2017

Omnichannel Operations Management

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Abstract
This dissertation studies how a firm could effectively make use of different selling channels to provide consumers with a seamless shopping experience. In the three essays, by analyzing stylized models where firms operates both online and offline channels and consumers strategically make channel choices, we examine the impacts of different types of omnichannel strategies in different industries. In the first essay, we focus on a specific omnichannel fulfillment strategy, i.e., buy-online-and-pick-up-in-store (BOPS). We find it may not be profitable to implement BOPS on products that sell well in stores. We also consider a decentralized retail system where store and online channels are managed separately, and find it is rarely efficient to allocate all BOPS revenue to a single channel. In the second essay, we study how retailers can effectively deliver product and inventory information to omnichannel consumers who strategically choose whether to gather information online/offline and whether to buy products online/offline. Specifically, we consider three information mechanisms: physical showrooms, virtual showrooms, and availability information. Our main result is that these information mechanisms may sometimes change customers' channel choice in a way such that total product returns increase and total retail profit decreases. In the third essay, we look at the restaurant industry. Specifically, we study the impacts of different self-order technologies on service operations. Online technology, through websites and mobile apps, allows customers to order and pay before coming to the store; offline technology, such as self-service kiosks, allows store customers to place orders without interacting with a human employee. We develop a stylized queueing model and study the impacts of self-order technologies on customer demand, employment levels, and restaurant profits. We find there could be a win-win-win situation, where everyone in the market, i.e., consumers (including those who do not use the technology), workers and the firm, could benefit from the implementation of the self-order technologies.

Degree Type
Dissertation

Degree Name
Doctor of Philosophy (PhD)

Graduate Group
Operations & Information Management

First Advisor
Xuanming Su

Keywords
Consumer Behavior, Market-Operations Interface, Omnichannel, Operations Management

OMNICHANNEL OPERATIONS MANAGEMENT

Fei Gao

A DISSERTATION

in

Operations, Information and Decisions

For the Graduate Group in Managerial Science and Applied Economics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2017

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OMNICHANNEL OPERATIONS MANAGEMENT

ACKNOWLEDGEMENT

First and foremost, I would like to express my sincere gratitude to my advisor, Professor Xuanming Su, for his guidance and support. I am truly fortunate to work with him. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my PhD study.

I am also extremely grateful to Professors Gerard Cachon and Morris Cohen, who are also on my dissertation committee, for their detailed and insightful comments on my dissertation. Moreover, I really appreciate their generous support and advice during my job market process.

Finally, and most importantly, I would like to thank my parents, Hong Liang and Jianshe Gao, for their support, encouragement and unwavering love throughout my life.

ABSTRACT

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Xuanming Su

This dissertation studies how a firm could effectively make use of different selling channels to provide consumers with a seamless shopping experience. In the three essays, by analyzing stylized models where firms operates both online and offline channels and consumers strategically make channel choices, we examine the impacts of different types of omnichannel strategies in different industries. In the first essay, we focus on a specific omnichannel fulfillment strategy, i.e., buy-online-and-pick-up-in-store (BOPS). We find it may not be profitable to implement BOPS on products that sell well in stores. We also consider a decentralized retail system where store and online channels are managed separately, and find it is rarely efficient to allocate all BOPS revenue to a single channel. In the second essay, we study how retailers can effectively deliver product and inventory information to omnichannel consumers who strategically choose whether to gather information online/offline and whether to buy products online/offline. Specifically, we consider three information mechanisms: physical showrooms, virtual showrooms, and availability information. Our main result is that these information mechanisms may sometimes change customers channel choice in a way such that total product returns increase and total retail profit decreases. In the third essay, we look at the restaurant industry. Specifically, we study the impacts of different self-order technologies on service operations. Online technology, through websites and mobile apps, allows customers to order and pay before coming to the store; offline technology, such as self-service kiosks, allows store customers to place orders without interacting with a human employee. We develop a stylized queueing model and study the impacts of self-order technologies on customer demand, employment levels, and restaurant profits. We find there could be a win-win-win situation, where everyone in the market, i.e., consumers

(including those who do not use the technology), workers and the firm, could benefit from the implementation of the self-order technologies.

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CHAPTER 1 : Introduction

The general topic of this dissertation is omnichannel operations management. Nowadays, firms have many different channels to reach their customers. Instead of treating different channels as independent silos, more and more firms realize the need to integrate different channels to provide customers with a seamless shopping experience. This dissertation studies the impacts of various omnichannel strategies on consumer behavior and firm operational efficiency through three essays.

In the first essay, we look at a specific omnichannel fulfillment strategy in the retail industry. Many retailers have recently started to offer customers the option to buy online and pick up in store (BOPS). We study the impact of the BOPS initiative on store operations. We build a stylized model where a retailer operates both online and offline channels. Consumers strategically make channel choices. The BOPS option affects consumer choice in two ways: by providing real-time information about inventory availability and by reducing the hassle cost of shopping. We obtain three findings. First, not all products are well-suited for in-store pickup; specifically, it may not be profitable to implement BOPS on products that sell well in stores. Second, BOPS enables retailers to reach new customers, but for existing customers, the shift from online fulfillment to store fulfillment may decrease profit margins when the latter is less cost effective. Finally, in a decentralized retail system where store and online channels are managed separately, BOPS revenue can be shared across channels to alleviate incentive conflicts; it is rarely efficient to allocate all the revenue to a single channel.

In the second essay, we study how retailers can effectively deliver online and offline information to omnichannel consumers who strategically choose whether to gather information online/offline and whether to buy products online/offline. Information resolves two types of uncertainty: product value uncertainty (i.e., consumers realize valuations when they inspect the product in store, but may end up returning the product when they purchase online) and availability uncertainty (i.e., store visits are futile when consumers encounter stockouts). We consider three information mechanisms: *physical showrooms* allow consumers to learn valuations anytime they visit the store, even during stockouts; *virtual showrooms* give consumers online access to an imperfect signal of their valuations; *availability information* provides real-time information about whether the store is in stock. Our main results follow. First, physical showrooms may prompt retailers to reduce store inventory, which increases availability risk and discourages store patronage. Second, virtual showrooms may increase online returns and hurt profits, if they induce excessive customer migration from store to online channels. Third, availability information may be redundant when availability risk is low, and may render physical showrooms ineffective when implemented jointly. Finally, when customers are homogeneous, these mechanisms may not exhibit significant complementarities and the optimal information structure may involve choosing only one of the three.

In the third essay, we shift focus to restaurant industry. Many restaurants have recently implemented self-order technologies across both online and offline channels. Online technology, through websites and mobile apps, allows customers to order and pay before coming to the store; offline technology, such as self-service kiosks, allows store customers to place orders without interacting with a human employee. In this essay, we develop a stylized theoretical model to study the impact of self-order technologies on customer demand, employment levels, and restaurant profits. Our main results follow. First, customers using self- order technologies experience reduced waiting cost and increased demand, and moreover, these benefits may even carry over to customers who do not use these technologies. Second, although public opinion suggests that self-order technologies facilitate job cuts, we find instead that some firms should increase employment levels, and paradoxically, this recommendation holds for firms with high labor costs. Finally, we find that firms should implement online (offline) self-order technology when customers have high (low) wait sensitivity.

CHAPTER 2 : Omnichannel Retail Operations with Buy-Online-and-Pick-up-in-Store

2.1. Introduction

As consumers become accustomed to online shopping, brick-and-mortar retailers have increasingly supplemented their shops with online businesses (Financial Times, 2013). The online channel has traditionally been viewed as a separate way to sell products. Today, however, many retailers have realized the need to integrate their existing channels to enrich customer value proposition and improve operational efficiency. As a result, there is an emerging focus on "omnichannel retailing" with the goal of providing consumers with a seamless shopping experience through all available shopping channels (Bell et al., 2014; Brynjolfsson et al., 2013; Rigby, 2011). When asked about omnichannel priorities, the retailers surveyed by Forrester Research reported that fulfillment initiatives ranked higher than any other channel integration program; moreover, among all omnichannel fulfillment initiatives, allowing customers to buy online and pick up in store (BOPS) is regarded as the most important one (Forrester, 2014). According to Retail Systems Research (RSR), as of June 2013, 64% of retailers have implemented BOPS (RSR, 2013).

Retailers benefit from allowing customers to pick up their online orders in store. Specifically, BOPS generates store traffic and potentially increases sales (New York Times, 2011). According to a recent UPS study, among those who have used an in-store pickup option, 45% of them have made a new purchase when picking up the purchase in store (UPS, 2015). Typically, a substantial amount of store sales is generated through such cross-selling: it is estimated that, on average, when a customer comes to the store intending to buy \$100 worth of merchandise, they leave with \$120 to \$125 worth of merchandise (Washington Post, 2015). Thus, unsurprisingly, more and more retailers are starting to offer the BOPS functionality on their websites (RSR, 2013).

A key challenge facing retailers is to choose the right set of products for BOPS. Most retailers generally agree that BOPS should not be a blanket functionality that is blindly applied to all products across all categories. According to senior vice president and general manager of Walmart.com, Steve Nave, one of the reasons for being selective is to focus on those products that would "drive more customers into the stores" (Time, 2011). On the websites of major retailers such as Toys R Us, customers will find that some products are not eligible for BOPS. For example, new releases such as the LEGO Star Wars Sith Infiltrator are not available for store pickup. These items are sold in stores, but shoppers need to check their local stores for availability if they do not wish to wait for online delivery. In contrast, for most of the extensive line of LEGO products sold on toysrus.com, the BOPS option is available. Since retailers typically carry large numbers of SKUs online, a key challenge is to understand the main criteria for selecting which product to allow for in-store pickup.

Many retailers regard BOPS as a way to reach new customers, as this new fulfillment option has become increasingly popular among shoppers (New York Times, 2011). With BOPS, consumers experience instant gratification, avoid shipping and delivery changes, and enjoy the convenience of hassle-free shopping (their items have already been picked and packed by store staff by the time they arrive). With this unique combination that has never been offered before, it is not unrealistic for retailers to expect market expansion. However, another more pessimistic view is that BOPS simply shifts customers from online fulfillment to store fulfillment; customers who use BOPS would have purchased online anyway. To understand the impact of BOPS on retailers' bottom-lines, it is useful to distinguish between the demand cannibalization and demand creation scenarios described above.

The advent of BOPS blurs the distinction between store and online operations. Although BOPS orders originate online, they are fulfilled using inventory in retail stores. Consequently, a successful BOPS implementation requires good coordination between the online and offline channels. Very few retailers have completely dismantled their online and offline channel silos, maintaining a single accounting ledger with an associated organizational structure for all sales regardless of channel (Forrester, 2014). When online and offline channels are operated by separate teams, the company needs to decide how to allocate BOPS revenue. On this subject, there is no consensus: 46% of retailers allow the online channel to receive full credit for the transaction, 31% award full credit to the store channel, and 23% divide them between channels (Forrester, 2014). This lack of consensus is not surprising, since both channels have legitimate reasons to claim credit. The store incurs operating costs for fulfilling demand, while the online channel is the source of demand in the first place.

In this chapter, we focus on the following research questions:

1. For what types of product will the BOPS option be profitable?

2. How does BOPS impact the retailer's customer base?

3. How should BOPS revenue be allocated between store and online channels?

To address these questions, we develop a stylized model that captures essential elements of omnichannel retail environments. There is a retailer who operates online and store channels, with the goal of maximizing total expected profit over both channels. Consumers strategically choose among buying online, buying in store, and buying online for store pickup, to maximize individual utility. We first analyze the centralized system; specifically, for a particular product with given financial parameters, we study optimal inventory decisions under BOPS and examine the impact of BOPS on total profits. Using these results, we determine whether a product should be carried in store and whether BOPS should be offered. Finally, we consider the decentralized system and examine how to allocate BOPS revenue between channels.

Our first main finding is that BOPS may not be suitable for all products. Specifically, for products that are bestsellers in retail stores, the benefit of BOPS may be outweighed by the drawback, which is as follows. Since products that are available for store-pickup must be in stock, BOPS indirectly discloses real-time store inventory status. Customers initiating a BOPS order online and finding that the desired item is out of stock will not visit the store.

In this way, stockouts of blockbuster products may drive customers away, thus reducing store traffic and cross-selling opportunities. In other words, BOPS may compromise the function of bestselling products in attracting customers to the store.

Our second result is that BOPS helps retailers expand their market coverage. As BOPS mitigates the stockout risk and hassle costs during the shopping journey, more people will be willing to consider buying from the retailer. However, apart from attracting new customers, BOPS can also sway the channel choices of existing customers. Among these existing customers, some who had waited for their orders to be shipped to them (i.e., online fulfillment) may now choose to pick up their orders (i.e., store fulfillment) instead. This shift is unprofitable if the profit margin is lower in stores compared to the online channel.

Finally, in decentralized systems where store and online channels are operated by separate entities, we identify a misalignment of incentives. Specifically, the store neglects the fact that potential BOPS customers may purchase online instead when there is no stock in the store for them to pick up. Online sales generate value for the company but the store is not explicitly compensated when they occur. Consequently, the store stocks too much inventory if they retain 100% of BOPS revenue. To correct for this potential incentive problem, it is optimal to give the store channel partial credit for fulfilling BOPS demand.

2.2. Literature Review

This chapter studies the management of online and offline channels. With the advent of e-commerce around the turn of the century, many manufacturers or suppliers introduced a direct online selling channel, which competes with their own retail partners. Much of the literature on channel management studies this type of business setting. Chiang et al. (2003) study a price setting game between a manufacturer and its independent retailer. They find the manufacturer is more profitable even if no sales occur in the direct channel, because the manufacturer can use the direct channel to improve the functioning of the retail channel by preventing the prices from being too high and thus leading to more sales or orders from the retailer. Some other papers also study the pricing game, but are more concerned with specific pricing mechanisms, e.g., price matching between channels (Cattani et al., 2006) and personalized pricing (Liu and Zhang, 2006). Apart from pricing, Tsay and Agrawal (2004) consider firms' sales effort and find that both parties can benefit from the addition of a direct channel. Chen et al. (2008) study service competition between the two channels and characterize the optimal channel strategy for the supplier. Netessine and Rudi (2006) study the practice of drop-shipping, where the supplier stocks and owns the inventory and ships products directly to customers at retailers' request. In contrast to this stream of work, we focus on the case where a single retailer manages both online and store channels, as commonly observed in retail environments today.

Omnichannel management has received a lot of attention in industry; the topic is broadly surveyed in Bell et al. (2014), Brynjolfsson et al. (2013) and Rigby (2011). In addition, there are a few other papers. Ansari et al. (2008) empirically study how customers migrate between channels in a multichannel environment and the role of marketers in shaping migration through their communications strategy. Chintagunta et al. (2012) study consumer channel choice in grocery stores and empirically quantify the relative transaction costs when households choose between the online and offline channels. Ofek et al. (2011) focus on the impact of product returns on a multichannel retailer, and use a theoretical model to examine how pricing strategies and physical store assistance levels change as a result of the additional online outlet. Gallino and Moreno (2014) empirically investigate the impact of BOPS on a retailer's sales in both online and offline channels. Interestingly, they find that instead of increasing online sales, the implementation of BOPS is associated with a reduction in online sales and an increase in store sales and traffic. While most papers in this area are empirical, we develop a tractable theoretical framework. Using our model, we study omnichannel inventory management and channel coordination within the firm.

There are models in operations management that study the role of inventory availability on demand, given that consumers have to incur hassle costs and bear the stockout risk when visiting stores. Dana and Petruzzi (2001) is the first paper that extends the classic newsvendor model by assuming demand is a function of both price and inventory level. Since then, many researchers have investigated how to attract people to pay the hassle cost to come to the store: Su and Zhang (2009) show that it is always beneficial to the retailer if he can credibly make an ex ante quantity commitment. Yin et al. (2009) compare the efficacy of two different in-store inventory display formats to manipulate consumer expectations on the availability. Allon and Bassamboo (2011) explore the issue of cheap talk when the information shared by the retailer is not verifiable. Alexandrov and Lariviere (2012) examines the role of reservations in the context of revenue management. Cachon and Feldman (2015) focus on retailer's pricing issue and find that a strategy that embraces frequent discounts is optimal. In this chapter, we study a new way to attract customers to store, BOPS, by which the retailer shares with customers the real-time information about store inventory status.

A critical feature of our model is that customers will make additional purchases once they enter the store. There are many papers in marketing (e.g., Li et al. (2005); Akçura and Srinivasan (2005); Li et al. (2011)) and operations management (e.g., Netessine et al. (2006); Gurvich et al. (2009); Armony and Gurvich (2010)) examining how to make use of the cross-selling opportunities during the interaction with consumers. In this chapter, instead of studying the design of a cross-selling strategy, we will focus on the impact of cross-selling benefits on the implementation of the new omnichannel fulfillment strategy, i.e., BOPS.

This chapter provides a different angle to the stream of work on strategic customer behavior in retail management. Su (2007) study a dynamic pricing problem with a heterogeneous population of strategic as well as myopic customers and show that optimal price paths could involve either markups or markdowns. Aviv and Pazgal (2008) study two types of markdown pricing policies (i.e., contingent and announced fixed-discount) in the presence of strategic consumers. There is a rich body of work on operational strategies that consider strategic customers: e.g., capacity rationing (Liu and van Ryzin, 2008), supply chain contracting

(Su and Zhang, 2008), quick response (Cachon and Swinney, 2009), opaque selling (Jerath et al., 2010), posterior price matching (Lai et al., 2010) and product rollovers (Liang et al., 2014). Most of the existing literature consider models in which consumers decide whether to make an immediate purchase or to wait for future discounts. In contrast, we concentrate on consumers' channel choice. In other words, instead of studying the decision of *when* to buy, we pay attention to the decision of *where* to buy.

The topic of decentralization has been studied by many researchers in the operations management area. There are two streams of literature, focusing on *interfirm* and *intrafirm* coordination. For the former, there is a large body of literature on the design of optimal contracts to optimize supply chain performance; see, for example, Lee and Whang (1999); Taylor (2002); Cachon and Lariviere (2005). Readers can refer to Cachon (2003) for a comprehensive review. The other research stream addresses conflicts of interest within a firm. Harris et al. (1982) study an intrafirm resource allocation problem where different divisional managers of the firm possess private information that is not available to the headquarters. Porteus and Whang (1991) examine different incentive plans to coordinate the marketing and manufacturing managers of the firm. A similar cross-functional coordination problem is also studied by Kouvelis and Lariviere (2000). In retail assortment planning, Cachon and Kök (2007) study category management, where each category is managed separately by different managers. In line with this research stream, we look at incentive conflicts between the store channel and the headquarters of an omnichannel retailer.

2.3. Model

There is a retailer who sells a product through two channels, store and online, at price p. In the store channel, the retailer faces a newsvendor problem: there is a single inventory decision q to be made before random demand is realized, so there may eventually be unmet demand or leftovers in the store. The unit cost of inventory is c, and the salvage value of leftover units is normalized to zero. The online channel is modeled exogenously: the retailer simply obtains a net profit margin w from each unit of online demand. This model focuses on store operations and can be separately applied to any particular product the retailer carries.

The market demand D is random and follows a continuous distribution F and density f. Consumers choose between store and online channels to maximize their utility. Each individual consumer has valuation v for the product. When shopping in store, each consumer incurs hassle cost h_s (e.g., traveling to the store or searching for the product in aisles); similarly, when shopping online, each consumer incurs hassle cost h_o (e.g., paying shipping fees or waiting for the product to arrive). There is a key difference between store and online hassle costs: h_s is incurred *before* customers find and purchase the product in the store, whereas h_o is incurred *after* customers make the purchase online. To ensure that consumers are willing to consider both channels, we assume that both hassle costs are smaller than the surplus $v - p$.

We first consider the scenario before BOPS is introduced; here, each consumer makes a choice between shopping online directly or going to the store. If she chooses to buy online directly, her payoff is simply given by

$$u_o = v - p - h_o.$$

On the other hand, if she chooses to go to the store, her payoff is

$$u_s = -h_s + \hat{\xi}(v - p) + (1 - \hat{\xi})(v - p - h_o).$$

To understand this expression, note that the consumer first incurs the hassle cost h_s upfront. Then, once she is in the store, she may encounter two possible outcomes: (1) if the store has inventory, then she can make a purchase on the spot and receive payoff $v - p$; (2) if the store is out of stock, she can go back to buying the product online and receive payoff $v - p - h_o$. The consumer expects the former to occur with probability $\hat{\xi}$. Based on this belief, the consumer compares the expected utility from each channel and chooses accordingly.

Next, we consider retailer's decision problem. First, the retailer anticipates that a fraction $\hat{\phi} \in [0, 1]$ of customers will visit the store; i.e., if total demand is D, the retailer expects that the number of customers coming to the store will be $\hat{\phi} D$. Given this belief, the retailer's profit function is

$$\pi(q) = pE \min\left(\hat{\phi} D, q\right) - cq + rE\left(\hat{\phi} D\right) + wE\left(\left(1 - \hat{\phi}\right) D\right) + wE\left(\hat{\phi} D - q\right)^{+}. \tag{2.1}$$

Given the store inventory level q, the newsvendor expected profit from selling the product in the store channel is shown in the first two terms above. In addition, since customers tend to make additional purchase when they come to the store (UPS, 2015; Washington Post, 2015), there is an additional profit r from every customer coming to the store; this is the third term in the profit function above. The last two terms above show the retailer's online profit, the fourth represents profit from customers who shop online directly and the last represents profit from customers who switch to online after encountering stockouts in store. With the profit function above, the retailer chooses q to maximize expected profit.

To study the strategic interaction between the retailer and the consumers, we shall use the notion of rational expectations (RE) equilibrium (see Su and Zhang (2008, 2009); Cachon and Swinney (2009)). One important feature of a RE equilibrium is that beliefs must be consistent with actual outcomes. In other words, the retailer's belief $\hat{\phi}$ must coincide with the true proportion ϕ of consumers choosing the store channel, and consumers' beliefs over in-store inventory availability probability $\hat{\xi}$ must agree with the actual in-stock probability corresponding to the retailer's chosen quantity q. According to Deneckere and Peck (1995) and Dana (2001), this probability is given by $A(q) = E \min(\phi D, q)/E(\phi D)$ where $\phi > 0$. The reason is as follows. Conditional on her own presence in the market, an individual consumer's posterior demand density is $g(x) = xf(x)/ED$. Therefore, given this posterior demand density, the availability probability is $\int \left(\min\left(\phi x, q\right)/\phi x\right) g\left(x\right) dx = A\left(q\right)$, since the product is available with probability $\min(\phi x, q)/\phi x$ when there are x consumers in the market. Then, we have the following definition for a RE equilibrium. Henceforth, we refer to the RE equilibrium as "equilibrium" for brevity.

Definition 1. *A RE equilibrium $(\phi, q, \hat{\xi}, \hat{\phi})$ satisfies the following:*

i Given $\hat{\xi}$, if $u_s \geq u_o$, then $\phi = 1$; otherwise $\phi = 0$;

ii. Given $\hat{\phi}$, $q = \arg\max_q \pi(q)$, where $\pi(q)$ is given in (2.1);

iii. $\hat{\xi} = A(q)$;

iv. $\hat{\phi} = \phi$.

Conditions (i) and (ii) state that under beliefs $\hat{\xi}$ and $\hat{\phi}$, consumers and the retailer are choosing the optimal decisions. Conditions (iii) and (iv) are the consistency conditions.

First, it is easy to see that there always exists a *nonparticipatory* equilibrium $(0, 0, 0, 0)$. If the retailer expects no one comes to the store to buy the product, he will stock nothing there, i.e., $q = 0$; If consumers believe there is no inventory in the store, they will not come, i.e., $\phi = 0$. In the end, we have a self-fulfilling prophecy and beliefs are trivially consistent with the actual outcome.

Is there any *participatory* equilibrium where consumers are willing to visit store and the retailer has stock in the store as well (i.e., $\phi = 1$ and $q > 0$)? When such an equilibrium exists, it Pareto-dominates the nonparticipatory equilibrium, because it generates positive payoffs for both the retailer and consumers. We shall adopt the Pareto dominance equilibrium selection rule. The following proposition gives the equilibrium result; we use the superscript $(\cdot^\circ)$ to denote the equilibrium outcome for this basic scenario. All proofs in this dissertation are presented in Appendix A.7.

Proposition 1. *If $h_s \leq \xi^\circ \cdot h_o$ and $p - c > w$, then customers visit store and $q^\circ = \bar{F}^{-1}(\frac{c}{p-w})$. Otherwise, no one comes to store and $q^\circ = 0$. Here, $\xi^\circ = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED}$ is the equilibrium in-stock probability at the store.*

According to Proposition 1, in order to have positive sales in the store, we need to ensure that the store channel is attractive to both the consumers and the retailer. As for the retailer, it is profitable for him to sell through the store channel only if he could get a higher margin by selling offline than online (i.e., $p - c > w$). Nonetheless, even if store fulfillment is attractive to the retailer, store sales can occur only if consumers are willing to pay a visit to the store. The first condition of Proposition 1 ensures that: (i) the store in-stock probability is large enough, and (ii) the online hassle cost dominates the store hassle cost, so that when put together, consumers are willing to risk encountering stockouts and come to the store. Under the conditions of Proposition 1, there is a participatory equilibrium.

Now, we turn to the scenario where the retailer implements BOPS on the product. With this added functionality, consumers assess information online and face one of two possible situations. The first possibility is that the product is out of stock at the store and BOPS is not an option; in this case, the consumer simply buys from the online channel. The other possibility is that the product is in stock and BOPS is feasible; in this case, the consumer chooses where to shop and we discuss this decision problem below. We stress that with the introduction of BOPS, consumers no longer have to form beliefs about inventory availability because this information is immediately accessible online. In other words, a useful by-product of BOPS is inventory availability information, which is provided on a real-time basis.

When BOPS is a viable option, the consumer faces a choice between three alternatives: buy online, buy in store, or use BOPS. To distinguish between the last two options, we introduce a new model parameter h_b, which is the hassle cost associated with using BOPS. Although BOPS consumers still need to go to the store after making their purchases online, the process is different from buying in store; for example, BOPS consumers do not search for products in store because their orders would already have been picked and packed by store staff. Therefore, the BOPS hassle cost h_b differs from the store and online hassle costs h_s, h_o. With this setup, all three alternatives yield the utility $v - p - h_i$, where the hassle cost h_i corresponds to the shopping mode chosen by the consumer. In other words, utility maximization boils down to choosing the shopping mode with the lowest cost. When the online channel offers the lowest hassle cost, consumers never go to the store. When the store hassle cost is lowest, consumers buy in store but only after verifying online that the product is in stock. When the BOPS hassle cost is lowest, consumers place orders online for store pickup.

We are now ready to write down the retailer's profit function with BOPS. When consumers choose to go to the store (i.e., when the online hassle cost h_o exceeds either the store hassle cost h_s or the BOPS hassle cost h_b), the profit function is

$$\pi(q) = pE \min(D, q) - cq + rE \min(D, q) + wE (D - q)^+ .$$

This is because when the store inventory level is q, there are on average $E \min(D, q)$ customers who come to the store, since they come to the store only when a corresponding unit is available. The first two terms above correspond to the newsvendor profit from selling the product and the third term corresponds to the additional cross-selling profit. Finally, when demand exceeds store inventory, customers who find that the store is out of stock can still choose to buy online; this yields the last term. In the other case where all consumers prefer shopping in the online channel (i.e., when h_o is the smallest hassle cost), the retailer will stock nothing in the store (i.e., $q = 0$) and earn an expected profit $\pi = wED$.

We use superscript $(\cdot^*)$ to denote the market outcome with BOPS, which is given in the following proposition:

Proposition 2. *If $\min(h_s, h_b) \leq h_o$ and $p - c > w - r$, then customers visit the store and $q^* = \bar{F}^{-1}\left(\frac{c}{p+r-w}\right)$. Otherwise, no one comes to store and $q^* = 0$.*

Proposition 2 (after BOPS) differs from Proposition 1 (before BOPS) in three significant ways. First, the condition $h_s \leq \xi^\circ \cdot h_o$ in Proposition 1 is weakened to $\min(h_s, h_b) \leq h_o$ in Proposition 2; in particular, the term corresponding to the in-stock probability ξ° vanishes. This discrepancy suggests that the risk of stockouts is no longer of concern after the introduction of BOPS. Indeed, BOPS provides real-time inventory information that essentially guarantees availability once an order is placed. In this way, BOPS attracts consumers to the store. The second difference is that the condition $p-c > w$ in Proposition 1 is weakened to $p - c > w - r$ in Proposition 2. The former condition requires the margin to be higher in store than online for the retailer to carry the product in store, but the latter condition combines the store margin with the cross-selling benefit. In other words, BOPS makes it more attractive for retailers to carry products in store; due to the cross-selling benefit r, the retailer may wish to carry a product in store even when the store margin is lower than the online margin. However, inventory now becomes more important: with BOPS, the retailer loses both the product margin $p-c$ as well as the cross-selling benefit r in the event of a stockout. This brings us to the third difference between Propositions 1 and 2: the critical fractile and hence the in-stock probability is higher with BOPS. This occurs because the underage cost increases from $p - c - w$ to $p - c - w + r$ after the introduction of BOPS. Consequently, the retailer has the incentive to increase inventory to lure more customers to the store.

In summary, we find that BOPS impacts store operations in two main ways. First, BOPS expands the set of products that may be offered at the store. For such products, omnichannel consumers who were previously unwilling to buy in store can be swayed by BOPS to visit the store. These products may originally be online exclusives but are now profitable to bring to retail stores. Second, for products that were originally carried at the store, BOPS leads to an increase in the in-stock probability. In other words, the store channel stocks more inventory, and would consequently end up with more leftover inventory. Excess inventory appears to be an inevitable downside of BOPS implementations (Reuters, 2014); despite this downside, the next section shows that BOPS may be accompanied by increased profits.

2.4. Information Effect and Convenience Effect

In this section, we compare the market outcomes before and after the introduction of BOPS (i.e., Propositions 1 and 2). With the following comparison results, we are able to identify two main effects of BOPS, i.e., the information effect and the convenience effect.

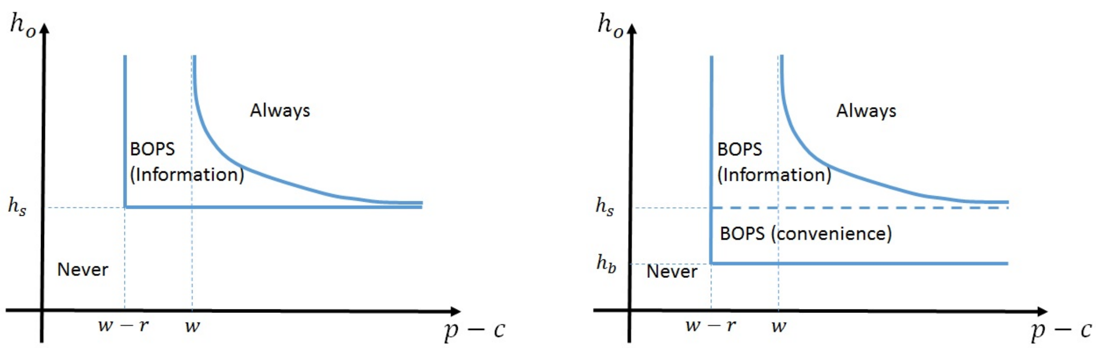
Let us first describe the framework for our analysis. Based on Propositions 1 and 2, we note there are three parameter regions. In some cases, consumers who were initially unwilling to visit the store will find the trip more appealing after the BOPS option is made available. In some other cases, BOPS will have no impact on channel choice: consumers always prefer a particular channel regardless of the BOPS option. These possibilities are summarized in Figure 1 below. Specifically, our three parameter regions are as follows:

i. In the "Always" regions (i.e., $h_s \leq \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED} h_o$ and $p-c > w$), consumers always buy the product in store, regardless of the implementation of BOPS;

ii. In the "Never" regions (i.e., $\min(h_s, h_b) > h_o$ or $p-c \leq w-r$), consumers never come to the store, preferring to buy the product online;

iii. In the "BOPS" regions (i.e., $h_s > \frac{E\min\left[D, \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)\right]}{ED} h_o$, $\min(h_s, h_b) \leq h_o$ and $p-c > w-r$), consumers come to the store only if BOPS is available. The "BOPS" regions are further labeled "Information" or "Convenience" as discussed below.

Figure 1: Do consumers buy the product in store?



(a) $h_b > h_s$ (b) $h_b \leq h_s$

In the following analysis, we will examine the impact of BOPS on the retailer's profit by separately considering different parameter regions in Figure 1. We begin with the following proposition.

Proposition 3. *If* $h_s \in \left(\frac{E \min\left[D, \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)\right]}{ED} h_o, h_o \right]$ *and* $p-c > w-r$*, then customers visit the store only if BOPS is available. Further, BOPS increases total profit (i.e.,* $\pi^* > \pi^\circ$*).*

The conditions in Proposition 3 correspond to the "BOPS" regions labeled "Information" in Figures 1a and 1b. In these parameter regions, BOPS influences consumer shopping behavior through the information sharing mechanism discussed in the previous section. By revealing real-time information about store inventory status, BOPS draws additional customers to the store; these customers were previously unwilling to visit the store because they were discouraged by the possibility of stockouts. In such cases, Proposition 3 confirms that BOPS leads to increased profit for the retailer. This increase in profit arises because the store profit margin $p - c$, combined with the cross-selling benefit r, exceeds the online margin w. In other words, through information provision, BOPS brings about a demand shift to the more profitable store channel.

There is a subtle difference between the two "BOPS (Information)" regions of Figures 1a and 1b. Although demand shifts to the store in both cases, they occur in different ways. In the "BOPS (Information)" region of Figure 1a, since the pickup hassle cost h_b exceeds the store hassle cost h_s, offering BOPS induces consumers to buy in store after verifying availability online, without actually using the BOPS functionality. On the other hand, in the corresponding region of Figure 1b, consumers indeed buy online and pick up in store when the option is available. We separately discuss these two behaviors in the next two paragraphs.

When consumers verify availability online without actually using the BOPS functionality, BOPS simply serves as a source of information. The same market outcome arises if the retailer simply provides real-time availability information on the website (i.e., directly showing whether or not store is in stock). This strategy has been adopted by retailers such as Gap and Levi's. Our model can be applied to study this pure information sharing mechanism, which can be regarded as a special case with $h_b > h_s$. In this special case, BOPS generates an interesting dynamic: after the implementation of BOPS as an added online functionality, online sales may *decrease*, while store sales may increase. This phenomenon was first identified by Gallino and Moreno (2014), who undertake a comprehensive empirical study of a US retailer with a recent BOPS implementation.

On the other hand, when the pickup process is relatively hassle-free, consumers will indeed buy online and pick up in store. In this case, apart from eliminating the risk of stockouts as described above, BOPS also provides consumers with a more convenient means of shopping. In this sense, comparing the two "BOPS (Information)" regions in Figures 1a and 1b, consumer surplus is higher in the latter than in the former.

The next proposition examines the "BOPS" region labeled "Convenience" in Figure 1b.

Proposition 4. *If $h_b \leq h_o < h_s$ and $p - c > w - r$, customers visit the store only if BOPS is available. Further, BOPS increases total profit (i.e., $\pi^* > \pi^\circ$).*

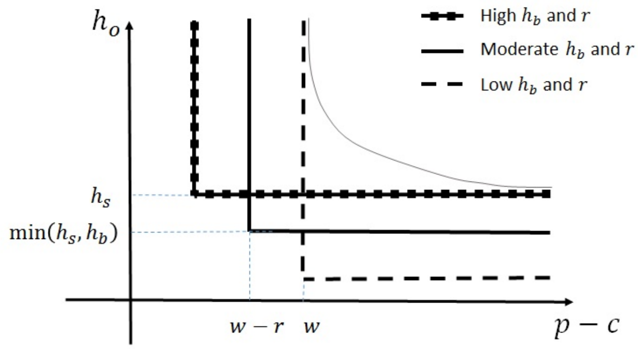
The above result highlights the importance of shopping convenience for BOPS to attract consumers to the store. By additionally providing convenience, BOPS becomes more powerful than a pure information sharing mechanism. In the "BOPS (Convenience)" region of Figure 1b, a pure information sharing mechanism can never attract customers to the store; even if customers are guaranteed availability in store, they still prefer to buy online because the online hassle cost is lower than the store hassle cost (i.e., $h_o < h_s$). However, once BOPS is available and provides convenience that trumps an online order (i.e., $h_b < h_o$), customers may prefer to buy online and pick up in store. This shopping mode benefits the retailer because customers may buy additional products (yielding profit r) when they pick up their products. As long as the store margin $p - c$ and cross-selling benefit r exceeds the online margin w, the convenience dimension of BOPS will lead to increased profit for the retailer.

Proposition 4 provides a word of caution for retailers. Although making the pickup process more convenient is potentially a good way to improve the profitability of BOPS, retailers should exercise care in preserving the cross-selling benefit. In particular, some retailers have introduced drive-through service that allows customers to receive their orders without leaving their cars (New York Times, 2012; Bloomberg, 2012). Although this will help to reduce hassle in the pickup process, it will also prevent people from entering the store and thus lead to a loss of the cross-selling benefit r. According to Proposition 4, if the margin from selling this particular product in the store is very high (i.e., $p - c > w$), then it is still profitable for the retailer to implement BOPS even if $r = 0$. However, if profit margins are lower in store than online, then the cross-selling profit r plays an important role; in this case, drive-through service may hurt the retailer's overall profit by neutralizing the advantages of cross-selling.

There is a delicate balance between pickup convenience and cross-selling potential. While consumers appreciate a more convenient pickup process, retail managers wishing to make the most out of the cross-selling opportunity may choose to locate the pick-up counter at far corners of the store so that shoppers have to walk through the entire store before picking up their online orders (Retail Dive, 2015). This tradeoff is illustrated in Figure 2. At one extreme, setting the pickup counter at the back of store maximizes both pickup cost h_b and cross-selling benefit r (i.e., the dotted L-line). At the other extreme, providing drive through pickup service minimizes both h_b and r (i.e., the dashed L-line). As both h_b and r increase, the L-line in Figure 1 moves up and left, and the "BOPS" region changes. The optimal location of the L curve depends on retailer's portfolio of products. According to Figure 2, if most of the retailer's products have high store profit margins (i.e., $p-c$ is large) and can be easily purchased online (i.e., h_o is small), then the retailer should seek to make the pickup process more convenient; in contrast, if most of the retailer's products have low store profit margins (i.e., $p-c$ is small) and are difficult to purchase online (i.e., h_o is large), then setting the pickup counter far from the store entrance is a better strategy.

The next proposition tells a different side of the story. Although BOPS brings about many benefits, it may lead to reduced profits in some cases.

Figure 2: Impacts of BOPS hassle cost h_b and cross-selling benefit r



Proposition 5. *If* $h_s \leq \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED} h_o$ *and* $p - c > w$*, customers visit the store regardless of the implementation of BOPS. Further, if* $r > 0$*, then BOPS decreases total profit (i.e.,* $\pi^* < \pi^\circ$*).*

The conditions of Proposition 5 correspond to the "Always" regions in Figures 1a and 1b. In these regions, consumers choose to visit the store regardless of BOPS. Here, BOPS has an important but easily overlooked effect. Prior to the introduction of BOPS, all customers were already willing to visit the store, but after BOPS is made available, fewer consumers will come to the store. This is because customers who attempt to place an order online but find that the item is not in stock for pickup will no longer go to the store. As store traffic decreases, the retailer loses the potential profit from cross-selling. The loss of cross-selling benefits (i.e., whenever $r > 0$) leads to a reduction in total profits, as shown in Proposition 5. Since BOPS can be selectively implemented, our result suggests that the BOPS option should not be offered on products that have been attracting considerable demand to the store.

Our results differ from existing findings in the literature because we study a different information sharing mechanism. In our model, BOPS provides real-time information about the store inventory status, i.e., customers are informed that the product is available until inventory runs out. However, in Su and Zhang (2009), the focus is on quantity commitment, i.e., the retailer commits to an initial inventory level in the store. With quantity commitment, the retailer may be able to use a small amount of store inventory to attract a large number of customers to visit the store. In particular, when the cross-selling benefit is very large, the retailer may still choose to stock the product in the store even if it is more profitable to sell online, with the hope of attracting customers to make additional purchases in store. Such a "loss leader strategy" is no longer feasible when the retailer implements BOPS, because customers have access to real-time store inventory information and will not visit the store after the product is out of stock. Therefore, we find that BOPS, by providing real-time inventory information, may decrease profits, while Su and Zhang (2009) find that quantity commitment is generally valuable.

In summary, BOPS has two effects: it provides customers with real-time information about in-store inventory availability and it introduces a new shopping mode that may add convenience to customers. The former effect (information effect) helps attract customers to the store by letting them know about inventory availability, but it is a double-edged sword in that when inventory is not available, it turns away customers who might be willing to visit the store. The latter effect (convenience effect) applies when customers use the store pickup functionality, as opposed to simply using BOPS as a source of availability information; it draws customers to the store and may even open up new sources of demand.

When put together, the information and convenience effects of BOPS yield different profit implications. Figures 1a and 1b present a clear distinction: in the "BOPS" regions, BOPS leads to higher profits, but in the "Always" regions, BOPS leads to lower profits. The difference between these two regions is that, prior to the introduction of BOPS, consumers were already willing to visit the store in the latter but not in the former. These results suggest that BOPS should be offered for products with weak store sales but not those with strong records to begin with. In other words, it is likely profitable to implement BOPS on in-store "underdogs" but may not be so for in-store "favorites."

2.5. Heterogeneous Customers

In this section, we incorporate customer heterogeneity. For example, some customers may reside further away from the store than others; some may be more impatient than others and are thus more averse to waiting for online delivery. In our model, the store and online hassle costs h_s, h_o may now differ across customers. Specifically, customers are uniformly distributed across the following "square" $\{(h_s, h_o)|h_s \in [0, H], h_o \in [0, H]\}$, where $H > v - p$ (i.e., some customers have a prohibitively high hassle cost in one channel). The goal is to study the impact of BOPS on a retailer's customer base in such a heterogeneous market.

We begin by considering the scenario in the absence of BOPS. In this case, each customer has three options: go to the store, buy online, or leave the market. The corresponding utilities are:

$$
\begin{aligned}
u_s &= -h_s + \hat{\xi}(v - p) + (1 - \hat{\xi})u_o^+, \\
u_o &= v - p - h_o, \\
u_l &= 0,
\end{aligned}
$$

where $\hat{\xi}$ denotes the belief about store inventory availability as before. Note that customers who find the store out of stock will buy online only if doing so is preferred over leaving the market. As customers make utility-maximizing choices (which depend on their hassle costs h_s, h_o), the market is divided into four segments, as depicted in Figure 3(a). Specifically, there are "pure online" customers (who buy online directly), "store→online" customers (who visit the store but switch online when the store is out of stock), "pure store" customers (who visit the store exclusively), as well as customers who simply leave the market. Denote the fractions of these four types of customer as α_o, α_{so}, α_s, and α_l, respectively. Given that consumers are uniformly distributed in $\{(h_s, h_o)|h_s \in [0, H], h_o \in [0, H]\}$, we can find that $\alpha_o = \frac{v-p}{H} - \frac{\hat{\xi}(v-p)^2}{2H^2}$, $\alpha_{so} = \frac{\hat{\xi}(v-p)^2}{2H^2}$, $\alpha_s = \frac{\hat{\xi}(v-p)(H-(v-p))}{H^2}$, and $\alpha_l = \frac{[H-(v-p)][H-\hat{\xi}(v-p)]}{H^2}$.

On the supply side, the retailer faces the following profit function:

$$\begin{aligned}\pi(q) = pE\min\left(\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)D, q\right) - cq + rE\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)D \\ +wE\hat{\alpha}_o D + wE\tfrac{\hat{\alpha}_{so}}{\hat{\alpha}_s+\hat{\alpha}_{so}}\left(\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)D-q\right)^+\end{aligned} \tag{2.2}$$

where $\hat{\alpha}_o$, $\hat{\alpha}_{so}$, $\hat{\alpha}_s$, and $\hat{\alpha}_l$ denote the retailer's beliefs over the α's above. Given the store inventory level q, the newsvendor expected profit from selling the product in the store channel is shown in the first two terms above. The third term captures the additional cross-selling profit r from every customer coming to the store. The last two terms above show the retailer's online profit. The fourth represents profit from customers who shop online directly; the last represents profit from customers who switch to online after encountering stockouts in store, in which case we assume store customers have equal chance of being rationed. With the profit function above, the retailer chooses q to maximize expected profit.

Definition 2. *A RE equilibrium $(\alpha_o, \alpha_{so},, \alpha_s, \alpha_l, q, \hat{\xi}, \hat{\alpha}_o, \hat{\alpha}_{so}, \hat{\alpha}_s, \hat{\alpha}_l)$ satisfies the following:*

i Given $\hat{\xi}$, then $\alpha_o = \frac{v-p}{H} - \frac{\hat{\xi}(v-p)^2}{2H^2}$, $\alpha_{so} = \frac{\hat{\xi}(v-p)^2}{2H^2}$, $\alpha_s = \frac{\hat{\xi}(v-p)(H-(v-p))}{H^2}$, and $\alpha_l = \frac{[H-(v-p)][H-\hat{\xi}(v-p)]}{H^2}$;

ii. Given $\hat{\alpha}_o, \hat{\alpha}_{so}, \hat{\alpha}_s$ and $\hat{\alpha}_l$, $q = \arg\max_q \pi(q)$, where $\pi(q)$ is given in (2.2);

iii. $\hat{\xi} = A(q)$, where $A(q) = \frac{E\min((\alpha_s+\alpha_{so})D,q)}{E(\alpha_s+\alpha_{so})D}$;

iv. $\hat{\alpha}_s = \alpha_s$, $\hat{\alpha}_o = \alpha_o$, $\hat{\alpha}_{so} = \alpha_{so}$ and $\hat{\alpha}_l = \alpha_l$.

The following proposition gives the RE equilibrium. As before, we use the superscripts $\cdot^\circ$ and $\cdot^*$ to denote the no-BOPS and BOPS scenario, respectively.

Proposition 6. *If $p - c > w\frac{v-p}{2H-(v-p)}$, then there are customers visiting store ($\alpha_s^\circ = \frac{\xi^\circ(v-p)(H-(v-p))}{H^2} > 0$, $\alpha_{so}^\circ = \frac{\xi^\circ(v-p)^2}{2H^2} > 0$) and $q^\circ = \left(\alpha_s^\circ + \alpha_{so}^\circ\right)\bar{F}^{-1}\left(\frac{c}{p-w\frac{v-p}{2H-(v-p)}}\right)$, where the equilibrium store in-stock probability is $\xi^\circ = \frac{\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w\frac{v-p}{2H-(v-p)}}\right)\right)}{ED}$. Otherwise, no one comes to store and $q^\circ = 0$, $\xi^\circ = 0$.*

Next, we turn to the scenario where the retailer implements BOPS on the product. When a customer uses BOPS, she experiences hassle in both online and offline worlds. For example, she needs to go through the online payment process and she also has to go to the store to pick up the product. As a result, we assume the hassle cost of using BOPS is given by $h_b = \beta_s h_s + \beta_o h_o$, where $\beta_s, \beta_o \in (0, 1)$. Further, as explained before, all customers have access to information about store inventory status before they visit the store.

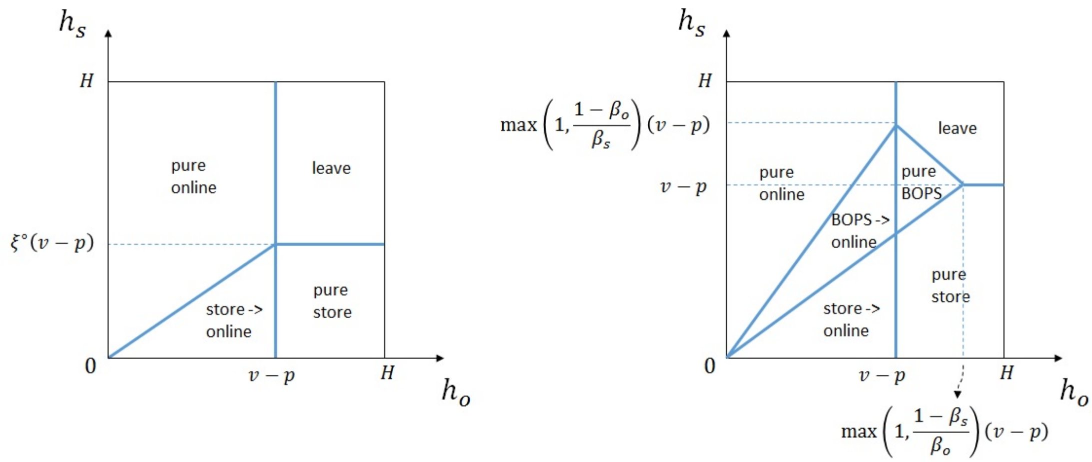
Now, we consider customer choice in the presence of BOPS. There are two cases to consider. First, when the store is in stock, the consumer faces a choice between four alternatives: buy online (with payoff $v-p-h_o$), buy in store (with payoff $v-p-h_s$), use BOPS (with payoff $v-p-h_b$), or leave (with payoff 0). Based on their individual hassle costs, consumers choose their shopping mode with the highest payoff. Second, when the store is out of stock, only customers with $h_o < v-p$ will choose to buy online, while the rest will leave. With the above decisions, the market is divided into six segments, as depicted in Figure 3(b). In addition to the four segments described before, we now see "BOPS→online" customers (who use BOPS if the store is in stock but switch online otherwise) and "pure BOPS" customers. Denote the fraction of these two new types of customer as α^*_{bo} and α^*_b. We can calculate the sizes of these six customer segments as follows:

$$\alpha^*_i = \iint_{A^*_i} \frac{1}{H^2} dh_s dh_o, \ i = o, so, s, l, bo, b$$

where

$$A^*_o = \{(h_s, h_o)\,|v-p-h_o > \max(v-p-h_s, v-p-h_b, 0)\}$$
$$A^*_{so} = \{(h_s, h_o)\,|v-p-h_s > \max(v-p-h_o, v-p-h_b, 0), v-p-h_o > 0\}$$
$$A^*_s = \{(h_s, h_o)\,|v-p-h_s > \max(v-p-h_o, v-p-h_b, 0), 0 > v-p-h_o\}$$
$$A^*_l = \{(h_s, h_o)\,|0 > \max(v-p-h_o, v-p-h_s, v-p-h_b)\}$$
$$A^*_{bo} = \{(h_s, h_o)\,|v-p-h_b > \max(v-p-h_o, v-p-h_s, 0), v-p-h_o > 0\}$$
$$A^*_b = \{(h_s, h_o)\,|v-p-h_b > \max(v-p-h_o, v-p-h_s, 0), 0 > v-p-h_o\}$$

Figure 3: Market Segmentation with and without BOPS



(a) Without BOPS

(b) With BOPS

Next, the retailer's profit function can be expressed as follows:

$$
\begin{aligned}
\pi(q) =& pE\min\left(\left(\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*\right)D, q\right) - cq + rE\min\left(\left(\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*\right)D, q\right) \\
& wE\alpha_o^* D + wE\frac{\alpha_{so}^* + \alpha_{bo}^*}{\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*}\left(\left(\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*\right)D - q\right)^+
\end{aligned}
$$

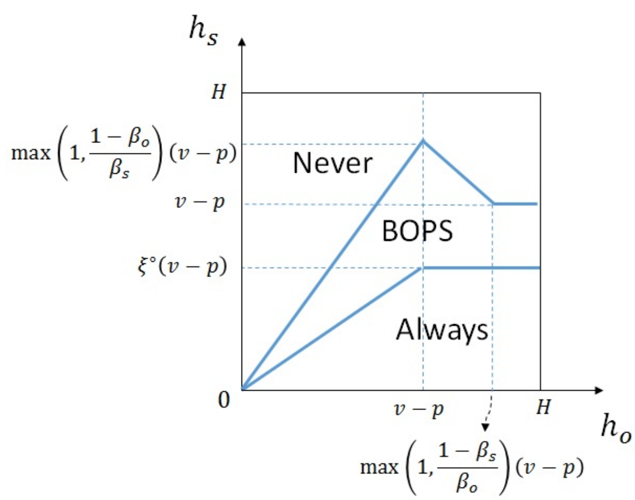
To understand this expression, note when the store inventory level is q, there are on average $E\min((\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*)D, q)$ customers who come to the store, since they come to the store only when a corresponding unit is available. The first two terms above correspond to the newsvendor profit from selling the product and the third term corresponds to the additional cross-selling profit. The last two terms represent the profit from the online channel: the fourth term is the profit from those who buy online directly, while the fifth term is the profit from those who prefer to go to store but buy online instead because of store stockouts.

Proposition 7. *When there is BOPS, the market outcome is given as follows:*

- *if* $p - c > w\frac{\alpha_{so}^* + \alpha_{bo}^*}{\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*} - r$*, then there are customers visiting store and* $q^* = \left(\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*\right)\bar{F}^{-1}\left(\frac{c}{p + r - w\frac{\alpha_{so}^* + \alpha_{bo}^*}{\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*}}\right)$*;*
- *if* $p - c \leq w\frac{\alpha_{so}^* + \alpha_{bo}^*}{\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*} - r$*, then no one ever comes to store and* $q^* = 0$*.*

With a homogeneous population of customers, we identified three possible scenarios (as shown in Figure 1 in Section 2.4). Consumers may: (i) always visit the store, (ii) never visit the store, or (iii) visit the store only after BOPS is implemented. With a heterogeneous consumer population, these types of behavior may coexist, as shown in Figure 4 (which is obtained from comparing Figures 3(a) and 3(b)). In other words, there are three types of customers, each exhibiting a specific response to BOPS. Moreover, the retailer's profit from each type of customer follows the same pattern as before. If a consumer always visits the store (as those in the "Always" region), then the retailer's profit from this customer decreases after BOPS is implemented; she stops coming to the store once she knows that the store is out of stock and thus the retailer loses the potential cross-selling benefit. Next, if a consumer visits the store only after BOPS is implemented (as those in the "BOPS" region), then BOPS increases retailer's profit. Finally, if a consumer never shops in the store (as those in the "Never" region), then offering BOPS does not affect the retailer's profit.

Figure 4: When do consumers go to store?



The next proposition shows the impact of BOPS on the retailer's overall customer base.

Proposition 8.

i. BOPS helps to expand market coverage, i.e., $\alpha_s^* + \alpha_o^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^* > \alpha_s^\circ + \alpha_o^\circ + \alpha_{so}^\circ$;

ii. Suppose $r = 0$. *If there are customers visiting store when there is no BOPS, then there* *exists $\bar{w}$ such that the implementation of BOPS decreases total profit (i.e., $\pi^* < \pi^\circ$) if $\beta_s + \beta_o < 1$ and $w > \bar{w}$.*

Before BOPS is implemented, customers who face high hassle costs in both store and online channels do not consider purchasing from the retailer. Part (i) of Proposition 8 shows that BOPS could provide a way for the retailer to reach these customers. By alleviating the risk of stockouts and reducing the hassle of shopping, BOPS could attract some new customers to join the market. This market expansion effect could also be seen from the reduction of the "leave" region in Figure 3(b) compared to Figure 3(a).

While reaching out to new customers, BOPS may change the behavior of existing customers. Specifically, the more convenient BOPS is, the more existing online customers will choose to pick up their orders in store; this shift will hurt profits if the store profit margin is lower than the online profit margin. This potential drawback of BOPS may exist even when $r = 0$, as shown in Proposition 8(ii). In other words, apart from possibly eliminating cross-selling opportunities (when $r > 0$) as discovered earlier, BOPS has another potential drawback of shifting demand to a less profitable store channel.

2.6. Decentralized System

In this section, we study the scenario where the store and online channels are operated by two separate teams, with the goal of understanding how BOPS revenue should be allocated. We begin by examining the case with homogeneous customers and then later extrapolate our findings to the case with heterogeneous customers. (A detailed analysis of the decentralized system for the heterogeneous market is given in Appendix A.1.) We assume that the conditions in Proposition 2 hold (i.e., $\min(h_s, h_b) \leq h_o$ and $p - c > w - r$) and BOPS hassle cost is lowest (i.e., $h_b < \min(h_s, h_o)$); otherwise, there would be nobody using BOPS and the issue of revenue allocation becomes irrelevant as the system collapses into two independent channels.

We use $\theta \in [0, 1]$ to denote the share of BOPS revenue that the store obtains. In other words, for every customer who purchases online and picks up in the store, the store earns θp from selling the product. Note that once the customer comes to the store, the store could also get an additional profit r through cross-selling. Therefore, the total revenue that the store could receive from fulfilling each unit of BOPS demand is $\theta p + r$. Then, the store's expected profit as a function of the stocking decision $\tilde{q}$ is given below. Here, we use the "tilde" symbol $(\tilde{\cdot})$ to denote the decentralized case.

$$\tilde{\pi}_s = (\theta p + r)\, E \min(D, \tilde{q}) - c\tilde{q}$$

Proposition 9. *In the decentralized system, the store will stock $\tilde{q}^*$ which is given as follows:*

- *If $\theta p + r - c > 0$, then $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\theta p + r}\right)$;*
- *If $\theta p + r - c \leq 0$, then $\tilde{q}^* = 0$.*

In practice, according to the survey conducted by Forrester Research (Forrester, 2014), the two most common revenue sharing schemes are either giving the store full credit (i.e., $\theta = 1$) or letting the online channel keep all the revenue (i.e., $\theta = 0$). According to Proposition 9, when $\theta = 1$, the store will definitely stock the product to serve BOPS users, because he can not only receive a positive profit margin from selling this particular product, but he may also be able to cross sell other products to those who come to store for pickup. However, if $\theta = 0$, BOPS customers represent a pure cost to the store; then, the store may choose not to stock the product in store, unless the cross-selling benefit is large enough to offset the loss from serving BOPS customers (i.e., $r > c$).

What is the optimal share of BOPS revenue that should be allocated to the store? We use $\tilde{\pi}^*(\theta)$ to denote total profit in the decentralized system with revenue allocation θ. For any given θ, the next proposition compares decentralized and centralized inventory decisions and shows that the store is usually either overstocked or understocked, relative to the centralized benchmark. However, there is an optimal revenue share, under which the decentralized system achieves the centralized optimal profit.

Proposition 10. *Total profit $\tilde{\pi}^*(\theta)$ is quasiconcave in θ. Moreover,*

- *if $\theta < \frac{p-w}{p}$, then $\tilde{q}^* < q^*$ and $\tilde{\pi}^*(\theta) < \pi^*$;*
- *if $\theta = \frac{p-w}{p}$, then $\tilde{q}^* = q^*$ and $\tilde{\pi}^*(\theta) = \pi^*$;*
- *if $\theta > \frac{p-w}{p}$, then $\tilde{q}^* > q^*$ and $\tilde{\pi}^*(\theta) < \pi^*$.*

Proposition 10 points out an incentive conflict between the store channel and the retail organization. If the store channel obtains a large share of BOPS revenue, they tend to stock too much. This is because the store channel considers only store profits but neglects the fact that customers may still be willing to shop online after the store runs out of inventory for customers to pick up. In contrast, if the store is allocated only a small share of BOPS revenue, they tend to stock too little. Since the store channel incurs the inventory cost for fulfilling BOPS demand, it is natural to suppress inventory to decrease exposure to potential losses. In general, it is optimal to give the store partial credit for the revenue earned from BOPS customers. In fact, our result also shows that such a simple revenue sharing mechanism is sufficient for the retailer to fully coordinate the store and online channels.

Proposition 10 shows that the optimal revenue share $\theta^* = \frac{p-w}{p}$ coordinates the decentralized system and achieves the centralized optimal profit π^*. With this optimal revenue share, the incentives of the store channel are aligned with the entire organization. According to Proposition 9, the decentralized store channel holds stock if and only if $\theta p + r - c > 0$. However, from the perspective of the entire organization, as we have shown in the previous section, it is optimal to stock the product in the store as long as $p-c > w-r$, i.e., whenever the store margin $p-c$, combined with the cross-selling benefit r, exceeds the online margin w. The revenue share θ that aligns both sets of incentives is precisely $\theta^* = \frac{p-w}{p}$.

Note that the optimal share for the store channel $\theta^* = \frac{p-w}{p}$ is decreasing in w. The reason is as follows. As the online margin w increases, it becomes more profitable to sell through the online channel. However, in the absence of BOPS, the store will continue to stock the same amount of inventory since they tend to neglect the profit from online orders. Through sharing the BOPS revenue, the retail organization has a natural way to correct for the misaligned incentive. By allocating less revenue to the store channel for fulfilling BOPS demand, the retail organization can induce the store channel to lower their inventory level. This compensates for the incentive conflict and allows more demand to flow to the more profitable online channel.

Although Proposition 10 shows that the optimal revenue sharing mechanism will achieve full centralized profits in a homogeneous market, Appendix A.1 presents a different result for a heterogeneous market. Specifically, when the proportion of customers who use the BOPS functionality is too low, the amount of BOPS revenue to be shared between the channels may be insignificant and as a result, a simple revenue sharing mechanism cannot fully coordinate the omnichannel retail system. The analysis in Appendix A.1 highlights the benefit as more customers adopt BOPS: the increased stream of BOPS revenue can provide headquarters with more leverage to alleviate incentive conflicts between the store and online channels.

Omnichannel retailers who recognize the incentive conflicts brought about by BOPS have begun to experiment with simple revenue sharing schemes. A common and simple approach is to assign full credit for the sale of a BOPS item to both store and online channels. In other words, there is some double counting on internal books that are subsequently adjusted for. Depending on accounting protocols, this method is usually akin to allocating equal revenue shares to each channel. Since the optimal revenue share may not be 50%, the simple heuristic described above has room for improvement, and our analysis provides a possible way to think about how to do so.

In practice, a retailer may carry a large number of SKUs with different prices and margins. Admittedly, it would be impractical to set a different revenue sharing parameter θ for each SKU. Instead, a retailer may want to have a common θ for a group of products. In such case, since the optimal profit function $\tilde{\pi}^*(\theta)$ is quasiconcave in θ, the optimal θ^*'s given in Proposition 10 for the group of products could serve as benchmarks for the retailer to find a common compromise θ that is close enough to each of the different θ^*'s.

2.7. Conclusion

In this chapter, we study a specific omnichannel fulfillment strategy: buy-online-and-pick-up-in-store (BOPS). We develop a stylized model that captures essential elements of omnichannel retail environments; in particular, consumers strategically choose channels for purchase and fulfillment. We find that BOPS attracts consumer demand through an *information* effect and a *convenience* effect. The former effect arises because BOPS reveals real-time information about store inventory availability. Products that are available for store-pickup must be in stock. With this assurance, customers are more willing to visit the store. The latter effect arises because BOPS offers a new and possibly more convenient mode of shopping. By helping consumers pick out items and move them to checkout counters, BOPS reduces the hassle of shopping.

Even though BOPS is a popular fulfillment option among consumers, we find that retailers need to be cautious when implementing it. Retailers can benefit from this new fulfillment strategy by being selective when choosing the set of products eligible for in-store pickup. BOPS can help attract more customers to the store and thus boost the sales of products that were previously not selling well; however, for store bestsellers, offering the BOPS option may have the unintended consequence of reducing store traffic. Moreover, although BOPS can be a good strategy for a retailer to build up its customer base, BOPS may at the same time drive existing online customers to the store channel where the profit margin might be lower compared to the online channel. Finally, retailers with decentralized operations can maximize profits by allocating BOPS revenue between the online and store channels appropriately, and giving full credit to either channel is seldom optimal.

To simplify the analysis, we have imposed two assumptions in our model: (1) The online channel is exogenous and always in stock, and (2) all customers check the information online when BOPS is offered. As robustness checks, we have built two model extensions to relax the assumptions above. Specifically, in Appendix A.2.1, we consider the case where the retailer has limited inventory in both the store and online channels; in Appendix A.2.2, we consider customers who simply head to the store by default (e.g., they may forget or simply not care to check websites beforehand). Our key results remain valid in these model extensions.

Beyond the scope of our current analysis is the potential impact of BOPS on operational costs. On the one hand, BOPS helps to reduce online shipping costs since it transfers the burden of last-mile delivery to customers. On the other hand, BOPS is accompanied by new fulfillment responsibilities wherein a retailer's comparative advantage may not lie. For example, stores need to train their workforce to perform pick-and-pack tasks in a timely fashion (Forrester, 2014), and to handle increased demand, stores need to hire more employees to deal with online orders (Business Insider, 2012). A more careful cost-based analysis of BOPS is left as an interesting topic for future research.

The omnichannel strategy discussed in this chapter, BOPS, addresses purchases that originate online but are completed in the store. In the spirit of Bell et al. (2014), who provide a framework for omnichannel retail, BOPS customers receive information online but their demand is fulfilled in the store. The reverse type of shopping behavior is where customers research the product in stores and then shop online, usually at lower prices. This is known as showrooming behavior and has been critiqued widely: showroomers and e-tailers are accused of free-riding on inventory displays at brick-and-mortar stores (Wall Street Journal, 2012b,a). Recent research by Balakrishnan et al. (2014) shows that showrooming behavior intensifies retail competition, and Mehra et al. (2013) studies how a brick-and-mortar retailer can counteract showrooming behavior through strategies such as price matching and retail club memberships. Taking the e-tailer's perspective, Bell et al. (2015) empirically studies the value of providing offline showrooms to mitigate customer uncertainty. We hope that our model in this chapter can contribute to this exciting line of research.

CHAPTER 3 : Online and Offline Information for Omnichannel Retailing

3.1. Introduction

In the present world of constant connectivity, the consumer's journey from product discovery to eventual purchase often involves multiple shopping channels (UPS, 2014). It is not uncommon for shoppers to browse products in stores and then place an order online, or to extensively research products online before completing the purchase in a physical store (Brynjolfsson et al., 2013). Consumers are becoming sophisticated enough to optimize their shopping experience by exhaustively considering all possible alternatives across all possible channels. As a result, retailers face immense pressure to integrate the best of both digital and physical worlds at each step of the customer experience (Rigby, 2011). This is the spirit of omnichannel retailing, which is fast becoming the norm in the industry (Forrester, 2014). One of the greatest challenges in the omnichannel environment is to effectively deliver information (Bell et al., 2014). As consumers actively seek information about product value and inventory availability, retailers can influence shopping paths by managing the sources of information. In this chapter, our goal is to understand how information influences omnichannel consumer behavior and retailer performance.

Displaying products in a showroom is one of the most common ways for retailers to convey product information to consumers. Typically, consumers can inspect the line of products on display whenever they come to the store. Although most commonly implemented in stores of home furniture and consumer electronics, the idea of showrooms has also been recently adopted by fashion e-tailers such as Bonobos and Warby Parker (New York Times, 2014; Wall Street Journal, 2014b). These companies have set up pure showrooms with products purely for display purposes: customers finding something they like in the showrooms can make a purchase by placing an order on the corresponding website. Although pure showrooms may carry a small amount of inventory, their primary function is information rather than fulfillment; Bell et al. (2015) demonstrate that the former can help generate profits by reducing online returns. In this sense, setting up showrooms can be a useful strategy for omnichannel retailers.

In recent years, showrooms have moved from physical into digital domains. With the development of virtual reality technology, online shoppers can now try on different products as if they were in the store (Financial Times, 2011). For example, on the website of BonLook, an eyewear retailer, consumers can upload their own photos to see how different frames will look on their digital faces. Similarly, on the website of UK luxury shirt brand Thomas Pink, consumers can check the fit of a shirt through a digital avatar. Many advanced technologies are now available from an increasing variety of providers. For example, Metail provides visualization technology that creates 3D models of shoppers based on a few customized measurements, while Shoefitr uses 3D scanning technology to measure the insides of shoes with accuracy up to a quarter of a millimeter (CNET, 2010). With these advances, virtual showrooms are no longer a figment of imagination as shoppers can now experience products first-hand anywhere and anytime. As online channels face more pressure from product returns, which are estimated to account for a third of all Internet sales (Wall Street Journal, 2013a), virtual showrooms will likely become a potential remedy (Bell et al., 2014; CNBC, 2015b).

While product uncertainty may discourage purchase, availability uncertainty may discourage patronage. Recognizing this customer concern, many retailers have started to provide real-time store availability information online. Shoppers can simply enter a zip code to check the current status of nearby stores (e.g., whether or not a product is in stock, or how much inventory is available). In fact, many retailers go a step further, allowing customers to complete the transaction online and pick up their order in the store shortly after; recent studies of this buy-online-pick-up-in-store option (Gallino and Moreno, 2014; Gao and Su, 2016a) highlight the importance of availability information. Simply put, some customers will shop in a store only if they know that a product is in stock.

In this chapter, we study the following three types of information mechanisms. *Physical* *showrooms* allow consumers to inspect products in the store even when it is out of stock; this may involve setting aside units as permanent display models. *Virtual showrooms*, by mimicking physical presence, enable online shoppers to evaluate products without coming to the store; product value uncertainty is reduced but still present because technology is never perfect. *Availability information* allows customers who have not visited the store to learn whether or not it has a product in stock, so that they may choose a different shopping path in each case. What is the impact of each of these three mechanisms? Which combination should a retailer implement? These are the research questions that we study in this chapter.

To address these questions, we develop a stylized model that captures essential elements of omni-channel retail environments. There is a retailer who operates online and store channels. Consumers strategically make channel choices to maximize individual utility. An important feature of our model is that consumers face product value uncertainty. Consumers who visit the store can inspect the product (if it is in stock) to fully learn their valuation before deciding whether to purchase; however, consumers who purchase online may later realize that they do not like the product and end up returning it. Demand is random. In the store channel, the retailer chooses a stocking quantity before demand is realized and may end up with leftovers or stockouts; however, in the online channel, the retailer obtains supply after demand is realized and is hence able to satisfy all orders. In our analysis, we first study each information mechanism (i.e., physical showrooms, virtual showrooms, or availability information), and then we examine potential interaction effects among them to determine the optimal information structure.

We summarize our main findings below.

First, physical showrooms attract customers to the store because they help customers resolve product value uncertainty regardless of whether the product is in or out of stock. In particular, during stockouts, shoppers who learn from the showroom display that they do not like the product can avoid making an unnecessary purchase online. In this way, physical showrooms help to reduce returns from customers who encounter a stockout in store, substitute online, and then return the product (which may lead to a net loss). Although physical showrooms reduce the cost of stockouts, they create a temptation for retailers to lower store inventory levels, and customers who are sensitive to availability risk may prefer to buy online instead. This leads to more returns and lower profit. Our results thus caution that physical showrooms are not always profitable, and retailers should make the best effort to maintain satisfactory inventory service levels in stores equipped with showrooms.

Second, virtual showrooms reduce product value uncertainty online, by screening out consumers who do not like the product before any purchase takes place. Understandably, returns go down and profits go up. However, our results raise another red flag. Sometimes, virtual showrooms make online shopping so attractive that customers who were originally shopping in store choose to migrate online. When there is significant customer migration online, total returns may go up (even as return rates go down) simply because many more customers are buying online before product value is completely ascertained (recall that virtual showrooms reduce but do not completely eliminate product value uncertainty). Therefore, our results suggest that virtual showrooms are profitable as long as they do not induce too much customer migration to the online channel.

Third, availability information eliminates availability risk and thus attracts customers to the store. Although availability information may sometimes be redundant (e.g., when consumers are already willing to visit the store even without availability information), it does not hurt profits when implemented on its own. However, when implemented together with showrooms, it may hamper their effectiveness. For example, suppose a retailer with physical showrooms now offers availability information as well. Then, when the store is in stock, customers may show up and be able to learn their valuation from the physical showroom; however, once customers realize that the store is out of stock, they may not show up. In this sense, providing availability information makes physical showrooms less useful.

Finally, the information mechanisms studied above do not exhibit significant complementarities because there are potential overlaps in how they impact consumer behavior. Specifically, physical showrooms and availability information both attract customers to the store, and both physical and virtual showrooms mitigate product value uncertainty. Given these overlaps, the optimal information structure often involves choosing one of the three mechanisms. For products that are traditionally afflicted with high online return rates, retailers should establish physical showrooms; however, for products with relatively more manageable online return rates, retailers should consider implementing virtual showrooms, and possibly also provide availability information to maintain the attractiveness of the store channel.

3.2. Literature Review

Similar to Chapter 2, this chapter also studies the management of online and offline channels. The traditional literature on channel management focuses on the situation where different channels are operated by different companies, e.g., Chiang et al. (2003), Cattani et al. (2006), Chen et al. (2008), and Netessine and Rudi (2006). In contrast to this stream of work, we focus on an omnichannel environment, where a retailer operates both online and offline channels in an integrated way (Bell et al., 2014; Brynjolfsson et al., 2013; Rigby, 2011). In this environment, most operations management papers focus on new omnichannel fulfillment methods. For example, Gallino and Moreno (2014) empirically test the impact of buy-online-and-pick-up-in-store (BOPS) on a retailer's sales in both online and offline channels, while Gao and Su (2016a) (which is based on Chapter 2 of this dissertation) study the implementation of BOPS and its implications on channel coordination from a theoretical perspective. Gallino et al. (2016) investigate another widely-used omnichannel fulfillment option, i.e., ship-to-store, and empirically demonstrate that within a group of SKUs, the share of sales of bottom-selling items increases. In contrast to these papers, we focus how to effectively provide information to consumers in the omnichannel environment. A closely related paper is Bell et al. (2015), which studies the impact of physical showrooms on consumers' channel choice. They find that with the introduction of physical showrooms, customers with a greater need for product information self-select into the physical channel, leading to reduced online returns and increased overall demand. In this chapter, we study physical showrooms, as well as virtual showrooms and availability information.

There is a large literature devoted to the question of how a firm can induce purchases from consumers facing product value uncertainty. The most widely studied mechanism is product returns. Che (1996) and Davis et al. (1995) consider the case where a seller offers full refunds (or money-back guarantees). Davis et al. (1998) focus on the consumer hassle costs during the return process, and Moorthy and Srinivasan (1995) show that full refunds can signal product quality. More recent papers study refund policies in conjunction with supply chain contracts (Su, 2009), overbooking (Gallego and Sahin, 2010), demand cannibalization (Akçay et al., 2013), and discount voucher (Gao and Chen, 2015); another paper by Hsiao and Chen (2012) finds that sellers of low-quality products may have to offer a refund that exceeds the selling price. The papers above study how returns/refunds can be used to limit the downside of a bad purchase, but in this chapter, one of our goals is to reduce the incidence of bad purchases in the first place. To this end, our physical and virtual showrooms directly reduce product value uncertainty by allowing consumers to experience products prior to purchase. Ultimately, showrooms and returns policies complement each other because they serve different functions: while returns policies provide insurance against consumer loss, showrooms provide product value information.

There is a stream of literature on using product availability as a strategic lever to attract demand. Being averse to stockouts, consumers may be reluctant to incur the hassle of going to a store. This practical issue has motivated much work in operations management. Following the classic paper by Dana and Petruzzi (2001), which extends the newsvendor model by endogenizing demand as a function of both price and inventory level, many papers have looked at different strategies to influence consumers' shopping decisions. Su and Zhang (2009) show that it is always beneficial to the retailer if he can credibly make an ex ante quantity commitment. Yin et al. (2009) investigate the impact of two different display formats, i.e., display-all versus display-one. Allon and Bassamboo (2011) study a situation where a retailer could share unverifiable and nonbinding (i.e., cheap talk) inventory information with consumers. Alexandrov and Lariviere (2012) examine the role of reservations in the context of revenue management. Cachon and Feldman (2015) study dynamic pricing and find that frequent discounts can attract customers to the store. In contrast to the previous studies, in this chapter, we consider a different mechanism where a retailer shares real-time store availability information that reflects the current inventory level available to each individual customer.

3.3. Base Model

There is a retailer who sells a product through two channels, store and online, at price p. In the store channel, the retailer faces a newsvendor problem: there is a single inventory decision q to be made before random demand is realized, so there may eventually be unmet demand or leftovers in the store. The unit cost of inventory is c, which includes the production/procurement cost and other store-related costs, e.g., the transportation cost of shipping the product to store and the inventory holding cost in store. Leftover units have no value. The online channel is modeled exogenously: for each unit sold online, the retailer obtains a net profit margin w if it is not returned, and incurs a net loss r otherwise.

The market demand D is random and follows distribution F. A fraction of $\theta \in (0, 1)$ of the population has positive value v for the product, and a fraction $1 - \theta$ has zero value. We refer to the former as "high types," the latter as "low types," and the parameter θ as the "high-type probability." Customers are homogeneous ex ante: they don't know their valuation (or type) beforehand, but θ and v are common knowledge. Customers may learn their valuations before purchase only if they examine the product in store; otherwise, customers learn valuations after purchase. In particular, online purchases may be returned, as described below.

Each consumer makes a choice between shopping online directly or going to the store. If she chooses to buy online directly, she incurs hassle cost h_o (e.g., paying shipping fees or waiting for the product to arrive), and realizes her valuation only after receiving the product. If she likes the product (i.e., she is high type), then she keeps it and receives payoff $v - p - h_o$; if she dislikes the product (i.e., she is low type), then she returns it. Returns are costly to both the retailer and the consumers: each returned unit generates net loss $r > 0$ to the retailer and an additional hassle cost $h_r > 0$ to the consumer. We assume that low type consumers prefer returning the product to keeping it, i.e., $h_r < p$. Therefore, the consumer's expected payoff from buying online directly is given by

$$u_o = \theta(v - p - h_o) - (1 - \theta)(h_o + h_r).$$

On the other hand, if the consumer chooses to go to the store, she has to first incur hassle cost h_s (e.g., traveling to the store or searching for the product in aisles). We stress that the store hassle cost h_s is incurred *before* customers find and purchase the product in the store, whereas the online hassle cost h_o above is incurred *after* customers make the purchase online. Once she is in the store, the customer may encounter two possible outcomes: (1) If the store has the product in stock, then she can evaluate the product on the spot: a high type makes a purchase and receives payoff $v - p$, while a low type leaves without any purchase and receives payoff 0. (2) If the store is out of stock, she cannot resolve her value uncertainty in store, but she can buy the product online and receive an expected payoff u_o instead. Let ξ denote the probability that the store is in stock, and let $\hat{\xi}$ denote consumers' beliefs about this probability; throughout this chapter, we use the hat notation $\hat{\cdot}$ to denote beliefs. We assume that customers arrive sequentially to the market, all customer permutations are equally likely, and customers do not know their order of arrival. As a result, everyone has the same belief $\hat{\xi}$. Then, given belief $\hat{\xi}$, each consumer's payoff from visiting the store can be expressed as follows:

$$u_s(\hat{\xi}) = -h_s + \hat{\xi}\theta\,(v - p) + (1 - \hat{\xi})u_o.$$

Consumers compare the expected utility from each channel and chooses accordingly. In the spirit of omnichannel choice, i.e., consumers are willing to consider both channels, we assume v is large enough so that $u_s(1) \geq 0$ and $u_o \geq 0$.

Next, we consider the retailer's decision problem. First, the retailer anticipates that a fraction $\hat{\phi} \in [0, 1]$ of customers will visit the store. Then, if total demand is D, the retailer expects that the number of customers coming to the store will be $\hat{\phi}D$; also, since only high-type customers will eventually make a purchase in the store, given the store inventory level q, the retailer expects that the number of store customers who find that the store has the product in stock is $D_{in}(q) = \min\left(\hat{\phi}D, \frac{q}{\theta}\right)$, and the remaining $D_{out}(q) = \left(\hat{\phi}D - \frac{q}{\theta}\right)^+$ will encounter stockouts when they come to the store. Note that even though the inventory is q, up to $\frac{q}{\theta}$ customers may examine the product in the store because only a fraction θ of those customers will buy. Then, the retailer's profit is as follows:

$$\pi(q) = p\theta ED_{in}(q) - cq \tag{3.1}$$

$$+ [w\theta - r(1-\theta)]ED_{out}(q) \tag{3.2}$$

$$+ [w\theta - r(1-\theta)](1-\hat{\phi})ED \tag{3.3}$$

Given the store inventory level q, the newsvendor expected profit from selling the product in the store channel is shown in the first term (3.1) above. The next two terms respectively represent profit from customers who switch online after encountering stockouts in the store and customers who buy online directly. For each unit of online demand, the expected profit is $w\theta - r(1-\theta)$, because a fraction $(1-\theta)$ of online sales are returned. Again, in the spirit of omnichannel retailing, we assume $w\theta - r(1-\theta) > 0$ so that the retailer is willing to operate both channels. Finally, the retailer chooses inventory level q to maximize expected profit.

To study the strategic interaction between the retailer and the consumers, we shall use the rational expectations (RE) equilibrium concept, as what we did in Chapter 2. One important feature of a RE equilibrium is that beliefs must be consistent with actual outcomes. In other words, the retailer's belief $\hat{\phi}$ must coincide with the true proportion ϕ of consumers choosing the store channel, and consumers' beliefs over in-store inventory availability $\hat{\xi}$ must agree with the actual in-stock probability corresponding to the retailer's chosen quantity q. This probability is given by $A(q) = E\min(\phi D, \frac{q}{\theta})/E(\phi D)$, where $\phi > 0$, using the derivation from Deneckere and Peck (1995) and Dana (2001). Then, we have the following definition for a RE equilibrium. Henceforth, we refer to the RE equilibrium as "equilibrium" for brevity.

Definition 3. *A RE equilibrium $(\phi, q, \hat{\xi}, \hat{\phi})$ satisfies the following:*

i Given $\hat{\xi}$, if $u_s(\hat{\xi}) > u_o$, then $\phi = 1$; otherwise $\phi = 0$;

ii. Given $\hat{\phi}$, $q = \arg\max_q \pi(q)$.

iii. $\hat{\xi} = A(q)$.

iv. $\hat{\phi} = \phi$.

Conditions (i) and (ii) state that under beliefs $\hat{\xi}$ and $\hat{\phi}$, the consumers and the retailer are choosing the optimal decisions. Conditions (iii) and (iv) are the consistency conditions.

First, note that there is always a *nonparticipatory* equilibrium $(0, 0, 0, 0)$. If the retailer expects that no one will come to the store, he will stock nothing there, i.e., $q = 0$; If consumers believe there is no inventory in the store, they will not come, i.e., $\phi = 0$. In the end, we have a self-fulfilling prophecy and beliefs are trivially consistent with the actual outcome.

Is there any *participatory* equilibrium where consumers are willing to visit the store and the retailer has stock in the store as well (i.e., $\phi = 1$ and $q > 0$)? When such an equilibrium exists, it Pareto-dominates the nonparticipatory equilibrium. The reasoning is as follows. In the participatory equilibrium, customers choose to visit the store rather than buying online directly (as what they would do in the nonparticipatory equilibrium), so they must be better off in the former. Similarly, the retailer prefers to stock $q > 0$ in the participatory equilibrium rather than choosing $q = 0$ (which leads customers to buy online and yields the same profit as the nonparticipatory equilibrium), so the retailer must also be better off in the participatory equilibrium. In this chapter, we shall adopt the Pareto dominance equilibrium selection rule. The following proposition gives the equilibrium result; we use the superscript $(\cdot^\circ)$ to denote the equilibrium outcome for this base scenario.

Proposition 11. *There exists a threshold $\psi^\circ \in [0,1]$ such that*

- *if $\theta < \psi^\circ$, then consumers visit the store and $q^\circ = \theta \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right) > 0$;*
- *if $\theta \geq \psi^\circ$, then consumers buy online directly and $q^\circ = 0$.*

Proposition 11 shows that consumers come to the store only if the high-type probability is low (i.e., $\theta < \psi^\circ$) and they recognize the need to touch and feel the product before making a purchase. In this case, the retailer prepares a positive amount of inventory in the store to accommodate consumer demand. To the retailer, the overage cost is simply the inventory cost c and the underage cost is given by $p - c - w + r\frac{1-\theta}{\theta}$. The latter expression is the combination of two effects. First, each unit of underage is a lost opportunity of selling to a high type, who buys online instead, leading to a difference in profit margin of $p - c - w$. Second, each unit of underage is also a lost opportunity for customers to realize their valuations (by inspecting that unit of inventory); corresponding to one high type who could have bought this unit are $\frac{1-\theta}{\theta}$ low types who could have realized their low valuations and thus avoided the return cost r. Thus, as the online return cost r increases, the retailer holds more inventory, purely for informational purposes.

3.4. Physical Showrooms

Previously, in the base model, consumers can evaluate the product in the store only if it is in stock. Now suppose that there is a physical showroom in the store, so that consumers can always inspect the product, even when it is out of stock. In this section, we study the equilibrium outcomes and profit implications of adding such a physical showroom.

As before, we start by considering the consumer's choice between shopping online and in store. The consumer's payoff from visiting the store is $u_s(\hat{\xi}) = -h_s + \hat{\xi}\theta(v-p) + (1-\hat{\xi})\theta(v-p-h_o)$, given her belief about store inventory availability $\hat{\xi}$. Note that the first two parts of u_s remain the same as in the base model. The only change occurs in the third term: when the consumer encounters stockouts in store, she is still able to realize her valuation from the physical showroom; she chooses to purchase online and receives payoff $v-p-h_o$ only if she is of high type. The consumer's payoff from buying online directly, on the other hand, remains unchanged as $u_o = \theta(v-p-h_o) - (1-\theta)(h_o+h_r)$.

The retailer has a belief $\hat{\phi}$ about the fraction of consumers who would visit the store. Given this belief, the retailer's profit is as follows.

$$\pi(q) = p\theta ED_{in}(q) - cq \tag{3.4}$$

$$+ w\theta ED_{out}(q) \tag{3.5}$$

$$+ [w\theta - r(1-\theta)](1-\hat{\phi})ED \tag{3.6}$$

Comparing this to the profit function in the base model, we note that the only difference occurs in the term (3.5): this is because the physical showroom allows low types who encounter stockouts to learn their types and thus avoid the trouble of buying online and then returning. However, the two profit functions become identical when $r = 0$. Nevertheless, the retailer may still obtain different profits in equilibrium because consumer channel choices may change after the implementation of physical showrooms (i.e., ϕ in equilibrium may be different), as shown below.

Similar to the analysis of the base model, we apply the notion of RE equilibrium to describe the market outcome. We use superscript $\cdot^p$ to denote physical showrooms.

Proposition 12. *With physical showrooms, there exists a threshold $\psi^p \in [0,1]$ such that*

- *if $\theta < \psi^p$, then consumers visit the store and $q^p = \theta\bar{F}^{-1}\left(\frac{c}{p-w}\right) \geq 0$;*
- *if $\theta \geq \psi^p$, then consumers buy online directly and $q^p = 0$.*

Moreover, $\psi^p > \psi^\circ$ if and only if $r \leq \bar{r}$ for some $\bar{r} \geq 0$.

Proposition 12 (after the provision of physical showrooms) differs from Proposition 11 (before the provision of physical showrooms) in two significant ways. First, with a physical showroom, it may be rational for consumers to visit the store even if they correctly expect that there is no inventory held in store. Specifically, note that when $p - c \leq w$ and $\theta < \psi^p$, the optimal store inventory is 0 but consumers still visit the store. In this case, the store serves as a pure showroom: Consumers can always evaluate their valuations in store, but they must make purchases online since there is no inventory in the store. The idea of pure showrooms has been pioneered by e-tailers such as Warby Parker and has proven to be effective in curbing high online return rates (Bell et al., 2015). It is not surprising that pure showrooms are mostly adopted by e-tailers (i.e., with w higher than $p - c$); as these e-tailers establish a brick-and-mortar presence, they start holding inventory in the store. This is consistent with Warby Parker's recent move: in their stores, consumers can now buy non-prescription glasses and sunglasses to take with them.

Another difference between Propositions 11 and 12 is the change in store inventory level. Because low-type consumers will never buy online when the store is out of stock (since they can learn that they are of low type via the physical showroom), the underage cost decreases from $p - c - w + r\frac{1-\theta}{\theta}$ to $p - c - w$. As a consequence, the retailer holds less inventory in the store (i.e., $q^p \leq q^\circ$).

A reduction in store inventory level has a large impact on the channel choice of consumers. Consumers face two types of uncertainty during the shopping journey: (1) product value uncertainty, i.e., the possibility of buying something they do not like, and (2) availability uncertainty, i.e., the possibility of going to the store but encountering a stockout. By allowing consumers to inspect the product any time they come to the store, physical showrooms help to resolve product value uncertainty, which could attract consumers to visit the store; however, at the same time, the physical showrooms prompt the retailer to lower the store inventory level, which would increase availability risk and thus push consumers to the online channel. As a result, whether consumers are more likely to visit the store after physical showrooms are set up depends on which one of the two opposing effects is more significant. When returns are not too costly for the retailer (i.e., r is small), the reduction of store inventory is small, so the first effect above dominates and physical showrooms attract some customers to the store (i.e., $\psi^p > \psi^\circ$). In contrast, when the return cost r is very large, the second effect dominates and physical showrooms may drive consumers online (i.e., $\psi^p < \psi^\circ$). This consumer equilibrium behavior can be summarized in Figure 5.

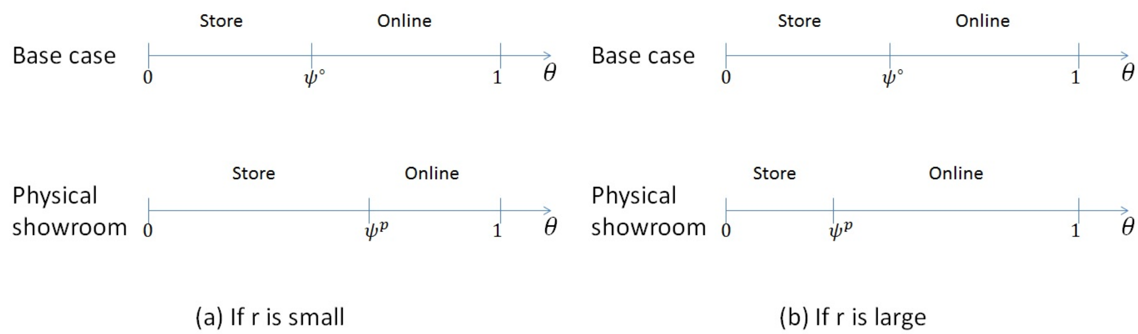


Figure 5: Comparison of consumer behavior with and without physical showroom

Proposition 13. *Compared to base model,*

- *if* $\theta < \psi^p$, *then providing physical showrooms increases total profit (i.e.,* $\pi^p > \pi^\circ$*);*
- *if* $\psi^p < \psi^\circ$ *and* $\theta \in [\psi^p, \psi^\circ)$, *then providing physical showrooms reduces total profit (i.e.,* $\pi^p < \pi^\circ$*);*
- *if* $\theta \geq \max(\psi^p, \psi^\circ)$, *then providing physical showrooms generates the same amount of profit (i.e.,* $\pi^p = \pi^\circ$*).*

According to Proposition 13, when the high-type probability is sufficiently large, i.e., $\theta \geq \max(\psi^p, \psi^\circ)$, then the physical showroom does not make a difference since consumers will always purchase online. Offering a physical showroom is profitable to the retailer as long as customers actually visit the showroom in store (as when $\theta < \psi^p$). Note that in this case, if the store is out of stock, consumers simply check the product in store but eventually complete the purchase online. This shopping behavior is known as showrooming, but is generated by stockouts rather than price differences. Though consumer showrooming behavior has been critiqued widely because showroomers often end up buying from a competitor's website (Wall Street Journal, 2012b,a), our result implies that it may still be beneficial to the retailer if consumers facing stockouts can be persuaded to make the purchase on the retailer's own online channel.

However, physical showrooms may have a negative effect on the retailer's profit when θ is moderate, i.e., $\theta \in [\psi^p, \psi^\circ)$. In this range, consumers would have come to the store in the base model, but now choose to buy online instead because they do not anticipate high in-stock probability in the store. In this case, the retailer would see an increase in returns and a decrease in profit; this is because the online purchases by low types will end up in returns. According to a comprehensive numerical study, the details of which are given in Appendix A.5, the implementation of physical showrooms leads to a reduction in profit for about 4.34% of 914,895 different parameter combinations, most of which arise when the store profit margin $p - c$ is lower than the online profit margin w.

3.5. Virtual Showrooms

Suppose the retailer implements virtual showrooms in the online channel. By trying on the product virtually, consumers receive an imperfect signal of their valuations. For modeling convenience, we assume that the signal S is binary, i.e., a group of consumers remain interested in the product (i.e., $S = 1$) while the remaining discover that the product is not for them and leave (i.e., $S = 0$). Further, we assume that the latter group consists of a fraction $\alpha \in (0, 1]$ of the low-type customers, i.e., the probabilities that a low-type and a high-type customer receives a signal $S = 0$ are $P(S = 0|L) = \alpha$ and $P(S = 0|H) = 0$, respectively, where L and H denotes the consumer type (i.e., *l*ow and *h*igh). In other words, the virtual showroom only screens out some of low-type customers, while potential high-type customers still have interest in the product after trying the product virtually. Note if $\alpha = 1$, the virtual showroom offers a perfect signal and all consumers learn their types, but if $\alpha < 1$, the virtual showroom screens out a fraction α of the low types. We thus interpret

α as the degree of informativeness of the virtual showroom.

We apply Bayesian updating to the beliefs of those consumers who remain interested in the product. This group includes all potential high-type customers and a fraction $1-\alpha$ of the potential low-type customers. Let θ' denote the posterior probability of a remaining customer (i.e., those who received signal $S=1$) being high-type. Then, given the priors $P(H)=\theta$ and $P(L)=1-\theta$, we have $\theta' = P(H|S=1) = \frac{P(S=1|H)P(H)}{P(S=1|H)P(H)+P(S=1|L)P(L)} = \frac{\theta}{1-\alpha(1-\theta)} > \theta$. Then, the two models, with and without virtual showrooms, are very similar. The only difference is that virtual showrooms generate a new consumer pool by filtering away some potential low-type consumers. As a result, the total demand size is $D' = P(S=1)D = [1-\alpha(1-\theta)]D$ (with cdf $F'(x) = F(\frac{x}{1-\alpha(1-\theta)})$ for any x) and a fraction θ' of them is of high type. Thus, similar to Proposition 11, the equilibrium outcome is given in the following proposition; we use the superscript $(\cdot^v)$ to denote the equilibrium outcome for the scenario with virtual showrooms. Henceforth, without further specification, we simply refer to this new pool of consumers as the retailer's consumers.

Proposition 14. *With virtual showrooms, there exists a threshold $\psi^v \in [0, \psi^\circ]$ such that*

- *if $\theta < \psi^v$, then consumers visit the store and $q^v = \theta' \bar{F}'^{-1}\left(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}\right) > 0$;*
- *if $\theta \geq \psi^v$, then consumers buy online directly and $q^v = 0$.*

Comparing Proposition 14 (after the provision of virtual showrooms) and Proposition 11 (before the provision of virtual showrooms), since $\psi^v \leq \psi^\circ$, virtual showrooms may attract customers to buy online instead of in the store (see Figure 6). This is not surprising. Virtual showrooms give consumers a similar hands-on experience as in the store. With decreased product value uncertainty, shopping online becomes more productive and appealing to consumers.

Interestingly, virtual showrooms are similar to consumer reviews; both provide online consumers with product information to help them alleviate value uncertainty. However, while reviews provide subjective information from other consumers, virtual showrooms provide

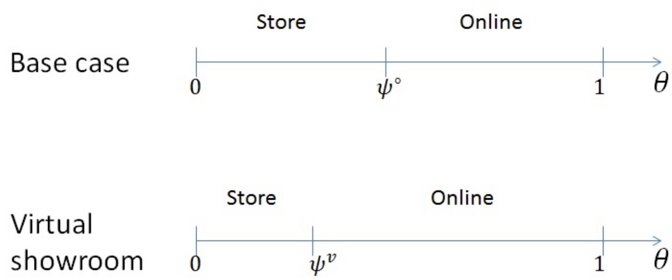


Figure 6: Comparison of consumer behavior with and without virtual showroom

objective information from the retailer. Therefore, consumer reviews may be subject to self-selection biases, e.g., Li and Hitt (2008), and they can also be influenced through dynamic pricing, as in Papanastasiou and Savva (2016); Yu et al. (2015). In contrast, virtual showrooms offer information content that is immune to strategic manipulation.

Proposition 15. *Compared to the base model,*

- *if* $\theta < \psi^v$ *or* $\theta \geq \psi^\circ$*, then providing virtual showrooms increases total profit, i.e.,* $\pi^v > \pi^\circ$*;*

- *if* $\theta \in [\psi^v, \psi^\circ)$*, there exists* $\bar{w}$ *such that providing virtual showrooms increases total profit (i.e.,* $\pi^v > \pi^\circ$*) if* $w > \bar{w}$*; but reduces total profit (i.e.,* $\pi^v < \pi^\circ$*) if* $w < \bar{w}$*.*

There are three cases discussed in Proposition 15. First, if consumers have a small high-type probability (i.e., $\theta < \psi^v$), then they always turn to the physical store for validation before making any purchasing decision. When the store is out of stock, consumers may choose to buy online directly before resolving product value uncertainty. In this case, virtual showrooms help by screening out some low-type consumers beforehand so that the potential number of returns will be smaller.

Second,if consumers have a large high-type probability (i.e., $\theta \geq \psi^\circ$), then they are comfortable buying online. In this case, virtual showrooms serve as the main source of product information for consumers. By screening out potential low-type customers before they make any purchase on the website, virtual showrooms help to avoid returns and increase profits.

However, when $\theta \in [\psi^v, \psi^\circ)$, implementing virtual showrooms may backfire. In this case, consumers originally visit the store in the base model, but virtual showrooms attract them to buy online instead. Although the online return rate decreases from $1 - \theta$ (without virtual showroom) to $1 - \theta'$ (with virtual showroom), total returns may increase. This is because more people now choose to buy online, including low types who are destined to return their purchases. The resulting increase in returns will drive down total profit. Unless the online profit margin is high enough, customer migration from store to online will be unprofitable for the retailer. This result offers a possible explanation for fashion retailer H&M's decision to remove their virtual showroom (called the Dressing Room) from their website even though it has been popular among consumers (H&M, 2010). Our result also suggests that retailers should be cautious when looking at online return rates and should not neglect the total number of returns as an informative companion metric. According to a comprehensive numerical study, the details of which are given in Appendix A.5, the implementation of virtual showrooms hurts profits for about 4.58% of 914,895 different parameter combinations, most of which arise when the online profit margin w is lower than the store profit margin $p - c$.

3.6. Availability Information

Suppose the retailer provides availability information on its website, so that consumers are able to check real-time store inventory status. With such information, the sequence of consumer arrivals matter: consumers who arrive early will see that the store has inventory in stock, but consumers who arrive late will encounter stockouts. In the former case, the consumer can go to the store and receive an expected payoff of $u_{s,in} = -h_s + \theta(v - p)$ because she will certainly obtain the product if she realizes high valuation. In the latter case, the consumer will receive nothing from visiting the store and thus will choose to buy online instead. In practice, some retailers may also reveal the number of units in inventory as part of the availability information. This does not change consumers' shopping behavior in our model, since consumers care only about whether the store is in stock or not when they arrive in the market.

Each consumer chooses between shopping online directly or going to the store, given the current store inventory availability status. Let ϕ_{in} denote the fraction of customers visiting the store when it is in stock. As before, only high-type consumers will absorb store inventory when they come to the store. Then, the expected number of customers who see that the store is in stock is $D_{in}(q) = \min\left(D, \frac{q}{\theta\phi_{in}}\right)$, and the remaining $D_{out}(q) = \left(D - \frac{q}{\theta\phi_{in}}\right)^{+}$ will find that the store is already out of stock when they check availability online. Note that the expressions for $D_{in}(q)$ and $D_{out}(q)$ are slightly different from before because all consumers, whether or not they choose to come to the store, will receive availability information. Thus, the retailer's profit function is as follows:

$$\pi(q) = p\theta\phi_{in} ED_{in}(q) - cq \tag{3.7}$$

$$+ [w\theta - r(1-\theta)](1-\phi_{in}) ED_{in}(q) \tag{3.8}$$

$$+ [w\theta - r(1-\theta)] ED_{out}(q) \tag{3.9}$$

The first two parts of the profit function correspond to the case when the store is in stock: (3.7) is the newsvendor profit from the store, and (3.8) is the profit from those who buy online directly. The last part (3.9) corresponds to the case when the store is out of stock and all consumers buy online.

We use superscript $(\cdot^{a})$ to denote the market outcome with availability information, which is given in the following proposition.

Proposition 16. *With availability information, there exists a threshold $\psi^{a} \in [\psi^{\circ}, 1]$ such that the market outcome is given as follows:*

- *If $\theta < \psi^{a}$, then consumers visit store if store is in stock, and buy online directly if store is out of stock; $q^{a} = \theta\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right) > 0$*
- *If $\theta \geq \psi^{a}$, then consumers always buy online directly; $q^{a} = 0$.*

Comparing Proposition 16 (after the provision of availability information) and Proposition 11 (before the provision of availability information), since $\psi^a \geq \psi^\circ$, we see that providing availability information may attract consumers to the store (see Figure 7). In the base scenario, consumers bear availability risk because they incur the hassle of going to the store before finding out whether the store is in stock. Now, providing availability information eliminates availability risk: if the store is in stock, consumers are guaranteed availability before incurring any sunk cost.

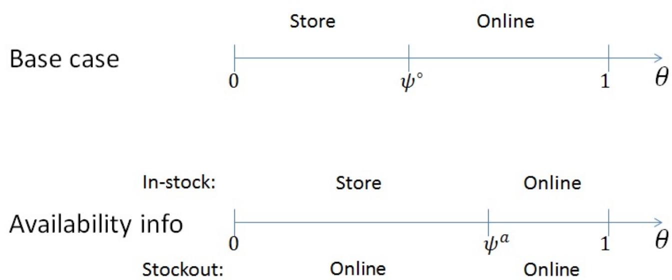


Figure 7: Comparison of consumer behavior with and without availability information

Su and Zhang (2009) study a different availability information sharing mechanism, i.e., quantity commitment, under which the retailer commits to an initial inventory level in the store. Note there are two key differences between quantity commitment and the availability information mechanism studied in this chapter. First, quantity commitment merely informs consumers about the retailer's initial inventory level, and thus consumers may still encounter stockouts when they visit the store; in contrast, with real-time availability information, consumers no longer need to face any availability uncertainty. Second, as the name suggests, quantity commitment requires a commitment device (e.g., reputation or legal institutions), but real-time availability information is easily verifiable (e.g., using mobile devices); in this sense, the latter is simpler to implement.

Proposition 17. *Compared to the base model,*

- *if $\theta < \psi^\circ$ or $\theta \geq \psi^a$, then providing availability information generates the same amount of profit, i.e., $\pi^a = \pi^\circ$;*

- *if $\theta \in [\psi^{\circ}, \psi^{a})$, then providing availability information increases total profit, i.e., $\pi^{a} > \pi^{\circ}$.*

According to Proposition 17, offering availability information has no negative effect on the retailer. Specifically, if availability risk is not a pivotal factor in consumers' shopping decision, e.g., they always visit the store because of high product value uncertainty (as when $\theta \leq \psi^{\circ}$) or they always feel comfortable buying online directly (as when $\theta > \psi^{a}$), then sharing availability information with consumers does not make a difference.

In contrast, if consumers care about availability risk and choose to buy online purely out of concern for such risk (as when $\theta \in [\psi^{\circ}, \psi^{a})$), then providing availability information will have a positive effect on profits. By assuring consumers about inventory availability in store, the retailer can attract more people to the store; consumers can then physically inspect the product and realize their valuation before making any purchase. The bottom-line is a decrease in product returns and thus an increase in total profit.

3.7. Joint Implementation

So far, we have separately looked at physical showrooms, virtual showrooms, and availability information. For each information mechanism, we have studied the individual impact on consumer behavior and retail operational efficiency. In this section, we focus on the interactions among them, searching for potential complementary effects when the retailer implements more than one of them at the same time. The goal of this section is to find out what combination of the three mechanisms the retailer should implement. In the following analysis, we combine the superscripts of corresponding types of information mechanisms to form the notation for the combined scenario. For example, superscript $\cdot^{pa}$ denotes the case where both physical showroom and availability information are provided. We start with pairwise combinations before considering implementing all three mechanisms simultaneously. Detailed analyses of the market outcome of all these scenarios are given in Appendix A.3.

We start with combining physical showrooms and availability information.

Proposition 18. $\pi^{pa} \leq \max(\pi^p, \pi^a)$, *and there exists* $\underline{\psi}^{pa} \in [0, \psi^p]$ *such that* $\pi^{pa} < \pi^p$ *if* $\theta \in [\underline{\psi}^{pa}, \psi^p)$.

According to Propositions 12 and 16, both physical showroom and availability information attract customers to the store. It is this overlap that creates redundancy if they are implemented together. As shown in Proposition 18, there is no complementary effect between these two mechanisms. In fact, offering availability information may hurt profits if the retailer has already implemented physical showrooms. We provide a brief explanation next.

When there are both physical showrooms and availability information, as shown in Appendix A.3.1, consumers always visit the store when the high-type probability is small (i.e., $\theta < \underline{\psi}^{pa}$), even when the store is already out of stock and consumers know that. In such cases, consumers still prefer to check the product in the physical showroom before making any purchase. However, if the high-type probability is slightly higher, then consumers would stop coming to store and instead choose to buy online directly once they know that the store is out of stock. As a result, the retailer can no longer use the physical showroom to screen out potential low-type consumers; this may lead to more returns and less profit.

Next, let us combine physical showrooms and virtual showrooms.

Proposition 19. $\pi^{pv} \leq \max(\pi^p, \pi^v)$, *and there exists* $\psi^{pv} \in [0, \psi^p]$ *such that (1)* $\pi^{pv} < \pi^p$ *if* $\theta \in [\psi^{pv}, \psi^p)$, *and (2)* $\pi^{pv} < \pi^v$ *if* $\psi^{pv} < \psi^v$ *and* $\theta \in [\psi^{pv}, \psi^v)$.

Both types of showrooms help consumers alleviate product value uncertainty. Given this overlap, Proposition 19 points out that it is sufficient to implement only one of them. Again, we see that a profit loss may result when both types of showrooms are used together compared to the case when they are implemented individually.

The reason why offering both showrooms would make profit strictly worse is because each one of them may bring about some negative effects as discussed in previous sections, and combining both does not help to solve these issues. Specifically, when $\theta \in [\psi^{pv}, \psi^p)$, compared to the case where there is only physical showrooms, similar to what we have found in Proposition 15, adding virtual showrooms would cause consumers (including some low types) to buy online instead of in store, and thus there will be more returns. When $\psi^{pv} < \psi^{v}$ and $\theta \in [\psi^{pv}, \psi^{v})$, compared to the case where there is only virtual showroom, similar to what we have found in Proposition 13, adding physical showrooms may prompt the retailer to decrease store inventory so much that consumers choose to buy online without first resolving their product value uncertainty in the store.

Finally, we consider the remaining combinations of: (i) putting together virtual showrooms and availability information, and (ii) putting together all three information mechanisms. These are respectively studied in the two parts of the following proposition.

Proposition 20.

- *There exists* $\psi^{va} \in [\psi^{v}, 1]$ *such that* $\pi^{va} > \max(\pi^{v}, \pi^{a})$ *if* $\theta \in [\psi^{v}, \psi^{va})$*;*
- $\pi^{pva} \leq \max(\pi^{pv}, \pi^{va})$*, and there exists* $\underline{\psi}^{pva} \in [0, \psi^{pv}]$ *such that* $\pi^{pva} < \pi^{pv}$ *if* $\theta \in [\underline{\psi}^{pva}, \psi^{pv})$*.*

Proposition 20 contains two parts. The first part shows that there may be a complementary effect between virtual showrooms and availability information. This happens if $\theta \in [\psi^{v}, \psi^{va})$, i.e., consumers visit the store when it is in stock and buy online if the store is out of stock (see Proposition 52 in Appendix A.3.3). In this case, the synergy between availability information and virtual showrooms works as follows: availability information helps to attract consumers to the store by eliminating availability uncertainty, and when the store is out of stock and consumers buy online, the virtual showroom serves as a filter to screen out low-type consumers. Out of the three information mechanisms studied in this chapter, virtual showrooms and availability information is the only pair that exhibits complementarity. This is because there are potential overlaps in the functions of these mechanisms: Both physical showrooms and availability information would attract customers to store, while both physical and virtual showrooms would mitigate consumers' product value uncertainty. Consequently, it is unsurprising that we find complementary effects only between virtual showrooms and availability information. In the same spirit, we also find that it is not necessary to combine all three different types of information mechanisms, as shown in the second part of Proposition 20.

To conclude this section, we examine the optimal information structure, based on the three mechanisms that we have studied. Since it may be costly to implement any one of them, we consider offering multiple mechanisms only when it generates strictly higher profit. Then, given Propositions 18 -20, there are five possible options: choose exactly one of the three mechanisms, implement both virtual showrooms and availability information, or "do nothing" as in the base model. Let π^* denote the optimal profit under the optimal information structure. We have the following result.

Proposition 21.

i. If $\theta < \psi^p$, then $\pi^* = \pi^p$;

ii. If $\theta \geq \psi^p$, then

a) if customers visit the store in the base model (i.e., $\theta < \psi^\circ$), then there exist thresholds $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3$ and

- *if $\alpha < \alpha_1$, then $\pi^* = \pi^v$*
- *if $\alpha \in (\alpha_1, \alpha_2)$, then $\pi^* = \pi^{va}$*
- *if $\alpha \in (\alpha_2, \alpha_3)$, then $\pi^* = \pi^\circ$*
- *if $\alpha > \alpha_3$, then $\pi^* = \pi^v$*

b) if customers buy online in the base model (i.e., $\theta \geq \psi^\circ$), then there exist thresholds $\alpha'_1, \alpha'_2 \in [0, 1]$ such that $\alpha'_1 \leq \alpha'_2$ and

- *if $\alpha < \alpha'_1$, then $\pi^* = \pi^{va}$*

* *if* $\alpha \in (\alpha'_1, \alpha'_2)$, *then* $\pi^* = \pi^a$

* *if* $\alpha > \alpha'_2$, *then* $\pi^* = \pi^v$

Proposition 21 suggests that the main driver behind the optimal information mechanism is the parameter θ, i.e., the high-type probability. Part (i) of the proposition shows that when θ is low (i.e., $\theta < \psi^p$), it is optimal to provide only physical showrooms; part (ii) shows that when θ is high (i.e., $\theta \geq \psi^p$), virtual showrooms or/and availability information work better. A reasonable proxy for the θ parameter is the online return rate, since products with a smaller high-type probability θ are more likely to be returned. Therefore, Proposition 21(i) suggests that for products that are traditionally afflicted with high online return rates (e.g., fashion apparel and accessories), physical showrooms should be implemented to attract customers into stores to touch and feel the product before making a purchase. Our analysis advocates physical showrooms rather than virtual showrooms, which tend to be a more intuitive choice for retailers plagued by online returns. Our finding is also consistent with the recent move by many fashion retailers' (e.g., Bonobos and Warby Parker's) to build more physical showrooms (New York Times, 2014; Wall Street Journal, 2014b).

Next, for products with relatively more manageable online return rates, such as home goods and electronics (Laseter and Rabinovich, 2011, pg.118), Proposition 21(ii) describes the optimal information mechanism. Depending on whether customers visit the store or buy online directly in the base model (i.e., without any information mechanism), the baseline scenarios are different and our results are separately reported in parts (ii)(a) and (ii)(b). In both cases, our results suggest that the retailer should consider implementing virtual showrooms to provide product value information to consumers, and possibly also provide availability information to maintain the attractiveness of the store channel. In part (ii)(a), we also find that when customers visit the store in the base model, it may be optimal to not implement any additional information mechanism at all.

Proposition 21 also demonstrates the importance of the α parameter, i.e., the informativeness of the virtual showroom. Some implementations of virtual showrooms involve relatively rudimentary functions (e.g., picture upload), while others use more advanced virtual reality technologies (e.g., 3D technology) which offer online shoppers a more vivid try-on experience. As virtual showrooms become more informative (i.e., as α increases), Proposition 21(ii) shows that it is possible for them to become less attractive. Specifically, in both parts (a) and (b), virtual showrooms are a part of the optimal information mechanism only when α is either sufficiently large or sufficiently small. In other words, attempts to enhance the informativeness of virtual showrooms, if not significant, may reduce profits. The reason is as follows: unless the virtual showroom becomes very effective in screening out low-type customers, it may simply end up attracting online transactions and thus increasing online returns.

3.8. Extensions

3.8.1. Endogenous Online Channel

In our original model, we have assumed that the online channel is exogenous; specifically, there is always enough inventory to satisfy online demand. We can extend our basic model to include an endogenous online channel as follows. Suppose the retailer has a one-time ordering opportunity to prepare the inventory level q_s and q_o in the store and online channel. The unit inventory costs in the store and online channels are c_s and c_o, respectively. The product price p is still the same across both channels. Then, for each unit of product sold online, if it is not returned, the retailer earns the revenue p; if it is returned, the retailer incurs a net cost k. When the store is out of stock, consumers may switch back to the retailer's online channel as before. When the online channel is out of stock, those who are willing to buy online will leave for other websites to buy the product at the same price. In many companies (e.g., Bonobos and Warby Parker), store employees are trained and equipped with digital devices to help store customers order online. However, when customers are shopping online at home, it is hard for a firm to persuade customers to come to store when online is out of stock. Our model setup is to capture this difference. Appendix A.4.1

presents a detailed analysis of this model. We find that the drivers behind consumer channel choice and the effects of the three information mechanisms remain unchanged.

3.8.2. Continuous Valuations

Our analysis has assumed that consumer valuation can take on only two values, either v or 0. As a result, there are only two types of customer in the market, i.e., high types (who like the product) and low types (who don't). Our model can be extended to a more general case. Suppose consumer valuation V follows a continuous distribution G. Similar as our basic model, consumers are homogeneous ex ante; they know the distribution G but not their individual valuation beforehand.

Consumers who buy online directly will realize their valuations after receiving the delivery, and there are three possible outcomes. Consumers with $V > p$ will find that they like the product and thus keep it, and consumers with $V < p - h_r$ will find that they do not like the product and thus return it. These two types of behavior correspond to the high type and low type in the basic model. Finally, we have a third possibility: consumers with $V \in (p - h_r, p)$ will find that they do not like the product (since $V - p < 0$) but they still choose to keep it rather than going through the hassles of returning it back to the retailer (since $-h_r < V - p$).

When there is a virtual showroom, similar as in the basic model, we assume that it helps to screen out some "low-type" customers in the online channel. Specifically, the signal consumers receive after checking with the virtual showroom is binary: those with valuation $V < \bar{v}$ will realize that their valuation is low and therefore leave the market without any purchase, while those with valuation $V \geq \bar{v}$ will realize that their valuation is greater than $\bar{v}$ and update their belief about the valuation distribution to G' such that $\forall v \geq \bar{v},\ G'(v) = \frac{G(v) - G(\bar{v})}{1 - G(\bar{v})}$, where the threshold $\bar{v} \leq p - h_r$. With some "low-type" customers being screened out, the total number of customers left in the market is $D' = [1 - G(\bar{v})]D$. Then, we obtain a new customer pool with total demand size D' and valuation distribution G'. A detailed analysis of this model extension is given in Appendix A.4.2, from which we find that our main insights remain valid.

3.8.3. Informed and Uninformed Consumers

So far, we have assumed that consumers are homogeneous at the beginning of their shopping journey, although they may have different realized valuations ex post. We can extend our basic model to incorporate ex ante heterogeneity in the consumer population. Specifically, suppose that a fraction λ of the customers are informed, i.e., they know that they are of high type (i.e., with valuation v) right from the beginning. The remaining fraction $1-\lambda$ of customers are uninformed as before, i.e., they do not know their types but they know the high-type probability θ. As a result, only uninformed customers will react to the implementation of physical/virtual showrooms, since informed customers do not need any product value information.

Appendix A.4.3 presents a detailed analysis of our extended model, which shows that most of our key insights continue to hold. However, there is a slight difference: specifically, we find that the retailer may sometimes wish to offer both physical showrooms and availability information (because the former caters to uninformed customers while the latter caters to informed customers). Nevertheless, we continue to find that physical and virtual showrooms do not exhibit any complementary effects.

3.8.4. Heterogeneous Valuations

In our analysis, we have assumed that customers have the same valuation v if they find they like the product. Our model can also be extended to the case where consumers have heterogeneous valuations. Suppose there are two types of customers and a fraction θ of the first (second) type customers has value v_1 (v_2), and a fraction $1-\theta$ has zero value. Then, as long as both types of customers are willing to consider both channels, i.e., $u_{si}(1) \geq 0$ and $u_{oi} \geq 0$, $i = 1, 2$, all of our previous results continue to hold. The reason is as follows: Each consumer's channel choice (which is determined by the sign of $u_{si} - u_{oi}$) is independent of their valuation. Also, because customers pay the same price and they have the same probability $1 - \theta$ of returning the product, the retailer obtains the same expected profit from each type of customers. In other words, from the retailer's perspective, the two types of customer are the same.

3.9. Conclusion

Consumers are no longer dedicated to one particular channel. Their shopping journey has become omnichannel; they strategically switch channels to best suit their personal convenience when evaluating and purchasing products. In response, more and more retailers are starting to transition from the traditional channel-specific management style to omnichannel operations, where different channels are integrated in a seamless way. In this chapter, we studied three omnichannel information mechanisms, i.e., physical showrooms, virtual showrooms, and availability information, which have been recently experimented and adopted by an increasing number of retailers.

The implementation of any omnichannel strategy is not easy. For example, to set up physical showrooms, a retailer may need to restructure its stores; similarly, virtual showrooms rely on the development of relevant technologies, and an integrated information system is needed to provide availability information online. However, we believe that these technical barriers will eventually be overcome. In this chapter, we have put aside considerations of implementation cost, and set our focus on the implications of these information mechanisms on retail operations. This way, we hope to help retailers understand the pros and cons if and when implementation becomes possible.

In this chapter, we build a stylized model that captures essential elements of omnichannel retail environments. We find the information mechanisms are generally profitable to a retailer by helping to alleviate consumer uncertainty about product value and inventory availability. However, in some cases, we find unexpected consequences that lead to more returns and less profit, echoing the well-known enigma that providing more information does not necessarily translate into higher profits. Furthermore, we find that the information mechanisms studied herein may be substitutes, so retailers should exercise caution when implementing multiple mechanisms simultaneously.

The current model focuses on a retailer's inventory decision, while price is assumed to be exogenous. This model setup applies to the case where the price is predetermined. For example, Warby Parker has kept the price of most of their glasses constant at $95, even after they have implemented showrooms. Another interesting and important question is how to integrate omnichannel initiatives with pricing strategies. Price differentiation has generated the phenomenon of showrooming: shoppers physically inspect a product in one retailer's store and then buy the same product from another online retailer at a lower price (Wall Street Journal, 2012a,b). Balakrishnan et al. (2014) and Mehra et al. (2013) provide some initial analyses in this area. When consumers engage in showrooming, one might expect that physical showrooms will become less profitable; however, there is anecdotal evidence suggesting that big-box retailers, the presumed victims of showrooming behavior, are in fact trying to make their stores a better showroom for consumers (CNBC, 2013; Wall Street Journal, 2013b). Future research is needed to understand this interesting phenomenon.

CHAPTER 4 : Omnichannel Service Operations with Online and Offline Self-Order Technologies

4.1. Introduction

Many merchants speak of the gains to be made by fusing in-store with digital commerce. Now, restaurants are turning to a variety of hardware to make it happen. Specifically, many firms in the restaurant industry are considering the implementation of self-order technologies, which are believed to be able to streamline transaction processes, reduce overhead, and potentially increase revenue (Kimes and Collier, 2015). In general, there are two types of self-order technologies, depending on whether self-service is offered online or offline.

Online platforms that allow customers to use services without the direct involvement of service employees have become ubiquitous with the development of Internet and mobile technology (Chung, 2013). Recently, many restaurants are also starting to realize the power of the Internet by providing online self-order service. For example, at the end of 2014, Starbucks introduced a feature called Mobile Order & Pay in their mobile app, through which customers can place an order and pay ahead using their mobile devices. This way, customers can skip the order line and no longer need to wait in the store for their order to be prepared. Instead, when they later walk into the store, they can go directly to the pickup area and ask the barista for their order. Besides mobile ordering apps, some other firms (e.g., Taco Bell) also allow customers to order through the web, although there is some evidence (e.g., eMarketer (2012)) that restaurant orders placed via mobile apps are 10 times higher than those placed via web.

In-store (offline) machines have been widely used in many service industries to enable customer self-service (e.g., ATMs in the banking industry and self checkout stations in the retail industry). In the restaurant industry, some firms recently installed self-order machines in their offline store channel. For example, McDonalds have in-store kiosks in about

45% of their restaurants in Europe, through which customers can place an order without interacting with a human cashier (Eater, 2016). Chili's has recently installed more than 45,000 tablets, which allows ordering directly at the table (the Atlantic, 2014). More advanced technologies, such as robot waiters, have also been adopted by some restaurants to provide customers with more convenient ordering experiences in the store (Mirror, 2016). Machines, compared to human cashiers, offer restaurants a cheaper way to take orders from customers; this is often touted as a major motivation behind many companies' move to offer offline self-order service (CNN, 2014; Yahoo Finance, 2016).

Both online and offline self-order technologies allow customers to place an order by themselves. However, there are some key differences between the two. On one hand, online customers have instant access to the online self-order system because they own the digital device and can also preorder before they arrive at the store. In contrast, with offline self-order technology, customers cannot preorder in advance and may even need to wait to use the ordering machines. Another major difference is that online self-order technology may have limited accessibility among customers, compared to its offline counterpart. This is because online self-ordering requires customers to have the digital device to access the platform. For example, Starbuck's Mobile Order & Pay app, which is regarded as one of the most popular mobile ordering apps in the industry, is used by only 10% of its customers (NBC News, 2016). In contrast, in-store machines are generally accessible to any customer who comes to the store.

Self-order technology is meant to speed up the ordering and purchase process, and thus helps to boost sales (Business Insider, 2015a). However, the whole restaurant service is not limited to ordering; a significant part of customer waits occurs while the firm prepares food in the kitchen area. It is not clear how the customer's total wait will change as a result of the implementation of the self-service technology at the ordering stage. Moreover, the popular press has focused attention on the benefit of self-order technology on those online or tech-savvy customers (e.g., Wall Street Journal (2014a); Time (2015)), while neglecting customers who may not be able to use or have no access to the technology. In this chapter, we adopt a broad view of the entire operational process and the entire market.

With self-order technology, customers no longer need a store employee to take the order for them. As a result, there is a lot of public concern about potential job cuts in the restaurant industry (CNN, 2014). The recent movement of minimum wage increases in the US further amplifies such concerns, and it is believed that restaurants are going to turn to automation to replace human workers in order to cut labor costs (Yahoo Finance, 2016). However, restaurants claim that they are not planning to pare down their work force (Business Insider, 2015a; Huffinton Post, 2015). In this chapter, we intend to reconcile these two groups of opinions and derive conditions under which total workforce level decreases (as the public fears) or not (as firms claim) after the implementation of self-order technologies.

The implementation of either type of self-order technologies requires significant investment. For example, implementing self-order kiosks at McDonalds' reportedly costs between \$120,000 and \$160,000; this is a significant financial burden to the store (Business Insider, 2015b). When it comes to online ordering platforms, for chain restaurants with multiple locations, the availability of such services at all of its locations is crucial to its success (Mobile Strategy 360, 2016); this typically requires an integration of IT systems of all stores, which is a technologically and financially challenging task. Given limited budget, it is important for firms to know which type of self-order technology is more profitable to pursue.

In this chapter we focus on the following research questions:

1. What are the impacts of both self-order technologies on customer shopping behavior?
2. How should a firm change its capacity and workforce level with self-order technology?
3. Which self-order technology (online or offline) will help a firm generate more profit?

To address these questions, we develop a stylized model. There is a restaurant serving wait-sensitive customers. The service system is modeled as a two-stage tandem queueing network, where customers first place an order through the register and then wait for their order to be prepared in the kitchen. The firm chooses the capacity level at both stages to maximize profit rate. We first study the impact of each of the online and offline self-order technology on the firm's optimal capacity choice and customer's shopping behavior. Then, we compare these results to the base scenario where there is no self-order service. Finally, we compare online and offline self-order technologies and study how a firm should choose between the two.

We summarize our main results. First, all customers can benefit from the implementation of self-order technology and thus increase their shopping rate. On the one hand, those customers who use the technology shop more frequently than before because of the reduction of wait cost: With online self-order technology, online customers not only skip the ordering line, but they can also order prior to arrival and thus do not need to wait in the store while their food is being prepared; with offline self-order technology, those tech-savvy customers who are willing to use machines enjoys a shorter wait time than before, as the firm can afford to provide a larger capacity due to the lower cost of machines. On the other hand, customers who do not use the self-order technologies may also benefit from their implementation and thus choose to come to the store more often than before; although they may still encounter a long line when placing an order, it is now faster to get the food after placing an order because it is profitable for the firm to increase capacity level in the kitchen.

Second, after the implementation of either type of self-order technology, the firm should shift workforce from the front-end to the back-end of the store; and sometimes the firm should even increase total workforce level. On the one hand, the staffing level should decrease at the front-end as self-order technologies allow customers to place an order without the help of human servers. On the other hand, as both self-order technologies help to reduce the cost to serve a customer, the firm may have the incentive to increase capacity in the kitchen area to speed up the food preparation process in order to attract more customers to the store. Combining the impact on both the front-end and back-end of the store, it may be profitable for the firm to increase jobs after implementing self-order technologies, although public opinion suggests otherwise (CNN, 2014; Yahoo Finance, 2016). In fact, we find that firms with high capacity costs (and hence low capacity levels) should have a greater incentive to hire more workers along with the implementation of self-order technologies, because it is more effective to increase capacity to attract demand from customers who previously had to experience long wait times.

Third, online self-order technology allows customers to place an order ahead of time using their own digital device and thus significantly reduces customer wait cost; however, it is hard for the firm to offer wide accessibility of the service to customers in the market. With offline self-order technology, the firm can ensure the availability of the service to any store customers who are willing to use the technology, but it does not provide customers with as much benefit of wait cost reduction as the online technology does, since customers still need to wait in the store for the self-order machines to be available and for the food to be prepared in the kitchen. As a result, if customers care significantly about wait time, online self-order technology helps to bring in more profit as it is more effective in relieving customer wait cost; otherwise, offline self-order technology is more profitable to implement, as the firm can maximize its impact by broadening its reach. Since customer wait sensitivity is positively related to income level (Campbell and Frei, 2011; Propper, 1995), our results indicate that a firm targeting a high-end (low-end) market should first consider implementing online (offline) self-order technology.

4.2. Literature Review

There is a large body of literature studying capacity management in a queueing system where customers are sensitive to wait, for example, Mendelson (1985); Chen and Frank (2004). Hassin and Haviv (2003)[Chapter 8] provides a detailed review of this stream of literature. One way to change service capacity is through the change of workforce level. Erlang (1917) first suggested the square-root staffing rule in the context of his study of circuit-switched telephony, which is later formally proved by Halfin and Whitt (1981) using fluid approximation. Bassamboo et al. (2010) study a staffing problem where mean arrival rate of work is random, and find that simple capacity prescriptions derived via a suitable newsvendor problem are surprisingly accurate. Please refer to Gans et al. (2003) and Aksin et al. (2007) for excellent reviews of literature on staffing decisions in services. We contribute to this stream of literature by studying the impacts of self-service technologies on the service rate and workforce level decisions, motivated by the recent move towards service automation in the restaurant industry. At the same time, many firms are shifting service workload from their employees to outside independent contractor, see Gurvich et al. (2015) and Ibrahim (2015) for some recent work on workforce management in the on-demand economy; self-service technology also helps to take some workload away from firms, shifting some service responsibilities that are traditionally fulfilled by store employees to machines or customers.

Although self-service system is relatively new for restaurants, it has been widely used in other industries. The implementation of self-service technologies in banks and retail stores have been studied extensively. Many empirical papers (e.g., Hitt and Frei (2002); Meuter et al. (2000); Curran et al. (2003); Iqbal et al. (2003); Marzocchi and Zammit (2006)) focus on identifying attitudinal, behavioral, and demographic factors associated with customer's adoption decision and their evaluation process after a firm opens the self-service channel (e.g., ATM, online banking, self checkout station, etc). Campbell and Frei (2010) and Xue et al. (2007, 2011) further test whether and how self-service technologies change customer behavior. In this chapter, we study the impact of self-order technology in the restaurant industry. A detailed literature review on restaurant management is provided by Thompson (2010). In particular, the link between self-order technology and job cuts has generated much debate. Since the adoption of self-order technology is still in its infancy in the restaurant industry (where its implementation is in the form of pilot programs for many firms), there is currently little analytical research. Susskind and Curry (2016) and Tan and Netessine (2016) provide some recent empirical work on the impact of tabletop ordering devices on restaurant performance. We contribute to the restaurant management area by analytically examining the impact of both online and offline self-order technologies on service operations for restaurants.

In this chapter, we consider a scenario where a firm with a physical store considers offering the self-order service in the online or offline channel. The integration of online and offline channels to provide customers seamless shopping experience is the core principle behind omnichannel management. The topic is broadly surveyed in Bell et al. (2015) and Brynjolfsson et al. (2013). Different omnichannel strategies have been studied in literature, e.g., ship-to-store and ship-from-store (Jasin and Sinha, 2015; Gallino et al., 2016), online and offline showrooms (Bell et al., 2015; Gao and Su, 2016b). Bernstein et al. (2008, 2009) study firm's channel choice in the face of competition. Gallino and Moreno (2014) empirically and Gao and Su (2016a) (which is based on Chapter 2 of this dissertation) analytically examine the impacts of a new omnichannel fulfillment method, i.e., buy-online-and-pick-up-in-store (BOPS), on retail operations. Online self-order technology serves a similar function as the BOPS fulfillment method, both allowing customers to place an order in the online channel and then pick it up in the offline channel. Most papers in the area of omnichannel management focus on retail companies, to which inventory and product management are important considerations. In contrast, we study a service system with queues, where service capacity management is crucial.

One of the most common ways of providing online self-order service is through mobile ordering apps. How to effectively use mobile devices as a marketing channel to send targeted promotion messages to customers has been a heated topic; see, for instance, Luo et al. (2013); Bart et al. (2014); Ghose et al. (2015); Andrews et al. (2015); Fong et al. (2015). Different from this stream of literature, instead of empirically examining the impacts of mobile technology on a firm's marketing efficiency, we analytically study how mobile ordering app (or online self-order technology in general) could influence operations in a service system.

4.3. Base Model

There is a restaurant serving food to customers. The service system is modeled as a 2-stage tandem queueing network: at stage 1, customers place their orders through the front-end cashiers/waiters, and then at stage 2, the order is transmitted to the back-end kitchen area where food is prepared. From each customer, the firm obtains net revenue r. The firm decides the service rate (or capacity) at each stage, denoted as μ_1 and μ_2. Suppose the firm incurs a cost at rate $c_i\mu_i$ for maintaining capacity μ_i at stage $i = 1, 2$, where unit capacity cost $c_i > 0$. To ensure it is profitable to serve a customer, we assume $r - c_1 - c_2 > 0$.

There is a number of M homogeneous customers in the market. Customers are assumed to be infinitesimal, i.e., every customer is small relative to the size of the market. Customers need to wait at both stages before getting their food. Denote $w_i(\mu_i, \Lambda)$ as the wait time at stage $i = 1, 2$ given the capacity μ_i and total demand rate Λ. Each customer's shopping rate λ is assumed to be linearly decreasing in the total wait time $w_1 + w_2$, i.e.,

$$\lambda = [\alpha - \beta(w_1(\mu_1, \Lambda) + w_2(\mu_2, \Lambda))]^+ ,$$

where coefficients $\alpha, \beta > 0$ represent the base shopping rate and wait sensitivity, respectively. In equilibrium, we should have $\Lambda = M\lambda$. Moreover, the wait time function $w(\mu, \Lambda)$ should be convexly decreasing in μ and convexly increasing in λ. For simplicity, we assume $w_1(\mu_1, \Lambda) = \frac{1}{\mu_1 - \Lambda}$ and $w_2(\mu_2, \Lambda) = \frac{1}{\mu_2 - \Lambda}$, which correspond to the average wait time in a M/M/1 queue.[1] Without loss of generality, we normalize market size to 1, i.e., $M = 1$.[2]

Given the demand function, the firm chooses the capacity level at each stage, i.e., μ_1 and

[1] We can think of the two-stage tandem queueing network as an open Jackson Network. Then, by Jackson's theorem, the queueing network behaves as if it were composed of independent M/M/1 queues (Wolff, 1989).

[2] The linear demand model can be interpreted in a different way (Cachon and Feldman, 2011): Suppose customers are heterogeneous in terms of their valuation v of the service. Assume v is uniformly distributed within $(0, \bar{v})$. Customer utility is given as $u = v - \tilde{\beta}(w_1 + w_2)$, and they will come and place an order if $u \geq 0$. Suppose there are α potential customers in the market. Then, total demand is $\lambda = \alpha \Pr(v \geq \tilde{\beta}(w_1 + w_2)) = \alpha - \beta(w_1 + w_2)$, where $\beta = \tilde{\beta}\alpha/\bar{v}$. With this interpretation, our results that λ increases after the implementation of self-order technologies (i.e., Propositions 25 and 30) indicate that these technologies can help restaurants attract more customers (who were not willing to come before due to long wait) to the store.

μ_2, to maximize the profit rate, i.e.,

$$\begin{aligned} &\max_{\lambda \leqslant \mu_1, \lambda \leqslant \mu_2} \; r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = [\alpha - \beta\left(w_1(\mu_1, \lambda) + w_2(\mu_2, \lambda)\right)]^+ \end{aligned} \tag{4.1}$$

The following proposition gives the optimal solution; we use superscript $\cdot^b$ to denote the base case.

Proposition 22. *$\exists \bar{\alpha}$ such that if and only if $\alpha > \bar{\alpha}$, then there exists a unique optimal solution that yields positive profit:*

- $\mu_1^b = \lambda^b + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$;
- $\mu_2^b = \lambda^b + \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}$,

where $\lambda^b = \alpha - \beta\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}} > 0$.

The expression of optimal capacity follows the rule of thumb for capacity planning (Bassamboo et al., 2010): it involves a "base capacity" to match the mean demand (i.e., λ^b) and augmenting that by a "safety capacity" (i.e., $\sqrt{\frac{\beta(r-c_1-c_2)}{c_i}}$ for stage $i = 1, 2$) that hedges against variability in realized arrivals. Note, when the capacity cost c_i is cheaper, the firm can afford to set a higher level of safety capacity. Also, if customers care more about wait (i.e., β is large), the firm will also set a higher safety capacity level to increase service level. To conduct meaningful comparisons in the following analysis, we assume $\alpha > \bar{\alpha}$ so that the firm is open and obtains positive profit in the base case.

4.4. Online Self-Order Technology

In this section, we consider the case where the firm implements self-order technology online, e.g., by providing online ordering website or mobile ordering app. Note that not every customer in the market will have access to the online self-order service at the moment when they want to place an order. At the very least, customers need a digital device, e.g., a computer or a smartphone, to enter the self-order platform. Furthermore, with mobile ordering apps, there is an "app overload" problem, since "no one wants to have to install a new app for every business or service that they want to interact with," as remarked by Facebook CEO Mark Zuckerberg (Wall Street Journal, 2016a). Thus, online self-orders are available only to digitally-connected customers who are sufficiently engaged with the firm (e.g., to download the app). In our model, the firm essentially deals with two types of customers: (i) online customers, who place the order via the online self-order channel, and (ii) store customers, who physically come to the store to place the order. Suppose the fraction of the online customers in the market is $\theta \in (0,1)$, and the remaining $1-\theta$ are store customers.

Our model captures two key effects of online self-order technology. First, online customers can instantaneously place the order through their own digital devices and do not need to wait in line for their turn to order the food. In other words, their wait time w_1 at stage 1 is essentially reduced to 0. We call this as the *instant-order effect* of the online self-order technology. However, customers still need to wait for their food to be prepared in the kitchen. Another benefit of the online self-order technology is that it allows people to order in advance, i.e., before they actually arrive at the store. This preorder feature allows online customers to be able to place an order on the go (and thus make better use of the wait time), instead of waiting in the store for their food to be ready for pickup. As a result, online customer's wait sensitivity for stage 2 is scaled down to $\xi\beta \leq \beta$, where $\xi \in (0,1]$. We call this as the *advance-order effect* of the online self-order technology. Therefore, online customers' shopping rate is given as follows:

$$\lambda_o = [\alpha - \xi\beta w_2(\mu_2, \Lambda)]^+ ,$$

where Λ is the total demand rate, including both online and store customers. Given that there is a fraction θ of online customers, the total demand rate from this market segment is $\theta\lambda_o$.

Next, we turn to store customers. Store customers need to wait in line at stage 1 to place their order as before. As a result, their total wait time in the store is still the sum of the wait time at both stages, i.e., $w_1(\mu_1, \Lambda_1) + w_2(\mu_2, \Lambda)$. Note, because online customers skip the ordering stage in the store, the demand at stage 1, i.e., Λ_1, includes only store customers. In addition, since restaurants typically don't distinguish online and offline orders in the kitchen[3], customer average wait time at stage 2, $w_2(\mu_2, \Lambda)$, only depends on the capacity level μ_2 and total demand rate Λ. As a result, store customers' shopping rate can be expressed as follows:

$$\lambda_s = [\alpha - \beta(w_1(\mu_1, \Lambda_1) + w_2(\mu_2, \Lambda))]^+ .$$

Given that there is a fraction $1 - \theta$ of store customers, the total demand rate from this market segment is $(1 - \theta)\lambda_s$.

In equilibrium, we should have $\Lambda_1 = (1 - \theta)\lambda_s$ and $\Lambda = \theta\lambda_o + (1 - \theta)\lambda_s$.

The firm chooses the capacity level at each stage, i.e., μ_1 and μ_2, to maximize profit rate, i.e.,

$$\begin{aligned} &\max_{(1-\theta)\lambda_s \le \mu_1, \theta\lambda_o + (1-\theta)\lambda_s \le \mu_2} \quad r\left(\theta\lambda_o + (1-\theta)\lambda_s\right) - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_o = [\alpha - \xi\beta w_2(\mu_2, \theta\lambda_o + (1-\theta)\lambda_s)]^+ \\ &\qquad \lambda_s = [\alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s) + w_2(\mu_2, \theta\lambda_o + (1-\theta)\lambda_s)\right)]^+ \end{aligned} \tag{4.2}$$

The following proposition gives the optimal solution; we use superscript $\cdot^o$ (for 'online') to denote the case with online self-order technology.

Proposition 23. *$\exists \bar{\alpha}' \ge \bar{\alpha}$ such that if $\alpha > \bar{\alpha}'$, then there exists a unique optimal solution:*

- $\mu_1^o = (1-\theta)\lambda_s^o + \sqrt{\frac{(1-\theta)\beta(r-c_1-c_2)}{c_1}}$;
- $\mu_2^o = (1-\theta)\lambda_s^o + \theta\lambda_o^o + \sqrt{\frac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}}$,

[3] For example, according to Taco Bell and Starbucks' employees, for both companies, the policy is that once the online order comes in, it gets injected into the order line like it was at the register. Sources: `https://www.reddit.com/r/tacobell/comments/2ssoop/horrible_first_mobile_order_experience` and `https://www.reddit.com/r/starbucks/comments/3vt3o7/question_on_ordering_on_mobile_vs_waiting_on_line/`

where

- $\lambda_s^o = \alpha - \beta\sqrt{\frac{c_1}{(1-\theta)\beta(r-c_1-c_2)}} - \beta\sqrt{\frac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}} > 0,$
- $\lambda_o^o = \alpha - \xi\beta\sqrt{\frac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}} > 0.$

Otherwise, there does not exist an optimal solution in which both online and store customers are served.

To focus on the most interesting and realistic scenario where the firm serves both online and store customers, we assume $\alpha > \bar{\alpha}'$ throughout the study. Note their α can be interpreted as customer's base preference for the product (i.e., the shopping rate if they do not need to wait). The assumption $\alpha > \bar{\alpha}'$ implies that customers value the product sufficiently to generate positive demand at zero wait.

The next two results characterize the changes in customer wait times and shopping rates after the implementation of online self-order technology.

Proposition 24. *With online self-order technology,*

i. $w_1^o > w_1^b$*;*

ii. given c_1 *and* c_2*,* $w_2^o < w_2^b$ *if and only if* $\frac{c_1+c_2}{r} > m_w$ *for some* $m_w \in (0,1)$*.*

Proposition 25. *With online self-order technology,*

i. online customers come more often than before, i.e., $\lambda_o^o > \lambda^b$*;*

ii. given c_1 *and* c_2*, store customers come to the store more often (i.e.,* $\lambda_s^o > \lambda^b$*) if and only if* $\frac{c_1+c_2}{r} > m_\lambda$ *and* $\theta \in (0, \psi_s)$ *for some* $m_\lambda \in [m_w, 1)$ *and* $\psi_s > 0$*;*

iii. total demand increases, i.e., $\theta\lambda_o^o + (1-\theta)\lambda_s^o > \lambda^b$*.*

According to Proposition 25(i), the firm sees an increase in demand from online customers. With the online self-order service, online customers can skip the line at stage 1 and easily place the order via their digital device. Moreover, as online customers can order before they actually arrive at the store (e.g., while they are still on the way to the store), they can make better use of the time while waiting for their food to be ready for pickup. With a reduction in disutility during the shopping process, online customers increase their shopping rates. This result is in line with the findings of ChowNow, one of the leading vendors of online ordering platforms: customers who order from mobile apps order nearly three times more frequently than before[4].

Next, according to Proposition 25(ii), the firm may also see an increase in demand from store customers. This means that store customers can also benefit from the implementation of online self-order technology and enjoy a shorter wait time, even though they do not use the service themselves. Interestingly, the shorter wait does not derive from a shorter line at stage 1 even though store customers no longer share capacity with online customers to place orders; on the contrary, as shown in Proposition 24(i), store customers end up waiting longer at stage 1 as the firm reduces the front-end capacity level μ_1 (see Proposition 26(i) below). Instead, as shown in Proposition 24(ii), the benefit comes from the fact that customers (including store customers) may wait shorter at stage 2. This is because the firm may find it profitable to increase workforce level in the kitchen (as shown in the following Proposition 26(ii)) and thus the faster food preparation stage more than compensates for the slower order placement stage.

Proposition 26. *With online self-order technology,*

i. $\mu_1^o < \mu_1^b$;

ii. *given* c_1 *and* c_2, *there exists a threshold* $m_\mu \in (0, m_w]$ *such that* $\mu_2^o > \mu_2^b$ *if* $\frac{c_1+c_2}{r} > m_\mu$ *and* $\mu_2^o < \mu_2^b$ *if* $\frac{c_1+c_2}{r} < m_\mu$.

Proposition 26 shows the impacts of online self-order technology on a firm's optimal capacity decision at both stages 1 and 2. Since capacity is positively related to the workforce level assigned to the particular stage, we can interpret the results as the change in optimal workforce level at each stage after the implementation of online self-order technology.

[4]Source: `https://www.chownow.com/online-ordering-system`

With online self-order technology, point (i) in Proposition 26 shows that the firm should reduce the workforce level in the front end of the store. This is mainly because of the self service feature of the technology. As online customers place the order by themselves via their digital device, the firm no longer needs to assign as many cashiers as before to take customers' order. This way, the firm is able to save some labor cost without significantly impairing service at stage 1.

At the same time, Proposition 26(ii) implies that the firm may also need to decrease the workforce level at stage 2 (i.e., $\mu_2^o < \mu_2^b$). In fact, as Proposition 24 shows, the wait at stage 2 may increase. This is driven by the fact that online customers can order ahead of time, and thus can tolerate a longer wait time during which they can conduct other business instead of waiting in the store, i.e., while the absolute wait w_2 increases, online customers' disutility from waiting decreases (due to the advance-order effect). Since the average wait sensitivity in the population decreases, a slower service speed in the kitchen will hurt demand to a lesser extent. Therefore, the firm may have incentive to further cut the workforce level at stage 2 to save more labor cost.

However, Proposition 26(ii) also shows that the firm sometimes may need to increase the workforce level at stage 2. This is because the average total service cost to serve a customer is cheaper as online customers can place the order by themselves. As a result, it would be worthwhile for the firm to attract more customers to the store through extra investment in the capacity in the kitchen area, which helps to increase the speed of service.

Moreover, Proposition 26(ii) indicates that firms with relatively high capacity costs c_1, c_2 should increase back-end capacity μ_2. The reason is as follows: If a firm has relative high capacity cost, then they tend to have a low initial capacity level μ_2^b (and thus a long wait time for customers) before the implementation of online self-order technology. With the online self-order technology, as shown in Proposition 26(i), the firm can cut the workforce level in the front end, which enables the firm to make more investment in the kitchen area to speed up the food preparation process to provide faster service to customers and thus boost total sales. Moreover, because of economies of scale in a queueing system, the same amount of increase in capacity will have a greater effect on the average wait time for a slower system. This explains why a firm with higher relative capacity cost has more incentive to increase stage 2 capacity level.

In general, according to Proposition 26, there are two possible outcomes after the implementation of online self-order technology: The firm should either (1) cut jobs throughout the entire store, or (2) shift workforce from the front end to the back end. The first scenario is what generates massive concern over job loss among workers (CNN, 2014). But the firm's desire to increase capacity at stage 2 to attract more customers may attenuate such negative effect on the total workforce level. This yields the second possible scenario, where some cashiers may simply be moved to the kitchen area. In fact, as we demonstrate below, sometimes a firm needs more than just a shift of workforce level within the firm: they may need to hire more workers.

Note that we cannot simply sum up the capacity level at each stage to capture the total workforce level, as different stages may involve different service requirements (e.g., taking an order may last fifty seconds but making a burger may take three minutes). Therefore, to study the change of total workforce level in the store, we need to specify the relationship between capacity μ and workforce level k. Specifically, suppose it takes $\tau_i > 0$ amount of time for one worker to serve a customer at stage i. Then, with k_i workers at stage i, the firm can (on average) finish serving $\frac{k_i}{\tau_i}$ customers per unit of time, which corresponds to the capacity level at that stage. In other words, we have $\mu_i = \frac{k_i}{\tau_i}$, where k_i and τ_i are the workforce level and the average customer service requirement (measured in time units) at stage i, respectively.

With the above normalization between capacity and workforce levels, we have the following result.

Proposition 27. *Given c_1 and c_2, there exists a threshold $m_k \in [m_\mu, 1)$ such that the firm should increase total workforce level after implementing online self-order technology (i.e.,*

$k_1^o + k_2^o > k_1^b + k_2^b$) *if and only if* $\frac{c_1+c_2}{r} > m_k$.

Despite a widespread perception that high labor costs encourage the adoption of self-order technology as a measure to cut jobs (CNN, 2014; Yahoo Finance, 2016), Proposition 27 shows that firms, particularly those with high capacity cost, should choose not to cut any jobs but rather increase total workforce level in the store along with the implementation of online self-order technology. This result supports restaurant managers' claim that the implementation of self-order technology will bring about minimal job cuts (Business Insider, 2015a; Huffinton Post, 2015).

The reason behind Proposition 27 is as follows: Online self-order technology has two effects on online customers: (1) It helps to eliminate their wait at stage 1 (i.e., instant-order effect); and (2) it allows them to preorder and thus reduces the negative impact of wait at stage 2 (i.e., advance-order effect). Both effects have a direct negative impact on workforce level: The instant-order effect reduces the need for human servers at stage 1, and the advance-order effect means service speed may not be as important as before since customers care less about wait. However, both effects also have an indirect impact on workforce level: They both drive up total demand (as shown in Proposition 25) and thus increase the need for capacity in the kitchen, especially when capacity cost is relatively high, as shown in Proposition 26. This relationship is depicted in Figure 8.

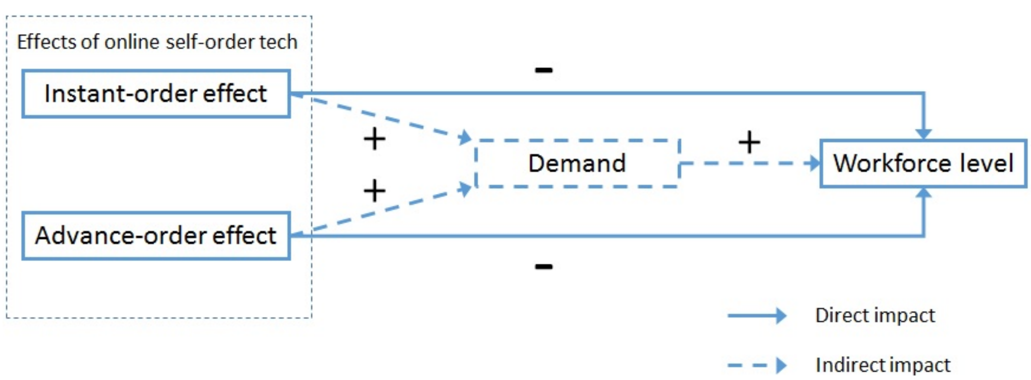


Figure 8: Impact of Online Self-Order Technology on Workforce Level

In general, our results above show that the firm is able to attract more demand and thus boost sales with online self-order technology. Such increase in total demand is mainly driven by the fact that online customers are willing to shop more frequently. Meanwhile, store customers may also choose to increase their shopping rate, as the firm speeds up the service by increasing the back-end capacity level. Moreover, according to Propositions 25 and 27, if $\frac{c_1+c_2}{r} > \max(m_k, m_\lambda)$ and $\theta \in (0, \psi_s)$, then we find that everyone in the economy could benefit from the implementation of online self-order technology: both store and online customers are willing to come to the store more often (i.e., $\lambda_s^o > \lambda^b$, $\lambda_o^o > \lambda^b$), the firm obtains more demand (i.e., $\theta\lambda_o^o + (1-\theta)\lambda_s^o > \lambda^b$), and more workers are hired by the firm (i.e., $k_1^o + k_2^o > k_1^b + k_2^b$).

4.5. Offline Self-Order Technology

In this section, we consider the case where the firm implements the self-order technology offline, e.g., by providing in-store kiosks and robot waiters. By replacing human servers with machines, the firm can lower the marginal cost of service interaction at stage 1; this is the main reason behind many companies' move to offer offline self-order service (CNN, 2014; Yahoo Finance, 2016). We model this innovation as follows.

There are two types of servers at stage 1: human (with capacity μ_{1h}) and machine (with capacity μ_{1m}). The cost of human capacity is the same as before, i.e., c_1. However, the cost of machine capacity is c_{1m}. As mentioned above, we assume that it is cheaper to serve each customer with machine capacity than with human capacity.

In addition, suppose there are two types of customers: A fraction $\eta \in (0, 1)$ of all customers are tech-savvy and they always prefer to use the self-order machines. On the other hand, the remaining $1-\eta$ are traditional customers who place an order only with human servers; they may not be comfortable or proficient with using self-order machines. For each type of customers, denote their respective wait times at stage 1 as $w_{1m}(\mu_{1m}, \Lambda_{1m})$ and $w_{1h}(\mu_{1h}, \Lambda_{1h})$, where Λ_{1m} and Λ_{1h} are the total demand rates for machine and human capacities at stage 1. Therefore, each type of customer's shopping rate is given as follows:

- tech-savvy customers: $\lambda_m = [\alpha - \beta\left(w_{1m}(\mu_{1m}, \Lambda_{1m}) + w_2(\mu_2, \Lambda)\right)]^+$;

- traditional customers: $\lambda_h = [\alpha - \beta\left(w_{1h}(\mu_{1h}, \Lambda_{1h}) + w_2(\mu_2, \Lambda)\right)]^+$.

Similar to before, we assume $w_{1m}(\mu_{1m}, \Lambda_{1m}) = \frac{1}{\mu_{1m}-\Lambda_{1m}}$ and $w_{1h}(\mu_{1h}, \Lambda_{1h}) = \frac{1}{\mu_{1h}-\Lambda_{1h}}$. Given that tech-savvy and traditional customers respectively account for η and $1-\eta$ of the total population, in equilibrium, we should have $\Lambda_{1m} = \eta\lambda_m$, $\Lambda_{1h} = (1-\eta)\lambda_h$, and $\Lambda = \eta\lambda_m + (1-\eta)\lambda_h$.

Now, we formulate the firm's optimization problem. The firm chooses the capacity level at each stage, i.e., μ_{1h}, μ_{1m} and μ_2, to maximize profit rate, as follows.

$$\begin{aligned} \max_{\substack{\eta\lambda_m \le \mu_{1m}, \\ (1-\eta)\lambda_h \le \mu_{1h}, \\ \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2}} \quad & r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\ s.t. \quad \lambda_m &= [\alpha - \beta\left(w_{1m}\left(\mu_{1m}, \eta\lambda_m\right) + w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right)]^+ \\ \lambda_h &= [\alpha - \beta\left(w_{1h}\left(\mu_{1h}, (1-\eta)\lambda_h\right) + w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right)]^+ \end{aligned} \tag{4.3}$$

The following proposition gives the optimal solution; we use superscript $\cdot^s$ (for 'store') to denote the case with offline self-order technology.

Proposition 28. *$\exists\bar{\alpha}'' \ge \bar{\alpha}'$ such that if $\alpha > \bar{\alpha}''$, then there exists a unique optimal solution:*

- $\mu^s_{1m} = \eta\lambda^s_m + \sqrt{\frac{\beta\eta(r-c_{1m}-c_2)}{c_{1m}}}$;

- $\mu^s_{1h} = (1-\eta)\lambda^s_h + \sqrt{\frac{\beta(1-\eta)(r-c_1-c_2)}{c_1}}$;

- $\mu^s_2 = \eta\lambda^s_m + (1-\eta)\lambda^s_h + \sqrt{\frac{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}{c_2}}$,

where

- $\lambda^s_m = \alpha - \beta\sqrt{\frac{c_{1m}}{\beta\eta(r-c_{1m}-c_2)}} - \beta\sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}} > 0$;

- $\lambda^s_h = \alpha - \beta\sqrt{\frac{c_1}{\beta(1-\eta)(r-c_1-c_2)}} - \beta\sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}} > 0$.

Otherwise, there does not exist an optimal solution in which both tech-savvy and traditional *customers are served.*

To focus on the most interesting scenario where the firm serves both tech-savvy and traditional customers, we assume $\alpha > \bar{\alpha}''$ throughout the study. This assumption implies that customers value the product sufficiently to generate positive demand at zero wait.

Our model highlights an important consequence of offline self-order technology: it may create separate waiting lines, one for human servers and one for machines. There are two reasons for such separation. First, offline technology may exclude a segment of traditional customers, who have no choice but to wait for human servers. Second, self-ordering innovations may be so popular that some customers are willing to wait behind long lines even when human cashiers are available (Wall Street Journal, 2016b). Consequently, separate queues may now emerge in place of the single queue in our base model. Although self-order technology reduces capacity cost and thus encourages capacity investment, the separate-queues system goes against conventional wisdom of resource pooling and may lead to higher waiting times. This is particularly so when the capacity costs of humans and machines are comparable, i.e., $c_{1m} \approx c_1$. In practice, without dramatic reductions in cost, it is unlikely that offline self-order technology will be implemented. After all, reducing lines and saving time at the ordering stage is one of the main reasons why many restaurants implement self-order machines in the first place (QSR Magazine, 2016).

Lemma 1. *There is a threshold $\bar{c}_{1m} \in (0, c_1)$ such that average customer wait at stage 1 is shorter than that in the base model (i.e., $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$) if and only if $c_{1m} < \bar{c}_{1m}$.*

The above result shows that as long as machine capacity cost is below a certain threshold, the firm can afford to provide ample machine capacity to ensure that the average customer wait at stage 1 is shorter compared to the base model. Henceforth, we make the following assumption.

Assumption 1. *The capacity cost for machines c_{1m} is sufficiently low. Specifically, $c_{1m} < \bar{c}_{1m}$.*

In practice, it is widely believed that machines cost significantly less than human employees (CNN, 2014; Yahoo Finance, 2016). For example, a study by Forrester Research showed that self-service check-in costs the airlines 16 cents a passenger, compared with $3.68 using ticket-counter agents (New York Times, 2004). We expect the same magnitude of cost reduction would apply for self-order machines in a restaurant as well.

The next two propositions characterize customer wait and customer demand in equilibrium, when offline self-order technology is provided.

Proposition 29. *With offline self-order technology,*

i. *tech-savvy customers wait shorter at stage 1, i.e.,* $w_{1m}^s < w_1^b$*;*

ii. *traditional customers wait longer at stage 1, i.e.,* $w_{1h}^s > w_1^b$*;*

iii. *everyone waits shorter at stage 2, i.e.,* $w_2^s < w_2^b$*.*

Proposition 30. *With offline self-order technology,*

i. *tech-savvy customers come to the store more often than before, i.e.,* $\lambda_m^s > \lambda^b$*;*

ii. *given* c_{1m}, c_1, c_2*, traditional customers come to the store more often (i.e.,* $\lambda_h^s > \lambda^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_\lambda$ *and* $\eta \in (0, \psi'_s)$ *for some* $m'_\lambda \in (0,1)$ *and* $\psi'_s > 0$*;*

iii. *total demand increases, i.e.,* $\eta\lambda_m^s + (1-\eta)\lambda_h^s > \lambda^b$*.*

Propositions 29 and 30 show the changes in customer wait times and shopping rates after the implementation of offline self-order technology, respectively. Similar to the case with online self-order technology, we find that the implementation of offline self-order technology helps to increase total demand. Moreover, the technology is not only beneficial to those tech-savvy customers who use the machines; it could also benefit traditional customers who do not adopt the technology.

According to Proposition 30(i), tech-savvy customers shop more frequently than before. Different from the case with online self-order technology, tech-savvy customers need to share the in-store self-order machines with fellow customers and thus they may still need to wait to place an order. However, the cheap cost of machines (compared to labor cost) allows the firm to afford more machine capacity, and thus tech-savvy customers can still enjoy a shorter wait time with the offline self-order technology compared to the base case.

Proposition 30(ii) implies that the demand from traditional customers may also increase. This means that traditional customers can also benefit from the implementation of offline self-order technology, even though they do not use the technology directly. Similar to the case with online self-order technology, this is mainly because of the potential wait time reduction at stage 2 rather than at stage 1 (see Proposition 29 (ii) and (iii)). As the next result confirms, capacity expansion occurs at stage 2 of the service process.

Proposition 31. *With offline self-order technology,*

i. $\mu_{1h}^s < \mu_1^b$ *and* $\mu_{1m}^s + \mu_{1h}^s > \mu_1^b$;

ii. $\mu_2^s > \mu_2^b$.

With offline self-order technology, the firm should replace some costly human servers with cheaper machine servers at stage 1. Therefore, as shown in Proposition 31(i), human capacity in the front end of the store should decrease. But the firm should also increase overall capacity (i.e., $\mu_{1m}^s + \mu_{1h}^s$) as the usage of machine capacity brings down average capacity cost.

Similar to online self-order technology, offline self-order technology could also prompt the firm to increase the capacity/workforce level at stage 2, as the service cost decreases and thus the firm can now allocate the extra resources saved from the implementation of cheap machines at stage 1 to increase the service speed at stage 2 to attract more customers to the store. However, different from online self-order technology, Proposition 31(ii) implies that the firm should never decrease the workforce level at stage 2 when the self-order service is provided offline. The reason is as follows: Online self-order service allows customers to order ahead of time, and thus online customers can tolerate a longer wait time as they can make better use of the wait time rather than standing in the store; in contrast, with offline self-order machines, customers still need to come to the store to place the order and physically wait there for their food to be ready for pickup. Therefore, the firm still needs to keep a high service rate at stage 2 with offline self-order technology, in order to avoid customers in the store waiting too long after they place the order.

Although self-order machines should replace some human cashiers at stage 1, Proposition 31(ii) shows that the firm should also increase capacity in the kitchen area to speed up the food preparation process. The overall impact on optimal workforce levels is characterized in the next proposition. Note that we employ a similar normalization: given customer's average service requirement τ_1 and τ_2, the corresponding workforce levels at each stage are $k_1^s = \mu_{1h}^s \tau_1$ and $k_2^s = \mu_2^s \tau_2$. In particular, at stage 1, human capacity μ_{1h} generates employment but not machine capacity μ_{1m}.

Proposition 32. *Given c_{1m}, c_1 and c_2, then there exists a threshold $m_k' \in (0,1)$ such that the firm should increase total workforce level after implementing offline self-order technology (i.e., $k_1^s + k_2^s > k_1^b + k_2^b$) if and only if $\frac{c_1+c_2}{r} > m_k'$.*

Proposition 32 indicates that the firm may need to increase total workfoce level along with the implementation of offline self-order technology. This corroborates with McDonald's practice in Europe, where they have added more jobs after the implementation of in-store kiosks (Huffinton Post, 2015). Similar to online self-order technology, Proposition 32 implies that firms, especially those with high capacity costs (i.e., $\frac{c_1+c_2}{r}$), should increase total workforce level after the implementation of offline self-order technology. The reason is as follows: In the case of high capacity cost, self-order machines help save a large amount of labor cost at stage 1. With these savings, the firm can invest more in manpower at stage 2 to further speed up the service and attract more demand to store. Such capacity investment is more beneficial when the original capacity level is low (due to high capacity cost), because capacity investments yield economies of scale in a queueing system as mentioned before.

Similar to the online self-order technology, the implementation of offline self-order technology could lead to a win-win-win situation as well: According to Propositions 30 and 32, if $\frac{c_1+c_2}{r} > \max(m'_k, m'_\lambda)$ and $\eta \in (0, \psi'_s)$, then both traditional and tech-savvy customers are willing to come to the store more often (i.e., $\lambda_h^s > \lambda^b$, $\lambda_m^s > \lambda^b$), the firm obtains more demand (i.e., $\eta\lambda_m^s + (1-\eta)\lambda_m^s > \lambda^b$), and more workers are hired by the firm (i.e., $k_1^s + k_2^s > k_1^b + k_2^b$).

4.6. Profit Implications

In this section, we look at the impact of both types of self-order technologies on firm's profit. Denote π^b, π^o and π^s as the optimal profit in the base model, and in the cases with online and offline self-order technology, respectively.

We first compare the optimal profit with and without self-order technologies. Since both types of technologies may require a substantial financial investment, it is important to study the magnitude of the benefit, i.e., $\pi^o - \pi^b$ and $\pi^s - \pi^b$. If these differences are large, then it means the implementation of self-order technologies is more likely to generate significant positive profit for the firm. Our analysis below focuses on the profit change brought about by each self-order technology, while abstracting away from the implementation costs that lie beyond the scope of our model.

Proposition 33. *Compared with the base model, both types of self-order technologies increase profit (i.e., $\pi^o > \pi^b$ and $\pi^s > \pi^b$). Moreover,*

- *with online self-order technology, (1) $\frac{\partial(\pi^o-\pi^b)}{\partial r} > 0$, (2) $\frac{\partial^2(\pi^o-\pi^b)}{\partial c_2 \partial \xi} < 0$ and $\frac{\partial(\pi^o-\pi^b)}{\partial c_2} < 0$ if $\xi = 1$, (3) $\frac{\partial(\pi^o-\pi^b)}{\partial \theta} > 0$, and (4) $\frac{\partial(\pi^o-\pi^b)}{\partial \xi} < 0$;*
- *with offline self-order technology, (1) $\frac{\partial(\pi^s-\pi^b)}{\partial r} > 0$, (2) $\frac{\partial(\pi^s-\pi^b)}{\partial c_2} < 0$, (3) $\frac{\partial(\pi^s-\pi^b)}{\partial \eta} > 0$, and (4) $\frac{\partial(\pi^s-\pi^b)}{\partial c_{1m}} < 0$.*

In general, both types of self-order technologies can generate more profit, which explains their popularity in the restaurant industry. Moreover, Proposition 33 sheds light on the impact of different model parameters on the profit improvement.

First, firms with high gross margin (i.e., large r) will enjoy more benefit from the implementation of both types of self-order technologies, because they have more incentive to provide large capacity and thus fast service to attract more customers to the store (as shown by the optimal solution in Proposition 22). In this case, self-order technologies help to save service cost as human capacity is replaced with cheaper devices, which is either owned by customers themselves as with the online technology or provided by the firm as with the offline technology.

Second, firms with a smaller capacity cost at stage 2 tend to have larger profit improvement from the implementation of offline self-order technology. A similar result holds for online self-order technology if there is no advance-order effect (i.e., when $\xi = 1$). This is because both types of technologies lead to orders being placed at a faster rate and thus create more demand at stage 2. To deal with the increase in demand, as shown in Propositions 26 and 31, the firm oftentimes needs to increase capacity at stage 2; a cheaper c_2 allows the firm to make such changes without increasing cost too much. Note c_2 can be regarded as the cost of serving one customer at stage 2, and thus it should be proportional to customer's service requirement τ_2. In other words, this result implies that firms with an simple food making process (e.g., quick-service restaurants) will benefit more from the implementation of self-order technologies. However, for online self-order technology, Proposition 33 shows that the advance-order effect will attenuate this impact (i.e., $\frac{\partial^2(\pi^o - \pi^b)}{\partial c_2 \partial \xi} < 0$), because waiting at stage 2 matters less as customers become less sensitive to wait.

In addition, Proposition 33 shows that online self-order technology helps to bring about more profit improvement when there are more people equipped with mobile/online ordering devices (i.e., a large θ) and when the advance-order effect is more significant (i.e., a small ξ). This helps to explain Taco Bell's decision to implement online self-order technology: their target audience is a demographic of predominantly male 18-to-34-year-olds (Forbes, 2012), who spend more money online in a given year than any other age group (Business Insider, 2015c). This result also implies that if a firm chooses to implement online self-order technology, then they can strive for a lower ξ. One way to increase the advance-order effect is to shorten customer's perceived wait time, which can be achieved by reducing uncertainty in the waiting process (Maister, 1984) through operational transparency, e.g., by keeping customer informed about the status of their online order (Buell and Norton, 2011).

Finally, Proposition 33 indicates that offline self-order technology helps to bring about more profit improvement when more people are willing to use machines by themselves (i.e., a high η). One way to encourage a higher adoption rate of the technology is through effective interface designs that make the ordering process is simple and intuitive for users. Another factor that could affect the profitability of offline self-order technology is the capacity cost c_{1m}. The firm may consider building the ordering machines in-house, rather than purchasing the service from a third-party kiosks provider, so as to avoid service, licensing, and maintenance fees.

We next compare the profit generated by online and offline self-order technologies. Currently, many firms have only one of the two technologies implemented in their stores. For example, Starbucks and Taco Bell only have online self-order platforms, whereas McDonald's and HoneyGrow only have kiosks installed in stores. What are the drivers behind this choice? This is an important question because implementing self-order technologies involves substantial financial investment and significant changes in store operations (Business Insider, 2015b; Mobile Strategy 360, 2016). With unlimited resources, the firm always prefers having both types of technologies, but with limited resources, understanding the relative benefits of online and offline technologies will help prioritize implementation efforts.

Proposition 34. *There exists $\bar{\beta} \geq 0$ such that online self-order technology generates more profit than offline self-order technology (i.e., $\pi^o > \pi^s$) if and only if $\beta > \bar{\beta}$.*

According to Proposition 34, whether online or offline self-order technology is preferred depends on customers' wait sensitivity (i.e., β). This is because the two types of self-order technologies have the following key difference: With online self-order technology, customers own the digital device (e.g., smartphones, personal computers); in contrast, with offline self-order technology, the ordering machines are provided by the restaurant and thus shared by customers in store. As a result, offline self-order technology is more accessible to customers as it does not require any digital connectivity on customer's side (i.e., we should expect $\eta > \theta$)[5], but online self-order technology is more effective in terms of reducing customer's wait time as those online customers owns the ordering device and thus no longer need to wait to place an order. Consequently, when customers are highly averse to waiting (i.e., large β), the firm should offer online self-order service to enable online customers to enjoy a larger benefit of wait cost reduction. In contrast, when customers are not very sensitive to delay (i.e., small β), the firm should implement offline self-order technology to maximize its impact by ensuring that more customers can use the self-order service.

Previous empirical studies (Campbell and Frei, 2011; Propper, 1995) show that high-income customers tend to have high wait sensitivity. In this light, Proposition 34 implies that firms targeting a high-income (low-income) market should implement online (offline) self-order technology. With this interpretation, Proposition 34 gives us a possible explanation of Starbucks' choice of mobile ordering app over offline self-order technology, since most Starbucks stores operate in high-income neighborhoods with a median income higher than \$50,000 (Eater, 2015). However, for McDonald's, Proposition 34 indicates that offline self-order technology, e.g., in-store kiosks, might be a better choice, since their typical customer profile falls within the lower income bracket (Marketplace, 2014). Our result is in line with McDonald's current strategy: up to April 2016, they have installed kiosks in 600 U.S. restaurants; by the end of 2016, the company plans to have them in 1,000 locations (Eater, 2016).

[5] If $\eta \leq \theta$, we can show that $\bar{\beta} = 0$, i.e., online self-order technology always generates more profit than offline self-order technology.

4.7. Extensions

4.7.1. *Customer Heterogeneity*

In our original model, we have assumed that customers have the same base shopping rate α and wait sensitivity β. We can extend our basic model to incorporate customer heterogeneity in these two demand parameters. Suppose tech-savvy customers and traditional customers have different base shopping rate and wait sensitivity, denoted as α_m and α_h, β_m and β_h, respectively. In the case with online self-order technology, suppose some tech-savvy customers have access to the online ordering platform and the remaining customers (e.g., other tech-savvy customers who may not wish to install the app, and all traditional customers) place an order in the store. In the case with offline self-order technology, all tech-savvy customers place an order through the machines while all traditional customers turn to human servers. This extended model can capture customer heterogeneity in terms of two aspects: (1) Customers may have different loyalty levels towards the firm (which is reflected by different base shopping rates), and (2) customers may have different sensitivity levels towards waiting in the store. In Appendix A.6.1, we present a detailed analysis of this model. Suppose that tech-savvy customers are more wait sensitive than traditional customers (i.e., $\beta_m \geq \beta_h$); this is natural since customers adopt self-order technology primarily because of their impatience with waiting lines (eMarketer, 2014). We find that our main insights remain valid.

4.7.2. *Convex Impact of Wait Time*

So far, we have assumed that demand is a linear function of wait time. Our model can be extended to a more general case where wait time has a convex impact on customer demand rate. In other words, there is decreasing sensitivity to each additional minute of waiting as the waiting spells last longer. Specifically, we consider the following demand function:

$$\lambda = \alpha - \beta w_1^{\phi} - \beta w_2^{\phi},$$

where w_1 and w_2 represent the wait times at stages 1 and 2, and parameter $\phi \in (0,1]$. Since $\phi \in (0,1]$, demand λ is convex with respect to wait time at each stage. Note, the linear demand model in the basic model is a special case of this general demand model with $\phi = 1$. Appendix A.6.2 presents a detailed analysis of this extended model, and we find our main insights continue to hold.

4.7.3. Alternative Wait Time Function

In our analysis, we have formulated the firm's optimization problem with respect to capacity μ. The three optimization problems (4.1, 4.2, 4.3) can also be reformulated with respect to the number of servers as follows:

- Basic model:

$$\begin{aligned} &\max_{\lambda \leqslant k_1/\tau_1, \lambda \leqslant k_2/\tau_2} \quad r\lambda - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda = [\alpha - \beta\left(w_1(k_1,\lambda) + w_2(k_2,\lambda)\right)]^+ \end{aligned} \tag{4.1$'$}$$

- Online self-order technology:

$$\begin{aligned} &\max_{(1-\theta)\lambda_s \leq k_1/\tau_1, \theta\lambda_o + (1-\theta)\lambda_s \leq k_2/\tau_2} \quad r\left(\theta\lambda_o + (1-\theta)\lambda_s\right) - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda_o = [\alpha - \xi\beta w_2(k_2, \theta\lambda_o + (1-\theta)\lambda_s)]^+ \\ &\qquad \lambda_s = [\alpha - \beta\left(w_1(k_1,(1-\theta)\lambda_s) + w_2(k_2,\theta\lambda_o + (1-\theta)\lambda_s)\right)]^+ \end{aligned} \tag{4.2$'$}$$

- Offline self-order technology:

$$\begin{aligned} &\max_{\substack{\eta\lambda_m \leq k_{1m}/\tau_1, \\ (1-\eta)\lambda_h \leq k_1/\tau_1, \\ \eta\lambda_m + (1-\eta)\lambda_h \leq k_2/\tau_2}} \quad r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - l_{1m}k_{1m} - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda_m = [\alpha - \beta\left(w_1\left(k_{1m}, \eta\lambda_m\right) + w_2\left(k_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right)]^+ \\ &\qquad \lambda_h = [\alpha - \beta\left(w_1\left(k_1, (1-\eta)\lambda_h\right) + w_2\left(k_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right)]^+ \end{aligned} \tag{4.3$'$}$$

where $l_1 = c_1/\tau_1$, $l_{1m} = c_{1m}/\tau_1$, $l_2 = c_2/\tau_2$, $w_i(k,\lambda) = \frac{1}{k/\tau_i - \lambda}$, and τ_i is the average service time at stage $i = 1,2$. Here, l_1 and l_2 can be interpreted as the labor cost per unit of time at stages 1 and 2, l_{1m} is the corresponding cost for machines. The number of machine servers is denoted by k_{1m}.

So far, we have assumed wait time function takes the following form: $w_i(k, \lambda) = \frac{1}{k/\tau_i - \lambda}$. In Appendix A.6.3, we numerically test the robustness of our main insights with a different wait time function, i.e.,

$$w_i(k, \lambda) = \left(\frac{\tau_i}{k}\right) \left(\frac{\rho_i^{\sqrt{2(k+1)}-1}}{1 - \rho_i}\right) + \tau_i$$

where $\rho_i = \frac{\lambda \tau_i}{k}$. This corresponds to the approximated average wait time in a M/M/k queue (Cachon and Terwiesch, 2009). The numerical results show that our main insights remain valid.

4.8. Conclusion

In 2016, California and New York approved a \$15 minimum wage, which doubles the current federal minimum. Dramatic increases in labor costs have a significant effect on the restaurant industry, where profit margins are pennies on the dollar and labor makes up about a third of total expenses (Wall Street Journal, 2016b). As a result, restaurants are turning to service automation to reduce cost while maintaining service quality. In this chapter, we study online and offline self-order technologies, which have been implemented by an increasing number of restaurants.

We find that self-order technology could benefit everyone in the economy. Customers, including those who may not have access to the self-order service, would benefit from a shorter amount of wait time. In addition, the firm is able to provide higher service capacity level at lower cost. Interestingly, contrary to the public fear that self-service technology is bound to replace workers, we find that it is sometimes optimal for firms, especially those with high capacity cost, to increase workforce level after introducing online ordering platforms or in-store machines. Especially for the quick-service restaurant industry, which is generally a low-margin and high-labor-cost business (Yahoo Finance, 2013), our results indicate that self-order technologies are more likely to bring about benefits to everyone including customers, workers, and firms.

Although both online and offline self-order technologies allow customers to place an order without interacting with a human cashier in the store, each type of technology has different pros and cons. Online self-order technology offers customers more convenience by allowing them to preorder using their own digital device, whereas offline self-order technology is more accessible to customers as the ordering machines are provided in store by the firm. Our results show that a higher customer wait sensitivity tilts the firm's optimal choice in favor of online over offline technologies. Since customers of higher income tend to have higher wait sensitivity (Campbell and Frei, 2011; Propper, 1995), our results indicate that firms targeting a high-end market (e.g., Starbucks) should implement online self-order technology (e.g., mobile ordering app or online ordering website); in contrast, for restaurants whose customers are mainly in the lower income bracket (e.g., McDonald's), offline self-order technology (e.g., in-store kiosks or robot waiters) is more profitable to implement.

Restaurant service includes both ordering and food preparation stages; both require customers to wait. Self-order technology helps to shift the workload at the ordering stage from the firm to customers. Nevertheless, the food preparation stage still requires ample staffing. This two-stage nature of service processes in the restaurant industry contributes to a shift of labor from front-end to back-end tasks and possibly to an increase in total workforce level. However, in some other industries, where self-service technology can cover the entire service process, it might be inevitable to see workers being replaced by technology. A typical example is the banking industry, where people can complete most financial transactions through ATMs and online/mobile channels; this has caused the decline of a large number of bank branches in the US over the past several years (CNBC, 2015a). We hope our model in this chapter can contribute to the study of self-service technology in more general service processes over a variety of industries. Moreover, some restaurants are even trying to automate the food preparation process using robots (Business Insider, 2016). It would also be interesting to study the impacts of service automation in the kitchen area, in contrast to the self-order technology in the front end of the store, on consumer shopping behavior and restaurant service operations.

APPENDIX

A.1. Decentralized System in Heterogeneous Market

Similar to what we did in Section 2.6, we only look at the case where there are some customers using BOPS, i.e., $p-c > w\frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} - r$ and $\alpha_b^*+\alpha_{bo}^* > 0$ (see Proposition 7); otherwise, there would be nobody using BOPS and the issue of revenue allocation becomes irrelevant as the system collapses into two independent channels.

With revenue sharing parameter θ, the store channel's profit is given as follows

$$\tilde{\pi}_s = \left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} + \frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right) pE\min\left(\left(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*\right)D, q\right) - cq$$
$$+ rE\min\left(\left(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*\right)D, q\right)$$

Proposition 35. *In the decentralized system, the store will stock $\tilde{q}^*$ which is given as follows:*

- *If* $\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} + \frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p + r - c > 0$, *then* $\tilde{q}^* = \left(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*\right)\bar{F}^{-1}\left(\frac{c}{\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} + \frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r}\right)$;
- *If* $\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} + \frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p + r - c \leq 0$, *then* $\tilde{q}^* = 0$.

Proposition 36. *Total profit $\tilde{\pi}^*(\theta)$ is quasiconcave in θ. Moreover,*

- *If* $\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_{so}^*+\alpha_{bo}^*} < \frac{w}{p}$, *then* $\forall\theta \in [0,1]$, $\tilde{q}^* > q^*$ *and* $\tilde{\pi}^*(\theta) < \pi^*$.
- *If* $\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_{so}^*+\alpha_{bo}^*} \geq \frac{w}{p}$, *then*
  - *if* $\theta < \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, *then* $\tilde{q}^* < q^*$ *and* $\tilde{\pi}^*(\theta) < \pi^*$;
  - *if* $\theta = \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, *then* $\tilde{q}^* = q^*$ *and* $\tilde{\pi}^*(\theta) = \pi^*$;
  - *if* $\theta > \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, *then* $\tilde{q}^* > q^*$ *and* $\tilde{\pi}^*(\theta) < \pi^*$.

Proposition 36 shows that if there are not many people using BOPS (i.e., $\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_{so}^*+\alpha_{bo}^*} < \frac{w}{p}$ or $\alpha_b^* p + \alpha_{bo}^*(p-w) < \alpha_{so}^* w$), then there is not enough BOPS revenue for the two channels to share and thus for the headquarters to fully correct store channels' incentive: Even if the store channel is given no credit for fulfilling BOPS demand, i.e., $\theta = 0$, they still stock more than the system optimal level.

A.2. Model Extensions to Chapter 2

A.2.1. Endogenous Online Channel

In this section, we relax the assumption in the original model that online is exogenous. Suppose both online and offline follow newsvendor setup. Retail price is p, which is the same across both channels. Cross-selling benefit in store is r. Unit inventory costs are c_s and c_o in the store and online channel, both of which are smaller than p. Retailer needs to decide inventory levels q_s and q_o in the store and online channel.

Homogeneous Market

Consumers setup is the same as before: They have valuation v for the product and hassle costs h_s, h_o, h_b. We assume when the online channel is out of stock, those who are willing to buy online will leave for other websites to buy the product at the same price. Therefore, given belief about store inventory availability $\hat{\xi}$, consumer's utility from visiting store is the same as before, i.e., $u_s = -h_s + \hat{\xi}(v-p) + (1-\hat{\xi})(v-p-h_o)$. Consumers compare the expected utility from each channel and choose accordingly.

Retailer has belief $\hat{\phi}$ about the fraction of customers who visit store. Given this belief, the retailer's profit function is

$$\pi(q_s, q_o) = pE\min\left(\hat{\phi}D, q_s\right) - c_s q_s + rE\hat{\phi}D + pE\min\left((1-\hat{\phi})D + \left(\hat{\phi}D - q_s\right)^+, q_o\right) - c_o q_o \tag{A.1}$$

Definition 4. *A RE equilibrium $(q_s, q_o, \phi, \hat{\phi}, \hat{\xi})$ satisfies the following:*

i *Given $\hat{\xi}$, if $u_s \geq u_o$, then $\phi = 1$; otherwise $\phi = 0$;*

ii. *Given $\hat{\phi}$, $(q_s, q_o) = \arg\max \pi(q_s, q_o)$, where $\pi(q_s, q_o)$ is given in (A.1);*

iii. *$\hat{\xi} = A(q_s)$;*

iv. *$\hat{\phi} = \phi$.*

The following proposition gives the RE equilibrium.

Proposition 37. *If $h_s \leq \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} h_o$ and $c_s < c_o$, then customers visit store and $q_s^\circ = \bar{F}^{-1}(\frac{c_s}{p})$ and $q_o^\circ = 0$. Otherwise, no one comes to store and $q_s^\circ = 0$ and $q_o^\circ = \bar{F}^{-1}(\frac{c_o}{p})$.*

With BOPS, retailer's profit function is as follows:

- If $\min(h_s, h_b) > h_o$, then no one comes to store, and thus

$$\pi = pE\min(D, q_o) - c_o q_o$$

- If $\min(h_s, h_b) \leq h_o$, then consumers come to store if it is in stock, and thus

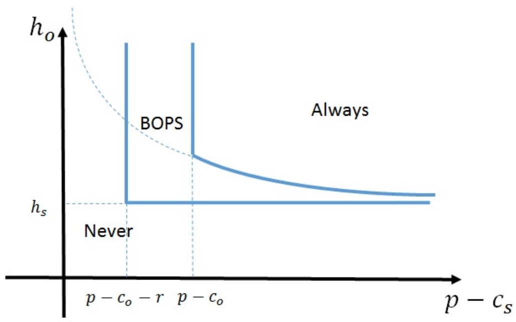
$$\pi = (p + r)\, E\min(D, q_s) - c_s q_s + pE\min\left((D - q_s)^+, q_o\right) - c_o q_o$$

Proposition 38. *When there is BOPS,*

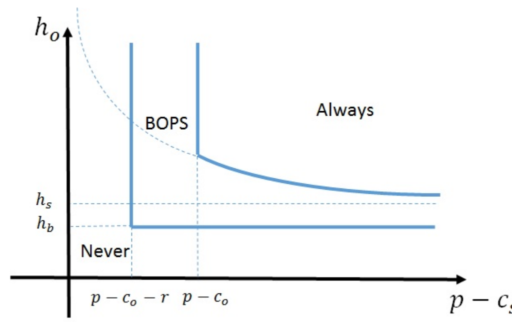
- *if $\min(h_s, h_b) \leq h_o$ and $c_s < c_o + r$, then customers visit store and*
  - *if $c_s \leq \frac{p+r}{p} c_o$, then $q_s^* = \bar{F}^{-1}\left(\frac{c_s}{p+r}\right)$ and $q_o^* = 0$;*
  - *if $c_s > \frac{p+r}{p} c_o$, then $q_s^* = \bar{F}^{-1}\left(\frac{c_s - c_o}{r}\right)$ and $q_o^* = \bar{F}^{-1}\left(\frac{c_o}{p}\right) - \bar{F}^{-1}\left(\frac{c_s - c_o}{r}\right)$;*
- *otherwise, no one comes to store and $q_s^* = 0$ and $q_o^* = \bar{F}^{-1}\left(\frac{c_o}{p}\right)$.*

Comparing Propositions 37 and 38, we can have the three regions as before, though the shape is different (see Figure 9).

Figure 9: Do consumers buy the product in store?



(a) $h_b > h_s$



(b) $h_b \leq h_s$

In the "BOPS" and "Always" region, we can get similar comparison results as before:

Proposition 39. *If* $h_s \in \left(\frac{E \min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} h_o, h_o \right]$ *and* $c_s < c_o + r$*, then customers visit the store only if BOPS is available. Further, BOPS increases total profit (i.e.,* $\pi^* > \pi^\circ$*).*

Proposition 40. *If* $h_b \leq h_o < h_s$ *and* $c_s < c_o + r$*, customers visit the store only if BOPS is available. Further, BOPS increases total profit (i.e.,* $\pi^* > \pi^\circ$*).*

Proposition 41. *If* $h_s \leq \frac{E \min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} h_o$ *and* $c_s < c_o$*, then customers visit store regardless of the implementation of BOPS. Further, if* $r > 0$*, then BOPS decreases total profit (i.e.,* $\pi^* < \pi^\circ$*).*

Heterogeneous Market

Similar as what we did for the homogeneous market, we assume when the online channel is out of stock, those who are willing to buy online will leave for other websites to buy the product at the same price. Thus, we could find that consumers behavior remains the same as before.

Given retailer's belief $\hat{\alpha}_o, \hat{\alpha}_{so}, \hat{\alpha}_s, \hat{\alpha}_l$, the retailer's profit function is

$$\begin{aligned} \pi\left(q_s, q_o\right) = & pE \min\left(\left(\hat{\alpha}_s + \hat{\alpha}_{so}\right) D, q_s\right) - c_s q_s + rE\left(\hat{\alpha}_s + \hat{\alpha}_{so}\right) D \\ & + pE \min\left(\hat{\alpha}_o D + \frac{\hat{\alpha}_{so}}{\hat{\alpha}_s + \hat{\alpha}_{so}} \left(\left(\hat{\alpha}_s + \hat{\alpha}_{so}\right) D - q_s\right)^+, q_o\right) - c_o q_o \end{aligned} \quad (A.2)$$

Definition 5. *A RE equilibrium* $(\alpha_o, \alpha_{so}, , \alpha_s, \alpha_l, q_s, q_o, \hat{\xi}, \hat{\alpha}_o, \hat{\alpha}_{so}, \hat{\alpha}_s, \hat{\alpha}_l)$ *satisfies the following:*

i *Given* $\hat{\xi}$, *then* $\alpha_o = \frac{v-p}{H} - \frac{\hat{\xi}(v-p)^2}{2H^2}$, $\alpha_{so} = \frac{\hat{\xi}(v-p)^2}{2H^2}$, $\alpha_s = \frac{\hat{\xi}(v-p)(H-(v-p))}{H^2}$, *and* $\alpha_l = \frac{[H-(v-p)][H-\hat{\xi}(v-p)]}{H^2}$;

ii. *Given* $\hat{\alpha}_o, \hat{\alpha}_{so}, \hat{\alpha}_s$ *and* $\hat{\alpha}_l$, $(q_s, q_o) = \arg\max \pi(q_s, q_o)$, *where* $\pi(q_s, q_o)$ *is given in (A.2);*

iii. $\hat{\xi} = A(q_s)$, *where* $A(q_s) = \frac{E\min((\alpha_s+\alpha_{so})D, q_s)}{E(\alpha_s+\alpha_{so})D}$;

iv. $\hat{\alpha}_s = \alpha_s$, $\hat{\alpha}_o = \alpha_o$, $\hat{\alpha}_{so} = \alpha_{so}$ *and* $\hat{\alpha}_l = \alpha_l$.

Proposition 42.

- *If* $c_s < \frac{v-p}{2H-(v-p)}c_o + \frac{2H-2(v-p)}{2H-(v-p)}p$, *then there are customers visiting store, specifically,*
  - *if* $c_s < c_o$, *then* $\alpha_s^\circ = \frac{\xi_1^\circ(v-p)(H-(v-p))}{H^2}$, $\alpha_{so}^\circ = \frac{\xi_1^\circ(v-p)^2}{2H^2}$, $\alpha_o^\circ = \frac{v-p}{H} - \frac{\xi_1^\circ(v-p)^2}{2H^2}$, $q_s^\circ = (\alpha_s^\circ + \alpha_{so}^\circ)\bar{F}^{-1}(\frac{c_s}{p})$ *and* $q_o^\circ = \alpha_o^\circ \bar{F}^{-1}(\frac{c_o}{p})$, *where* $\xi_1^\circ = \frac{\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}$
  - *if* $c_s \geq c_o$, *then* $\alpha_s^\circ = \frac{\xi_2^\circ(v-p)(H-(v-p))}{H^2}$, $\alpha_{so}^\circ = \frac{\xi_2^\circ(v-p)^2}{2H^2}$, $\alpha_o^\circ = \frac{v-p}{H} - \frac{\xi_2^\circ(v-p)^2}{2H^2}$, $q_s^\circ = (\alpha_s^\circ+\alpha_{so}^\circ)\bar{F}^{-1}\left(\frac{c_s - \frac{v-p}{2H-(v-p)}c_o}{\frac{2H-2(v-p)}{2H-(v-p)}p}\right)$ *and* $q_o^\circ = (\alpha_o^\circ+\alpha_{so}^\circ)\bar{F}^{-1}(\frac{c_o}{p}) - \alpha_{so}^\circ F^{-1}\left(\frac{c_s - \frac{v-p}{2H-(v-p)}c_o}{\frac{2H-2(v-p)}{2H-(v-p)}p}\right)$, *where* $\xi_2^\circ = \frac{\min\left(D, F^{-1}\left(\frac{c_s - \frac{v-p}{2H-(v-p)}c_o}{\frac{2H-2(v-p)}{2H-(v-p)}p}\right)\right)}{ED}$
- *If* $c_s \geq \frac{v-p}{2H-(v-p)}c_o + \frac{2H-2(v-p)}{2H-(v-p)}p$, *then no one ever comes to store and* $q_s^\circ = 0$, $q_o^\circ = \frac{v-p}{H}\bar{F}^{-1}(\frac{c_o}{p})$, $\xi^\circ = 0$.

With BOPS, retailer's profit function is as follows

$$\begin{aligned}\pi(q_s, q_o) =& pE\min\left((\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*)D, q_s\right) - c_s q_s + rE\min\left((\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*)D, q_s\right) \\ &+ pE\min\left(\alpha_o^* D + \frac{\alpha_{so}^* + \alpha_{bo}^*}{\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*}((\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*)D - q_s)^+, q_o\right) - c_o q_o\end{aligned}$$

Proposition 43. *Suppose* $r = 0$. *When there is BOPS, market outcome is given as follows:*

- *If* $c_s < \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}c_o + \frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}p$, *then there are customers visiting store, and*

– *if* $c_s < c_o$*, then* $q_s^* = (\alpha_s^* + \alpha_{so}^*)\bar{F}^{-1}(\frac{c_s}{p})$ *and* $q_o^* = \alpha_o^* \bar{F}^{-1}(\frac{c_o}{p})$

– *if* $c_s \geq c_o$*, then* $q_s^* = (\alpha_s^* + \alpha_{so}^*)\bar{F}^{-1}\left(\frac{c_s - \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}c_o}{\frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}p}\right)$ *and* $q_o^* = (\alpha_o^* + \alpha_{so}^*)\bar{F}^{-1}(\frac{c_o}{p}) - \alpha_{so}^* F^{-1}\left(\frac{c_s - \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}c_o}{\frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}p}\right)$,

- *If* $c_s \geq \frac{v-p}{2H-(v-p)}c_o + \frac{2H-2(v-p)}{2H-(v-p)}p$*, then no one ever comes to store and* $q_s^* = 0$*,* $q_o^* = \frac{v-p}{H}\bar{F}^{-1}(\frac{c_o}{p})$*.*

Proposition 44.

i. BOPS helps to expand market coverage, i.e., $\alpha_s^* + \alpha_o^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^* > \alpha_s^\circ + \alpha_o^\circ + \alpha_{so}^\circ$*;*

ii. Suppose $r = 0$*. If there are customers visiting store when there is no BOPS, then there exists* $\bar{c}_s$ *such that the implementation of BOPS decreases total profit (i.e.,* $\pi^* < \pi^\circ$*) if* $\beta_s + \beta_o < 1$ *and* $c_s > \bar{c}_s$*.*

This shows that although BOPS could expand market coverage, it may still reduce total profit even if $r = 0$.

Decentralized System

Finally, let's look at decentralized system. Here, for simplicity of exposition, we only look at the homogeneous market. Same as before, we only consider the situation where consumers use BOPS when it is available. Then, with the revenue sharing parameter θ, store and online's profits are given as follows:

$$\tilde{\pi}_s = (\theta p + r)\, E \min(D, \tilde{q}_s) - c_s \tilde{q}_s$$

$$\tilde{\pi}_o = (1-\theta)\, pE \min(D, \tilde{q}_s) + pE \min\left((D - \tilde{q}_s)^+, \tilde{q}_o\right) - c_o \tilde{q}_o$$

Note the BOPS revenue is just free money to the online channel. So we only need to consider store's incentive. Comparing $\tilde{\pi}_s$ with the centralized profit funtion π, we can still find that the store channel ignores the fact that customers would buy online instead in case of stockouts (i.e., the second term in $\tilde{\pi}_o$), and thus there will be a profit loss in the decentralized system compared to the centralized case. Moreover, it is easy to find the following revenue sharing parameter will correct store's incentive and coordinate both channels:

- If $c_s \leq \frac{p+r}{p} c_o$, then $\theta^* = 1$
- If $c_s > \frac{p+r}{p} c_o$, then $\theta^* = \frac{c_o r}{(c_s - c_o)p} \in (0, 1)$.

A.2.2. Default Channel Choice

Suppose a fraction λ of customers are nonstrategic, who head to store by default (because they may forget or they may not care about checking the website beforehand) and may consider buying from the online channel only if store is out of stock; the rest $1 - \lambda$ are strategic as before. As a result, when there is BOPS, only a fraction $1 - \lambda$ of customers will check online for store inventory information.

Homogeneous Market

Let's first consider the case when there is no BOPS. Assume all customers (including both nonstrategic and strategic customers) have the same online hassle cost h_o. Then, since $v - p - h_o \geq 0$, all nonstrategic customers will buy online if store is out of stock.

The retailer has a belief about the fraction of strategic customers who visit store, denoted as $\hat{\phi}$. Given this belief, his total profit is

$$\begin{aligned}\pi =& pE \min\left(\left(\lambda + \hat{\phi}(1-\lambda)\right) D, q\right) - cq + rE\left(\lambda + \hat{\phi}(1-\lambda)\right) D \\ &+ wE(1-\hat{\phi})(1-\lambda)D + wE\left(\left(\lambda + \hat{\phi}(1-\lambda)\right) D - q\right)^+\end{aligned}$$

The following proposition gives the RE equilibrium.

Proposition 45. *If* $h_s \leq \frac{E \min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED} h_o$ *and* $p - c > w$*, then strategic customers* *visit store and* $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w}\right)$*. Otherwise, no strategic customer comes to store and* $q^\circ = \lambda \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)$.

When there is BOPS, strategic customers choose their shopping channel given the current store inventory status.

- If $\min(h_s, h_b) \leq h_o$, strategic customers come to store if it is in stock and the retailer's profit is

$$\pi = pE\min(D,q) - cq + rE\lambda D + rE(1-\lambda)\min(D,q) + wE(D-q)^+$$

- If $\min(h_s, h_b) > h_o$, no strategic customers come to store and the retailer's profit is

$$\pi = pE\min(\lambda D,q) - cq + rE\lambda D + wE(1-\lambda)D + wE(\lambda D-q)^+$$

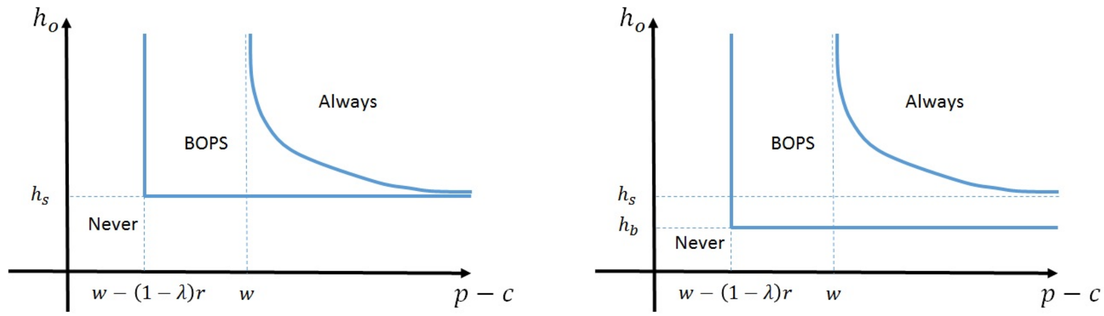
Proposition 46. *When there is BOPS, if* $\min(h_s, h_b) \leq h_o$ *and* $p - c > w - (1-\lambda)r$*, then strategic customers visit store and* $q^* = \bar{F}^{-1}\left(\frac{c}{p+(1-\lambda)r-w}\right)$*; otherwise, no strategic customer comes to store and* $q^* = \lambda \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)$.

Comparing Propositions 45 and 46, Figure 10 shows the impact of BOPS on strategic customer's channel choice. Note it is similar to Figure 1 in Section 2.4, and it is easy to verify that the same insights still hold in this case, i.e., BOPS increases profit in the "BOPS" region as it helps to persuade strategic customers to the more profitable store channel, but BOPS decreases profit in the "Always" region because strategic customers do not come to store once it is out of stock and thus the retailer loses some cross-selling profits.

Heterogeneous Market

Strategic customer's behavior is the same as before. As for nonstrategic customers, they always first go to store. When nonstrategic customers are in store, if store is in stock, they

Figure 10: Do strategic consumers buy the product in store?



(a) $h_b > h_s$ (b) $h_b \le h_s$

buy on the spot; if store is out of stock, they can either buy online instead (and obtain payoff $v - p - h_o$) or leave (and obtain payoff 0). Assume, nonstrategic consumers are heterogeneous in terms of h_o, and the distribution is the same as strategic customers, i.e., all nonstrategic customers are distributed uniformly on the line $[0, H]$. Then, when store is out of stock, a fraction $\frac{v-p}{H}$ of nonstrategic customers (i.e., those with $h_o \le v - p$) will substitute to the online channel.

When there is no BOPS, we keep using notation α_o, α_s and α_{so} to denote the fraction of the pure online, pure store and store-to-online customers among strategic customers. The retailer has belief over these α's, denoted as $\hat{\alpha}_o$, $\hat{\alpha}_s$, and $\hat{\alpha}_{so}$. Given this belief, the retailer's profit is

$$\begin{aligned}\pi =& pE\min\left(\left(\lambda + \left(\hat{\alpha}_s + \hat{\alpha}_{so}\right)(1-\lambda)\right)D, q\right) - cq + rE\left(\lambda + \left(\hat{\alpha}_s + \hat{\alpha}_{so}\right)(1-\lambda)\right)D \\ &+ wE\hat{\alpha}_o(1-\lambda)D + w\frac{\frac{v-p}{H}\lambda + \hat{\alpha}_{so}(1-\lambda)}{\lambda + \left(\hat{\alpha}_s + \hat{\alpha}_{so}\right)(1-\lambda)}E(\left(\lambda + \left(\hat{\alpha}_s + \hat{\alpha}_{so}\right)(1-\lambda)\right)D - q)^+\end{aligned}$$

Proposition 47. *When there is no BOPS, the RE equilibrium is given as follows:*

- *If equation* $\xi = \frac{\min\left(D, \bar{F}^{-1}\left(\frac{c}{p - w\Delta^\circ(\xi)} \wedge 1\right)\right)}{ED}$ *has a solution* $\xi^\circ > 0$, *then there are strategic consumers visiting store and* ($\alpha_s^\circ = \frac{\xi^\circ(v-p)(H-(v-p))}{H^2} > 0$, $\alpha_{so}^\circ = \frac{\xi^\circ(v-p)^2}{2H^2} > 0$) *and* $q^\circ = (\lambda + (\alpha_s^\circ + \alpha_{so}^\circ)(1-\lambda))\bar{F}^{-1}\left(\frac{c}{p - w\Delta^\circ(\xi^\circ)}\right)$, *where* $\Delta^\circ(\xi) = \frac{v-p}{2H-(v-p)} + \frac{(v-p)(H-(v-p))\lambda}{H(2H-(v-p))\left[\lambda + \frac{2H-(v-p)}{2H^2}(v-p)\xi(1-\lambda)\right]}$;

- *otherwise, no strategic customer comes to store and* $q^\circ = 0$.

Note a sufficient condition for the equation $\xi = \frac{\min\left(D, \bar{F}^{-1}\left(\frac{c}{p - w\Delta^\circ(\xi)} \wedge 1\right)\right)}{ED}$ having a positive solution is $p - w\Delta^\circ(0) > c$. The reason is as follows: If $\xi = 1$, the left hand side of the equation is strictly larger than the right hand side; if $\xi = 0$, when $p - w\Delta^\circ(0) > c$, the left hand side of the equation is strictly smaller than the right hand side. Since both sides of the equation are continuous and increasing in ξ, the equation must have a solution $\xi^\circ \in (0, 1)$.

Next, let's look at the case when there is BOPS. Note nonstrategic customers will always first visit store, so they will not be influenced by the information effect. Let's keep using our original notations, $\alpha_s^*, \alpha_o^*, \alpha_b^*, \alpha_{so}^*, \alpha_{bo}^*$, to denote the fraction of pure store, pure online, pure BOPS, store-to-online and BOPS-to-online customers among strategic customers. Then, the retailer's profit function is

$$\begin{aligned}
\pi =& pE \min\left(\left(\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)\right) D, q\right) - cq + rE\lambda D \\
&+ rE \frac{\left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)}{\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)} \min\left(\left(\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)\right) D, q\right) \\
&+ wE\alpha_o^*(1-\lambda)D \\
&+ wE \frac{\frac{v-p}{H}\lambda + \left(\alpha_{so}^* + \alpha_{bo}^*\right)(1-\lambda)}{\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)} \left(\left(\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)\right) D - q\right)^+
\end{aligned}$$

Proposition 48. *With BOPS, the market outcome is given as follows:*

- *if* $p - c > w\Delta^* - r\frac{\left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)}{\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)}$, *then there are strategic customers visiting store and*

  $$q^* = \left(\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)\right) \bar{F}^{-1}\left(\frac{c}{p - w\Delta^* + r\frac{\left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)}{\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)}}\right),$$

  where $\Delta^* = \frac{\frac{v-p}{H}\lambda + \left(\alpha_{so}^* + \alpha_{bo}^*\right)(1-\lambda)}{\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)}$;

- *otherwise, no strategic customer comes to store and* $q^* = 0$.

Proposition 49.

i. BOPS helps to expand market coverage, i.e., $\lambda + (\alpha_s^* + \alpha_o^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*)(1-\lambda) > \lambda + (\alpha_s^\circ + \alpha_o^\circ + \alpha_{so}^\circ)(1-\lambda)$*;*

ii. Suppose $r = 0$*. If* $p - w\Delta^\circ(0) > c$*, i.e., there are customers visiting store when there is no BOPS, then there exist* $\bar{w}$ *and* $\bar{\beta}_s > 0$ *such that the implementation of BOPS decreases total profit (i.e.,* $\pi^* < \pi^\circ$*) if* $\beta_s < \bar{\beta}_s$*,* $w > \bar{w}$ *and* $H > \frac{v-p}{\beta_o}$*.*

This shows that although BOPS could expand market coverage, it may still reduce total profit even if $r = 0$.

Decentralized System

For simplicity of exposition, let's only look at the homogeneous market. In a decentralized system, assuming there are strategic customers using BOPS, the store's profit is

$$\tilde{\pi}_s = \underbrace{pE\lambda \min(D, \tilde{q}) + rE\lambda D}_{from\ nonstrategic\ customers} + \underbrace{(\theta p + r)\, E(1-\lambda) \min(D, \tilde{q})}_{from\ strategic\ customers} - c\tilde{q}$$

Comparing it with the centralized profit function π described above, we find the store's incentive is not fully aligned with the system's. With proper choice of the revenue sharing parameter θ, we should be able to correct store's incentive. However, we find that if λ is very large, we may not have enough BOPS revenue to correct store's incentive to overstock. Thus, a simple θ may not be enough to make sure we achieve the centralized profit level.

A.3. Detailed Analyses of Scenarios When Multiple Mechanisms are Provided Simultaneously

A.3.1. Providing both Physical Showrooms and Availability Information

With physical showrooms and availability information, consumer utilities are as follows

- utility from buying online directly: $u_o = \theta(v - p - h_o) - (1-\theta)(h_o + h_r)$
- utility from visiting store when store is in stock: $u_{s,in} = -h_s + \theta(v - p)$
- utility from visiting store when store is out of stock: $u_{s,out} = -h_s + \theta(v - p - h_o)$

Then, if $u_o < u_{s,out}$, i.e.,

$$h_s < (1-\theta)(h_o + h_r) \tag{A.3}$$

consumers would always visit store, even if it is shown that store is out of stock. In this case, retailer profit is $\pi = pE\min(\theta D, q) - cq + wE(\theta D - q)^+ = (p-w)E\min(\theta D, q) - cq + wE\theta D$, and thus the optimal store inventory level $q^{pa} = \theta \bar{F}^{-1}\left(\frac{c}{p-w}\right)$. Note condition (A.3) is equivalent to $\theta < \underline{\psi}^{pa}$ where $\underline{\psi}^{pa} = \min(\max(\frac{h_o+h_r-h_s}{h_o+h_r}, 0), 1)$, given $\theta \in (0,1)$.

If $u_o \in [u_{s,out}, u_{s,in})$, i.e.,

$$h_s \in [(1-\theta)(h_o + h_r), h_o + (1-\theta)h_r) \tag{A.4}$$

consumers would visit store if store is in stock but buy online directly if store is out of stock. In this case, retailer profit $\pi = pE\min(\theta D, q) - cq + wE(\theta D - q)^+ - rE\frac{1-\theta}{\theta}(\theta D - q)^+ = (p - w + r\frac{1-\theta}{\theta})E\min(\theta D, q) - cq + (w\theta - r(1-\theta))ED$, and thus the optimal store inventory level $q^{pa} = \theta \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$. Note condition (A.4) is equivalent to $\theta \in [\underline{\psi}^{pa}, \tilde{\psi}^{pa})$ where $\tilde{\psi}^{pa} = \min(\max(\frac{h_o+h_r-h_s}{h_r}, 0), 1)$. Moreover, note if the critical fractile $\frac{c}{p-w+r\frac{1-\theta}{\theta}} \geq 1$ ($\Leftrightarrow \theta \geq \frac{r}{(w+r-p+c)^+}$, given $\theta \in (0,1)$), then $q^{pa} = 0$, i.e., the store never has the product in stock and thus consumers actually always buy online as a result.

If $u_o \geq u_{s,in}$, i.e., $h_s \geq h_o + (1-\theta)h_r$ or $\theta \geq \tilde{\psi}^{pa}$, then consumers always buy online directly. In this case, retailer profit $\pi = wE\theta D - rE(1-\theta)D$. The optimal inventory level is just $q^{pa} = 0$.

The following proposition summaries the discussion above, where $\overline{\psi}^{pa} = \min(\frac{r}{(w+r-p+c)^+}, \tilde{\psi}^{pa})$

Proposition 50. *With physical showrooms and availability information, the market out-*

come is given as follows:

- *If $\theta \leq \underline{\psi}^{pa}$, then consumers always visit store; $q^{pa} = \theta \bar{F}^{-1}\left(\frac{c}{p-w}\right)$;*
- *If $\theta \in [\underline{\psi}^{pa}, \overline{\psi}^{pa})$, then consumers visit the store if it has the product in stock but buy online directly if store is out of stock; $q^{pa} = \theta \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$;*
- *If $\theta \geq \overline{\psi}^{pa}$, then consumers always buy online directly; $q^{pa} = 0$.*

A.3.2. Providing both Physical Showrooms and Virtual Showrooms

With virtual showrooms, there is a new customer pool. Then, having both physical showrooms and virtual showrooms is very similar to the case when there is only physical showrooms, simply replacing with a new set of parameters (θ' and D' (or F')). Thus, similar to the proof of Proposition 12, we can easily find the RE equilibrium (the proof of which is omitted):

Proposition 51. *With physical showrooms and virtual showrooms, there exists a threshold ψ^{pv} such that*

- *if $\theta < \psi^{pv}$, then consumers visit store; $q^{pv} = \theta' \bar{F'}^{-1}\left(\frac{c}{p-w}\right)$*
- *if $\theta \geq \psi^{pv}$, then consumers buy online directly; $q^{pv} = 0$.*

A.3.3. Providing both Virtual Showrooms and Availability Information

Again, note that with virtual showrooms, there is a new customer pool. Then, having both virtual showrooms and availability information is very similar to the case when there is only availability information, simply replacing with a new set of parameters (θ' and D' (or F')). Thus, similar to the proof of Proposition 16, we can easily find the market outcome (the proof of which is omitted):

Proposition 52. *With virtual showrooms and availability information, there exists a threshold $\psi^{va} \in [\psi^{v}, 1]$ such that the market outcome is given as follows:*

- *If $\theta < \psi^{va}$, then consumers visit store if store is in stock, and buy online directly if store is out of stock; $q^{va} = \theta' \bar{F'}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}\right)$*
- *If $\theta \geq \psi^{va}$, then consumers always buy online directly; $q^{va} = 0$.*

A.3.4. Providing All Three Mechanisms

Similar to the previous two sections, we note that with virtual showrooms, there is a new customer pool. Then, having all three mechanisms is very similar to the case when there are both physical showrooms and availability information, simply replacing with a new set of parameters (θ' and D' (or F')). Thus, similar to the discussion in Section A.3.1, we can easily find the market outcome (the proof of which is omitted):

Proposition 53. *With all three types of information, the market outcome is given as follows:*

- *If $\theta < \underline{\psi}^{pva}$, then consumers always visit store; $q^{pva} = \theta' \bar{F'}^{-1}\left(\frac{c}{p-w}\right)$;*
- *If $\theta \in [\underline{\psi}^{pva}, \overline{\psi}^{pva})$, then consumers visit store if it is in stock but buy online directly if store is out of stock; $q^{pva} = \theta' \bar{F'}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}\right)$;*
- *If $\theta \geq \overline{\psi}^{pva}$, then consumers always buy online directly; $q^{pva} = 0$.*

A.4. Model Extensions to Chapter 3

In Chapter 3, we have the following main findings:

1. Adding physical showrooms may reduce profits.
2. Adding virtual showrooms may reduce profits.
3. Providing only availability information never reduces profits.
4. There is no complementary effect between physical showrooms and availability information.

5. There is no complementary effect between physical showrooms and virtual showrooms.

In this section, we extend our basic model in three different ways. The goal is to show the robustness of the results listed above.

A.4.1. Endogenous Online Channel

In this section, we relax the assumption in Chapter 2 that the online channel is exogenous. Suppose both online and offline channels follow the standard newsvendor setup. The retail price is p, which is the same across both channels. Unit inventory costs are c_s and c_o in the store and online channel, both of which are smaller than p. For each unit sold online, if it is not returned, the retailer can get the revenue p, i.e., the price of the product; if it is returned, the retailer incurs net cost k (i.e., the retailer cannot make money from dealing with returns). The retailer decides the inventory levels in both channels, q_s and q_o, in the beginning.

Consumers setup is the same as before. We assume when the online channel is out of stock, those who are willing to buy online will leave for other websites to buy the product at the same price and obtain the same utility u_o. Note, when consumers encounter a stockout in store, they will buy from the retailer's online channel first. In many companies (e.g., Bonobos and Warby Parker), store employees are trained and equipped with digital devices to help store customers order online. However, when customers are shopping online at home, it is hard for a firm to persuade customers to come to store when online is out of stock. Our model setup is to capture this difference.

With this model setup, we can find that consumer's utility functions, and thus their channel choices, remain unchanged.

Base Model

Given belief $\hat{\phi}$, the retailer's total profit is

- If $\hat{\phi} = 1$, then

$$\pi(q_s, q_o) = pE\min(\theta D, q_s) - c_s q_s + [p\theta - k(1-\theta)]E\min\left(\left(D - \frac{q_s}{\theta}\right)^+, q_o\right) - c_o q_o$$

- If $\hat{\phi} = 0$, then

$$\pi(q_s, q_o) = [p\theta - k(1-\theta)]E\min(D, q_o) - c_o q_o$$

Definition 6. *A RE equilibrium $(q_s, q_o, \phi, \hat{\phi}, \hat{\xi})$ satisfies the following:*

i Given $\hat{\xi}$, if $u_s > u_o$, then $\phi = 1$; otherwise $\phi = 0$;

ii. Given $\hat{\phi}$, $(q_s, q_o) = \arg\max \pi(q_s, q_o)$;

iii. $\hat{\xi} = A(q_s)$;

iv. $\hat{\phi} = \phi$.

Proposition 54. *In the base model, the RE equilibrium is given as follows:*

- *If $h_s < \xi_1^\circ[h_o + (1-\theta)h_r]$ and $c_o \geq [p\theta - k(1-\theta)]\frac{c_s}{p}$, then consumers visit store (i.e., $\phi^\circ = 1$) and $q_s^\circ = \theta\bar{F}^{-1}(\frac{c_s}{p})$, $q_o^\circ = 0$;*

- *if $h_s < \xi_2^\circ[h_o + (1-\theta)h_r]$ and $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s \leq k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, then consumers visit store (i.e., $\phi^\circ = 1$) and $q_s^\circ = \theta\bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)$, $q_o^\circ = \bar{F}^{-1}\left(\frac{c_o}{p\theta - k(1-\theta)}\right) - \bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)$;*

- *otherwise, no one comes to store (i.e., $\phi^\circ = 0$) and $q_s^\circ = 0$, $q_o^\circ = \bar{F}^{-1}\left(\frac{c_o}{p\theta - k(1-\theta)}\right)$,*

where $\xi_1^\circ = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}$, $\xi_2^\circ = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)\right)}{ED}$.

Physical Showrooms

Suppose there is a physical showroom in the store. Given belief $\hat{\phi}$, the retailer's total profit is

- If $\hat{\phi} = 1$, then

$$\pi = pE\min\left(\theta D, q_s\right) - c_s q_s + pE\min\left(\left(\theta D - q_s\right)^+, q_o\right) - c_o q_o$$

- If $\hat{\phi} = 0$, then

$$\pi = [p\theta - k(1-\theta)]E\min\left(D, q_o\right) - c_o q_o$$

Proposition 55. *With physical showrooms, the RE equilibrium is given as follows:*

- *If* $h_s < \frac{E\min\left(D,\bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}\theta h_o + (1-\theta)\left(h_o + h_r\right)$ *and* $c_s < c_o$*, then consumers come to store (i.e.,* $\phi^p = 1$*) and* $q_s^p = \theta\bar{F}^{-1}(\frac{c_s}{p})$, $q_o^p = 0$*;*

- *if* $h_s < (1-\theta)(h_o + h_r)$ *and* $c_s \geq c_o$*, then consumers come to store (i.e.,* $\phi^p = 1$*) and* $q_s^p = 0$, $q_o^p = \theta\bar{F}^{-1}(\frac{c_o}{p})$*;*

- *otherwise, no one comes to store (i.e.,* $\phi^p = 0$*) and* $q_s^p = 0$, $q_o^p = \bar{F}^{-1}\left(\frac{c_o}{p\theta - k(1-\theta)}\right)$*.*

Proposition 56. *If* $(1-\theta)\left(h_o + h_r\right) < h_s < \frac{E\min\left(D,\bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}\left[h_o + (1-\theta)\,h_r\right]$ *and* $[p\theta - k(1-\theta)]\frac{c_s}{p} < c_o < c_s$*, then consumers visit store if there is no physical showroom (i.e.,* $\phi^\circ = 1$*) and they buy online directly if there is physical showroom (i.e.,* $\phi^p = 0$*). Also, in this case, physical showroom decreases profit, i.e.,* $\pi^\circ > \pi^p$*.*

This shows that our first finding, i.e., physical showrooms could backfire, is robust.

Virtual Showroom

Suppose there is a virtual showroom online. This is just a special case of the base model, with D' and θ'.

Proposition 57. *With virtual showrooms, the RE equilibrium is given as follows:*

- *If* $h_s < \xi_1^v[h_o + (1-\theta')h_r]$ *and* $c_o \geq [p\theta' - k(1-\theta')]\frac{c_s}{p}$*, then consumers visit store (i.e.,* $\phi^v = 1$*) and* $q_s^v = \theta'\bar{F}'^{-1}(\frac{c_s}{p})$, $q_o^v = 0$*;*

- *if* $h_s < \xi_2^v[h_o + (1-\theta')h_r]$ *and* $c_o < [p\theta' - k(1-\theta')]\frac{c_s}{p}$ *and* $c_s \le k\frac{1-\theta'}{\theta'} + \frac{c_o}{\theta'}$, *then consumers visit store (i.e.,* $\phi^v = 1$*) and* $q_s^v = \theta'\bar{F}'^{-1}\left(\frac{\theta' c_s - c_o}{k(1-\theta')}\right)$, $q_o^v = \bar{F}'^{-1}\left(\frac{c_o}{p\theta' - k(1-\theta')}\right) - \bar{F}'^{-1}\left(\frac{\theta' c_s - c_o}{k(1-\theta')}\right)$*;*

- *otherwise, no one comes to store (i.e.,* $\phi^v = 0$*) and* $q_s^v = 0$, $q_o^v = \bar{F}'^{-1}\left(\frac{c_o}{p\theta' - k(1-\theta')}\right)$*,*

where $\xi_1^v = \frac{E\min\left(D', \bar{F}'^{-1}\left(\frac{c_s}{p}\right)\right)}{ED'}$, $\xi_2^v = \frac{E\min\left(D', \bar{F}'^{-1}\left(\frac{\theta' c_s - c_o}{k(1-\theta')}\right)\right)}{ED'}$.

Proposition 58. *If* $\frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}\left[h_o + (1-\theta')\,h_r\right] < h_s < \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}[h_o + (1-\theta)h_r]$ *and* $[p\theta' - k(1-\theta')]\frac{c_s}{p} < c_o$*, then consumers visit store if there is no virtual showroom (i.e.,* $\phi^\circ = 1$*) and they buy online directly if there is virtual showroom (i.e.,* $\phi^v = 0$*). Also, in this case, there exists* $\bar{c}_o$ *such that virtual showroom decreases profit (i.e.,* $\pi^\circ > \pi^v$*) if* $c_o > \bar{c}_o$*.*

This shows that our second finding, i.e., virtual showrooms could backfire, is robust.

Availability Information

Suppose there is availability information.

- If $h_s < h_o + (1-\theta)h_r$, then consumers will come to store only if it is in stock. Then, the retailer's profit is

$$\pi = pE\min(\theta D, q_s) - c_s q_s + [p\theta - k(1-\theta)]E\min\left(\left(D - \frac{q_s}{\theta}\right)^+, q_o\right) - c_o q_o$$

- If $h_s \ge h_o + (1-\theta)h_r$, then consumers never come to store. Then, the retailer's profit is $\pi = [p\theta - k(1-\theta)]E\min(D, q_o) - c_o q_o$.

Proposition 59. *With availability information, the market outcome is given as follows:*

- *If* $h_s < h_o + (1-\theta)h_r$ *and* $c_o \ge [p\theta - k(1-\theta)]\frac{c_s}{p}$*, then consumers visit store if store is in stock and* $q_s^a = \theta\bar{F}^{-1}(\frac{c_s}{p})$, $q_o^a = 0$*;*

- *if* $h_s < h_o + (1-\theta)h_r$ *and* $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ *and* $c_s \le k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$*, then con-*

sumers visit store if store is in stock and $q_s^a = \theta \bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)$, $q_o^a = \bar{F}^{-1}\left(\frac{c_o}{p\theta - k(1-\theta)}\right) - \bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)$;

- *otherwise, no one ever comes to store and* $q_s^a = 0$, $q_o^a = \bar{F}^{-1}\left(\frac{c_o}{p\theta - k(1-\theta)}\right)$,

Proposition 60. *Availability info never reduces profit, i.e.,* $\pi^a \geq \pi^\circ$.

This shows that our third finding, i.e., availability information never reduces profit, is robust.

Joint Implementation

Here, we only look at the two pairs, i.e., physical showrooms and availability information, and physical showrooms and virtual showrooms. Our goal is to check if there is any complementary effect between them.

Suppose there are both physical showrooms and availability information.

- If $h_s < (1-\theta)(h_o + h_r)$, then consumers would always visit store. Thus,

$$\pi = pE\min(\theta D, q_s) - c_s q_s + pE\min\left((\theta D - q_s)^+, q_o\right) - c_o q_o$$

- if $(1-\theta)(h_o + h_r) \leq h_s < h_o + (1-\theta)h_r$, then consumers come to store only if it is in stock. Thus,

$$\pi = pE\min(\theta D, q_s) - c_s q_s + [p\theta - k(1-\theta)]\, E\min\left(\left(D - \frac{q_s}{\theta}\right)^+, q_o\right) - c_o q_o$$

- if $h_s \geq h_o + (1-\theta)h_r$, then no one ever comes to store. Thus,

$$\pi = [p\theta - k(1-\theta)]\, E\min(D, q_o) - c_o q_o$$

Proposition 61. $\pi^{pa} \leq \max(\pi^p, \pi^a)$

This shows that there is no complementary effect between physical showroom and availability information.

Suppose there are both physical and virtual showrooms. This is just the same as the physical showroom only scenario with a different customer pool D' and θ'.

Proposition 62. $\pi^{pv} \leq \max(\pi^p, \pi^v)$

This shows that there is no complementary effect between these two types of showrooms.

A.4.2. Continuous Valuation

Suppose consumer valuation V is continuously distributed on $[0, +\infty)$. Let $G(v)$ be the proportion of consumers with a valuation v or lower. Ex ante, consumers know the distribution G but not their valuations, so they are homogeneous. Ex post, consumers will learn their valuations after purchase or by checking the product in store.

All other model elements remain the same as before.

Base Model

If consumers buy online directly, after they get the delivery, they can realize their valuation. For a consumer with V, if she keeps it, she gets payoff $V - p - h_o$; if she returns it, her payoff is $-h_o - h_r$. Then, only those with $V < p - h_r$ will return the product. The ex ante expected payoff of buying online directly is

$$u_o = E_V \max(V - p - h_o, -h_o - h_r)$$

Note those with $V \in (p - h_r, p)$ will not like the product (since $V < p$) but they don't return it because the return cost is too high (since $V - p > -h_r$).

In the store, if there is stock, then consumers can realize their valuation, and only those with $V \geq p$ will make a purchase on the spot, and the others will leave. If store is out of stock, they can buy online. So the expected payoff is

$$u_s(\hat{\xi}) = -h_s + \hat{\xi} E_V \max(V-p,0) + (1-\hat{\xi})u_o$$

Here, as what we did in Chapter 2, we assume consumers would consider both channels, i.e., $u_o \geq 0$ and $u_s(1) \geq 0$.

Denote the fraction of customers who visit store as ϕ. In an equilibrium, this could be either 0 or 1, since consumers are ex ante homogeneous.

Given belief $\hat{\phi}$, the retailer's profit function is

$$\pi(q) = p\bar{G}(p) E_D D_{in}(q) - cq \tag{A.5}$$
$$+ \left[w\bar{G}(p-h_r) - rG(p-h_r)\right] E_D D_{out}(q) \tag{A.6}$$
$$+ \left[w\bar{G}(p-h_r) - rG(p-h_r)\right] \left(1-\hat{\phi}\right) E_D D \tag{A.7}$$

where $D_{in}(q) = \min(\hat{\phi}D, \frac{q}{\bar{G}(p)})$, and $D_{out}(q) = (\hat{\phi}D - \frac{q}{\bar{G}(p)})^+$.

Definition 7. *A RE equilibrium $(q, \phi, \hat{\phi}, \hat{\xi})$ satisfies the following:*

i Given $\hat{\xi}$, if $u_s > u_o$, then $\phi = 1$; otherwise $\phi = 0$;

ii. Given $\hat{\phi}$, $q = \arg\max \pi(q)$;

iii. $\hat{\xi} = A(q)$, where $A(q) = \frac{\min(\phi D, q/\bar{G}(p))}{E\phi D}$;

iv. $\hat{\phi} = \phi$.

Proposition 63. *If $h_s < h_s^\circ$, then consumers visit store (i.e., $\phi^\circ = 1$) and $q^\circ = \bar{G}(p)\bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}}\right)$, where $h_s^\circ = \xi^\circ(E_V\max(V-p,0) + h_o - E_V\max(V-p,-h_r))$ and $\xi^\circ = \frac{E_D \min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w\frac{\bar{G}(p-h_r)}{\bar{G}(p)}+r\frac{G(p-h_r)}{\bar{G}(p)}}\right)\right)}{E_D D}$; otherwise, no one comes to store (i.e., $\phi^\circ = 0$), and $q^\circ = 0$.*

Physical Showrooms

With physical showrooms, when store is out of stock, consumers are still able to realize their valuation. In such case, only those store customers with $V - p - h_o \geq 0$ will keep on buying online. So

$$u_s(\hat{\xi}) = -h_s + \hat{\xi} E_V \max(V-p,0) + (1-\hat{\xi}) E_V \max(V-p-h_o,0)$$

Then, the retailer's profit is

$$\pi = p\bar{G}\left(p\right) E_D D_{in}\left(q\right) - cq \tag{A.8}$$

$$+ w\bar{G}(p+h_o) E_D D_{out}\left(q\right) \tag{A.9}$$

$$+ \left[w\bar{G}\left(p-h_r\right) - rG\left(p-h_r\right)\right]\left(1-\hat{\phi}\right) E_D D \tag{A.10}$$

where $D_{in}(q) = \min(\hat{\phi}D, \frac{q}{\bar{G}(p)})$, and $D_{out}(q) = (\hat{\phi}D - \frac{q}{\bar{G}(p)})^+$.

Proposition 64. *With physical showrooms, if $h_s < h_s^p$, then consumers visit store (i.e., $\phi^p = 1$), and $q^p = \bar{G}\left(p\right)\bar{F}^{-1}\left(\frac{c}{p-w\frac{\bar{G}(p+h_o)}{\bar{G}(p)}}\right)$, where $h_s^p = \xi^p E_v \max(V-p,0) + (1-\xi^p)E_V \max(V-p-h_o,0) + h_o - E_V \max(V-p,-h_r)$ and $\xi^p = \frac{E_D \min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w\frac{\bar{G}(p+h_o)}{\bar{G}(p)}}\right)\right)}{E_D D}$; otherwise, no one comes to store (i.e., $\phi^p = 0$), and $q^p = 0$. Moreover, $h_s^p > h_s^\circ$ if and only if r is small enough.*

Proposition 65. *If $h_s^p < h_s^\circ$ and $h_s \in [h_s^p, h_s^\circ)$, then consumers come to store in the base case but buy online directly if there is physical showroom. As a result, providing physical showrooms reduces total profit (i.e., $\pi^p < \pi^\circ$).*

This shows that our first finding, i.e., physical showrooms could backfire, is robust.

Virtual Showrooms

Similar to what we did in Chapter 3, we assume virtual showrooms help to screen out some "low-type" customers. Specifically, the signal consumers receive after checking with the virtual showroom is still binary: those whose valuation $V < \bar{v}$ will realize their valuation is low and therefore leave the market without any purchase, while those whose valuation $V \geq \bar{v}$ will know their valuation is greater than $\bar{v}$ and update their belief about the valuation distribution to G' such that $\forall v \geq \bar{v}$, $G'(v) = \frac{G(v)-G(\bar{v})}{1-G(\bar{v})}$. Here, we assume threshold $\bar{v} \leq p - h_r$, i.e., virtual showrooms would screen out only those who really don't like the product and are bound to return the product if they buy it online. The total number of customers left in the market is $D' = [1 - G(\bar{v})]D$.

Thus, the model is the same as base model except for a new customer pool, D' and G'. We denote E'_V as the expectation over V given G'.

Proposition 66. *With virtual showrooms, if $h_s < h_s^v$, then consumers visit store (i.e., $\phi^v = 1$), and $q^v = \bar{G}(p)\bar{F}^{-1}\left(\frac{c}{p-w\frac{\bar{G}'(p-h_r)}{\bar{G}'(p)}+r\frac{G'(p-h_r)}{\bar{G}'(p)}}\right)$, where $h_s^v = \xi^v(E'_V \max(V-p,0) + h_o - E'_V \max(V-p,-h_r))$ and $\xi^v = \frac{E_D \min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w\frac{\bar{G}'(p-h_r)}{\bar{G}'(p)}+r\frac{G'(p-h_r)}{\bar{G}'(p)}}\right)\right)}{E_D D}$; otherwise, no one comes to store (i.e., $\phi^v = 0$), and $q^v = 0$. Moreover, $h_s^v < h_s^\circ$.*

Proposition 67. *If $h_s \in [h_s^v, h_s^\circ)$, then consumers come to store in the base case but buy online directly if there is virtual showroom. As a result, there exists $\bar{w}$ such that providing virtual showrooms reduces total profit (i.e., $\pi^v < \pi^\circ$) if $w < \bar{w}$.*

This shows that our second finding, i.e., virtual showrooms could backfire, is robust.

Availability Information

With availability information, if it is shown that store is in stock, then $u_{s,in} = -h_s + E_V \max(V-p,0)$. Apparently, consumer will not go to store if store is out of stock.

Denote ϕ_{in} as the fraction of customers visiting store when it is in stock. Then, the retailer's profit function is

$$
\begin{aligned}
\pi(q) = {} & p\bar{G}(p)\phi_{in} ED_{in}(q) - cq \\
& + [w\bar{G}(p-h_r) - rG(p-h_r)](1-\phi_{in}) ED_{in}(q) \\
& + [w\bar{G}(p-h_r) - rG(p-h_r)] ED_{out}(q)
\end{aligned}
$$

where $D_{in}(q) = \min(D, \frac{q}{\theta\phi_{in}})$ and $D_{out}(q) = (D - \frac{q}{\theta\phi_{in}})^+$.

Proposition 68. *With availability information, if $h_s \le h_s^a$, then consumers come to store if store is in stock and buy online if store is out of stock, where $h_s^a = E_V \max(V-p,0) + h_o - E_V \max(V-p,-h_r)$ and $q^a = \bar{G}(p)\bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}}\right)$; otherwise, no one comes to store, and $q^a = 0$.*

Proposition 69. *Compared to the base case, providing availability information never decreases profit, i.e., $\pi^a \ge \pi^\circ$.*

This shows that our third finding, i.e., availability never reduces profit, is robust.

Joint Implementation

Here, we only look at the two pairs, i.e., physical showrooms and availability information, and physical showrooms and virtual showrooms. Our goal is to check if there is any complementary effect between them.

Suppose there are both physical showrooms and availability information. Now even if store is out of stock, consumers may still want to visit store, $u_{s,out} = -h_s + E_V \max(V-p-h_o, 0)$; if store is in stock, $u_{s,in} = -h_s + E_V \max(V-p,0)$.

- If $h_s \le E_V \max(V-p-h_o,0) + h_o - E_V \max(V-p,-h_r)$, then consumers come to store even if store is out of stock. Thus,

$$\pi = pE_D \min\left(\bar{G}(p)D, q\right) - cq + w\bar{G}(p+h_o)E_D\left(D - \frac{q}{\bar{G}(p)}\right)^+$$

- If $E_V \max(V-p-h_o,0)+h_o-E_V \max(V-p,-h_r) < h_s \le E_V \max(V-p,0)+h_o-E_V \max(V-p,-h_r)$, then consumers come to store only if it is in stock. Thus,

$$\pi = pE_D \min\left(\bar{G}(p)\, D, q\right) - cq + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right] E_D \left(D - \frac{q}{\bar{G}(p)}\right)^+$$

- If $h_s > E_V \max(V-p,0)+h_o-E_V \max(V-p,-h_r)$, then no one comes to store. Thus,

$$\pi = \left[w\bar{G}(p-h_r) - rG(p-h_r)\right] E_D D$$

Proposition 70. $\pi^{pa} \le \max(\pi^p, \pi^a)$

This shows that there is no complementary effect between physical showrooms and availability information.

Suppose there are both physical and virtual showrooms. This is just the same as the physical showrooms only scenario with a different customer pool D' and G'.

Proposition 71. $\pi^{pv} \le \max(\pi^p, \pi^v)$.

This shows that there is no complementary effect between these two types of showroom.

A.4.3. Informed and Uninformed Customers

Suppose that there are two groups of customers: a fraction λ of them are informed and a fraction $1-\lambda$ of them are uninformed. Informed customers know they are high type (v). But uninformed customers don't know their type ex ante; they just know θ of them are high type (v) while the others are low type (0).

Everything else remains the same as the base model. Note the uninformed customers' behavior remains the same as what we described in the simple model. So, here we only describe informed customers' behavior. In this section, we use subscripts $\cdot_u$ and $\cdot_i$ to denote the parameters regarding uninformed and informed customers, respectively.

Base Model

For uninformed customers, their utility functions are given in Chapter 3 and are also presented here:

- $u_{o,u} = \theta(v - p - h_o) - (1 - \theta)(h_o + h_r)$
- $u_{s,u}(\hat{\xi}) = -h_s + \hat{\xi}\theta(v - p) + (1 - \hat{\xi})u_{o,u}$

For informed customers, they don't have valuation uncertainty, but still they need to face availability uncertainty. Specifically, their utility functions are

- $u_{o,i} = v - p - h_o$
- $u_{s,i}(\hat{\xi}) = -h_s + \hat{\xi}(v - p) + (1 - \hat{\xi})u_{o,i}$

Denote the fraction of informed customers who visit store as ϕ_i, and the fraction of uninformed customers who visit store as ϕ_u. We only consider the cases where people from the same group choose the same channel. Therefore, we have four possible equilibrium outcomes:

1. $\phi_i = 0, \phi_u = 0$
2. $\phi_i = 1, \phi_u = 0$
3. $\phi_i = 0, \phi_u = 1$
4. $\phi_i = 1, \phi_u = 1$

Note if $u_{s,i}(\hat{\xi}) > u_{o,i}$, then we must have $u_{s,u}(\hat{\xi}) > u_{o,u}$. Thus, $\phi_i = 1, \phi_u = 0$ cannot be an equilibrium. So we have three left.

Given beliefs $\hat{\phi}_i$ and $\hat{\phi}_u$, the retailer's profit function is

- if $\hat{\phi}_i = 1$ and $\hat{\phi}_u = 1$, then

$$\begin{aligned}\pi(q) =& pE\min\left(\left(\lambda + (1-\lambda)\,\theta\right) D, q\right) - cq \\ &+ \left[w\theta - r\,(1-\theta)\right] E\,(1-\lambda)\left[D - \frac{q}{\lambda + (1-\lambda)\,\theta}\right]^+ \\ &+ wE\lambda\left(D - \frac{q}{\lambda + (1-\lambda)\,\theta}\right)^+\end{aligned}$$

- if $\hat{\phi}_i = 0$ and $\hat{\phi}_u = 1$, then

$$\begin{aligned}\pi(q) =& pE\min\left((1-\lambda)\,\theta D, q\right) - cq \\ &+ \left[w\theta - r\,(1-\theta)\right] E\Big[(1-\lambda)\,D - \frac{q}{\theta}\Big]^+ \\ &+ wE\lambda D\end{aligned}$$

- if $\hat{\phi}_i = 0$ and $\hat{\phi}_u = 0$, then

$$\pi(q) = [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D$$

Definition 8. *A RE equilibrium $(\phi_u, \phi_i, q, \hat{\xi}, \hat{\phi}_u, \hat{\phi}_i)$ satisfies the following:*

i Given $\hat{\xi}$,

$$\phi_u = \begin{cases} 1 & \text{if } u_{s,u} > u_{o,u} \\ 0 & \text{if } u_{s,u} \le u_{o,u} \end{cases} \quad \text{and} \quad \phi_i = \begin{cases} 1 & \text{if } u_{s,i} > u_{o,i} \\ 0 & \text{if } u_{s,i} \le u_{o,i} \end{cases}$$

ii. Given $\hat{\phi}_u, \hat{\phi}_i$, $q = \arg\max_q \pi(q)$;

iii. $\hat{\xi} = A(q)$, where $A(q) = \frac{E\min((\lambda\phi_i + (1-\lambda)\phi_u\theta)D, q)}{E(\lambda\phi_i + (1-\lambda)\phi_u\theta)D}$;

iv. $\hat{\phi}_u = \phi_u, \hat{\phi}_i = \phi_i$.

Proposition 72. *In the base model, the RE equilibrium is as follows:*

- *if* $h_s < \xi_1^\circ h_o$, *then all customers come to store, i.e.,* $\phi_i^\circ = 1, \phi_u^\circ = 1$

- *if* $\xi_2^\circ h_o \leq h_s < \xi_2^\circ [h_o + (1-\theta) h_r]$, *then only uninformed customers come to store, i.e.,* $\phi_i^\circ = 0, \phi_u^\circ = 1$*;*

- *otherwise, no one comes to store, i.e.,* $\phi_i^\circ = 0, \phi_u^\circ = 0$*,*

where $\xi_1^\circ = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{(1-\theta)(1-\lambda)}{\lambda+(1-\lambda)\theta}}\right)\right)}{ED}$ *and* $\xi_2^\circ = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)\right)}{ED}$.

Physical Showrooms

Suppose there is a physical showroom. The informed customers' utility functions remain the same as in the base model, since showrooms do not have any effect on them.

Given beliefs $\hat{\phi}_i$ and $\hat{\phi}_u$, the retailer's profit function is

- if $\hat{\phi}_i = 1$ and $\hat{\phi}_u = 1$, then

$$\begin{aligned}\pi =& pE\min\left(\left(\lambda + (1-\lambda)\,\theta\right) D, q\right) - cq \\ &+ wE[(\lambda + (1-\lambda)\,\theta)\, D - q]^+\end{aligned}$$

- if $\hat{\phi}_i = 0$ and $\hat{\phi}_u = 1$, then

$$\begin{aligned}\pi =& pE\min\left((1-\lambda)\,\theta D, q\right) - cq \\ &+ w\theta E\Big[(1-\lambda)\, D - \frac{q}{\theta}\Big]^+ \\ &+ wE\lambda D\end{aligned}$$

- if $\hat{\phi}_i = 0$ and $\hat{\phi}_u = 0$, then

$$\pi = [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D$$

Proposition 73. *With physical showrooms, the RE equilibrium is as follows:*

- *if* $h_s < \xi^p h_o$, *then all customers come to store, i.e.,* $\phi_i^p = 1, \phi_u^p = 1$
- *if* $\xi^p h_o \leq h_s < \xi^p \theta h_o + (1-\theta)(h_o + h_r)$, *then only uninformed customers come to store, i.e.,* $\phi_i^p = 0, \phi_u^p = 1$;
- *otherwise, no one comes to store, i.e.,* $\phi_i^p = 0, \phi_i^p = 0$,

where $\xi^p = \frac{E \min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED}$.

Proposition 74. *If* $\max\{\xi_2^\circ h_o, \xi^p \theta h_o + (1-\theta)(h_o + h_r)\} < h_s < \xi_2^\circ[h_o + (1-\theta)h_r]$, *then uninformed customers come to store in the base case and buy online if there is physical showroom while informed customers always buy online. As a result, physical showroom decreases total profit, i.e.,* $\pi^\circ > \pi^p$.

This shows that our first finding, i.e., physical showrooms could backfire, is robust.

Virtual Showrooms

Suppose there is a virtual showroom online. Note virtual showrooms do not affect informed customers. It simply screens out α of low-type uninformed customers. Then $D' = [1 - \alpha(1-\theta)(1-\lambda)]D$, $\theta' = \frac{\theta}{1-\alpha(1-\theta)}$ and $\lambda' = \frac{\lambda}{1-\alpha(1-\theta)(1-\lambda)}$.

Here, we refer to those remaining in the market as the customers.

Proposition 75. *With virtual showroom, the RE equilibrium is as follows:*

- *if* $h_s < \xi_1^v h_o$, *then all customers come to store, i.e.,* $\phi_i^v = 1, \phi_u^v = 1$
- *if* $\xi_2^v h_o \leq h_s < \xi_2^v[h_o + (1-\theta)h_r]$, *then only uninformed customers come to store, i.e.,* $\phi_i^v = 0, \phi_u^v = 1$;
- *otherwise, no one comes to store, i.e.,* $\phi_i^v = 0, \phi_i^v = 0$,

where $\xi_1^v = \frac{E\min\left(D', \bar{F}'^{-1}\left(\frac{c}{p-w+r\frac{(1-\theta')(1-\lambda')}{\lambda'+(1-\lambda')\theta'}}\right)\right)}{ED'}$ *and* $\xi_2^v = \frac{E\min\left(D', \bar{F}'^{-1}\left(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}\right)\right)}{ED'}$.

Proposition 76. *If* $\max\{\xi_2^\circ h_o, \xi_2^v[h_o+(1-\theta')h_r]\} < h_s \leq \xi_2^\circ[h_o+(1-\theta)h_r]$*, then uninformed customers come to store in the base case and buy online if there is virtual showroom while informed customers always buy online. As a result, there exists* $\bar{w}$ *such that providing virtual showroom decreases total profit (i.e.,* $\pi^\circ > \pi^v$*) if* $w < \bar{w}$.

This shows that our second finding, i.e., virtual showrooms could backfire, is robust.

Availability Information

With availability info, customers (including informed) don't need to form beliefs.

- If $h_s < h_o$, then both informed and uninformed customers go to store if in stock. Thus, retailer's profit

$$\begin{aligned}\pi =& pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q\right) - cq \\ &+ \left[w\theta - r(1-\theta)\right]E(1-\lambda)\left[D - \frac{q}{\lambda+(1-\lambda)\theta}\right]^+ \\ &+ wE\lambda\left(D - \frac{q}{\lambda+(1-\lambda)\theta}\right)^+\end{aligned}$$

- if $h_o \leq h_s < h_o + (1-\theta)h_r$, then only uninformed customer will go to store if in stock. Thus,

$$\begin{aligned}\pi =& pE\min\left((1-\lambda)\theta D, q\right) - cq \\ &+ \left[w\theta - r(1-\theta)\right]E\left[(1-\lambda)D - \frac{q}{\theta}\right]^+ \\ &+ wE\lambda D\end{aligned}$$

- if $h_s \geq h_o + (1-\theta)h_r$, then no one ever comes to store. Thus,

$$\pi = [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D$$

Proposition 77. *With availability information,*

- *If $h_s < h_o$, then both informed and uninformed customers go to store if store is in stock, and* $q^a = (\lambda + (1-\lambda)\theta)\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{(1-\lambda)(1-\theta)}{\lambda+(1-\lambda)\theta}}\right)$*;*
- *If $h_o \leq h_s < h_o + (1-\theta)h_r$, then only uninformed customer will go to store if in stock, and* $q^a = (1-\lambda)\theta\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$*;*
- *If $h_s \geq h_o + (1-\theta)h_r$, then no one ever comes to store, and $q^a = 0$.*

Proposition 78. *Compared to the base case, providing availability information never decreases profit, i.e., $\pi^a \geq \pi^\circ$.*

This shows that our third finding, i.e., availability information never reduces profit, is robust.

Joint Implementation

Here, we only look at the two pairs, i.e., physical showrooms and availability information, and physical showrooms and virtual showrooms. Our goal is to check if there is any complementary effect between them.

Suppose there are both physical showrooms and availability information.

- If $h_s < \min(h_o, (1-\theta)(h_o + h_r))$, then informed customers go to store if in-stock and buy online if stockouts; uninformed customers always go to store. Thus,

$$\pi = pE\min\left(\left(\lambda + (1-\lambda)\theta\right)D, q\right) - cq + wE(\left(\lambda + (1-\lambda)\theta\right)D - q)^+$$

- If $h_o < (1-\theta)(h_o+h_r)$ and $h_o \le h_s < (1-\theta)(h_o+h_r)$, then informed customers always buy online; uninformed customers always go to store. Thus,

$$\pi = pE\min\left((1-\lambda)\,\theta D, q\right) - cq + wE((1-\lambda)\,\theta D - q)^{+} + wE\lambda D$$

- If $h_o > (1-\theta)(h_o+h_r)$ and $(1-\theta)(h_o+h_r) \le h_s < h_o$, then informed customers go to store if in stock and buy online if stockouts; uninformed customers go to stock if in stock and buy online if stockouts. Thus,

$$\begin{aligned}\pi =& pE\min\left(\left(\lambda + (1-\lambda)\,\theta\right)D, q\right) - cq \\ &+ \left[w\theta - r\,(1-\theta)\right]E\,(1-\lambda)\left[D - \frac{q}{\lambda + (1-\lambda)\,\theta}\right]^{+} \\ &+ wE\lambda\left(D - \frac{q}{\lambda + (1-\lambda)\,\theta}\right)^{+}\end{aligned}$$

- If $\max(h_o, (1-\theta)(h_o+h_r)) \le h_s < h_o + (1-\theta)h_r$, then informed customers always buy online; uninformed customers go to store if in stock and buy online if out of stock. Thus,

$$\begin{aligned}\pi =& pE\min\left((1-\lambda)\,\theta D, q\right) - cq \\ &+ \left[w\theta - r\,(1-\theta)\right]E\left[(1-\lambda)\,D - \frac{q}{\theta}\right]^{+} \\ &+ wE\lambda D\end{aligned}$$

- If $h_s \ge h_o + (1-\theta)h_r$, then no one ever comes to store. Thus,

$$\pi = [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D$$

Proposition 79. *$\pi^{pa} > \max(\pi^p, \pi^a)$ if and only if $\xi^p h_o < h_s < \min(h_o, (1-\theta)(h_o+h_r))$.*

Let us compare consumer equilibrium behavior in the three scenarios when $\xi^p h_o < h_s <$

$\min(h_o, (1-\theta)(h_o + h_r))$:

i. Physical showrooms only:

   - informed customer: buy online
   - uninformed customer: go to store

ii. Availability information only:

   - informed customer: go to store if in stock, buy online if stockouts
   - uninformed customer: go to store if in stock, buy online if stockouts

iii. Physical showrooms and availability information:

   - informed customer: go to store if in stock, buy online if stockouts
   - uninformed customer: always go to store

Proposition 79 shows that when different types of customers require different types of information, we may need to offer both physical showroom and availability info. Specifically, we need physical showrooms to attract uninformed customer, and availability information to attract informed customers to the store.

Suppose there are both physical and virtual showrooms. This is similar to the physical showrooms only scenario with D', θ' and λ'.

Proposition 80. $\pi^{pv} \leq \max(\pi^p, \pi^v)$.

This shows that there is no complementary effect between these two types of showrooms.

A.5. Numerical Study for Chapter 3

The goals of this numerical study are as follows:

1. Investigate the likelihood of $\pi^\circ > \pi^p$;

2. Investigate the likelihood of $\pi^\circ > \pi^v$.

Here, we consider the following parameter values:

- $v = 60$
- $p = 30$, $w = 15$
- $\frac{p-c}{w} = \{0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4\}$
- $r = \{1, 3, 5, 7, 9\}$
- $h_o = \{1, 2, 3, 4, 5\}$
- $h_s = \{1, 2, 3, 4, 5\}$
- $h_r = \{1, 2, 3, 4, 5\}$
- $D \sim N(\mu, \sigma^2)$, where $\mu = 100$, $\frac{\sigma}{\mu} = \{\frac{1}{4}, \frac{1}{4.5}, \frac{1}{5}, \frac{1}{5.5}, \frac{1}{6}, \frac{1}{6.5}, \frac{1}{7}, \frac{1}{7.5}, \frac{1}{8}\}$
- $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9\}$
- $\theta = \{0.1, 0.3, 0.5, 0.7, 0.9\}$

There are 1,265,625 cases in total. After checking with the assumptions we made in Chapter 3, we end up having 914,895 cases. For each case, we calculate the equilibrium profit π°, π^p and π^v based on the equilibrium outcomes described in Chapter 3.

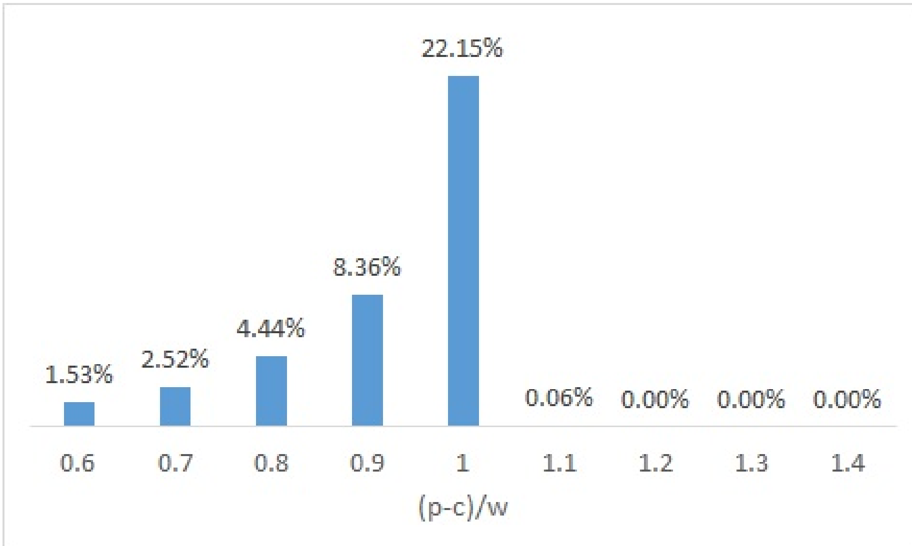
We are interested in how frequent $\pi^\circ > \pi^p$ and $\pi^\circ > \pi^v$ happen.

Among all the 914,895 cases, we find that 4.34% of them have $\pi^\circ > \pi^p$, and 4.58% of them have $\pi^\circ > \pi^v$.

Figures 11 and 12 show how the results change given different profit margin ratios. Specifically, given $\frac{p-c}{w}$, we calculate the fraction of cases where $\pi^\circ > \pi^p$ (see Figure 11) and

$\pi^{\circ} > \pi^{v}$ (see Figure 12). From these two figures, we find a general pattern: physical showrooms are more likely to reduce profits when the store channel is less profitable, while virtual showrooms are more likely to reduce profits when the online channel is less profitable. An implication to retailers: showrooms should be implemented in their strong channels.

Figure 11: Proportion of instances that physical showrooms backfire (i.e., $\pi^{o} > \pi^{p}$)



A.6. Model Extensions to Chapter 4

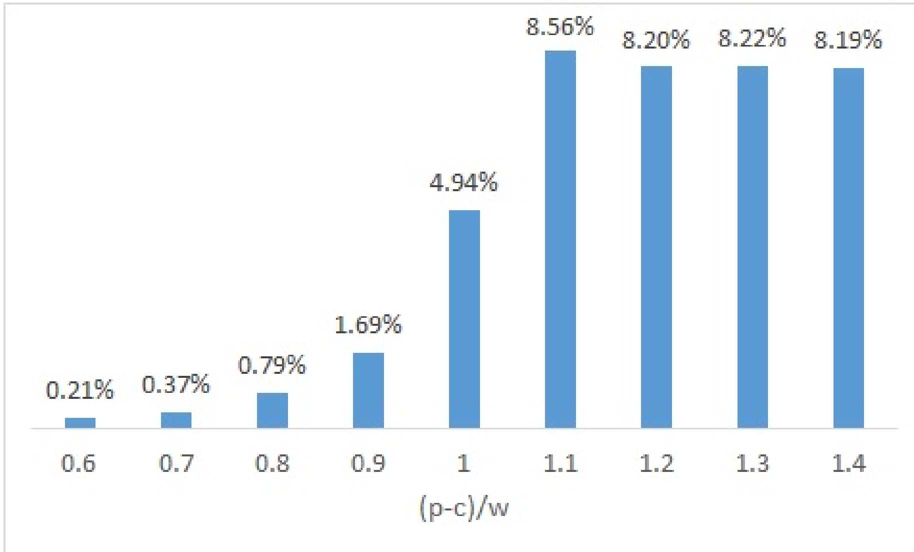
In Chapter 4, we have three main insights:

1. [Demand] With self-order technologies, total demand increases; and those who don't use the technology may also benefit from their implementation and choose to visit store more often. (I.e., Propositions 25 and 30)

2. [Workforce Level] With self-order technologies, total workforce level may increase, especially for firms with high cost-revenue ratio. (I.e., Propositions 27 and 32)

3. [Profit] Online self-order technology generates more profit than offline self-order technology if and only if customer wait sensitivity is large. (I.e., Proposition 34)

Figure 12: Proportion of instances that virtual showrooms backfire (i.e., $\pi^o > \pi^v$)



Below, we extend our basic model in three different ways. The goal is to show the robustness of the results listed above.

A.6.1. Model Extension: Customer Heterogeneity

Suppose there are two types of customers. A fraction η are tech-savvy customers, and the rest $1-\eta$ are traditional customers. Tech-savvy customers and traditional customers have different base shopping rate and wait sensitivity, denoted as α_m and α_h, β_m and β_h, respectively. This general demand model can capture customer heterogeneity in terms of two aspects: (1) Customers may have different loyalty levels towards the firm (which is reflected by different base shopping rates), and (2) customers may have different sensitivity levels towards wait in store. The model presented in Chapter 4 is a special case of this general model with $\alpha_h = \alpha_m$ and $\beta_h = \beta_m$. We assume tech-savvy customers are more wait sensitive than traditional customers (i.e., $\beta_m \geq \beta_h$); this is natural since customers adopt self-order technology primarily because of their impatience with waiting lines (eMarketer, 2014).

Similar to what we did in Chapter 4, in the following analysis, we focus on the case where the firm serves both types of customers, which is a valid assumption when α_m and α_h are large.

Base Model

Given the demand rate function of each type of customers, i.e., $\lambda_m = [\alpha_m - \beta_m(w_1 + w_2)]^+$ and $\lambda_h = [\alpha_h - \beta_h(w_1 + w_2)]^+$, the firm chooses the capacity level at each stage, μ_1 and μ_2, to maximize the profit rate, i.e.,

$$\begin{aligned} &\max_{(1-\eta)\lambda_h+\eta\lambda_m \leqslant \mu_1, (1-\eta)\lambda_h+\eta\lambda_m \leqslant \mu_2} r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_m = [\alpha_m - \beta_m (w_1(\mu_1, (1-\eta)\lambda_h + \eta\lambda_m) + w_2(\mu_2, (1-\eta)\lambda_h + \eta\lambda_m))]^+ \\ &\qquad \lambda_h = [\alpha_h - \beta_h (w_1(\mu_1, (1-\eta)\lambda_h + \eta\lambda_m) + w_2(\mu_2, (1-\eta)\lambda_h + \eta\lambda_m))]^+ \end{aligned} \tag{A.11}$$

Proposition 81. *The firm's optimal solution is given as follows:*

- $\mu_1^b = \lambda^b + \sqrt{\frac{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}{c_1}}$;
- $\mu_2^b = \lambda^b + \sqrt{\frac{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}{c_2}}$,

where $\lambda^b = (1-\eta)\lambda_h^b + \eta\lambda_m^b$, *where* $\lambda_h^b = \alpha_h - \beta_h\sqrt{\frac{c_1}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}}$ *and* $\lambda_m^b = \alpha_m - \beta_m\sqrt{\frac{c_1}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+\eta\beta_m)(r-c_1-c_2)}}$

Online Self-Order Technology

Suppose there is online self-order technology, then some tech-savvy customers will use the technology and become online customers. Suppose a fraction $\theta \leq \eta$ of all customers are online customers. Because of the instant-order and advance-order effects, their shopping rate is given by $\lambda_{mo} = [\alpha_m - \xi\beta_m w_2]^+$, where $\xi \in (0, 1]$. As for store customers, there are two types: traditional customers (with demand function $\lambda_h = [\alpha_h - \beta_h(w_1 + w_2)]^+$) and the remaining tech-savvy customers who don't have access to the online technology (with demand function $\lambda_{ms} = [\alpha_m - \beta_m(w_1 + w_2)]^+$).

The firm chooses the capacity level at each stage, i.e., μ_1 and μ_2, to maximize profit rate:

$$\begin{aligned}
\max_{\substack{(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\le\mu_1,\\ \theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\le\mu_2}} \quad & r\left(\theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\right)-c_1\mu_1-c_2\mu_2\\
s.t.\quad & \lambda_{mo}=[\alpha_m-\xi\beta_m w_2(\mu_2,\theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms})]^+\\
& \lambda_h=[\alpha_h-\beta_h\left(w_1(\mu_1,(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms})+w_2(\mu_2,\theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms})\right)]^+\\
& \lambda_{ms}=[\alpha_m-\beta_m(w_1(\mu_1,(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms})+w_2(\mu_2,\theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}))]^+
\end{aligned} \tag{A.12}$$

Proposition 82. *With online self-order technology, the firm's optimal solution is given as follows:*

- $\mu_1^o=(1-\eta)\lambda_h^o+(\eta-\theta)\lambda_{ms}^o+\sqrt{\frac{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}{c_1}}$;
- $\mu_2^o=(1-\eta)\lambda_h^o+(\eta-\theta)\lambda_{ms}^o+\theta\lambda_{mo}^o+\sqrt{\frac{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}}$,

where

- $\lambda_{mo}^o=\alpha_m-\xi\beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$;
- $\lambda_h^o=\alpha_h-\beta_h\sqrt{\frac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}}-\beta_h\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$;
- $\lambda_{ms}^o=\alpha_m-\beta_m\sqrt{\frac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}}-\beta_m\sqrt{\frac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}$.

Proposition 83. *With online self-order technology,*

- *online customers come more often than before, i.e.,* $\lambda_{mo}^o>\lambda_m^b$*;*
- *given* c_1 *and* c_2*, store customers come to store more often (i.e.,* $\lambda_h^o>\lambda_h^b$ *and* $\lambda_{ms}^o>\lambda_m^b$*) if and only if* $\frac{c_1+c_2}{r}>m_\lambda$ *and* $\theta\in(0,\psi_s)$ *for some* $m_\lambda<1$ *and* $\psi_s>0$*;*
- *total demand increases, i.e.,* $\theta\lambda_{mo}^o+(1-\eta)\lambda_h^o+(\eta-\theta)\lambda_{ms}^o>(1-\eta)\lambda_h^b+\eta\lambda_m^b$.

This shows that our original Proposition 25 still holds in this case.

Proposition 84. *Given* c_1 *and* c_2*, then there exists a threshold* $m_k<1$ *such that the firm increases total workforce level after implementing online self-order technology (i.e.,*

$k_1^o + k_2^o > k_1^b + k_2^b$) *if and only if* $\frac{c_1+c_2}{r} > m_k$.

This shows that our original Proposition 27 still holds in this case.

Offline Self-Order Technology

With offline self-order technology, same as the base model, tech-savvy customers always prefer to use the self-order machines, while traditional customers place an order only with human servers. Then, the firm's optimization problem is given as follows:

$$\begin{aligned} \max_{\substack{\eta\lambda_m \le \mu_{1m}, \\ (1-\eta)\lambda_h \le \mu_{1h}, \\ \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2}} \quad & r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\ s.t. \quad \lambda_m &= \left[\alpha_m - \beta_m w_{1m}\left(\mu_{1m}, \eta\lambda_m\right) - \beta_m w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right]^+ \\ \lambda_h &= \left[\alpha_h - \beta_h w_{1h}\left(\mu_{1h}, (1-\eta)\lambda_h\right) - \beta_h w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right)\right]^+ \end{aligned} \tag{A.13}$$

Proposition 85. *With offline self-order technology, the firm's optimal solution is given as follows:*

- $\mu_{1m}^s = \eta\lambda_m^s + \sqrt{\frac{\beta_m\eta(r-c_{1m}-c_2)}{c_{1m}}}$;
- $\mu_{1h}^s = (1-\eta)\lambda_h^s + \sqrt{\frac{\beta_h(1-\eta)(r-c_1-c_2)}{c_1}}$;
- $\mu_2^s = \eta\lambda_m^s + (1-\eta)\lambda_h^s + \sqrt{\frac{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}{c_2}}$,

where

- $\lambda_m^s = \alpha_m - \beta_m\sqrt{\frac{c_{1m}}{\beta_m\eta(r-c_{1m}-c_2)}} - \beta_m\sqrt{\frac{c_2}{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}}$;
- $\lambda_h^s = \alpha_h - \beta_h\sqrt{\frac{c_1}{\beta_h(1-\eta)(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)}}$.

Similar to what we did in Chapter 4, in the following analysis, we assume the machine capacity cost c_{1m} is small enough such that the average wait time at stage 1 is shorter with self-order technology compared to the base case, i.e., $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$.

Proposition 86. *With offline self-order technology,*

- *tech-savvy customers come more often than before, i.e.,* $\lambda_m^s > \lambda_m^b$*;*

- *given* c_{1m}, c_1, c_2*, traditional customers come to store more often (i.e.,* $\lambda_h^s > \lambda_h^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_\lambda$ *and* $\eta \in (0, \psi'_s)$ *for some* $m'_\lambda < 1$ *and* $\psi'_s > 0$*;*

- *total demand increases, i.e.,* $\eta\lambda_m^s + (1-\eta)\lambda_h^s > (1-\eta)\lambda_h^b + \eta\lambda_m^b$*.*

This shows that our original Proposition 30 still holds in this case.

Proposition 87. *Given* c_{1m}*,* c_1 *and* c_2*, then there exists a threshold* $m'_k < 1$ *such that the firm increases total workforce level after implementing offline self-order technology (i.e.,* $k_1^s + k_2^s > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_k$*.*

This shows that our original Proposition 32 still holds in this case.

Profit Implications

Suppose $\beta_h = bb_h$ and $\beta_m = bb_m$, i.e., b measures the base wait sensitivity in the market.

Proposition 88. *There exists* $\bar{b} \geq 0$ *such that online self-order technology generates more profit than offline self-order technology (i.e.,* $\pi^o > \pi^s$*) if and only if* $b > \bar{b}$*.*

This shows that our original Proposition 34 still holds in this case.

A.6.2. Convex Impact of Wait Time

In Chapter 4, we assumed demand is a linear function of wait time. In this extension, we relax this assumption. Specifically, we consider the following demand function: $\lambda = \alpha - \beta w_1^\phi - \beta w_2^\phi$, where w_1 and w_2 represent the wait times at stages 1 and 2, and parameter $\phi \in (0, 1]$. Since $\phi \in (0, 1]$, demand λ is convex with respect to wait time at each stage. Note, the linear demand model presented in Chapter 4 is a special case of this general demand model with $\phi = 1$.

Similar to what we did in Chapter 4, in the following analysis, we focus on the case where the firm serves all types of customers (including online, store, tech-savvy, traditional), which is a valid assumption when α is large.

Base Model

Given the demand function, the firm chooses the capacity level at each stage, μ_1 and μ_2, to maximize the profit rate, i.e.,

$$\begin{aligned} &\max_{\lambda\leqslant\mu_1,\lambda\leqslant\mu_2} r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = \left[\alpha - \beta\left(w_1(\mu_1,\lambda)\right)^{\phi} - \beta\left(w_2(\mu_2,\lambda)\right)^{\phi}\right]^{+} \end{aligned} \tag{A.14}$$

Proposition 89. *The firm's optimal solution is given as follows:*

- $\mu_1^b = \lambda^b + \left(\frac{\phi\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$;
- $\mu_2^b = \lambda^b + \left(\frac{\phi\beta(r-c_1-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$,

where $\lambda^b = \alpha - \beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$.

Online Self-Order Technology

With online self-order technology, because of the instant-order and advance-order effects, online customer's demand function is given as $\lambda_o = \alpha - \xi\beta w_2^{\phi}$. The store customer's demand function remains the same as before, i.e., $\lambda_s = \alpha - \beta w_1^{\phi} - \beta w_2^{\phi}$. Therefore, the firm's optimization problem is as follows:

$$\begin{aligned} &\max_{(1-\theta)\lambda_s\leq\mu_1,(1-\theta)\lambda_s+\theta\lambda_o\leq\mu_2} r\left((1-\theta)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_o = \left[\alpha - \xi\beta\left(w_2(\mu_2,(1-\theta)\lambda_s+\theta\lambda_o)\right)^{\phi}\right]^{+} \\ &\qquad \lambda_s = \left[\alpha - \beta\left(w_1(\mu_1,(1-\theta)\lambda_s)\right)^{\phi} - \beta\left(w_2(\mu_2,(1-\theta)\lambda_s+\theta\lambda_o)\right)^{\phi}\right]^{+} \end{aligned} \tag{A.15}$$

Proposition 90. *With online self-order technology, the firm's optimal solution is given as follows:*

- $\mu_1^o = (1-\theta)\lambda_s^o + \left(\frac{\phi(1-\theta)\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$;

- $\mu_2^o = (1-\theta)\lambda_s^o + \theta\lambda_o^o + \left(\frac{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$,

where

- $\lambda_s^o = \alpha - \beta\left(\frac{c_1}{\phi(1-\theta)\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}}$,

- $\lambda_o^o = \alpha - \xi\beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}}$.

Proposition 91. *With online self-order technology,*

- *online customers come more often than before, i.e.,* $\lambda_o^o > \lambda^b$;

- *given* c_1 *and* c_2, *store customers come to store more often (i.e.,* $\lambda_s^o > \lambda^b$*) if and only if* $\frac{c_1+c_2}{r} > m_\lambda$ *and* $\theta \in (0, \psi_s)$ *for some* $m_\lambda < 1$ *and* $\psi_s > 0$;

- *total demand increases, i.e.,* $(1-\theta)\lambda_s^o + \theta\lambda_o^o > \lambda^b$.

This shows that our original Proposition 25 still holds in this case.

Proposition 92. *Given* c_1 *and* c_2, *then there exists a threshold* $m_k < 1$ *such that the firm increases total workforce level after implementing online self-order technology (i.e.,* $k_1^o + k_2^o > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m_k$.

This shows that our original Proposition 27 still holds in this case.

Offline Self-Order Technology

With self-order technology, the firm's optimization problem is given as follows:

$$\begin{aligned}
\max_{\substack{\eta\lambda_m \le \mu_{1m},\\ (1-\eta)\lambda_h \le \mu_{1h}\\ \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2}} \quad & r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\
s.t. \quad \lambda_m &= \left[\alpha - \beta(w_{1m}\left(\mu_{1m}, \eta\lambda_m\right))^\phi - \beta(w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right))^\phi\right]^+ \\
\lambda_h &= \left[\alpha - \beta(w_{1h}\left(\mu_{1h}, (1-\eta)\lambda_h\right))^\phi - \beta(w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\lambda_h\right))^\phi\right]^+
\end{aligned} \tag{A.16}$$

Proposition 93. *With offline self-order technology, the firm's optimal solution is given as follows:*

- $\mu_{1m}^s = \eta\lambda_m^s + \left(\frac{\phi\beta\eta(r-c_{1m}-c_2)}{c_{1m}}\right)^{\frac{1}{1+\phi}}$;
- $\mu_{1h}^s = (1-\eta)\,\lambda_h^s + \left(\frac{\phi\beta(1-\eta)(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$;
- $\mu_2^s = \eta\lambda_m^s + (1-\eta)\,\lambda_h^s + \left(\frac{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$,

where

- $\lambda_m^s = \alpha - \beta\left(\frac{c_{1m}}{\phi\beta\eta(r-c_{1m}-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$;
- $\lambda_h^s = \alpha - \beta\left(\frac{c_1}{\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$.

Similar to what we did in Chapter 4, in the following analysis, we assume the machine capacity cost c_{1m} is small enough such that the average wait time at stage 1 is shorter with self-order technology compared to the base case, i.e., $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$.

Proposition 94. *With offline self-order technology,*

- *tech-savvy customers come more often than before, i.e.,* $\lambda_m^s > \lambda^b$;
- *given* c_{1m}, c_1, c_2, *traditional customers come to store more often (i.e.,* $\lambda_h^s > \lambda^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_\lambda$ *and* $\eta \in (0, \psi'_s)$ *for some* $m'_\lambda < 1$*) and* $\psi'_s > 0$;
- *total demand increases, i.e.,* $\eta\lambda_m^s + (1-\eta)\lambda_h^s > \lambda^b$.

This shows that our original Proposition 30 still holds in this case.

Proposition 95. *Given* c_{1m}, c_1 *and* c_2, *then there exists a threshold* $m'_k < 1$ *such that the firm increases total workforce level after implementing offline self-order technology (i.e.,* $k_1^s + k_2^s > k_1^b + k_2^b$*) if and only if* $\frac{c_1+c_2}{r} > m'_k$.

This shows that our original Proposition 32 still holds in this case.

Profit Implications

Proposition 96. *There exists* $\bar{\beta} \geq 0$ *such that online self-order technology generates more profit than offline self-order technology (i.e.,* $\pi^o > \pi^s$*) if and only if* $\beta > \bar{\beta}$*.*

This shows that our original Proposition 34 still holds in this case.

A.6.3. Alternative Wait Time Function

In Chapter 4, we formulated firm's optimization problem with respect to capacity μ. The three optimization problems (4.1, 4.2, 4.3) can also be reformulated with respect to the number of servers k as follows:

- Basic model:

$$\begin{aligned} &\max_{\lambda \leqslant k_1/\tau_1, \lambda \leqslant k_2/\tau_2} \quad r\lambda - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda = [\alpha - \beta\,(w_1(k_1, \lambda) + w_2(k_2, \lambda))]^+ \end{aligned} \tag{4.1$'$}$$

- Online self-order technology:

$$\begin{aligned} &\max_{(1-\theta)\lambda_s \leq k_1/\tau_1, \theta\lambda_o + (1-\theta)\lambda_s \leq k_2/\tau_2} \quad r\,(\theta\lambda_o + (1-\theta)\lambda_s) - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda_o = [\alpha - \xi\beta w_2(k_2, \theta\lambda_o + (1-\theta)\lambda_s)]^+ \\ &\qquad \lambda_s = [\alpha - \beta\,(w_1(k_1, (1-\theta)\lambda_s) + w_2(k_2, \theta\lambda_o + (1-\theta)\lambda_s))]^+ \end{aligned} \tag{4.2$'$}$$

- Offline self-order technology:

$$\begin{aligned} &\max_{\substack{\eta\lambda_m \leq k_{1m}/\tau_1, \\ (1-\eta)\lambda_h \leq k_1/\tau_1, \\ \eta\lambda_m + (1-\eta)\lambda_h \leq k_2/\tau_2}} \quad r\,(\eta\lambda_m + (1-\eta)\,\lambda_h) - l_{1m} k_{1m} - l_1 k_1 - l_2 k_2 \\ &s.t. \quad \lambda_m = [\alpha - \beta\,(w_1\,(k_{1m}, \eta\lambda_m) + w_2\,(k_2, \eta\lambda_m + (1-\eta)\,\lambda_h))]^+ \\ &\qquad \lambda_h = [\alpha - \beta\,(w_1\,(k_1, (1-\eta)\,\lambda_h) + w_2\,(k_2, \eta\lambda_m + (1-\eta)\,\lambda_h))]^+ \end{aligned} \tag{4.3$'$}$$

where $l_1 = c_1/\tau_1$, $l_{1m} = c_{1m}/\tau_1$, $l_2 = c_2/\tau_2$, $w_i(k,\lambda) = \frac{1}{k/\tau_i - \lambda}$, and τ_i is the average service time at stage $i = 1, 2$. Here, l_1 and l_2 can be interpreted as the labor cost per unit of time at stages 1 and 2, l_{1m} is the corresponding cost for machines. The number of machine servers is denoted by k_{1m}.

In Chapter 4, we have assumed wait time function takes the following form: $w_i(k,\lambda) = \frac{1}{k/\tau_i - \lambda}$. In this section, we numerically test the robustness of our main insights with a different wait time function, i.e.,

$$w_i(k,\lambda) = \left(\frac{\tau_i}{k}\right)\left(\frac{\rho_i^{\sqrt{2(k+1)}-1}}{1-\rho_i}\right) + \tau_i \tag{A.17}$$

where $\rho_i = \frac{\lambda \tau_i}{k}$. This corresponds to the approximated average wait time in a M/M/k queue (Cachon and Terwiesch, 2009).

In the numerical study, we consider the following parameter values:

- $\tau_1 = \{\frac{0.5}{60}, \frac{1}{60}\}$. We assume the average service requirement is 0.5/60 or 1/60 hour (i.e., 30 seconds or 1 min) to place an order.
- $\tau_2 = t\tau_1$, where $t = \{3, 5, 7\}$. Here, we only look at the case where $\tau_2 > \tau_1$ because it generally takes longer cooking food at stage 2 than processing an order at stage 1.
- $\alpha = 100$. Here, we assume the maximum traffic (i.e., if there is no wait) in a store is 100 people per hour.
- $\beta = \{300, 400, 500, 600\}$. This implies that the longest amount of wait time people can tolerate (after which their shopping rate is 0) is $\alpha/\beta = \{1/3, 1/4, 1/5, 1/6\}$ hours.
- $\eta = \{0.6, 0.7, 0.8, 0.9\}$
- $\theta = \zeta\eta$, where $\zeta = \{0.2, 0.4, 0.6, 0.8\}$.
- $\xi = \{0.2, 0.4, 0.6, 0.8\}$

- $l_1 = l_2 = \{8, 9, 10, 11\}$ The range of the hourly wage is consistent with the data provided by the Bureau of Labor Statistics (`http://www.bls.gov/oes/current/oes353021.htm`).

- $l_{1m} = xl_1$, where $x = \{0.01, 0.1\}$.

- $\frac{l_1\tau_1 + l_2\tau_2}{r} = \{0.1, 0.3, 0.5, 0.7\}$. The range of the cost-revenue ratio is selected based on the following fact: According to National Restaurant Association (2010), for a restaurant, the median cost of food and beverage sales is 31.9%, and the median cost of salaries and wages is 29.4%. Note, $l_1\tau_1 + l_2\tau_2$ is the cost to serve one customer; r is the sales revenue from each customer net of cost of food and beverage. Then, the data above implies that the median of the cost-revenue ratio $\frac{l_1\tau_1 + l_2\tau_2}{r}$ should be around $29.4\%/(1 - 31.9\%) = 43.2\%$.

There are 49.152 cases in total. After checking with the assumptions we made in Chapter 4, we end up having 35,620 cases. For each case, we solve the three optimization problems above with the wait time function (A.17). To simplify calculation, we assume $k_1, k_2, k_{1m} \in \mathbb{R}_+$.

First, we check the impact of self-order technology on demand. Here are the results:

- With online technology, compared to the base scenario:
  - total demand increases (i.e., $\theta\lambda_o^o + (1-\theta)\lambda_s^o > \lambda^b$) in all cases;
  - online customers come more often (i.e., $\lambda_o^o > \lambda^b$) in all cases;
  - store customers come more often (i.e., $\lambda_s^o > \lambda^b$) in about 21.7% of cases.
- With offline technology, compared to the base scenario:
  - total demand increases (i.e., $\eta\lambda_m^s + (1-\eta)\lambda_h^s > \lambda^b$) in all cases;
  - tech-savvy customers come more often (i.e., $\lambda_m^s > \lambda^b$) in all cases;

- traditional customers come more often (i.e., $\lambda_h^s > \lambda^b$) in about 5.4% of cases.

These results are consistent with Propositions 25 and 30 in Chapter 4.

Next, we check the impact of self-order technology on workforce level:

- Figure 13 shows the proportion of instances that $k_1^o + k_2^o > k_1^b + k_2^b$ given the cost-revenue ratio $(l_1\tau_1 + l_2\tau_2)/r$. It shows that a firm with higher cost-revenue ratio will be more likely to increase workforce level after the implementation of online self-order technology, which is consistent with Proposition 27.

- Figure 14 shows the proportion of instances that $k_1^s + k_2^s > k_1^b + k_2^b$ given the cost-revenue ratio $(l_1\tau_1 + l_2\tau_2)/r$. It shows that a firm with higher cost-revenue ratio will be more likely to increase workforce level after the implementation of offline self-order technology, which is consistent with Proposition 32.

Finally, we check the optimal choice between online and offline self-order technologies: Figure 15 shows the proportion of instances that $\pi^o > \pi^s$ given β. It implies that online self-order technology is more profitable if β is large, which is consistent with Proposition 34.

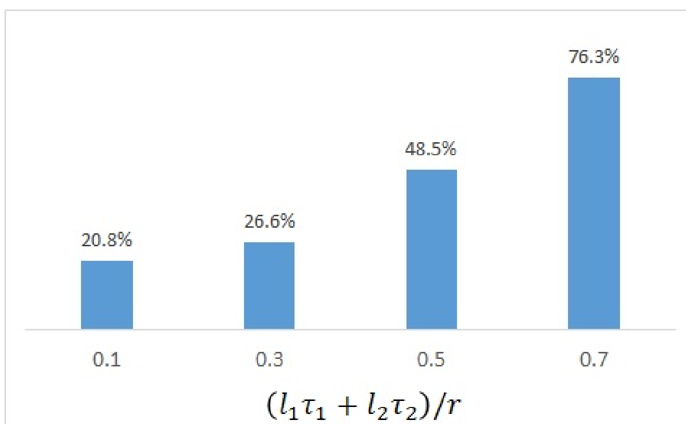


Figure 13: Proportion of instances that total workforce level increases after the implementation of online self-order technology (i.e., $k_1^o + k_2^o > k_1^b + k_2^b$)

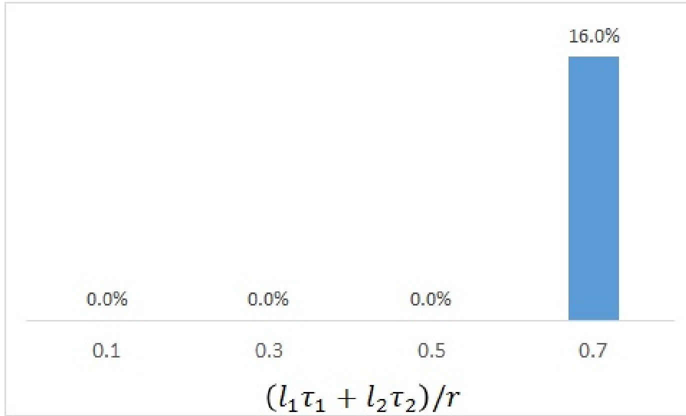


Figure 14: Proportion of instances that total workforce level increases after the implementation of offline self-order technology (i.e., $k_1^s + k_2^s > k_1^b + k_2^b$)

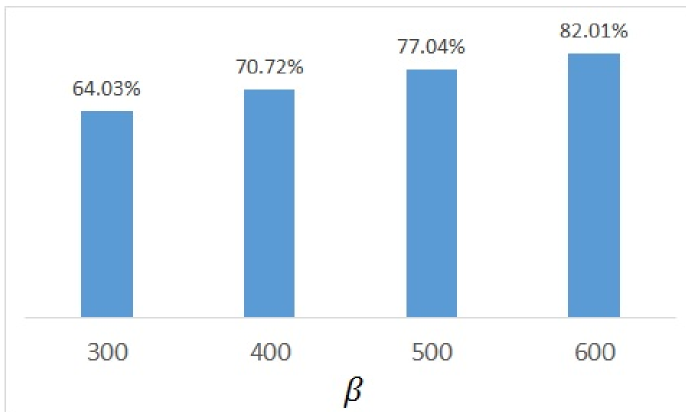


Figure 15: Proportion of instances that online self-order technology generates more profit than offline self-order technology (i.e., $\pi^o > \pi^s$)

A.7. Proofs

Proof of Proposition 1: Let's look for participatory RE equilibrium, where $\phi = 1$ and $q > 0$. All we need to do is to check the four conditions specified in Definition 1.

First, we look at retailer's problem: Given belief $\hat{\phi}$, the retailer maximizes total profit $\pi = pE\min(\hat{\phi}D, q) - cq + rE(\hat{\phi}D) + wE((1-\hat{\phi})D) + wE(\hat{\phi}D - q)^+ = (p-w)E\min(\hat{\phi}D, q) - cq + (\hat{\phi}r + w)ED$, which is a typical newsvendor problem (plus a constant $(\hat{\phi}r + w)ED$), and therefore the optimal order quantity q° is given by $\bar{F}\left(\frac{q^\circ}{\hat{\phi}}\right) = \frac{c}{p-w} \wedge 1$, where $x \wedge y$ means $\min(x, y)$.

Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\phi} = \phi = 1$. Thus, $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)$. Since $q > 0$ in the participatory equilibrium, we must have $p - c > w$ and thus $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q^\circ) = E\min(\phi D, q^\circ)/E(\phi D) = E\min(D, \bar{F}^{-1}(\frac{c}{p-w}))/ED$.

Finally, we go back to consumer's decision. To ensure $\phi = 1$, we need $u_s \geq u_o$, i.e., $h_s \leq \hat{\xi}^\circ h_o$. Therefore, we need $h_s \leq \frac{E\min(D,\bar{F}^{-1}(\frac{c}{p-w}))}{ED} h_o$.

Based on the analysis above, we find the conditions for a participatory equilibrium are $h_s \leq \frac{E\min(D,\bar{F}^{-1}(\frac{c}{p-w}))}{ED} h_o$ and $p - c > w$. And the equilibrium outcome is $\phi = 1$ and $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w}\right)$. □

Proof of Proposition 2: If $\min(h_s, h_b) \leq h_o$, then the profit function $\pi = pE\min(D, q) - cq + rE\min(D, q) + wE(D - q)^+ = (p + r - w)E\min(D, q) - cq + wED$, which is a typical newsvendor problem (plus a constant wED). Thus, the optimal order quantity q^* is given by $\bar{F}(q^*) = \frac{c}{p+r-w} \wedge 1$. Then, if $p - c > w - r$, we have $q^* = \bar{F}^{-1}\left(\frac{c}{p+r-w}\right)$; otherwise, $q^* = 0$.

If $\min(h_s, h_b) > h_o$, then no customer comes to store and thus the retailer will stock nothing in the store, i.e., $q^* = 0$. □

Proof of Proposition 3: If $h_s \in \left(\frac{E\min\left[D,\bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)\right]}{ED} h_o, h_o\right]$ and $p - c > w - r$, then

$$
\begin{aligned}
\pi^* &= pE\min(D, q^*) - cq^* + rE\min(D, q^*) + wE(D - q^*)^+ \\
&= (p + r - w)E\min(D, q^*) - cq^* + wED \\
&> wED \\
&= \pi^\circ
\end{aligned}
$$

where the inequality is due to the fact that $q^* > 0$. This completes the proof. □

Proof of Proposition 4: If $h_b \le h_o < h_s$ and $p - c > w - r$, then $q^* = \bar{F}^{-1}\left(\frac{c}{p+r-w}\right) > 0$ because of Proposition 2. Then,

$$\begin{aligned}
\pi^* &= pE\min(D, q^*) - cq^* + rE\min(D, q^*) + wE(D - q^*)^+ \\
&= (p + r - w)\, E\min(D, q^*) - cq^* + wED \\
&> wED \\
&= \pi^\circ
\end{aligned}$$

where the inequality is due to the fact that $q^* > 0$. This completes the proof. □

Proof of Proposition 5: If $h_s \le \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED} h_o$ and $p - c > w$, we have $q^\circ \neq q^*$ when $r > 0$. Then

$$\begin{aligned}
\pi^\circ &= pE\min(D, q^\circ) - cq^\circ + rED + wE(D - q^\circ)^+ \\
&= (p^\circ - w)\, E\min(D, q^\circ) - cq^\circ + rED + wED \\
&> (p^* - w)\, E\min(D, q^*) - cq^* + rED + wED \\
&> (p^* - w)\, E\min(D, q^*) - cq^* + rE\min(D, q^*) + wED \\
&= p^* E\min(D, q^*) - cq^* + rE\min(D, q^*) + wE(D - q^*)^+ \\
&= \pi^*
\end{aligned}$$

where the first inequality is due to the strict concavity of the newsvendor profit function and q° is the unique maximizer of π°, and the second inequality is because $ED > E\min(D, q)$ for any q. This completes the proof. □

Proof of Proposition 6: Note nonparticipatory equilibrium, $(\frac{v-p}{H}, 0, 0, 0, 0, 0, \frac{v-p}{H}, 0, 0, 0)$, always exists. Same as what we did in the base model, we are going to look for participatory equilibrium, where $\alpha_s^\circ > 0, \alpha_{so}^\circ > 0$ and $q^\circ > 0$. All we need to do is to check the four conditions specified in Definition 2.

First, we look at retailer's problem: Given belief $\hat{\alpha}_o, \hat{\alpha}_s, \hat{\alpha}_{so}$, the retailer maximizes total profit

$$\begin{aligned}\pi &= pE\min\left(\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)D,q\right)-cq+rE\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)D\\&\quad+wE\hat{\alpha}_oD+wE\tfrac{\hat{\alpha}_{so}}{\hat{\alpha}_s+\hat{\alpha}_{so}}\left(\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)D-q\right)^+\\&=\left(p-w\tfrac{\hat{\alpha}_{so}}{\hat{\alpha}_s+\hat{\alpha}_{so}}\right)E\min((\hat{\alpha}_s+\hat{\alpha}_{so})D,q)-cq\\&\quad+((\hat{\alpha}_s+\hat{\alpha}_{so})r+(\hat{\alpha}_o+\hat{\alpha}_{so})w)ED\end{aligned}$$

which is a typical newsvendor problem (plus a constant $((\hat{\alpha}_s+\hat{\alpha}_{so})r+(\hat{\alpha}_o+\hat{\alpha}_{so})w)ED$), and therefore the optimal order quantity q° is given by $\bar{F}\left(\frac{q^\circ}{\hat{\alpha}_s+\hat{\alpha}_{so}}\right)=\frac{c}{p-w\frac{\hat{\alpha}_{so}}{\hat{\alpha}_s+\hat{\alpha}_{so}}}\wedge 1$.

Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\alpha}_i=\alpha_i^\circ$, $i=o,s,so$. Note, in any participatory equilibrium, $\xi>0$. Then, $\frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}=\frac{v-p}{2H-(v-p)}$. Thus, $q^\circ=(\alpha_s^\circ+\alpha_{so}^\circ)\bar{F}^{-1}\left(\frac{c}{p-w\frac{v-p}{2H-(v-p)}}\wedge 1\right)$. Since $q^\circ>0$ in the participatory equilibrium, we must have $p-c>w\frac{v-p}{2H-(v-p)}$ and thus $q^\circ=(\alpha_s^\circ+\alpha_{so}^\circ)\bar{F}^{-1}\left(\frac{c}{p-w\frac{v-p}{2H-(v-p)}}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}=A(q^\circ)=E\min((\alpha_s^\circ+\alpha_{so}^\circ)D,q^\circ)/E((\alpha_s^\circ+\alpha_{so}^\circ)D)=E\min(D,\bar{F}^{-1}(\frac{c}{p-w\frac{v-p}{2H-(v-p)}}))/ED$, which is indeed greater than 0. Thus, there are customers who are willing to come to store.

Based on the analysis above, we find the condition for a participatory equilibrium is $p-c>w\frac{v-p}{2H-(v-p)}$. And the equilibrium store inventory level $q^\circ=\bar{F}^{-1}\left(\frac{c}{p-w\frac{v-p}{2H-(v-p)}}\right)$. $\square$

Proof of Proposition 7: Note the retailer's profit can be expressed as

$$\begin{aligned}\pi &= \left(p+r-w\tfrac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\right)E\min\left(\left(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*\right)D,q\right)-cq\\&\quad+w(\alpha_o^*+\alpha_{so}^*+\alpha_{bo}^*)ED\end{aligned}$$

which is a typical newsvendor problem (plus a constant $w(\alpha_o^*+\alpha_{so}^*+\alpha_{bo}^*)ED$). Thus, the optimal store inventory level is given by

$$q^*=\left(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*\right)\bar{F}^{-1}\left(\frac{c}{p+r-w\frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}}\wedge 1\right)$$

$\square$

Proof of Proposition 8: Denote $A_l^\circ = \{(h_s, h_o) | \max(v - p - h_o, -h_s + \xi^\circ(v-p)) < 0\}$ and $A_l^* = \{(h_s, h_o) | \max(v - p - h_o, v - p - h_s, v - p - \beta_s h_s - \beta_o h_o) < 0\}$. To prove part (i), we simply need to show that $A_l^* \subset A_l^\circ$. Since $\xi^\circ < 1$, for any $(h_s, h_o) \in A_l^*$, we will have $(h_s, h_o) \in A_l^\circ$. Thus, $A_l^* \subset A_l^\circ$.

Next, let's look at part (ii). To simplify notation, we denote $\Delta^\circ = \frac{v-p}{2H-(v-p)}$ and $\Delta^* = \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}$. Note if $\beta_s + \beta_o > 1$, we have $\Delta^\circ < \Delta^*$. Thus, $\frac{p-c}{\Delta^\circ} > \frac{p-c}{\Delta^*}$.

If there are customers visiting store when there is no BOPS, we need to have $w < \frac{p-c}{\Delta^\circ}$. The following analysis assumes this condition holds.

Suppose $r = 0$. If $w \geq \frac{p-c}{\Delta^*}$ (this is possible since $\frac{p-c}{\Delta^\circ} > \frac{p-c}{\Delta^*}$), then no one comes to store when there is BOPS . In this case,

$$\begin{aligned}
\pi^* &= wE\left(\alpha_{so}^* + \alpha_{bo}^* + \alpha_o^*\right) D \\
&= wE\left(\alpha_{so}^\circ + \alpha_o^\circ\right) D \\
&< pE\min\left(\left(\alpha_s^\circ + \alpha_{so}^\circ\right) D, q\right) - cq + rE\left(\alpha_s^\circ + \alpha_{so}^\circ\right) D + wE\alpha_o^\circ D \\
&\quad + wE\frac{\alpha_{so}^\circ}{\alpha_s^\circ + \alpha_{so}^\circ}\left(\left(\alpha_s^\circ + \alpha_{so}^\circ\right) D - q\right)^+ \\
&= \pi^\circ
\end{aligned}$$

where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case where there is no BOPS. If $w < \frac{p-c}{\Delta^*}$, then there are consumers visiting store when there is BOPS. Note (1) $\pi^\circ - \pi^*$ is continuous in w, and (2) $\pi^\circ > \pi^*$ when $w = \frac{p-c}{\Delta^*}$. Thus, the analysis above implies that there exists $\bar{w} < \frac{p-c}{\Delta^*}$ such that $\pi^\circ > \pi^*$ if $w > \bar{w}$. $\square$

Proof of Proposition 9: Note $\tilde{\pi}_s$ is a typical newsvendor profit function. Then, the optimal order quantity $\tilde{q}^*$ is given by $\bar{F}(\tilde{q}^*) = \frac{c}{\theta p + r} \wedge 1$. Thus, if $\theta p + r - c > 0$, we have $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\theta p + r}\right)$, otherwise, $\tilde{q}^* = 0$. $\square$

Proof of Proposition 10: In the decentralized system, for any θ, denote optimal stock levels in the store as $\tilde{q}^*(\theta)$. Suppose we have the same stock level in the centralized system. Then clearly we will achieve exactly the same total profit as $\tilde{\pi}^*(\theta)$, since the BOPS revenue is just shared between channels in the decentralized case. Thus, we must have $\tilde{\pi}^*(\theta) \leq \pi^*$. And due to the fact that π is strictly concave in q and $q^* > 0$ (because of the assumptions that $h_b < \min(h_s, h_o)$ and $p - c > w - r$), we have $\tilde{\pi}^*(\theta) = \pi^*$ if only if $\tilde{q}^* = q^*$.

Next, let's compare the optimal store inventory levels in both the centralized and decentralized systems:

- If $\theta < \frac{p-w}{p}$, then $\frac{c}{\theta p+r} > \frac{c}{p+r-w}$. Thus, $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\theta p+r}\right) < \bar{F}^{-1}\left(\frac{c}{p+r-w}\right) = q^*$. Since $\tilde{q}^* \neq q^*$, $\tilde{\pi}^* < \pi^*$.

- If $\theta = \frac{p-w}{p}$, then $\frac{c}{\theta p+r} = \frac{c}{p+r-w}$. Thus, $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\theta p+r}\right) = \bar{F}^{-1}\left(\frac{c}{p+r-w}\right) = q^*$. Since $\tilde{q}^* = q^*$, $\tilde{\pi}^* = \pi^*$.

- If $\theta > \frac{p-w}{p}$, then $\frac{c}{\theta p+r} < \frac{c}{p+r-w}$. Thus, $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\theta p+r}\right) > \bar{F}^{-1}\left(\frac{c}{p+r-w}\right) = q^*$. Since $\tilde{q}^* \neq q^*$, $\tilde{\pi}^* < \pi^*$.

Finally, let's prove $\tilde{\pi}^*$ is quasiconcave in θ.

Let's first show that $\tilde{\pi}^*$ is nondecreasing in $\theta < \frac{p-w}{p}$. If $\theta \leq \frac{c-r}{p}$, then $\tilde{q}^* = 0$ and thus $\tilde{\pi}^* = wED$, which is independent of θ. If $\theta \in (\frac{c-r}{p}, \frac{p-w}{p})$, then $\frac{\partial \tilde{\pi}^*}{\partial \theta} = \frac{\partial \tilde{\pi}^*}{\partial q}\frac{\partial \tilde{q}^*}{\partial \theta} > 0$, because $\frac{\partial \tilde{\pi}^*}{\partial q} > 0$ if $q < \tilde{q}^*$ and $\frac{\partial \tilde{q}^*}{\partial \theta} > 0$.

Then, let's show that $\tilde{\pi}^*$ is nonincreasing in $\theta > \frac{p-w}{p}$. Note $\frac{\partial \tilde{\pi}^*}{\partial \theta} = \frac{\partial \tilde{\pi}^*}{\partial q}\frac{\partial \tilde{q}^*}{\partial \theta} < 0$, because $\frac{\partial \tilde{\pi}^*}{\partial q} < 0$ if $q > \tilde{q}^*$ and $\frac{\partial \tilde{q}^*}{\partial \theta} > 0$.

Thus, we can conclude that $\tilde{\pi}^*$ is quasiconcave in θ. $\square$

Proof of Proposition 35: Note $\tilde{\pi}_s$ is a typical newsvendor profit function. Then, the optimal order quantity $\tilde{q}^*$ is given by $\bar{F}(\frac{\tilde{q}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}) = \frac{c}{\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} + \frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r} \wedge 1.$

Thus, if

$\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}+\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r-c>0$, then

$\tilde{q}^* = (\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*)\,\bar{F}^{-1}\left(\frac{c}{\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}+\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r}\right)$, otherwise, $\tilde{q}^* = 0$.

□

Proof of Proposition 11: Let's look for participatory RE equilibrium, where $\phi = 1$. All we need to do is to check the four conditions specified in Definition 1.

First, we look at the retailer's problem: Note the profit function can be expressed as $\pi = (p - w + r\frac{1-\theta}{\theta})E\min(\hat{\phi}\theta D, q) - cq + (w\theta - r(1-\theta))ED$, which is a typical newsvendor problem (plus a constant $(w\theta + r(1-\theta))ED$), and therefore the optimal order quantity q° is given by $\bar{F}\left(\frac{q^\circ}{\hat{\phi}\theta}\right) = \frac{c}{p-w+r\frac{1-\theta}{\theta}}$.

Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\phi} = \phi = 1$. Thus, $q^\circ = \theta\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q^\circ) = E\min(\phi\theta D, q^\circ)/E(\phi\theta D) = E\min(D, \bar{F}^{-1}(\frac{c}{p-w+r\frac{1-\theta}{\theta}}))/ED$.

Finally, we need to check, given $\hat{\xi} = E\min(D, \bar{F}^{-1}(\frac{c}{p-w+r\frac{1-\theta}{\theta}}))/ED$, consumers are indeed willing to visit store, i.e., $u_s > u_o$, which gives us the condition

$$h_s < \frac{E\min(D, \bar{F}^{-1}(\frac{c}{p-w+r\frac{1-\theta}{\theta}}))}{ED}[h_o + (1-\theta)h_r]. \tag{SA.1}$$

Note the right hand side of condition (SA.1) is decreasing in θ and is also boundless for $\theta \in \mathbb{R}$. Therefore, there exists a unique $\theta^\circ \in \mathbb{R}$ such that $h_s = \frac{E\min(D, \bar{F}^{-1}(\frac{c}{p-w+r\frac{1-\theta^\circ}{\theta^\circ}}))}{ED}[h_o + (1-\theta^\circ)h_r]$ and condition (SA.1) is equivalent to $\theta < \theta^\circ$. Since $\theta \in (0,1)$ in our model, we truncate the cutoff point and set $\psi^\circ = \min(\max(\theta^\circ, 0), 1)$.

Based on the analysis above, we find the condition for a participatory equilibrium is $\theta < \psi^\circ$.

In such case, the equilibrium store inventory level is $q^\circ = \theta \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$, which is strictly positive, since condition (SA.1) will not hold otherwise. $\square$

Proof of Proposition 12: Let's look for participatory RE equilibrium, where $\phi = 1$.

First, we look at the retailer's problem: Note the profit function can be expressed as $\pi = (p-w)E\min(\hat{\phi}\theta D, q) - cq + (w\theta - r(1-\hat{\phi})(1-\theta))ED$, which is a typical newsvendor problem (plus a constant $(w\theta - r(1-\hat{\phi})(1-\theta))ED$), and therefore the optimal order quantity q^p is given by $\bar{F}\left(\frac{q^p}{\hat{\phi}\theta}\right) = \frac{c}{p-w}$.

Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\phi} = \phi = 1$. Thus, $q^p = \theta \bar{F}^{-1}\left(\frac{c}{p-w}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^p = A(q^p) = E\min(\phi\theta D, q^p)/E(\phi\theta D) = E\min(D, \bar{F}^{-1}(\frac{c}{p-w}))/ED$.

Finally, we need to check given $\hat{\xi} = E\min(D, \bar{F}^{-1}(\frac{c}{p-w}))/ED$, consumers are indeed willing to visit store, i.e., $u_s > u_o$, which gives us the condition

$$h_s < \frac{E\min(D, \bar{F}^{-1}(\frac{c}{p-w}))}{ED}\theta h_o + (1-\theta)(h_o + h_r). \tag{SA.2}$$

Note the left hand side of condition (SA.2) is decreasing in θ and is also boundless for $\theta \in \mathbb{R}$. Therefore, there exists a unique $\theta^p \in \mathbb{R}$ such that $h_s = \frac{E\min(D, \bar{F}^{-1}(\frac{c}{p-w}))}{ED}\theta^p h_o + (1-\theta^p)(h_o + h_r)$ and condition (SA.2) is equivalent to $\theta < \theta^p$. Since $\theta \in (0,1)$ in our model, we truncate the cutoff point and set $\psi^p = \min(\max(\theta^p, 0), 1)$ and thus we get the equilibrium result.

Note $\frac{\partial \psi^o}{\partial r} \geq 0$ and $\frac{\partial \psi^p}{\partial r} = 0$. Also, when $r = 0$, condition (SA.1) implies condition (SA.2), and thus $\theta^p \geq \theta^\circ$ or $\psi^p \geq \psi^\circ$. Therefore, $\exists \bar{r} \geq 0$ such that $\psi^p > \psi^\circ$ if and only if $r < \bar{r}$. $\square$

Proof of Proposition 13: If $\theta < \psi^p$, then we have

$$\begin{aligned}\pi^\circ &= pE\min(\theta D, q^\circ) - cq^\circ + wE(\theta D - q^\circ)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^\circ)^+ \\ &< pE\min(\theta D, q^\circ) - cq^\circ + wE(\theta D - q^\circ)^+ \\ &< pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ = \pi^p\end{aligned}$$

where the second inequality is because q^p rather than q° is the optimal solution that maximizes the newsvendor profit function π^p.

If $\psi^p < \psi^\circ$ and $\theta \in [\psi^p, \psi^\circ)$, then we have

$$\begin{aligned}\pi^\circ &= pE\min(\theta D, q^\circ) - cq^\circ + wE(\theta D - q^\circ)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^\circ)^+ \\ &= (p - w - r\frac{1-\theta}{\theta})E\min(\theta D, q^\circ) - cq^\circ + w\theta ED - rE(1-\theta)D \\ &> wE\theta D - rE(1-\theta)D = \pi^p\end{aligned}$$

If $\theta \geq \max(\psi^p, \psi^\circ)$, then $\pi^\circ = wE\theta D - rE(1-\theta)D = \pi^p$. □

Proof of Proposition 14: Since this is just the base model with a new set of parameter θ' and D', similar to the proof of Proposition 1, we can show that the participatory equilibrium exists if and only if

$$h_s < \frac{E\min(D', \bar{F}'^{-1}(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}))}{ED'}[h_o + (1-\theta')h_r]. \tag{SA.3}$$

and in the participatory equilibrium, we have $\phi^v = 1$ and $q^v = \theta'\bar{F}'(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}) > 0$. Since $D' = [1-\alpha(1-\theta)]D$ and $F'(x) = F(\frac{x}{1-\alpha(1-\theta)})$, the right hand side of condition (SA.3) equals to $\frac{E\min\left(D,\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta'}{\theta'}}\right)\right)}{ED}[h_o + (1-\theta')h_r]$, which is decreasing in θ and is also boundless for $\theta \in \mathbb{R}$. Therefore, there exists a unique $\theta^v \in \mathbb{R}$ such that $h_s = \frac{E\min(D,\bar{F}^{-1}(\frac{c}{p-w+r\frac{1-\theta'(\theta^v)}{\theta'(\theta^v)}}))}{ED}[h_o + (1-\theta'(\theta^v))h_r]$ and condition (SA.3) is equivalent to $\theta < \theta^v$, where $\theta'(\theta^v) = \frac{\theta^v}{1-\alpha(1-\theta^v)}$, i.e., the posterior fraction of high-type customers given prior θ^v. Since $\theta \in (0,1)$ in our model, we truncate the cutoff point and set $\psi^v = \min(\max(\theta^v, 0), 1)$ and thus we get the equilibrium result.

Since $\theta' > \theta$, it is easy to show that condition (SA.3) implies condition (SA.1), and thus $\theta^\circ \geq \theta^v$. Therefore, their truncated counterparts $\psi^\circ \geq \psi^v$. $\square$

Proof of Proposition 15: If $\theta < \psi^v$, then we have

$$\begin{aligned}
\pi^\circ &= pE\min(\theta D, q^\circ) - cq^\circ + wE(\theta D - q^\circ)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^\circ)^+ \\
&< pE\min(\theta D, q^\circ) - cq^\circ + wE(\theta D - q^\circ)^+ - rE\frac{1-\theta'}{\theta'}(\theta D - q^\circ)^+ \\
&= pE\min(\theta' D', q^\circ) - cq^\circ + wE(\theta' D' - q^\circ)^+ - rE\frac{1-\theta'}{\theta'}(\theta' D' - q^\circ)^+ \\
&< pE\min(\theta' D', q^v) - cq^v + wE(\theta' D' - q^v)^+ - rE\frac{1-\theta'}{\theta'}(\theta' D' - q^v)^+ = \pi^v
\end{aligned}$$

where the first inequality is because of $\theta' > \theta$, the second inequality is because q^v rather than q° is the optimal solution that maximizes the newsvendor profit function π^v.

If $\theta \geq \psi^\circ$, then we have $\pi^\circ = wE\theta D - rE(1-\theta)D < wE\theta D - rE(1-\alpha)(1-\theta)D = wE\theta' D' - r(1-\theta')D' = \pi^v$.

If $\theta \in [\psi^v, \psi^\circ)$, then $\pi^\circ = (p - w + r\frac{1-\theta}{\theta})E\min(\theta D, q^\circ) - cq^\circ + (w\theta - r(1-\theta))ED$, while $\pi^v = wE\theta' D' - rE(1-\theta')D' = wE\theta D - rE(1-\alpha)(1-\theta)D$. Then, by Envelop Theorem, $\frac{\partial(\pi^\circ - \pi^v)}{\partial w} = \frac{\partial \pi^\circ}{\partial w} - \frac{\partial \pi^v}{\partial w} = -E\min(\theta D, q^\circ) < 0$. To conclude the result, we just need to note that it is indeed possible to have $\pi^\circ > \pi^v$, e.g., when α is very close to 0. $\square$

Proof of Proposition 16: Comparing $u_{s,in}$ and u_o, we have

- $u_{s,in} > u_o$ if the following condition holds

$$h_s < h_o + (1-\theta)h_r \tag{SA.4}$$

  Then, consumers visit store if store is in stock. Thus, retailer profit function is $\pi =$

$pE\min(\theta D, q) - cq + wE(\theta D - q)^+ - rE\frac{1-\theta}{\theta}(\theta D - q)^+ = (p - w + r\frac{1-\theta}{\theta})E\min(\theta D, q) - cq + (w\theta - r(1-\theta))ED$, which is a typical newsvendor problem (plus a constant $(w\theta + r(1-\theta))ED$), and therefore the optimal order quantity $q^a = \theta\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$. Note condition (SA.4) is equivalent to $\theta < \tilde{\psi}^a$, where $\tilde{\psi}^a = \min(\max(\frac{h_o+h_r-h_s}{h_r}, 0), 1)$, given $\theta \in (0,1)$. Moreover, note if the critical fractile $\frac{c}{p-w+r\frac{1-\theta}{\theta}} \geq 1$ ($\Leftrightarrow \theta \geq \frac{r}{(w+r-p+c)^+}$, given $\theta \in (0,1)$), then $q^a = 0$, i.e., store is never in stock and thus consumers actually always buy online as a result.

- If condition (SA.4) does not hold, i.e., $\theta \geq \tilde{\psi}^a$, then $u_{s,in} \leq u_o$ and consumers buy online directly. Then, it is easy to see that the optimal store inventory level is $q^a = 0$.

By setting $\psi^a = \min(\frac{r}{(w+r-p+c)^+}, \tilde{\psi}^a)$, we get the market outcome.

Finally, to prove $\psi^a \geq \psi^\circ$, we only need to note the following two facts: First, since condition (SA.1) implies condition (SA.4), we have $\tilde{\psi}^a \geq \psi^\circ$; second, since $\frac{c}{p-w+r\frac{1-\theta}{\theta}} \geq 1$ (or $\theta \geq \frac{r}{(w+r-p+c)^+}$) implies that $q^\circ = 0$, we have $\frac{r}{(w+r-p+c)^+} \geq \psi^\circ$. $\square$

Proof of Proposition 17: If $\theta < \psi^\circ$, then we have

$$\begin{aligned}
\pi^\circ &= pE\min(\theta D, q^\circ) - cq^\circ + wE(\theta D - q^\circ)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^\circ)^+ \\
&= pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ = \pi^a
\end{aligned}$$

If $\theta \geq \psi^a$, then we have $\pi^\circ = wE\theta D - rE(1-\theta)D = \pi^a$.

If $\theta \in [\psi^\circ, \psi^a)$, then we have

$$\begin{aligned}
\pi^\circ &= wE\theta D - rE(1-\theta)D \\
&< (p - w + r\frac{1-\theta}{\theta})E\min(\theta D, q^a) - cq^a + wE\theta D - rE(1-\theta)D \\
&= pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ = \pi^a
\end{aligned}$$

$\square$

Proof of Proposition 18: First, note condition A.3 in Appendix A.3.1 implies condition (SA.2), thus $\underline{\psi}^{pa} \leq \psi^p$. Moreover, it is easy to check that $\overline{\psi}^{pa} = \psi^a$. Since $\underline{\psi}^{pa} \leq \overline{\psi}^{pa}$, we have $\underline{\psi}^{pa} \leq \psi^a$.

1. If $\theta < \underline{\psi}^{pa}$, then $\pi^p = pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ = pE\min(\theta D, q^{pa}) - cq^{pa} + wE(\theta D - q^{pa})^+ = \pi^{pa}$ and

$$\begin{aligned}
\pi^p &= pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ \\
&> pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ \\
&> pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ = \pi^a
\end{aligned}$$

   where the first inequality is because q^p is the unique maximal of π^p. Thus, we have $\pi^p = \pi^{pa} > \pi^a$.

2. If $\theta \in [\underline{\psi}^{pa}, \min(\psi^p, \psi^a))$, then

$$\begin{aligned}
\pi^p &= pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ \\
&> pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ \\
&> pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ = \pi^a
\end{aligned}$$

   and

$$\begin{aligned}
\pi^a &= pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ \\
&= pE\min(\theta D, q^{pa}) - cq^{pa} + wE(\theta D - q^{pa})^+ - rE\frac{1-\theta}{\theta}(\theta D - q^{pa})^+ = \pi^{pa}
\end{aligned}$$

   Thus, we have $\pi^p > \pi^a = \pi^{pa}$

3. If $\theta \in [\min(\psi^p, \psi^a), \max(\psi^p, \psi^a))$,

- if $\psi^p < \psi^a$, then

$$\begin{aligned}\pi^a &= pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ \\ &= pE\min(\theta D, q^{pa}) - cq^{pa} + wE(\theta D - q^{pa})^+ - rE\frac{1-\theta}{\theta}(\theta D - q^{pa})^+ = \pi^{pa}\end{aligned}$$

and

$$\begin{aligned}\pi^a &= pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ \\ &= \left(p - w + r\frac{1-\theta}{\theta}\right)E\min(\theta D, q^a) - cq^a + wE\theta D - rE(1-\theta)D \\ &> wE\theta D - rE(1-\theta)D = \pi^p\end{aligned}$$

Thus, we have $\pi^a = \pi^{pa} > \pi^p$;

- if $\psi^p > \psi^a$, then

$$\begin{aligned}\pi^a &= pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ \\ &= pE\min(\theta D, q^{pa}) - cq^{pa} + wE(\theta D - q^{pa})^+ - rE\frac{1-\theta}{\theta}(\theta D - q^{pa})^+ = \pi^{pa}\end{aligned}$$

and

$$\begin{aligned}\pi^p &= pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ \\ &> pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ \\ &> pE\min(\theta D, q^a) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ = \pi^a\end{aligned}$$

Thus, we have $\pi^p > \pi^a = \pi^{pa}$.

4. If $\theta \geq \max(\psi^p, \psi^a)$, then $\pi^p = \pi^a = \pi^{pa} = wE\theta D - rE(1-\theta)D$.

□

Proof of Proposition 19: The addition of virtual showrooms only creates a new customer pool with D' and θ'. Then since ψ^p is determined by condition (SA.2), ψ^{pv} should be determined by condition $h_s < \frac{E\min(D',\bar{F}'^{-1}(\frac{c}{p-w}))}{ED'}\theta h_o + (1-\theta')(h_o+h_r)$, which is implied by condition (SA.2). Thus, we have $\psi^{pv} \leq \psi^p$.

If $\theta < \psi^{pv}$, then $\pi^p = pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ = pE\min(\theta' D', q^{pv}) - cq^{pv} + wE(\theta' D' - q^{pv})^+ = \pi^{pv}$ and

$$\begin{aligned}
\pi^p &= pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ \\
&> \max\big(pE\min(\theta D, q^v) - cq^v + wE(\theta D - q^v)^+, wE\theta D\big) \\
&> \max\Bigg(pE\min\big(\theta' D', q^v\big) - cq^v + wE\big(\theta' D' - q^v\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^v\big)^+, \\
&\qquad\qquad wE\theta' D' - rE\left(1-\theta'\right)D'\Bigg) \\
&= \pi^v
\end{aligned}$$

If $\theta \in [\psi^{pv}, \psi^p)$, then $\pi^p = pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ = (p-w)\,E\min(\theta D, q^p) - cq^p + wE\theta D > wE\theta' D' - rE\,(1-\theta')\,D = \pi^{pv}$ and

$$\begin{aligned}
\pi^p &= pE\min(\theta D, q^p) - cq^p + wE(\theta D - q^p)^+ \\
&> \max\big(pE\min(\theta D, q^v) - cq^v + wE(\theta D - q^v)^+, wE\theta D\big) \\
&> \max\Bigg(pE\min\big(\theta' D', q^v\big) - cq^v + wE\big(\theta' D' - q^v\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^v\big)^+, \\
&\qquad\qquad wE\theta' D' - rE\left(1-\theta'\right)D'\Bigg) \\
&= \pi^v
\end{aligned}$$

If $\theta \in [\psi^p, \max(\psi^p, \psi^v))$, then $\pi^p = wE\theta D - rE(1-\theta)D$ and $\pi^{pv} = wE\theta' D' - r(1-\theta')D'$, and

$$
\begin{aligned}
\pi^v &= pE\min\left(\theta' D', q^v\right) - cq^v + wE\big(\theta' D' - q^v\big)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^v\right)^+ \\
&= \left(p - w + r\frac{1-\theta'}{\theta'}\right) E\min\left(\theta' D', q^v\right) - cq^v + wE\theta' D' - rE\left(1-\theta'\right) D' \\
&> wE\theta' D' - rE\left(1-\theta'\right) D' \\
&= wE\theta D - r\left(1-\alpha\right) E\left(1-\theta\right) D \\
&\geq \max\left(\pi^{pv}, \pi^p\right)
\end{aligned}
$$

If $\theta \geq \max(\psi^p, \psi^v)$, $\pi^v = \pi^{pv} = wE\theta' D' - r(1-\theta')D' = wE\theta D - r\left(1-\alpha\right) E\left(1-\theta\right) D > wE\theta D - rE(1-\theta)D = \pi^p$.

$\square$

Proof of Proposition 20: Let's first prove the part (i):

Think of the case with virtual showroom as a new base model with a new customer pool. Then, since $\psi^a \geq \psi^\circ$, we should also have $\psi^{va} \geq \psi^v$.

If $\theta \in [\psi^v, \psi^{va})$, then

$$
\begin{aligned}
\pi^{va} &= pE\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\big(\theta' D' - q^{va}\big)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^{va}\right)^+ \\
&= \left(p - w + r\frac{1-\theta'}{\theta'}\right) E\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\theta' D' - rE\left(1-\theta'\right) D' \\
&> wE\theta' D' - rE\left(1-\theta'\right) D' = \pi^v
\end{aligned}
$$

and

$$
\begin{aligned}
\pi^{va} &= pE\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\big(\theta' D' - q^{va}\big)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^{va}\right)^+ \\
&> pE\min\left(\theta' D', q^{a}\right) - cq^{a} + wE\big(\theta' D' - q^{a}\big)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^{a}\right)^+ \\
&> pE\min\left(\theta D, q^{a}\right) - cq^{a} + wE(\theta D - q^{a})^+ - rE\frac{1-\theta}{\theta}(\theta D - q^{a})^+ = \pi^a
\end{aligned}
$$

Thus, we get part (i).

Next, let's look at part (ii):

Since we can think of the case with virtual showroom as a new base model with new customer pool (θ' and D'), the proof should be similar to the proof of Proposition 18 and thus omitted.

$\square$

Proof of Proposition 21: Let's first look at the case when $\theta < \psi^p$. In Proposition 13, we have shown that $\pi^p > \pi^\circ$. Thus, all we need to show is that physical showroom is strictly better than the other three options as well.

$$
\begin{aligned}
\pi^p &= pE\min\left(\theta D, q^p\right) - cq^p + wE(\theta D - q^p)^+ \\
&> \max\left(pE\min\left(\theta D, q^v\right) - cq^v + wE(\theta D - q^v)^+, wE\theta D\right) \\
&> \max\Bigg(pE\min\left(\theta' D', q^v\right) - cq^v + wE\left(\theta' D' - q^v\right)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^v\right)^+, \\
&\qquad\qquad wE\theta' D' - rE\left(1-\theta'\right)D'\Bigg) \\
&= \pi^v
\end{aligned}
$$

A similar argument could be used to prove $\pi^p > \pi^{va}$, by simply replacing the superscript $\cdot^v$ with $\cdot^{va}$ in the equation above. Finally, compared with the case where there is only availability information

$$
\begin{aligned}
\pi^p &= pE\min\left(\theta D, q^p\right) - cq^p + wE(\theta D - q^p)^+ \\
&> \max\left(pE\min\left(\theta D, q^a\right) - cq^a + wE(\theta D - q^a)^+, wE\theta D\right) \\
&> \max\left(pE\min\left(\theta D, q^a\right) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+, wE\theta D - rE\left(1-\theta\right)D\right) \\
&= \pi^a
\end{aligned}
$$

So, we can conclude $\pi^* = \pi^p$.

Next, let's look at the case where $\theta \geq \psi^p$. Let's first show the suboptimality of the option of offering only physical showroom:

- if $\theta < \max(\psi^p, \psi^\circ)$, then

$$\begin{aligned}
\pi^\circ &= pE\min\left(\theta D, q^\circ\right) - cq^\circ + wE(\theta D - q^\circ)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^\circ)^+ \\
&= \left(p - w + r\frac{1-\theta}{\theta}\right)E\min\left(\theta D, q^\circ\right) - cq^\circ + wE\theta D - rE\left(1-\theta\right)D \\
&> wE\theta D - rE\left(1-\theta\right)D = \pi^p
\end{aligned}$$

- if $\theta > \max(\psi^p, \psi^\circ)$, then $\pi^\circ = wE\theta D - rE\left(1-\theta\right)D = \pi^p$

So in the following analysis, we only need to look at the other four types of information.

- Case 1: $\theta < \psi^\circ$. Note that if $\psi^p \geq \psi^\circ$, we don't even have this case. So in the following proof, we only focus on the case when $\psi^p < \psi^\circ$. Note when $\alpha = 0$, we have $\psi^v = \psi^\circ$ and $\psi^{va} = \psi^a$. Also, it is easy to check that $\frac{\partial \psi^v}{\partial \alpha} \leq 0$ and $\frac{\partial \psi^{va}}{\partial \alpha} \leq 0$. Then, let's set $\alpha_1 = \underset{\alpha\in[0,1]}{\arg\min}\left|\theta - \psi^v\left(\alpha\right)\right|$ and $\alpha_2 = \underset{\alpha\in[0,1]}{\arg\min}\left|\theta - \psi^{va}\left(\alpha\right)\right|$. Since $\forall\alpha\ \psi^v \leq \psi^{va}$, we have $\alpha_1 \leq \alpha_2$.

  - If $\alpha < \alpha_1$, then $\theta < \psi^v \leq \min(\psi^\circ, \psi^{va}) \leq \psi^a$. Therefore, $\pi^v = \pi^{va}$ and $\pi^a = \pi^\circ$. Moreover,

$$\begin{aligned}
\pi^v &= pE\min\left(\theta' D', q^v\right) - cq^v + wE\left(\theta' D' - q^v\right)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^v\right)^+ \\
&> pE\min\left(\theta' D', q^a\right) - cq^a + wE\left(\theta' D' - q^a\right)^+ - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^a\right)^+ \\
&> pE\min\left(\theta D, q^a\right) - cq^a + wE(\theta D - q^a)^+ - rE\frac{1-\theta}{\theta}(\theta D - q^a)^+ = \pi^a
\end{aligned}$$

    Thus, $\pi^* = \pi^v$.

– If $\alpha \in (\alpha_1, \alpha_2)$, then $\psi^v < \theta < \min(\psi^\circ, \psi^{va}) \leq \pi^a$. Therefore, $\pi^a = \pi^\circ$. Also,

$$\begin{aligned}
\pi^{va} &= pE\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\big(\theta' D' - q^{va}\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^{va}\big)^+ \\
&> pE\min\left(\theta' D', q^{a}\right) - cq^{a} + wE\big(\theta' D' - q^{a}\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^{a}\big)^+ \\
&> pE\min\left(\theta D, q^{a}\right) - cq^{a} + wE(\theta D - q^{a})^+ - rE\frac{1-\theta}{\theta}(\theta D - q^{a})^+ = \pi^a
\end{aligned}$$

and

$$\begin{aligned}
\pi^{va} &= pE\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\big(\theta' D' - q^{va}\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^{va}\big)^+ \\
&= \left(p - w + r\frac{1-\theta'}{\theta'}\right)E\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\theta' D' - rE\left(1-\theta'\right)D' \\
&> wE\theta' D' - rE\left(1-\theta'\right)D' = \pi^v
\end{aligned}$$

Thus, we can conclude $\pi^* = \pi^{va}$.

– If $\alpha > \alpha_2$, then $\psi^v \leq \psi^{va} < \theta < \psi^\circ \leq \psi^a$. Therefore, $\pi^a = \pi^\circ$ and $\pi^v = \pi^{va}$. Thus, we only need to compare π° and π^v. Because $\frac{\partial(\pi^v - \pi^\circ)}{\partial\alpha} = r(1-\theta)ED > 0$, there exists $\alpha_3 \geq \alpha_2$ such that $\pi^\circ > \pi^v$ if and only if $\alpha \in (\alpha_2, \alpha_3)$. Thus, $\pi^* = \pi^\circ$ if $\alpha \in (\alpha_2, \alpha_3)$; $\pi^* = \pi^v$ if $\alpha > \alpha_3$.

- Case 2: $\theta > \psi^\circ$.

  – If $\theta \in [\max(\psi^\circ, \psi^p), \max(\psi^a, \psi^p)]$, then let's set $\alpha_1' = \arg\min_{\alpha\in[0,1]} |\theta - \psi^{va}(\alpha)|$.

    * If $\alpha < \alpha_1'$, then $\psi^v \leq \psi^\circ < \theta < \psi^{va} \leq \psi^a$. Therefore, $\pi^v = wE\theta' D' - rE(1-\theta')D' > wE\theta D - r(1-\theta)D = \pi^\circ$. Also,

$$\begin{aligned}
\pi^{va} &= pE\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\big(\theta' D' - q^{va}\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^{va}\big)^+ \\
&> pE\min\left(\theta' D', q^{a}\right) - cq^{a} + wE\big(\theta' D' - q^{a}\big)^+ - rE\frac{1-\theta'}{\theta'}\big(\theta' D' - q^{a}\big)^+ \\
&> pE\min\left(\theta D, q^{a}\right) - cq^{a} + wE(\theta D - q^{a})^+ - rE\frac{1-\theta}{\theta}(\theta D - q^{a})^+ = \pi^a
\end{aligned}$$

and

$$
\begin{aligned}
\pi^{va} &= pE\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\left(\theta' D' - q^{va}\right)^{+} - rE\frac{1-\theta'}{\theta'}\left(\theta' D' - q^{va}\right)^{+} \\
&= \left(p - w + r\frac{1-\theta'}{\theta'}\right) E\min\left(\theta' D', q^{va}\right) - cq^{va} + wE\theta' D' - rE\left(1-\theta'\right) D' \\
&> wE\theta' D' - rE\left(1-\theta'\right) D' = \pi^{v}
\end{aligned}
$$

Thus, we can conclude $\pi^{*} = \pi^{va}$.

* If $\alpha > \alpha'$, then $\psi^{v} \leq \min(\psi^{\circ}, \pi^{va}) \leq \max(\psi^{\circ}, \psi^{va}) < \theta < \psi^{a}$. Therefore, $\pi^{va} = \pi^{v}$ and

$$
\begin{aligned}
\pi^{a} &= pE\min\left(\theta D, q^{a}\right) - cq^{a} + wE(\theta D - q^{a})^{+} - rE\frac{1-\theta}{\theta}(\theta D - q^{a})^{+} \\
&= \left(p - w + r\frac{1-\theta}{\theta}\right) E\min\left(\theta D, q^{a}\right) - cq^{a} + wE\theta D - rE\left(1-\theta\right) D \\
&> wE\theta D - rE\left(1-\theta\right) D = \pi^{\circ}
\end{aligned}
$$

Thus, we conly need to compare π^{v} and π^{a}. Because $\frac{\partial(\pi^{v}-\pi^{a})}{\partial\alpha} = r(1-\theta)ED > 0$, there exists $\alpha_2' \geq \alpha_1'$ such that $\pi^{a} > \pi^{v}$ if and only if $\alpha \in (\alpha_1', \alpha_2')$. Thus $\pi^{*} = \pi^{a}$ if $\alpha \in (\alpha_1', \alpha_2')$; $\pi^{*} = \pi^{v}$ if $\alpha > \alpha_2'$.

– If $\theta > \max(\psi^{a}, \psi^{p})$, then since $\psi^{v} \leq \psi^{\circ}$ and $\psi^{a} \geq \psi^{\circ}$, we have $\psi^{a} \geq \psi^{v}$. Also, note ψ^{a} is determined by condition (SA.4), which is implied by condition $h_s < h_o + (1-\theta')h_r$, which could be shown to determine the cutoff ψ^{va}. Thus, we also have $\psi^{va} \leq \psi^{a}$. Then, if $\theta \geq \max(\psi^{p}, \psi^{a})$, we have $\pi^{v} = \pi^{va} = wE\theta' D' - rE(1-\theta')D' > wE\theta D - rE(1-\theta)D = \pi^{a} = \pi^{p} = \pi^{\circ}$. So we can conclude $\pi^{*} = \pi^{v}$. In this case, we can simply set $\alpha_1' = \alpha_2' = 0$.

$\square$

Proof of Proposition 22: First, consider the following optimization problem:

$$\begin{aligned} &\max_{0\leq\lambda\leq\mu_1\leq\frac{r\alpha}{c_1},0\leq\lambda\leq\mu_2\leq\frac{r\alpha}{c_2}} \quad r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = \alpha - \beta\left(w_1(\mu_1,\lambda) + w_2(\mu_2,\lambda)\right) \end{aligned} \tag{A.18}$$

Since the constraint set is compact and the objective function is continuous, by Weierstass Theorem, there is a maximum to (A.18). Let's denote the optimal solution to (A.18) as $\lambda^*, \mu_1^*, \mu_2^*$, then the corresponding optimal objective value $\pi^* = r\lambda^* - c_1\mu_1^* - c_2\mu_2^*$.

Lemma 2. *If $\pi^* > 0$, then $(\lambda^*, \mu_1^*, \mu_2^*)$ must be the optimal solution to the optimization problem (4.1); if $\pi^* \leq 0$, then there is no solution to (4.1) that yields positive profit.*

Proof of Lemma 2: If $\pi^* > 0$, then we must have $\lambda^* > 0$. Thus, $(\lambda^*, \mu_1^*, \mu_2^*)$ is also a feasible solution to (1). Suppose the optimal solution to (1) is $(\lambda^b, \mu_1^b, \mu_2^b) \neq (\lambda^*, \mu_1^*, \mu_2^*)$. Then, it means that $\pi^b = r\lambda^b - c_1\mu_1^b - c_2\mu_2^b > \pi^* > 0$. Then, it is easy to check that $(\lambda^b, \mu_1^b, \mu_2^b)$ is a feasible solution to (A.18) as well. But it leads to a higher objective value π^b for (A.18), which contradicts to the fact that $(\lambda^*, \mu_1^*, \mu_2^*)$ is the optimal solution to (A.18). Thus, we must have $(\lambda^b, \mu_1^b, \mu_2^b) = (\lambda^*, \mu_1^*, \mu_2^*)$.

If $\pi^* \leq 0$, suppose there exists a solution $(\lambda^b, \mu_1^b, \mu_2^b)$ to (1) such that $\pi^b = r\lambda^b - c_1\mu_1^b - c_2\mu_2^b > 0$. Then, it is easy to check that $(\lambda^b, \mu_1^b, \mu_2^b)$ should also be a feasible solution to (A.18), which leads to a positive objective value π^b for (A.18). This contradicts to the fact that $\pi^* \leq 0$. Thus, there is no solution to (4.1) that yields positive profit. $\square$

By Lemma 2, finding the optimal solution to (4.1) that yields positive profit is equivalent to finding the optimal solution to (A.18) that yields positive profit.

The Lagrangian of (A.18) is defined as follows:

$$L(\lambda, \mu_1, \mu_2, \rho) = r\lambda - c_1\mu_1 - c_2\mu_2 + \rho\left[\lambda - \alpha + \beta\left(\frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}\right)\right]$$

where $\rho \in \mathbb{R}$ is the Lagrange multiplier. To find the critical points of $L(\lambda, \mu_1, \mu_2, \rho)$, we solve the following equation set:

$$\begin{aligned}
&\tfrac{\partial L}{\partial \lambda} = r + \rho + \rho\beta \tfrac{1}{(\mu_1-\lambda)^2} + \rho\beta \tfrac{1}{(\mu_2-\lambda)^2} = 0\\
&\tfrac{\partial L}{\partial \mu_i} = -c_i - \rho\beta \tfrac{1}{(\mu_i-\lambda)^2} = 0, i = 1, 2\\
&\lambda - \alpha + \beta\left(\tfrac{1}{\mu_1-\lambda} + \tfrac{1}{\mu_2-\lambda}\right) = 0\\
&0 \leq \lambda \leq \mu_1 \leq \tfrac{r\alpha}{c_1}, 0 \leq \lambda \leq \mu_2 \leq \tfrac{r\alpha}{c_2}
\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.18) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the boundary conditions that $0 \leq \lambda$ and $\mu_i \leq \frac{r\alpha}{c_i}$, and solve the equation set above, which gives us a unique solution:

$$\begin{aligned}
&\lambda^* = \alpha - \beta\sqrt{\tfrac{c_1}{\beta(r-c_1-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}}\\
&\mu_1^* = \lambda^* + \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}}\\
&\mu_2^* = \lambda^* + \sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}
\end{aligned} \tag{A.19}$$

So, if $\lambda^* \geq 0$ and $\mu_i^* \leq \frac{r\alpha}{c_i}$, $i = 1, 2$, then $(\lambda^*, \mu_1^*, \mu_2^*)$ must be the optimal solution to (A.18); otherwise, the optimal solution to (A.18) must be corner solution (i.e., at least one of the follow holds, $\lambda^* = 0$ and $\mu_i^* = \frac{r\alpha}{c_i}$, $i = 1, 2$), in which case the optimal value is clearly nonpositive. Note, with $(\lambda^*, \mu_1^*, \mu_2^*)$ defined in (A.19), we have $\pi^* = r\lambda^* - c_1\mu_1^* - c_2\mu_2^* = (r - c_1 - c_2)\alpha - 2\sqrt{\beta(r-c_1-c_2)c_1} - 2\sqrt{\beta(r-c_1-c_2)c_2}$, which is positive if $\alpha > \frac{2\sqrt{\beta}(\sqrt{c_1}+\sqrt{c_2})}{\sqrt{r-c_1-c_2}}$. It is easy to check that if $\alpha > \frac{2\sqrt{\beta}(\sqrt{c_1}+\sqrt{c_2})}{\sqrt{r-c_1-c_2}}$, we have $\lambda^* \geq 0$ and $\mu_i^* \leq \frac{r\alpha}{c_i}$, $i = 1, 2$. This implies that there is an optimal solution $(\lambda^*, \mu_1^*, \mu_2^*)$ that yields positive profit for (A.18) if and only if $\alpha > \bar{\alpha} = \frac{2\sqrt{\beta}(\sqrt{c_1}+\sqrt{c_2})}{\sqrt{r-c_1-c_2}}$. Then, by Lemma 2, we can conclude that same result holds for the optimization problem (4.1). $\square$

Proof of Proposition 23: Suppose we set the two capacity levels at μ_1^b, μ_2^b (i.e., the optimal solution in the base model). Then, the demand rates are

$$\lambda_o^\star = \left[\alpha - \xi\beta w_2(\mu_2^b, (1-\theta)\lambda_s^\star + \theta\lambda_o^\star)\right]^+$$
$$\lambda_s^\star = \left[\alpha - \beta\left(w_1(\mu_1^b, (1-\theta)\lambda_s^\star) + w_2(\mu_2^b, (1-\theta)\lambda_s^\star + \theta\lambda_o^\star)\right)\right]^+$$

It is easy to check $(1-\theta)\lambda_s^\star + \theta\lambda_o^\star > \lambda^b$. This implies that the profit with μ_1^b, μ_2^b and online self-order technology is greater than π^b. Therefore, under optimal solution, there must be customers coming to store after the implementation of online self-order technology, i.e., at least one of λ_s^o and λ_o^o is positive. Note, if $\lambda_s^o > 0$, we must have $\lambda_o^o > 0$. Thus, we must have $\lambda_o^o > 0$ in the optimal solution. So the optimization problem (4.2) is equivalent to the following problem

$$\begin{aligned} &\max_{0\le\lambda_o,(1-\theta)\lambda_s\le\mu_1,(1-\theta)\lambda_s+\theta\lambda_o\le\mu_2} r\left(\left(1-\theta\right)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda_o = \alpha - \xi\beta w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o) \\ &\qquad \lambda_s = \left[\alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s) + w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)\right]^+ \end{aligned} \tag{A.20}$$

Then, the firm has two choices:

i. Shut down stage 1 (i.e., $\mu_1 = 0$) and sell only to online customers. In this case, the optimal solution is obtained by solving the following optimization problem:

$$\begin{aligned} &\max_{0\le\lambda_o,\theta\lambda_o\le\mu_2\le\frac{r\alpha}{c_2}} r\theta\lambda_o - c_2\mu_2 \\ &s.t. \quad \lambda_o = \alpha - \xi\beta w_2(\mu_2, \theta\lambda_o) \end{aligned} \tag{A.21}$$

The Lagrangian of (A.21) is defined as follows:

$$L\left(\lambda_o, \mu_2, \rho\right) = r\theta\lambda_o - c_2\mu_2 + \rho\left[\lambda_o - \alpha + \xi\beta\frac{1}{\mu_2 - \theta\lambda_o}\right]$$

where $\rho \in \mathbb{R}$ is the Lagrange multiplier. To find the critical points of $L\left(\lambda_o, \mu_2, \rho\right)$, we solve the following equation set:

$$\begin{aligned}
&\frac{\partial L}{\partial \lambda_o} = r\theta + \rho + \rho\xi\beta\theta\frac{1}{(\mu_2-\theta\lambda_o)^2} = 0\\
&\frac{\partial L}{\partial \mu_2} = -c_2 - \rho\xi\beta\frac{1}{(\mu_2-\theta\lambda_o)^2} = 0\\
&\lambda_o - \alpha + \xi\beta\frac{1}{\mu_2-\theta\lambda_o} = 0\\
&0 \le \lambda_o, \theta\lambda_o \le \mu_2 \le \frac{r\alpha}{c_2}
\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.21) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the conditions that $0 \le \lambda_o$ and $\mu_2 \le \frac{r\alpha}{c_2}$, and solve the equation set above, which gives us a unique solution:

$$\begin{aligned}
\lambda_o^* &= \alpha - \xi\beta\sqrt{\frac{c_2}{\theta\xi\beta(r-c_2)}}\\
\mu_2^* &= \lambda_o^* + \sqrt{\frac{\theta\xi\beta(r-c_2)}{c_2}}
\end{aligned} \tag{A.22}$$

So, if $\lambda^* \ge 0$ and $\mu_2^* \le \frac{r\alpha}{c_2}$, then (λ_o^*, μ_2^*) together with $\lambda_1^* = 0, \mu_1^* = 0$ must be the optimal solution to (A.21); otherwise, the optimal solution to (A.21) must be corner solution (i.e., at least one of the follow holds, $\lambda_o^* = 0$ and $\mu_2^* = \frac{r\alpha}{c_2}$), in which case the optimal value is clearly nonpositive, and thus the firm won't choose this option at all.

ii. Sell to both types of customers. In this case, the optimal solution is obtained by solving the following optimization problem:

$$\begin{aligned}
&\max_{0\le\lambda_o, 0\le(1-\theta)\lambda_s\le\mu_1\le\frac{r\alpha}{c_1}, (1-\theta)\lambda_s+\theta\lambda_o\le\mu_2\le\frac{r\alpha}{c_2}} r\left(\left(1-\theta\right)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2\\
&s.t. \quad \lambda_o = \alpha - \xi\beta w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\\
&\qquad \lambda_s = \alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s) + w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)
\end{aligned} \tag{A.23}$$

The Lagrangian of (A.23) is defined as follows:

$$\begin{aligned}\tilde{L}\left(\lambda_s,\lambda_o,\mu_1,\mu_2,\rho_1,\rho_2\right)=&r\left(\left(1-\theta\right)\lambda_s+\theta\lambda_o\right)-c_1\mu_1-c_2\mu_2\\&+\rho_1\left[\lambda_o-\alpha+\xi\beta\frac{1}{\mu_2-\left(1-\theta\right)\lambda_s-\theta\lambda_o}\right]\\&+\rho_2\left[\lambda_s-\alpha+\beta\frac{1}{\mu_1-\left(1-\theta\right)\lambda_s}+\beta\frac{1}{\mu_2-\left(1-\theta\right)\lambda_s-\theta\lambda_o}\right]\end{aligned}$$

where $\rho_1,\rho_2\in\mathbb{R}$ are the Lagrange multipliers. To find the critical points of $\tilde{L}\left(\lambda_s,\lambda_o,\mu_1,\mu_2,\rho_1,\rho_2\right)$, we solve the following equation set:

$$\begin{aligned}&\tfrac{\partial\tilde{L}}{\partial\lambda_s}=r\left(1-\theta\right)+\rho_2+\rho_1\xi\beta\left(1-\theta\right)\tfrac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^2}+\rho_2\beta\left(1-\theta\right)\tfrac{1}{\left(\mu_1-(1-\theta)\lambda_s\right)^2}\\&\qquad+\rho_2\beta\left(1-\theta\right)\tfrac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^2}=0\\&\tfrac{\partial\tilde{L}}{\partial\lambda_o}=r\theta+\rho_1+\rho_1\xi\beta\theta\tfrac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^2}+\rho_2\beta\theta\tfrac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^2}=0\\&\tfrac{\partial\tilde{L}}{\partial\mu_1}=-c_1-\rho_2\beta\tfrac{1}{\left(\mu_1-(1-\theta)\lambda_s\right)^2}=0\\&\tfrac{\partial\tilde{L}}{\partial\mu_2}=-c_2-\rho_1\xi\beta\tfrac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^2}-\rho_2\beta\tfrac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^2}=0\\&\lambda_o-\alpha+\xi\beta\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o}=0\\&\lambda_s-\alpha+\beta\tfrac{1}{\mu_1-(1-\theta)\lambda_s}+\beta\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o}=0\\&0\leq\lambda_o,0\leq\left(1-\theta\right)\lambda_s\leq\mu_1\leq\tfrac{r\alpha}{c_1},\left(1-\theta\right)\lambda_s+\theta\lambda_o\leq\mu_2\leq\tfrac{r\alpha}{c_2}\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.23) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the boundary conditions (i.e., $0\leq\lambda_o$, $0\leq\lambda_s$ and $\mu_i\leq\frac{r\alpha}{c_i}$ $i=1,2$), and solve the equation set above, which gives us a unique solution:

$$\begin{aligned}&\tilde{\lambda}_s^*=\alpha-\beta\sqrt{\tfrac{c_1}{(1-\theta)\beta(r-c_1-c_2)}}-\beta\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}\\&\tilde{\lambda}_o^*=\alpha-\xi\beta\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}\\&\tilde{\mu}_1^*=(1-\theta)\tilde{\lambda}_s^o+\sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)}{c_1}}\\&\tilde{\mu}_2^*=(1-\theta)\tilde{\lambda}_s^o+\theta\tilde{\lambda}_o^o+\sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}}\end{aligned}$$

So, if $(\tilde{\lambda}_o^*,\tilde{\lambda}_s^*,\tilde{\mu}_1^*,\tilde{\mu}_2^*)$ is an interior solution, then it must be the optimal solution to

(A.23); otherwise, the solution to (A.23) must be corner solution, in which case we can check the optimal value is either nonpositive or strictly less than that under option (i), and thus the firm never chooses this option.

Therefore, the optimal solution to (A.20) (and thus to (4.2)) must be chosen from the interior solutions under the two options, whichever gives higher optimal value. For option (i), under the interior solution $(\lambda_o^*, \lambda_s^*, \mu_1^*, \mu_2^*)$, the optimal value is $\pi^* = r((1-\theta)\lambda_s^* + \theta\lambda_o^*) - c_1\mu_1^* - c_2\mu_2^* = (r-c_2)\theta\alpha - 2\sqrt{\theta\xi\beta(r-c_2)c_2}$; for option (ii), under the interior solution $(\tilde{\lambda}_o^*, \tilde{\lambda}_s^*, \tilde{\mu}_1^*, \tilde{\mu}_2^*)$, the optimal value is $\tilde{\pi}^* = r((1-\theta)\tilde{\lambda}_s^* + \theta\tilde{\lambda}_o^*) - c_1\tilde{\mu}_1^* - c_2\tilde{\mu}_2^* = (r - c_2 - c_1(1-\theta))\alpha - 2\sqrt{(1-\theta)\beta(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta(r-c_1-c_2) + \theta\xi\beta(r-c_2)]c_2}$. Thus, $\pi^* < \tilde{\pi}^*$ if and only if $\alpha > \bar{\alpha}'' = \frac{2\sqrt{\beta c_1}}{\sqrt{(1-\theta)(r-c_1-c_2)}} + \frac{2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}}{(r-c_1-c_2)(1-\theta)} - \frac{2\sqrt{\theta\xi\beta(r-c_2)c_2}}{(r-c_1-c_2)(1-\theta)}$. Then, we conclude the proof by setting $\bar{\alpha}' = \max(\bar{\alpha}, \bar{\alpha}'')$. □

Proof of Proposition 24: From Propositions 22 and 23, we get $w_1^b = \sqrt{\frac{c_1}{\beta(r-c_1-c_2)}}$, $w_2^b = \sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}$, $w_1^o = \sqrt{\frac{c_1}{(1-\theta)\beta(r-c_1-c_2)}}$, and $w_2^o = \sqrt{\frac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}$. Since $\theta > 0$, we have $w_1^o > w_1^b$. Also, note that $\frac{\partial(w_2^o - w_2^b)}{\partial r} > 0$ and $\lim\limits_{r\to c_1+c_2} \left(w_2^o - w_2^b\right) < 0$. Thus, given c_1 and c_2, $\exists \bar{r} > c_1 + c_2$ such that $w_2^o < w_2^b$ if and only if $\frac{c_1+c_2}{r} > \frac{c_1+c_2}{\bar{r}} = m_w$.

□

Proof of Proposition 25: $\lambda_o^o = \alpha - \sqrt{\frac{\xi^2\beta^2 c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}} \geq \alpha - \sqrt{\frac{\beta c_2}{(1-\theta)(r-c_1-c_2)+\theta(r-c_2)}} > \alpha - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} = \lambda^b$ where the first inequality is because of $\xi \leq 1$.

Next, let's prove the second bullet point in Proposition 25. Note,

$$\left(\lambda_s^o - \lambda_s^b\right) \frac{\sqrt{r-c_1-c_2}}{\beta} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{(1-\theta)\,\beta}} - \sqrt{\frac{c_2}{(1-\theta)\,\beta + \theta\xi\beta\frac{r-c_2}{r-c_1-c_2}}}$$

which is decreasing in r for $r \geq c_1 + c_2$. When $r \to c_1 + c_2$, $\left(\lambda_s^o - \lambda_s^b\right) \frac{\sqrt{r-c_1-c_2}}{\beta} \to \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{(1-\theta)\beta}} > 0$ if $\theta < \psi_s = 1 - \left(\frac{\sqrt{c_1}}{\sqrt{c_1}+\sqrt{c_2}}\right)^2$. Then, we can conclude the result.

By Proposition 24, $\lambda_s^o > \lambda^b$ must happen when $w_2^o < w_2^b$, and thus we have $m_\lambda > m_w$.

Finally, let's prove the third point in Proposition 25. Note $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o = \alpha - \sqrt{\frac{(1-\theta)\beta c_1}{r-c_1-c_2}} - \frac{\sqrt{c_2}\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\sqrt{r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}}$. Also, $\frac{\partial\sqrt{(1-\theta)\beta c_1}}{\partial\theta} < 0$, $\frac{\partial\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta} \le 0$, and $\frac{\partial\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta} < 0$. Thus, $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o$ is increasing in θ. Note when $\theta = 0$, we have $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o = \lambda^b$. Thus, $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o > \lambda^b$ for all $\theta > 0$. □

Proof of Proposition 26: Note $\mu_1^o = \sqrt{1-\theta}\left(\sqrt{1-\theta}\alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{(r-c_1-c_2)+\theta/(1-\theta)\xi(r-c_2)}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}\right)$, which is decreasing in θ. Since when $\theta = 0$ we have $\mu_1^o = \mu_1^b$, we can conclude $\mu_1^o < \mu_1^b$ for any $\theta > 0$.

Next, let's prove part (ii). Note that

$$\frac{\mu_2^o-\mu_2^b}{\sqrt{\beta(r-c_1-c_2)}} = \frac{\sqrt{c_1}-\sqrt{(1-\theta)c_1}}{r-c_1-c_2} + \frac{\sqrt{c_2}-(1-\theta+\theta\xi)\sqrt{\frac{c_2}{1-\theta+\theta\xi\frac{r-c_2}{r-c_1-c_2}}}}{r-c_1-c_2} + \sqrt{\frac{1-\theta+\theta\xi\frac{r-c_2}{r-c_1-c_2}}{c_2}} - \sqrt{\frac{1}{c_2}} \tag{A.24}$$

It is easy to find the 1st and 3rd terms in (A.24) are decreasing in r. Also, since $\xi \le 1$, we have $\sqrt{c_2} - (1-\theta+\theta\xi)\sqrt{\frac{c_2}{1-\theta+\theta\xi\frac{r-c_2}{r-c_1-c_2}}} > 0$. Thus, we find the 2nd term in (A.24) is also decreasing in r. In sum, we find $\frac{\mu_2^o-\mu_2^b}{\sqrt{\beta(r-c_1-c_2)}}$ is decreasing in r. Note, when $-\beta\,(r-c_1-c_2) + \xi\beta\,(r-c_2) = 0$ (i.e., $r = \frac{c_1+c_2-\xi c_2}{1-\xi} > c_1+c_2$). then $\mu_2^o - \mu_2^b$ $= \left(1-\sqrt{1-\theta}\right)\sqrt{\frac{\beta c_1}{r-c_1-c_2}} + (1-\xi)\,\theta\sqrt{\frac{\beta c_2}{r-c_1-c_2}} > 0$. Thus, there exists $\bar{r} > \frac{c_1+c_2-\xi c_2}{1-\xi}$ such that $\mu_2^o > \mu_2^b$ if and only if $r < \bar{r}$. Then, we can define $m_\mu = \frac{c_1+c_2}{\bar{r}}$ and conclude the result. Because $(1-\theta)\lambda_s^o + \theta\lambda_o^o > \lambda^b$, $w_2^o < w_2^b$ implies $\mu_2^o > \mu_2^b$. Thus, we must have $m_\mu < m_w$. □

Proof of Proposition 27:

$$\begin{aligned}k_1^o+k_2^o-k_1^b-k_2^b = &-\theta\alpha\tau_1+\sqrt{\tfrac{\beta c_1}{r-c_1-c_2}}\left(\tau_1+\tau_2\right)\left(1-\sqrt{1-\theta}\right)\\&-\left(\tau_1+\tau_2\right)\left[\left(\left(1-\theta\right)\beta+\theta\xi\beta\right)\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}-\beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}}\right]\\&-\sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}}\tau_1\left(1-\sqrt{1-\theta}\right)\\&+\tau_2\left(\sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}}-\sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}\right)\\&+\tau_1\theta\xi\beta\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}\end{aligned}\tag{A.25}$$

Let's first show the 3rd term in (A.25), i.e.,
$-\left(\tau_1+\tau_2\right)\left[\left(\left(1-\theta\right)\beta+\theta\xi\beta\right)\sqrt{\tfrac{c_2}{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}-\beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}}\right]$ (denoted as $f_3(r)$) is decreasing in r:

$$\frac{\partial f_3}{\partial r}=-\left(\tau_1+\tau_2\right)\left[-\frac{\sqrt{c_2}}{2}\frac{\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\left(r-c_2-\frac{(1-\theta)\beta c_1}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{3}{2}}}+\frac{\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}}\right]$$

Since $\frac{\partial[(1-\theta)\beta+\theta\xi\beta]}{\partial\theta}\leq 0$ and $\frac{\partial\left[r-c_2-\frac{(1-\theta)\beta c_1}{(1-\theta)\beta+\theta\xi\beta}\right]}{\partial\theta}>0$, we can find that $\frac{\partial f_3}{\partial r}$ is decreasing in θ. Note when $\theta=0$, $\frac{\partial f_3}{\partial r}=0$. Thus, $\frac{\partial f_3}{\partial r}<0$ for all $\theta>0$.

Next, let's show the 5th term in (A.25), i.e., $\tau_2\left(\sqrt{\tfrac{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}{c_2}}-\sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}\right)$ (denoted as $f_5(r)$) is decreasing in r: Note, $\frac{\partial\sqrt{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}{\partial r}=\frac{\sqrt{(1-\theta)\beta+\theta\xi\beta}}{2\sqrt{r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}}$. Since $\frac{\partial\sqrt{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta}\leq 0$ and $\frac{\partial\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}}{\partial\theta}<0$, we find $\frac{\partial\sqrt{(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)}}{\partial r}$ is decreasing in θ, and thus $\frac{\partial f_5}{\partial r}$ is decreasing in θ. Note when $\theta=0$, we have $\frac{\partial f_5}{\partial r}=0$. Thus, $\frac{\partial f_5}{\partial r}<0$ for all $\theta>0$.

Therefore, we can conclude that $\frac{\partial(k_1^o+k_2^o-k_1^b-k_2^b)}{\partial r}<0$. Note, if $r\to c_1+c_2$, we have $(k_1^o+k_2^o-k_1^b-k_2^b)\sqrt{r-c_1-c_2}\to(\tau_1+\tau_2)\sqrt{\beta c_2}+\sqrt{\beta c_1}(\tau_1+\tau_2)(1-\sqrt{(1-\theta)})>0$, which implies $k_1^o+k_2^o-k_1^b-k_2^b>0$ if r is very close to c_1+c_2. Thus, there exists $\bar{r}>c_1+c_2$ such that $k_1^o+k_2^o-k_1^b-k_2^b>0$ if and only if $r<\bar{r}$. Then, we can define $m_k=\frac{c_1+c_2}{\bar{r}}$. Since to have $k_1^o+k_2^o-k_1^b-k_2^b>0$, we must have $\mu_2^o>\mu_2^b$, this implies $m_k>m_\mu$. $\square$

Proof of Proposition 28: The retailer has three options to choose from:

1. Serve only tech savvy customers (i.e., by setting $\mu_{1h} = 0$), and the optimization problem is

$$\begin{aligned} &\max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}}\\ 0\le\eta\lambda_m\le\mu_2\le\frac{r\alpha}{c_2}}} r\eta\lambda_m - c_{1m}\mu_{1m} - c_2\mu_2 \\ &s.t. \lambda_m = \alpha - \beta\frac{1}{\mu_{1m}-\eta\lambda_m} - \beta\frac{1}{\mu_2-\eta\lambda_m} \end{aligned}$$

   if we ignore the boundary conditions, we can derive the unique optimal solution as

$$\begin{aligned} \tilde{\lambda}_m^* &= \alpha - \beta\sqrt{\frac{c_{1m}}{\eta\beta(r-c_{1m}-c_2)}} - \beta\sqrt{\frac{c_2}{\eta\beta(r-c_{1m}-c_2)}} \\ \tilde{\mu}_{1m}^* &= \eta\tilde{\lambda}_m^* + \sqrt{\frac{\eta\beta(r-c_{1m}-c_2)}{c_{1m}}} \\ \tilde{\mu}_2^* &= \eta\tilde{\lambda}_m^* + \sqrt{\frac{\eta\beta(r-c_{1m}-c_2)}{c_2}} \end{aligned} \tag{A.26}$$

   and the corresponding optimal value is $\tilde{\pi}^* = (r-c_{1m}-c_2)\eta\alpha-2\sqrt{\eta\beta(r-c_{1m}-c_2)c_{1m}}-2\sqrt{\eta\beta(r-c_{1m}-c_2)c_2}$. So, if $(\tilde{\lambda}_m^*, \tilde{\mu}_{1m}^*, \tilde{\mu}_2^*)$ is an interior solution, then it must be the optimal solution to (A.26); otherwise, the solution to (A.26) must be corner solution, in which case we can check the optimal value is clearly nonpositive, and thus the firm never chooses this option.

2. Serve only traditional customers (i.e., by setting $\mu_{1m} = 0$), and the optimization problem is

$$\begin{aligned} &\max_{\substack{0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1}\\ 0\le(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}} r\left(1-\eta\right)\lambda_h - c_{1h}\mu_{1h} - c_2\mu_2 \\ &s.t.\ \lambda_h = \alpha - \beta\frac{1}{\mu_{1h}-(1-\eta)\lambda_h} - \beta\frac{1}{\mu_2-(1-\eta)\lambda_h} \end{aligned} \tag{A.27}$$

   if we ignore the boundary conditions, we can derive the unique optimal solution as

$$\begin{aligned} \hat{\lambda}_h^* &= \alpha - \beta\sqrt{\frac{c_1}{(1-\eta)\beta(r-c_1-c_2)}} - \beta\sqrt{\frac{c_2}{(1-\eta)\beta(r-c_1-c_2)}} \\ \hat{\mu}_1^* &= (1-\eta)\hat{\lambda}_m^* + \sqrt{\frac{(1-\eta)\beta(r-c_1-c_2)}{c_1}} \\ \hat{\mu}_2^* &= (1-\eta)\hat{\lambda}_m^* + \sqrt{\frac{(1-\eta)\beta(r-c_1-c_2)}{c_2}} \end{aligned}$$

   and the corresponding optimal value is $\hat{\pi}^* = (r-c_1-c_2)(1-\eta)\alpha$

$-2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1} - 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_2}$. So, if $(\hat{\lambda}_h^*, \hat{\mu}_1^*, \hat{\mu}_2^*)$ is an interior solution, then it must be the optimal solution to (A.27); otherwise, the solution to (A.27) must be corner solution, in which case we can check the optimal value is clearly nonpositive, and thus the firm never chooses this option.

3. Serve both types of customers. In this case, the optimal solution is obtained by solving the following optimization problem:

$$\begin{aligned}
&\max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1},\\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}} \quad r\left(\eta\lambda_m+(1-\eta)\lambda_h\right)-c_{1m}\mu_{1m}-c_1\mu_{1h}-c_2\mu_2 \\
&s.t. \quad \lambda_m=\alpha-\beta w_{1m}\left(\mu_{1m},\eta\lambda_m\right)-\beta w_2\left(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h\right) \\
&\qquad\; \lambda_h=\alpha-\beta w_{1h}\left(\mu_{1h},(1-\eta)\lambda_h\right)-\beta w_2\left(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h\right)
\end{aligned} \tag{A.28}$$

Since the constraint set is compact and the objective function is continuous, by Weierstrass Theorem, there is a maximum to (A.28).

The Lagrangian of (A.28) is defined as follows:

$$\begin{aligned}
L(\lambda_m,\lambda_h,\mu_{1m},\mu_{1h},\mu_2,\rho_m,\rho_h) = {} & r\left(\eta\lambda_m+(1-\eta)\lambda_h\right)-c_{1m}\mu_{1m}-c_1\mu_{1h}-c_2\mu_2 \\
& -\rho_m\left(\lambda_m-\alpha+\beta\frac{1}{\mu_{1m}-\eta\lambda_m}+\beta\frac{1}{\mu_2-\eta\lambda_m-(1-\eta)\lambda_h}\right) \\
& -\rho_h\left(\lambda_h-\alpha+\beta\frac{1}{\mu_{1h}-(1-\eta)\lambda_h}+\beta\frac{1}{\mu_2-\eta\lambda_m-(1-\eta)\lambda_h}\right)
\end{aligned}$$

where $\rho_m, \rho_h \in \mathbb{R}$ are the Lagrange multiplier. To find the critical points of $L(\lambda_m, \lambda_h, \mu_{1m}, \mu_{1h}, \mu_2, \rho_m, \rho_h)$, we solve the following equation set:

$$\begin{aligned}
&\frac{\partial L}{\partial \lambda_m} = r\eta-\rho_m-\frac{\rho_m\beta\eta}{(\mu_{1m}-\eta\lambda_m)^2}-\frac{(\rho_m+\rho_h)\beta\eta}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0 \\
&\frac{\partial L}{\partial \lambda_h} = r\left(1-\eta\right)-\rho_h-\frac{\rho_h\beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^2}-\frac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0 \\
&\frac{\partial L}{\partial \mu_{1m}} = -c_{1m}+\frac{\rho_m\beta}{(\mu_{1m}-\eta\lambda_m)^2}=0 \\
&\frac{\partial L}{\partial \mu_{1h}} = -c_1+\frac{\rho_h\beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^2}=0 \\
&\frac{\partial L}{\partial \mu_2} = -c_2+\frac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^2}=0 \\
&0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}}, 0\le\left(1-\eta\right)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1}, \eta\lambda_m+\left(1-\eta\right)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}
\end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.28) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Let's first ignore the boundary conditions, and solve the equation set above, which gives us a unique solution:

$$\begin{aligned}
\lambda_m^* &= \alpha - \beta\sqrt{\tfrac{c_{1m}}{\beta\eta(r-c_{1m}-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}\\
\lambda_h^* &= \alpha - \beta\sqrt{\tfrac{c_1}{\beta(1-\eta)(r-c_1-c_2)}} - \beta\sqrt{\tfrac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}\\
\mu_{1m}^* &= \eta\lambda_m^* + \sqrt{\tfrac{\beta\eta(r-c_{1m}-c_2)}{c_{1m}}}\\
\mu_{1h}^* &= (1-\eta)\,\lambda_h^* + \sqrt{\tfrac{\beta(1-\eta)(r-c_1-c_2)}{c_1}}\\
\mu_2^* &= \eta\lambda_m^* + (1-\eta)\,\lambda_h^* + \sqrt{\tfrac{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}{c_2}}
\end{aligned}$$

Note it is easy to check that if $\lambda_m^* \geq 0$ and $\lambda_h^* \geq 0$, then $\mu_i \leq \frac{r\alpha}{c_i}$. Thus, if $\lambda_m^* \geq 0$ and $\lambda_h^* \geq 0$ (or α is large), then $(\lambda_m^*, \lambda_h^*, \mu_{1m}^*, \mu_{1h}^*, \mu_2^*)$ is the optimal solution to (A.28), and the corresponding optimal value is $\pi^* = (r-c_{1m}-c_2)\eta\alpha + (r-c_1-c_2)(1-\eta)\alpha -$ $2\sqrt{\eta\beta c_{1m}(r-c_{1m}-c_2)} - 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1}$ $-\, 2\sqrt{\beta c_2[(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)]}$.

So, if $(\lambda_m^*, \lambda_h^*, \mu_{1m}^*, \mu_{1h}^*, \mu_2^*)$ is an interior solution, then it must be the optimal solution to (A.28); otherwise, the solution to (A.28) must be corner solution, in which case we can check the optimal value is either nonpositive or strictly less than that under the first two options, and thus the firm never chooses this option.

Note $\pi^* - \tilde{\pi}^*$ and $\pi^* - \hat{\pi}^*$ are linearly increasing in α, and thus if and only if α is large enough, we have $\pi^* > \max(\tilde{\pi}^*, \hat{\pi}^*)$, and thus it is optimal for the firm to serve both types of customers.

□

Proof of Lemma 1: From the optimal solution given in Proposition 28, we know $w_{1h}^s = \sqrt{\frac{c_1}{\beta(1-\eta)(r-c_1-c_2)}}$, $w_{1m}^s = \sqrt{\frac{c_{1m}}{\beta\eta(r-c_{1m}-c_2)}}$, and $w_2^s = \sqrt{\frac{c_2}{\beta[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}$.

Then, $\eta w_{1m}^s + (1-\eta) w_{1h}^s = \sqrt{\frac{\eta c_{1m}}{\beta(r-c_{1m}-c_2)}} + \sqrt{\frac{(1-\eta)c_1}{\beta(r-c_1-c_2)}}$ which is increasing in c_{1m}. Also, when $c_{1m}=0$, $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$. Thus, there exists $\bar{c}_{1m} > 0$ such that $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$ if and only if $c_{1m} < \bar{c}_{1m}$. □

Proof of Proposition 29: Because $\eta < 1$, we can find $w_{1h}^s > w_1^b$. Since $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, we must have $w_{1m}^s < w_1^b$. Also, since $c_{1m} < c_1$, we have $w_2^s < w_2^b$. □

Proof of Proposition 30: First, because $w_{1m}^s < w_1^b$ and $w_2^s < w_2^b$, we find $\lambda_m^s = \alpha - \beta w_{1m}^s - \beta w_2^s > \alpha - \beta w_1^b - \beta w_2^b = \lambda^b$.

Second, note

$$\frac{\left(\lambda_h^s - \lambda^b\right)\sqrt{r-c_1-c_2}}{\beta} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{\beta\left(1-\eta\right)}} - \sqrt{\frac{c_2}{\beta\eta\frac{r-c_{1m}-c_2}{r-c_1-c_2} + \beta\left(1-\eta\right)}}$$

which is decreasing in r. Then, if $\lim_{r\to c_1+c_2} \frac{\left(\lambda_h^s-\lambda^b\right)\sqrt{r-c_1-c_2}}{\beta} = \sqrt{\frac{c_1}{\beta}} + \sqrt{\frac{c_2}{\beta}} - \sqrt{\frac{c_1}{\beta(1-\eta)}} > 0$ (i.e., η is small enough), then there exists $\bar{r} > c_1 + c_2$ such that $\lambda_h^s > \lambda^b$ if and only if $r < \bar{r}$ (or $\frac{c_1+c_2}{r} < \frac{c_1+c_2}{\bar{r}} = m'_\lambda$).

Finally, note

$$\begin{aligned}
\eta\lambda_m^s + \left(1-\eta\right)\lambda_h^s = \alpha &- \sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{\left(1-\eta\right)\beta c_1}{r-c_1-c_2}} \\
&- \sqrt{\frac{\beta c_2}{\left(r-c_{1m}-c_2\right)\eta + \left(r-c_1-c_2\right)\left(1-\eta\right)}} \\
> \alpha &- \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{\left(r-c_1-c_2\right)}} \\
= \lambda^b &
\end{aligned}$$

where the inequality is due to $c_{1m} < c_1$ and Assumption 1, which implies $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$ (by Lemma 1) or $\sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$). □

Proof of Proposition 31: Note $\mu_{1h}^s = \sqrt{1-\eta}\Bigg(\sqrt{1-\eta}\alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$
$- \sqrt{\frac{\beta c_2}{(r-c_1-c_2)+\eta/(1-\eta)(r-c_{1m}-c_2)}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}\Bigg)$, which is decreasing in η. Since when $\eta = 0$ we have $\mu_{1h}^s = \mu_1^b$, we can conclude $\mu_{1h}^s < \mu_1^b$ for any $\eta > 0$. Also, note

$$\begin{aligned}\mu_{1m}^s + \mu_{1h}^s =& \alpha - \sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} \\ & - \sqrt{\frac{\beta c_2}{(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)}} \\ & + \sqrt{\frac{\beta\eta(r-c_{1m}-c_2)}{c_{1m}}} + \sqrt{\frac{\beta(1-\eta)(r-c_1-c_2)}{c_1}}\end{aligned}$$

Because of Assumption 1 and Lemma 1, we have $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$, and thus $\sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$. Because $c_{1m} < c_1$, we have $\sqrt{\frac{\beta c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}} < \sqrt{\frac{\beta c_2}{r-c_1-c_2}}$. Also, define function $f(\eta) = \sqrt{\frac{\beta\eta(r-c_{1m}-c_2)}{c_{1m}}} + \sqrt{\frac{\beta(1-\eta)(r-c_1-c_2)}{c_1}}$. We can easily check that f is a concave function. Note $f(0) = \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$ and $f(1) > \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$. This implies that $f(\eta) > \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}$ for any $\eta \in (0,1]$. Based on the results above, we can conclude that $\mu_{1m}^s + \mu_{1h}^s > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} - \sqrt{\frac{\beta c_2}{r-c_1-c_2}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}} = \mu_1^s$.

Second, note

$$\begin{aligned}\mu_2^s =& \alpha - \sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} \\ & - \sqrt{\frac{\beta c_2}{(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)}} \\ & + \sqrt{\frac{\beta\left[(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)\right]}{c_2}}\end{aligned}$$

Because of Assumption 1 and Lemma 1, we have $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$, and thus $\sqrt{\frac{\eta\beta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)\beta c_1}{r-c_1-c_2}} < \sqrt{\frac{\beta c_1}{r-c_1-c_2}}$. Also, define function
$g(\eta) = -\sqrt{\frac{\beta c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}} + \sqrt{\frac{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}{c_2}}$. We can easily check that g is an increasing function. Thus, for any $\eta \in (0,1]$, $g(\eta) > g(0) = \sqrt{\frac{\beta c_2}{r-c_1-c_2}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}$. Based on the results above, we can conclude that $\mu_2^s > \alpha - \sqrt{\frac{\beta c_1}{r-c_1-c_2}} -$

$\sqrt{\frac{\beta c_2}{r-c_1-c_2}} + \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}} = \mu_2^s.$ □

Proof of Proposition 32: Note

$$\frac{\partial k_1^s}{\partial r} = \frac{\tau_1\sqrt{\beta\left(1-\eta\right)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\tau_1\left(1-\eta\right)\sqrt{\beta c_2}}{2[(r-c_{1m}-c_2)\,\eta+(r-c_1-c_2)\left(1-\eta\right)]^{\frac{3}{2}}} + \frac{\tau_1\sqrt{\beta\left(1-\eta\right)}}{2\sqrt{c_1\left(r-c_1-c_2\right)}}$$

which is decreasing in η. Since $k_1^s = k_1^b$ when $\eta = 0$, we can conclude that $\frac{\partial k_1^s}{\partial r} < \frac{\partial k_1^b}{\partial r}$ as $\eta > 0$.

Note

$$\begin{aligned}\frac{\partial k_2^s}{\partial r} =& \frac{\sqrt{\eta\beta c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta\left(1-\eta\right)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} \\ &+ \frac{\sqrt{\beta c_2}}{2[(r-c_{1m}-c_2)\,\eta+(r-c_1-c_2)\left(1-\eta\right)]^{\frac{3}{2}}} \\ &+ \frac{\sqrt{\beta}}{2\sqrt{c_2\left[(r-c_{1m}-c_2)\,\eta+(r-c_1-c_2)\left(1-\eta\right)\right]}}\end{aligned}$$

Because of $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$ (by Assumption 1 and Lemma 1) and $c_{1m} < c_1$, we have $\frac{\sqrt{\eta\beta c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} < \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}}$. Also, define function $f\left(\eta\right) = \frac{\sqrt{\beta c_2}}{2[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\sqrt{\beta}}{2\sqrt{c_2[(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)]}}$, which is a decreasing function. Thus, for any $\eta > 0$, $f(\eta) < f(0)$. Based on the results above, we can conclude that $\frac{\partial k_2^s}{\partial r} < \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta}}{2\sqrt{c_2(r-c_1-c_2)}} = \frac{\partial k_2^b}{\partial r}$.

Because $\frac{\partial k_2^s}{\partial r} < \frac{\partial k_2^b}{\partial r}$ and $\frac{\partial k_1^s}{\partial r} < \frac{\partial k_1^b}{\partial r}$, we can conclude that $\frac{\partial(k_1^s+k_2^s)}{\partial r} < \frac{\partial(k_1^b+k_2^b)}{\partial r}$. Note when $r \to c_1 + c_2$, then $k_1^s + k_2^s - k_1^b - k_2^b > 0$. Thus, there exists $\bar{r} > c_1 + c_2$ such that $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if and only if $r < \bar{r}$. Then, define $m_k' = (c_1 + c_2)/\bar{r}$ and we get the result. □

Proof of Proposition 33: Based on the optimal solutions, we have

- $\pi^b = r\lambda^b - c_1\mu_1^b - c_2\mu_2^b = (r-c_1-c_2)\alpha - 2\sqrt{\beta(r-c_1-c_2)c_1} - 2\sqrt{\beta(r-c_1-c_2)c_2}$

- $\pi^o = r((1-\theta)\lambda_1^o+\theta\lambda_2^o)-c_1\mu_1^o-c_2\mu_2^o = [r-c_2-c_1(1-\theta)]\alpha-2\sqrt{(1-\theta)\beta(r-c_1-c_2)c_1}-2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}$

- $\pi^s = r\left((1-\eta)\lambda_h^s+\eta\lambda_m^s\right)-c_{1m}\mu_{1m}^s-c_1\mu_1^s-c_2\mu_2^s = (r-c_{1m}-c_2)\,\eta\alpha + (r-c_1-c_2)(1-\eta)\,\alpha-2\sqrt{\eta\beta c_{1m}(r-c_{1m}-c_2)}-2\sqrt{(1-\eta)\,\beta\,(r-c_1-c_2)\,c_1} - 2\sqrt{\beta c_2\left[(r-c_{1m}-c_2)\,\eta+(r-c_1-c_2)(1-\eta)\right]}$

Let's first compare π^o and π^b.

Suppose we keep using μ_1^b and μ_2^b in the case of online self-order technology. Then, we want to show that total demand rate $\lambda > \lambda^b$. Suppose not, i.e., $\lambda \le \lambda^b$. Then, we have $w_2 = 1/(\mu_2^b-\lambda) \le w_2^b$. Because $\xi \le 1$, we have $\lambda_o = \alpha-\xi\beta w_2 > \lambda_o^b$. This means that $\lambda_s < \lambda_s^b$ (otherwise we cannot have $\lambda = (1-\theta)\lambda_s+\theta\lambda_o \le \lambda^b$). Then, $w_1 = 1/(\mu_1^b-(1-\theta)\lambda_s) < w_1^b$. This implies that $\lambda_s = \alpha-\beta w_1-\beta w_2 > \lambda_s^b$. This contradicts the result $\lambda_s < \lambda_s^b$ that we just obtained. So we must have $\lambda > \lambda^b$. Then, with this feasible solution μ_1^b and μ_2^b with online self-order technology, the total profit is $r\lambda-c_1\mu_1^b-c_2\mu_2^b > \pi^b$. Therefore, the optimal profit $\pi^o > \pi^b$.

Next, we look at how $\pi^o-\pi^b$ changes as different model parameters change.

(1) $\frac{\partial\left(\pi^o-\pi^b\right)}{\partial r} = \sqrt{\frac{c_1\beta}{r-c_1-c_2}}+\sqrt{\frac{c_2\beta}{r-c_1-c_2}}-\sqrt{\frac{(1-\theta)c_1\beta}{r-c_1-c_2}}-\sqrt{\frac{(1-\theta+\theta\xi)c_2\beta}{r-\frac{1-\theta}{1-\theta+\theta\xi}c_1-c_2}} > 0$, because $\theta \le 1$ and $\xi \le 1$.

(2) $\frac{\partial\left(\pi^o-\pi^b\right)}{\partial c_2} = \mu_2^b-\mu_2^o$. Since $\frac{\partial\mu_2^o}{\partial\xi} > 0$, we have $\frac{\partial^2(\pi^o-\pi^b)}{\partial c_2\partial\xi} < 0$. Also, note that if $\xi = 1$, then $\mu_2^b < \mu_2^o$, and thus $\frac{\partial(\pi^o-\pi^b)}{\partial c_2} < 0$.

(3) Let's first prove the following lemma:

Lemma 3. *π^o is increasing in θ.*

Proof of Lemma 3: For $\theta_1, \theta_2 \in [0,1]$, suppose $\theta_1 < \theta_2$. Denote the optimal solution when $\theta = \theta_1$ as $\mu_1^\triangle$, $\mu_2^\triangle$, $\lambda_s^\triangle$, $\lambda_o^\triangle$. The corresponding total demand and profit are $\lambda^\triangle = (1-\theta_1)\lambda_s^\triangle+\theta_1\lambda_o^\triangle$ and $\pi^\triangle$. It is easy to check that $\lambda_s^\triangle < \lambda_o^\triangle$.

When $\theta = \theta_2$, suppose we keep using $\mu_1^{\triangle}$ and $\mu_2^{\triangle}$. Then, under this feasible solution, we want to show that total demand rate $\lambda > \lambda^{\triangle}$. Suppose not, i.e., $\lambda \leq \lambda^{\triangle}$. Then, $w_2 = \frac{1}{\mu_2^{\triangle} - \lambda} \leq \frac{1}{\mu_2^{\triangle} - \lambda^{\triangle}} = w_2^{\triangle}$, which implies that $\lambda_o \geq \lambda_o^{\triangle}$. Since $\lambda \leq \lambda^{\triangle}$, $\theta_1 < \theta_2$ and $\lambda_o \geq \lambda_o^{\triangle}$, we can find $\lambda_s \leq \lambda_s^{\triangle}$. Then, $w_1 = \frac{1}{\mu_1^{\triangle} - (1-\theta_2)\lambda_s} < \frac{1}{\mu_1^{\triangle} - (1-\theta_1)\lambda_s^{\triangle}} = w_1^{\triangle}$. However, if $w_2 < w_2^{\triangle}$ and $w_1 \leq w_1^{\triangle}$, then we must have $\lambda_s > \lambda_s^{\triangle}$, which contradicts to what we just found (i.e., $\lambda_s \leq \lambda_s^{\triangle}$). Thus, we must have $\lambda > \lambda^{\triangle}$. This implies that when $\theta = \theta_2$, with the feasible solution $\mu_1^{\triangle}$ and $\mu_2^{\triangle}$, we have more demand and thus more profit. Therefore, the optimal profit when $\theta = \theta_2$, denoted as $\pi^{\triangle\triangle}$, must be greater than $\pi^{\triangle}$. This completes the proof. $\square$

According to Lemma 3, we have $\frac{\partial \pi^o}{\partial \theta} > 0$. Therefore, $\frac{\partial(\pi^o - \pi^s)}{\partial \theta} > 0$.

(4) It is easy to find that $\frac{\partial \pi^o}{\partial \xi} < 0$. Thus, $\frac{\partial(\pi^o - \pi^b)}{\partial \xi} < 0$.

Nest, let's compare π^s and π^b.

Note

$$\begin{aligned}
\frac{\partial\left(\pi^s - \pi^b\right)}{\partial \eta} &= \left(c_1 - c_{1m}\right)\alpha - \sqrt{\frac{\beta c_{1m}\left(r - c_{1m} - c_2\right)}{\eta}} + \sqrt{\frac{\beta c_1\left(r - c_1 - c_2\right)}{1-\eta}} \\
&\quad - \frac{\sqrt{\beta c_2}\left(c_1 - c_{1m}\right)}{\sqrt{\left(r - c_{1m} - c_2\right)\eta + \left(r - c_1 - c_2\right)(1-\eta)}} \\
&> \left(c_1 - c_{1m}\right)\sqrt{\frac{\beta c_{1m}}{\eta\left(r - c_{1m} - c_2\right)}} - \sqrt{\frac{\beta c_{1m}\left(r - c_{1m} - c_2\right)}{\eta}} + \sqrt{\frac{\beta c_1\left(r - c_1 - c_2\right)}{1-\eta}} \\
&= \sqrt{\frac{r - c_1 - c_2}{r - c_{1m} - c_2}}\sqrt{c_1 c_{1m}\beta}\left(\sqrt{\frac{r - c_{1m} - c_2}{(1-\eta)c_{1m}}} - \sqrt{\frac{r - c_1 - c_2}{\eta c_1}}\right) \\
&> 0
\end{aligned}$$

where the first inequality is because of $\lambda_m^s > 0$ (and thus $\alpha > \beta\sqrt{\frac{c_{1m}}{\beta\eta(r - c_{1m} - c_2)}} + \beta\sqrt{\frac{c_2}{\beta[(r - c_{1m} - c_2)\eta + (r - c_1 - c_2)(1-\eta)]}}$), and the second inequality is because $w_{1m}^s < w_{1h}^s$ (and thus $\sqrt{\frac{r - c_{1m} - c_2}{(1-\eta)c_{1m}}} - \sqrt{\frac{r - c_1 - c_2}{\eta c_1}} > 0$).

Next, we look at how $\pi^s - \pi^b$ changes as different model parameters change.

(1) Note

$$\frac{\partial\left(\pi^s-\pi^b\right)}{\partial r} = \sqrt{\frac{c_1}{r-c_1-c_2}} + \sqrt{\frac{c_2}{r-c_1-c_2}} - \sqrt{\frac{\eta c_{1m}}{r-c_{1m}-c_2}} - \sqrt{\frac{(1-\eta)c_1}{r-c_1-c_2}}$$
$$-\sqrt{\frac{c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}}$$

Because of Assumption 1 and Lemma 1, we have $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, and thus $\sqrt{\frac{\eta c_{1m}}{r-c_{1m}-c_2}} + \sqrt{\frac{(1-\eta)c_1}{r-c_1-c_2}} < \sqrt{\frac{c_1}{r-c_1-c_2}}$. Also, because $c_{1m} < c_1$, we have $\sqrt{\frac{c_2}{r-c_1-c_2}} > \sqrt{\frac{c_2}{(r-c_{1m}-c_2)\eta+(r-c_1-c_2)(1-\eta)}}$. Thus, $\frac{\partial\left(\pi^s-\pi^b\right)}{\partial r} > 0$.

(2) $\frac{\partial(\pi^s-\pi^b)}{\partial c_2} = \mu_2^b - \mu_2^s < 0$.

(3) Let's first prove the following lemma:

Lemma 4. π^s *is increasing in* η.

Proof of Lemma 4: For $\eta_1, \eta_2 \in [0,1]$, suppose $\eta_1 < \eta_2$. Denote the optimal solution when $\eta = \eta_1$ as $\mu_{1m}^*,\ \mu_{1h}^*,\ \mu_2^*,\ \lambda_s^*,\ \lambda_o^*$. The corresponding total demand and profit are $\lambda^* = (1-\eta_1)\lambda_h^* + \eta_1\lambda_m^*$ and π^*. Since $w_{1m}^* < w_{1h}^*$, we have $\lambda_m^* > \lambda_h^*$.

When $\eta = \eta_2$, consider the following feasible solution: $\lambda_m^\triangle = \lambda_m^*,\ \lambda_h^\triangle = \lambda_h^*,\ \mu_{1m}^\triangle = \mu_{1m}^* + (\eta_2-\eta_1)\lambda_m^*,\ \mu_{1h}^\triangle = \mu_{1h}^* - (\eta_2-\eta_1)\lambda_h^*,\ \mu_2^\triangle = \mu_2^* + (\eta_2-\eta_1)(\lambda_m^*-\lambda_h^*)$. Then, the corresponding profit $\pi^\triangle = \pi^* + r(\lambda_m^*-\lambda_h^*) - c_{1m}(\eta_2-\eta_1)\lambda_m^* + c_1(\eta_2-\eta_1)\lambda_h^* - c_2(\eta_2-\eta_1)(\lambda_m^*-\lambda_h^*) > \pi^* + (r-c_1-c_2)(\eta_2-\eta_1)(\lambda_m^*-\lambda_h^*) > 0$, where the first inequality is due to $c_{1m} < c_1$. Therefore, the optimal profit with $\eta = \eta_2$ must be greater than π^*. This concludes the proof. $\square$

By Lemma 4, we find $\frac{\partial(\pi^s-\pi^b)}{\partial \eta} > 0$.

(4) It is easy to find that $\frac{\partial \pi^s}{\partial c_{1m}} < 0$. Thus, $\frac{\partial(\pi^s-\pi^b)}{\partial c_{1m}} < 0$.

$\square$

Proof of Proposition 34:

$$\begin{aligned}\pi^o - \pi^s =& - c_1 (1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha \\ &+ 2\sqrt{\eta\beta(r-c_{1m}-c_2)c_{1m}} + 2\sqrt{(1-\eta)\beta(r-c_1-c_2)c_1} \\ &+ 2\sqrt{\beta c_2[(r-c_{1m}-c_2)\eta + (r-c_1-c_2)(1-\eta)]} \\ &- 2\sqrt{(1-\theta)\beta(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta(r-c_1-c_2)+\theta\xi\beta(r-c_2)]c_2}\end{aligned}$$

Note that $\frac{\partial^2(\pi^o-\pi^s)}{\partial\eta^2} < 0$. When $\eta = 0$, $\pi^s = \pi^b < \pi^o$. Define $\bar{\eta}$ such that $c_1(1-\theta) = c_{1m}\bar{\eta} + c_1(1-\bar{\eta})$. When $\eta = \bar{\eta}$, then we can check that $\pi^o - \pi^s > 0$. Thus, for $\eta \in (0, \bar{\eta}]$, we have $\pi^o - \pi^s > 0$, or $\bar{\beta} = 0$. [It is easy to check that $\bar{\eta} > \theta$. Thus, if $\eta \in (0, \theta]$, we must have $\pi^o > \pi^s$ or $\bar{\beta} = 0$.]

If $\eta > \bar{\eta}$, we have $-c_1(1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha < 0$. Then, $\frac{\partial\frac{\pi^o-\pi^s}{\sqrt{\beta}}}{\partial\beta} = \frac{\partial\frac{-c_1(1-\theta)\alpha+[c_{1m}\eta+c_1(1-\eta)]\alpha}{\sqrt{\beta}}}{\partial\beta}$ > 0. Thus, there exists $\bar{\beta} \geq 0$ such that $\pi^o - \pi^s > 0$ if and only if $\beta > \bar{\beta}$.

$\square$

Proof of Proposition 36: In the decentralized system, for any θ, denote optimal stock levels in the store as $\tilde{q}^*(\theta)$. Suppose we have the same stock level in the centralized system. Then clearly we will achieve exactly the same total profit as $\tilde{\pi}^*(\theta)$, since the BOPS revenue is just shared between channels in the decentralized case. Thus, we must have $\tilde{\pi}^*(\theta) \leq \pi^*$. And due to the fact that π is strictly concave in q and $q^* > 0$ (because of the assumptions that $p - c > w\frac{\alpha^*_{so}+\alpha^*_{bo}}{\alpha^*_s+\alpha^*_b+\alpha^*_{so}+\alpha^*_{bo}} - r$), we have $\tilde{\pi}^*(\theta) = \pi^*$ if only if $\tilde{q}^* = q^*$.

Next, let's compare the optimal store inventory levels in both the centralized and decentralized systems:

- If $\frac{\alpha^*_b+\alpha^*_{bo}}{\alpha^*_{so}+\alpha^*_{bo}} < \frac{w}{p}$, then $\forall\theta \in [0,1]$, we have $\left(\frac{\alpha^*_s+\alpha^*_{so}}{\alpha^*_s+\alpha^*_b+\alpha^*_{so}+\alpha^*_{bo}} + \frac{\alpha^*_b+\alpha^*_{bo}}{\alpha^*_s+\alpha^*_b+\alpha^*_{so}+\alpha^*_{bo}}\theta\right)p + r > p + r - w\frac{\alpha^*_{so}+\alpha^*_{bo}}{\alpha^*_s+\alpha^*_b+\alpha^*_{so}+\alpha^*_{bo}}$. Therefore, $\tilde{q}^* > q^*$. Since $\tilde{q}^* \neq q^*$, $\tilde{\pi}^* < \pi^*$.

- If $\frac{\alpha^*_b+\alpha^*_{bo}}{\alpha^*_{so}+\alpha^*_{bo}} \geq \frac{w}{p}$, then

– if $\theta < \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, then $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}+\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r}\right) < \bar{F}^{-1}\left(\frac{c}{p+r-w\frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}}\right) = q^*$. Since $\tilde{q}^* \neq q^*$, $\tilde{\pi}^* < \pi^*$.

– if $\theta = \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, then $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}+\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r}\right) = \bar{F}^{-1}\left(\frac{c}{p+r-w\frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}}\right) = q^*$. Since $\tilde{q}^* = q^*$, $\tilde{\pi}^* = \pi^*$.

– if $\theta > \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, then $\tilde{q}^* = \bar{F}^{-1}\left(\frac{c}{\left(\frac{\alpha_s^*+\alpha_{so}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}+\frac{\alpha_b^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\theta\right)p+r}\right) > \bar{F}^{-1}\left(\frac{c}{p+r-w\frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}}\right) = q^*$. Since $\tilde{q}^* \neq q^*$, $\tilde{\pi}^* < \pi^*$.

Finally, let's prove $\tilde{\pi}^*$ is quasiconcave in θ.

Let's first show $\tilde{\pi}^*$ is nondecreasing in $\theta < \frac{p-w}{p}$. If $\theta \leq \frac{(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{so}^*)(c-r)-(\alpha_s^*+\alpha_{so}^*)p}{(\alpha_b^*+\alpha_{bo}^*)p}$, then $\tilde{q}^* = 0$ and thus $\tilde{\pi}^* = wE(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*)D$, which is independent of θ. If $\frac{(\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{so}^*)(c-r)-(\alpha_s^*+\alpha_{so}^*)p}{(\alpha_b^*+\alpha_{bo}^*)p} < \theta < \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$, then $\frac{\partial\tilde{\pi}^*}{\partial\theta} = \frac{\partial\tilde{\pi}^*}{\partial q}\frac{\partial\tilde{q}^*}{\partial\theta} > 0$, because $\frac{\partial\tilde{\pi}^*}{\partial q} > 0$ if $q < \tilde{q}^*$ and $\frac{\partial\tilde{q}^*}{\partial\theta} > 0$.

Then, let's show that $\tilde{\pi}^*$ is nonincreasing in $\theta > \frac{(\alpha_b^*+\alpha_{bo}^*)p-(\alpha_{so}^*+\alpha_{bo}^*)w}{(\alpha_b^*+\alpha_{bo}^*)p}$. Note $\frac{\partial\tilde{\pi}^*}{\partial\theta} = \frac{\partial\tilde{\pi}^*}{\partial q}\frac{\partial\tilde{q}^*}{\partial\theta} < 0$, because $\frac{\partial\tilde{\pi}^*}{\partial q} < 0$ if $q > \tilde{q}^*$ and $\frac{\partial\tilde{q}^*}{\partial\theta} > 0$.

Thus, we can conclude that $\tilde{\pi}^*$ is quasiconcave in θ. $\square$

Proof of Proposition 37: Note nonparticipatory equilibrium, $(0, \bar{F}^{-1}(\frac{c_o}{p}), 0, 0, 0)$, always exists. Same as what we did in the base model, we are going to look for participatory equilibrium, where $\phi^\circ = 1$ and $q_s^\circ > 0$. All we need to do is to check the four conditions specified in Definition 4.

First, we look at retailer's problem: In the participatory equilibrium, retailer's belief is consistent, i.e., $\hat{\phi} = \phi^\circ = 1$. Then, the retailer's profit is $\pi = pE\min(D, q_s) - c_s q_s + rED +$

$pE\min((D-q_s)^+,q_o)-c_oq_o = rED+pE\min(D,q_s+q_o)-c_sq_s-c_oq_o$, the optimal solution of which is easy to find as the follows

- if $c_s < c_o$, then $q_s^\circ = \bar{F}^{-1}(\frac{c_s}{p})$ and $q_o^\circ = 0$
- if $c_s \geq c_o$, then $q_s^\circ = 0$ and $q_o^\circ = \bar{F}^{-1}(\frac{c_o}{p})$.

Thus, to ensure participatory equilibrium, we need $c_s < c_o$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q_s^\circ) = E\min(\phi^\circ D, q_s^\circ)/E(\phi^\circ D) = E\min(D,\bar{F}^{-1}(\frac{c_s}{p}))/ED$.

Finally, we go back to consumer's decision. To ensure $\phi^\circ = 1$, we need $u_s \geq u_o$, i.e., $h_s \leq \hat{\xi}^\circ h_o$. Therefore, we need $h_s \leq \frac{E\min(D,\bar{F}^{-1}(\frac{c_s}{p}))}{ED}h_o$.

Based on the analysis above, we find the conditions for a participatory equilibrium are $h_s \leq \frac{E\min(D,\bar{F}^{-1}(\frac{c_s}{p}))}{ED}h_o$ and $c_s < c_o$. And the equilibrium outcome is $\phi^\circ = 1$, $q_s^\circ = \bar{F}^{-1}\left(\frac{c_s}{p}\right)$ and $q_o^\circ = 0$. $\square$

Proof of Proposition 38: First, let's look at the case where $\min(h_s,h_b) \leq h_o$. In this case, consumers come to store if it is in stock. Thus, the retailer's profit is

$$\pi = (p+r)\,E\min(D,q_s) - c_sq_s + pE\min\left((D-q_s)^+,q_o\right) - c_oq_o$$

First, by checking the Hessian matrix, we can easily find the profit function is jointly concave in (q_s,q_o). Moreover, note that $\frac{\partial\pi}{\partial q_s} = r\bar{F}(q_s)+p\bar{F}(q_s+q_o)-c_s$ and $\frac{\partial\pi}{\partial q_o} = p\bar{F}(q_s,q_o)-c_o$. Thus, by the first order condition and taking into account the constraints that $q_s \geq 0$ and $q_o \geq 0$, we can find the optimal solution that maximizes the profit function as follows:

- if $c_s \leq \frac{p+r}{p}c_o$, then $\bar{F}(q_s^*) = \frac{c_s}{p+r} \Rightarrow q_s^* = \bar{F}^{-1}(\frac{c_s}{p+r})$, and $q_o^* = 0$
- if $c_s \in (\frac{p+r}{p}c_o, c_o+r)$, then $\bar{F}(q_s^*) = \frac{c_s-c_o}{r} \Rightarrow q_s^* = \bar{F}^{-1}(\frac{c_s-c_o}{r})$, and $\bar{F}(q_s^*+q_o^*) = \frac{c_o}{p} \Rightarrow$ $q_o^* = \bar{F}^{-1}(\frac{c_o}{p}) - \bar{F}^{-1}(\frac{c_s-c_o}{r})$

- if $c_s \geq c_o + r$, then $q_s^* = 0$ and $\bar{F}(q_o^*) = \frac{c_o}{p} \Rightarrow q_o^* = \bar{F}^{-1}(\frac{c_o}{p})$, in which case customers actually never come to store because store does not have any inventory.

Then, let's look at the case where $\min(h_s, h_b) > h_o$. Then, consumers never come to store and the retailer's profit is $\pi = pE\min(D, q_o) - c_o q_o$, which is a newsvendor problem. Thus, $q_s^* = 0$ and $q_o^* = \bar{F}^{-1}(\frac{c_o}{p})$. □

Proof of Proposition 39: If customers visit the store only if BOPS is available, then

$$\pi^* = (p + r)\, E \min(D, q_s^*) - c_s q_s^* > pE \min(D, q_o^\circ) - c_o q_o^\circ = \pi^\circ$$

where the inequality is because (q_s^*, q_o^*) rather than (q_s°, q_o°) maximizes π^*. □

Proof of Proposition 40: Since consumer behavior is the same as Proposition 39, the proof is also very similar to the one for Proposition 39, and thus omitted. □

Proof of Proposition 41: If $h_s \leq \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} h_o$ and $c_s < c_o$, then $q_s^\circ = \bar{F}^{-1}(\frac{c_s}{p})$, $q_o^\circ = 0$, $q_s^* = \bar{F}^{-1}(\frac{c_s}{p+r})$ and $q_o^* = 0$. Then,

$$\begin{aligned}
\pi^\circ &= rED + pE\min(D, q_s^\circ) - c_s q_s^\circ \\
&> rED + pE\min(D, q_s^*) - c_s q_s^* \\
&> rE\min(D, q_s^*) + pE\min(D, q_s^*) - c_s q_s^* = \pi^*
\end{aligned}$$

where the first inequality is due to the fact that q_s° rather than q_s^* maximizes π°. □

Proof of Proposition 42: Note nonparticipatory equilibrium always exists. Same as what we did in the base model, we are going to look for participatory equilibrium, where $\alpha_s > 0$, $\alpha_{so} > 0$ and $q_s^\circ > 0$.

First, let's look at retailer's problem. It is easy to verify that π is jointly concave in (q_s, q_o). Also, note that with $\hat{\alpha}_i = \alpha_i^\circ$, $i = s, o, so, l$,

- if $\frac{q_s}{\alpha_s^\circ+\alpha_{so}^\circ} > \frac{q_o}{\alpha_o^\circ}$, then

$$\frac{\partial\pi}{\partial q_s} = p\bar{F}\left(\frac{q_s}{\alpha_s^\circ+\alpha_{so}^\circ}\right) - c_s$$
$$\frac{\partial\pi}{\partial q_o} = p\bar{F}\left(\frac{q_o}{\alpha_o^\circ}\right) - c_o$$

- if $\frac{q_s}{\alpha_s^\circ+\alpha_{so}^\circ} \leq \frac{q_o}{\alpha_o^\circ}$, then

$$\frac{\partial\pi}{\partial q_s} = \frac{\alpha_s^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p\bar{F}\left(\frac{q_s}{\alpha_s+\alpha_{so}^\circ}\right) + \frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p\bar{F}\left(\frac{q_o+\frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}}q_s}{\alpha_o^\circ+\alpha_{so}^\circ}\right) - c_s$$
$$\frac{\partial\pi}{\partial q_o} = p\bar{F}\left(\frac{q_o+\frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}q_s}{\alpha_o^\circ+\alpha_{so}^\circ}\right) - c_o$$

Then, given the constraints $q_s \geq 0$ and $q_o \geq 0$, we can easily get the solution as follows:

- If $c_s < c_o$, then

$$q_s^\circ = (\alpha_s^\circ + \alpha_{so}^\circ)\,\bar{F}^{-1}\left(\frac{c_s}{p}\right)$$
$$q_o^\circ = \alpha_o^\circ\bar{F}^{-1}\left(\frac{c_o}{p}\right)$$

- if $c_s \geq c_o$ and $c_s < \frac{\alpha_{so}^\circ}{\alpha_s+\alpha_{so}^\circ}c_o + \frac{\alpha_s^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p$, then

$$q_s^\circ = (\alpha_s^\circ + \alpha_{so}^\circ)\,\bar{F}^{-1}\left(\frac{c_s-\frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}c_o}{\frac{\alpha_s^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p}\right)$$
$$q_o^\circ = (\alpha_o^\circ + \alpha_{so}^\circ)\,\bar{F}^{-1}\left(\frac{c_o}{p}\right) - \alpha_{so}^\circ\bar{F}^{-1}\left(\frac{c_s-\frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}c_o}{\frac{\alpha_s^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p}\right)$$

- if $c_s \geq \frac{\alpha_{so}^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}c_o + \frac{\alpha_s^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p$, then

$$q_s^\circ = 0$$
$$q_o^\circ = (\alpha_o^\circ + \alpha_{so}^\circ)\,\bar{F}^{-1}\left(\frac{c_o}{p}\right)$$

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q_s^\circ) = E\min((\alpha_s^\circ + \alpha_{so}^\circ)D, q_s^\circ)/E((\alpha_s^\circ + \alpha_{so}^\circ)D)$. Thus, if $c_s < c_o$, then $\hat{\xi}^\circ = \xi_1^\circ$; if $c_s \geq c_o$ and $c_s < \frac{\alpha_{so}^\circ}{\alpha_s+\alpha_{so}^\circ}c_o + \frac{\alpha_s^\circ}{\alpha_s^\circ+\alpha_{so}^\circ}p$, then $\hat{\xi}^\circ = \xi_2^\circ$; otherwise, $\hat{\xi}^\circ = 0$. Here, we used the fact that for any $\hat{\xi} > 0$, we have $\frac{\alpha_{so}}{\alpha_s+\alpha_{so}} = \frac{v-p}{2H-(v-p)}$ and $\frac{\alpha_s}{\alpha_s+\alpha_{so}} = \frac{2H-2(v-p)}{2H-(v-p)}$.

Note $\hat{\xi}^\circ$ is independent of α's. Thus, to ensure equilibrium, we just need to replace $\hat{\xi}$ with $\hat{\xi}^\circ$ and set all the αs to satisfy (i) in Definition 5. □

Proof of Proposition 43: First, when $r = 0$, we can find that π is jointly concave in (q_s, q_o). Also,

- if $\frac{q_s}{\alpha_s^*+\alpha_b^*+\alpha_{bo}^*+\alpha_{so}^*} > \frac{q_o}{\alpha_o^*}$, then

$$\begin{aligned}\frac{\partial \pi}{\partial q_s} &= p\bar{F}\left(\frac{q_s}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\right) - c_s \\ \frac{\partial \pi}{\partial q_o} &= p\bar{F}\left(\frac{q_s}{\alpha_o^*}\right) - c_o\end{aligned}$$

- if $\frac{q_s}{\alpha_s^*+\alpha_b^*+\alpha_{bo}^*+\alpha_{so}^*} \leq \frac{q_o}{\alpha_o^*}$, then

$$\begin{aligned}\frac{\partial \pi}{\partial q_s} &= \frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p\bar{F}\left(\frac{q_s}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*}\right) \\ &\quad + \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p\bar{F}\left(\frac{q_o + \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} q_s}{\alpha_o^*+\alpha_{so}^*+\alpha_{bo}^*}\right) - c_s \\ \frac{\partial \pi}{\partial q_o} &= p\bar{F}\left(\frac{q_o + \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} q_s}{\alpha_o^*+\alpha_{so}^*+\alpha_{bo}^*}\right) - c_o\end{aligned}$$

Then, given the constraints $q_s \geq 0$ and $q_o \geq 0$, we can easily get the solution as follows:

- If $c_s < c_o$, then

$$\begin{aligned}q_s^* &= \left(\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*\right) \bar{F}^{-1}\left(\frac{c_s}{p}\right) \\ q_o^* &= \alpha_o^* \bar{F}^{-1}\left(\frac{c_o}{p}\right)\end{aligned}$$

- if $c_s \geq c_o$ and $c_s < \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} c_o + \frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p$, then

$$q_s^* = \left(\alpha_s^* + \alpha_b^* + \alpha_{so}^* + \alpha_{bo}^*\right) \bar{F}^{-1}\left(\frac{c_s - \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} c_o}{\frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p}\right)$$
$$q_o^* = \left(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*\right) \bar{F}^{-1}\left(\frac{c_o}{p}\right) - \left(\alpha_{so}^* + \alpha_{bo}^*\right) \bar{F}^{-1}\left(\frac{c_s - \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} c_o}{\frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p}\right)$$

- if $c_s \geq \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} c_o + \frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p$, then

$$q_s^* = 0$$
$$q_o^* = \left(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*\right) \bar{F}^{-1}\left(\frac{c_o}{p}\right)$$

$\square$

Proof of Proposition 44: Since consumer behavior remains the same as before, part (i) remains valid as before.

Next, let's look at part (ii). To simplify notation, we denote $\Delta^\circ = \frac{v-p}{2H-(v-p)} c_o + \frac{2H-2(v-p)}{2H-(v-p)} p$ and $\Delta^* = \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} c_o + \frac{\alpha_s^*+\alpha_b^*}{\alpha_s^*+\alpha_b^*+\alpha_{so}^*+\alpha_{bo}^*} p$. Note if $\beta_s + \beta_o < 1$, we have $\Delta^\circ > \Delta^*$.

If there are customers visiting store when there is no BOPS, we need to have $c_s < \Delta^\circ$. The following analysis assumes this condition holds.

Suppose $r = 0$. If $c_s \geq \Delta^*$ (this is possible since $\Delta^\circ > \Delta^*$), then no one comes to store when there is BOPS. In this case,

$$
\begin{aligned}
\pi^* &= pE\min\left(\left(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*\right)D, q_o^*\right) - c_o q_o^* \\
&= pE\min\left(\left(\alpha_o^\circ + \alpha_{so}^\circ\right)D, q_o^*\right) - c_o q_o^* \\
&< pE\min\left(\left(\alpha_s^\circ + \alpha_{so}^\circ\right)D, q_s^\circ\right) - c_s q_s^\circ + rE\left(\alpha_s^\circ + \alpha_{so}^\circ\right)D \\
&\qquad + pE\min\left(\alpha_o^\circ D + \frac{\alpha_{so}^\circ}{\alpha_s^\circ + \alpha_{so}^\circ}\left(\left(\alpha_s^\circ + \alpha_{so}^\circ\right)D - q_s^\circ\right)^+, q_o^\circ\right) - c_o q_o^\circ \\
&= \pi^\circ
\end{aligned}
$$

where the inequality is because $q_s = 0, q_o = q_o^*$ is also a feasible but not the optimal solution to the case where there is no BOPS. If $c_s < \Delta^*$, then there are consumers visiting store when there is BOPS. Note (1) $\pi^\circ - \pi^*$ is continuous in c_s, and (2) $\pi^\circ > \pi^*$ when $c_s = \Delta^*$. Thus, the analysis above implies that there exists $\bar{c}_s < \Delta^*$ such that $\pi^\circ > \pi^*$ if $c_s > \bar{c}_s$. $\square$

Proof of Proposition 45: Note there are two possible equilibrium outcomes

i. Participatory equilibrium: $\phi^\circ = 1$

ii. Nonparticipatory equilibrium: $\phi^\circ = 0$.

Let's first look for participatory equilibrium: First, we look at retailer's problem: Given belief $\hat{\phi}$, the retailer maximizes total profit

$$
\begin{aligned}
\pi &= pE\min((\lambda + \hat{\phi}(1-\lambda))D, q) - cq + rE((\lambda + \hat{\phi}(1-\lambda))D) \\
&\quad + wE((1-\hat{\phi})(1-\lambda)D) + wE((\lambda + \hat{\phi}(1-\lambda))D - q)^+ \\
&= (p-w)E\min((\lambda + \hat{\phi}(1-\lambda))D, q) - cq + ((\lambda + \hat{\phi}(1-\lambda))r + w)ED
\end{aligned}
$$

which is a typical newsvendor problem (plus a constant $((\lambda + \hat{\phi}(1-\lambda))r + w)ED$), and therefore the optimal order quantity q° is given by $\bar{F}\left(\frac{q^\circ}{\lambda + \hat{\phi}(1-\lambda)}\right) = \frac{c}{p-w} \wedge 1$. Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\phi} = \phi^\circ = 1$. Thus, $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)$. Since $q > 0$ in the participatory equilibrium, we must have $p - c > w$ and thus $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w}\right)$. Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q^\circ) = E\min(D, q^\circ)/E(D) = E\min(D, \bar{F}^{-1}(\frac{c}{p-w}))/ED$. Finally, we go back to consumer's decision. To ensure $\phi^\circ = 1$, we need $u_s \geq u_o$, i.e., $h_s \leq \hat{\xi}^\circ h_o$. Therefore, we need $h_s \leq \frac{E\min(D,\bar{F}^{-1}(\frac{c}{p-w}))}{ED} h_o$. Based on the analysis above, we find the conditions for a participatory equilibrium are $h_s \leq \frac{E\min(D,\bar{F}^{-1}(\frac{c}{p-w}))}{ED} h_o$ and $p - c > w$. And the equilibrium outcome is $\phi^\circ = 1$ and $q^\circ = \bar{F}^{-1}\left(\frac{c}{p-w}\right)$.

Next, let's look for nonparticipatory equilibrium, i.e., $\phi^\circ = 0$. Note, given belief $\hat{\phi}$, the retailer's optimal store inventory level is given by $\bar{F}\left(\frac{q^\circ}{\lambda+\hat{\phi}(1-\lambda)}\right) = \frac{c}{p-w} \wedge 1$ as shown above. Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\phi} = \phi^\circ = 0$. Thus, $q^\circ = \lambda\bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)$. Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q^\circ) = E\min(\lambda D, q^\circ)/E(\lambda D) = E\min(D, \bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right))/ED$. Finally, we go back to consumer's decision. To ensure $\phi^\circ = 0$, we need $u_s < u_o$, i.e., $h_s > \hat{\xi}^\circ h_o$. Therefore, we need $h_s \leq \frac{E\min(D,F^{-1}\left(\frac{c}{p-w}\wedge 1\right))}{ED} h_o$. Based on the analysis above, we find the conditions for a nonparticipatory equilibrium are $h_s \leq \frac{E\min(D,F^{-1}\left(\frac{c}{p-w}\wedge 1\right))}{ED} h_o$. And the equilibrium outcome is $\phi^\circ = 0$ and $q^\circ = \lambda\bar{F}^{-1}\left(\frac{c}{p-w} \wedge 1\right)$.

Finally, note the conditions for the participatory equilibrium and nonparticipatory equilibrium are mutually exclusive and collectively exhaustive. Thus, the equilibrium always exists and is unique. □

Proof of Proposition 46: If $\min(h_s, h_b) \leq h_o$, then the profit function $\pi = pE\min(D, q) - cq + rE\lambda D + rE(1-\lambda)\min(D, q) + wE(D - q)^+ = (p + (1-\lambda)r - w)E\min(D, q) - cq + wED$, which is a typical newsvendor problem (plus a constant wED). Thus, the optimal order quantity q^* is given by $\bar{F}(q^*) = \frac{c}{p+(1-\lambda)r-w} \wedge 1$. Then, if $p - c > w - (1-\lambda)r$, we have $q^* = \bar{F}^{-1}\left(\frac{c}{p+r-w}\right)$; otherwise, $q^* = 0$.

If $\min(h_s, h_b) > h_o$, then no strategic customer comes to store and thus the retailer's profit is $\pi = pE\min(\lambda D, q) - cq + rE\lambda D + wE(1-\lambda)D + wE(\lambda D - q)^+ = (p - w)E\min(\lambda D, q) - cq + (r\lambda + w)ED$, which is a typical newsvendor problem (plus a constant $(r\lambda + w)ED$). Thus, the optimal order quantity $q^* = \lambda\bar{F}^{-1}(\frac{c}{p-w} \wedge 1)$. □

Proof of Proposition 47: Let's first look for participatory equilibrium:

First, we look at retailer's problem: The retailer's profit can be expressed as

$$
\begin{aligned}
\pi &= pE\min\left(\left(\lambda+\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)(1-\lambda)\right)D,q\right)-cq+rE\left(\lambda+\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)(1-\lambda)\right)D \\
&\quad +wE\hat{\alpha}_o(1-\lambda)D+w\frac{\frac{v-p}{H}\lambda+\hat{\alpha}_{so}(1-\lambda)}{\lambda+(\hat{\alpha}_s+\hat{\alpha}_{so})(1-\lambda)}E\left(\left(\lambda+\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)(1-\lambda)\right)D-q\right)^+ \\
&= \left(p-w\frac{\frac{v-p}{H}\lambda+\hat{\alpha}_{so}(1-\lambda)}{\lambda+(\hat{\alpha}_s+\hat{\alpha}_{so})(1-\lambda)}\right)E\min\left(\left(\lambda+\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)(1-\lambda)\right)D,q\right)-cq \\
&\quad +\left(w\left(\tfrac{v-p}{H}\lambda+\left(\hat{\alpha}_o+\hat{\alpha}_{so}\right)(1-\lambda)\right)+r\left(\lambda+\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)(1-\lambda)\right)\right)ED
\end{aligned}
$$

which is a typical newsvendor problem (plus a constant $\left(w\left(\frac{v-p}{H}\lambda+\left(\hat{\alpha}_o+\hat{\alpha}_{so}\right)(1-\lambda)\right)+r\left(\lambda+\left(\hat{\alpha}_s+\hat{\alpha}_{so}\right)(1-\lambda)\right)\right)ED$), and therefore the optimal order quantity q° is given by $\bar{F}\left(\frac{q^\circ}{\lambda+(\hat{\alpha}_s+\hat{\alpha}_{so})(1-\lambda)}\right)=\frac{c}{p-w\frac{\frac{v-p}{H}\lambda+\hat{\alpha}_{so}(1-\lambda)}{\lambda+(\hat{\alpha}_s+\hat{\alpha}_{so})(1-\lambda)}}\wedge 1$.

Since in equilibrium, the retailer's belief is consistent with the outcome, we have $\hat{\alpha}_i=\alpha_i^\circ$, $i=o,s,so$. Thus, $\bar{F}\left(\frac{q^\circ}{\lambda+(\alpha_s^\circ+\alpha_{so}^\circ)(1-\lambda)}\right)=\frac{c}{p-w\Delta^\circ(\xi)}\wedge 1$

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}=\xi^\circ=A(q^\circ)=E\min((\lambda+\left(\alpha_s^\circ+\alpha_{so}^\circ\right)(1-\lambda))D,q^\circ)/E((\lambda+\left(\alpha_s^\circ+\alpha_{so}^\circ\right)(1-\lambda))D)=E\min(D,\bar{F}^{-1}(\frac{c}{p-w\Delta^\circ(\xi^\circ)}\wedge 1)/ED$. To ensure the existence of a participatory equilibrium, we need to make sure there exists $\xi^\circ>0$ such that $\xi^\circ=E\min(D,\bar{F}^{-1}(\frac{c}{p-w\Delta^\circ(\xi^\circ)}\wedge 1)/ED$.

What's left is to show that when equation $\xi=\frac{\min\left(D,\bar{F}^{-1}\left(\frac{c}{p-w\Delta^\circ(\xi)}\wedge 1\right)\right)}{ED}$ does not have a positive solution, nonparticipatory outcome (i.e., no strategic customers comes to store and $q=0$) is an equilibrium.

Note a necessary condition for the equation $\xi=\frac{\min\left(D,\bar{F}^{-1}\left(\frac{c}{p-w\Delta^\circ(\xi)}\wedge 1\right)\right)}{ED}$ not having a positive solution is $p-w\Delta^\circ(0)\leq c$. Note $\Delta^\circ(0)=\frac{v-p}{H}$. Given $p-w\frac{v-p}{H}\leq c$, it is easy to find that the retailer will set $q=0$ if no strategic customers visit store. Then, given $q=0$, strategic customers will indeed not come to store. Thus, this nonparticipatory outcome is a RE equilibrium. □

Proof of Proposition 48: The profit function can be expressed as

$$\pi = \left[p - w\Delta^* + r\frac{\left(\alpha_s^*+\alpha_{so}^*+\alpha_b^*+\alpha_{bo}^*\right)(1-\lambda)}{\lambda+\left(\alpha_s^*+\alpha_{so}^*+\alpha_b^*+\alpha_{bo}^*\right)(1-\lambda)}\right] E\min\left(\left(\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)\right)D, q\right)$$
$$-cq + \left(w\left(\tfrac{v-p}{H}\lambda + \left(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*\right)(1-\lambda)\right) + r\lambda\right)ED$$

which is a typical newsvendor problem (plus a constant $\left(w\left(\frac{v-p}{H}\lambda + \left(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*\right)(1-\lambda)\right) + r\lambda\right)ED$). Thus, the optimal order quantity

$$q^* = \left(\lambda + \left(\alpha_s^* + \alpha_{so}^* + \alpha_b^* + \alpha_{bo}^*\right)(1-\lambda)\right)\bar{F}^{-1}\left(\frac{c}{p-w\Delta^*+r\frac{\left(\alpha_s^*+\alpha_{so}^*+\alpha_b^*+\alpha_{bo}^*\right)(1-\lambda)}{\lambda+\left(\alpha_s^*+\alpha_{so}^*+\alpha_b^*+\alpha_{bo}^*\right)(1-\lambda)}} \wedge 1\right) \qquad \square$$

Proof of Proposition 49: If $\frac{v-p}{\beta_o} < H$, then $\lim\limits_{\beta_s \to 0} \frac{\alpha_{so}^*+\alpha_{bo}^*}{\alpha_s^*+\alpha_{so}^*+\alpha_b^*+\alpha_{bo}^*} > \frac{v-p}{H}$. Therefore, $\lim\limits_{\beta_s\to 0} \Delta^* > \frac{v-p}{H}$. Since Δ^* is continuous in β_s, there exists $\bar{\beta}_s > 0$ such that $\Delta^* > \frac{v-p}{H}$ if $\beta_s < \bar{\beta}_s$. Note $\Delta^\circ(0) = \frac{v-p}{H}$.

Suppose $r = 0$. If $w \geq \frac{p-c}{\Delta^*}$ (this is possible since $\frac{p-c}{\Delta^\circ} > \frac{p-c}{\Delta^*}$), then no one comes to store when there is BOPS . In this case,

$$\begin{aligned}
\pi^* &= \left(w\left(\frac{v-p}{H}\lambda + \left(\alpha_o^* + \alpha_{so}^* + \alpha_{bo}^*\right)(1-\lambda)\right) + r\lambda\right)ED \\
&= \left(w\left(\frac{v-p}{H}\lambda + \left(\alpha_o^\circ + \alpha_{so}^\circ\right)(1-\lambda)\right) + r\lambda\right)ED \\
&< pE\min\left(\left(\lambda + \left(\alpha_s^\circ + \alpha_{so}^\circ\right)(1-\lambda)\right)D, q^\circ\right) - cq^\circ + rE\left(\lambda + \left(\alpha_s^\circ + \alpha_{so}^\circ\right)(1-\lambda)\right)D \\
&\qquad + wE\alpha_o^\circ(1-\lambda)D + w\frac{\frac{v-p}{H}\lambda + \alpha_{so}^\circ(1-\lambda)}{\lambda + \left(\alpha_s^\circ + \alpha_{so}^\circ\right)(1-\lambda)}E\left(\left(\lambda + \left(\alpha_s^\circ + \alpha_{so}^\circ\right)(1-\lambda)\right)D - q^\circ\right)^+ \\
&= \pi^\circ
\end{aligned}$$

where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case where there is no BOPS. If $w < \frac{p-c}{\Delta^*}$, then there are consumers visiting store when there is BOPS. Note (1) $\pi^\circ - \pi^*$ is continuous in w, and (2) $\pi^\circ > \pi^*$ when $w = \frac{p-c}{\Delta^*}$. Thus, the analysis above implies that there exists $\bar{w} < \frac{p-c}{\Delta^*}$ such that $\pi^\circ > \pi^*$ if $w > \bar{w}$. $\square$

Proof of Proposition 54: Note nonparticipatory equilibrium, $(0, \bar{F}^{-1}(\frac{c_o}{p\theta - k(1-\theta)}), 0, 0, 0)$, always exists. Same as what we did in Chapter 3, we are going to look for participatory equilibrium, where $\phi^\circ = 1$ and $q_s^\circ > 0$. All we need to do is to check the four conditions specified in Definition 6.

First, we look at retailer's problem: In the participatory equilibrium, retailer's belief is consistent, i.e., $\hat{\phi} = \phi^\circ = 1$. Then, the retailer's profit is

$$\pi = pE\min(\theta D, q_s) - c_s q_s + [p\theta - k(1-\theta)]\, E\min\left(\left(D - \frac{q_s}{\theta}\right)^+, q_o\right) - c_o q_o.$$

It is easy to verify the profit function is jointly concave in (q_s, q_o). Note the first order derivatives are given as follows: $\frac{\partial \pi}{\partial q_s} = k\frac{1-\theta}{\theta}\bar{F}\left(\frac{q_s}{\theta}\right) + \left(p - k\frac{1-\theta}{\theta}\right)\bar{F}\left(\frac{q_s}{\theta} + q_o\right) - c_s$ and $\frac{\partial \pi}{\partial q_o} = [p\theta - k(1-\theta)]\,\bar{F}\left(\frac{q_s}{\theta} + q_o\right) - c_o$. Given that $q_s \geq 0$ and $q_o \geq 0$, it is easy to find the optimal solution is given as follows:

- if $c_o \geq [p\theta - k(1-\theta)]\frac{c_s}{p}$, then $\bar{F}(\frac{q_s}{\theta}) = \frac{c_s}{p}$ and $q_o = 0$
- if $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s \leq k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, then $\bar{F}(\frac{q_s}{\theta}) = \frac{\theta c_s - c_o}{k(1-\theta)}$ and $\bar{F}(\frac{q_s}{\theta} + q_o) = \frac{c_o}{p\theta - k(1-\theta)}$
- if $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s > k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, then $q_s = 0$ and $\bar{F}(q_o) = \frac{c_o}{p\theta - k(1-\theta)}$

Thus, to ensure participatory equilibrium, we need (i) $c_o \geq [p\theta - k(1-\theta)]\frac{c_s}{p}$ or (ii) $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s \leq k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$

Let's first look at case (i): In equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q_s^\circ) = E\min(\phi^\circ D, q_s^\circ/\theta)/E(\phi^\circ D) = E\min(D, \bar{F}^{-1}(\frac{c_s}{p}))/ED$. Then, we go back to consumer's decision. To ensure $\phi^\circ = 1$, we need $u_s > u_o$, i.e., $h_s < \frac{E\min(D, \bar{F}^{-1}(\frac{c_s}{p}))}{ED}[h_o + (1-\theta)h_r]$.

Next, let's look at case (ii): In equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q_s^\circ) = E\min(\phi^\circ D, q_s^\circ/\theta)/E(\phi^\circ D) = \frac{E\min(D, \bar{F}^{-1}(\frac{\theta c_s - c_o}{k(1-\theta)}))}{ED}$. Then, we go back to consumer's decision. To ensure $\phi^\circ = 1$, we need $u_s > u_o$, i.e.,

$h_s < \frac{E\min(D,\bar{F}^{-1}(\frac{\theta c_s - c_o}{k(1-\theta)}))}{ED}[h_o + (1-\theta)h_r]$.

Based on the analysis above, we find the conditions for a participatory equilibrium are

- $h_s < \frac{E\min(D,\bar{F}^{-1}(\frac{c_s}{p}))}{ED}[h_o+(1-\theta)h_r]$ and $c_o \geq [p\theta - k(1-\theta)]\frac{c_s}{p}$, in which the equilibrium outcome is $\phi^\circ = 1$, $q_s^\circ = \theta\bar{F}^{-1}\left(\frac{c_s}{p}\right)$ and $q_o^\circ = 0$.

- $h_s < \frac{E\min(D,\bar{F}^{-1}(\frac{\theta c_s - c_o}{k(1-\theta)}))}{ED}[h_o+(1-\theta)h_r]$ and $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s \leq k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, in which the equilibrium outcome is $\phi^\circ = 1$, $q_s^\circ = \theta\bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)$, $q_o^\circ = \bar{F}^{-1}\left(\frac{c_o}{p\theta - k(1-\theta)}\right) - \bar{F}^{-1}\left(\frac{\theta c_s - c_o}{k(1-\theta)}\right)$.

□

Proof of Proposition 55: Note nonparticipatory equilibrium, $(0, \bar{F}^{-1}(\frac{c_o}{p\theta - k(1-\theta)}), 0, 0, 0)$, always exists. Same as what we did in Chapter 3, we are going to look for participatory equilibrium, where $\phi^p = 1$.

First, we look at retailer's problem: In the participatory equilibrium, retailer's belief is consistent, i.e., $\hat{\phi} = \phi^p = 1$. Then, retailer's profit is $\pi = pE\min(\theta D, q_s) - c_s q_s + pE\min((\theta D - q_s)^+, q_o) - c_o q_o = pE\min(\theta D, q_s + q_o) - c_s q_s - c_o q_o$. Then, it is easy to find that the optimal solution is as follows

- if $c_s < c_o$, then $q_s^p = \theta\bar{F}^{-1}(\frac{c_s}{p})$ and $q_o^p = 0$

- if $c_s \geq c_o$, then $q_s^p = 0$ and $q_o^p = \theta\bar{F}^{-1}(\frac{c_o}{p})$.

Let's first look at the case when $c_s < c_o$. In equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^p = A(q_s^p) = E\min(\phi^p D, q_s^p/\theta)/E(\phi^p D) = E\min(D, \bar{F}^{-1}(\frac{c_s}{p}))/ED$. Then, we go back to consumer's decision. To ensure $\phi^p = 1$, we need $u_s > u_o$, i.e., $h_s < \hat{\xi}^p\theta h_o + (1-\theta)(h_o + h_r)$. Therefore, we need $h_s < \frac{E\min(D,\bar{F}^{-1}(\frac{c_s}{p}))}{ED}\theta h_o + (1-\theta)(h_o + h_r)$.

Next let's look at the case when $c_s \geq c_o$. In equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^p = A(q_s^p) = 0$. Then, we go back to consumer's decision. To ensure $\phi^p = 1$, we need $u_s > u_o$, i.e., $h_s < \hat{\xi}^p \theta h_o + (1-\theta)(h_o + h_r)$. Therefore, we need $h_s < (1-\theta)(h_o + h_r)$.

Based on the analysis above, we find the conditions for a participatory equilibrium are

- $h_s < \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} \theta h_o + (1-\theta)(h_o + h_r)$ and $c_s < c_o$, in which the equilibrium outcome is $\phi^p = 1$, $q_s^p = \theta \bar{F}^{-1}(\frac{c_s}{p})$, $q_o^p = 0$;
- $h_s < (1-\theta)(h_o + h_r)$ and $c_s \geq c_o$, in which the equilibrium outcome is $\phi^p = 1$, $q_s^p = 0$, $q_o^p = \theta \bar{F}^{-1}(\frac{c_o}{p})$;

$\square$

Proof of Proposition 56: If $(1-\theta)(h_o + h_r) < h_s < \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} [h_o + (1-\theta) h_r]$ and $[p\theta - k(1-\theta)]\frac{c_s}{p} < c_o < c_s$, according to Propositions 54 and 55, we can easily find that $\phi^\circ = 1, q_s^\circ = \theta \bar{F}^{-1}(\frac{c_s}{p}), q_o^\circ = 0$ and $\phi^p = 0, q_s^p = 0, q_o^p = \bar{F}^{-1}(\frac{c_o}{p\theta - k(1-\theta)})$. Then,

$$\begin{aligned} \pi^\circ &= pE\min(\theta D, q_s^\circ) - c_s q_s^\circ \\ &> [p\theta - k(1-\theta)] E\min(D, q_o^p) - c_o q_o^p = \pi^p \end{aligned}$$

where the inequality is because $q_s = 0$ and $q_o = q_o^p$ is a feasible but not the optimal solution to the case where there is no physical showroom. $\square$

Proof of Proposition 57: The proof is similar to the proof of Proposition 54. We only need to replace D and θ with D' and θ'. Details are omitted. $\square$

Proof of Proposition 58: If $\frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} [h_o + (1-\theta') h_r] < h_s < \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED} [h_o + (1-\theta) h_r]$ and $[p\theta' - k(1-\theta')]\frac{c_s}{p} < c_o$, then $\phi^\circ = 1, q_s^\circ = \theta \bar{F}^{-1}(\frac{c_s}{p}), q_o^\circ = 0$ and $\phi^v = 0, q_s^v = 0, q_o^v = \bar{F}'^{-1}(\frac{c_o}{p\theta' - k(1-\theta')})$. Then, $\pi^\circ = pE\min(\theta D, q_s^\circ) - c_s q_s^\circ$ and $\pi^v = [p\theta' - k(1-\theta')]E\min(D, q_s^v) - c_o q_s^v$. Note by Envelop Theorem, we have $\frac{\partial(\pi^\circ - \pi^v)}{\partial c_o} = q_s^v > 0$. Also, note when $c_o = p\theta' - k(1-\theta')$, $\pi^v = 0 < \pi^o$. Thus, there exists $\bar{c}_o$ such that $\pi^\circ > \pi^v$ if $c_o > \bar{c}_o$. $\square$

Proof of Proposition 59: If $h_s < h_o + (1-\theta)h_r$, then consumers come to store only if it is in stock. For the profit function

$$\pi = pE\min(\theta D, q_s) - c_s q_s + [p\theta - k(1-\theta)]E\min\left(\left(D - \frac{q_s}{\theta}\right)^+, q_o\right) - c_o q_o$$

it is easy to check it is jointly concave in (q_s, q_o). Note the first order derivatives are $\frac{\partial \pi}{\partial q_s} = k\frac{1-\theta}{\theta}\bar{F}\left(\frac{q_s}{\theta}\right) + \left(p - k\frac{1-\theta}{\theta}\right)\bar{F}\left(\frac{q_s}{\theta} + q_o\right) - c_s$ and $\frac{\partial \pi}{\partial q_o} = [p\theta - k(1-\theta)]\bar{F}\left(\frac{q_s}{\theta} + q_o\right) - c_o$. Given that $q_s \geq 0$ and $q_o \geq 0$, it is easy to find the optimal solution is given as follows:

- if $c_o \geq [p\theta - k(1-\theta)]\frac{c_s}{p}$, then $\bar{F}(\frac{q_s^a}{\theta}) = \frac{c_s}{p}$ and $q_o^a = 0$
- if $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s \leq k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, then $\bar{F}(\frac{q_s^a}{\theta} + q_o^a) = \frac{c_o}{p\theta - k(1-\theta)}$ and $\bar{F}(\frac{q_s^a}{\theta}) = \frac{\theta c_s - c_o}{k(1-\theta)}$
- if $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s > k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, then $q_s^a = 0$ and $\bar{F}(q_o^a) = \frac{c_o}{p\theta - k(1-\theta)}$

If $h_s \geq h_o + (1-\theta)h_r$, then all customers always buy online. Then, profit function is $\pi = [p\theta - k(1-\theta)]E\min(D, q_o) - c_o q_o$. Thus, $q_s^a = 0$, $q_o^a = \bar{F}^{-1}(\frac{c_o}{p\theta - k(1-\theta)})$. $\square$

Proof of Proposition 60:

- If $h_s < h_o + (1-\theta)h_r$ and $c_o \geq [p\theta - k(1-\theta)]\frac{c_s}{p}$, then $\pi^a = pE\min(\theta D, q_s^a) - c_s q_s^a$.
  - If $h_s < \xi_1^\circ[h_o + (1-\theta)h_r]$, then $\pi^\circ = pE\min(\theta D, q_s^\circ) - c_s q_s^\circ = \pi^a$;
  - If $h_s \geq \xi_1^\circ[h_o + (1-\theta)h_r]$, then $\pi^\circ = [p\theta + k(1-\theta)]E\min(D, q_o^\circ) - c_o q_o^\circ < \pi^a$, where the inequality is because $q_s = 0$ and $q_o = q_o^\circ$ is also a feasible but not the optimal solution to the case where there is availability information;

- If $h_s < h_o + (1-\theta)h_r$ and $c_o < [p\theta - k(1-\theta)]\frac{c_s}{p}$ and $c_s \leq k\frac{1-\theta}{\theta} + \frac{c_o}{\theta}$, then $\pi^a = pE\min(\theta D, q_s^a) - c_s q_s^a + [p\theta + k(1-\theta)]\, E\min\left(\left(D - \frac{q_s^a}{\theta}\right)^+, q_o^a\right) - c_o q_o^a$
  - If $h_s \leq \xi_2^\circ[h_o + (1-\theta)h_r]$, then $\pi^\circ = pE\min(\theta D, q_s^\circ) - c_s q_s^\circ + [p\theta - k(1-\theta)]\, E\min\left(\left(D - \frac{q_s^\circ}{\theta}\right)^+, q_o^\circ\right) - c_o q_o^\circ = \pi^a$;
  - $h_s > \xi_2^\circ[h_o + (1-\theta)h_r]$, then $\pi^\circ = [p\theta - k(1-\theta)]\, E\min(D, q_o^\circ) - c_o q_o^\circ < \pi^a$, where the inequality is because $q_s = 0$ and $q_o = q_o^\circ$ is also a feasible but not the optimal solution to the case where there is availability information;
- Otherwise, then $\pi^a = [p\theta - k(1-\theta)]\, E\min(D, q_o^a) - c_o q_o^a = \pi^\circ$.

$\square$

Proof of Proposition 61.

- If $h_s \leq (1-\theta)(h_o + h_r)$, then

$$\begin{aligned}\pi^{pa} &= \max_{q_s, q_o \geq 0} \left\{pE\min(\theta D, q_s) - c_s q_s + pE\min\left((\theta D - q_s)^+, q_o\right) - c_o q_o\right\} \\ &= pE\min(\theta D, q_s^p) - c_s q_s^p + pE\min\left((\theta D - q_s^p)^+, q_o^p\right) - c_o q_o^p = \pi^p\end{aligned}$$

- If $(1-\theta)(h_o + h_r) \leq h_s < h_o + (1-\theta)h_r$, then

$$\begin{aligned}\pi^{pa} &= \max_{q_s, q_o \geq 0} \left\{pE\min(\theta D, q_s) - c_s q_s + [p\theta - k(1-\theta)]\, E\min\left(\left(D - \frac{q_s}{\theta}\right)^+, q_o\right) - c_o q_o\right\} \\ &= pE\min(\theta D, q_s^a) - c_s q_s^a + [p\theta - k(1-\theta)]\, E\min\left(\left(D - \frac{q_s^a}{\theta}\right)^+, q_o^a\right) - c_o q_o^a = \pi^a\end{aligned}$$

- If $h_s > h_o + (1-\theta)h_r$, then

$$\begin{aligned}\pi^{pa} &= \max_{q_o \geq 0} \left\{[p\theta - k(1-\theta)]\, E\min(D, q_o) - c_o q_o\right\} \\ &= [p\theta + k(1-\theta)]\, E\min(D, q_o^a) - c_o q_o^a = \pi^a\end{aligned}$$

$\square$

Proof of Proposition 62: First, similar to the proof of Proposition 55, we can find the RE equilibrium for the case when there are both physical and virtual showrooms. We simply replace D and θ with D' and θ'. The results are given as follows:

- If $h_s < \frac{E\min\left(D', \bar{F}'^{-1}\left(\frac{c_s}{p}\right)\right)}{ED'}\theta' h_o + (1-\theta')(h_o + h_r)$ and $c_s < c_o$, then consumers come to store (i.e., $\phi^{pv} = 1$) and $q_s^{pv} = \theta' \bar{F}'^{-1}(\frac{c_s}{p})$, $q_o^{pv} = 0$;

- if $h_s < (1-\theta)(h_o + h_r)$ and $c_s \geq c_o$, then consumers come to store (i.e., $\phi^{pv} = 1$) and $q_s^{pv} = 0$, $q_o^{pv} = \theta' \bar{F}'^{-1}(\frac{c_o}{p})$;

- otherwise, no one comes to store (i.e., $\phi^{pv} = 0$) and $q_s^{pv} = 0$, $q_o^{pv} = \bar{F}'^{-1}\left(\frac{c_o}{p\theta' - k(1-\theta')}\right)$.

Then, let's compare profits.

- If $h_s < \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c_s}{p}\right)\right)}{ED}\theta' h_o + (1-\theta')(h_o + h_r)$ and $c_s < c_o$, then

$$\begin{aligned}\pi^{pv} &= pE\min(\theta' D', q_s^{pv}) - c_s q_s^{pv} + pE\min\left((\theta' D' - q_s^{pv})^+, q_o^{pv}\right) - c_o q_o^{pv} \\ &= pE\min(\theta' D', q_s^{p}) - c_s q_s^{p} + pE\min\left((\theta' D' - q_s^{p})^+, q_o^{p}\right) - c_o q_o^{p} = \pi^p\end{aligned}$$

- If $h_s < (1-\theta')(h_o + h_r)$ and $c_s > c_o$, then

$$\begin{aligned}\pi^{pv} &= pE\min(\theta' D', q_o^{pv}) - c_o q_o^{pv} \\ &= pE\min(\theta' D', q_o^{p}) - c_o q_o^{p} = \pi^p\end{aligned}$$

- Otherwise,

$$\begin{aligned}\pi^{pv} &= [p\theta' - k(1-\theta')]\,E\min(D', q_o^{pv}) - c_o q_o^{pv} \\ &\leq pE\min(\theta' D', q_s^{v}) - c_s q_s^{v} + [p\theta' - k(1-\theta')]\,E\min\left(\left(D' - \frac{q_s^v}{\theta'}\right)^+, q_o^v\right) - c_o q_o^v \\ &= \pi^v\end{aligned}$$

  where the inequality is because $q_s = 0, q_o = q_o^{pv}$ is also a feasible solution to the case where there is only virtual showroom.

$\square$

Proof of Proposition 63: Note nonparticipatory equilibrium, $(0, 0, 0, 0)$, always exists. Same as what we did in the simple model, we are going to look for participatory equilibrium, where $\phi^\circ = 1$ and $q^\circ > 0$. All we need to do is to check the four conditions specified in Definition 7.

First, we look at retailer's problem: Given belief $\hat{\phi} = 1$, the retailer maximizes total profit $\pi = pE\min(\bar{G}(p)D, q) - cq + [w\bar{G}(p-h_r) - rG(p-h_r)]E(D - \frac{q}{\bar{G}(p)})^+$
$= (p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)})E\min(\bar{G}(p)D, q) - cq + (w\bar{G}(p-h_r) - rG(p-h_r))ED$, which is a typical newsvendor problem (plus a constant $(w\bar{G}(p-h_r) - rG(p-h_r))ED$), and therefore the optimal order quantity $q^\circ = \bar{G}(p)\bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = A(q^\circ) = E\min(\phi D, q^\circ/\bar{G}(p))/E(\phi D) = E\min(D, \bar{F}^{-1}(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}}))/ED$.

Finally, we go back to consumer's decision. To ensure $\phi = 1$, we need $u_s > u_o$, which gives us the condition $h_s < h_s^\circ$. $\square$

Proof of Proposition 64: Let's look for participatory RE equilibrium, where $\phi = 1$.

First, we look at retailer's problem: Given belief $\hat{\phi} = 1$, the retailer maximizes total profit $\pi = pE\min(\bar{G}(p)D, q) - cq + w\bar{G}(p+h_o)E(D - \frac{q}{\bar{G}(p)})^+ = (p - w\frac{\bar{G}(p+h_o)}{\bar{G}(p)})E\min(\bar{G}(p)D, q) - cq + (w\bar{G}(p+h_o))ED$, which is a typical newsvendor problem (plus a constant $((w\bar{G}(p+h_o))ED)$, and therefore the optimal order quantity $q^p = \bar{G}(p)\bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p+h_o)}{\bar{G}(p)}}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^p = A(q^p) = E\min(\phi D, q^p/\bar{G}(p))/E(\phi D) = E\min(D, \bar{F}^{-1}(\frac{c}{p - w\frac{\bar{G}(p+h_o)}{\bar{G}(p)}}))/ED$.

Finally, we go back to consumer's decision. To ensure $\phi = 1$, we need $u_s > u_o$, which gives us the condition $h_s < h_s^p$.

Note $\frac{\partial h_s^\circ}{\partial r} \geq 0$ and $\frac{\partial h_s^p}{\partial r} = 0$. Also, when $r = 0$, we have $h_s^\circ < h_s^p$. Therefore, $\exists \bar{r} \geq 0$ such that $h_s^p > h_s^\circ$ if and only if $r < \bar{r}$. □

Proof of Proposition 65: If $h_s^p < h_s^\circ$ and $h_s \in [h_s^p, h_s^\circ]$, then

$$\begin{aligned}\pi^\circ &= pE\min\left(\bar{G}(p)\,D, q^\circ\right) - cq^\circ + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E\left(D - \frac{q^\circ}{\bar{G}(p)}\right)^+ \\ &> \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]ED = \pi^p\end{aligned}$$

where the inequality is because $q = 0$ is a feasible but not the optimal solution to the case where there is no physical showroom. □

Proof of Proposition 66: Since this is just the base model with a new set of parameters G' and D', similar to the proof of Proposition 63, we can show that the participatory equilibrium exists if and only if $h_s < h_s^v$.

Next, let's show that $h_s^v < h_s^\circ$: Note that

$$\begin{aligned}\xi^v &= \frac{E_{D'}\min\left(D', \bar{F}'^{-1}\left(\frac{c}{p - w\frac{\bar{G}'(p-h_r)}{\bar{G}'(p)} + r\frac{G'(p-h_r)}{\bar{G}'(p)}}\right)\right)}{E_{D'}D'} \\ &= \frac{E_D\min\left(D, \bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)-G(\bar{v})}{\bar{G}(p)}}\right)\right)}{E_D D} \\ &< \frac{E_D\min\left(D, \bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}}\right)\right)}{E_D D} = \xi^\circ\end{aligned}$$

and then,

$$\begin{aligned}h_s^v &= \xi^v\left(E'_V\max(V-p,0) + h_o - E'_V\max(V-p,-h_r)\right) \\ &= \xi^v\left(\frac{E_V\max(V-p,0) - E_V\max(V-p+h_r,0)}{1-G(\bar{v})} + h_o + h_r\right) \\ &< \xi^o\left(\frac{E_V\max(V-p,0) - E_V\max(V-p+h_r,0)}{1-G(\bar{v})} + h_o + h_r\right) \\ &< \xi^o\left(E_V\max(V-p,0) - E_V\max(V-p+h_r,0) + h_o + h_r\right) = h_s^\circ\end{aligned}$$

$\square$

Proof of Proposition 67: If $h_s \in [h_s^v, h_s^\circ)$, then

$$\pi^\circ - \pi^v = pE\min\left(\bar{G}(p)\,D, q^\circ\right) - cq^\circ + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E\left(D - \frac{q^o}{\bar{G}(p)}\right)^+$$
$$- \left[w\bar{G}(p-h_r) - r\left(G(p-h_r) - G(\bar{v})\right)\right]ED$$
$$= \left[p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}\right]E\min\left(\bar{G}(p)\,D, q^\circ\right) - cq^\circ - rG(\bar{v})\,ED$$

By Envelop Theorem, we have $\frac{\partial(\pi^\circ - \pi^v)}{\partial w} = -\frac{\bar{G}(p-h_r)}{\bar{G}(p)}E\min\left(\bar{G}(p)\,D, q^\circ\right) < 0$. To conclude the result, we just need to note that it is indeed possible to have $\pi^\circ > \pi^v$, e.g., when α is very close to 0. $\square$

Proof of Proposition 68: Solving for $u_{s,in} > u_o$, we have the condition $h_s < h_s^a$. In this case, the retailer's profit is

$$\pi = pE\min\left(\bar{G}(p)\,D, q\right) - cq + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E\left(D - \frac{q}{\bar{G}(p)}\right)^+$$

Thus, $q^a = \bar{G}(p)\,\bar{F}^{-1}\left(\frac{c}{p - w\frac{\bar{G}(p-h_r)}{\bar{G}(p)} + r\frac{G(p-h_r)}{\bar{G}(p)}}\right)$.

If $u_{s,in} \leq u_o$, i.e., $h_s \geq h_s^a$, then no one comes to store. So clearly, $q^a = 0$. $\square$

Proof of Proposition 69:

- If $h_s < \xi^\circ(E_V \max(V-p,0) + h_o - E_V\max(V-p,-h_r))$,

$$\pi^a = pE\min\left(\bar{G}(p)\,D, q^a\right) - cq^a + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E\left(D - \frac{q^a}{\bar{G}(p)}\right)^+$$
$$= pE\min\left(\bar{G}(p)\,D, q^\circ\right) - cq^\circ + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E\left(D - \frac{q^\circ}{\bar{G}(p)}\right)^+ = \pi^\circ$$

- If $\xi^\circ(E_V\max(V-p,0) + h_o - E_V\max(V-p,-h_r)) \leq h_s <_V E\max(V-p,0) + h_o -$

$E_V \max(V-p,-h_r)$, then

$$\begin{aligned}
\pi^a &= pE\min\left(\bar{G}(p)D, q^a\right) - cq^a + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E\left(D - \frac{q^a}{\bar{G}(p)}\right)^+ \\
&> \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]ED = \pi^\circ
\end{aligned}$$

where the inequality is because $q=0$ is also a feasible but not the optimal solution to the case where there is availability information.

- If $h_s \geq E_V \max(V-p,0) + h_o - E_V \max(V-p,-h_r)$, then $\pi^a = \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]ED = \pi^o$.

□

Proof of Proposition 70:

- If $h_s \leq E_V \max(V-p-h_o,0) + h_o - E_V \max(V-p,-h_r)$, then

$$\begin{aligned}
\pi^{pa} &= \max_{q\geq 0}\left\{pE_D\min\left(\bar{G}(p)D, q\right) - cq + w\bar{G}(p+h_o)E_D\left(D - \frac{q}{\bar{G}(p)}\right)^+\right\} \\
&= pE_D\min\left(\bar{G}(p)D, q^p\right) - cq^p + w\bar{G}(p+h_o)E_D\left(D - \frac{q^p}{\bar{G}(p)}\right)^+ = \pi^p
\end{aligned}$$

- If $E_V \max(V-p-h_o,0) + h_o - E_V \max(V-p,-h_r) < h_s \leq E_V \max(V-p,0) + h_o - E_V \max(V-p,-h_r)$, then,

$$\begin{aligned}
\pi^{pa} &= \max_{q\geq 0}\left\{pE_D\min\left(\bar{G}(p)D, q\right) - cq\right. \\
&\qquad \left. + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E_D\left(D - \frac{q}{\bar{G}(p)}\right)^+\right\} \\
&= pE_D\min\left(\bar{G}(p)D, q^a\right) - cq^a + \left[w\bar{G}(p-h_r) - rG(p-h_r)\right]E_D\left(D - \frac{q^a}{\bar{G}(p)}\right)^+ \\
&= \pi^a
\end{aligned}$$

- If $h_s > E_V \max(V-p,0) + h_o - E_V \max(V-p,-h_r)$, then

$$\pi = \left[w\bar{G}\left(p-h_r\right) - rG\left(p-h_r\right)\right] E_D D = \pi^a.$$

$\square$

Proof of Proposition 71: Similar to the proof of Proposition 64, we can get the RE equilibrium of the case when there are both physical and virtual showrooms. We just need to replace D and G with D' and G'. Thus, the RE equilibrium is given as follows:

With physical and virtual showrooms, if $h_s \leq h_s^{pv}$, then consumers visit store (i.e., $\phi^{pv} = 1$), and $q^{pv} = \bar{G}'\left(p\right)\bar{F}'^{-1}\left(\frac{c}{p-w\frac{\bar{G}'(p+h_o)}{\bar{G}'(p)}}\right)$, where $h_s^{pv} = \xi^{pv}E'_V \max(V-p,0)+(1-\xi^{pv})E'_V \max(V-p-h_o,0)+h_o-E'_V \max(V-p,-h_r)$ and $\xi^{pv} = \frac{E_{D'}\min\left(D', \bar{F}'^{-1}\left(\frac{c}{p-w\frac{\bar{G}'(p+h_o)}{\bar{G}'(p)}}\right)\right)}{E_{D'}D'}$; otherwise, no one comes to store (i.e., $\phi^{pv} = 0$), and $q^{pv} = 0$.

Note that $\frac{\bar{G}'(p+h_o)}{\bar{G}'(p)} = \frac{\bar{G}(p+h_o)}{\bar{G}(p)}$. Then, we can find $\xi^{pv} = \xi^p$.

$$\begin{aligned}
h_s^{pv} &= \xi^{pv}E'_V \max(V-p,0) + (1-\xi^{pv})E'_V \max(V-p-h_o,0) + h_o - E'_V \max(V-p,-h_r) \\
&= \frac{\xi^{pv}E_V \max\left(V-p,0\right) + \left(1-\xi^{pv}\right)E_V \max\left(V-p-h_o,0\right) - E_V \max\left(V-p+h_r,0\right)}{1-G\left(\bar{v}\right)} \\
&\quad + h_o + h_r \\
&< \xi^{pv}E_V \max\left(V-p,0\right) + \left(1-\xi^{pv}\right)E_V \max\left(V-p-h_o,0\right) - E_V \max\left(V-p+h_r,0\right) \\
&\quad + h_o + h_r \\
&= \xi^{p}E_V \max\left(V-p,0\right) + \left(1-\xi^{v}\right)E_V \max\left(V-p-h_o,0\right) - E_V \max\left(V-p+h_r,0\right) \\
&\quad + h_o + h_r \\
&= h_s^p
\end{aligned}$$

- If $h_s \le h_s^{pv}$, then

$$\begin{aligned}\pi^{pv} &= pE\min\left(\bar{G}'(p)\,D', q^{pv}\right) - cq^{pv} + w\bar{G}'(p+h_o)\,E\left(D' - \frac{q^{pv}}{\bar{G}'(p)}\right)^+ \\ &= pE\min\left(\bar{G}(p)\,D, q^{pv}\right) - cq^{pv} + w\bar{G}(p+h_o)\,E\left(D - \frac{q^{pv}}{\bar{G}(p)}\right)^+ \\ &= pE\min\left(\bar{G}(p)\,D, q^{p}\right) - cq^{p} + w\bar{G}(p+h_o)\,E\left(D - \frac{q^{p}}{\bar{G}(p)}\right)^+ = \pi^p\end{aligned}$$

- If $h_s \in (h_s^{pv}, \max(h_s^{pv}, h_s^v)]$, then

$$\begin{aligned}\pi^{pv} &= \left[w\bar{G}'(p-h_r) - rG'(p-h_r)\right]E_{D'}D' \\ &< pE\min\left(\bar{G}'(p)\,D', q^{v}\right) - cq^{v} + \left[w\bar{G}'(p-h_r) - rG'(p-h_r)\right]E_{D'}\left(D' - \frac{q^{v}}{\bar{G}'(p)}\right)^+ \\ &= \pi^v\end{aligned}$$

  where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case when there is only virtual showroom.

- If $h_s > \max(h_s^{pv}, h_s^v)$, then $\pi^{pv} = \left[w\bar{G}'(p-h_r) - rG'(p-h_r)\right]E_{D'}D' = \pi^v$.

$\square$

Proof of Proposition 72: Note nonparticipatory equilibrium, $(0,0,0,0,0,0)$, always exists. Same as what we did in Chapter 3, we are going to look for participatory equilibrium, where at least one of ϕ_i and ϕ_u is 1.

- Let's first look for conditions where $\phi_u = \phi_i = 1$ is an equilibrium.

Given $\hat{\phi}_u = \hat{\phi}_i = 1$, then the retailer's profit function is

$$\begin{aligned}
\pi(q) =& pE\min\left(\left(\lambda + (1-\lambda)\,\theta\right)D, q\right) - cq \\
&+ \left[w\theta - r\,(1-\theta)\right]E\,(1-\lambda)\left[D - \frac{q}{\lambda + (1-\lambda)\,\theta}\right]^+ + wE\lambda\left(D - \frac{q}{\lambda + (1-\lambda)\,\theta}\right)^+ \\
=& \left[p - w + r\frac{(1-\theta)(1-\lambda)}{\lambda + (1-\lambda)\theta}\right]E\min\left(\left(\lambda + (1-\lambda)\,\theta\right)D, q\right) - cq \\
&+ \left[w\left(\theta\,(1-\lambda) + \lambda\right) - r\,(1-\theta)\,(1-\lambda)\right]ED
\end{aligned}$$

which is a typical newsvendor problem (plus a constant $\left[w\left(\theta\,(1-\lambda) + \lambda\right) - r\,(1-\theta)\,(1-\lambda)\right]ED$), and therefore the optimal order quantity $q^\circ = \left(\lambda + (1-\lambda)\,\theta\right)\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{(1-\theta)(1-\lambda)}{\lambda+(1-\lambda)\theta}}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^\circ = \frac{E\min((\lambda+(1-\lambda)\theta)D, q^\circ)}{E(\lambda+(1-\lambda)\theta)D} = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{(1-\theta)(1-\lambda)}{\lambda+(1-\lambda)\theta}}\right)\right)}{ED} = \xi_1^\circ$.

Finally, we go back to consumer's decision. To ensure $\phi_u = \phi_i = 1$, we need $u_{s,u} \geq u_{o,u}$ and $u_{s,i} > u_{o,i}$, which gives us the condition $h_s < \xi_1^\circ h_o$.

- Next, let's look for the condition where $\phi_i = 0$ and $\phi_u = 1$.

Given $\hat{\phi}_i = 0$ and $\hat{\phi}_u = 1$, the retailer's profit is

$$\begin{aligned}
\pi(q) =& pE\min\left((1-\lambda)\,\theta D, q\right) - cq + \left[w\theta - r\,(1-\theta)\right]E\left[(1-\lambda)\,D - \frac{q}{\theta}\right]^+ + wE\lambda D \\
=& \left[p - w + r\frac{1-\theta}{\theta}\right]E\min\left((1-\lambda)\,\theta D, q\right) - cq \\
&+ \left[\left(w\theta - r\,(1-\theta)\right)(1-\lambda) + w\lambda\right]ED
\end{aligned}$$

which is a typical newsvendor problem (plus a constant $\left[\left(w\theta - r\,(1-\theta)\right)(1-\lambda) + w\lambda\right]ED$), and therefore the optimal order quantity $q^\circ = (1-\lambda)\,\theta\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^{\circ} = \frac{E\min((1-\lambda)\theta D, q^{\circ})}{E(1-\lambda)\theta D} = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)\right)}{ED} = \xi_2^{\circ}$.

Finally, we go back to consumer's decision. To sure $\phi_u = 1$ and $\phi_i = 0$, we need $u_{s,u} > u_{o,u}$ and $u_{s,i} \leq u_{o,i}$, which gives us the condition $\xi_2^{\circ} \leq h_s < \xi_2^{\circ}[h_o + (1-\theta)h_r]$.

In the end, we note that $\xi_2^{\circ} \geq \xi_1^{\circ}$. Thus, the conditions for the two types of participatory RE equilibrium is disjoint. Then, the participatory RE equilibrium is unique. Also, for all the other cases, we only have nonparticipatory equilibrium. □

Proof of Proposition 73: Let's look for participatory RE equilibrium,, where at least one of ϕ_i and ϕ_u is 1.

- Let's first look for conditions where $\phi_u = \phi_i = 1$ is an equilibrium.

  Given $\hat{\phi}_u = \hat{\phi}_i = 1$, the retailer's profit is

$$\begin{aligned}\pi &= pE\min\left(\left(\lambda + (1-\lambda)\,\theta\right)D, q\right) - cq + wE(\left(\lambda + (1-\lambda)\,\theta\right)D - q)^{+} \\ &= (p-w)\,E\min\left(\left(\lambda + (1-\lambda)\,\theta\right)D, q\right) - cq + w\left(\lambda + (1-\lambda)\,\theta\right)ED\end{aligned}$$

  which is a typical newsvendor problem (plus a constant $w\left(\lambda + (1-\lambda)\,\theta\right)ED)$, and therefore the optimal order quantity $q^p = (\lambda + (1-\lambda)\theta)\bar{F}^{-1}(\frac{c}{p-w})$.

  Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^{p} = \frac{E\min((\lambda+(1-\lambda)\theta)D, q^p)}{E(\lambda+(1-\lambda)\theta)D} = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED}$.

  Finally, we go back to consumer's decision. To ensure $\phi_u = \phi_i = 1$, we need $u_{s,u} \geq u_{o,u}$ and $u_{s,i} > u_{o,i}$, which gives us the condition $h_s < \xi^p h_o$.

- Next, let's look for the condition where $\phi_i = 0$ and $\phi_u = 1$ is an equilibrium.

Given $\hat{\phi}_i = 0$ and $\hat{\phi}_u = 1$, the retailer's profit is

$$\begin{aligned}\pi(q) =& pE\min\left(\left(1-\lambda\right)\theta D, q\right) - cq + wE[\left(1-\lambda\right)D - q]^+ + wE\lambda D \\ =& \left(p-w\right)E\min\left(\left(1-\lambda\right)\theta D, q\right) - cq + \left[w(1-\lambda)\theta + w\lambda\right]ED\end{aligned}$$

which is a typical newsvendor problem (plus a constant $\left[w(1-\lambda)\theta + w\lambda\right]ED$), and therefore the optimal order quantity $q^p = \left(1-\lambda\right)\theta\bar{F}^{-1}\left(\frac{c}{p-w}\right)$.

Note in equilibrium, we also need consumer's belief to be consistent with the outcome, i.e., $\hat{\xi}^p = \frac{E\min((1-\lambda)\theta D, q^p)}{E(1-\lambda)\theta D} = \frac{E\min\left(D, \bar{F}^{-1}\left(\frac{c}{p-w}\right)\right)}{ED}$.

Finally, we go back to consumer's decision. To sure $\phi_u = 1$ and $\phi_i = 0$, we need $u_{s,u} > u_{o,u}$ and $u_{s,i} \leq u_{o,i}$, which gives us the condition $\xi^p \leq h_s < \xi^p\theta h_o + (1-\theta)(h_o + h_r)$.

□

Proof of Proposition 74: If $\max\{\xi_2^\circ h_o, \xi^p\theta h_o + (1-\theta)(h_o + h_r)\} < h_s < \xi_2^\circ[h_o + (1-\theta)h_r]$, then

$$\begin{aligned}\pi^p &= [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D \\ &< pE\min\left(\left(1-\lambda\right)\theta D, q^\circ\right) - cq^\circ + \left[w\theta - r\left(1-\theta\right)\right]E\left[\left(1-\lambda\right)D - \frac{q^\circ}{\theta}\right]^+ + wE\lambda D = \pi^\circ\end{aligned}$$

where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case where there is no physical showroom. □

Proof of Proposition 75: Since this is just the base model with a new set of parameters D', θ', λ', the proof is similar to the proof of Proposition 72, and thus omitted. □

Proof of Proposition 76: If $\max\{\xi_2^{\circ}h_o, \xi_2^{v}[h_o+(1-\theta')h_r]\} \le h_s < \xi_2^{\circ}[h_o+(1-\theta)h_r]$, then

$$\begin{aligned}
\pi^{\circ}-\pi^{v} &= \left(p-w+r\frac{1-\theta}{\theta}\right)E\min\left(\left(1-\lambda\right)\theta D, q^{\circ}\right)-cq^{\circ} \\
&\quad + \left(\left(w\theta-r\left(1-\theta\right)\right)\left(1-\lambda\right)+w\lambda\right)ED-\left(\left(w\theta'-r\left(1-\theta'\right)\right)\left(1-\lambda'\right)+w\lambda'\right)ED' \\
&= \left(p-w+r\frac{1-\theta}{\theta}\right)E\min\left(\left(1-\lambda\right)\theta D, q^{\circ}\right)-cq^{\circ}-r\alpha\left(1-\theta\right)\left(1-\lambda\right)ED
\end{aligned}$$

By Envelop Theorem, we have $\frac{\partial(\pi^{\circ}-\pi^{v})}{\partial w} = -E\min\left(\left(1-\lambda\right)\theta D, q^{\circ}\right) < 0$. To conclude the result, we just need to note that it is indeed possible to have $\pi^{\circ} > \pi^{v}$, e.g., when α is very close to 0. □

Proof of Proposition 77:

- If $h_s < h_o$, then the retailer's profit can be expressed as

$$\begin{aligned}
\pi &= \left[p-w+r\frac{\left(1-\theta\right)\left(1-\lambda\right)}{\lambda+\left(1-\lambda\right)\theta}\right]E\min\left(\left(\lambda+\left(1-\lambda\right)\theta\right)D, q\right)-cq \\
&\quad + \left[w\left(\theta\left(1-\lambda\right)+\lambda\right)-r\left(1-\theta\right)\left(1-\lambda\right)\right]ED,
\end{aligned}$$

  which is a newsvendor profit (plus a constant $\left[w\left(\theta\left(1-\lambda\right)+\lambda\right)-r\left(1-\theta\right)\left(1-\lambda\right)\right]ED$). Thus, the optimal inventory level is $q^a = \left(\lambda+\left(1-\lambda\right)\theta\right)\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{(1-\lambda)(1-\theta)}{\lambda+(1-\lambda)\theta}}\right)$;

- If $h_o \le h_s < h_o+(1-\theta)h_r$, then the retailer's profit can be expressed as

$$\begin{aligned}
\pi &= \left[p-w+r\tfrac{1-\theta}{\theta}\right]E\min\left(\left(1-\lambda\right)\theta D, q\right)-cq \\
&\quad + \left[w\left(\theta\left(1-\lambda\right)+\lambda\right)-r\left(1-\theta\right)\left(1-\lambda\right)\right]ED
\end{aligned}$$

  which is a newsvendor profit (plus a constant $\left[w\left(\theta\left(1-\lambda\right)+\lambda\right)-r\left(1-\theta\right)\left(1-\lambda\right)\right]ED$). Thus, the optimal inventory level is $q^a = \left(1-\lambda\right)\theta\bar{F}^{-1}\left(\frac{c}{p-w+r\frac{1-\theta}{\theta}}\right)$;

- If $h_s \ge h_o+(1-\theta)h_r$, then no one ever comes to store, and thus $q^a = 0$.

□

Proof of Proposition 78:

- If $h_s < \xi_1^\circ h_o$, then

$$\begin{aligned}
\pi^a &= \max_{q\geq 0}\Bigg\{pE\min\left(\left(\lambda+(1-\lambda)\,\theta\right)D,q\right)-cq \\
&\qquad +\left[w\theta-r\,(1-\theta)\right]E\,(1-\lambda)\left[D-\frac{q}{\lambda+(1-\lambda)\,\theta}\right]^+ + wE\lambda\left(D-\frac{q}{\lambda+(1-\lambda)\,\theta}\right)^+\Bigg\} \\
&= pE\min\left(\left(\lambda+(1-\lambda)\,\theta\right)D,q^\circ\right)-cq^\circ+\left[w\theta-r\,(1-\theta)\right]E\,(1-\lambda)\left[D-\frac{q^\circ}{\lambda+(1-\lambda)\,\theta}\right]^+ \\
&\qquad + wE\lambda\left(D-\frac{q^\circ}{\lambda+(1-\lambda)\,\theta}\right)^+ \\
&= \pi^\circ
\end{aligned}$$

- If $\xi_1^\circ h_o < h_s < h_o$,

  - if $\xi_2^\circ < h_s \leq \xi_2^\circ[h_o+(1-\theta)h_r]$, then

$$\begin{aligned}
\pi^a &= \max_{q\geq 0}\Bigg\{pE\min\left(\left(\lambda+(1-\lambda)\,\theta\right)D,q\right)-cq \\
&\qquad +\left[w\theta-r\,(1-\theta)\right]E\,(1-\lambda)\left[D-\frac{q}{\lambda+(1-\lambda)\,\theta}\right]^+ \\
&\qquad + wE\lambda\left(D-\frac{q}{\lambda+(1-\lambda)\,\theta}\right)^+\Bigg\} \\
&\geq pE\min\left(\left(\lambda+(1-\lambda)\,\theta\right)D,q^\circ\right)-cq^\circ \\
&\qquad +\left[w\theta-r\,(1-\theta)\right]E\,(1-\lambda)\left[D-\frac{q^\circ}{\lambda+(1-\lambda)\,\theta}\right]^+ \\
&\qquad + wE\lambda\left(D-\frac{q^\circ}{\lambda+(1-\lambda)\,\theta}\right)^+ \\
&= \left[p-w+r\,(1-\theta)\,\frac{1-\lambda}{\lambda+(1-\lambda)\,\theta}\right]E\min\left(\left(\lambda+(1-\lambda)\,\theta\right)D,q^\circ\right)-cq^\circ \\
&\qquad +\left[\left(w\theta-r\,(1-\theta)\right)(1-\lambda)+w\lambda\right]ED \\
&\geq \left[p-w+r\,(1-\theta)\,\frac{1}{\theta}\right]E\min\left((1-\lambda)\,\theta D,q^\circ\right)-cq^\circ \\
&\qquad +\left[\left(w\theta-r\,(1-\theta)\right)(1-\lambda)+w\lambda\right]ED \\
&= \pi^\circ
\end{aligned}$$

– otherwise, then

$$\begin{aligned}
\pi^a &= \max_{q\geq 0}\Big\{pE\min\left(\left(\lambda+\left(1-\lambda\right)\theta\right)D, q\right)-cq \\
&\quad +\left[w\theta-r\left(1-\theta\right)\right]E\left(1-\lambda\right)\left[D-\frac{q}{\lambda+\left(1-\lambda\right)\theta}\right]^+ + wE\lambda\left(D-\frac{q}{\lambda+\left(1-\lambda\right)\theta}\right)^+\Big\} \\
&\geq [w\theta-r(1-\theta)]E(1-\lambda)D+wE\lambda D \\
&= \pi^\circ
\end{aligned}$$

where the inequality is because $q=0$ is a feasible but not the optimal solution to the case where there is availability information.

- If $h_o < h_s < h_o+(1-\theta)h_r$, then

– if $\xi_2^\circ < h_s \leq \xi_2^\circ[h_o+(1-\theta)h_r]$, then

$$\begin{aligned}
\pi^a &= \max_{q\geq 0}\Big\{pE\min\left(\left(1-\lambda\right)\theta D, q\right)-cq \\
&\quad +\left[w\theta-r\left(1-\theta\right)\right]E\Big[\left(1-\lambda\right)D-\frac{q}{\theta}\Big]^+ + wE\lambda D\Big\} \\
&= pE\min\left(\left(1-\lambda\right)\theta D, q^\circ\right)-cq^\circ+\left[w\theta-r\left(1-\theta\right)\right]E\left[\left(1-\lambda\right)D-\frac{q^\circ}{\theta}\right]^+ \\
&\quad + wE\lambda D \\
&= \pi^\circ
\end{aligned}$$

– otherwise, then

$$\begin{aligned}
\pi^a &= \max_{q\geq 0}\Big\{pE\min\left(\left(1-\lambda\right)\theta D, q\right)-cq \\
&\quad +\left[w\theta-r\left(1-\theta\right)\right]E\Big[\left(1-\lambda\right)D-\frac{q}{\theta}\Big]^+ + wE\lambda D\Big\} \\
&\geq [w\theta-r(1-\theta)]E(1-\lambda)D+wE\lambda D \\
&= \pi^\circ
\end{aligned}$$

where the inequality is because $q=0$ is a feasible but not the optimal solution to the case where there is availability information.

- If $h_s > h_o + (1-\theta)h_r$, then $\pi^a = [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D = \pi^\circ$.

$\square$

Proof of Proposition 79:

- If $h_s < \min(\xi^p h_o, h_o, (1-\theta)(h_o+h_r))$, then

$$\begin{aligned}
\pi^{pa} &= \max_{q\geq 0}\left\{pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q\right) - cq + wE((\lambda+(1-\lambda)\theta)D-q)^+\right\} \\
&= pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q^p\right) - cq^p + wE((\lambda+(1-\lambda)\theta)D-q^p)^+ \\
&= \pi^p
\end{aligned}$$

- If $\xi^p h_o < h_s < \min(h_o, (1-\theta)(h_o+h_r))$, then

$$\begin{aligned}
\pi^{pa} &= \max_{q\geq 0}\left\{pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q\right) - cq + wE((\lambda+(1-\lambda)\theta)D-q)^+\right\} \\
&\geq pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q^p\right) - cq^p + wE((\lambda+(1-\lambda)\theta)D-q^p)^+ \\
&= (p-w)E\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q^p\right) - cq^p + w(\lambda+(1-\lambda)\theta)ED \\
&> (p-w)E\min\left(\left(1-\lambda\right)\theta D, q^p\right) - cq^p + w(\lambda+(1-\lambda)\theta)ED \\
&= \pi^p
\end{aligned}$$

$$\begin{aligned}
\pi^{pa} &= \max_{q\geq 0}\left\{pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q\right) - cq + wE((\lambda+(1-\lambda)\theta)D-q)^+\right\} \\
&\geq pE\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q^a\right) - cq^a + wE((\lambda+(1-\lambda)\theta)D-q^a)^+ \\
&= (p-w)E\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q^a\right) - cq^a + w(\lambda+(1-\lambda)\theta)ED \\
&> (p-w)E\min\left(\left(\lambda+(1-\lambda)\theta\right)D, q^a\right) - cq^a + w(\lambda+(1-\lambda)\theta)ED \\
&\qquad - r(1-\theta)(1-\lambda)[ED - E\min(D, \frac{q^a}{\lambda+(1-\lambda)\theta})] \\
&= \pi^a
\end{aligned}$$

- If $h_o < (1-\theta)(h_o+h_r)$ and $h_o < h_s < (1-\theta)(h_o+h_r)$, then

$$\begin{aligned}
\pi^{pa} &= \max_{q\geq 0}\left\{pE\min\left(\left(1-\lambda\right)\theta D, q\right) - cq + wE((1-\lambda)\theta D - q)^+ + wE\lambda D\right\} \\
&= pE\min\left(\left(1-\lambda\right)\theta D, q^p\right) - cq^p + wE((1-\lambda)\theta D - q^p)^+ + wE\lambda D \\
&= \pi^p
\end{aligned}$$

- If $h_o > (1-\theta)(h_o+h_r)$ and $(1-\theta)(h_o+h_r) < h_s < h_o$, then

$$\begin{aligned}
\pi^{pa} = \max_{q\geq 0}\Bigg\{ & pE\min\left(\left(\lambda+\left(1-\lambda\right)\theta\right)D, q\right) - cq \\
& + \left[w\theta - r\left(1-\theta\right)\right]E\left(1-\lambda\right)\left[D - \frac{q}{\lambda+\left(1-\lambda\right)\theta}\right]^+ + wE\lambda\left(D - \frac{q}{\lambda+\left(1-\lambda\right)\theta}\right)^+\Bigg\} \\
= {} & pE\min\left(\left(\lambda+\left(1-\lambda\right)\theta\right)D, q^a\right) - cq^a \\
& + \left[w\theta - r\left(1-\theta\right)\right]E\left(1-\lambda\right)\left[D - \frac{q^a}{\lambda+\left(1-\lambda\right)\theta}\right]^+ + wE\lambda\left(D - \frac{q^a}{\lambda+\left(1-\lambda\right)\theta}\right)^+ \\
= {} & \pi^a
\end{aligned}$$

- If $\max(h_o, (1-\theta)(h_o+h_r)) < h_s < h_o + (1-\theta)h_r$, then

$$\begin{aligned}
\pi &= \max_{q\geq 0}\left\{pE\min\left(\left(1-\lambda\right)\theta D, q\right) - cq + \left[w\theta - r\left(1-\theta\right)\right]E\left[\left(1-\lambda\right)D - \frac{q}{\theta}\right]^+ + wE\lambda D\right\} \\
&= pE\min\left(\left(1-\lambda\right)\theta D, q^a\right) - cq^a + \left[w\theta - r\left(1-\theta\right)\right]E\left[\left(1-\lambda\right)D - \frac{q^a}{\theta}\right]^+ + wE\lambda D \\
&= \pi^a
\end{aligned}$$

- If $h_s > h_o + (1-\theta)h_r$, then $\pi^{pa} = [w\theta - r(1-\theta)]E(1-\lambda)D + wE\lambda D = \pi^a$.

□

Proof of Proposition 80: Similar to the proof of Proposition 73, we can get the RE equilibrium of the case when there are both physical and virtual showrooms. We just need to replace D, θ, λ with D', θ', λ'. Thus, the RE equilibrium is given as follows:

- if $h_s \leq \xi^v h_o$, then all customers come to store, i.e., $\phi_i^v = 1, \phi_u^v = 1$

- if $\xi^v h_o < h_s \leq \xi^v \theta' h_o + (1-\theta')(h_o + h_r)$, then only uninformed customers come to store, i.e., $\phi_i^v = 0, \phi_u^v = 1$;

- otherwise, no one comes to store, i.e., $\phi_i^v = 0, \phi_u^v = 0$,

where $\xi^v = \frac{E \min\left(D', \bar{F}'^{-1}\left(\frac{c}{p-w}\right)\right)}{ED'}$.

It is easy to find that $\xi^v = \xi^p$.

- If $h_s < \xi^p h_o$, then

$$\begin{aligned}
\pi^{pv} &= \max_{q \geq 0} \left\{ pE \min\left(\left(\lambda' + \left(1-\lambda'\right)\theta'\right) D', q\right) - cq + wE\left(\left(\lambda' + \left(1-\lambda'\right)\theta'\right) D' - q\right)^+ \right\} \\
&= \max_{q \geq 0} \left\{ pE \min\left(\left(\lambda + \left(1-\lambda\right)\theta\right) D, q\right) - cq + wE\left(\left(\lambda + \left(1-\lambda\right)\theta'\right) D - q\right)^+ \right\} \\
&= pE \min\left(\left(\lambda + \left(1-\lambda\right)\theta\right) D, q^p\right) - cq^p + wE(\left(\lambda + \left(1-\lambda\right)\theta\right) D - q^p)^+ \\
&= \pi^p
\end{aligned}$$

- If $\xi^p h_o < h_s < \xi^p \theta' h_o + (1-\theta')(h_o + h_r)$, then

$$\begin{aligned}
\pi^{pv} &= \max_{q \geq 0} \left\{ pE \min\left(\left(1-\lambda'\right)\theta' D', q\right) - cq + w\theta' E\left(\left(1-\lambda'\right) D - \frac{q}{\theta'}\right)^+ + wE\lambda' D' \right\} \\
&= \max_{q \geq 0} \left\{ pE \min\left(\left(1-\lambda\right)\theta D, q\right) - cq + w\theta E\left(\left(1-\lambda\right) D - \frac{q}{\theta}\right)^+ + wE\lambda D \right\} \\
&= pE \min\left(\left(1-\lambda\right)\theta D, q^p\right) - cq^p + w\theta E\left(\left(1-\lambda\right) D - \frac{q^p}{\theta}\right)^+ + wE\lambda D \\
&= \pi^p
\end{aligned}$$

- If $\xi^p \theta' h_o + (1-\theta')(h_o+h_r) < h_s < \xi^p \theta h_o + (1-\theta)(h_o+h_r)$, then

$$\begin{aligned}
\pi^{pv} &= \left[w\theta' - r\left(1-\theta'\right)\right] E\left(1-\lambda'\right) D + wE\lambda' D' \\
&\leq w\theta' E(1-\lambda')D' + wE\lambda' D' \\
&= w\theta E(1-\lambda)D + wE\lambda D \\
&\leq pE\min\left((1-\lambda)\theta D, q^p\right) - cq^p + w\theta E\left((1-\lambda)D - \frac{q^p}{\theta}\right)^+ + wE\lambda D \\
&= \pi^p
\end{aligned}$$

  where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case where there is only physical showroom.

- If $h_s > \xi^p \theta h_o + (1-\theta)(h_o+h_r)$, then $\pi^{pv} = [w\theta' - r(1-\theta')] E(1-\lambda') D' + wE\lambda' D'$. Let's compare π^{pv} with π^v. Note π^v can take only three values:

  – if $\phi_i^v = \phi_u^v = 1$, then

$$\begin{aligned}
\pi^v = & pE\min\left(\left(\lambda' + \left(1-\lambda'\right)\theta'\right) D', q^v\right) - cq^v \\
& + \left[w\theta' - r\left(1-\theta'\right)\right] E\left(1-\lambda'\right)\left[D' - \frac{q^v}{\lambda' + (1-\lambda')\theta'}\right]^+ \\
& + wE\lambda'\left(D' - \frac{q^v}{\lambda' + (1-\lambda')\theta'}\right)^+ \\
\geq & \left[w\theta' - r\left(1-\theta'\right)\right] E\left(1-\lambda'\right) D' + wE\lambda' D' = \pi^{pv}
\end{aligned}$$

  where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case where there is only virtual showroom;

  – if $\phi_i^v = 0$ and $\phi_u^v = 1$, then

$$\begin{aligned}
\pi^v = & pE\min\left(\left(1-\lambda'\right)\theta' D', q^v\right) - cq^v \\
& + \left[w\theta' - r\left(1-\theta'\right)\right] E\left[\left(1-\lambda'\right) D' - \frac{q^v}{\theta'}\right]^+ + wE\lambda' D' \\
\geq & \left[w\theta' - r\left(1-\theta'\right)\right] E\left(1-\lambda'\right) D' + wE\lambda' D' = \pi^{pv}
\end{aligned}$$

where the inequality is because $q = 0$ is also a feasible but not the optimal solution to the case where there is only virtual showroom;

– if $\phi_i^v = \phi_u^v = 0$, then $\pi^v = \left[w\theta' - r\left(1-\theta'\right)\right] E\left(1-\lambda'\right) D' + wE\lambda' D' = \pi^{pv}$.

$\square$

Proof of Proposition 81: Let's define $\alpha = (1-\eta)\alpha_h + \eta\alpha_m$ and $\beta = (1-\eta)\beta_h + \eta\beta_m$.

Since we only focus on the case where the retailer serves both types, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned}
\max_{\substack{\lambda_m, \lambda_h \geq 0, \\ (1-\eta)\lambda_h + \eta\lambda_m \leqslant \mu_1 \leqslant \frac{r\alpha}{c_1}, \\ (1-\eta)\lambda_h + \eta\lambda_m \leqslant \mu_2 \leqslant \frac{r\alpha}{c_2}}} \quad & r((1-\eta)\lambda_h + \eta\lambda_m) - c_1\mu_1 - c_2\mu_2 \\
s.t. \quad \lambda_m &= \alpha_m - \beta_m\left(w_1(\mu_1, (1-\eta)\lambda_h + \eta\lambda_m) + w_2(\mu_2, (1-\eta)\lambda_h + \eta\lambda_m)\right) \\
\lambda_h &= \alpha_h - \beta_h\left(w_1(\mu_1, (1-\eta)\lambda_h + \eta\lambda_m) + w_2(\mu_2, (1-\eta)\lambda_h + \eta\lambda_m)\right)
\end{aligned}$$

the optimal solution of which can be obtained by the following optimization problem:

$$\begin{aligned}
\max_{0 \leq \lambda \leq \mu_i \leq \frac{r\alpha}{c_i}} \quad & r\lambda - c_1\mu_1 - c_2\mu_2 \\
s.t. \quad & \lambda = \alpha - \beta\left(w_1(\mu_1, \lambda) + w_2(\mu_2, \lambda)\right)
\end{aligned} \tag{A.29}$$

where $\lambda = (1-\eta)\lambda_h + \eta\lambda_m$. The Lagrangian of (A.29) is defined as follows:

$$L\left(\lambda, \mu_1, \mu_2, \rho\right) = r\lambda - c_1\mu_1 - c_2\mu_2 + \rho\left[\lambda - \alpha + \beta\left(\frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}\right)\right]$$

where $\rho \in \mathbb{R}$ is the Lagrange multiplier. To find the critical points of $L\left(\lambda, \mu_1, \mu_2, \rho\right)$, we solve the following equation set:

$$
\begin{aligned}
&\frac{\partial L}{\partial \lambda} = r + \rho + \rho\beta\frac{1}{(\mu_1-\lambda)^2} + \rho\beta\frac{1}{(\mu_2-\lambda)^2} = 0\\
&\frac{\partial L}{\partial \mu_i} = -c_i - \rho\beta\frac{1}{(\mu_i-\lambda)^2} = 0, i = 1, 2\\
&\lambda - \alpha + \beta\left(\frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2-\lambda}\right) = 0\\
&0 \le \lambda \le \mu_1 \le \frac{r\alpha}{c_1}, 0 \le \lambda \le \mu_2 \le \frac{r\alpha}{c_2}
\end{aligned}
$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.29) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$
\begin{aligned}
&\lambda^b = \alpha - \beta\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}\\
&\mu_1^b = \lambda^b + \sqrt{\frac{\beta(r-c_1-c_2)}{c_1}}\\
&\mu_2^b = \lambda^b + \sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}
\end{aligned}
$$

Then, we can find $\lambda_h^b = \alpha_h - \beta_h w_1(\mu_1^b, \lambda^b) - \beta_h w_2(\mu_2^b, \lambda^b) = \alpha_h - \beta_h\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta_h\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}$ and $\lambda_m^b = \alpha_m - \beta_m w_1(\mu_1^b, \lambda^b) - \beta_m w_2(\mu_2^b, \lambda^b) = \alpha_m - \beta_m\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}} - \beta_m\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}$. □

Proof of Proposition 82: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$
\begin{aligned}
&\max_{\substack{\lambda_{mo},\lambda_h,\lambda_{ms}\ge 0,\\ (1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\le\mu_1\le\frac{r\alpha}{c_1},\\ \theta\lambda_{mo}+(1-\eta)\lambda_h+(\eta-\theta)\lambda_{ms}\le\mu_2\le\frac{r\alpha}{c_2}}} \quad r\left(\theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}\right) - c_1\mu_1 - c_2\mu_2\\
&s.t. \quad \lambda_{mo} = \alpha_m - \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms})\\
&\qquad \lambda_h = \alpha_h - \beta_h\left(w_1(\mu_1, (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) + w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms})\right)\\
&\qquad \lambda_{ms} = \alpha_m - \beta_m(w_1(\mu_1, (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}) + w_2(\mu_2, \theta\lambda_{mo} + (1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}))
\end{aligned}
$$

the optimal solution of which can be obtained by solving the following optimization problem:

$$
\begin{aligned}
\max \quad & r\left(\theta\lambda_{mo} + (1-\theta)\lambda_s\right) - c_1\mu_1 - c_2\mu_2 \\
& 0 \le (1-\theta)\lambda_s \le \mu_1 \le \tfrac{r\alpha}{c_1}, \\
& \theta\lambda_{mo} + (1-\theta)\lambda_s \le \mu_2 \\
s.t. \quad & \lambda_{mo} = \alpha_m - \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s) \\
& \lambda_s = \alpha_s - \beta_s\left(w_1(\mu_1, \lambda_s) + w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s)\right)
\end{aligned} \tag{A.30}
$$

where $\lambda_s = \frac{(1-\eta)\lambda_h + (\eta-\theta)\lambda_{ms}}{1-\theta}$, $\alpha_s = \frac{(1-\eta)\alpha_h + (\eta-\theta)\alpha_m}{1-\theta}$ and $\beta_s = \frac{(1-\eta)\beta_h + (\eta-\theta)\beta_m}{1-\theta}$. The Lagrangian of (A.30) is

$$
\begin{aligned}
L\left(\lambda_{mo}, \lambda_s, \mu_1, \mu_2, \rho_1, \rho_2\right) =& r\left(\theta\lambda_{mo} + (1-\theta)\lambda_s\right) - c_1\mu_1 - c_2\mu_2 \\
& + \rho_1\left(\lambda_{mo} - \alpha_m + \xi\beta_m w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s)\right) \\
& + \rho_2\left(\lambda_s - \alpha_s + \beta_s\left(w_1(\mu_1, \lambda_s) + w_2(\mu_2, \theta\lambda_{mo} + (1-\theta)\lambda_s)\right)\right)
\end{aligned}
$$

To find the critical points of $L\left(\lambda_{mo}, \lambda_s, \mu_1, \mu_2, \rho_1, \rho_2\right)$, we solve the following equation set:

$$
\begin{aligned}
& \tfrac{\partial L}{\partial \lambda_s} = r(1-\theta) + \rho_2 + \rho_1\xi\beta_m(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} + \rho_2\beta_s(1-\theta)\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^2} \\
& \qquad +\rho_2\beta_s(1-\theta)\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o)^2} = 0 \\
& \tfrac{\partial L}{\partial \lambda_{mo}} = r\theta + \rho_1 + \rho_1\xi\beta_m\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} + \rho_2\beta_s\theta\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} = 0 \\
& \tfrac{\partial L}{\partial \mu_1} = -c_1 - \rho_2\beta_s\tfrac{1}{(\mu_1-(1-\theta)\lambda_s)^2} = 0 \\
& \tfrac{\partial L}{\partial \mu_2} = -c_2 - \rho_1\xi\beta_m\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} - \rho_2\beta_s\tfrac{1}{(\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo})^2} = 0 \\
& \lambda_{mo} - \alpha_m + \xi\beta_m\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo}} = 0 \\
& \lambda_s - \alpha_s + \beta_s\tfrac{1}{\mu_1-(1-\theta)\lambda_s} + \beta_s\tfrac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_{mo}} = 0 \\
& 0 \le (1-\theta)\lambda_s \le \mu_1 \le \tfrac{r\alpha}{c_1}, 0 \le \lambda_{mo}, (1-\theta)\lambda_s + \theta\lambda_{mo} \le \mu_2 \le \tfrac{r\alpha}{c_2}
\end{aligned}
$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.30) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve all types of customers, the solution must be interior, which gives us a unique solution:

$$\begin{aligned}
\mu_1^o &= (1-\theta)\,\lambda_s^o + \sqrt{\tfrac{\beta_s(r-c_1-c_2)}{c_1}} \\
\mu_2^o &= (1-\theta)\,\lambda_s^o + \theta\lambda_{mo}^o + \sqrt{\tfrac{\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}} \\
\lambda_{mo}^o &= \alpha_m - \xi\beta_m\sqrt{\tfrac{c_2}{\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}} \\
\lambda_s^o &= \alpha_s - \beta_s\sqrt{\tfrac{c_1}{\beta_s(r-c_1-c_2)}} - \beta_h\sqrt{\tfrac{c_2}{\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}
\end{aligned}$$

Then,

$$\begin{aligned}
\lambda_s^o &= \alpha_h - \beta_h w_1(\mu_1^o,(1-\theta)\lambda_s^o) - \beta_h w_2(\mu_2^o,\theta\lambda_{mo}^o+(1-\theta)\lambda_s^o) \\
&= \alpha_h - \beta_h\sqrt{\tfrac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}} - \beta_h\sqrt{\tfrac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{ms}^o &= \alpha_m - \beta_m w_1(\mu_1^o,(1-\theta)\lambda_s^o) - \beta_m w_2(\mu_2^o,\theta\lambda_{mo}^o+(1-\theta)\lambda_s^o) \\
&= \alpha_m - \beta_m\sqrt{\tfrac{c_1}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)}} - \beta_m\sqrt{\tfrac{c_2}{((1-\eta)\beta_h+(\eta-\theta)\beta_m)(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}
\end{aligned}$$

$\square$

Proof of Proposition 83: Note

$$\begin{aligned}
\lambda_{mo}^o &= \alpha_m - \xi\beta_m\sqrt{\tfrac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}} \\
&\geq \alpha_m - \beta_m\sqrt{\tfrac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\beta_m(r-c_2)}} \\
&> \alpha_m - \beta_m\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}} \\
&> \alpha_m - \beta_m\sqrt{\tfrac{c_1}{\beta(r-c_1-c_2)}} - \beta_m\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}} = \lambda_m^b
\end{aligned}$$

where the first inequality is because of $\xi \leq 1$, and the second inequality is because $\beta = (1-\eta)\beta_h + \eta\beta_m = (1-\theta)\beta_s + \theta\beta_m$.

Note

$$\begin{aligned}&\left(\lambda_h^o-\lambda_h^b\right)\tfrac{\sqrt{r-c_1-c_2}}{\beta_h}\\ &=\left(\lambda_{ms}^o-\lambda_{ms}^b\right)\tfrac{\sqrt{r-c_1-c_2}}{\beta_m}\\ &=\sqrt{\tfrac{c_1}{\beta}}+\sqrt{\tfrac{c_2}{\beta}}-\sqrt{\tfrac{c_1}{(1-\theta)\beta_s}}-\sqrt{\tfrac{c_2}{(1-\theta)\beta_s+\theta\beta_m\frac{r-c_2}{r-c_1-c_2}}}\end{aligned}$$

which is decreasing in r. Then if $\sqrt{\frac{c_1}{\beta}}+\sqrt{\frac{c_2}{\beta}}-\sqrt{\frac{c_1}{(1-\theta)\beta_s}}>0$ (or θ is small enough), then given c_1 and c_2, there exists $\bar{r}>c_1+c_2$ such that $\lambda_h^o-\lambda_h^b>0$ and $\lambda_{ms}^o-\lambda_m^b>0$ if and only if $\frac{c_1+c_2}{r}>\frac{c_1+c_2}{\bar{r}}$. □

Proof of Proposition 84:

$$\begin{aligned}&k_1^o+k_2^o-k_1^b-k_2^b\\ &=-\theta\alpha\tau_1+\sqrt{\tfrac{\beta c_1}{r-c_1-c_2}}\left(\tau_1+\tau_2\right)\left(1-\sqrt{1-\theta}\right)\\ &\quad-\left(\tau_1+\tau_2\right)\left[\left(\left(1-\theta\right)\beta_s+\theta\xi\beta_m\right)\sqrt{\tfrac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}-\beta\sqrt{\tfrac{c_2}{\beta(r-c_1-c_2)}}\right]\\ &\quad-\sqrt{\tfrac{\beta(r-c_1-c_2)}{c_1}}\tau_1\left(1-\sqrt{1-\theta}\right)+\tau_2\left(\sqrt{\tfrac{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}}-\sqrt{\tfrac{\beta(r-c_1-c_2)}{c_2}}\right)\\ &\quad+\tau_1\theta\xi\beta_m\sqrt{\tfrac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}\end{aligned}\tag{A.31}$$

Let's first show the 3rd term in (A.31), i.e., $-\left(\tau_1+\tau_2\right)\left[\left(\left(1-\theta\right)\beta_s+\theta\xi\beta_m\right)\sqrt{\frac{c_2}{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}}-\beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}\right]$ (denoted as $f_3(r)$) is decreasing in r:

$$\frac{\partial f_3}{\partial r}=-\left(\tau_1+\tau_2\right)\left[-\frac{\sqrt{c_2}}{2}\frac{\sqrt{(1-\theta)\beta_s+\theta\xi\beta_m}}{\left(r-c_2-\frac{(1-\theta)\beta_s c_1}{(1-\theta)\beta_s+\theta\xi\beta_m}\right)^{\frac{3}{2}}}+\frac{\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}}\right]$$

Since $\frac{\partial[(1-\theta)\beta_s+\theta\xi\beta_m]}{\partial\theta}\leq 0$ and $\frac{\partial\left[r-c_2-\frac{(1-\theta)\beta_s c_1}{(1-\theta)\beta_s+\theta\xi\beta_m}\right]}{\partial\theta}>0$ and $\frac{\partial\beta}{\partial\theta}=\beta_m-\beta_s\geq 0$ (because $\beta_m\geq\beta_h$), we can find that $\frac{\partial f_3}{\partial r}$ is decreasing in θ. Note when $\theta=0$, $\frac{\partial f_3}{\partial r}=0$. Thus, $\frac{\partial f_3}{\partial r}<0$ for all $\theta>0$.

Next, let's show the 5th term in (A.31), i.e., $\tau_2\left(\sqrt{\frac{(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_m(r-c_2)}{c_2}}-\sqrt{\frac{\beta(r-c_1-c_2)}{c_2}}\right)$

(denoted as $f_5(r)$) is decreasing in r:

$$\frac{\partial f_5}{\partial r} = \frac{\tau_2}{2\sqrt{c_2}} \left(\frac{\sqrt{(1-\theta)\,\beta_s + \theta\xi\beta_m}}{\sqrt{r - \frac{(1-\theta)\beta_s(c_1+c_2)+\theta\xi\beta_m c_2}{(1-\theta)\beta_s+\theta\xi\beta_m}}} - \frac{\sqrt{\beta}}{\sqrt{r-c_1-c_2}} \right) < 0$$

Therefore, we can conclude that $\frac{\partial(k_1^o+k_2^o-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1 + c_2$, we have $(k_1^o + k_2^o - k_1^b - k_2^b)\sqrt{r-c_1-c_2} \to (\tau_1+\tau_2)\sqrt{\beta c_2} + \sqrt{\beta c_1}(\tau_1+\tau_2)(1-\sqrt{1-\theta}) > 0$, which implies $k_1^o + k_2^o - k_1^b - k_2^b > 0$ if r is very close to $c_1 + c_2$. Thus, there exists $\bar{r} > c_1 + c_2$ such that $k_1^o + k_2^o - k_1^b - k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k = \frac{c_1+c_2}{\bar{r}}$.

□

Proof of Proposition 85: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned} \max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1},\\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}} \quad & r\left(\eta\lambda_m + (1-\eta)\,\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\ s.t. \quad \lambda_m &= \alpha_m - \beta_m w_{1m}\left(\mu_{1m}, \eta\lambda_m\right) - \beta_m w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\,\lambda_h\right) \\ \lambda_h &= \alpha_h - \beta_h w_{1h}\left(\mu_{1h}, (1-\eta)\,\lambda_h\right) - \beta_h w_2\left(\mu_2, \eta\lambda_m + (1-\eta)\,\lambda_h\right) \end{aligned} \tag{A.32}$$

The Lagrangian of (A.32) is

$$\begin{aligned} & L\left(\lambda_m, \lambda_h, \mu_{1h}, \mu_{1m}, \mu_2, \rho_h, \rho_m\right) \\ &= r\left(\eta\lambda_m + (1-\eta)\,\lambda_h\right) - c_1\mu_{1h} - c_{1m}\mu_{1m} - c_2\mu_2 \\ &\quad - \rho_h\left(\lambda_h - \alpha_h + \beta_h\frac{1}{\mu_{1h} - (1-\eta)\,\lambda_h} + \beta_h\frac{1}{\mu_2 - (1-\eta)\,\lambda_h - \eta\lambda_m}\right) \\ &\quad - \rho_m\left(\lambda_m - \alpha_m + \beta_m\frac{1}{\mu_{1m} - \eta\lambda_m} + \beta_m\frac{1}{\mu_2 - (1-\eta)\,\lambda_h - \eta\lambda_m}\right) \end{aligned}$$

To find the critical points, we solve the following equation set:

$$
\begin{aligned}
&\frac{\partial L}{\partial \lambda_m} = r\eta - \rho_m - \frac{\rho_m \beta_m \eta}{(\mu_{1m} - \eta\lambda_m)^2} - \frac{(\rho_m\beta_m + \rho_h\beta_h)\eta}{(\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h)^2} = 0 \\
&\frac{\partial L}{\partial \lambda_h} = r(1-\eta) - \rho_h - \frac{\rho_h\beta_h(1-\eta)}{(\mu_{1h} - (1-\eta)\lambda_h)^2} - \frac{(\rho_m\beta_m + \rho_h\beta_h)(1-\eta)}{(\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h)^2} = 0 \\
&\frac{\partial L}{\partial \mu_{1m}} = -c_{1m} + \frac{\rho_m\beta_m}{(\mu_{1m} - \eta\lambda_m)^2} = 0 \\
&\frac{\partial L}{\partial \mu_{1h}} = -c_1 + \frac{\rho_h\beta_h(1-\eta)}{(\mu_{1h} - (1-\eta)\lambda_h)^2} = 0 \\
&\frac{\partial L}{\partial \mu_2} = -c_2 + \frac{(\rho_m\beta_m + \rho_h\beta_h)(1-\eta)}{(\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h)^2} = 0 \\
&0 \le \eta\lambda_m \le \mu_{1m} \le \frac{r\alpha}{c_{1m}}, 0 \le (1-\eta)\lambda_h \le \mu_{1h} \le \frac{r\alpha}{c_1}, \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2 \le \frac{r\alpha}{c_2}
\end{aligned}
$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.30) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve all types of customers, the solution must be interior, which gives us a unique solution:

$$
\begin{aligned}
&\lambda_m^s = \alpha_m - \beta_m\sqrt{\frac{c_{1m}}{\beta_m\eta(r - c_{1m} - c_2)}} - \beta_m\sqrt{\frac{c_2}{\beta_m(r - c_{1m} - c_2)\eta + \beta_h(r - c_1 - c_2)(1-\eta)}} \\
&\lambda_h^s = \alpha_h - \beta_h\sqrt{\frac{c_1}{\beta_h(1-\eta)(r - c_1 - c_2)}} - \beta_h\sqrt{\frac{c_2}{\beta_m(r - c_{1m} - c_2)\eta + \beta_h(r - c_1 - c_2)(1-\eta)}} \\
&\mu_{1m}^s = \eta\lambda_m^s + \sqrt{\frac{\beta_m\eta(r - c_{1m} - c_2)}{c_{1m}}} \\
&\mu_{1h}^s = (1-\eta)\lambda_h^s + \sqrt{\frac{\beta_h(1-\eta)(r - c_1 - c_2)}{c_1}} \\
&\mu_2^s = \eta\lambda_m^s + (1-\eta)\lambda_h^s + \sqrt{\frac{\beta_m(r - c_{1m} - c_2)\eta + \beta_h(r - c_1 - c_2)(1-\eta)}{c_2}}
\end{aligned}
$$

□

Proof of Proposition 86: First, note

$$
\begin{aligned}
\frac{(\lambda_m^s - \lambda_m^b)\sqrt{\beta}}{\beta_m} = &\sqrt{\frac{c_1}{r - c_1 - c_2}} + \sqrt{\frac{c_2}{r - c_1 - c_2}} \\
&- \sqrt{\frac{c_{1m}\beta}{\beta_m\eta(r - c_{1m} - c_2)}} - \sqrt{\frac{c_2\beta}{\beta_m\eta(r - c_{1m} - c_2) + \beta_h(1-\eta)(r - c_1 - c_2)}}
\end{aligned}
$$

Because $c_{1m} < c_1$, we have $\sqrt{\frac{c_2}{r-c_1-c_2}} - \sqrt{\frac{c_2\beta}{\beta_m\eta(r-c_{1m}-c_2)+\beta_h(1-\eta)(r-c_1-c_2)}} > 0$. Also, we can easily check that $w_{1h}^s > w_1^b$. Then, since $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, we must have $w_{1m}^s < w_1^b$, which implies $\sqrt{\frac{c_1}{r-c_1-c_2}} - \sqrt{\frac{c_{1m}\beta}{\beta_m\eta(r-c_{1m}-c_2)}} > 0$. Thus, we have $\frac{(\lambda_m^s - \lambda_m^b)\sqrt{\beta}}{\beta_m} > 0$ or $\lambda_m^s > \lambda_m^b$.

Second, note

$$\frac{\left(\lambda_h^s-\lambda_h^b\right)\sqrt{\beta\left(r-c_1-c_2\right)}}{\beta_h}=\sqrt{c_1}+\sqrt{c_2}-\sqrt{\frac{c_1\beta}{\beta_h\left(1-\eta\right)}}-\sqrt{\frac{c_2\beta}{\beta_m\eta\frac{r-c_{1m}-c_2}{r-c_1-c_2}+\beta_h\left(1-\eta\right)}}$$

which is decreasing in r. Then if $\sqrt{c_1}+\sqrt{c_2}-\sqrt{\frac{c_1\beta}{(1-\eta)\beta_h}}>0$ (or η is small enough), then there exists $\bar{r}>c_1+c_2$ such that $\lambda_h^s-\lambda_h^b>0$ if and only if $\frac{c_1+c_2}{r}>\frac{c_1+c_2}{\bar{r}}$.

Finally, let's look at total demand rate. We first prove the following lemma:

Lemma 5. *If* $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, *then* $\sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}}+\sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}<\sqrt{\frac{\beta c_1}{r-c_1-c_2}}$.

Proof of Lemma 5: Because $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, we have $\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}}+\sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}<\sqrt{\frac{\beta c_1}{r-c_1-c_2}}$. So all we need show is $\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}}+\sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}\geq\sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}}+\sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}$. Note

$$\begin{aligned}
&\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}}+\sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}-\sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}}-\sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}\\
&\geq\sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}-\sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}-\left[\sqrt{\beta_m^2}-\sqrt{\beta^2}\right]\left[\sqrt{\frac{c_1}{\beta(r-c_1-c_2)}}-\sqrt{\frac{(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}\right]\\
&=\sqrt{\frac{(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}\left(\beta_m-\beta_h\right)\left(1-\eta\right)\left[\frac{1}{1-\eta}-\sqrt{\frac{\beta_h}{(1-\eta)\beta}}\right]\\
&\geq 0
\end{aligned}$$

where the first inequality is because $\sqrt{\frac{\beta^2\eta c_{1m}}{\beta_m(r-c_{1m}-c_2)}}+\sqrt{\frac{\beta^2(1-\eta)c_1}{\beta_h(r-c_1-c_2)}}-\sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}}-\sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}$ is decreasing in c_{1m} and $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, and the second inequality is because $\beta_m\geq\beta_h$ and $\beta>(1-\eta)\beta_h$. This completes the proof. □

Because $\eta w_{1m}^s+(1-\eta)w_{1h}^s<w_1^b$, Lemma 5 and $c_{1m}<c_1$, we can check that $\eta\lambda_m^s+(1-\eta)\,\lambda_h^s=\alpha-\sqrt{\frac{\beta_m\eta c_{1m}}{r-c_{1m}-c_2}}-\sqrt{\frac{\beta_h(1-\eta)c_1}{r-c_1-c_2}}-\beta\sqrt{\frac{c_2}{\beta_m\eta(r-c_{1m}-c_2)+\beta_h(1-\eta)(r-c_1-c_2)}}>\alpha-\sqrt{\frac{\beta c_1}{r-c_1-c_2}}-\beta\sqrt{\frac{c_2}{\beta(r-c_1-c_2)}}=\lambda^b$ □

Proof of Proposition 87: Note $\frac{\partial k_1^s}{\partial r}=\frac{\tau_1\sqrt{\beta_h(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}}+\frac{\tau_1(1-\eta)\beta_h\beta\sqrt{c_2}}{2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}}+\frac{\tau_1\sqrt{\beta_h(1-\eta)}}{2\sqrt{c_1(r-c_1-c_2)}}<\frac{\tau_1\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}}+\frac{\tau_1\sqrt{\beta c_2}}{2(r-c_1-c_2)^{\frac{3}{2}}}+\frac{\tau_1\sqrt{\beta}}{2\sqrt{c_1(r-c_1-c_2)}}=\frac{\partial k_1^b}{\partial r}$, where the inequality is because $c_{1m} < c_1$ and $\beta > (1-\eta)\beta_h$.

Also, note

$$\begin{aligned}\frac{\partial k_2^s}{\partial r} =& \frac{\sqrt{\eta\beta_m c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta_h(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} \\ &+ \frac{\beta^2\sqrt{c_2}}{2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\beta}{2\sqrt{c_2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]}}\end{aligned}$$

By Lemma 5 and $c_{1m} < c_1$, we have $\frac{\sqrt{\eta\beta_m c_{1m}}}{2(r-c_{1m}-c_2)^{\frac{3}{2}}} + \frac{\sqrt{\beta_h(1-\eta)c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} < \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}}$. Also, define function $f(c_{1m}) = \frac{\beta^2\sqrt{c_2}}{2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]^{\frac{3}{2}}} + \frac{\beta}{2\sqrt{c_2[\beta_m(r-c_{1m}-c_2)\eta+\beta_h(r-c_1-c_2)(1-\eta)]}}$, which is an increasing function. Thus, $f(c_{1m}) < f(c_1)$. Since $\frac{\partial k_2^b}{\partial r} = \frac{\sqrt{\beta c_1}}{2(r-c_1-c_2)^{\frac{3}{2}}} + f(c_1)$, we have $\frac{\partial k_2^s}{\partial r} < \frac{\partial k_2^b}{\partial r}$.

Therefore, we can conclude that $\frac{\partial(k_1^s+k_2^s-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1 + c_2$, we have $(k_1^s + k_2^s - k_1^b - k_2^b)\sqrt{r-c_1-c_2} \to (\tau_1+\tau_2)(\sqrt{\beta c_1} + \sqrt{\beta c_2} - \sqrt{\beta_h(1-\eta)c_1}) > 0$, which implies $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if r is very close to $c_1 + c_2$. Thus, there exists $\bar{r} > c_1 + c_2$ such that $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k' = \frac{c_1+c_2}{\bar{r}}$. $\square$

Proof of Proposition 88:

$$\begin{aligned}\pi^o - \pi^s =& -c_1(1-\theta)\alpha + [c_{1m}\eta + c_1(1-\eta)]\alpha \\ &+ 2\sqrt{\eta\beta_m(r-c_{1m}-c_2)c_{1m}} + 2\sqrt{(1-\eta)\beta_h(r-c_1-c_2)c_1} \\ &+ 2\sqrt{c_2[\beta_m(r-c_{1m}-c_2)\eta + \beta_h(r-c_1-c_2)(1-\eta)]} \\ &- 2\sqrt{(1-\theta)\beta_s(r-c_1-c_2)c_1} - 2\sqrt{[(1-\theta)\beta_s(r-c_1-c_2)+\theta\xi\beta_o(r-c_2)]c_2}\end{aligned}$$

Note that $\frac{\partial^2(\pi^o-\pi^s)}{\partial\theta^2} < 0$. Also note that when $\theta = \eta$, we must have the $\pi^o > \pi^s$, the proof of which is as follows: Suppose the optimal solution for the offline model is $\mu_{1m}^s, \mu_{1h}^s, \mu_2^s, \lambda_m^s, \lambda_h^s$. Consider the following feasible solution for the online model: $\mu_1^\triangle, \mu_2^\triangle, \lambda_m^\triangle, \lambda_h^\triangle$, where $\mu_1^\triangle = \mu_{1h}^s$ and $\mu_2^\triangle = \mu_2^s$. Then, suppose $\eta\lambda_m^\triangle + (1-\eta)\lambda_h^\triangle \le \eta\lambda_m^s + (1-\eta)\lambda_h^s$. Then, we must have $w_2^\triangle \le w_2^s$. Then, because online customers don't wait at stage 1, we have $\lambda_m^\triangle > \lambda_m^s$. Then, $\lambda_h^\triangle < \lambda_h^s$. Then, $w_1^\triangle < w_1^s$. However, if $w_1^\triangle < w_1^s$ and $w_2^\triangle < w_2^s$, then this means that

$\lambda_h^{\triangle} > \lambda_h^s$. We get a contradiction. Thus, we must have $\eta\lambda_m^{\triangle} + (1-\eta)\lambda_h^{\triangle} > \eta\lambda_m^s + (1-\eta)\lambda_h^s$. This implies that $\pi^{\triangle} > \pi^s$. Thus, we must have $\pi^o \geq \pi^{\triangle} > \pi^s$. The results above implies that $\exists\bar{\theta} \leq \eta$ such that $\pi^o > \pi^s$ if and only if $\theta > \bar{\theta}$.

Note when $c_{1m}\eta = c_1(\eta - \theta)$ (or $\theta = \frac{(c_1 - c_{1m})\eta}{c_1}$), we have $\pi^o - \pi^s > 0$ for any $b > 0$. This implies that $\bar{\theta} \leq \frac{(c_1 - c_{1m})\eta}{c_1}$. Thus, for any $b > 0$, if $\theta \geq \frac{(c_1 - c_{1m})\eta}{c_1}$ (or $c_{1m}\eta - c_1(\eta-\theta) \geq 0$), we must have $\pi^o - \pi^s > 0$, i.e., $\bar{b} = 0$. If $c_{1m}\eta - c_1(\eta-\theta) < 0$, then $\frac{\partial \frac{\pi^o - \pi^s}{\sqrt{b}}}{\partial b} = \frac{\partial \frac{(c_{1m}\eta - c_1(\eta-\theta))\alpha}{\sqrt{b}}}{\partial b} > 0$. Thus, there exists $\bar{b} \geq 0$ such that $\pi^o - \pi^s > 0$ if and only if $b > \bar{b}$. $\square$

Proof of Proposition 89: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned} &\max_{0 \leqslant \lambda \leqslant \mu_1 \leqslant \frac{r\alpha}{c_1}, \lambda \leqslant \mu_2 \leqslant \frac{r\alpha}{c_2}} r\lambda - c_1\mu_1 - c_2\mu_2 \\ &s.t. \quad \lambda = \alpha - \beta\left(w_1(\mu_1, \lambda)\right)^{\phi} - \beta\left(w_2(\mu_2, \lambda)\right)^{\phi} \end{aligned} \tag{A.33}$$

The Lagrangian of (A.33) is

$$L(\lambda, \mu_1, \mu_2, \rho) = r\lambda - c_1\mu_1 - c_2\mu_2 + \rho\left(\lambda - \alpha + \beta\frac{1}{(\mu_1 - \lambda)^{\phi}} + \beta\frac{1}{(\mu_2 - \lambda)^{\phi}}\right)$$

To find the critical points, we solve the following equation set:

$$\begin{aligned} &\tfrac{\partial L}{\partial \lambda} = r + \rho + \tfrac{\phi\rho\beta}{(\mu_1 - \lambda)^{1+\phi}} + \tfrac{\phi\rho\beta}{(\mu_2 - \lambda)^{1+\phi}\partial} = 0 \\ &\tfrac{\partial L}{\partial \mu_1} = -c_1 - \tfrac{\phi\rho\beta}{(\mu_1 - \lambda)^{1+\phi}} = 0 \\ &\tfrac{\partial L}{\partial \mu_2} = -c_2 - \tfrac{\phi\rho\beta}{(\mu_2 - \lambda)^{1+\phi}} = 0 \\ &\lambda - \alpha + \beta\tfrac{1}{(\mu_1 - \lambda)^{\phi}} + \beta\tfrac{1}{(\mu_2 - \lambda)^{\phi}} \\ &\lambda \leq \mu_1, \lambda \leq \mu_2 \end{aligned}$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.29) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$\lambda^b = \alpha - \beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}$$
$$\mu_1^b = \lambda^b + \left(\frac{\phi\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}}$$
$$\mu_2^b = \lambda^b + \left(\frac{\phi\beta(r-c_1-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}$$

□

Proof of Proposition 90: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned} \max_{\substack{0\le(1-\theta)\lambda_s\le\mu_1\le\frac{r\alpha}{c_1},\\ 0\le\lambda_o,\\ (1-\theta)\lambda_s+\theta\lambda_o\le\mu_2\le\frac{r\alpha}{c_2}}} \quad & r\left(\left(1-\theta\right)\lambda_s + \theta\lambda_o\right) - c_1\mu_1 - c_2\mu_2 \\ s.t. \quad & \lambda_o = \alpha - \xi\beta\left(w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)^{\phi} \\ & \lambda_s = \alpha - \beta\left(w_1(\mu_1, (1-\theta)\lambda_s)\right)^{\phi} - \beta\left(w_2(\mu_2, (1-\theta)\lambda_s + \theta\lambda_o)\right)^{\phi} \end{aligned} \tag{A.34}$$

The Lagrangian of (A.34) is given as follows:

$$\begin{aligned} L\left(\lambda_o, \lambda_s, \mu_1, \mu_2, \rho_o, \rho_s\right) =& r\left(\theta\lambda_o + \left(1-\theta\right)\lambda_s\right) - c_1\mu_1 - c_2\mu_2 \\ & - \rho_1\left(\lambda_o - \alpha + \xi\beta(w_2(\mu_2, \theta\lambda_o + \left(1-\theta\right)\lambda_s))^{\phi}\right) \\ & - \rho_2\left(\lambda_s - \alpha + \beta(w_1(\mu_1, \lambda_s))^{\phi} + \beta(w_2(\mu_2, \theta\lambda_{mo} + \left(1-\theta\right)\lambda_s))^{\phi}\right) \end{aligned}$$

To find the critical points, we solve the following equation set:

$$
\begin{aligned}
\frac{\partial L}{\partial \lambda_s} &= r\left(1-\theta\right)+\rho_2+\phi\rho_1\xi\beta\left(1-\theta\right)\frac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^{1+\phi}}+\phi\rho_2\beta\left(1-\theta\right)\frac{1}{\left(\mu_1-(1-\theta)\lambda_s\right)^{1+\phi}} \\
&\qquad +\phi\rho_2\beta\left(1-\theta\right)\frac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^{1+\phi}}=0 \\
\frac{\partial L}{\partial \lambda_o} &= r\theta+\rho_1+\phi\rho_1\xi\beta\theta\frac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^{1+\phi}}+\phi\rho_2\beta\theta\frac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^{1+\phi}}=0 \\
\frac{\partial L}{\partial \mu_1} &= -c_1-\phi\rho_2\beta\frac{1}{\left(\mu_1-(1-\theta)\lambda_s\right)^{1+\phi}}=0 \\
\frac{\partial L}{\partial \mu_2} &= -c_2-\phi\rho_1\xi\beta\frac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^{1+\phi}}-\rho_2\beta\frac{1}{\left(\mu_2-(1-\theta)\lambda_s-\theta\lambda_o\right)^{1+\phi}}=0 \\
&\lambda_o-\alpha+\xi\beta\frac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o}=0 \\
&\lambda_s-\alpha+\beta\frac{1}{\mu_1-(1-\theta)\lambda_s}+\beta\frac{1}{\mu_2-(1-\theta)\lambda_s-\theta\lambda_o}=0 \\
&0\le\left(1-\theta\right)\lambda_s\le\mu_1\le\frac{r\alpha}{c_1},0\le\lambda_o,\left(1-\theta\right)\lambda_s+\theta\lambda_o\le\mu_2\le\frac{r\alpha}{c_2}
\end{aligned}
$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.29) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$
\begin{aligned}
\lambda_s^o &= \alpha-\beta\left(\frac{c_1}{\phi(1-\theta)\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}} \\
\lambda_o^o &= \alpha-\beta_o\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{\phi}{1+\phi}} \\
\mu_1^o &= (1-\theta)\lambda_s^o+\left(\frac{\phi(1-\theta)\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}} \\
\mu_2^o &= (1-\theta)\lambda_s^o+\theta\lambda_o^o+\left(\frac{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}{c_2}\right)^{\frac{1}{1+\phi}}
\end{aligned}
$$

□

Proof of Proposition 91:

$$
\begin{aligned}
\lambda_o^o &= \alpha-\left[\frac{c_2(\xi\beta)^{1+\phi}}{\phi\xi\beta\theta(r-c_2)+\phi\beta(r-c_1-c_2)(1-\theta)}\right]^{\frac{\phi}{1+\phi}} \\
&\ge \alpha-\left[\frac{c_2\beta^{\phi}}{\phi\theta(r-c_2)+\phi(r-c_1-c_2)(1-\theta)}\right]^{\frac{\phi}{1+\phi}} \\
&> \alpha-\left[\frac{c_2\beta^{\phi}}{\phi(r-c_1-c_2)}\right]^{\frac{\phi}{1+\phi}} \\
&> \alpha-\beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}-\beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}}=\lambda^b
\end{aligned}
$$

where the first inequality is because of $\xi \leq 1$.

Next, let's prove the second bullet point. Note

$$
\begin{aligned}
&\left(\lambda_s^o - \lambda^b\right) \frac{(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}{\beta} \\
&= \left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}} + \left(\frac{c_2}{\phi\beta}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_1}{\phi\beta(1-\theta)}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_2}{\phi\beta(1-\theta)+\phi\xi\beta\theta\frac{r-c_2}{r-c_1-c_2}}\right)^{\frac{\phi}{1+\phi}}
\end{aligned}
$$

which is decreasing in r. When $r \to c_1+c_2$, $\left(\lambda_s^o - \lambda^b\right) \frac{(r-c_1-c_2)^{\frac{\phi}{1+\phi}}}{\beta} \to \left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}} + \left(\frac{c_2}{\phi\beta}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_1}{\phi\beta(1-\theta)}\right)^{\frac{\phi}{1+\phi}} > 0$ if θ is small enough. Then, we can conclude the result.

Finally, let's prove the third point. Note $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o = \alpha - ((1-\theta)\,\beta)^{\frac{1}{1+\phi}} \left(\frac{c_1}{\phi(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \frac{c_2^{\frac{\phi}{1+\phi}}((1-\theta)\beta+\theta\xi\beta)^{\frac{1}{1+\phi}}}{\left(r-\frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{\phi}{1+\phi}}}$, which is increasing in θ. Note when $\theta = 0$, we have $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o = \lambda^b$. Thus, $(1-\theta)\,\lambda_s^o + \theta\lambda_o^o > \lambda^b$ for all $\theta > 0$. $\square$

Proof of Proposition 92:

$$
\begin{aligned}
&k_1^o + k_2^o - k_1^b - k_2^b \\
&= -\theta\alpha\tau_1 + (\tau_1+\tau_2)\left(1-(1-\theta)^{\frac{\phi}{1+\phi}}\right)\beta\left(\frac{c_1}{\phi\beta(r-c_1-c_2)}\right)^{\frac{1}{1+\phi}} \\
&\quad - (\tau_1+\tau_2)\left[((1-\theta)\,\beta+\theta\xi\beta)\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{1}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{1}{1+\phi}}\right] \\
&\quad - \left(\frac{\phi\beta(r-c_1-c_2)}{c_1}\right)^{\frac{1}{1+\phi}} \tau_1 \left(1-(1-\theta)^{\frac{1}{1+\theta}}\right) \\
&\quad + \tau_2\left(\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{-\frac{1}{1+\phi}} - \left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{-\frac{1}{1+\phi}}\right) \\
&\quad + \tau_1\theta\xi\beta\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{1}{1+\phi}}
\end{aligned}
\tag{A.35}
$$

Let's first show the 3rd term in (A.35), i.e.,
$-(\tau_1+\tau_2)\left[((1-\theta)\,\beta+\theta\xi\beta)\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)}\right)^{\frac{1}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta(r-c_1-c_2)}\right)^{\frac{1}{1+\phi}}\right]$ (denoted as $f_3(r)$) is decreasing in r:

$$
\frac{\partial f_3}{\partial r} = -(\tau_1+\tau_2)\left[-\frac{c_2^{\frac{1}{1+\phi}}}{(1+\phi)\,\phi^{\frac{1}{1+\phi}}} \frac{((1-\theta)\beta+\theta\xi\beta)^{\frac{\phi}{1+\phi}}}{\left(r-c_2-\frac{(1-\theta)\beta c_1}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{1}{1+\phi}+1}} + \frac{\beta^{\frac{\phi}{1+\phi}} c_2^{\frac{1}{1+\phi}}}{(1+\phi)\,\phi^{\frac{1}{1+\phi}}(r-c_1-c_2)^{\frac{1}{1+\phi}+1}}\right]
$$

which is decreasing in θ. Note when $\theta = 0$, $\frac{\partial f_3}{\partial r} = 0$. Thus, $\frac{\partial f_3}{\partial r} < 0$ for all $\theta > 0$.

Next, let's show the 5th term in (A.35), i.e.,
$\tau_2 \left(\left(\frac{c_2}{\phi(1-\theta)\beta(r-c_1-c_2)+\phi\theta\xi\beta(r-c_2)} \right)^{-\frac{1}{1+\phi}} - \left(\frac{c_2}{\phi\beta(r-c_1-c_2)} \right)^{-\frac{1}{1+\phi}} \right)$ (denoted as $f_5(r)$) is decreasing in r:

$$\frac{\partial f_5}{\partial r} = \frac{c_2^{-\frac{1}{1+\phi}}}{(1+\phi)\,\phi^{-\frac{1}{1+\phi}}} \left(\frac{((1-\theta)\beta+\theta\xi\beta)^{\frac{1}{1+\phi}}}{\left(r - \frac{(1-\theta)\beta(c_1+c_2)+\theta\xi\beta c_2}{(1-\theta)\beta+\theta\xi\beta}\right)^{\frac{\phi}{1+\phi}}} - \frac{\beta^{\frac{1}{1+\phi}}}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}}} \right) < 0$$

Therefore, we can conclude that $\frac{\partial(k_1^o+k_2^o-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1+c_2$, we have $(k_1^o+k_2^o-k_1^b-k_2^b)(r-c_1-c_2)^{\frac{1}{1+\phi}} \to (\tau_1+\tau_2)\beta\left(\frac{c_2}{\phi\beta}\right)^{\frac{1}{1+\phi}} + \left(1-(1-\theta)^{\frac{\phi}{1+\phi}}\right)\beta\left(\frac{c_1}{\phi\beta}\right)^{\frac{1}{1+\phi}}(\tau_1+\tau_2)(1-\sqrt{(1-\theta)}) > 0$, which implies $k_1^o+k_2^o-k_1^b-k_2^b > 0$ if r is very close to c_1+c_2. Thus, there exists $\bar{r} > c_1+c_2$ such that $k_1^o+k_2^o-k_1^b-k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k = \frac{c_1+c_2}{\bar{r}}$. □

Proof of Proposition 93: Since we only focus on the case where the retailer serves all types of customers, the optimal solution can be obtained by solving the following optimization problem:

$$\begin{aligned}
&\max_{\substack{0\le\eta\lambda_m\le\mu_{1m}\le\frac{r\alpha}{c_{1m}},\\ 0\le(1-\eta)\lambda_h\le\mu_{1h}\le\frac{r\alpha}{c_1}\\ \eta\lambda_m+(1-\eta)\lambda_h\le\mu_2\le\frac{r\alpha}{c_2}}} \quad r\left(\eta\lambda_m+(1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\
&s.t. \quad \lambda_m = \alpha - \beta(w_{1m}(\mu_{1m},\eta\lambda_m))^\phi - \beta(w_2(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h))^\phi \\
&\qquad\quad \lambda_h = \alpha - \beta(w_{1h}(\mu_{1h},(1-\eta)\lambda_h))^\phi - \beta(w_2(\mu_2,\eta\lambda_m+(1-\eta)\lambda_h))^\phi
\end{aligned} \tag{A.36}$$

The Lagrangian of (A.36) is given as follows:

$$
\begin{aligned}
&L(\lambda_m, \lambda_h, \mu_{1h}, \mu_{1m}, \mu_2, \rho_m, \rho_h) \\
&= r\left(\eta\lambda_m + (1-\eta)\lambda_h\right) - c_{1m}\mu_{1m} - c_1\mu_{1h} - c_2\mu_2 \\
&\quad - \rho_m\left(\lambda_m - \alpha + \beta\Big(\frac{1}{\mu_{1m} - \eta\lambda_m}\Big)^{\phi} + \beta\Big(\frac{1}{\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h}\Big)^{\phi}\right) \\
&\quad - \rho_h\left(\lambda_h - \alpha + \beta\Big(\frac{1}{\mu_{1h} - (1-\eta)\lambda_h}\Big)^{\phi} + \beta\Big(\frac{1}{\mu_2 - \eta\lambda_m - (1-\eta)\lambda_h}\Big)^{\phi}\right)
\end{aligned}
$$

To find the critical points, we solve the following equation set:

$$
\begin{aligned}
&\tfrac{\partial L}{\partial \lambda_m} = r\eta - \rho_m - \tfrac{\rho_m\beta\eta}{(\mu_{1m}-\eta\lambda_m)^{\phi+1}} - \tfrac{(\rho_m+\rho_h)\beta\eta}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \lambda_h} = r(1-\eta) - \rho_h - \tfrac{\rho_h\beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^{\phi+1}} - \tfrac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \mu_{1m}} = -c_{1m} + \tfrac{\rho_m\beta}{(\mu_{1m}-\eta\lambda_m)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \mu_{1h}} = -c_1 + \tfrac{\rho_h\beta(1-\eta)}{(\mu_{1h}-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\tfrac{\partial L}{\partial \mu_2} = -c_2 + \tfrac{(\rho_m+\rho_h)\beta(1-\eta)}{(\mu_2-\eta\lambda_m-(1-\eta)\lambda_h)^{\phi+1}} = 0 \\
&\eta\lambda_m \le \mu_{1m}, (1-\eta)\lambda_h \le \mu_{1h}, \eta\lambda_m + (1-\eta)\lambda_h \le \mu_2
\end{aligned}
$$

By Proposition 5.6 in (Sundaram, 1996)[page 122], we know the optimal solution to (A.29) is one of the critical points. (Note the constraint qualification holds everywhere on the feasible set.) Since the firm finds it optimal to serve both types of customers, the solution must be interior, which gives us a unique solution:

$$
\begin{aligned}
&\lambda_m^s = \alpha - \beta\Big(\tfrac{c_{1m}}{\phi\beta\eta(r-c_{1m}-c_2)}\Big)^{\frac{\phi}{1+\phi}} - \beta\Big(\tfrac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} \\
&\lambda_h^s = \alpha - \beta\Big(\tfrac{c_1}{\phi\beta(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} - \beta\Big(\tfrac{c_2}{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}\Big)^{\frac{\phi}{1+\phi}} \\
&\mu_{1m}^s = \eta\lambda_m^s + \Big(\tfrac{\phi\beta\eta(r-c_{1m}-c_2)}{c_{1m}}\Big)^{\frac{1}{1+\phi}} \\
&\mu_{1h}^s = (1-\eta)\lambda_h^s + \Big(\tfrac{\phi\beta(1-\eta)(r-c_1-c_2)}{c_1}\Big)^{\frac{1}{1+\phi}} \\
&\mu_2^s = \eta\lambda_m^s + (1-\eta)\lambda_h^s + \Big(\tfrac{\phi\beta\eta(r-c_{1m}-c_2)+\phi\beta(1-\eta)(r-c_1-c_2)}{c_2}\Big)^{\frac{1}{1+\phi}}
\end{aligned}
$$

□

Proof of Proposition 94: First note

$$\frac{\left(\lambda_m^s-\lambda^b\right)\phi^{\frac{\phi}{1+\phi}}}{\beta^{\frac{1}{1+\phi}}} = \left(\frac{c_1}{r-c_1-c_2}\right)^{\frac{\phi}{1+\phi}} + \left(\frac{c_2}{r-c_1-c_2}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_{1m}}{\eta\left(r-c_{1m}-c_2\right)}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_2}{\eta\left(r-c_{1m}-c_2\right)+(1-\eta)\left(r-c_1-c_2\right)}\right)^{\frac{\phi}{1+\phi}}$$

Because $c_{1m} < c_1$, we have $\left(\frac{c_2}{r-c_1-c_2}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_2}{\eta(r-c_{1m}-c_2)+(1-\eta)(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} > 0$. Also, we can easily check that $w_{1h}^s > w_1^b$. Then, since $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, we must have $w_{1m}^s < w_1^b$, which implies $\left(\frac{c_1}{r-c_1-c_2}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_{1m}}{\eta(r-c_{1m}-c_2)}\right)^{\frac{\phi}{1+\phi}} > 0$. Thus, we have $\frac{\left(\lambda_m^s-\lambda^b\right)\phi^{\frac{\phi}{1+\phi}}}{\beta^{\frac{1}{1+\phi}}} > 0$, or $\lambda_m^s > \lambda^b$.

Second, note

$$\frac{\left(\lambda_h^s-\lambda_h^b\right)(r-c_1-c_2)^{\frac{\phi}{1+\phi}}\phi^{\frac{\phi}{1+\phi}}}{\beta^{\frac{1}{1+\phi}}} = c_1^{\frac{\phi}{1+\phi}} + c_2^{\frac{\phi}{1+\phi}} - \left(\frac{c_1}{1-\eta}\right)^{\frac{\phi}{1+\phi}} - \left(\frac{c_2}{\eta\frac{r-c_{1m}-c_2}{r-c_1-c_2}+(1-\eta)}\right)^{\frac{\phi}{1+\phi}}$$

which is decreasing in r. Then if $c_1^{\frac{\phi}{1+\phi}} + c_2^{\frac{\phi}{1+\phi}} - \left(\frac{c_1}{1-\eta}\right)^{\frac{\phi}{1+\phi}} > 0$ (or η is small enough), then there exists $\bar{r} > c_1 + c_2$ such that $\lambda_h^s - \lambda_h^b > 0$ if and only if $\frac{c_1+c_2}{r} > \frac{c_1+c_2}{\bar{r}}$.

Finally, let's look at total demand rate. We first prove the following lemma:

Lemma 6. *If $\eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, then $\eta(w_{1m}^s)^\phi + (1-\eta)(w_{1h}^s)^\phi < (w_1^b)^\phi$.*

Proof of Lemma 6: Because $0 < \eta w_{1m}^s + (1-\eta)w_{1h}^s < w_1^b$, we have $(\eta w_{1m}^s + (1-\eta)w_{1h}^s)^\phi < (w_1^b)^\phi$. Note $(\eta w_{1m}^s + (1-\eta)w_{1h}^s)^\phi \geq \eta^\phi(w_{1m}^s)^\phi + (1-\eta)^\phi(w_{1h})^\phi \geq \eta(w_{1m}^s)^\phi + (1-\eta)(w_{1h})^\phi$, where the second inequality is because of $\eta \in (0,1)$ and $\phi \in (0,1]$. Thus, we have $\eta(w_{1m}^s)^\phi + (1-\eta)(w_{1h}^s)^\phi < (w_1^b)^\phi$. $\square$

Because $\eta w_{1m}^s + (1-\eta) w_{1h}^s < w_1^b$, Lemma 5 and $c_{1m} < c_1$, we can check that

$$\begin{aligned}
\eta\lambda_m^s + (1-\eta)\,\lambda_h^s = \alpha - \eta\beta\left(\frac{c_{1m}}{\phi\eta\beta\,(r-c_{1m}-c_2)}\right)^{\frac{\phi}{1+\phi}} \\
-(1-\eta)\,\beta\left(\frac{c_1}{\phi\,(1-\eta)\,\beta\,(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} \\
-\beta\left(\frac{c_2}{\phi\beta\eta\,(r-c_{1m}-c_2)+\phi\beta\,(1-\eta)\,(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} \\
> \alpha - \beta\left(\frac{c_1}{\phi\beta\,(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} - \beta\left(\frac{c_2}{\phi\beta\,(r-c_1-c_2)}\right)^{\frac{\phi}{1+\phi}} = \lambda^b
\end{aligned}$$

□

Proof of Proposition 95: First note

$$\begin{aligned}
\frac{\partial k_1^s}{\partial r} &= \frac{\phi^{\frac{1}{1+\phi}}\beta^{\frac{1}{1+\phi}}(1-\eta)^{\frac{1}{1+\phi}}c_1^{\frac{\phi}{1+\phi}}\tau_1}{(1+\phi)(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}} + \frac{\tau_1(1-\eta)\beta^2\phi^2 c_2^{\frac{\phi}{1+\phi}}}{(1+\phi)[\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta)]^{\frac{\phi}{1+\phi}+1}} \\
&\quad + \frac{\tau_1(\beta\phi(1-\eta))^{\frac{1}{1+\phi}}}{(1+\phi)c_1^{\frac{1}{1+\phi}}(r-c_1-c_2)^{\frac{\phi}{1+\phi}}} \\
&< \frac{\phi^{\frac{1}{1+\phi}}\beta^{\frac{1}{1+\phi}}c_1^{\frac{\phi}{1+\phi}}\tau_1}{(1+\phi)(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}} + \frac{\tau_1\beta^2\phi^2 c_2^{\frac{\phi}{1+\phi}}}{(1+\phi)[\beta(r-c_1-c_2)]^{\frac{\phi}{1+\phi}+1}} + \frac{\tau_1(\beta\phi)^{\frac{1}{1+\phi}}}{(1+\phi)c_1^{\frac{1}{1+\phi}}(r-c_1-c_2)^{\frac{\phi}{1+\phi}}} = \frac{\partial k_1^b}{\partial r}
\end{aligned}$$

Also, note

$$\begin{aligned}
\frac{\partial k_2^s}{\partial r} &= \eta\beta\left(\frac{c_{1m}}{\phi\beta\eta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_{1m}-c_2)^{\frac{\phi}{1+\phi}+1}} + (1-\eta)\,\beta\left(\frac{c_1}{\phi\beta(1-\eta)}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}} \\
&\quad + \phi\beta^2 c_2^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta))^{\frac{\phi}{1+\phi}+1}} \\
&\quad + \frac{\phi\beta}{(1+\phi)c_2^{\frac{1}{1+\phi}}}\frac{1}{(\beta(r-c_{1m}-c_2)\eta+\beta(r-c_1-c_2)(1-\eta))^{\frac{\phi}{1+\phi}}}
\end{aligned}$$

By Lemma 6 and $c_{1m} < c_1$, we have

$$\begin{aligned}
&\eta\beta\left(\frac{c_{1m}}{\phi\beta\eta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_{1m}-c_2)^{\frac{\phi}{1+\phi}+1}} + (1-\eta)\,\beta\left(\frac{c_1}{\phi\beta(1-\eta)}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}} \\
&< \beta\left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}}\frac{\phi}{1+\phi}\frac{1}{(r-c_1-c_2)^{\frac{\phi}{1+\phi}+1}}.
\end{aligned}$$

Also, define function

$$
\begin{aligned}
f\left(c_{1m}\right) =& \phi\beta^2 c_2^{\frac{\phi}{1+\phi}} \frac{\phi}{1+\phi} \frac{1}{\left(\beta\left(r-c_{1m}-c_2\right)\eta+\beta\left(r-c_1-c_2\right)\left(1-\eta\right)\right)^{\frac{\phi}{1+\phi}+1}} \\
&+ \frac{\phi\beta}{\left(1+\phi\right)c_2^{\frac{1}{1+\phi}}} \frac{1}{\left(\beta\left(r-c_{1m}-c_2\right)\eta+\beta\left(r-c_1-c_2\right)\left(1-\eta\right)\right)^{\frac{\phi}{1+\phi}}},
\end{aligned}
$$

which is an increasing function. Thus, $f(c_{1m}) < f(c_1)$. Since $\frac{\partial k_2^b}{\partial r} = \beta\left(\frac{c_1}{\phi\beta}\right)^{\frac{\phi}{1+\phi}} \frac{\phi}{1+\phi} \frac{1}{\left(r-c_1-c_2\right)^{\frac{\phi}{1+\phi}+1}} + f\left(c_1\right)$, we have $\frac{\partial k_2^s}{\partial r} < \frac{\partial k_2^b}{\partial r}$.

Therefore, we can conclude that $\frac{\partial(k_1^s+k_2^s-k_1^b-k_2^b)}{\partial r} < 0$. Note, if $r \to c_1+c_2$, we have $(k_1^s + k_2^s - k_1^b - k_2^b)(r-c_1-c_2)^{\frac{\phi}{1+\phi}} \to (\tau_1+\tau_2)\beta^{\frac{1}{1+\phi}}\phi^{\frac{-\phi}{1+\phi}}(c_1^{\frac{\phi}{1+\phi}} + c_2^{\frac{\phi}{1+\phi}} - c_1^{\frac{\phi}{1+\phi}}(1-\eta)^{\frac{1}{1+\phi}}) > 0$, which implies $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if r is very close to c_1+c_2. Thus, there exists $\bar{r} > c_1+c_2$ such that $k_1^s + k_2^s - k_1^b - k_2^b > 0$ if and only if $r < \bar{r}$. Then, we can define $m_k' = \frac{c_1+c_2}{\bar{r}}$. □

Proof of Proposition 96: Note that

$$
\begin{aligned}
\pi^o =& \left[r-c_1\left(1-\theta\right)-c_2\right]\alpha - \left(\frac{c_1}{\phi}\right)^{\frac{\phi}{1+\phi}}\left[\beta\left(r-c_1-c_2\right)\left(1-\theta\right)\right]^{\frac{1}{1+\phi}} \\
&-\phi^{\frac{1}{1+\phi}} c_1^{\frac{\phi}{1+\phi}}\left[\beta\left(r-c_1-c_2\right)\left(1-\theta\right)\right]^{\frac{1}{1+\phi}} \\
&-\left(\frac{c_2}{\phi}\right)^{\frac{\phi}{1+\phi}}\left[\left(1-\theta\right)\beta\left(r-c_1-c_2\right)+\theta\xi\beta\left(r-c_2\right)\right]^{\frac{1}{1+\phi}} \\
&-\phi^{\frac{1}{1+\phi}} c_2^{\frac{\phi}{1+\phi}}\left[\left(1-\theta\right)\beta\left(r-c_1-c_2\right)+\theta\xi\beta\left(r-c_2\right)\right]^{\frac{1}{1+\phi}}
\end{aligned}
$$

and

$$
\begin{aligned}
\pi^s =& \left(r-c_2\right)\alpha - c_{1m}\eta\alpha - c_1\left(1-\eta\right)\alpha \\
&-\left[\phi^{-\frac{\phi}{1+\phi}}+\phi^{\frac{1}{1+\phi}}\right] c_{1m}^{\frac{\phi}{1+\phi}}\left[\eta\beta\left(r-c_{1m}-c_2\right)\right]^{\frac{1}{1+\phi}} \\
&-\left[\phi^{-\frac{\phi}{1+\phi}}+\phi^{\frac{1}{1+\phi}}\right] c_1^{\frac{\phi}{1+\phi}}\left[\left(1-\eta\right)\beta\left(r-c_1-c_2\right)\right]^{\frac{1}{1+\phi}} \\
&-\left[\phi^{-\frac{\phi}{1+\phi}}+\phi^{\frac{1}{1+\phi}}\right] c_2^{\frac{\phi}{1+\phi}}\left[\eta\beta\left(r-c_{1m}-c_2\right)+\beta\left(1-\eta\right)\left(r-c_1-c_2\right)\right]^{\frac{1}{1+\phi}}
\end{aligned}
$$

Note that $\frac{\partial^2(\pi^o-\pi^s)}{\partial\theta^2} < 0$. Also note that when $\theta = \eta$, we must have the $\pi^o > \pi^s$, the proof of which is as follows: Suppose the optimal solution for the offline model is $\mu_{1m}^s, \mu_{1h}^s, \mu_2^s, \lambda_m^s, \lambda_h^s$. Consider the following feasible solution for the online model: $\mu_1^\triangle, \mu_2^\triangle, \lambda_m^\triangle, \lambda_h^\triangle$, where $\mu_1^\triangle =$

μ_{1h}^s and $\mu_2^{\triangle} = \mu_2^s$. Then, suppose $\eta\lambda_m^{\triangle} + (1-\eta)\lambda_h^{\triangle} \leq \eta\lambda_m^s + (1-\eta)\lambda_h^s$. Then, we must have $w_2^{\triangle} \leq w_2^s$. Then, because online customers don't wait at stage 1, we have $\lambda_m^{\triangle} > \lambda_m^s$. Then, $\lambda_h^{\triangle} < \lambda_h^s$. Then, $w_1^{\triangle} < w_1^s$. However, if $w_1^{\triangle} < w_1^s$ and $w_2^{\triangle} < w_2^s$, then this means that $\lambda_h^{\triangle} > \lambda_h^s$. We get a contradiction. Thus, we must have $\eta\lambda_m^{\triangle} + (1-\eta)\lambda_h^{\triangle} > \eta\lambda_m^s + (1-\eta)\lambda_h^s$. This implies that $\pi^{\triangle} > \pi^s$. Thus, we must have $\pi^o \geq \pi^{\triangle} > \pi^s$. The results above implies that $\exists\bar{\theta} \leq \eta$ such that $\pi^o > \pi^s$ if and only if $\theta > \bar{\theta}$.

Note when $c_{1m}\eta = c_1(\eta - \theta)$ (or $\theta = \frac{(c_1 - c_{1m})\eta}{c_1}$), we have $\pi^o - \pi^s > 0$ for any $\beta > 0$. This implies that $\bar{\theta} \leq \frac{(c_1 - c_{1m})\eta}{c_1}$. Thus, for any $\beta > 0$, if $\theta \geq \frac{(c_1 - c_{1m})\eta}{c_1}$ (or $c_{1m}\eta - c_1(\eta-\theta) \geq 0$), we must have $\pi^o - \pi^s > 0$, i.e., $\bar{b} = 0$. If $c_{1m}\eta - c_1(\eta-\theta) < 0$, then $\frac{\partial \frac{\pi^o - \pi^s}{\beta^{\frac{1}{1+\phi}}}}{\partial\beta} = \frac{\partial \frac{(c_{1m}\eta - c_1(\eta-\theta))\alpha}{\beta^{\frac{1}{1+\phi}}}}{\partial\beta} > 0$. Thus, there exists $\bar{\beta} \geq 0$ such that $\pi^o - \pi^s > 0$ if and only if $\beta > \bar{\beta}$. $\square$

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