APPLICATION OF GEOMETRY IN GEODETIC INSTRUMENTS AND MEASUREMENT TECHNICS

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ABSTRACT

Historically, both words ”geometry” and “geodesy” refer to the measurement of land. Although geometry became part of mathematics who studies the properties and mutual relations of figures its principles stayed grate part of geodesy through geometrical methods of measurement and working principle of geodetic instruments. The majority of geodetic instruments are optical and their principles are based on geometrical optics, which use the ray concept as foundation. There are lot of geodetic measurement methods that are based only on geometrical laws, such as cartographical and photogrammetric projections, trigonometric and geometric levelling methods, triangulation and trilateration as technics of determination position of points on Earth surface. Global Positioning System (GPS), satellite geodesy and Very Long Baseline Interferometry (VLBI) are basically geometric too. In this paper all these methods are described and explained in order to emphasize connection between these two disciplines.

Keywords: applied geometry; geometrical geodetic methods; geometrical optics; geodesy

INTRODUCTION

The very word “geometry” literally means "measurement of land" and is derived from two Greek words γεα (geo) - "land" and μετεω (metreo) - "measure". According to Herodotus, Egyptian geometry was created because of the need to measure land plots. In a further development, the geometry is gone much farther and became part of mathematics who studies the properties and mutual relations of figures. For the measurement of land Aristotle has already introduced another name- geodesy [11].

Geodesy is the science which has two main areas, “Higher Geodesy”, which studies the shape and dimensions of the planet Earth and its gravitational field, and "Practical Geodesy" or "Engineering Geodesy", which describes the measurement of land, instruments and technics of surveying, performance of various types of projected facilities and cartography. Both areas of Geodesy apply different methods based on geometrical laws, called geometrical methods.

Like methods, principles as well as the development of geodetic instruments, are based on geometry. The first geodetic instrument, utilized for astronomical observations, used the principles of stereographic projection and it has been constructed by Hipparchus in the 2th century BC, while Erastothenes used geometrical rules to measure radius of the Earth [11]. Further development of surveying instruments was based on the laws of geometrical optics, starting from magnifying lenses to field glasses and telescopes, which have become an integral part of modern dumpy levels, theodolites, total stations and scanners.

In the next chapters the implementation of geometry in geodetic instruments and measuring methods is presented and explained.

2. RAY TRACING IN OPTICAL INSTRUMENTES

Every optical system consists of one or more reflecting and refracting surfaces, and its basic function is to transform diverging optical wavefront coming from the object points to the converging spherical wavefront towards image points and form suitable image of an object for specific application. Since the used wavelengths of light waves are much smaller than the dimensions of optical elements in the systems, the passage of wavefronts is solving using ray concept. The optical ray is a line drawn in space corresponding to the direction of flow of radiant energy. The passage of rays through an optical system may be determined by purely geometrical procedures and it is the base of geometrical optics [5].

A ray travels between two points of a homogeneous medium as a straight line and it could be presented also with propagating vector $\vec{k}_m$ where n indikates m-th medium. If an incident ray travels towards the interface plane between two mediums of different index of refraction n_1 and n_2 it is partially reflected from the plane and partially transmitted to the next medium as refracted ray (Figure 1). The incident ray and the normal to the interface in the point of incidence (represented with vector $\vec{N}$) determine the plain of incidence. The incident, reflected and refracted rays, and their corresponding propagating vectors $\vec{k}_i$, $\vec{k}_r$ and $\vec{k}_t$ are coplanar, laying in the plane of incidence. The reflected ray obey the law of reflection and the incident angle ε_i between incident ray and the normal is the same as reflecting angle ε_r, between reflecting ray and the normal. The refracted ray obeys the law of refraction, or Snell law. It can be presented in a plane with simple equation

$$n_1 \cdot sin\varepsilon_i = n_2 \cdot sin\varepsilon_t \tag{Eq.1}$$

or in general with vector equation [5],

$$n_1 \cdot (\vec{k}_i \times \vec{N}) = n_2(\vec{k}_t \times \vec{N}). \tag{Eq.2}$$

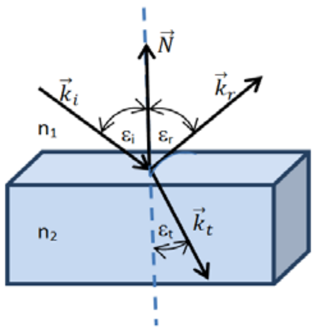


Figure. 1: The incident reflecting and refracting rays

Ray tracing and computing the passage of rays through an optical system is purely geometrical problem and is the basic tool of optical system design. The principles of ray tracing are the same for simple and complex optical systems and the geometrical construction or computer graphics can be used owing to system complexity.

Most optical instruments are completely or partially rotationally symmetrical about single axis named optical axis of a system, and are considered as centred. The usual methods of ray tracing are developed for such systems. Aspheric rotationally symmetric surfaces are difficult to produce although their application improved quality of optical systems and ray tracing methods can be applied for them, too [1]. The ray tracing from one surface to another can be constructed in a pure geometrical way. An example, geometrical construction of refracting rays at spherical interface will be described and presented in Figure 2.

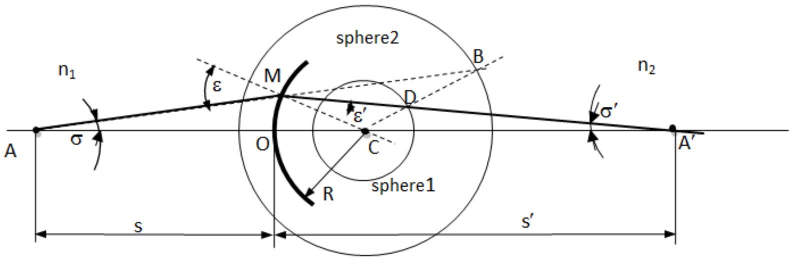


Figure. 2: Geometric construction of a refracted ray on spherical surface

A ray (AM) is traveling in a medium with refractive index n_1 and striking a spherical interface of radius R with another medium of refractive index n_2 and it is supposed that $n_2>n_1$. Incident ray emanated from point A of the optical axis hit the interface at point M. In order to obtain the direction of the refractive ray two mutually concentric spheres and concentric with the interface are constructed. First one with radius Rn_1/n_2 (smaller sphere 1) and the other (sphere2) with radius Rn_2/n_1 [1].The incident ray is continued and intersects the sphere 2 at point B, and then B is line connected to center C. The point D where BC intersects the sphere1 defines the direction MD od the refracted ray. Since CD/MD= n_1/n_2 and also CM/CB= n_1/n_2 the triangles ΔMCD and ΔBCM are similar having the mutual angle ∠MCB. Applying the law of sines to those triangles the relationship

$$\frac{CM}{CB} = \frac{\sin\varepsilon'}{\sin\varepsilon} = \frac{n_1}{n_2} \qquad \text{(Eq.3)}$$

proves that this construction is valid. This method enables to determine the image distance A′O =s' if an initial object distance is AO=s, without having to consider height of point M above the optical axis. From the triangles ΔAMC and ΔCMA′ and the sine theorem the relations (4-6) are obtained

$$\sin\varepsilon' = \frac{n_1}{n_2}\, sin\varepsilon = \frac{R+s}{R}\sin\sigma \qquad \text{(Eq.4)}$$

$$\sigma' = \sigma - \varepsilon - \varepsilon' \qquad \text{(Eq.5)}$$

$$\boldsymbol{s' - R = R\frac{\sin\varepsilon'}{\sin\sigma'}} \qquad \text{(Eq.6)}$$

And s' can be calculated as

$$\boldsymbol{s' = R\left(1 + \frac{sin\varepsilon'}{sin\sigma'}\right)} \qquad \text{(Eq.7)}$$

If the optical system involves number of refracting surfaces, to assess the effect of these surfaces the described geometrical procedure and the equations from (3-6) should be applied to each surface in succession, as presented in Figure 3. for two successive spherical surfaces of opposite curvature that is representation of a single lens. The image obtained on refraction of the k-th surface A'_k becomes the object A_{k+1} to the next, (k+1)-th refracting surface and the angle $\sigma_{k+1} = \sigma'_k$. Also, the distance s_{k+1} from (k+1)-th surface to A_{k+1} can be calculated using as $s_{k+1} = s'_k - d_k$, where d_k is distance between surfaces [1].

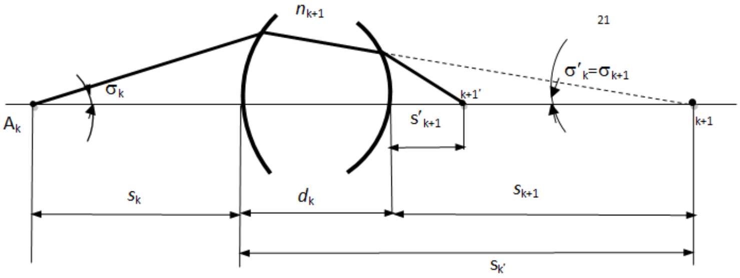


Figure. 3: Geometric ray tracing on successive refracting surfaces

The described geometric construction is suitable for the systems containing few surfaces and for tracing of the meridional rays that intersects the optical axis and practically lay in one plane. Among them the paraxial rays, passing near optical axis at small angles are always treated with simplified geometrical construction. The non-meridional or skew rays do not intersects the optical axis and travel in a broken lines around it and they are much more complicated for ray tracing. Their propagation is in fact three-dimensional problem.

Automatic numerical ray tracing is applicable to the propagating of all types of rays, paraxial meridional and skew. This ray tracing procedure consists of two parts, the transfer procedure and the refraction procedure. The transfer procedure involves computing of the intersection point of the ray on the surface from known optical direction and the intersection coordinates of ray at the previous surface. The refraction procedure involves computing the optical direction of a refracted ray from the intersection point data and the ray direction in the previous ray segment. Than those data are successively applied to the next surfaces.

In order to illustrate this procedures the simplest case for the paraxial meridional rays traversing few mediums separated with spherical interfaces is presented [5]. In general, for the k-th surface radius R_k , the indices of refraction of mediums before and after the surface n_k and n_{k+1} , the thickness of the medium after surface d_k are input data. The intersection point of the incident ray and the interface, M_K has vertical coordinate y_k and known direction angle.

At first the Snell law approximation for small angles is applied in the point M_k

$$n_k \cdot \varepsilon_{ik} = n_{k+1} \cdot \varepsilon_{tk} \quad \text{(Eq.8)}$$

in the form of

$$n_k \cdot (\alpha_{ik} + \alpha_k) = n_{k+1} \cdot (\alpha_{tk} + \alpha_k). \quad \text{(Eq.9)}$$

The incident ray direction is given with the angle α_{ik} between the ray and the optical axis and the direction of the refracting ray with the angle αtk. The small angle α_k connected to the direction of normal to the surface is calculated as

$$\alpha_k \approx tg\alpha_k = \frac{y_k}{R_k} \quad \text{(Eq.10)}$$

and introduced in equation (9) as

$$n_k \cdot (\alpha_{ik} + \frac{y_k}{R_k}) = n_{k+1} \cdot (\alpha_{tk} + \frac{y_k}{R_k}). \quad \text{(Eq.11)}$$

After separating the incident and refracting terms the equation becomes

$$n_{k+1} \cdot \alpha_{tk} = n_k \alpha_{ik} - \frac{(n_{k+1} - n_k)}{R_k} y_k = n_k \alpha_{ik} - D_k y_k$$
(Eq.12)

where D_k represents an optical power of the k-th surface . The equation (11) is the refracting equation of the k-th surface. From this equation the $\alpha_{ik+1}=\alpha_{tk}$, the direction of the incident ray to the (k+1)-th interface is determined.

After refraction in point M_k the ray propagates through homogeneous medium to the point M_{k+1} on the (k+1)-th interface as a straight line. The y_{k+1}, coordinate of point M_{k+1}, can be calculated as

$$y_{k+1} = y_k + d_{kh} \cdot \alpha_{tk} \qquad \text{(Eq.13)}$$

where d_{kh} is horizontal distance between the M_k and M_{k+1}. This equation is known as transfer equation and it follows the ray in homogeneous medium. For small angles $d_{kh} \approx d_k = O_k O_{k+1}$. After computing the direction angle α_{ik+1} and y_{k+1} the equation (12) and (13) are used successively on the next surfaces.

The optical elements such as prisms, spherical or aspherical lenses, plane and spherical mirrors are basic parts of optical instruments. In geodesy, these basic optical elements and their combination are used for construction of refracting prisms, oculars and objectives of field glasses, telescopes , microscopes which are the integral part or more complex instruments such as dumpy levels, theodolites, total stations, scanners, projecting and imaging instruments. Ray tracing enables precise construction of optical part of these instruments, elimination of images distortions, calculating the correction factor and evaluation of measuring uncertainness. As an example, the theodolite will be presented as complex optical instrument.

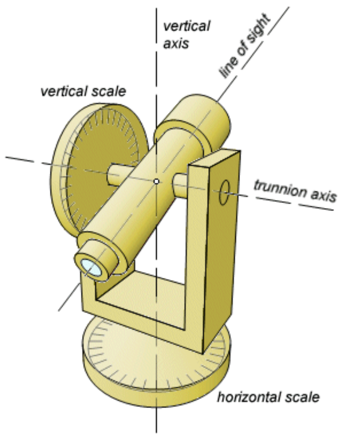


Figure. 4: The working principle of theodolite

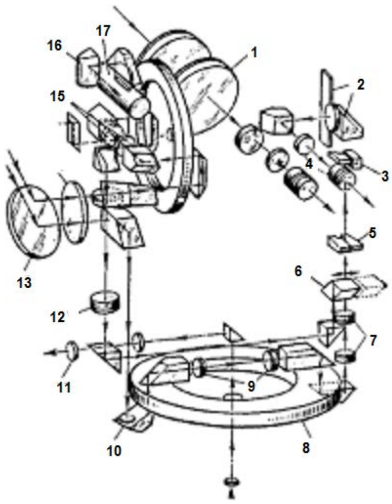


Figure. 5: Theodolite optical parts

The theodolite is an instrument for precision measuring of horizontal and vertical angles [4]. Its general simplified look is presented in Figure 4. and basically consists of a sighting telescope that rotates on a vertical axis, a horizontal circular scale that rotates on this same axis to measure the horizontal angle. The second axis, the trunnion axis is perpendicular to the vertical axis and moves with the instrument. The trunnion axis allows the telescope to pivot up and down, and it has a scale to measure the vertical angles. The resolution of the theodolite angular measuring units is the initial limiting factor and currently available theodolites have resolutions of 0.3 seconds of arc. Looking inside the theodolite as presented in Figure 5. shows that it is complex optical instrument, with lot of groups of optical elements that has specific functions. Objective and complex ocular lenses of telescope and its optical axis are denoted as 1 in Figure 5. For the horizontal and vertical angle readings is used reading microscope numbered as 4 together with objectives for vertical and horizontal scale numbered as 12 and 7. Lot of prisms redirect light in horizontal or vertical direction or temporary block some rays. The lenses and prisms in part denoted as 11 are used as optical centring device. The light mirror denote as

13 brings additional light to the theodolite optics. Similar objectives and oculars combinations for magnifying the objects, reflecting and refractive prisms are parts of other geodetic instruments in the similar way.

3. GEOMETRY IN CARTOGRAPHICAL AND PHOTOGRAMMETRIC PROJECTIONS

A cartographical map projections are mathematical procedures which enable the converting (mathematically speaking, mapping) curved surface (sphere or rotational ellipsoid) of Earth and other celestial bodies to the plane. The aim of studying map projections is to create mathematical basis for making maps and solving theoretical and practical tasks in cartography, geodesy, geography, astronomy, navigation and other related disciplines.

The points on the surface of an ellipsoid or sphere are determined by the intersection of meridians and parallels. Network of meridians and parallels in the plane of projection is called the basic network and network whose shape in the observed map projection is the simplest is a normal network [7].

The main task of cartographic mapping is to establish the dependence between the coordinates of points on the Earth's ellipsoid or sphere and the coordinates of their images in a projection. That dependence is often defined by equations which include latitude φ and longitude λ as well as rectangular coordinates in the plane of projection, x and y.

Stereographic and orthographic projections are some of the oldest, and these projections were used by Hipparchus in second century B.C. [11] to create maps of the celestial sphere. Today there are hundreds of map projections which are classified according to shape of normal network as cylindrical, conic, and azimuthal as presented in Figure 6.

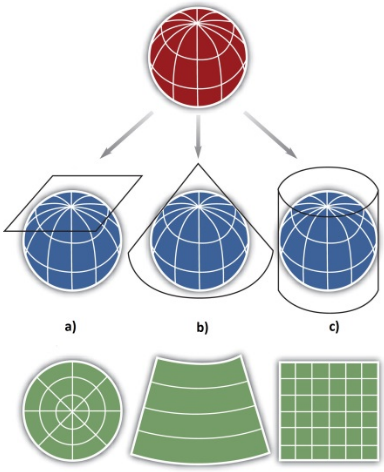


Figure. 6: Classification of map projections according to the shape of normal network:
(a) Azimuthal, (b) Conic, and (c) Cylindrical
(Source: Campbell and Shin (2012))

Map projections are used to reproduce parts or all of the Earth's surface with the least possible distortion. According to the types of deformation map projections are divided into conformal (keep equal angles), equivalent (keep equity of area) and equidistant (keep equal length). The geodetic projections are the special types, used for the needs of state surveying and official topographic maps. In this category are the most commonly used Universal Transverse Mercator (UTM), Transverse Mercator or Gauss-Kruger and the Lambert Conformal Conic projection.

Each of these projections can be determined analytically and graphically based on geometrical laws. Graphical determination becomes more complicated in the case of presenting certain parts of the Earth's surface and with using certain scales. Thanks to appearance of computer programs that enable automatic calculating and drawing map networks, today it is possible to present any part of the Earth's sphere or ellipsoid in any projection, and any scale.

Since that the largest implementation in geodesy have different variations of Mercator projection, it is geometrical determination will be presented in this paper.

The Mercator projection is a normal, cylindrical, conformal projection, presented by the Flemish geographer and cartographer Gerardus Mercator in 1569. Parallels and meridians are straight lines intersecting at right angles, with a requirement for conformality. Meridians are equally spaced and the parallel spacing increases with distance from the equator.

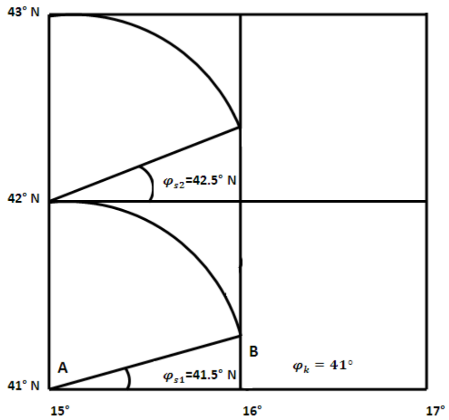


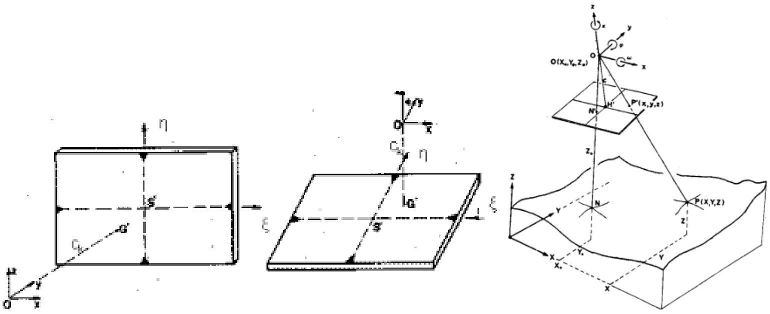
Figure. 7: Graphic design of Mercator projection

Mercator map can be constructed graphically using following procedure. First, on the bottom of available paper a line which represents the bottom edge of the map is drawn. Then, equally spaced vertical lines which represents the meridians in chosen intervals are added. In Figure 7. the parallel on 41° N represents bottom edge of the map and vertical lines are meridians in the interval from 15°-17° at distance of 1°. In order to draw the next parallel (42° N) it is necessary to raise the direction at an angle φ_S =41.5° from the point A till the intersection with next meridian (point B). This angle is chosen as an arithmetic mean of latitude for which has already drawn parallel and latitude for which we want to get next parallel. Then the length AB has to be transposed to the initial meridian and parallel line is constructed from those point. For the next parallel, process is repeating.

With this type of construction map scale (M) is not the specified, but can subsequently be calculated from known linear distance for interval between meridians equals 1°, which is explaned in detailes in [7].

Except in cartography, geometric projections are an integral part of another branch of geodesy, photogrammetry. Photogrammetry is the science of obtaining reliable information about objects and of measuring and interpreting this information. Major task of photogrammetry is concerned with reconstructing the object space from images. The geometrical relationship between image and object space can best be established by projection [6].

If points situated on a straight line in a plane or in a three-dimensional figure, are projected upon a single point located outside the figure in question, a so called central projection, central perspective or perspective projection occurs. The point located outside the figures, is called the projection centre or perspective centre. The geometric relationship between the perspective centre and the image plane is established by the interior or inner orientation. The geometric relationship between image and object is established by the exterior orientation [3]. During the establishing both of these orientation it's used central projection, as shown in Figures 8.



(a) (b)

Figure. 8: (a) Inner orientation, an (b) Exterior orientation
(Source: Kraus (1986))

4. APPLICATION OF GEOMETRY IN GEODETIC METHODS OF SURVEYING

One of the basic tasks of geodesy is to position the points in space, which means that they must be specified by their coordinates *x*, *y* and *z*. The coordinates *x* and *y* refer to the position in horizontal plane, while the coordinate *z* gives the vertical position of points. In geodesy, the height of a point is almost never determined directly, due to the unreliability of such measurements, as well as, a large number of errors which in this case may arise. Instead of that, the measuring the height it is done indirectly. The set of all operations, measurements and computations used for obtaining the height difference between two points is called levelling. There are several types of levelling, however, in the next chapter only trigonometric and geometric levelling will be presented, as well as that are types fully based on the lows of geometry.

4.1. Trigonometric levelling

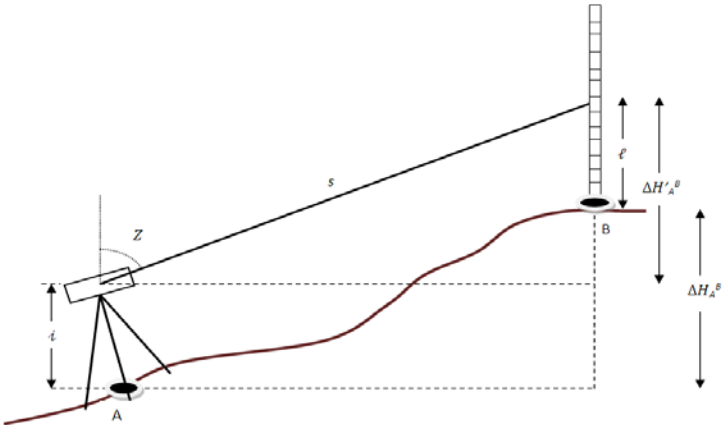


Figure. 9: General principles of trigonometric levelling

Trigonometric levelling method was created in the early years of the 19th century and represents the determination of height difference between two points using the trigonometric functions. In fact, applying of basic trigonometric identities is very easy to come up with a formula for determining the vertical distance between two points, which can be deduced from Figure 9. If height difference ΔH_{AB} between points A and B is unknown, it can be evaluated using trigonometric levelling by measuring the angle in the vertical plane (vertical angle α or zenith distance *Z*) and the length *s* between these two points. Observing triangle CDE we can notice that:

$$cosZ = \frac{\Delta H_A^{\prime B}}{s} \quad \text{(Eq.13)}$$

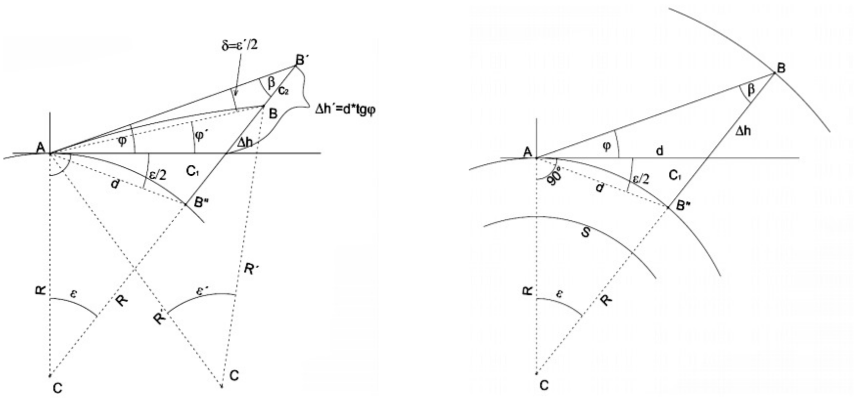
$$\Delta H_A^{\prime B} = s * cosZ \quad \text{(Eq.14)}$$

$$\Delta H_A^B = \Delta H_A^{\prime B} + i - \ell \quad \text{(Eq.15)}$$

Equations (13-14) can be applied in that form only when the distance between the points is up to few hundred metres, otherwise different effects have to be account, such as refraction and the curvature of Earth. To eliminate errors that they cause, it is necessary to calculate the corrections for these effects, using geometry, too.

If it assumed that the Earth is a sphere, according to Figure 10 (a), the angle β from the triangle ABC can be expressed through the angles φ and ε, and then value Δh is calculated by application of the sine theorem in the triangle ABB '.

$$\boldsymbol{\Delta h} = \frac{d\sin(\varphi+\frac{\varepsilon}{2})}{\sin\beta} = \frac{d\sin(\varphi+\frac{\varepsilon}{2})}{\cos(\varphi+\varepsilon)} \quad \text{(Eq.16)}$$



(a) (b)

Figure. 10: (a) The effect of the curvature of the Earth, and (b) The influence of refraction (Source: Dzapo (2008))

Since ε and $\varepsilon/2$ can be considered as small angles and the length of the tangent is approximately equal to the length of the arc AB ' across the angle ε, the equation for Δh become:

$$\Delta h = d * tg\varphi + \frac{d^2}{2R}, \quad \text{(Eq.17)}$$

where $d^2/2R$ is correction for the influence of the curvature of the Earth [4].

In practice distance *s* represents the optical ray propagation. Different densities of air layers between points A and B (Figure. 10 (b)) caused their different indices of refraction. The optical ray propagating from B to A is curved due to refraction and it seems that straight optical ray is propagating from point B'.

If it is surmised that the density of air layers decreases with height, the curve has the form of a circle with concave side facing towards the Earth. Let δ be the difference between the measured vertical angle φ and real angle φ ', and the radius of the Earth R and the refractive curve *R'* are related as *R = kR'*. By applying the sine theorem in the triangle ABC " it is possible to express the angle ε '/ 2, and then with a further developments of the equation, shown in detail in (-), the real height difference between A and B can be determined.

$$\Delta h = d * tg\varphi + \frac{d^2}{2R} - k\frac{d^2}{2R} = d * tg\varphi + (1-k)\frac{d^2}{2R}. \quad \text{(Eq.18)}$$

Correction member for to the impact of refraction is [4]:

$$c_{2=-}\ k\frac{d^2}{2R} \quad \text{(Eq.19)}$$

4.2. Geometric levelling

Geometric levelling is a method of determining the height difference between two points on the physical surface of the Earth by means of a horizontal line of sight, originated in the mid19th century. Height difference is determined based on the difference of reading the two vertical level staff. By definition, the height difference between points A and B represents the vertical distance between the levels of surface pulled through these two points with presuming that the levels of surface are mutually parallel and that the Earth is spherical, from Figure. 11 can be derived [4]:

$$\Delta h = h'_B - h'_A \qquad \text{(Eq.8)}$$

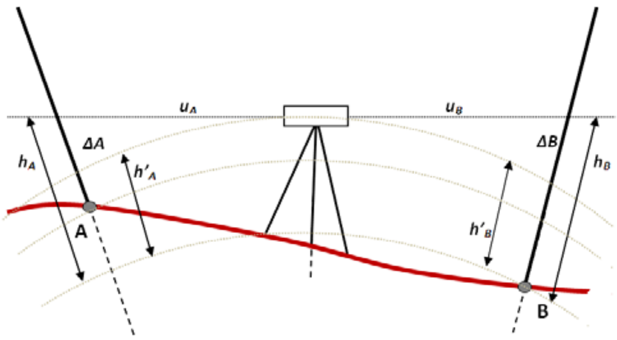


Figure. 11: The basic principle of geometric levelling

However, the measured readings are h_A and h_B. Dashed lines represents curved optical rays due to curvature of Earth. If the distances of points A and B from the instrument (u_A and $u_{B)}$ are equal, the differences ΔA and ΔB, given as:

$$\Delta A = h_{A-}h'_A \qquad \text{(Eq.9)}$$

$$\Delta B = h_{B-}h'_B \qquad \text{(Eq.10)}$$

will be the same.

In this case, the real height difference between points A and B will be obtain directly from level staff readings regardless of the curvature of Earth:

$$\Delta h = h_B - h_A \qquad \text{(Eq.11)}$$

4.3. Triangulation and Trilateration

Earlier in the past it was not possible to accurately measure very long distances, but it was manageable to measure the angles between points many kilometres apart very accurately. For that reason, 1615. Willebrord Snellius is developed a method of determining the position of points called triangulation. It is based on the trigonometric proposition that if one side and three angles of a triangle are known, the remaining sides can be computed. Moreover, if the direction of one side is known, the directions of the remaining sides can be determined. A triangulation system is formed of a series of joined or overlapping triangles in which one side is measured and remaining sides are calculated from angles measured at the vertices of the triangles. The vertices of the triangles are termed as triangulation stations and the side of the triangle whose length is known, is called the base line.

Since the entire process of determining the coordinates uses connectivity of geometric figures, trigonometric relations and the theorems, we can see that essence of triangulation lies in geometry [2].

Forms of trigonometric networks can be numerous (chain of triangles, braced quadrilaterals, centred triangles and polygons etc.). The shape of the triangles is significant as there is a lot of inaccuracy in a long skinny triangle, but one with base angles of about 45 degrees is ideal.

The idea of triangulation is that if we know the value of a one side and all angles in a triangle, unknown sides can be computed, and then use that information to get coordinates of points. Linear measurements in triangulation are used to determine the scale of the network and to prevent deformation.

All computations in the trigonometric network are carried out mainly by the rules of spherical and planar trigonometry. Since the network is located at the physical surface of the Earth which is not an ideal surface, it is necessary to connect trigonometric network with the Earth as a celestial body. This is achieved by knowing of ellipsoidal coordinates of a one trigonometric point and azimuth θ of a one trigonometric side.

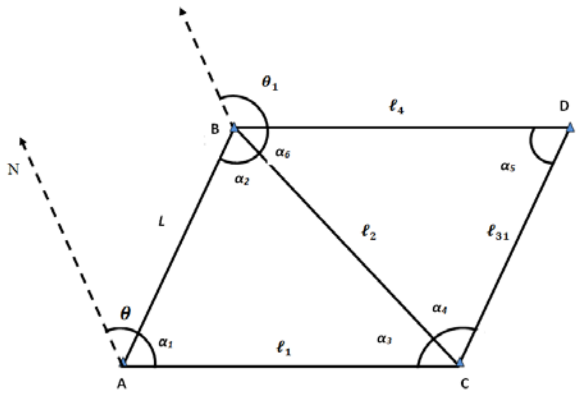


Figure. 12 : The principle of triangulation

If all the angles in triangles ABC and BCD are measured (Figure 2.), as well as the length L and the azimuth θ, and ellipsoidal coordinates of the point A are familiar, calculation of all sides of the triangles ($l_{1,}$ $l_{2,}$ etc.) can be done by applying sine theorem. Then the azimuths of every side (like θ_{BA}) are calculated and using procedure described in details in [2] the the coordinate X and Y of all points in triangle can be determined as:

$$X_B = X_A + Lsin\theta \quad \text{(Eq.14)}$$

$$Y_B = Y_A + Lcos\theta \quad \text{(Eq.15)}$$

Than the coordinates of other points in a triangle can be obtained successively relative to a predetermined point.

When we speak about triangulation, it's unavoidable to mention a method which is, in fact, a contrast of triangulation, and is also based on the geometry of triangles.

Middle of 20th century brings us development of accurate methods of measuring long distances. Thanks to this, a new method, trilateration, is appeared. Unlike triangulation, which uses the measurement of angles to determine location, trilateration uses measured distances. The distances in a triangle could then be measured directly, and the angles can be calculated using the cosine, sine and tangent theorem [2]. The process of calculating coordinates of points through the chain of triangles is then the same as for triangulation.

It has the following advantages:

- Because the fact that distances can be measured more accurately than angles, it is more accurate than the triangulation,
- It is less expensive than triangulation.

A combined triangulation and trilateration system consists of a network of triangles in which all the angles and all the lengths are measured, which allows to check the observations and improve the accuracy of the calculations in the triangles.

Triangulation is used for many purposes in geodesy, including surveying, navigation, astrometry and metrology. With the advancement of technology, these methods have evolved and adapted to modern instruments and techniques. Thus, trilateration found a place in the modern applicationas as basic principle of global positioning system.

4.4. Global Positioning System and Satellite Geodesy

The Global Positioning System (GPS) is a space-based navigation system that provides location and time information anywhere on or near the Earth where there is an unobstructed line of sight to four or more GPS satellites.

Global Positioning System is currently the only completely functional global navigation satellite system. GPS consists of 24 satellites deployed in Earth orbit, which send a radio signal to Earth [8]. GPS receivers based on these radio signals can determine its exact position - altitude, latitude and longitude - at any place on the planet, day and night, in all weather conditions.

Each satellite emits microwave radio signal sequence that is known to the receiver. While the receiver receives the signal, it is able to determine the time that elapses from the transmission of radio signals from the satellites to the reception at his position. Distance from the satellite receiver is calculated from this time, since the radio signal travels at well known speed. The signal also carries information about the current position of satellites from which to broadcast. If you know the distance from the satellite receiver and position of satellite, it is known that the receiver is located somewhere in the sphere of specific dimensions with center in a satellite. Since the positions of the three satellite receivers and the distance from each of them is known, by the trilateration method you can determine the position of the receiver. Trilateration is based on the fact that the three spheres intersect at most two points (one of them is out of Earth surface).The three-dimensional position of an unknown point is determined by the intersection of such a three spheres. However, as measured lengths containing the same unknown error of synchronization clock on the receiver and on the satellite, for a complete solution they need to be at least four spheres.

The selection of the four satellites, or their geometrical configuration with respect to the station, affects directly the positioning accuracy. An indicator of the quality of the geometry of the satellite constellation is called the Dilution of Precision or DOP.DOP only depends on the position of the satellites: how many satellites you can see, how high they are in the sky, and the bearing towards them [8].

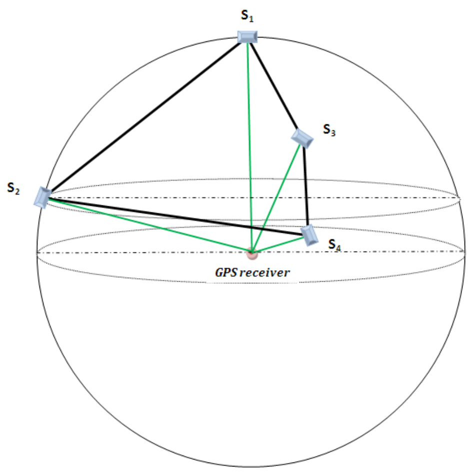


Figure. 13: Tetrahedron form of GPS satellites

According to [12] the best accuracy and the best DOP factor is realized when the volume of the tetrahedron formed by the four satellites is maximized. Theoretically, the largest possible tetrahedron is one for which one satellite is at the zenith and tree satellites are below the Earth's horizon at an elevation angle of -19.47° and equally spaced in azimuth. GPS reciver on or near Earth cannot observe the below-horizon satellites, so in this case, the best possible real DOP can be achieved with one satellite at the zenith and three satellites equally-spaced on the horizon (Figure 13).

GPS is nowadays one of the most common methods in use and belongs into the domain of satellite geodesy. Satellite geodesy comprises the observational and computational techniques which allow the solution of geodetic problems by the use of precise measurements to, from, or between artificial, mostly near-Earth, satellites. That geometry has tremendous applications in satellite geodesy is best reflected in the fact that in each method of this discipline geometry and arrangement of the satellites are the foundation [8].

Let's say that satellites can be used as high orbiting targets, which are visible over large distances. Following the classical concepts of Earth-bound trigonometric networks, the satellites may be regarded as "reference" control points within large-scale. If the satellites are observed simultaneously from different ground stations, it is of no importance that the orbits of artificial satellites are governed by gravitational forces. This purely geometric consideration leads to the geometrical method of satellite geodesy.

Compared with classical techniques, the main advantage of the satellite methods is that they can bridge large distances, and thus establish geodetic ties between continents and islands.

4.5. Very Long Baseline Interferometry

Geodetic Very Long Baseline Interferometry (VLBI) is one of the few major world-wide positioning techniques. It is probably the most accurate one over large distances. It is a pure geometric technique, i.e. it is not sensitive to the gravity field of the Earth (except for, mostly negligible relativistic effects), and it yields therefore - by definition - no geocentric coordinates. The measurement concept is tied to a quasi-inertial frame of very distant and compact extra-galactic radio sources (mostly quasars) and in this coordinate system VLBI is able to measure baseline vectors (and their changes in time) between distant stations on Earth [10].

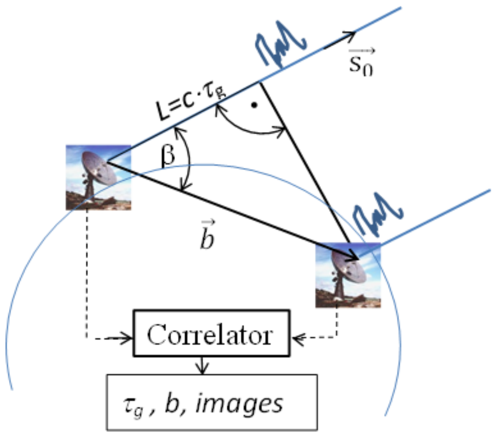


Figure. 14: The configuration of VLBI

As it is presented in Figure 14. the basic observational part of a VLBI configuration consists of two radio telescopes, two atomic clocks and two data storage units. The basic concept of VLBI consists of an incoming planar wave front that propagates along the unit vector to the radio source $\vec{s_0}$ and arrives at two antennas, which are pointed simultaneously at the same radio source and are separated by the baseline vector $\vec{b}$. The scalar product of $\vec{b}$ and $\vec{s_0}$ divided by the speed of light c determines the primary geodetic observable, geometric delay τ_g [9] :

$$\tau_g = t_2 - t_1 = \frac{\vec{b}\cdot\vec{s_0}}{c} = \frac{b\cos\beta}{c} \quad \text{(Eq.16)}$$

It contains all the information for the geodetic analysis as it is dependent on the position of the telescopes, the position of the source, etc. Also, observable only depends on fundamental physics as it is derived from a realization of the atomic second and a clock synchronization convention. The second independent observable is delay rate $\dot{\tau}_g$, the time derivative of τ_g. Their value for each observation is derived via a correlation process. The correlator is located at a central institute, to which all recorded data are sent.

Using measurements from different sources the precision of 1mm for horizontal and 2-3mm in vertical direction are achieved.

Generally, radio telescope consists of radio antenna of parabolic shape (parabolic dishes) and radio receiver. One or more antennas collect the incoming waves, in the same way as a curved mirror can focus visible light to a point. The amount of incoming radiation that can be collected depends on the size of its dish , and the size of dishes are up to 70 meters.

The analysis of VLBI data for astrometric and geodetic purposes requires the estimation of source positions, station positions, Earth orientation parameters (EOP), and parameters characterizing the behaviour of the clock and atmosphere at each station.

The angular resolution achieved by this technique is presently superior to any other technique using astronomical observations. The resolution of the Earth-based VLBI is, however, limited by the physical dimension of the Earth. This limitation has now been overcome by Space Very Long Baseline Interferometry (SVLBI), in which the orbiting radio telescope is operated in conjunction with the ground-based radio telescopes. The SVLBI main observables, the delay and delay rate, contain information related to the geometry of ground station – satellite – radio source positions [9].

CONCLUSION

The described geodetic methods and instrumental principles show that geometry and geodesy are tightly connected from their first appearances till nowadays. The development of more accurate instruments and combining of different geodetic methods encourages linking and practical applications of different areas of geometry in geodesy.

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