } \gamma_{k-1}^i = 0 \end{cases}. \tag{6.35}$$

That is, if a transmission occurred at the last time step the sensor knows that the plant state evolves according to the closed loop mode A_c^i and expects a measurement x_k^i with mean $\bar{x}_k^i = A_c^i x_{k-1}^i$. The case of no transmission is similar. The sensor can keep track of this evolving mean value assuming it knows all past measurements and transmission successes by acknowledgments.

Our approach is to guarantee that, conditioned on the current locally available information, sensor i transmits with the prescribed access rate *on average*, that is

$$\mathbb{E}[\alpha_k^i \mid \bar{x}_k^i] \leq \tilde{\alpha}^i. \tag{6.36}$$

At time k the sensor may adapt its channel access decision α_k^i to the measured plant and channel states x_k^i, h_k^i respectively. Hence we allow in general a policy of the form $\alpha_k^i = \boldsymbol{\alpha}^i(x_k^i, h_k^i)$, where the set of all such transmission functions at any given mean state value $\bar{x}_k$ is denoted by

$$\mathcal{A} = \{\boldsymbol{\alpha}^i : \mathbb{R}^{n_i} \times \mathbb{R}_+ \to [0,1]\}. \tag{6.37}$$

We propose then to select *dynamically* at each time step k the transmission functions $\boldsymbol{\alpha}^i, i = 1, \dots, m$ as a solution to a constrained optimization problem. The sensor would like to maximize the expected gains in control performance due to transmission, while at the same time adheres to the prescribed access rates according to (6.36). Formally this optimization problem is

$$\underset{\boldsymbol{\alpha}^i \in \mathcal{A}}{\text{minimize}} \quad \mathbb{E}[-\boldsymbol{\alpha}^i(x_k^i, h_k^i)\, q(h_k^i, p^i) \prod_{j \neq i}(1 - \tilde{\alpha}_j)\, {x_k^i}^T M^i x_k^i \mid \bar{x}_k^i] \tag{6.38}$$

$$\text{subject to} \quad \mathbb{E}[\boldsymbol{\alpha}^i(x_k^i, h_k^i) \mid \bar{x}_k^i] \leq \tilde{\alpha}^i. \tag{6.39}$$

In this problem the expectations are with respect to the current channel distribution of h_k^i, as well as the plant state distribution x_k^i which conditioned on the current mean value $\bar{x}_k^i$ just depends on the current noise w_{k-1}^i. We denote the optimal solutions, i.e., the optimal transmission functions as $\boldsymbol{\alpha}^{*i}$ for each system i.

At each time step the sensor selects the current transmit function as a solution to the constrained optimization problem (6.38)-(6.39) where both the objective and the constraint are expressed *in expectation over the current plant and channel conditions* x_k, h_k to be measured. The expected objective to be minimized in (6.38) entails the quadratic form ${x_k^i}^T M^i x_k^i$ of the local plant state x_k^i. As explained in Section 6.1 this represents the relative gains in control performance if a successful transmission occurs. The objective also involves the probability of successful transmission $q(h_k^i, p^i)$. So the overall objective indeed represents the expected gains in control performance of the system.

For the proposed policy we can establish the following result which characterizes the control performance using the proposed constrained rollout policies.

Theorem 6.2 (Rollout Random Access Policy Performance)**.** *Let Assumption 6.1 hold. Suppose every sensor uses the policy $\boldsymbol{\alpha}^{*i}$ optimizing (6.38)-(6.39) at each time step. Then the control cost of each system $i = 1, \ldots, m$ satisfies*

$$J^i_{control}(\boldsymbol{\alpha}^*) \leq J^i_{control}(\tilde{\boldsymbol{\alpha}}). \tag{6.40}$$

where $J^i_{control}(\tilde{\boldsymbol{\alpha}})$ is the control cost using the reference policies.

This result is important as it demonstrates that despite the collisions arising in the shared medium between transmissions, each sensor can adapt to their own local plant state measurements and channel conditions and guarantee an improved control performance (compared to the reference). As a side remark we note that by construction the sensors transmit at the same rate as the reference policies. This is because (6.39) is enforced at each time step, hence it holds in the long run too. The average rate of collisions remains the same as for the reference policy as well. What changes is that transmissions occur when it is opportunistically more important for the system, hence successful transmissions also occur when it is important.

To find the transmission function $\boldsymbol{\alpha}^{*i}$ at each time step given its current mean state $\bar{x}_k^i \in \mathbb{R}^{n_i}$, the sensor needs to solve problem (6.38)-(6.39). This is an optimization problem over the space of transmission functions, i.e., an infinite-dimensional optimization problem. However as we show in the following result it enjoys simple threshold-based optimal solutions.

Proposition 6.3. *Consider the transmission function optimization problem in (6.38)-(6.39) given any fixed $\bar{x}_k^i \in \mathbb{R}^{n_i}$. Suppose the distributions ϕ_{h^i} of channel states $\{h_k^i,\, k \geq 0\}$ and ϕ_{w^i} of plant disturbances $\{w_k^i,\, k \geq 0\}$ of system i are absolutely continuous, i.e., have probability density functions. Then there exists a non-negative constant $\nu^i \geq 0$, depending on $\bar{x}_k^i$, such that the function*

$$\boldsymbol{\alpha}^{*i}(x_k^i, h_k^i) = \begin{cases} 1, & \text{if } q(h_k^i, p^i) \prod_{j \neq i}(1 - \tilde{\alpha}_j)\, {x_k^i}^T M^i x_k^i \geq \nu^i \\ 0 & \text{otherwise.} \end{cases} \tag{6.41}$$

is an optimal solution.

The above proposition shows first that the proposed decision is not randomized but deterministic. Given the current plant and channel conditions the sensor will either transmit or not. Furthermore it is of a threshold form. The sensor transmits if the product between the current channel success $q(h^i_k, p^i) \prod (1 - \tilde{\alpha}_j)$ and value of transmission ${x^i_k}^T M^i x^i_k$ exceeds some threshold. Here the packet success assumes that all other sensors $j \neq i$ will adhere to their reference access rates $\tilde{\alpha}^j$.

We note that threshold policies with respect to the plant state frequently appear in the event-based control framework of, e.g., Xu and Hespanha (2004); Molin and Hirche (2009); Heemels et al. (2012). The threshold value here however is not a free parameter, but it depends on the current mean value $\bar{x}^i_k$ which dynamically varies at each time step (cf. (6.35)). As a result the transmission decision α^i_k is not just a time-invariant function of states x^i_k, h^i_k but it also depends on the mean value $\bar{x}^i_k$. This feature distinguishes our policies from other event-based control policies and is by construction added to ensure that the sensor adheres to the prescribed safe access rate $\tilde{\alpha}^i$.

Since Prop. 6.3 simplifies the search for general transmission functions in Theorem 6.2 to the search for threshold functions, we describe next a computationally efficient procedure to find the appropriate threshold. We note that in order to implement the proposed policy, a transmission function solving problem (6.38)-(6.39) is needed for any possible mean value $\bar{x}^i_k \in \mathbb{R}^{n_i}$. This requires the solution of an infinite number of such optimization problems. In practice, as well as in the numerical simulations that follow, the space of mean values $\bar{x}^i_k \in \mathbb{R}^n$ can be discretized and we can solve instead a large number of optimization problems at the discrete points.

6.3.2. *Computing Transmission Functions*

For notational simplicity we now drop the time index k. From the proof of Prop. 6.3 we see that the threshold ν^i, as a function of the current mean $\bar{x}^i$, corresponds to the optimal Lagrange dual variable of problem (6.38)-(6.39). We briefly describe a dual subgradient algorithm to find this optimal dual point. A subgradient direction for the dual problem is typically given by the constraint slack of the primal problem evaluated at a primal Lagrangian minimizer (cf. (Bertsekas et al., 2003, Ch.8)). More precisely, given some dual variable $\nu_t \geq 0$ at iteration t, a corresponding primal solution is given by substituting ν_t in place of ν^i in (6.41).

The constraint slack of this solution with respect to (6.39) is given by the difference

$$s_t := \mathbb{P}\Big[q(h^i, p^i) \prod_{j \neq i} (1 - \tilde{\alpha}_j)\; {x^i}^T M^i x^i \geq \nu^t \;\big|\; \bar{x}^i\Big] - \tilde{\alpha}^i. \tag{6.42}$$

Hence a dual subgradient ascent algorithm to compute the point ν_{t+1} for the next iteration is given by

$$\nu_{t+1} = \max\Big\{0, \nu_t + \varepsilon_t\, s_t\Big\} \tag{6.43}$$

where $\varepsilon_t \geq 0$ is a stepsize, and the maximum is taken so that the new dual variable is projected to the non-negatives. Iterating (6.43), the dual variable ν_t converges to the optimal ν^i – see, e.g., (Bertsekas et al., 2003, Ch. 8.2).

6.3.3. Numerical Simulations

We consider a random access architecture involving the control of two ($m = 2$) scalar systems of the form (6.30). The first system is described by

$$x_{k+1}^1 = \begin{cases} 0.6\, x_k^1 + w_k^1, & \text{if } \gamma_k^1 = 1 \\ 1.1\, x_k^1 + w_k^1, & \text{if } \gamma_k^1 = 0 \end{cases}, \tag{6.44}$$

i.e., its open loop is unstable. The second system is assumed to have open loop integrator dynamics and a controller resetting the state to zero, i.e.,

$$x_{k+1}^2 = \begin{cases} 0\, x_k^2 + w_k^2, & \text{if } \gamma_k^2 = 1 \\ 1\, x_k^2 + w_k^2, & \text{if } \gamma_k^2 = 0 \end{cases}. \tag{6.45}$$

Due to plant noise, taken here to be standard Gaussian, both system states will grow unbounded unless communication is performed often enough to close the loops. Moreover we assume the same quadratic state costs for both systems $Q_o^i = Q_c^1 = 1, i = 1, 2$ (cf. (6.23)). We also assume the channel fading states for the two systems are identically distributed, $h_k^i \sim \exp(1), i = 1, 2, k \geq 0$, while the probability of success is modeled as in Fig. 2.

We begin our approach by considering reference policies. For simplicity we assume both sensors access the shared wireless channel with a constant probability under the reference policies. In particular suppose system 1 which is the unstable one accesses the channel with probability $\tilde{\boldsymbol{\alpha}}^1 = 0.40$, while system 2 which is the marginally stable one with probability $\tilde{\boldsymbol{\alpha}}^2 = 0.15$. Despite the collisions arising when both sensors transmit simultaneously the two systems remain stable. Their control performances can be evaluated either theoretically following Section 6.3, i.e., by $J_{\text{control}}^i(\tilde{\boldsymbol{\alpha}}) = \text{Tr}(P^i W^i)$ where P^i satisfies (6.34), or in simulations. In Fig. 24 we plot the empirical control cost from simulations

$$\begin{aligned} J_{\text{control}}^1(\tilde{\boldsymbol{\alpha}}) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} (x_k^1)^2 \approx 32, \\ J_{\text{control}}^2(\tilde{\boldsymbol{\alpha}}) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} (x_k^2)^2 \approx 13. \end{aligned} \tag{6.46}$$

In other words using the reference policies the more unstable system experiences worse control performance.

We now employ the proposed state-aware policies described in the previous section. In particular each sensor $i = 1, 2$ keeps track of its own plant state statistics, and applies the transmission rule $\boldsymbol{\alpha}^{*i}$ solving problem (6.38)-(6.39). This is a threshold-based policy as shown in (6.41) which can be computed following Section 6.3.2. Sensor i has to solve this problem for all possible current mean

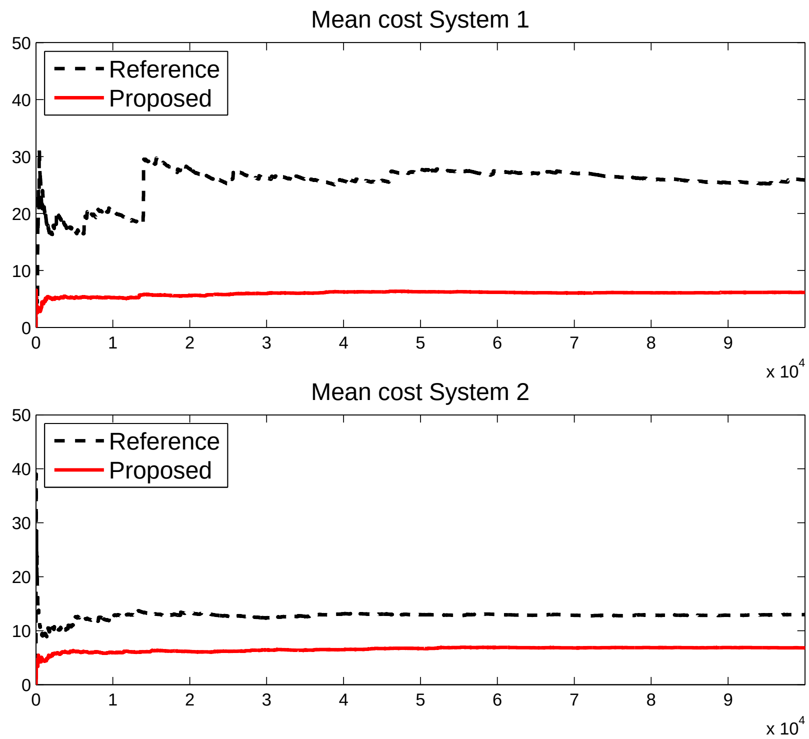

Figure 24: Simulation of reference random access policies and proposed state-aware random access policies. By opportunistically adapting to system states the proposed policies exhibit a significantly improved control performance.

values $\bar{x}_k^i \in \mathbb{R}$ of the plant state. In simulations we discretize the space $\bar{x}_k^i \in \mathbb{R}$, find the policies at the discrete points, and during run-time the sensor selects the policy of the closest discrete point.

We simulate the random access wireless architecture using the proposed state-aware policies and in Fig. 24 we present the average empirical control cost, i.e.,

$$\begin{aligned} J^1_{\text{control}}(\boldsymbol{\alpha}^*) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} (x_k^1)^2 \approx 6.3, \\ J^2_{\text{control}}(\boldsymbol{\alpha}^*) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} (x_k^2)^2 \approx 6.6 \end{aligned} \tag{6.47}$$

during simulation. As verified by Theorem 6.2 the performance using the proposed state-aware policies is improved compared to the reference. The theoretical result claims just an improvement, however the simulations illustrate significant improvements at the order of 80% for plant 1 and 50% for system 2. The reason for these improvements is that the proposed policies adapt online in an opportunistic fashion to both plant and channel conditions. This way a sensor accesses the shared channel only when favorable channel states or necessary control requirements appear. We note that it seems that the more unstable system has higher performance improvements. We also note that control performances are improved even though collisions still occur, in fact at the same

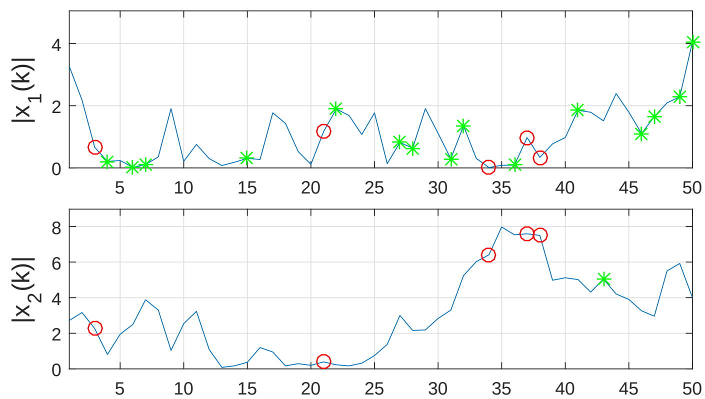

Figure 25: Reference random access. Green stars indicate successful transmissions for each system. Red circles indicate collisions between the two systems. Successful transmissions happen in a random fashion independent of the system states.

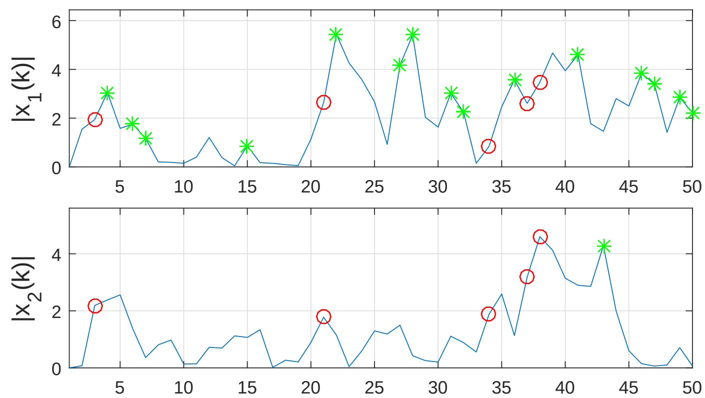

Figure 26: Constrained rollout random access. Green stars indicate successful transmissions for each system. Red circles indicate collisions between the two systems. Successful transmissions typically occur when the systems states are large, meaning that control performance is at stake.

rate as in the reference policies.

The large performance improvements can also be understood by looking at sample system traces (Fig. 25, Fig. 26. In the reference policy successful transmissions happen in a random fashion, independent of system states. In the constrained rollout policies successful transmissions are be design correlated with times when the system states are large, i.e., control regulation is needed.

Finally, we would also like to point out that the plant adaptation of the proposed rollout policy is so significant that in many cases it can outperform non-state-aware centralized policies. In other words it overcomes the performance losses due to collisions. For example, consider the two systems given above and consider a centralized scheduler that gives access to system 1 with probability 0.4 at each time step, and gives access to system 2 with probability 0.15 at each time step. These are the same rates as the reference random access rates, but in a centralized scheduling setup. We can compute the control performances for systems 1 and 2 to be respectively 13.7, 8 which are worse than the control performances achieved by the proposed state-aware random access policy.

6.4. Proofs

6.4.1. Notation used for the proofs of Proposition 6.2 and Theorem 6.1

We introduce some notation that will be used within the proofs of this chapter. When the current plant and channel states take some values $x \in \mathbb{R}^n, h \in \mathbb{R}_+$ and the sensor takes a decision $p \in [0, p_{\max}]$, let $\gamma \in \{0,1\}$ denote the random success of the current transmission modeled by (6.2). Also let $x_+ \in \mathbb{R}^n, h_+ \in \mathbb{R}_+$ denote the random value of the plant and channel states at the next time step. The distribution of the former depends on γ and the current plant disturbance $w \sim \phi_w$ according to $x_+ = \gamma A_c x + (1-\gamma) A_o x + w$ as follows from (6.1). The next channel state is independent by assumption. We denote the integration with respect to the distribution of γ and x_+, h_+ given the values of x, h, p as $\mathbb{E}[. \mid x, h, p]$. This expectation depends just on the control and communication system model, not on the employed policy.

Moreover consider the function

$$V(x,h) := x^T P(h) x \tag{6.48}$$

where $P(h)$ is a symmetric positive semidefinite matrix defined as

$$P(h) := q(h, \tilde{\mathbf{p}}(h)) \left(Q_c + A_c^T P A_c\right) + (1 - q(h, \tilde{\mathbf{p}}(h)))(Q_o + A_o^T P A_o), \tag{6.49}$$

where P is defined in (6.9). It is also easy to see from (6.49) that the expected value of the matrix $P(h)$ with respect to the channel distribution ϕ_h equals the matrix P, i.e.,

$$\mathbb{E}_h P(h) = P. \tag{6.50}$$

Define also for any $x \in \mathbb{R}^n, h \in \mathbb{R}_+$ and $p \in [0, p_{\max}]$ the function

$$F(x,h,p) := \mathbb{E}[x^T \left(\gamma Q_c + (1-\gamma) Q_o\right) x + V(x_+, h_+) \mid x, h, p] - V(x,h) - Tr(PW) \tag{6.51}$$

Substituting $x_+ = \gamma A_c x + (1-\gamma) A_o x + w$ in $V(x_+, h_+)$ at the right hand side and by expanding the expectation term we have

$$\begin{aligned} \mathbb{E}[x^T \left(\gamma Q_c + (1-\gamma) Q_o\right) x + V(x_+, h_+) \mid x, h, p] &= q(h,p)\, x^T (Q_c + A_c^T P A_c) x \\ &\quad + (1 - q(h,p)) x^T (Q_o + A_o^T P A_o) x + \mathrm{Tr}(P\, W) \end{aligned} \tag{6.52}$$

where the latter term $\mathrm{Tr}(P\,W)$ is due to the variance of the system disturbance and (6.50). Substituting (6.52) and (6.48)-(6.49) in the definition of the function $F(x,h,p)$ in (6.51) we derive an equivalent expression for the function as

$$F(x,h,p) = -(q(h,p) - q(h,\tilde{\mathbf{p}}(h))\,)\; x^T M x \tag{6.53}$$

All the above expressions will be used for the proofs of our results. In particular we will make use of both expressions (6.51) and (6.53). We also note for future reference that substituting the reference policy function $\tilde{\mathbf{p}}(h)$ in (6.53) we directly get

$$F(x,h,\tilde{\mathbf{p}}(h)) = 0 \tag{6.54}$$

6.4.2. *Proof of Proposition 6.2*

We are now ready to prove the result of this Proposition. First we will show that there is a unique symmetric positive semidefinite matrix solution P of (6.9). To do this we can describe the solution P as the limit of the matrix recursion

$$P_{k+1} = \tilde{q}\,(Q_c + A_c^T P_k A_c) + \,(1-\tilde{q})(Q_o + A_o^T P_k A_o), \tag{6.55}$$

for some initial matrix condition P_0. This recursion is convergent. To see this, we can perform the vectorization operation (see also Remark 6.2) to rewrite the recursion as

$$\mathrm{vec}(P_{k+1}) = (\tilde{q}\,A_c^T \otimes A_c^T - (1-\tilde{q})\,A_o^T \otimes A_o^T)\;\mathrm{vec}(P_k) + \;\mathrm{vec}(\tilde{q}\,Q_c + (1-\tilde{q})\,Q_o). \tag{6.56}$$

By assumption the employed reference policy $\tilde{\mathbf{p}}$ leads to mean square stability, which implies (6.8), i.e., that the matrix in (6.56) is asymptotically stable. Hence $\mathrm{vec}(P_k)$ converges to a unique limit point, and the matrix P_k converges to a unique matrix P solving equation (6.9).

To show that P is also a symmetric positive semidefinite matrix, let us pick the initial condition $P_0 = \tilde{q}\,Q_c + (1-\tilde{q})Q_o$. It can be easily argued by induction that $P_{k+1} \succeq P_k \succeq 0$ for all $k \geq 0$. Hence the limit also satisfies $P = \lim_{k\to\infty} P_k \succeq 0$.

We are now ready to prove the second part of the proposition. That is, if x_k, h_k denote the random variables describing the plant and channel states of the system at time k when using the reference policy $p_k = \tilde{\mathbf{p}}(h_k)$, we will show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}[x_k^T\,(\gamma_k Q_c + (1-\gamma_k) Q_o)\,x_k] = \mathrm{Tr}(P\,W). \tag{6.57}$$

If x_k, h_k are the random variables denoting the plant and channel states of the system at time k using the reference policy $p_k = \tilde{\mathbf{p}}(h_k)$, then by (6.54) we have that

$$F(x_k, h_k, p_k) = F(x_k, h_k, \tilde{\mathbf{p}}(h_k)) = 0 \tag{6.58}$$

holds almost surely for all $k \geq 0$. Since the function F is equal to zero, by its definition (6.51) we conclude that

$$\mathbb{E}[x_k^T(\gamma_k Q_c + (1-\gamma_k)Q_o)x_k + V(x_{k+1}, h_{k+1}) \,|\, x_k, h_k, p_k] = V(x_k, h_k) + \mathrm{Tr}(P\,W) \tag{6.59}$$

holds almost surely for all $k \geq 0$.

Let us take the expectation of (6.59) with respect to the distribution of x_k, h_k to conclude that

$$\mathbb{E}[x_k^T(\gamma_k Q_c + (1-\gamma_k)Q_o)x_k + V(x_{k+1}, h_{k+1})] = \mathbb{E}[V(x_k, h_k)] + \mathrm{Tr}(P\,W) \tag{6.60}$$

holds for all $k \geq 0$. Summing up (6.60) for $k = 0, \ldots, N-1$, removing the telescoping terms $\sum_{k=0}^{N-1} \mathbb{E}V(x_k, h_k)$, and dividing by N we get that

$$\frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}[x_k^T\,(\gamma_k Q_c + (1-\gamma_k)Q_o)\,x_k] + \frac{1}{N}\mathbb{E}[V(x_N, h_N)] = \frac{1}{N}\mathbb{E}V(x_0, h_0) + \mathrm{Tr}(P\,W) \tag{6.61}$$

Consider now the terms $\mathbb{E}V(x_k, h_k) = \mathbb{E}\,x_k^T\,P(h_k)\,x_k$ for any time $k \geq 0$. Since the channel state h_k at time k is by the model independent of the plant state x_k and distributed by ϕ_h, we have that

$$\mathbb{E}V(x_k, h_k) = \mathrm{Tr}(\mathbb{E}P(h_k)\,\mathbb{E}\,x_k\,x_k^T) = \mathrm{Tr}(P\,\mathbb{E}\,x_k\,x_k^T) < \infty \tag{6.62}$$

Here in the second equality we used (6.50) and in the inequality we used the mean square stability of the system by assumption ($\limsup_{k\to\infty} \mathbb{E}x_k x_k^T < \infty$). Hence the terms $\mathbb{E}V(x_0, h_0)$, $\mathbb{E}[V(x_N, h_N)]$ in (6.61) are uniformly bounded over N. Hence taking the limit in (6.61) as $N \to \infty$ verifies (6.57) and completes the proof.

6.4.3. Proof of Theorem 6.1

We use the notation introduced in the proof of Prop. 6.2. For any constant $\nu \geq 0$, and any plant and channel state value $x \in \mathbb{R}^n, h \in \mathbb{R}_+$ consider the minimization of the expression $F(x, h, p) + \nu\, p$ over power choices $p \in [0, p_{\max}]$. Since the reference choice $\tilde{\mathbf{p}}(h)$ is one feasible solution we have

$$\min_{p\in[0,p_{\max}]} F(x, h, p) + \nu\, p \;\leq\; F(x, h, \tilde{\mathbf{p}}(h)) + \nu\,\tilde{\mathbf{p}}(h) = \;\nu\,\tilde{\mathbf{p}}(h). \tag{6.63}$$

where the last equality follows from (6.54). We then analyze the left hand side of the above inequality (6.63). By the expression for the function F given in (6.53) we have that

$$\min_{p\in[0,p_{\max}]} F(x, h, p) + \nu\, p = \min_{p\in[0,p_{\max}]} \nu\, p - (q(h, p) - q(h, \tilde{\mathbf{p}}(h))\,)\; x^T M x \tag{6.64}$$

This minimization is exactly the same as the one appearing in (6.12), hence the optimal solution to this minimization problem is $\mathbf{p}_\nu^{\mathrm{roll}}(x, h)$. As a result (6.63) implies the inequality

$$F(x, h, \mathbf{p}_\nu^{\mathrm{roll}}(x, h)) + \nu\,\mathbf{p}_\nu^{\mathrm{roll}}(x, h) \leq \nu\,\tilde{\mathbf{p}}(h). \tag{6.65}$$

We will use this inequality to prove the Theorem.

Suppose x_k, h_k are the random variables representing the plant and channel states at the kth time step after employing the policy $p_k = \mathbf{p}_\nu^{\text{roll}}(x_k, h_k)$ given in (6.12). By (6.65) we get then that

$$F(x_k, h_k, p_k) + \nu\, p_k \leq \nu\, \tilde{\mathbf{p}}(h_k). \tag{6.66}$$

holds almost surely for all $k \geq 0$. Substituting then the definition of the function F by (6.51) in (6.66) we have that

$$\begin{aligned}\mathbb{E}[x_k^T\,(\gamma_k Q_c + (1-\gamma_k)Q_o)\, x_k + V(x_{k+1}, h_{k+1}) \,\big|\, x_k, h_k, p_k] + \nu\, p_k \\ \leq V(x_k, h_k) \,+ Tr(PW) \,+ \nu\, \tilde{\mathbf{p}}(h_k)\end{aligned} \tag{6.67}$$

holds almost surely for all $k \geq 0$. By taking the expectation of the above inequality with respect to the processes x_k, h_k, p_k we have that

$$\begin{aligned}\mathbb{E}[x_k^T\,(\gamma_k Q_c + (1-\gamma_k)Q_o)\, x_k + \nu\, p_k + V(x_{k+1}, h_{k+1})] \\ \leq \mathbb{E}V(x_k, h_k) \,+ Tr(PW) \,+ \nu\, \mathbb{E}\tilde{\mathbf{p}}(h_k)\end{aligned} \tag{6.68}$$

holds for all $k \geq 0$. Then summing up all the above inequalities (6.68) for $k = 0, \ldots, N-1$, removing the telescoping terms $\sum_{k=0}^{N-1} \mathbb{E}V(x_k, h_k)$, and dividing by N we get

$$\begin{aligned}\frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}[x_k^T\,(\gamma_k Q_c + (1-\gamma_k)Q_o)\, x_k + \nu\, p_k] + \frac{1}{N}\mathbb{E}[V(x_N, h_N)] \leq \\ \frac{1}{N}\mathbb{E}V(x_0, h_0) + \mathrm{Tr}(P\,W) + \nu\, \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}\tilde{\mathbf{p}}(h_k)\end{aligned} \tag{6.69}$$

Now since the function V is non-negative (cf.(6.49)), the term $\mathbb{E}[V(x_N, h_N)]$ is also non-negative and can be omitted from (6.69). Moreover $\mathbb{E}V(x_0, h_0)$ is bounded as the initial plant state x_0 has finite variance by assumption. Hence by taking the limit in (6.69) as $N \to \infty$ we conclude that

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}[x_k^T\,(\gamma_k Q_c + (1-\gamma_k)Q_o)\, x_k + \nu\, p_k] \leq \\ \mathrm{Tr}(P\,W) + \nu\, \limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}\tilde{\mathbf{p}}(h_k)\end{aligned} \tag{6.70}$$

As established in Proposition 6.2 the value $\mathrm{Tr}(P\,W)$ is equal to the average control performance $J_{\text{control}}(\tilde{\mathbf{p}})$ of the reference policy $\tilde{\mathbf{p}}$. Also the average sum at the right hand side of (6.70) is equal to the average power consumption of the reference policy $\tilde{\mathbf{p}}$ given in (6.7). Hence (6.70) verifies (6.13) and completes the proof.

6.4.4. *Proof of Theorem 6.2*

For simplicity the arguments to follow prove (6.40) for some fixed system i. A symmetric argument can be performed for each system $i = 1, \dots, m$. We will use notation similar to the one introduced in the proof of Proposition 6.2, however the present setup is more complex as there are many systems.

We will refer to the current mean value for each system i by $\bar{x}^i \in \mathbb{R}^{n_i}$ as introduced in (6.35). Then the current plant state $x^i \in \mathbb{R}^{n_i}$ at system i depends on the current mean $\bar{x}^i$ and the current random plant disturbance $w^i \in \mathbb{R}^{n_i}$ with distribution ϕ_{w^i}, according to $\bar{x}^i = x^i + w^i$. When the current channel states for each system i take some values $h^i \in \mathbb{R}_+$ and the sensors access the channel with probabilities $\alpha^i \in [0, 1]$, let $\gamma^i \in \{0, 1\}$ denote the random transmission success on link i modeled by (6.31). Also let $x^i_+ \in \mathbb{R}^{n_i}, h^i_+ \in \mathbb{R}_+$ denote the random value of the plant and channel states for system i at the next time step. The distribution of the former depends on γ^i and the next plant disturbance $w^i_+ \sim \phi_{w^i}$ according to (6.30). The next channel state h^i_+ is independent of all other variables by assumption. In a vector form we will denote the above variables as $\bar{x} \in \mathbb{R}^n, x \in \mathbb{R}^n, h \in \mathbb{R}^m_+, \alpha \in [0, 1]^m, \gamma \in 0, 1^m, x_+ \in \mathbb{R}^n, h_+ \mathbb{R}^m_+$. We denote the integration with respect to the distributions of these variables given the values of x, h, α as $\mathbb{E}[. \mid x, h, \alpha]$. This expectation depends just on the control and communication system model, not on the employed policy. The expectation of any function of just the current plant and channel states x, h given the current mean values $\bar{x}$ will be denoted by $\mathbb{E}[. \mid \bar{x}]$.

Moreover, similar to the proof of Proposition 6.2, define the functions

$$V^i(x^i, h^i) := {x^i}^T P^i(h^i) x^i \tag{6.71}$$

for i= $1, \dots, m$, where $P^i(h^i)$ are symmetric positive semidefinite matrices defined as [6]

$$\begin{aligned} P^i(h^i) :=& \tilde{\boldsymbol{\alpha}}^i(h^i)\, q(h^i) \prod_{j \neq i} [1 - \tilde{\boldsymbol{\alpha}}^j(h^j)]\, (Q^i_c + {A^i_c}^T P^i A^i_c) \\ &+ \Big(1 - \tilde{\boldsymbol{\alpha}}^i(h^i)\, q(h^i) \prod_{j \neq i} [1 - \tilde{\boldsymbol{\alpha}}^j(h^j)]\Big)(Q^i_o + {A^i_o}^T P^i A^i_o), \end{aligned} \tag{6.72}$$

where P^i is defined in (6.34). It is also easy to see from (6.72) and the definition of P^i in (6.34) that the expected value of the matrix $P^i(h^i)$ with respect to the channel distribution ϕ_h equals the matrix P^i, i.e., $\mathbb{E}_h P^i(h^i) = P^i$.

Similar to the proof of Proposition 6.2, define also for any $x^i \in \mathbb{R}^{n_i}, h^i \in \mathbb{R}_+$ and $\alpha \in [0, 1]^m$ the function

$$F(x^i, h^i, \alpha) := \mathbb{E}[{x^i}^T (\gamma^i Q^i_c + (1 - \gamma^i) Q^i_o)\, x^i + V^i(x^i_+, h^i_+) \mid x^i, h^i, \alpha] - V^i(x^i, h^i) - Tr(P^i W^i) \tag{6.73}$$

Note that the function depends on the whole vector α, i.e., the transmissions of all sensors. Fol-

[6] As the powers p^i are fixed for all systems, for brevity we will denote the function $q(h^i_k, p^i)$ as $q(h^i_k)$.

lowing arguments similar to the ones leading from (6.51) to (6.53) we can equivalently write the function F as

$$F(x^i, h^i, \alpha) = -\left[\alpha^i \prod_{j\neq i}[1-\alpha^j] - \tilde{\boldsymbol{\alpha}}^i(h^i) \prod_{j\neq i}[1-\tilde{\boldsymbol{\alpha}}^j(h^j)]\right] q(h^i)\, {x^i}^T\, M^i\, x^i \tag{6.74}$$

Now substituting α^i in (6.74) from any local transmission function $\boldsymbol{\alpha}^i(x^i, h^i)$ for each system $i = 1, \ldots, m$ we get an expression $F(x^i, h^i, \boldsymbol{\alpha}(x, h))$ which depends only on the current plant and channel conditions x, h of all systems, where with a slight abuse of notation here we use

$$\boldsymbol{\alpha}(x,h) := \begin{bmatrix} \vdots \\ \boldsymbol{\alpha}^i(x^i, h^i) \\ \vdots \end{bmatrix}. \tag{6.75}$$

We also note for future reference that substituting the reference transmission functions $\tilde{\boldsymbol{\alpha}}^i(h^i)$ in (6.74) we directly get

$$F(x^i, h^i, \tilde{\boldsymbol{\alpha}}(h)) = 0 \tag{6.76}$$

for any values of $x \in \mathbb{R}^n, h \in \mathbb{R}^m_+$.

The proof of the theorem relies on the following technical lemma.

Lemma 6.1. *Let Assumption 6.1 hold. Then for any value of $\bar{x} \in \mathbb{R}^n$ it holds that*

$$\mathbb{E}\left[F\left(x^i, h^i, \boldsymbol{\alpha}^*(x,h)\right) \mid \bar{x}\right] \leq 0 \tag{6.77}$$

where $\boldsymbol{\alpha}^*$ are the optimal solutions of problems (6.38)-(6.39) for all systems $i = 1, \ldots, m$.

Before proceeding to the proof of the lemma let us show how it can establish the desired result. Suppose x_k, h_k are the random variables representing the plant and channel states of all systems at the kth time step after each system $j = 1, \ldots, m$ employs the proposed policy $\alpha_k^j = \boldsymbol{\alpha}^{*j}(x_k^j, h_k^j)$ solving (6.38)-(6.39) . By (6.77) we get then that

$$\mathbb{E}\left[F\left(x_k^i, h_k^i, \alpha_k\right) \mid \bar{x}_k\right] \leq 0. \tag{6.78}$$

holds almost surely for all $k \geq 0$. Substituting then the definition of the function F by (6.73) in (6.78) we have that

$$\mathbb{E}[{x_k^i}^T\,(\gamma_k^i Q_c^i + (1-\gamma_k^i)Q_o^i)\,x_k^i + V^i(x_{k+1}^i, h_{k+1}^i) - V^i(x_k^i, h_k^i) \mid \bar{x}_k] \leq Tr(P^i W^i) \tag{6.79}$$

(6.80)

holds almost surely for all $k \geq 0$. By taking the expectation of the above inequality with respect to the processes x_k, h_k, α_k we have that

$$\mathbb{E}[x_k^{i\,T}\,(\gamma_k^i Q_c^i + (1-\gamma_k^i)Q_o^i)\,x_k^i] \;+\; \mathbb{E}V^i(x_{k+1}^i, h_{k+1}^i) - \mathbb{E}V^i(x_k^i, h_k^i) \leq Tr(P^i W^i) \tag{6.81}$$

holds for all $k \geq 0$. Then summing up all the above inequalities (6.81) for $k = 0, \dots, N-1$, removing the telescoping terms $\sum_{k=0}^{N-1} \mathbb{E}V^i(x_k^i, h_k^i)$, and dividing by N we get

$$\frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}[x_k^{i\,T}\,(\gamma_k^i Q_c^i + (1-\gamma_k^i)Q_o^i)\,x_k^i] + \frac{1}{N}\mathbb{E}[V^i(x_N^i, h_N^i)] \leq \frac{1}{N}\mathbb{E}V(x_0^i, h_0^i) + \mathrm{Tr}(P^i\,W^i) \tag{6.82}$$

Now since the function V is non-negative (cf.(6.72)), the term $\mathbb{E}[V^i(x_N^i, h_N^i)]$ is also non-negative and can be omitted from (6.82). Moreover $\mathbb{E}V(x_0^i, h_0^i)$ is bounded as the initial plant states have finite variance by assumption. Hence by taking the limit in (6.82) as $N \to \infty$ we conclude that

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{E}[x_k^{i\,T}\,(\gamma_k^i Q_c^i + (1-\gamma_k^i)Q_o^i)\,x_k^i] \leq \mathrm{Tr}(P^i\,W^i) \tag{6.83}$$

As established in Proposition 6.2 the value $\mathrm{Tr}(P^i\,W^i)$ is equal to the average control performance $J^i_{\text{control}}(\tilde{\boldsymbol{\alpha}})$ of the reference policy $\tilde{\boldsymbol{\alpha}}$. Hence (6.83) verifies (6.40) and completes the proof.

Proof of Lemma 6.1. First for notational convenience for any transmission functions $\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^m$ let us define the function

$$\mathbf{F}(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^{-i}) := \mathbb{E}\left[F\left(x^i, h^i, \boldsymbol{\alpha}(x,h)\right) \mid \bar{x}\right] \tag{6.84}$$

where $\boldsymbol{\alpha}^{-i}$ is the set of all transmission functions for systems $j \neq i$. To prove the lemma we need to show that

$$\mathbf{F}(\boldsymbol{\alpha}^{*i}, \boldsymbol{\alpha}^{*-i}) \leq 0. \tag{6.85}$$

Let us substitute the expression for the function F given in (6.74) in (6.84) to rewrite

$$\mathbf{F}(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^{-i}) = \mathbb{E}\left[-\left[\boldsymbol{\alpha}^i(x^i, h^i)\prod_{j\neq i}[1-\boldsymbol{\alpha}^j(x^j,h^j)] - \tilde{\boldsymbol{\alpha}}^i(h^i)\prod_{j\neq i}[1-\tilde{\boldsymbol{\alpha}}^j(h^j)]\right] q(h^i)\,x^{i\,T}\,M^i\,x^i \;\mid \bar{x}\right] \tag{6.86}$$

Now note that the plant and channel states for all systems $j \neq i$ conditioned on the current mean values $\bar{x}$ become independent among systems, because transmission functions are localized on each system. As a result we have

$$\begin{aligned}\mathbf{F}(\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^{-i}) = -\mathbb{E}&\left[\boldsymbol{\alpha}^i(x^i,h^i)\;q(h^i)\,x^{i\,T}\,M^i\,x^i \;\mid \bar{x}^i\right]\prod_{j\neq i}\mathbb{E}\left[1-\boldsymbol{\alpha}^j(x^j,h^j)\mid \bar{x}^j\right] \\ &+\mathbb{E}\left[\tilde{\boldsymbol{\alpha}}^i(h^i)\;q(h^i)\,x^{i\,T}\,M^i\,x^i \;\mid \bar{x}^i\right]\prod_{j\neq i}\mathbb{E}\left[1-\tilde{\boldsymbol{\alpha}}^j(h^j)\mid \bar{x}^j\right]\Big]\end{aligned} \tag{6.87}$$

Then note that by construction, the policies $\boldsymbol{\alpha}^{*j}$ for all systems $j \neq i$ satisfy the constraint (6.39), i.e.,

$$\mathbb{E}[\boldsymbol{\alpha}^{*j}(x^j,h^j) \mid \bar{x}^j] \leq \tilde{\alpha}^j = \mathbb{E}[\tilde{\boldsymbol{\alpha}}^j(h^j)] \tag{6.88}$$

where the last equality follows from (6.32). Since all the above quantities are probabilities, i.e., take values in $[0,1]$ we conclude that

$$-\prod_{j\neq i} \mathbb{E}\left[1-\boldsymbol{\alpha}^{*j}(x^j,h^j) \mid \bar{x}\right] \leq -\prod_{j\neq i} \mathbb{E}\left[1-\tilde{\boldsymbol{\alpha}}^j(h^j) \mid \bar{x}\right]. \tag{6.89}$$

Applying this inequality to (6.87), and since the matrix M^i is positive semidefinite by Assumption 6.1, we have that

$$\mathbf{F}(\boldsymbol{\alpha}^i,\boldsymbol{\alpha}^{*-i}) \leq \mathbf{F}(\boldsymbol{\alpha}^i,\tilde{\boldsymbol{\alpha}}^{-i}) \tag{6.90}$$

holds for any transmission function $\boldsymbol{\alpha}^i$.

Now consider the problem of minimizing $\mathbf{F}(\boldsymbol{\alpha}^i,\tilde{\boldsymbol{\alpha}}^{-i})$ over functions $\boldsymbol{\alpha}^i$ such that

$$\mathbb{E}[\boldsymbol{\alpha}^i(x^i,h^i) \mid \bar{x}^i] \leq \tilde{\alpha}^i \tag{6.91}$$

holds. By the expression (6.87) we have that the objective equals

$$\mathbf{F}(\boldsymbol{\alpha}^i,\tilde{\boldsymbol{\alpha}}^{-i}) = -\mathbb{E}\left[(\boldsymbol{\alpha}^i(x^i,h^i)-\tilde{\boldsymbol{\alpha}}^i(h^i))\, q(h^i)\, {x^i}^T M^i x^i \mid \bar{x}\right] \prod_{j\neq i}\left[1-\tilde{\alpha}^j\right] \tag{6.92}$$

This is exactly the same objective as the one appearing in the optimization problem (6.38)-(6.39), whose optimal solution is $\boldsymbol{\alpha}^{*i}$. Hence the optimal solution is the same for both problems and it satisfies

$$\mathbf{F}(\boldsymbol{\alpha}^{*i},\tilde{\boldsymbol{\alpha}}^{-i}) = \min_{\boldsymbol{\alpha}^i \text{ s.t. (6.91)}} \mathbf{F}(\boldsymbol{\alpha}^i,\tilde{\boldsymbol{\alpha}}^{-i}). \tag{6.93}$$

Now note that the reference transmission function $\tilde{\boldsymbol{\alpha}}^i$ is by construction a feasible solution to this problem, in general suboptimal, hence it must be that

$$\mathbf{F}(\boldsymbol{\alpha}^{*i},\tilde{\boldsymbol{\alpha}}^{-i}) \leq \mathbf{F}(\tilde{\boldsymbol{\alpha}}^i,\tilde{\boldsymbol{\alpha}}^{-i}) \tag{6.94}$$

Recall however that $\mathbf{F}(\tilde{\boldsymbol{\alpha}}^i,\tilde{\boldsymbol{\alpha}}^{-i}) = 0$ due to (6.76). Hence

$$\mathbf{F}(\boldsymbol{\alpha}^{*i},\tilde{\boldsymbol{\alpha}}^{-i}) \leq 0 \tag{6.95}$$

Combining (6.90) with (6.95) verifies (6.85) and completes the proof of the lemma. ■

6.4.5. Proof of Proposition 6.3

To simplify notation we will drop the index i within the proof and we will also omit the constant term $\prod_{j\neq i}(1-\tilde{\alpha}_j)$. More specifically we will prove under the same assumptions that the following optimization problem

$$\underset{\boldsymbol{\alpha}(.)\in\mathcal{A}}{\text{minimize}} \quad \mathbb{E}[-\boldsymbol{\alpha}(x,h)\, q(h,p)\, x^T M x \;\big|\; \bar{x}] \tag{6.96}$$

$$\text{subject to} \quad \mathbb{E}[\boldsymbol{\alpha}(x,h) \;\big|\; \bar{x}] \leq \tilde{\alpha}. \tag{6.97}$$

for any fixed value $\bar{x} \in \mathbb{R}^n$ has a solution of the form

$$\boldsymbol{\alpha}^*(x,h) = \begin{cases} 1, & \text{if } \; q(h,p)\, x^T M x \geq \nu \\ 0 & \text{otherwise.} \end{cases} \tag{6.98}$$

for some non-negative $\nu \geq 0$ which depends on $\bar{x}$.

We will show that problem (6.96)-(6.97) has zero duality gap and that ν corresponds to the optimal Lagrange dual variable. This is based on the results of Ribeiro (2012) where stochastic optimization problems of a similar mathematical structure are considered.

First, note that the expectations in (6.96)-(6.97) are well defined for any measurable function $\boldsymbol{\alpha} \in \mathcal{A}$. This is true for (6.97) as $\boldsymbol{\alpha}$ takes values in the finite interval $[0,1]$, and for (6.96) as the function $q(.)$ is bounded and the distribution of x has finite second moments by assumption. Second, the problem is strictly feasible. For example a constant policy $\boldsymbol{\alpha}(x,h) = \hat{\alpha}$ for all $x \in \mathbb{R}^n, h \in \mathbb{R}_+$ for some $\hat{\alpha} \in [0, \tilde{\alpha})$ strictly satisfies (5.21). Third, note that the random variables x, h have an absolutely continuous probability distribution given by the product of the plant disturbance distribution ϕ_w and the channel distribution ϕ_h which are abs. continuous by assumption.

Let us denote the optimal value of (6.96)-(6.97) by P^* and let us define the Lagrange dual problem. Consider a dual variable $\nu \in \mathbb{R}_+$. Then the Lagrangian function can be written as

$$L(\boldsymbol{\alpha}, \nu) = \mathbb{E}[-\boldsymbol{\alpha}(x,h)\, q(h,p)\, x^T M x] + \nu(\mathbb{E}[\boldsymbol{\alpha}(x,h)] - \tilde{\alpha}) \tag{6.99}$$

where for simplicity we dropped the conditioned expectation notation as $\bar{x}$ is fixed. Note then that by additivity of expectation we can rewrite the Lagrangian as

$$L(\boldsymbol{\alpha}, \nu) = \mathbb{E}[-\boldsymbol{\alpha}(x,h)\, (q(h,p)\, x^T M x - \nu)] - \nu\tilde{\alpha} \tag{6.100}$$

The Lagrange dual problem whose optimal value we denote by D^* is posed as

$$D^* = \underset{\nu\in\mathbb{R}_+}{\text{maximize}} \; \min_{\boldsymbol{\alpha}\in\mathcal{A}} L(\boldsymbol{\alpha}, \nu). \tag{6.101}$$

Then we can establish the following results.

By (Ribeiro, 2012, Theorem 1) the problem (6.96)-(5.21) has zero duality gap, i.e., $P^* = D^*$. Moreover by (Ribeiro, 2012, Theorem 4), if α^* is an optimal primal solution and ν^* is an optimal solution for the dual problem (6.101), then

$$\boldsymbol{\alpha}^* \in \underset{\boldsymbol{\alpha}\in\mathcal{A}}{\operatorname{argmin}} \; L(\boldsymbol{\alpha}, \nu^*). \tag{6.102}$$

These results hold under the abs. continuous probability measures and strict feasibility which we verified above.

The above characterization (6.102) suggests that we can recover the optimal variables α^* by just minimizing the unconstrained Lagrangian function. Given the form of the Lagrangian given in (6.100) we rewrite (6.102) as

$$\boldsymbol{\alpha}^* \in \underset{\boldsymbol{\alpha}\in\mathcal{A}}{\operatorname{argmin}} \; \mathbb{E}[-\boldsymbol{\alpha}(x,h)\,(q(h,p)\,x^T M x - \nu^*)]. \tag{6.103}$$

Moreover minimizing over the function $\boldsymbol{\alpha}$ defined for all arguments $x \in \mathbb{R}^n, h \in \mathbb{R}_+$ above is equivalent to minimizing at each argument separately, that is

$$\boldsymbol{\alpha}^*(x,h) \in \underset{\boldsymbol{\alpha}(x,h)\in[0,1]}{\operatorname{argmin}} \; -\boldsymbol{\alpha}(x,h)\,(q(h,p)\,x^T M x - \nu^*). \tag{6.104}$$

In other words we have exchanged the expectation and the minimization. This is done almost surely without loss of optimality, i.e., an optimal solution $\boldsymbol{\alpha}^*$ needs to satisfy (6.104) at all points x, h except perhaps for a measure zero set. The optimal value of $\boldsymbol{\alpha}(x,h) \in [0,1]$ at each point x, h takes value either 0 or 1 according to (6.98) given in the statement. Technically, since (6.104) is an inclusion we have to argue that the minimizers at the right hand side are almost surely unique. Indeed the set of points x, h where the minimizer is not unique is the set defined as

$$S = \{x, h : q(h,p)\,x^T M x = \nu^*\}. \tag{6.105}$$

Note then that

$$\mathbb{P}(S) = \mathbb{E}[\,\mathbb{P}[q(h,p) \,= \nu^*/x^T M x \mid x]\,] = \mathbb{E}[0] = 0. \tag{6.106}$$

That is because the function $q(h,p)$ is assumed strictly increasing in h, hence there is at most a single point h meeting the equality, and since the distribution of h is abs. continuous the integral over that set must be zero.

To sum up, we have shown that for each value $\bar{x}$ there exists a constant $\nu \geq 0$ such that the optimal solution is given by (6.98). This concludes the proof.

Chapter 7: Conclusion

The goal of this work was to design resource allocation policies for control systems implemented over wireless channels. Towards this goal, appropriate abstractions of wireless control systems were developed in Chapter 2. On the one hand this includes the mathematical description of dynamical systems involving sensors and actuators communicating between different physical locations. The main feature of these systems is that they operate on different modes, open and closed loop, depending on whether transmissions occur successfully or not. Appropriate abstractions of long term control performance are also introduced. On the other hand, the uncertainties in wireless communication are also explicitly modeled with wireless channel models. These models not only account for the randomly varying fading channel conditions, but also explicitly capture the allocation of transmit power resources. This model facilitates the analysis and design of resource allocation algorithms in wireless control systems.

The fundamental problem of co-designing transmit powers and control inputs for a wireless sensor-actuator system is considered in Chapter 3. The goal of the design is to minimize jointly average (linear quadratic) control performance and average power utilization. Separation of the two designs is made possible by a suboptimal decoupled information structure. The resulting optimal power allocation is characterized qualitatively and an opportunistic adaptation nature is revealed. Transmission should be avoided under adverse channel conditions or when the control system is under desirable conditions, while power allocation should increase as the system deviates. This introduces a novel paradigm for operating power-constrained wireless control systems. Related publications include Gatsis et al. (2013b, 2014c). Preliminary extensions to the problem of managing receiver power resources are presented in Gatsis et al. (2013a).

Channel-aware schedulers for wireless control systems with multiple loops closing over a shared wireless medium are considered in Chapter 4. This is posed as a constrained stochastic optimization problem where desired control performance requirements need to be guaranteed for each system, while the overall transmit power utilization is to be minimized. The structure of the optimal solution is characterized. The scheduler opportunistically decides one plant to close the loop at each time step, while transmit powers can be decoupled among the different sensors. We develop an offline optimization algorithm to solve the problem, as well as an online communication algorithm that converges to the optimal performance using only random observed channel sequences. Related publications include Gatsis et al. (2014a, 2015a). An extension of the approach to scheduling inter-dependent control tasks is considered in Gatsis et al. (2014b).

With decentralized channel access algorithms, sensors can independently decide whether to access the shared medium, without the need of a centralized scheduler. However the emerging wireless interferences cause control performance degradation, hence need to be mitigated. In Chapter 5 we develop a random access mechanism for control systems. The proposed policies are decoupled among systems and balance the control performance gains from transmissions with the losses due to interferences. Related publications include Gatsis et al. (2015c, 2016a). A different algorithm based on a game-theoretic approach is considered in Gatsis et al. (2015b) allowing the sensors to learn their policies in a decentralized fashion.

Finally in Chapter 6 we design dynamic sensor transmission policies adapting online to the varying control system state as well as the wireless channel state. Based on approximate dynamic programming the proposed policies come with improved control performance and resource utilization, as compared to simple policies used as a reference. The approach is illustrated in the problem of transmit power allocation, in the problem of scheduling, as well as in the problem of decentralized channel access. The latter case is particularly complex as the sensor policies are coupled globally over the shared medium but are allowed to adapt only to locally available information. Related publications include Gatsis et al. (2016b,c).

7.1. Open Problems and Future Research Directions

The future of modern smart infrastructures entails large numbers of interconnected devices in interaction with the physical world. As the control inputs applied to the system by actuators rely on wirelessly received data collected by sensors, the joint design of efficient control and wireless communication policies remains an important challenge. Despite recent advances towards this end (e.g. Gatsis et al. (2014c); Nayyar et al. (2013)), the determination of appropriate information structures as well as computationally tractable policy design algorithms requires further consideration.

The efficient allocation of communication resources, as in the problems examined in this work, needs to account for a large number of interconnected closed loop systems. To enable such a large scale design appropriate optimization procedures need to be developed, for example, distributed optimization algorithms and efficient spectrum management. As wireless sensors and actuators become available to the rest of the system in a time varying fashion, i.e., randomly connect or disconnect, the communication algorithms need to respond and adapt online to these changes and maintain efficiency.

As control systems change dynamically over time, opportunistically adapted mechanisms such as those presented in Chapter 6 become apparent. These mechanisms can respond to sensor measurements that are critical for the performance of the system, however determining when such critical measurements take place is challenging because of the large scale nature of the systems and the fact that information is only locally available. Moreover, to achieve efficient utilization of communication resources over a long planning horizon, appropriate abstractions of control performance need to be taken into account.

Furthermore, with increased connectivity an increased number of security concerns arise. Sensor measurements may not always be desired to be shared with other connected systems due to privacy concerns. In that case wireless mechanisms need not only account for reliable communication but also safe communication, e.g., in the presence of eavesdroppers. Alternatively if malicious agents are able to intrude and inject false measurements over the communication channel, it is essential to determine defense/controller strategies that would maintain control system integrity.

Appendix A: Imperfect Channel State Information

According to the wireless communication model of Chapter 2 the sensor/transmitter relies on channel state information to decide whether to access the channel to transmit and close the loop. Perfect channel state information is difficult to acquire at the transmitter side in practice, hence some channel estimation procedure needs to be followed – see, e.g., Hassibi and Hochwald (2003) for a discussion. In this section we illustrate how our developments can be adapted in this case.

A common model for imperfect channel state information in the literature is that *channel outage* occurs when the attempted transmission rate is higher than the maximum rate supported by the current channel conditions (Vakili et al. (2006); Zheng et al. (2008); Ouyang et al. (2010); Hu and Ribeiro (2013)). More specifically, suppose at some time step the channel state is given by $h_k \in \mathbb{R}_+$ while the sensor only knows a channel estimate $\hat{h}_k \in \mathbb{R}_+$. We do not examine a specific channel estimation procedure but instead characterize the estimation quality by a conditional probability distribution $\mathbb{P}(h \,|\, \hat{h})$.

The current channel state h_k can support a packet success rate described by the function $q(h_k\, p)$ depending on received power level $h_k p$ as described in Section 2.2. To achieve this rate however the transmitter needs to select a code appropriately adapted to the channel state h_k which is not perfectly known. Suppose then that the sensor attempts to transmit at the *estimated packet success rate* $q(\hat{h}_k\, p)$ instead. The probability of successfully receiving the message is then modeled as

$$\mathbb{P}(\gamma_k = 1 \,|\, h_k, \hat{h}_k) = \begin{cases} q(\hat{h}_k\, p) & \text{if } \hat{h}_k \leq h_k, \\ 0 & \text{otherwise} \end{cases} \tag{A.1}$$

That is, the rate equals the attempted rate if the channel fading gain is higher than the estimated gain, otherwise a channel outage occurs and the rate is zero. Depending on how reliable the channel estimation procedure is, the channel outage may result in significant performance losses.

A commonly proposed solution to mitigate this effect of channel outage is to employ a *backoff* rate. Given the current channel estimate the transmitter can attempt to communicate at a lower rate than the estimated rate to protect against the case where the actual channel state is worse than the estimated one. Instead of transmitting at the estimate rate $q(\hat{h}_k\, p)$ the sensor can alternatively select a backoff rate, or equivalently a backoff channel gain $b(\hat{h}_k)$ depending on the current channel estimate. The transmitter then attempts the backoff packet success rate $q(b(\hat{h}_k)\, p)$. The resulting probability of successful transmission then becomes

$$\mathbb{P}(\gamma_k = 1 \,|\, h_k, \hat{h}_k) = \begin{cases} q(b(\hat{h}_k)\, p) & \text{if } b(\hat{h}_k) \leq h_k, \\ 0 & \text{otherwise} \end{cases} \tag{A.2}$$

Integrating (A.2) with respect to the estimation quality model $\mathbb{P}(h_k \,|\, \hat{h}_k)$ we conclude that

$$\mathbb{P}(\gamma_k = 1 \,|\, \hat{h}_k) = q(b(\hat{h}_k))\; \mathbb{P}(\, b(\hat{h}_k) \leq h_k \,|\, \hat{h}_k). \tag{A.3}$$

The backoff channel rate $b(\hat{h}_k)$ can then be selected at each channel estimate $\hat{h}_k$ in order to maximize the probability of success given in the above expression. There is a tradeoff in this design. Low backoff rates imply fewer outages, as the second term in the product above increases, but also lower achieved rates, as the first term decreases. Typically backoff channels/rates are selected lower than the channel estimates, $b(\hat{h}_k) \leq \hat{h}_k$, so that the outage event $b(\hat{h}_k) > h_k$ happens with lower probability (Hu and Ribeiro (2013)). After a backoff mechanism is selected, we can denote (A.3) as a general function of the form

$$\mathbb{P}(\gamma_k = 1 \,\big|\, \hat{h}_k) = \hat{q}(\hat{h}_k). \tag{A.4}$$

The policies based on channel states presented in this work are still valid by replacing the general channel model (2.17) with the modified one in (A.4) expressed with respect to channel estimates. For example, consider a sensor obtaining channel state estimates $\hat{h}_k, k \geq 0$ which are i.i.d. over time and deciding whether to communicate over the channel in order to close its loop. Let us indicate this choice by a variable $\alpha_k \in [0, 1]$. Similar to the development of rollout policies (6.12), (6.28) in Sections 6.1, 6.2 we can propose the following policy

$$\alpha_k = \operatorname*{argmin}_{\alpha \in [0,1]} -\alpha \, \hat{q}(\hat{h}_k) \, x_k^T M x_k + \nu \, \alpha. \tag{A.5}$$

Here $\nu \geq 0$ is a non-negative constant parameter and the matrix M corresponds to the relative value-of-transmitting the current state x_k, as introduced in (6.11). Due to linearity of the objective the solution α_k to (A.5) takes values either 1 or 0, transmit or not. The sensor transmits when the expected value of transmission $\hat{q}(\hat{h}_k) \, x_k^T M x_k$ exceeds the parameter ν. We note that here the expected value of transmission takes into account the possible current channel state values given the current channel state estimate.

Appendix B: Control over Markov Fading Channels

In this section we briefly present how our resource allocation approach needs to be modified in cases where the channel fading conditions between sensors and actuators are not independent over time but follow instead a Markov process (Mushkin and Bar-David (1989); Ahmad et al. (2009); Ouyang et al. (2012)). To simplify the exposition suppose the channel can take one of two states, h_0, h_1 and the transition probability between states h_i and h_j is denoted as p_{ij}, for $i, j \in \{1, 2\}$. The resulting Markov chain is assumed to be irreducible and aperiodic so that it possesses a stationary distribution. As introduced in 2.2 the probability of successful transmission at a channel state h_i when transmitting with power p is modeled by the relationship $q(h_i, p)$.

Consider now the problem of optimal power allocation according to Chapter 3 where we are interested in minimizing the objective

$$\limsup_{N\to+\infty} \frac{1}{N}\, \mathbb{E} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \; + \lambda p_k. \tag{B.1}$$

Assuming that an optimal solution exists and that it technically satisfies the Bellman equation, as is the case in Theorem 3.2, then we can characterize its form as follows. When the current plant conditions are given by $\varepsilon_k \in \mathbb{R}^n$ (the innovation term in (3.11)) and the current channel conditions are given by $h_k \in \{h_0, h_1\}$, the optimal transmit power at the sensor is given by

$$p_k^* = \operatorname*{argmin}_{p\in[0,p_{\max}]} \; \lambda p - \, q(h_k, p)\, R(\varepsilon_k, h_k) \tag{B.2}$$

for some function $R(\varepsilon, h)$. Comparing this with the optimal power allocation for the i.i.d. channel case presented in (3.34) we see that here the channel h_k structurally plays a dual role. The current channel state h_k not only shows how easy it is to transmit at the current time step (by the term $q(h_k, p)$) but also reveals information about future gains (by the term $R(\varepsilon_k, h_k)$) since the channel state at the next time step is correlated.

We also briefly present how the Markovian channel structure affects the design of rollout power allocation policies as presented in Section 6.1. To begin with, suppose a reference policy is available which selects a power level $\tilde{p}_i \in [0, p_{\max}]$ under channel conditions h_i, for $i = 0, 1$. We denote the corresponding packet success probabilities at the two possible channel states according to the reference policy as

$$\tilde{q}_i := q(h_i, \tilde{p}_i), \quad i = 0, 1. \tag{B.3}$$

Suppose then the control system dynamics are given by

$$x_{k+1} = \begin{cases} A_c\, x_k + w_k, & \text{if } \gamma_k = 1 \\ A_o\, x_k + w_k, & \text{if } \gamma_k = 0 \end{cases} \tag{B.4}$$

as in Section 6.1. Also for simplicity suppose control system states are penalized at each time step by a quadratic cost of the form $x_k^T Q x_k$, i.e., the cost does not depend on the current packet success (cf. (6.3)).

Similar to (6.9), we first solve for matrices corresponding to the future control cost-to-go of the reference policy. Due to the Markovian structure, here we have to actually solve for a set of matrices $\{P_i, i = 0, 1\}$, one for each possible channel state value. Intuitively that is because the future expected control cost depends the current channel state. In particular one needs to solve

$$P_i = Q + \sum_{j=0,1} p_{ij} \left\{ \tilde{q}_j A_c^T P_j A_c + (1 - \tilde{q}_j) A_o^T P_j A_o \right\}, \text{ for } i = 0, 1. \tag{B.5}$$

These are a set of coupled linear matrix equalities that have to be jointly solved, e.g., via the vectorization approach in Remark 6.2.

Given the above reference policy, we can construct rollout power allocation policies that improve upon the reference performance, similar to the approach in Section 6.1. It turns out that the power allocation takes the following form. If the current plant state is $h_k = h_i$, for $i = 0, 1$ and current control system state is x_k, the transmit power according to the rollout policy becomes

$$p_k^{\text{roll}} = \underset{p \in [0, p_{\max}]}{\operatorname{argmin}} \ \nu p - q(h_i, p)\, x_k^T (A_o^T P_i A_o - A_c^T P_i A_c) x_k \tag{B.6}$$

where $\nu \geq 0$ is a scalar parameter. This power choice is in general of the same form as in the i.i.d. channel state (cf. (6.12)). A closer look reveals again the dual role of the current channel state as it appears both in the term $q(h_i, p)$ and in the quadratic form $x_k^T (A_o^T P_i A_o - A_c^T P_i A_c) x_k$. The latter quadratic form can be interpreted as the value of transmitting under the current control state x_k and the current channel state $h_k = h_i$. The reason for the latter is that the current channel conditions reveal information about how easy it will be to close the loop in the future, as channel states are coupled over time in a Markovian fashion. Hence current channel conditions are taken into account in evaluating future control performance gains.

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