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Stability Analysis of Higher-Order Delta-Sigma Modulators for Dual Sinusoidal Inputs

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Abstract- **The aim of this paper is to determine the stability of higher-order Δ-Σ modulators for sinusoidal inputs. The nonlinear gains for the single bit quantizer for a dual sinusoidal input have been derived and the maximum stable input limits for a fifth-order Chebyshev Type II based Δ-Σ modulators are established. These results are useful for optimising the design of higher-order Δ-Σ modulators.**

I. INTRODUCTION

The stable input amplitude limits for Δ-Σ modulators is complicated to predict due to the non-linearity introduced by the quantizer in the feedback loop. Various approaches have been employed to explain this nonlinear behaviour. Using quasilinear modeling, a new interpretation of the instability mechanism for Δ-Σ modulators based on the noise amplification curve is given in [1]. This is restricted for DC inputs and unity quantizer gains. The quasilinear method can be extended to more than one input with each input represented by a separate equivalent gain. This concept forms the basis for the Describing Function (DF) method [2]. In [3] the stability analysis for higher-order Δ-Σ modulators based on the noise amplification curve was performed using the DF method for DC and (single-tone) sinusoidal inputs for non-unity quantizer gain values. In this paper the analysis is extended for multiple (dual) tone sinusoidal inputs.

II. QUASILINEAR STABILITY ANALYSIS OF Δ-Σ MODULATORS

A generic Δ-Σ modulator having its quantizer replaced by a gain factor *K* followed by additive quantization noise *q(k)* [1] is shown in Figure 1.

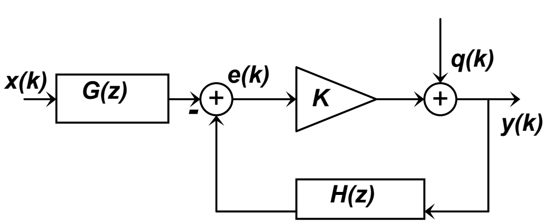

Figure 1. Quasilinear Δ-Σ modulator Quantizer Model.

The output of the modulator in the z-domain is given by :

$$Y(z) = STF(z)X(z) + NTF(z)Q(z) \qquad (1)$$

where, *Y(z), X(z)* and *Q(z)* are the z-transforms of the output, input and quantizer noise signals respectively. Also, *STF(z)* and *NTF(z)* are the Signal and Noise Transfer functions of the Δ-Σ modulator derived from Figure 1.

$$STF(z) = \frac{K.G(z)}{1 + K.H(z)} \qquad (2)$$

$$NTF(z) = \frac{1}{1 + K.H(z)} \qquad (3)$$

Since the poles of the denominator *(1+KH(z))* determine the stability of the modulator, for a given *H(z)*, there will be a certain interval [K_{min}, K_{max}] for which the modulator is stable [4]. Assuming *q(k)* to be Gaussian white stochastic $G(0, \sigma_q^2)$ and the transfer function between *q(k)* and *y(k)* to be known, then the output noise variance is given by

$$Var\{y(k)\} = \sigma_q^{\,2} \int_0^1 \left| NTF(e^{j\pi f}) \right|^2 df = \sigma_q^{\,2} A(K) \qquad (4)$$

where, σ_q^2 is the variance of *q(k)* and *A(K)* is the total output noise power amplification factor. Using Parseval's relation, *A(K)* can be found in the time domain as [1]:

$$A(K) = \sum_{k=0}^{\infty} \left| ntf(k) \right|^2 \underline{\underline{\Delta}} \left\| ntf \right\|_2^2 \qquad (5)$$

where *ntf(k)* is the impulse response corresponding to *NTF(z)* and *A(k)* is the squared two-norm of NTF(z). The *A(K)* curves of the loop-filter are crucial for the stability analysis of the Δ-Σ modulators. Typical curves for Type II Chebyshev 3rd and 4th order are shown in Figure2.

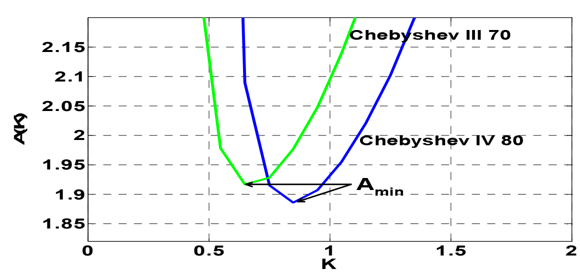

Figure 2. *A(K)* Curves for Type II Chebyshev NTF.

The A_{min} value is the global minimum of the curve. It has been shown in [1] that for stable operation $A(k) > A_{min}$.

III. NOISE AMPLIFICATION CURVES – DF METHOD

The quasilinear quantizer model in Figure 1 can be extended using separate gains K_x and K_n for the DF model as shown in Figures 3 and 4 [5].

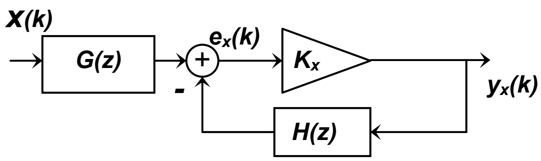

Figure 3. Δ-Σ modulator Quantizer Signal-Model

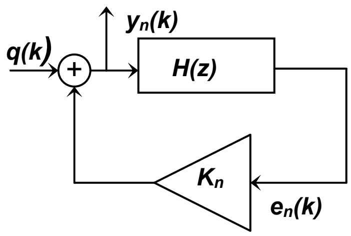

Figure 4. Δ-Σ modulator Quantizer Noise-Model

Figure 3 describes the model for the input signal with linear gain K_x whereas Figure 4 describes the noise signal model with linear gain K_n. The combined output signal is given by:

$$y(k) = y_x(k) + y_n(k) \tag{6}$$

The linearised gains for two sinusoidal input signals $x_a(t)=aCos(\omega_1(t)+\phi_1)$, $x_b(t)=bCos(\omega_2(t)+\phi_2)$ (where a, b are constants, ω_1, ω_2 the sinusoidal frequencies, ϕ_1 and ϕ_2 random phases) and a random Gaussian signal representing the feedback components have been solved for the case of a one-bit quantizer with an output $\pm\Delta$ in Appendix A where the final expressions are shown below:

$$K_a = \left(\frac{2}{\pi}\right)^{\frac{5}{2}}\left(\frac{\Delta}{\sigma}\right)\left(\frac{b}{a}\right)\left(\frac{1}{\frac{1}{2}-\rho_b^2}\right)\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_a^2\right)+\psi_a\right\} \tag{7}$$

$$K_b = \left(\frac{2}{\pi}\right)^{\frac{5}{2}}\left(\frac{\Delta}{\sigma}\right)\left(\frac{a}{b}\right)\left(\frac{1}{\frac{1}{2}-\rho_a^2}\right)\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_b^2\right)+\psi_b\right\} \tag{8}$$

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma}\right)e^{-\rho_a^2}e^{-\rho_b^2}\zeta \tag{9}$$

where
$$\psi_a = \left\{\frac{4}{3}\rho_a^2 - \frac{16}{45}\rho_a^4 + \frac{16}{175}\rho_a^6 - \frac{128}{6615}\rho_a^8 + ..\right\} \tag{10}$$

$$\psi_b = \left\{\frac{4}{3}\rho_b^2 - \frac{16}{45}\rho_b^4 + \frac{16}{175}\rho_b^6 - \frac{128}{6615}\rho_b^8 + ..\right\} \tag{11}$$

$$\zeta = \left\{1+\rho_a^2\rho_b^2 + \frac{\rho_a^4\rho_b^4}{4} + \frac{\rho_a^6\rho_b^6}{36} + \frac{\rho_a^8\rho_b^8}{576} + ...\right\} \tag{12}$$

and $\rho_a^2=(1/2)(a^2/\sigma^2)$, $\rho_b^2=(1/2)(b^2/\sigma^2)$. $F(.)$ is the confluent hypergeometric function [6]. The output noise variance is given by:

$$Var\{y(k)\} = \sigma_{e_n}^2 K_n^2 + \sigma_{q_{ab}}^2 \tag{13}$$

where σ^2_{qab} is the quantization noise power for the two uncorrelated sinusoidal inputs $x_a(t)$ and $x_b(t)$. Therefore from (4), (9) and (13) the noise amplification factor is given by:

$$A_{ab}(K) = \frac{\left(\frac{2}{\pi}\right)\left\{e^{-\rho_a^2}e^{-\rho_b^2}\right\}^2\zeta^2 + \sigma_{q_{ab}}^2}{\sigma_{q_{ab}}^2} \tag{14}$$

Since $x_a(t)$ and $x_b(t)$ are uncorrelated, the power of the output signal is given by:

$$E\{y^2(k)\} = \sigma_{e_n}^2 K_n^2 + \sigma_{q_{ab}}^2 + \sigma_{e_b}^2 K_b^2 + \sigma_{e_a}^2 K_a^2 \tag{15}$$

where σ^2_{eb} and σ^2_{ea} are the powers of the sinusoidal inputs at the quantizer input which are given by:

$$\sigma_{e_b}^2 = \frac{1}{K_b^2}\sigma_b^2 \quad \text{and} \quad \sigma_{e_a}^2 = \frac{1}{K_a^2}\sigma_a^2 \tag{16}$$

From (9), (15) and (16) we get:

$$\Delta^2 = \frac{2}{\pi}\Delta^2\{e^{-\rho_a^2}e^{-\rho_b^2}\}^2\zeta^2 + \sigma_{q_{ab}}^2 + \frac{b^2}{2} + \frac{a^2}{2} \tag{17}$$

Rearranging (17), the quantization noise is given by:

$$\sigma_{q_{ab}}^2 = \Delta^2\left[1 - \frac{a^2}{2\Delta^2} - \frac{b^2}{2\Delta^2} - \frac{2}{\pi}\{e^{-\rho_a^2}e^{-\rho_b^2}\}^2\zeta^2\right] \tag{18}$$

From (8) and (16) we get:

$$\left(\frac{2}{\pi}\right)^5\left(\frac{a^2}{b^2}\right)\frac{\rho_b^2}{\left[\frac{1}{2}-\rho_a^2\right]^2}\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_b^2\right)+\psi_b\right\}^2 = \frac{b^2}{2} \tag{19}$$

Similarly from (7) and (16) for the sinusoid $x_a(t)$ we have:

$$\left(\frac{2}{\pi}\right)^5\left(\frac{b^2}{a^2}\right)\frac{\rho_a^2}{\left[\frac{1}{2}-\rho_b^2\right]^2}\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_a^2\right)+\psi_a\right\}^2 = \frac{a^2}{2} \tag{20}$$

The two simultaneous equations (19) and (20) were solved by deploying the MATLAB symbolic toolbox in order to get the values of ρ_a and ρ_b for various values of a and b.

IV. RESULTS & SIMULATIONS

From (19) and (20), the values of ρ_b have been plotted in Figure 5. It is seen that ρ_b gets bigger as the amplitude b increases. However, the increase in ρ_b gets attenuated as the signal amplitude a increases from 0.2 to 0.8.

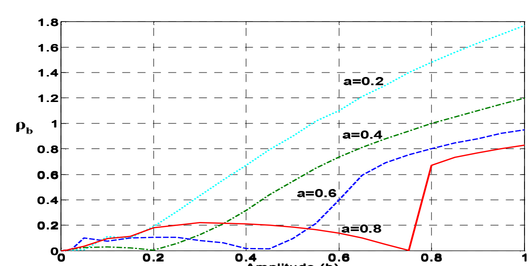

Figure 5. Variation of ρ_b versus b for different a amplitudes.

Using (18) the quantization noise σ^2_{qab} is plotted in Figure 6. The σ^2_{qab} in the regions $b < 0.2$, $b < 0.4$ and $b < 0.6$ for the curves I ($a = 0.2$), II ($a = 0.4$) and III ($a = 0.6$) respectively increases mainly due to ρ_a. As ρ_a becomes bigger when the amplitude a increases from 0.2 to 0.6, so does σ^2_{qab}. The increase in σ^2_{qab} in the regions $b > 0.2$, $b > 0.4$ and $b > 0.6$ for the curves I, II, and III respectively is mainly attributed to the increase in ρ_b. As ρ_b increases with a reduction in the amplitude a from 0.6 to 0.2 in Figure 5, so does σ^2_{qab}.

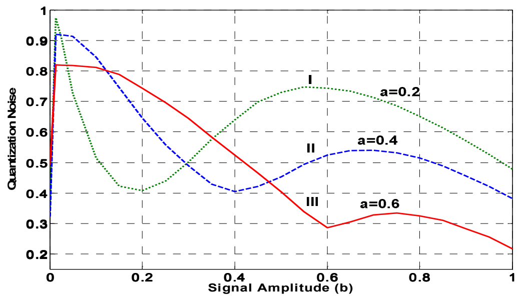

Figure 6. Variation of quantization noise versus the two sine amplitudes.

Figure 7 shows the noise amplification curves obtained from (40) for $a = 0.2$, 0.4 and 0.6.

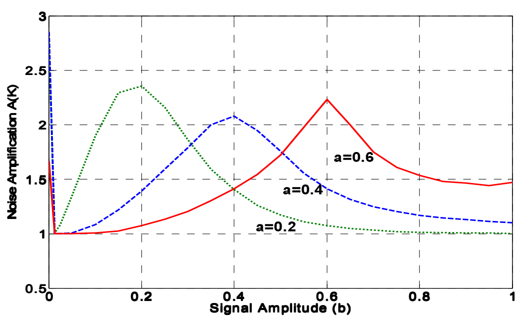

Figure 7. A(k) variation versus the two sine amplitudes.

Using the values obtained for $A_{ab}(k)$, the stable amplitude limits for b have been plotted for the 5th- Chebyshev Type II based NTF for $a = 0.2$ in Figure 8.

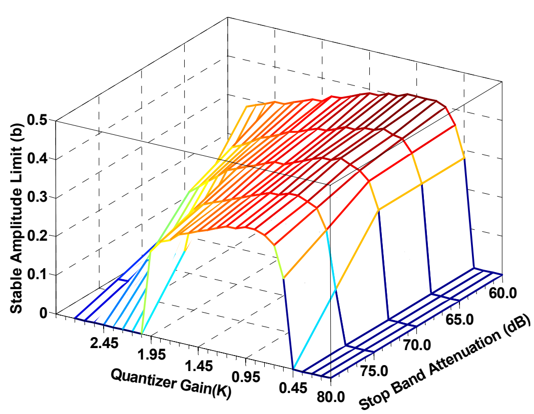

Figure 8. Stable limits of amplitude b of 5th-order for $a = 0.2$.

Simulations for the 5th-order Chebyshev Type II based Δ-Σ modulator shown in Figure 9 were performed for 1638400 samples where the input amplitude was increased in steps of 0.1. The maximum stable amplitude limits were obtained and compared with simulations as shown in Figure 9. Results obtained in [3] were used for the DC and single sinusoidal inputs.

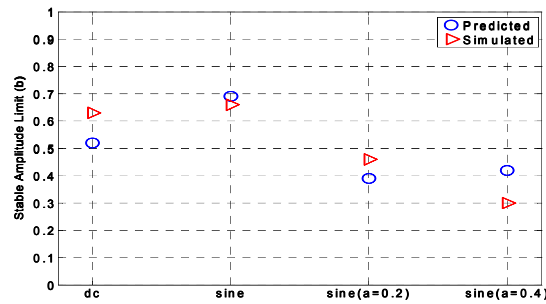

Figure 9. Simulation results for dc, sine & two sinusoidal inputs.

The reason for variation can be attributed to the fact that the derivation of the three gains (i.e. 2 sinusoids and one Gaussian) is based on the modified non-linearity concept. In order to compute the gain for any of the 3 inputs, it is assumed that the non-linear function has been modified in turn by each of the 2 remaining inputs. However, in real-time this may not be the case as all the 3 inputs coexist simultaneously.

V. CONCLUSION

The stability of higher-order Δ-Σ modulators for dual tone sinusoidal inputs using the Describing Function Method has been predicted. The nonlinear gains for the single bit quantizer for a dual sinusoidal input have been derived and the maximum stable input limits for 5th-order Chebyshev Type II based Δ-Σ modulator have been established. Accurate results for the stable amplitude curves can be obtained for a range of values of quantizer gain K in which the Δ-Σ modulators are likely to operate.

APPENDIX A

In this Appendix, the derivation of the gains for two inputs (a dual-tone sinusoidal one Gaussian) for a single-bit quantizer is made. If the inputs to the nonlinearity are of different (Probability Density Functions) PDFs or of different magnitudes of similar waveforms, the output component from one of these inputs depends not only on the magnitude of this particular input but also on the magnitudes of all the other inputs. The concept used here is the modified linearity concept [7], whereby to determine the response to a particular input, the nonlinear characteristic is modified in turn by each of the input signals present to obtain a modified nonlinearity to which the input is applied.

Sinusoidal Gains

The two sinusoidal inputs considered here are $x_a(t)=aCos(\omega_1(t)+\phi_1)$ and $x_b(t)=bCos(\omega_2(t)+\phi_2)$ where a, b are constants, ω_1, ω_2 the sinusoidal frequencies, assumed to be incommensurate, ϕ_1 and ϕ_2 are RVs each having a uniform PDF in the interval $[0, 2\pi]$. The second input is the quantization noise assumed to be Gaussian G$(0, \sigma)$ i.e. with zero mean and variance σ^2.The modified nonlinearity of single-bit quantizer with a random input is given by [8]:

$$n_1(\gamma) = 2\Delta\int_0^{\gamma} q(y)dy \quad \text{(A1)}$$

where $\pm\Delta$ is the output of the quantizer and $q(y)$ is the PDF of the random input. Therefore for a Gaussian input:

$$n_1(\gamma) = 2\Delta\int_0^{\gamma}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)e^{\frac{-y^2}{2\sigma^2}}dy \quad \text{(A2)}$$

On integration (A2) simplifies to:

$$n_1(\gamma) = \Delta erf\left(\frac{\gamma}{\sigma\sqrt{2}}\right) \quad \text{(A3)}$$

The non-linearity $n_1(\gamma)$ further modified to $n_2(\gamma)$ by one of the sinusoidal signals say $x_a(t)$ which is given by [7]:

$$n_2(\gamma) = \int_{-a}^{a} p(x)n_1(x+\gamma)dx \quad \text{(A4)}$$

where $p(x)$ is the PDF of $x_a(t)$. Therefore:

$$n_2(\gamma) = \int_{-a}^{a}\frac{1}{\pi}\frac{1}{\sqrt{a^2-x^2}}\Delta erf\left(\frac{x+\gamma}{\sigma\sqrt{2}}\right)dx \quad \text{(A5)}$$

$n_2(\gamma)$ is now the nonlinearity of the quantizer which has been modified by the sinusoidal input $x_a(t)$ and the quantization noise $G(0,\sigma)$. The next step is to evaluate the gain for $x_b(t)$ to this modified nonlinearity. The gain K_b of the sinusoidal input $x_b(t)$ to this non-linearity $n_2(\gamma)$ is given by [8]:

$$K_b = \frac{1}{\sigma_b^2}\int_{-b}^{b} xn_2(x)r(x)dx \quad \text{(A9)}$$

where $\sigma_b^2 = b^2/2$, is the variance and $r(x)$ the PDF of $x_b(t)$. On integrating (A9) we get the gain K_b for b as:

$$K_b = \left(\frac{2}{\pi}\right)^{\frac{5}{2}}\left(\frac{\Delta}{\sigma}\right)\left(\frac{a}{b}\right)\left(\frac{1}{\frac{1}{2}-\rho_a^2}\right)\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_b^2\right)+\psi_b\right\} \quad \text{(A17)}$$

where,

$$\psi_b = \left\{\frac{4}{3}\rho_b^2 - \frac{16}{45}\rho_b^4 + \frac{16}{175}\rho_b^6 - \frac{128}{6615}\rho_b^8 + ..\right\} \quad \text{(A18)}$$

In order to obtain the gain for $x_a(t)$, we proceed as in above to get:

$$K_a = \left(\frac{2}{\pi}\right)^{\frac{5}{2}}\left(\frac{\Delta}{\sigma}\right)\left(\frac{b}{a}\right)\left(\frac{1}{\frac{1}{2}-\rho_b^2}\right)\left\{{}_1F_1\left(1,\frac{3}{2},-\rho_a^2\right)+\psi_a\right\} \quad \text{(A19)}$$

Noise Gain

The modified nonlinearity of order 1 for a Gaussian input to a single bit quantizer is given by [8]:

$$n(\sigma,\gamma)_1 = \int_{-\infty}^{\infty} n(y+\gamma)H_1\left(\frac{y}{\sigma}\right)q(y)dy \quad \text{(A20)}$$

where H_1 is the Hermite Polynomial of the first order. Substituting for $q(y)$ and $n(y+\gamma)$ in (A20):

$$n(\sigma,\gamma)_1 = \frac{\Delta}{\sigma^2\sqrt{2\pi}}\int_{-\infty}^{\infty} ye^{-\frac{y^2}{2\sigma^2}}dy = \sqrt{\tfrac{2}{\pi}}\Delta e^{-\frac{\gamma^2}{2\sigma^2}} \quad \text{(A21)}$$

The noise gain K_n in the presence of another random input with PDF $p(r)$ is given by [8]

$$K_n = \frac{1}{\sigma}\int_{-\infty}^{\infty} n(\sigma,r)_1\, p(r)dr \quad \text{(A22)}$$

Here we consider the additional random input as a combination of two uncorrelated sinusoidal inputs. The joint PDF $p(r)$ of the two sinusoidal signals having amplitudes a and b, with incommensurate frequencies is: $p(r)=(r/\pi ab)(1/\sin\theta)$, where $\theta=\cos^{-1}\{[a^2+b^2-r^2]/2ab\}$. Solving the integral above we get the noise gain as:

$$K_n = \sqrt{\frac{2}{\pi}}\left(\frac{\Delta}{\sigma}\right)e^{-\rho_a^2}e^{-\rho_b^2}\zeta \quad \text{(A27)}$$

where,

$$\zeta = \left\{1+\rho_a^2\rho_b^2+\frac{\rho_a^4\rho_b^4}{4}+\frac{\rho_a^6\rho_b^6}{36}+\frac{\rho_a^8\rho_b^8}{576}+...\right\} \quad \text{(A28)}$$

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