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PRIMENA SLOJEVITIH KONAČNIH ELEMENATA U NUMERIČKOJ ANALIZI LAMINATNIH KOMPOZITNIH I SENDVIČ-PLOČA I LJUSKI S DELAMINACIJAMA

APPLICATION OF LAYERED FINITE ELEMENTS IN THE NUMERICAL ANALYSIS OF LAMINATED COMPOSITE AND SANDWICH STRUCTURES WITH DELAMINATIONS

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1 UVOD

Laminatni kompoziti su moderni materijali koji se široko primenjuju u različitim granama industrije, a najviše u mašinstvu i građevinarstvu. Ovi materijali, kao što su ugljenična vlakna, staklena vlakna ili polimeri ojačani vlaknima, imaju izuzetno visoku čvrstoću i krutost, uz relativnu malu sopstvenu težinu. Na primer, konstrukcije koje se primenjuju u avio i svemirskoj industriji izgrađene su od tankozidnih kompozitnih cilindričnih ili sferičnih delova [1], koji imaju odličnu otpornost na zamor ili koroziju. Zbog velikog potencijala za različite primene, laminatni kompoziti neprekidno privlače pažnju mnogih istraživača [2–4]. Drugi tip materijala koji se mogu analizirati primenom predloženog numeričkog modela jesu sendvič-paneli s mekim jezgrom, koji se zbog male težine primenjuju u građevinarstvu u vidu lakih krovnih i fasadnih termoizolacionih panela.

Osnovni faktor koji skraćuje životni vek laminatnih kompozitnih konstrukcija jeste prisustvo delaminacije koja nastaje kao posledica grešaka pri spajanju lamina u fabričkoj proizvodnji. U slučaju sendvič-panela veoma je bitno da idealna veza između obloge panela i mekog jezgra ne bude narušena, kako bi se panel ponašao u skladu s projektovanim zahtevima. Autori su ranije pokazali da prisustvo delaminacije ozbiljno utiče na mehaničke karakteristike laminatnih kompozitnih i sendvič-ploča [5–6].

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1 INTRODUCTION

Laminar composites are modern engineering materials applied widely in different industries, mostly in the mechanical and civil engineering. These materials, such as carbon-fiber, glass-fiber or fiber-reinforced polymers allow for high strength and stiffness at a relatively low weight. For example, the aerospace structures are generally made of thin-walled composite cylindrical or spherical shell components [1], which have an excellent fatigue and corrosion resistance properties. Because of a great potential for the structural applications, laminar composites continuously attract the attention of many researchers [2, 3, 4]. Another type of plate structures relevant for the considered numerical model is a soft-core sandwich panel, which low weight property makes them applicable in civil engineering as light roof and wall panels to provide the thermal isolation of buildings.

A major limiting factor for the lifetime of laminated composite structures is the presence of embedded delamination, resulting from the different fabrication-induced faults in the joining of laminas. In the case of sandwich panels it is important that the perfect bond between the face sheets and the soft-core remain intact for the panel to perform on the designed level. Authors have already shown that the presence of embedded delamination seriously influences the mechanical properties of laminated composite and sandwich plates [5, 6].

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Odgovor tankih laminatnih konstrukcija može se precizno sračunati primenom dvodimenzionalnih teorija ploča i ljuski. Globalni odgovor može se odrediti primenom relativno jednostavnih Equivalent Single Layer (ESL) teorija, koje ipak nisu potpuno adekvatne iz dva razloga: (1) smičuće deformacije su izraženije kod laminatnih kompozita u poređenju sa izotropnim pločama i (2) ESL teorije daju previše krut odgovor zbog uprošćenja koja se odnose na klasičnu kinematiku ploča i ljuski [7–9]. Klasična teorija ljuski bazirana na Kirchhoff-Love-ovim pretpostavkama potcenjuje vrednosti ugiba i precenjuje veličine sopstvenih frekvencija [10]. U smičućoj teoriji ploča prvog reda (FSDT) efekti smičuće deformacije uzimaju se u obzir primenom smičućih korekcionih faktora. Reddy je primenjivao konačni element baziran na FSDT za analizu savijanja laminatnh kompozitnih ljuski [11] kao i za geometrijski nelinearnu analizu neoštećenih laminatnih kompozita [12]. U radu [13] konačni element baziran na FSDT podeljen je na oštećene i neoštećene delove pri proračunu sopstvenih frekvencija i tonova oscilovanja laminatnih kompozitnih ploča s delaminacijom. Kako bi se preciznije uzelo u obzir krivljenje poprečnog preseka, Reddy [14] je razvio smičuću teoriju ploča višeg reda (HSDT), gde je polje pomeranja po visini poprečnog preseka aproksimirano kubnom funkcijom koordinate u pravcu debljine ploče. U prethodnim radovima [15], Vuksanović je koristio konačne elemente bazirane na jednoslojnom modelu ploča višeg reda za proračun statičkog i dinamičkog odgovora neoštećenih kompozitnih ploča. Pored primene u analizi laminatnih kompozitnih ploča, konačni elementi bazirani na HSDT primenjeni su i u analizi kompozitnih sendvič-panela, u radovima [16–17]. Metod diferencijalne kvadrature (DQ) kao i opšti metod diferencijalne kvadrature (GDQ) takođe su primenjeni za proračun sopstvenih frekvencija laminatnih kompozita [18–22], uz neka analitička rešenja [23–27].

Ograničenja u ESL teorijama motivisala su istraživače da razviju složene (slojevite) teorije ploča i ljuski, koje svaki materijalni sloj razmatraju posebno, pretpostavljajući jedinstveno polje pomeranja po debljini svakog sloja. Reddy-eva opšta laminatna teorija ploča (GLPT) [28], koju su kasnije unapredili Barbero i Reddy [29], postala je osnova za razvoj familije slojevitih konačnih elemenata koji imaju mogućnost da opišu nezavisno kretanje svakog sloja posebno. Ćetković i Vuksanović su razvili analitičko i numeričko slojevito rešenje opšte laminatne teorije ploča za analizu laminatnih kompozitnih i sendvič-ploča bez oštećenja [30] i verifikovali su model primenom postojećih eksperimentalnih i numeričkih rezultata [31–34]. Alnefaie [35] je koristio potpuni slojeviti numerički model baziran na MKE za proračun dinamičkih karakteristika laminatnih ploča uz uzimanje u obzir oštećenja između slojeva. Kontakt između slojeva tokom fenomena „disanja” delaminacije kod „pametnih” kompozitnih ploča uveden je u radovima [36–37]. Korišćen je slojeviti model ploče za ispitivanje linearnih i nelinearnih odgovora „pametnih” kompozitnih ploča s delaminacijom.

Istraživači se uglavnom fokusiraju na analizu laminatnih kompozitnih ploča primenom slojevitih konačnih elemenata, dok još uvek postoji mali broj istraživanja mogućnosti primene slojevitih konačnih elemenata u analizi laminatnih kompozitnih ljuski s delaminacijama. Başar sa saradnicima [38–39] razvio je

The structural response of thin laminated structures can be accurately determined by the use of the two-dimensional plate and shell theories. The global structural response can be determined by the use of relatively simple Equivalent Single Layer (ESL) laminate theories, which are partially inadequate for two reasons: (1) the transverse shear deformations for composite laminates are more pronounced compared to those of isotropic structures and (2) ESL theories give too stiff response because of the simplifications associated with the classical plate and shell kinematics [7-9]. The classical shell theory based on Kirchhoff-Love’s assumptions underpredicts the deflections and overpredicts the natural frequencies [10]. In the First-Order Shear Deformation Theory (FSDT) transverse shear effects are taken into account by means of the shear correction factors. Reddy used the shear deformable finite element based on the FSDT for the bending analysis of laminated composite shells [11] and also for the geometrically nonlinear analysis of intact laminated composites [12]. Ju et al. [13] divided the FSDT finite element into delaminated and intact segments and calculated the natural frequencies and mode shapes of delaminated composite plates. In order to account for more accurate cross-sectional warping Reddy [14] developed a Higher-Order Shear Deformation laminate theory (HSDT) where displacement expansion through the plate thickness was approximated using the cubic series expansion of thickness coordinates. In previous works [15], Vuksanović has used finite elements based on the single layer models of higher order for the calculation of the static and dynamic response of intact composite plates. Beside the application to the analysis of laminated composite plates, finite elements based on the Higher-Order Shear Deformation Theory were applied in the analysis of composite sandwich plates, in works of Nayak et al. [16-17]. The method of differential quadrature as well as Generalized Differential Quadrature Method (GDQ) was also used for the calculation of natural frequencies of laminated composite structures [18-22], along with some analytical solutions [23-27].

The limitations of the ESL theories motivated the researchers to derive refined (layerwise) plate and shell theories, which consider each material layer separately by assuming the unique displacement field through the thickness of each layer. The Generalized Layerwise Plate Theory (GLPT) proposed by Reddy [28] and further improved by Barbero and Reddy [29] became the basis for the development of family of layered finite elements capable to describe the independent motion of each layer separately. Ćetković and Vuksanović have derived both the analytical and numerical layerwise solution of the GLPT for the analysis of intact laminated composite and sandwich plates [30] and verified the model using the existing experimental and numerical results [31-34]. Alnefaie [35] used a full layerwise finite element model, for calculation of the fundamental dynamic characteristics of laminated plates considering interlaminar damage. Ghoshal et al. [36-37] incorporated interlaminar contact during the "breathing" phenomena in the delaminated zone of smart composite plates. They have used a layered plate model to investigate linear and nonlinear responses of smart composite structures with delamination.

While researchers mostly focused their attention on familiju slojevitih elemenata ljuske za proračun slobodnih vibracija laminatnih kompozitnih cilindričnih i hiperboličnih ljuski, bez razmatranja delaminacija. Botello je sa saradnicima [40] razvio trougaoni slojeviti konačni element baziran na Reddy-evoj GLP Teoriji, za analizu neoštećenih kompozitnih ploča i ljuski. Oni su takođe uveli i tehniku podstruktura tokom procesa formiranja karakterističnih globalnih matrica, kako bi eliminisali stepene slobode u ravni ploče.

U ovom radu prikazani su određeni moderni pristupi u numeričkoj analizi oštećenih kompozitnih i sendvič-ploča primenom slojevitih konačnih elemenata. Na početku je dato poređenje između dinamičkih karakteristika laminatnih kompozitnih i sendvič-ploča različitih oblika, s prisustvom ili bez prisustva delaminacije. Slojeviti konačni element prikazan u ovom radu primenjen je u uporednim numeričkim proračunima slobodnih vibracija laminatnih kompozitnih ploča s delaminacijama. Nakon toga predloženi model je primenjen u dinamičkoj analizi oštećenih kompozitnih i sendvič-panela. Zatim je model ploče proširen na numeričku analizu laminatnih kompozitnih ljuski s delaminacijama. Primenjena je Reddy-eva GLP Teorija, u kojoj je pretpostavljena nezavisna interpolacija komponenata pomeranja u ravni ploče i upravno na ravan, kao i mogućnost pojave diskontinuiteta na granicama susednih slojeva. U obzir je uzeta linearna promena pomeranja u ravni od sloja do sloja, kao i konstantan ugib po visini ploče. Laminatne kompozitne ljuske modelirane su kao skup malih trougaonih pločastih konačnih elemenata [41]. Krivljenje poprečnog preseka uzeto je u obzir slojevitim razvojem pomeranja po visini laminata. Konzistentna matrica masa uvedena je integracijom odgovarajućih inercionih članova po visini laminata. Konačni elementi implementirani su u originalni računarski program napisan u MATLAB®-u. Za generisanje modela i očitavanje rezultata korišćen je GiD® Pre/Post Processing program [42] razvijen u CIMNE-u, Barselona.

2 FORMULACIJA TEORIJE

2.1 Osnovne pretpostavke

U ovom radu razmatraćemo laminatne kompozitne i sendvič-ploče i ljuske konstantne debljine, sačinjene od *n* ortotropnih slojeva (Slika 1). Globalni koordinatni sistem je Dekartov koordinatni sistem, označen sa *xyz*, kao na Slici 1. Orijentacija vlakana svakog sloja definiše lokalnu *x*-osu materijalnih koordinata. Ugao između globalne *x*-ose i lokalne materijalne *x*-ose definiše orijentaciju vlakana svakog sloja koja će biti od značaja pri određivanju matrica transformacije. *N* predstavlja broj dodirnih površina između materijalnih slojeva (uključujući i spoljne) u kojima se nalaze čvorovi po debljini laminata (obično se usvaja $N=n+1$), dok *ND* predstavlja broj spojeva u kojima postoji delaminacija. Ukupna debljina laminata označava se sa *h*, dok se debljina *k*-tog sloja

the analysis of laminated composite plate structures using the layered finite elements there is still a lack of investigations regarding the applicability of layered finite elements for the analysis of laminated composite shells with delaminations. Başar et al. [38-39] developed a family of layered shell elements to calculate the free vibration response of laminated composite cylindrical and hyperboloid shells, without considering delamination. Botello et al. [40] derived the triangular layered finite element based on the Reddy's GLPT for the analysis of intact composite plate and shell structures, and also presented the substructuring technique to eliminate the in-plane degrees of freedom during the assembly process.

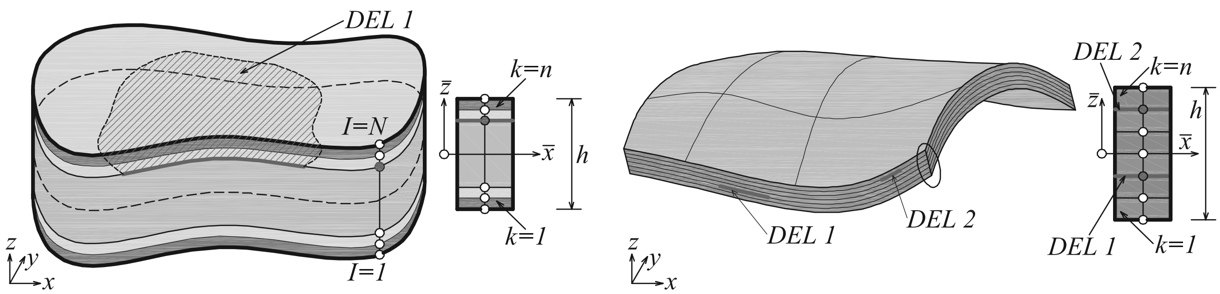
In this paper some recent advances in the numerical analysis of delaminated composite and sandwich plates using layered finite elements are presented. At first, the comparison between fundamental dynamic characteristics of laminated composite and sandwich plates of different shapes, with or without the presence of embedded delamination, is investigated numerically. The layered finite element developed in this paper is used to perform the comparative numerical calculation of free vibrations of laminated composite plates with embedded delaminations. After that the proposed model is used to perform the transient analysis of delaminated composite and sandwich plates. The plate model is then extended for the numerical analysis of laminated composite shells with embedded delaminations. Reddy's GLP Theory is used, which assumes independent interpolation of in-plane and out-of-plane displacement components, as well as possible discontinuities along the interfaces between adjacent layers. Piece-wise linear variation of in-plane displacement components and constant transverse displacement through the thickness are imposed. The laminated composite shells are modelled as the assembly of small triangular flat elements [41]. Cross-sectional warping is taken into account using the layerwise expansion of the displacements. A consistent mass matrix is employed by the integration of inertia terms through the thickness of the laminate. The finite elements have been implemented into the original program coded in MATLAB. The GiD® Pre/Post Processing software [42] developed in CIMNE, Barcelona is employed for generation of models and results of numerical examples.

2 FORMULATION OF THE THEORY

2.1 Basic assumptions

In this work we will consider laminated composite and sandwich plates and shells of constant height, composed of *n* orthotropic laminas (Figure 1). The global coordinate system is Cartesian coordinate system, denoted as *xyz*, as shown in Figure 1. The fiber direction of each lamina coincides with the local *x*-axis of material coordinates. The angle between the global *x*-axis and the local material *x*-axis defines the fiber direction of each lamina, which will serve for the calculation of the transformation matrices. *N* is the number of interfaces between the material layers (including the outer surfaces) in which nodes through the thickness are located (usually adopted as $N=n+1$), while *ND* represents the number of delaminated numerical layers. The overall označava sa h_k.

laminate thickness is denoted as h, while the thickness of the k^{th} lamina is denoted as h_k.



Slika 1. Kompozitna sendvič-ploča (levo) i laminatna kompozitna ljuska (desno) s delaminacijama, u globalnom koordinatnom sistemu xyz

Figure 1. Composite sandwich plate (left) and laminated composite shell (right) with embedded delaminations, in the global coordinate system xyz

Prošireni oblik opšte laminatne teorije ploča [28] zasnovan je na sledećim pretpostavkama:

1. Svi slojevi su međusobno idealno spojeni, osim u prethodno definisanoj zoni delaminacije, gde se mogu javiti skokovi u polju pomeranja u tri ortogonalna pravca.
2. Materijal je linearno elastičan i poseduje tri ravni simetrije. Svi slojevi su homogeni, bez mogućnosti pojave poprečnih prslina.
3. Prethodno definisana zona delaminacije je konstantna tokom mehaničkih procesa. Sprečeno je međusobno prodiranje sloja u sloj (overlapping).
4. Primenjene su nelinearne Von Kármán-ove kinematičke relacije kako bi se u obzir uzele umerene rotacije i male dilatacije.
5. Uzeta je u obzir neistegljivost linijskog elementa upravnog na srednju ravan ploče pre deformacije.

The GLP Theory [28] in its extended version is based on the following assumptions:

1. All layers are perfectly bonded together, except in the previously imposed delaminated area, where the jump discontinuities in three orthogonal directions may occur.
2. The material is linearly elastic and has three planes of material symmetry. All layers are considered as homogeneous materials, without the possibility of transverse cracking.
3. The previously imposed delaminated zone is kept constant during the mechanical process. The overlapping of layers is prevented.
4. Nonlinear kinematics according to von Kármán is incorporated to account for moderately large rotations and small strains.
5. Inextensibility of the transverse normal is imposed.

2.2 Polje pomeranja

Komponente pomeranja (u_1,u_2,u_3) u proizvoljnoj tački laminata (x,y,z) i proizvoljnom trenutku vremena t mogu se napisati kao:

2.2 Displacement field

The displacement components (u_1,u_2,u_3) at an arbitrary point (x,y,z) of the laminate and arbitrary time instant t can be written as:

$$
\begin{aligned}
u_1(x,y,z,t) &= u(x,y,t) + \sum_{I=1}^{N} u^I(x,y,t)\Phi^I(z) + \sum_{I=1}^{ND} U^I(x,y,t)H^I(z) \\
u_2(x,y,z,t) &= v(x,y,t) + \sum_{I=1}^{N} v^I(x,y,t)\Phi^I(z) + \sum_{I=1}^{ND} V^I(x,y,t)H^I(z) \\
u_3(x,y,z,t) &= w(x,y,t) + \sum_{I=1}^{ND} W^I(x,y,t)H^I(z)
\end{aligned}
\tag{1}
$$

U jednačini (1), (u,v,w) su apsolutne komponente pomeranja u srednjoj ravni laminata, (u^I,v^I) su relativne vrednosti pomeranja u I-tom numeričkom sloju u odnosu na pomeranja srednje ravni, (U^I,V^I,W^I) su skokovi u polju pomeranja u I-tom sloju gde postoji delaminacija. Promenljiva W^I predstavlja otvaranje delaminacije (mod I) u I-tom sloju gde postoji delaminacija. Kako bi se sprečilo prodiranje sloja u sloj u dinamičkoj analizi usvojen je uslov $W^I \geq 0$. Front delaminacije uvodi se zadavanjem graničnih uslova po pomeranjima U^I=V^I=W^I=0 na granici I-te delaminacije. $\Phi^I(z)$ su

In Eq. (1), (u,v,w) are the absolute displacement components in the middle plane of the laminate, (u^I,v^I) are the relative displacements in the I^{th} numerical layer in relation to the mid-plane displacements, (U^I,V^I,W^I) are jump discontinuities in the displacement field in the I^{th} delaminated layer. The variable W^I represents delamination opening (mode I) in the I^{th} delaminated interface. The condition $W^I \geq 0$ is adopted in the transient analysis to provide the non-penetration condition for delaminated surfaces within the I^{th} interface. The delamination front is represented by setting the essential linearne funkcije z-koordinate, kontinualne od sloja do sloja, kojima se interpoliraju komponente pomeranja u ravni. $H^I(z)$ su Heaviside-ove funkcije koje opisuju kinematiku delaminacije u I-tom sloju gde postoji oštećenje [5, 6, 8, 29]. Predloženim modelom ploče moguće je razmatrati proizvoljan broj delaminacija primenom odgovarajućeg broja dodatnih funkcija $H^I(z)$ u polju pomeranja.

boundary conditions $U^I=V^I=W^I=0$ on the I^{th} crack boundary. $\Phi^I(z)$ are linear layerwise continuous functions of the z-coordinate for interpolation of in-plane displacement components. H^I are Heaviside step functions to describe the delamination kinematics in I^{th} delaminated layer [5, 6, 8, 29]. The proposed plate model allows for the consideration of an arbitrary number of delaminations by using an arbitrary number of additional delamination expansions in the displacement field. In-plane displacements are piece-wise continuous through the thickness of the laminate in the intact region with discontinuities at the delaminated interfaces.

2.3 Kinematičke relacije

Kinematičke relacije kojima se pretpostavljaju male dilatacije i umereno male rotacije u skladu sa Von Kármán-ovim pretpostavkama definišu polje deformacija, koje se može podeliti na linearan (L) i geometrijski nelinearan (NL) deo:

2.3 Kinematic relations

Kinematic relations assuming the small strains and moderately large rotations according to von Kármán's assumptions define the strain field which can be divided into a linear (L) and geometrically nonlinear (NL) part:

$$
\begin{aligned}
\varepsilon_x^L &= \frac{\partial u}{\partial x} + \sum_{I=1}^{N} \frac{\partial u^I}{\partial x}\Phi^I + \sum_{I=1}^{ND} \frac{\partial U^I}{\partial x} H^I \\
\varepsilon_y^L &= \frac{\partial v}{\partial y} + \sum_{I=1}^{N} \frac{\partial v^I}{\partial y}\Phi^I + \sum_{I=1}^{ND} \frac{\partial V^I}{\partial y} H^I \\
\gamma_{xy}^L &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N} \left(\frac{\partial u^I}{\partial y} + \frac{\partial v^I}{\partial x} \right)\Phi^I + \sum_{I=1}^{ND} \left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x} \right) H^I \\
\gamma_{xz}^L &= \frac{\partial w}{\partial x} + \sum_{I=1}^{N} u^I \frac{d\Phi^I}{dz} + \sum_{I=1}^{ND} \frac{\partial W^I}{\partial x} H^I \\
\gamma_{yz}^L &= \frac{\partial w}{\partial y} + \sum_{I=1}^{N} v^I \frac{d\Phi^I}{dz} + \sum_{I=1}^{ND} \frac{\partial W^I}{\partial y} H^I
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
\varepsilon_x^{NL} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \sum_{I=1}^{ND} \frac{\partial W^I}{\partial x} H^I + \frac{1}{2} \sum_{I=1}^{ND} \sum_{J=1}^{ND} \frac{\partial W^I}{\partial x} \frac{\partial W^J}{\partial x} H^I H^J \\
\varepsilon_y^{NL} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \sum_{I=1}^{ND} \frac{\partial W^I}{\partial y} H^I + \frac{1}{2} \sum_{I=1}^{ND} \sum_{J=1}^{ND} \frac{\partial W^I}{\partial y} \frac{\partial W^J}{\partial y} H^I H^J \\
\gamma_{xy}^{NL} &= \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \sum_{I=1}^{ND} \sum_{J=1}^{ND} \frac{\partial W^I}{\partial x} \frac{\partial W^J}{\partial y} H^I H^J + \frac{\partial w}{\partial x} \sum_{I=1}^{ND} \frac{\partial W^I}{\partial y} H^I + \frac{\partial w}{\partial y} \sum_{I=1}^{ND} \frac{\partial W^I}{\partial x} H^I \\
\gamma_{xz}^{NL} &= \gamma_{yz}^{NL} = 0
\end{aligned}
\tag{3}
$$

2.4 Konstitutivne relacije pojedinačnog sloja

U ovom radu je usvojen konstitutivni model koji pretpostavlja ravno stanje napona u ploči, jer je u slučaju tankih ploča 3D konstitutivni model numerički nestabilan [5, 6]. Uz pretpostavku Hooke-ovog zakona linearne elastičnosti, konstitutivne relacije k-tog sloja za ravno stanje napona, u globalnom koordinatnom sistemu, mogu se prikazati kao:

2.4 Constitutive relations of the individual layer

A plane stress constitutive model is adopted because thin plates 3D constitutive model suffer the numerical instabilities [5, 6]. The constitutive equations of the k^{th} orthotropic lamina for the plane stress state that follows linear elastic Hooke's law, in the global coordinate system, can be written as:

$$\{\sigma\}^{(k)} = \left[\bar{Q}\right]^{(k)} \{\varepsilon\}^{(k)} \tag{4}$$

U jednačini (4), $\left[\bar{Q}\right]^{(k)}$ je redukovana matrica krutosti k-tog sloja u globalnom koordinatnom sistemu, dobijena iz matrične jednačine ($[\boldsymbol{T}]^{(k)}$-matrica transformacije):

In Eq. (4), $\left[\bar{Q}\right]^{(k)}$ is the matrix of reduced stiffness components of the k^{th} lamina in the global coordinate system, derived using the matrix relation ($[\boldsymbol{T}]^{(k)}$– transformation matrix):

$$\left[\bar{Q}\right]^{(k)} = \left[T\right]^{(k)-1} \left[Q\right]^{(k)} \left[T\right]^{(k)} \tag{5}$$

2.5 Princip virtualnog rada

Princip virtualnog rada za dinamički opterećene konstrukcije u vremenskom intervalu [0, T] izveden je primenom Hamilton-ovog principa:

2.5 Virtual work statement

The virtual work statement for dynamically loaded structures within the time interval [0, T] is derived using Hamilton's principle:

$$\int_0^T \left\{ \int_V \left[\sigma_{ij}\delta\varepsilon_j - q(x,y,t) - \rho\left(\ddot{u}_1\delta u_1 + \ddot{u}_2\delta u_2 + \ddot{u}_3\delta u_3\right) \right] dV \right\} dt = 0 \tag{6}$$

U jednačini (6) q predstavlja poprečno raspodeljeno opterećenje u srednjoj ravni ploče, $\ddot{u}$ je vektor ubrzanja, ρ je gustina i V je zapremina razmatranog domena. Ukoliko uvedemo presečne sile kao integrale komponentalnih napona po visini ploče dobijamo princip virtualnog rada u sledećem obliku:

In Eq. (6) q denotes the transverse mid-plane loading, $\ddot{u}$ denotes the acceleration vector, ρ is the mass density and V is the volume of the considered domain. If we introduce the stress resultants as the integrals of the componential stresses through the plate thickness we obtain the following form of the virtual work principle:

$$\int_0^T \left(\delta U^L + \delta U^{NL} + \delta V - \delta K \right) dt = 0 \tag{7a}$$

$$\delta U^L = \int_\Omega \left\{ \begin{aligned} & N_x \frac{\partial \delta u}{\partial x} + N_y \frac{\partial \delta v}{\partial y} + N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}\right) + Q_x \frac{\partial \delta w}{\partial x} + Q_y \frac{\partial \delta w}{\partial y} + \\ & + \sum_{I=1}^{N}\left(N_x^I \frac{\partial \delta u^I}{\partial x} + N_y^I \frac{\partial \delta v^I}{\partial y} + N_{xy}^I\left(\frac{\partial \delta u^I}{\partial y} + \frac{\partial \delta v^I}{\partial x}\right) + Q_x^I \delta u^I + Q_y^I \delta v^I \right) + \\ & + \sum_{I=1}^{ND}\left(\bar{N}_x^I \frac{\partial \delta U^I}{\partial x} + \bar{N}_y^I \frac{\partial \delta V^I}{\partial y} + \bar{N}_{xy}^I\left(\frac{\partial \delta U^I}{\partial y} + \frac{\partial \delta V^I}{\partial x}\right) + \bar{Q}_x^I \frac{\partial \delta W^I}{\partial x} + \bar{Q}_y^I \frac{\partial \delta W^I}{\partial y} \right) \end{aligned} \right\} d\Omega \tag{7b}$$

$$\delta U^{NL} = \int_\Omega \left\{ \begin{aligned} & N_x \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + N_y \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} + N_{xy}\left(\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y}\right) + \\ & + \bar{N}_x^I\left(\frac{\partial \delta w}{\partial x}\sum_{I=1}^{ND}\frac{\partial W^I}{\partial x} + \frac{\partial w}{\partial x}\sum_{I=1}^{ND}\frac{\partial \delta W^I}{\partial x}\right) + \bar{N}_y^I\left(\frac{\partial \delta w}{\partial y}\sum_{I=1}^{ND}\frac{\partial W^I}{\partial y} + \frac{\partial w}{\partial y}\sum_{I=1}^{ND}\frac{\partial \delta W^I}{\partial y}\right) + \\ & + \bar{N}_{xy}^I\left(\frac{\partial \delta w}{\partial x}\sum_{I=1}^{ND}\frac{\partial W^I}{\partial y} + \frac{\partial w}{\partial x}\sum_{I=1}^{ND}\frac{\partial \delta W^I}{\partial y} + \frac{\partial \delta w}{\partial y}\sum_{I=1}^{ND}\frac{\partial W^I}{\partial x} + \frac{\partial w}{\partial y}\sum_{I=1}^{ND}\frac{\partial \delta W^I}{\partial x}\right) + \\ & + \bar{N}_x^{IJ}\sum_{I,J=1}^{ND}\frac{1}{2}\left(\frac{\partial W^J}{\partial x}\frac{\partial \delta W^I}{\partial x} + \frac{\partial W^I}{\partial x}\frac{\partial \delta W^J}{\partial x}\right) + \bar{N}_y^{IJ}\sum_{I,J=1}^{ND}\frac{1}{2}\left(\frac{\partial W^J}{\partial y}\frac{\partial \delta W^I}{\partial y} + \frac{\partial W^I}{\partial y}\frac{\partial \delta W^J}{\partial y}\right) + \\ & + \bar{N}_{xy}^{IJ}\sum_{I,J=1}^{ND}\left(\frac{\partial W^J}{\partial y}\frac{\partial \delta W^I}{\partial x} + \frac{\partial W^I}{\partial x}\frac{\partial \delta W^J}{\partial y}\right) \end{aligned} \right\} d\Omega \tag{7c}$$

$$\delta V = -\int_{\Omega} \left(q \delta w \right) d\Omega \tag{7d}$$

$$\delta K = -\int_{\Omega} \left\{ \begin{aligned} & I_0 \left(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w \right) + \sum_{I=1}^{N} I^I \left(\ddot{u}^I \delta u + \ddot{v}^I \delta v + \ddot{u}\delta u^I + \ddot{v}\delta v^I \right) + \sum_{I=1}^{N}\sum_{J=1}^{N} I^{IJ} \left(\ddot{u}^I \delta u^J + \ddot{v}^I \delta v^J \right) + \\ & + \sum_{I=1}^{ND} \bar{I}^I \left(\ddot{U}^I \delta u + \ddot{V}^I \delta v + \ddot{W}^I \delta w + \ddot{u}\delta U^I + \ddot{v}\delta V^I + \ddot{w}\delta W^I \right) + \\ & + \sum_{I=1}^{N}\sum_{J=1}^{ND} \left(\bar{I}^{IJ} \left(\ddot{u}^I \delta U^J + \ddot{v}^I \delta V^J \right) + \bar{I}^{JI} \left(\ddot{U}^I \delta u^J + \ddot{V}^I \delta v^J \right) \right) + \sum_{I=1}^{ND}\sum_{J=1}^{ND} \tilde{I}^{IJ} \left(\ddot{U}^I \delta U^J + \ddot{V}^I \delta V^J + \ddot{W}^I \delta W^J \right) \end{aligned} \right\} d\Omega \tag{7e}$$

U jednačinama (7), $N_x, N_y, N_{xy}, Q_x, Q_y$ su presečne sile u srednjoj ravni ploče, $N_x^I, N_y^I, N_{xy}^I, Q_x^I, Q_y^I$ su relativne vrednosti presečnih sila u I-tom numeričkom sloju, $\overline{N}_x^I, \overline{N}_y^I, \overline{N}_{xy}^I, \overline{Q}_x^I, \overline{Q}_y^I$ su presečne sile koje razdvojene čvorove u zoni delaminacije u I-toj ravni drže sastavljenim i $I_0, I^I, I^{IJ}, \bar{I}^{IJ}, \tilde{I}^{IJ}$ su faktori inercije dobijeni integracijom gustine materijala po visini ploče. Presečne sile i faktori inercije su dati u [29]. Konstitutivne matrice laminata dobijene su integracijom linearno elastičnih ortotropnih konstitutivnih matrica svakog pojedinačnog sloja. One se označavaju sa [***A***], [***B***^{I}], [***E***^{I}], [***D***^{IJ}], [***L***^{IJ}], [***F***^{IJ}], [***L***^{IJK}], [***F***^{IJK}], [***F***^{IJKL}] i date su u radu [29].

3 NUMERIČKI MODEL MKE

3.1 Polje pomeranja

Posle određivanja „weak" forme, numerički model je dobijen prostornom i vremenskom diskretizacijom opisanom u ovom poglavlju. Slojeviti numerički model konačnog elementa baziran na prethodnim razmatranjima sastoji se od srednje ravni, *N* dodirnih površi između materijalnih slojeva po visini laminata (osim srednje ravni, uključujući i spoljne površi) i konačno *ND* numeričkih slojeva u kojima se nalazi delaminacija. Predložena teorija omogućava da se za generalisana pomeranja u čvorovima usvoje samo translacije u tri ortogonalna pravca. Promenljive u čvorovima (stepeni slobode) komponente su pomeranja (u,v,w) u srednjoj ravni, relativna pomeranja (u^I,v^I) u I-tom numeričkom sloju i skokovi u polju pomeranja (U^I,V^I,W^I) u I-tom numeričkom sloju gde postoji delaminacija. Prirodni koordinatni sistem pojedinačnog elementa ξ-η nalazi se u težištu elementa, Slika 2.

S obzirom na to što se proizvoljna geometrija ljuske ne može potpuno diskretizovati primenom četvorougaonih elemenata, primenjuju se trougaoni elementi. Interpolacione funkcije definišu se u prirodnom koordinatnom sistemu (na master elementu), jer se primenjuju trougaoni elementi proizvoljnog oblika i veličine. Prirodne koordinate ξ_i za proizvoljnu materijalnu tačku trougla (x,y,z) predstavljaju površine parcijalnih trouglova A_i, dobijenih povezivanjem tačke (x,y,z) sa uglovima trougla, kao na Slici 2. Zato važe sledeći uslovi:

In Eqs. (7), $N_x, N_y, N_{xy}, Q_x, Q_y$ are mid-plane stress resultants, $N_x^I, N_y^I, N_{xy}^I, Q_x^I, Q_y^I$ are relative values of the stress resultants in the I^{th} numerical layer, $\overline{N}_x^I, \overline{N}_y^I, \overline{N}_{xy}^I, \overline{Q}_x^I, \overline{Q}_y^I$ are the forces to hold the delaminated nodes together in the I^{th} delaminated interface and $I_0, I^I, I^{IJ}, \bar{I}^{IJ}, \tilde{I}^{IJ}$ are inertia terms derived by the integration of mass density through the plate thickness. The force resultants as well as inertia terms can be found in [29]. The constitutive matrices of the laminated plate are obtained by the integration of the linear elastic orthotropic matrices over all layers. They are denoted as [***A***], [***B***^{I}], [***E***^{I}], [***D***^{IJ}], [***L***^{IJ}], [***F***^{IJ}], [***L***^{IJK}], [***F***^{IJK}], [***F***^{IJKL}] and can be found in [29].

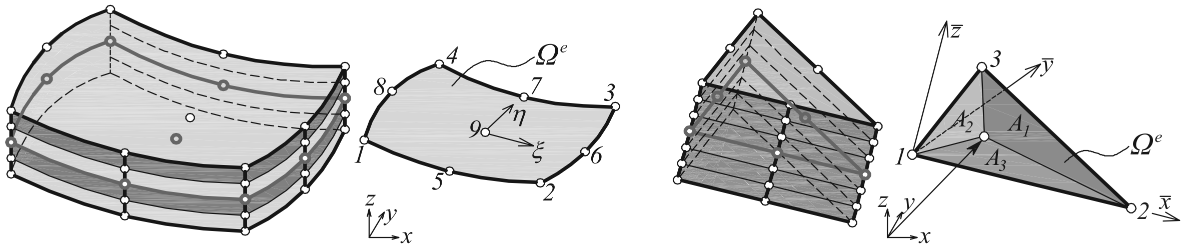
3 FINITE ELEMENT MODEL

3.1 Displacement field

After the derivation of the weak form, the numerical model is obtained by the spatial and temporal discretization described in this section. The layered numerical model of the single element, based on the previous considerations, consists of the middle plane, *N* interfaces between the laminas through the plate thickness (except the middle plane, including the outer surfaces), and finally *ND* interfaces in which delamination is present. The proposed theory allows adopting only translation components in three orthogonal directions as generalized displacements in the nodes. Nodal variables (degrees of freedom) are the displacement components (u,v,w) in the mid-plane, relative displacements (u^I,v^I) in I^{th} numerical layer and displacement jumps (U^I,V^I,W^I) in I^{th} delaminated numerical layer. The natural coordinate system of the single FE is located at its centroid, as shown in Figure 2.

Since the arbitrary shell geometry cannot be fully discretized using quadrilateral elements, triangular layered finite elements are used in this case. The shape functions are formulated in the natural coordinate system (on a master or unit triangle) because we are dealing with the triangular elements of arbitrary shape and size. The natural coordinates ξ_i for an arbitrary material point (x,y,z) of the triangle are the areas of partial triangles A_i, created by connecting the point (x,y,z) with triangle corners, as shown in Figure 2. Since there exist the following conditions:

$$\sum_{i=1}^{3} A_i = \sum_{i=1}^{3} \xi_i A = A \qquad \sum_{i=1}^{3} \xi_i = \xi_1 + \xi_2 + \xi_3 = 1 \qquad \xi_3 = 1 - \xi_1 - \xi_2 \tag{8}$$



Slika 2. Lagrange-ov četvorougaoni konačni element sa devet čvorova (levo) i trougaoni slojeviti konačni elementi sa tri čvora (desno) u prirodnom (ξ-η) i globalnom (xyz) koordinatnom sistemu ($\bar{x}\,\bar{y}\,\bar{z}$ - lokalni koordinatni sistem trougaonog elementa)

Figure 2. Lagrange 9-node quadrilateral (left) and 3-node triangular (right) layered finite elements in natural (ξ-η) and global (xyz) coordinate systems ($\bar{x}\,\bar{y}\,\bar{z}$ - local coordinate system of the triangular element)

Mreža konačnih elemenata generiše se u 2D ravni, a usvojene interpolacione funkcije po visini ploče se koriste za interpolaciju nepoznatih upravno na ravan laminata (čime se iz proračuna eliminiše z-koordinata). Ova pretpostavka omogućava da se interpolacija nepoznatih u ravni i upravno na ravan laminata vrši nezavisno. Nepoznate komponente pomeranja interpoliraju se u lokalnoj ravni pojedinačnog konačnog elementa, definisanoj s tri ugla trougla, dok lokalna *x*-osa povezuje čvorove 1–2. Lokalna *y*-osa nalazi se u ravni elementa i upravna je na *x*-osu, dok je lokalna *z*-osa upravna na ravan elementa I formira Dekartov koordinatni sistem ($\bar{x}\;\bar{y}\;\bar{z}$) lociran u čvoru 1. Radi jednostavnosti, za sva generalisana pomeranja koriste se iste interpolacione funkcije:

The FE mesh is generated in the 2D plane, and the adopted interpolation functions through the plate thickness are used for out-of-plane interpolation of the unknown variables (which eliminates the *z*-coordinate from the calculation). This assumption allows interpolating the unknown field variables independently for the in-plane and out-of-plane distribution. The unknown displacement components are interpolated in the local plane of the single finite element, defined by three corner nodes, while the local *x*-axis connects nodes 1-2. The local *y*-axis is positioned in the element plane and it is perpendicular to the local *x*-axis, while the local *z*-axis is perpendicular to the element plane and forms the local Cartesian orthogonal coordinate system ($\bar{x}\;\bar{y}\;\bar{z}$), located in the node 1. For the sake of simplicity, the same interpolation functions are used for the interpolation of all generalized displacements:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{m} u_i \psi_i \\ \sum_{i=1}^{m} v_i \psi_i \\ \sum_{i=1}^{m} w_i \psi_i \end{Bmatrix} = \left[\boldsymbol{\Psi}\right]\left\{\Delta\right\}, \quad \begin{Bmatrix} u^I \\ v^I \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{m} u_i^I \psi_i \\ \sum_{i=1}^{m} v_i^I \psi_i \end{Bmatrix} = \left[\bar{\boldsymbol{\Psi}}\right]\left\{\Delta^I\right\}, \quad \begin{Bmatrix} U^I \\ V^I \\ W^I \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{m} U_i^I \psi_i \\ \sum_{i=1}^{m} V_i^I \psi_i \\ \sum_{i=1}^{m} W_i^I \psi_i \end{Bmatrix} = \left[\boldsymbol{\Psi}\right]\left\{\bar{\Delta}^I\right\} \tag{9}$$

U jednačinama (9), $\{\Delta\}$, $\{\Delta^I\}$ i $\left\{\bar{\Delta}^I\right\}$ predstavljaju vektore pomeranja u srednjoj ravni, *I*-tom numeričkom sloju *I*-tom numeričkom sloju u kome postoji delaminacija, respektivno. Indeks *m* označava broj čvorova konačnog elementa. Primenjeni su pravougaoni Lagrange-ovi konačni elementi sa četiri ili devet čvorova, kao i trougaoni slojeviti konačni elementi s tri čvora. $\left[\boldsymbol{\Psi}\right]$ i $\left[\bar{\boldsymbol{\Psi}}\right]$ su matrice Lagrange-ovih interpolacionih funkcija:

In Eqs. (9), $\{\Delta\}$, $\{\Delta^I\}$ and $\left\{\bar{\Delta}^I\right\}$ denote displacement vectors in the middle plane, the I^{th} numerical layer and the I^{th} delaminated layer, respectively. Index *m* denotes the number of nodes per element. Four- and nine-node Lagrange quadrilateral, as well as three-node triangular layered finite elements are derived. $\left[\boldsymbol{\Psi}\right]$ and $\left[\bar{\boldsymbol{\Psi}}\right]$ are matrices of Lagrangian interpolation functions:

$$[\Psi]=\begin{bmatrix}\psi_1 & 0 & 0 & \\ 0 & \psi_1 & 0 & \cdots \\ 0 & 0 & \psi_1 & \end{bmatrix}_{3\times 3m} \quad , \quad [\bar{\Psi}]=\begin{bmatrix}\psi_1 & 0 & \\ 0 & \psi_1 & \cdots\end{bmatrix}_{2\times 2m} \tag{10}$$

3.2 Jednačine kretanja

Zamenom interpolacije komponenata pomeranja iz jednačine (9) u princip virtualnih pomeranja (jednačine (7)), dobijamo potpuno diskretizovan numerički model MKE [5, 6, 29]. Na ovaj način dobijamo sistem jednačina kretanja na nivou konstrukcije:

$$[\boldsymbol{M}]\{\ddot{\boldsymbol{d}}\}+[\boldsymbol{C}]\{\dot{\boldsymbol{d}}\}+\left[\boldsymbol{K}^{\boldsymbol{L}}+\boldsymbol{K}^{\boldsymbol{NL}}\right]\{\boldsymbol{d}\}=\{\boldsymbol{F}\} \tag{11}$$

Matrica masa [***M***], matrica prigušenja [***C***], linearne i nelinearne matrice krutosti [$\boldsymbol{K}^L$] i [$\boldsymbol{K}^{NL}$] i globalni vektor sila {***F***} dobijaju se sabiranjem odgovarajućih članova u karakterističnim matricama i vektorima pojedinačnih elemenata.

Sve matrice dobijene su primenom Gauss-Legendre-ove integracije na domenu pojedinačnog konačnog elementa, označenog sa Ω^e. Kako bi se iz proračuna eliminisala nepostojeća smičuća krutost (fenomen „shear locking, primenjena je selektivna integracija. Gauss-Legendre-ova integracija na trougaonom domenu sračunava se prema izrazu:

$$\int_{\Omega^e} F(\bar{x},\bar{y})d\Omega^e=\int_0^0\int_0^{1-\xi_2} F(\xi,\eta)d\xi d\eta\cdot\det[J]=\sum_{i=1}^{n_p}W_i f(\xi^i,\eta^i) \tag{12}$$

U jednačini (12), F je funkcija koju treba numerički sračunati, [J] je Jacobi-eva matrica, n_p je broj integracionih tačaka, ξ^i,η^i su koordinate i-te integracione tačke i W_i je odgovarajući težinski koeficijent [43]. Submatrice matrica krutosti i masa pojedinačnog konačnog elementa date su u [29]. Sledeće jednačine opisuju dva problema koji su razmatrani u ovom radu:

1. Linearne slobodne vibracije $\left(\left[\boldsymbol{K}^L\right]-\omega^2[\boldsymbol{M}]\right)\{\boldsymbol{d}\}=0$
2. Geometrijski nelinearna dinamička analiza $[\boldsymbol{M}]\{\ddot{\boldsymbol{d}}\}+[\boldsymbol{C}]\{\dot{\boldsymbol{d}}\}+\left[\boldsymbol{K}^{\boldsymbol{L}}+\boldsymbol{K}^{\boldsymbol{NL}}\right]\{\boldsymbol{d}\}=\{\boldsymbol{F}\}$

3.3 Transformacija u globalne koordinate i formiranje matrica sistema

Kada se proizvoljna ljuska deli na trougaone elemente, svaki element ima proizvoljnu orijentaciju u globalnom koordinatnom sistemu, pa je iz tog razloga od značaja da pogodno definišemo matrice tranformacije svakog konačnog elementa – potrebno je definisati kosinuse uglova koje lokalne ose zaklapaju s globalnim osama. Uzimajući u obzir ranija razmatranja, broj čvornih stepeni slobode u lokalnom koordinatnom sistemu je: $n_{DOF,LOCAL}$ = 3+2×N+3×ND. Ove lokalne komponente pomeranja transformišu se primenom matrice transformacije konačnog elementa $\left[\hat{\boldsymbol{T}}\right]$ u globalne

3.2 Equations of motion

When substituting the interpolation of displacement components from Eq. (9) into the principle of virtual displacements (Eqs. (7)), the fully discretized finite element model [5, 6, 29] has been obtained. This leads to the system of equations of motion on the structural level:

The mass matrix [***M***], the damping matrix [***C***], the linear and nonlinear structural stiffness matrices [$\boldsymbol{K}^L$] and [$\boldsymbol{K}^{NL}$] and the global force vector {***F***} are obtained from the assembly of the respective element matrices and element load vector.

All element matrices are derived using Gauss-Legendre quadrature over single finite element domain, denoted as Ω^e. Selective integration is used for the elimination of spurious shear stiffness from calculation (shear locking phenomenon). The Gauss-Legendre quadrature for triangular domain is written as:

In Eq. (12), F is the function to be calculated numerically, [J] is the Jacobi matrix, n_p is the number of integration points, ξ^i,η^i are the coordinates of the i^{th} integration point and W_i is the corresponding weighting factor [43]. The submatrices of the element stiffness and mass matrices are derived in [29]. Two problems investigated in this paper are described using the following equations:

1. Linear free vibrations $\left(\left[\boldsymbol{K}^L\right]-\omega^2[\boldsymbol{M}]\right)\{\boldsymbol{d}\}=0$
2. Geometrically nonlinear transient analysis $[\boldsymbol{M}]\{\ddot{\boldsymbol{d}}\}+[\boldsymbol{C}]\{\dot{\boldsymbol{d}}\}+\left[\boldsymbol{K}^{\boldsymbol{L}}+\boldsymbol{K}^{\boldsymbol{NL}}\right]\{\boldsymbol{d}\}=\{\boldsymbol{F}\}$

3.3 Transformation to global coordinates and assembly procedure

When the arbitrary shell is divided into triangular elements, each element has an arbitrary orientation in the global coordinate system, so it is now important to conveniently define the transformation matrices for each element - we need to define the cosines of each finite element. Following the preceding considerations, the number of nodal degrees of freedom in the local coordinate system is: $n_{DOF,LOCAL}$ = 3+2×N+3×ND. These local displacement components are transformed using the element transformation matrix $\left[\hat{\boldsymbol{T}}\right]$ into the global komponente pomeranja – čvorne stepene slobode u globalnom koordinatnom sistemu $n_{DOF,GLOBAL}$ = 3+3×N+3×ND. Matrice krutosti ili masa pojedinačnog konačnog elementa u globalnom koordinatnom sistemu sračunavaju se kao:

displacement components – nodal degrees of freedom in global coordinate system $n_{DOF,GLOBAL}$ = 3+3×N+3×ND. The global stiffness/mass matrices of the triangular finite element can be calculated as follows:

$$\begin{aligned}\left[\boldsymbol{K}\right]^{e} &= \left[\hat{\boldsymbol{T}}\right]^{T}\left[\boldsymbol{K}\right]\left[\hat{\boldsymbol{T}}\right]\\ \left[\boldsymbol{M}\right]^{e} &= \left[\hat{\boldsymbol{T}}\right]^{T}\left[\boldsymbol{M}\right]\left[\hat{\boldsymbol{T}}\right]\end{aligned} \tag{13}$$

4 REŠENJE VREMENSKOG PROBLEMA I KONTAKTNI ALGORITAM

Za integraciju u vremenu je primenjena Newmark-ova implicitna integraciona šema [8]. Ubrzanja i brzine su aproksimirani primenom redukovanih Taylor-ovih redova. Pomeranja i brzine određeni su primenom rekurentnih formula [8], a početni granični uslovi su homogeni. S obzirom na to što je matrica [K^{NL}] funkcija nepoznatog pomeranja $\{d\}_{n+1}$, sistem uslovnih jednačina sistema mora se rešavati iterativno sve dok kriterijum konvergencije ne bude zadovoljen. Picard-ov metod [8] primenjuje se sve dok greška ne bude manja ili jednaka vrednosti neke unapred usvojene tolerancije (recimo $\varepsilon \leq$ 1%). Prigušenje je zanemareno.

Tokom dinamičkog odgovora laminatnih kompozitnih ili sendvič-ploča s delaminacijom, može se formirati mali međuprostor između susednih slojeva u zoni delaminacije. Nakon toga razdvojeni slojevi tokom kretanja mogu ponovo da dođu u kontakt i na taj način zatvore postojeći međuprostor. Ovaj fenomen se u literaturi naziva „disanje” delaminacije [36–37], koje se može modelirati primenom kontaktnih uslova između čvorova, bez uzimanja u obzir trenja [6]. Kontaktni algoritam uspešno sprečava međusobno prodiranje sloja u sloj tokom dinamičkog odgovora kompozitnih ploča s delaminacijom. Ovo važi i za linearnu i za geometrijski nelinearnu analizu.

4 SOLUTION OF THE TIME DEPENDENT PROBLEM AND CONTACT ALGORITHM

For integration in time, an implicit Newmark’s integration scheme is employed [8]. The accelerations and velocities are approximated using truncated Taylor’s series. Displacements and velocities are approximated using recursive formulae [8], while the homogenous initial conditions are prescribed. The assembled equation must be solved iteratively until the convergence criterion is satisfied since the matrix [K^{NL}] is the function of displacements $\{d\}_{n+1}$. The Picard method [8] is employed until the error is less than or equal to some prescribed tolerance (say $\varepsilon \leq$ 1%). The structural damping is neglected.

During the transient response of delaminated composite or sandwich structures, a small gap may be formed between the adjacent layers in delaminated zones of the plate. After that the separated layers may unload and again contact each other at that delaminated interface. This phenomenon is referred to as "breathing" of a delamination [36-37], which can be modelled using the node-to-node frictionless contact conditions [6]. The contact algorithm successfully "corrects" the interlaminar penetration during the transient response of the delaminated composite plate both in the linear and geometrically nonlinear analysis.

5 NUMERIČKI PRIMERI I DISKUSIJA

U ranijim radovima [5, 6, 15, 30] autori su dokazali da ESL teorije precenjuju vrednosti sopstvenih frekvencija i potcenjuju vrednosti ugiba kod pravougaonih kompozitnih i sendvič-ploča bez oštećenja. U ovom radu su proširene mogućnosti primene modela na analizu slobodnih vibracija kružnih kompozitnih ploča s delaminacijama.

5.1 Analiza linearnih slobodnih vibracija

Primer 5.1.1. U prvom primeru [44] razmatra se neoštećena uklještena (CC) četvoroslojna kružna kompozitna ploča simetrične (θ/-θ/-θ/θ) šeme laminacije. Prečnik ploče označen je sa a, a ukupna debljina ploče sa h. Svi slojevi su jednake debljine h_k. Za sve slojeve su pretpostavljeni ortotropni konstitutivni modeli sa sledećim materijalnim karakteristikama: E_1/E_2 = 40, G_{12}/E_2 = = G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, υ_{12} = υ_{13} = υ_{23} = 0.25,

5 NUMERICAL EXAMPLES AND DISCUSSION

In previous works [5, 6, 15, 30], authors have proven that ESL theories overestimate the fundamental frequencies and underestimate the transverse deflections of rectangular intact composite and sandwich plates. The applicability of the model is extended here for the free vibrations analysis of circular composite plates with delaminations.

5.1 Linear free vibrations analysis

Example 5.1.1. The first benchmark example [44] is concerned with an intact 4-layer clamped (CC) circular composite plate with symmetric (θ/-θ/-θ/θ) stacking sequence. Plate diameter is denoted as a, while the overall plate thickness is denoted as h. All laminas are of equal thickness h_k. The following material parameters are assumed for orthotropic constitutive models of all laminas: E_1/E_2 = 40, G_{12}/E_2 = G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, υ_{12} = υ_{13} = υ_{23} = 0.25, ρ = const. The boundary conditions

ρ = const. Granični uslovi su zadati na uklještenim ivicama sprečavanjem svih generalisanih pomeranja u čvorovima. U ovom proračunu su zanemareni članovi koji odgovaraju velikim rotacijama. Rezultati su dobijeni primenom nestrukturirane mreže sa 931 slojevitim konačnim elementom sa devet čvorova, s redukovanom integracijom (broj čvorova je 3849). Sračunate su bezdimenzionalne sopstvene frekvencije neoštećenih kružnih laminatnih kompozitnih ploča prema izrazu $\Omega = \omega \cdot a^2/h(\rho/E_2)^{1/2}$ i izvršeno je poređenje s rezultatima dobijenim primenom različitih ESL teorija: smičuće teorije ploča [44] i smičuće teorije ploča prvog reda [45]. Rezultati su objedinjeni u Tabeli 1.

are prescribed along clamped boundaries by constraining all generalized displacements in edge nodes. The terms related to the large rotations in the kinematic equations are omitted in this calculation. The results are obtained using the unstructured FE mesh of 931 9-node layered elements with reduced integration (number of nodes is 3849). The nondimenzionalized fundamental frequencies $\Omega = \omega \cdot a^2/h(\rho/E_2)^{1/2}$ of intact circular laminated composite plates is calculated and compared with results obtained using different ESL theories: transverse shear deformation theory [44] and First-Order Shear Deformation Theory [45]. The results are elaborated in Table 1.

Tabela 1. Bezdimenzionalne sopstvene frekvencije četvoroslojne uklještene kružne kompozitne ploče simetrične angle-ply (θ/-θ/-θ/θ) šeme laminacije

Table 1. The nondimenzionalized fundamental frequencies of the 4-layer clamped circular laminated composite plates with symmetric (θ/-θ/-θ/θ) angle-ply stacking sequence

Numerical model	θ = 0	θ = 30	θ = 45
SDT [36]	23.130	24.063	24.557
FSDT [37]	22.211	24.071	24.752
GLPT, Present	22.913	25.107	25.684

Na osnovu Tabele 1 očigledno je da je predloženi model u stanju da predvidi sopstvene frekvencije kružnih laminatnih kompozitnih ploča, čak i uz korišćenje četvorougaonih elemenata za modeliranje kružne geometrije. Za sve šeme laminacije dobijen je nešto krući odgovor (2–4%). Sopstvene frekvencije očigledno rastu s povećanjem nivoa ortotropije.

Primer 5.1.2. U drugom primeru [13] razmatra se kvadratna kompozitna ploča sa osam slojeva, simetrične $(0/90/45/90)_s$ šeme laminacije. Ivice ploče imaju dužinu $a = 250mm$, dok je ukupna visina ploče $h = 2.12mm$. Svi slojevi su jednake debljine. Za ortotropni konstitutivni model svih slojeva pretpostavljeni su sledeći parametri materiala: $E_1 = 132GPa$, $E_2 = 5.35GPa$, $G_{12} = G_{13} = 2.79GPa$, $\upsilon_{12} = \upsilon_{13} = 0.291$, $\upsilon_{23} = 0.300$, $\rho = 1446.2kg/m^3$. Kvadratna delaminacija, stranice $a_{del} = a/2$ ranije je ubačena u srednjoj ravni (između slojeva 4 i 5), u centru ploče. Granični uslovi zadati su duž ivica ploče na sledeći način:

- slobodno oslonjene (SS_x) ivice: za $x = 0$ i $x = a$: $v = w = v^I = 0$,
- slobodno oslonjene (SS_y) ivice: za $y = 0$ i $y = b$: $u = w = u^I = 0$,
- uklještene (CC) ivice: $u = v = w = u^I = v^I = 0$.

Ploča je analizirana kako bi se ispitao uticaj graničnih uslova na sopstvene frekvencije kompozitnih ploča s prethodno ubačenom delaminacijom konstantne površine. Ploča je diskretizovana pomoću mreže od 6×6 slojevitih konačnih elemenata s devet čvorova, s redukovanom integracijom. Sračunate su sopstvene frekvencije za prva četiri tona oscilovanja neoštećenih i oštećenih ploča i upoređene su s rezultatima iz [13] u Tabeli 2.

From Table 1 it is obvious that the proposed model is capable to predict the fundamental frequencies of circular laminated composite plates, even by using the quadrilateral FE to describe the circular plate geometry. A slightly stiffer response (2-4%) is obtained for all stacking sequences. Natural frequencies obviously increase with the level of orthotropy.

Example 5.1.2. The second benchmark example [13] is concerned with an 8-layer square composite plate with symmetric $(0/90/45/90)_s$ stacking sequence. Side length of the plate is $a = 250mm$, while overall plate thickness is $h = 2.12mm$. All laminas are of equal thickness. The following material parameters are assumed for orthotropic constitutive models of all laminas: $E_1 = 132GPa$, $E_2 = 5.35GPa$, $G_{12} = G_{13} = 2.79GPa$, $\upsilon_{12} = \upsilon_{13} = 0.291$, $\upsilon_{23} = 0.300$, $\rho = 1446.2kg/m^3$. Square delamination of side $a_{del} = a/2$ is prescribed in the mid-plane (between layers 4 and 5) in the centre of the plate. The boundary conditions are prescribed along boundary edges as follows:

- simply supported (SS_x) edges: at $x = 0$ and $x = a$: $v = w = v^I = 0$,
- simply supported (SS_y) edges: at $y = 0$ and $y = b$: $u = w = u^I = 0$,
- clamped (CC) edges: $u = v = w = u^I = v^I = 0$.

The plate is analyzed to check the influence of boundary conditions on fundamental frequencies of composite plates with previously imposed delaminated zone of constant area. The plate is discretized by 6×6 9-node layered finite elements with reduced integration. Natural frequencies for first 4 modes for intact and delaminated plates are calculated and compared with the results from [13] in Table 2.

Tabela 2. Sopstvene frekvencije (Hz) neoštećenih i oštećenih $(0/90/45/90)_s$ kompozitnih ploča s različitim uslovima oslanjanja, za prva četiri tona oscilovanja

Table 2. Natural frequencies (Hz) for intact and delaminated (0/90/45/90)s composite plates with different boundary conditions for first four modes

	Model	State	1	2	3	4
Simply Supported Plate	**FSDT**	**Intact**	164.37	404.38	492.29	658.40
		Damaged	161.58	348.27	371.19	637.48
	Present	**Intact**	169.81	409.78	504.22	672.69
		Damaged	167.04	347.88	374.62	611.08
Clamped Plate	**FSDT**	**Intact**	346.59	651.51	781.06	1017.20
		Damaged	334.67	579.43	653.25	851.27
	Present	**Intact**	346.81	643.44	777.93	982.16
		Damaged	316.88	529.34	554.81	783.80

Rezultati prikazani u Tabeli 2 potvrđuju da je predloženi model u stanju da precizno predvidi sopstvene frekvencije neoštećenih i oštećenih kompozitnih ploča. Redukcija sopstvene frekvencije usled prisustva delaminacija veća je u slučaju viših tonova oscilovanja, za oba razmatrana slučaja.

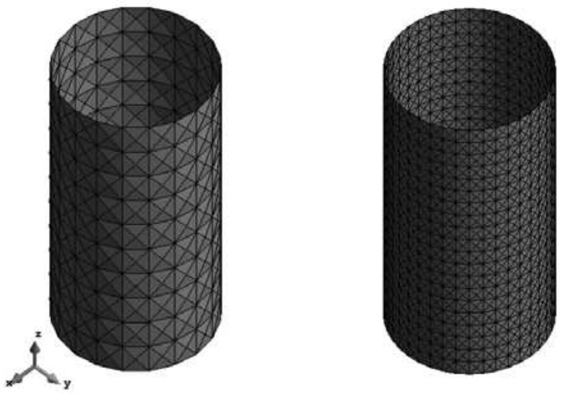
Primer 5.1.3. U trećem primeru razmatraju se cilindrične cross-ply laminatne kompozitne ljuske uklještene na oba kraja. Dužina razmatranih ljuski je L=12m, a poluprečnik R=3m. Ljuske su sačinjene od tri ortotropna sloja, pojedinačne debljine h_k=0.02m, tako da je ukupna debljina ljuske h=0.06m (odnos dužina/poluprečnik je L/R=4, a odnos debljina/poluprečnik je h/R=0.02, što odgovara tankim i umereno dugim ljuskama). Za sve slojeve su pretpostavljeni materijalni parametri koji odgovaraju grafit-epoksidu [27]: E_1 = 138 GPa, $E_2 = E_3$ = 8.96 GPa, $G_{12} = G_{13}$ = 7.1 GPa, G_{23} = 3.45 GPa, υ_{12} = 0.30, ρ = 1645 kg/m^3. Primenjeni su slojeviti trougaoni konačni elementi s tri čvora. Ljuska je diskretizovana strukturiranom mrežom s dve različite gustine (Mreža 1 – 800 elemenata i Mreža 2 – 3200 elemenata, videti Sliku 3). Granični uslovi su zadati duž uklještenih ivica sprečavanjem svih generalisanih pomeranja u čvorovima. Vrednosti bezdimenzionalnih frekvencija $\Omega = \omega \cdot 100R(\rho/E_2)^{1/2}$ dobijene u analizama poređene su s rezultatima iz [46] dobijenim primenom 2D prstenastih konačnih elemenata, kao i s rezultatima iz [27] dobijenim primenom kontinualnih elemenata baziranih na dinamičkoj matrici krutosti. Vrednosti sopstvenih frekvencija grafički su prikazane na Slici 4, za različite šeme laminacije.

Ovaj primer jasno pokazuje da se progušćenjem mreže dobijaju niže vrednosti sopstvenih frekvencija (konvergencija ka tačnom rešenju). Za **0/90/0** šemu laminacije dobijeno je odlično poklapanje u svim tonovima oscilovanja. Za **90/90/90** šemu laminacije dobijene su nešto niže vrednosti sopstvenih frekvencija u svim tonovima, zbog uticaja deformacije smicanja i idealizacija geometrije ljuske. Uzimanjem ovih napomena u obzir, predloženi model je u stanju da potpuno tačno predvidi sopstvene frekvencije laminatnih kompozitnih ljuski bez oštećenja.

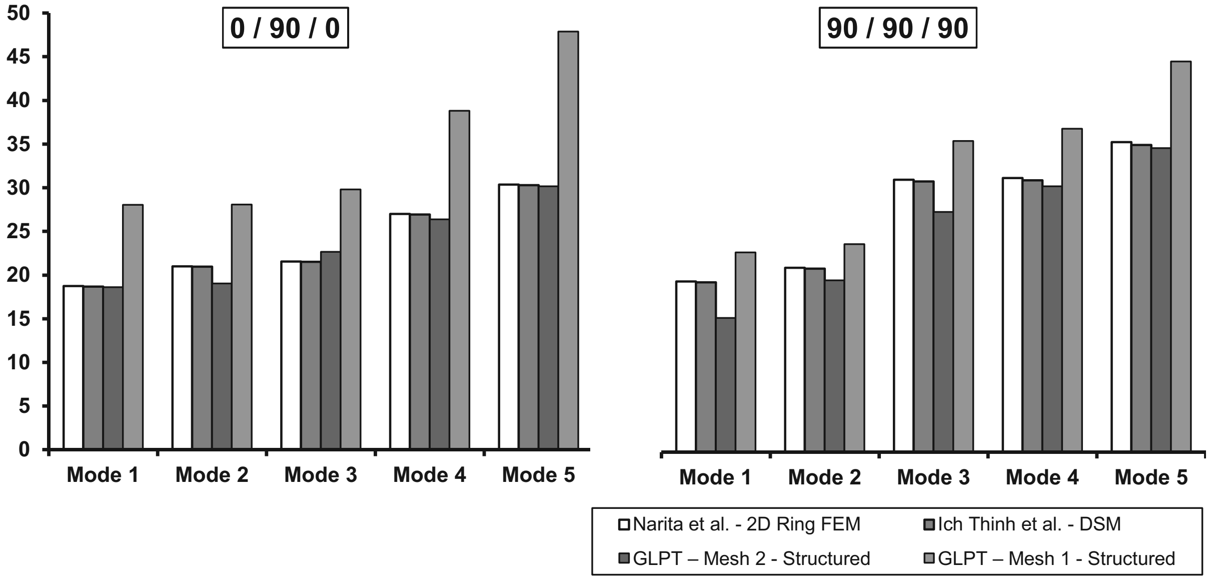
The results presented in Table 2 confirm that the proposed model is capable to accurately predict the fundamental frequencies of intact and delaminated composite plates. The reduction of the fundamental frequency caused by the presence of the delaminated zone is more pronounced for higher modes, for both examined cases.

Example 5.1.3. The third benchmark example is concerned with the cylindrical cross-ply laminated composite shells clamped at both ends. The length of the analyzed shells is L=12m, and the shell radius is R=3m. The shells are composed from three orthotropic layers, each of thickness h_k=0.02m, so the total shell thickness is h=0.06m (length-to-radius ratio L/R=4 and thickness-to-radius ratio h/R=0.02, which is related to thin and moderately long shells). The material parameters (Graphite-Epoxy) for all layers are assumed as [27]: E_1 = 138 GPa, $E_2 = E_3$ = 8.96 GPa, $G_{12} = G_{13}$ = 7.1 GPa, G_{23} = 3.45 GPa, υ_{12} = 0.30, ρ = 1645 kg/m^3. Layered triangular 3-node elements are used. The shell is discretized using the structured mesh of two different densities (Mesh 1 – 800 elements and Mesh 2 - 3200 elements, see Figure 3). The boundary conditions are prescribed along clamped edges by constraining all generalized displacements in edge nodes. The values of non-dimenzionalized frequency parameters $\Omega = \omega \cdot 100R(\rho/E_2)^{1/2}$ obtained from analyses are compared with the results by Narita et al. [46] using 2D ring FE model and Ich Thinh et al. [27] using continuous element constructed from the dynamic stiffness matrix. The results for fundamental frequencies are graphically interpreted in Figure 4, for different lamination schemes.

This example clearly shows that the mesh refinement leads to the lower values of frequency parameter (convergence to the exact solution). For the **0/90/0** lamination scheme, excellent agreement is obtained for all modes. For the **90/90/90** scheme, the slightly lower frequency parameters are obtained for all modes, because of the influence of the transverse shear deformation and the idealizations regarding the shell geometry. By taking these remarks into account, the presented model is fully capable to accurately predict the fundamental frequencies of intact laminated composite shells.



Slika 3. Strukturirane mreže trougaonih konačnih elemenata
Figure 3. Structured meshes of triangular finite elements



Slika 4. Poređenje bezdimenzionalnih frekvencija $\Omega = \omega \cdot 100R(\rho/E_2)^{1/2}$ cross-ply cilindričnih ljuski uklještenih na oba kraja, dobijenih primenom različitih numeričkih modela
Figure 4. Comparison of frequency parameters $\Omega = \omega \cdot 100R(\rho/E_2)^{1/2}$ of cross-ply cylindrical shells clamped at both ends, obtained using different numerical models

5.2 Geometrijski nelinearna dinamička analiza

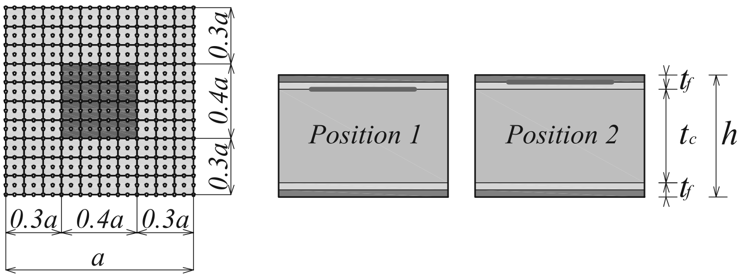
Primer 5.2.1. Poslednji primer ilustruje sposobnost predloženog modela za određivanje nezavisnog kretanja susednih slojeva u zoni delaminacija u sendvič-panelu. Numerički su ispitani linearni i geometrijski nelinearni dinamički odgovori sendvič-ploče s delaminacijom usled naglog eksponencijalnog opterećenja zbog eksplozije. Analizirana je petoslojna (0/90/jezgro/0/90) antisimetrična slobodno oslonjena (SS) kvadratna sendvič-ploča [6, 30]. Panel je napravljen od cross-ply obloge debljine t_f i mekog jezgra debljine t_c, gde je t_c/t_f = 10. Dužina strane ploče je a = 250 *mm*, a njena debljina je h = 2.50 *mm* (a/h = 100). Obloga panela napravljena je od grafit-epoksida T300/934 sa sledećim mehaničkim karakteristikama: $E_{1,f}$ = 131 *GPa*, $E_{2,f} = E_{3,f}$ = 10.34 *GPa*,

5.2 Geometrically nonlinear transient analysis

Example 5.2.1. The final benchmark example illustrates the capability of the proposed model to represent the independent motions of adjacent delaminated interfaces in a sandwich panel. The linear and geometrically nonlinear transient responses of a delaminated sandwich plate under exponential blast pulse loading are investigated numerically. A five layer (0/90/core/0/90) anti-symmetric simply supported (SS) square sandwich plate [6, 30] is analyzed. The plate is composed from cross-ply face sheets, with thickness t_f and a soft core with thickness t_c, where t_c/t_f = 10. The side length of the plate is a = 250 *mm* and its height is h = 2.50 *mm* (a/h = 100). The face sheets are made of Graphite-Epoxy T300/934 with the following mechanical characteristics: $E_{1,f}$ = 131 *GPa*, $E_{2,f} = E_{3,f}$ = 10.34 *GPa*,

$G_{12,f}$ = $G_{23,f}$ = 6.895 *GPa*, $G_{13,f}$ = 6.205 *GPa*, $\upsilon_{12,f}$ = $\upsilon_{13,f}$ = 0.22, $\upsilon_{23,f}$ = 0.49, ρ_f = 1627 kg/m^3. Izotropno meko jezgro napravljeno je od materijala sledećih karakteristika: E_c = 6.89 *MPa*, G_c = 6.895 *MPa*, υ_c = 0, ρ_c = 1550 kg/m^3. Ploča je diskretizovana pomoću mreže od 10×10 slojevitih konačnih elemenata sa devet čvorova s redukovanom integracijom. Jednako podeljeno poprečno opterećenje q_0 = 1.0 kN/m^2 zadato je u vidu eksponencijalnog pulsa $q(t) = q_0 \times e^{-\alpha t}$, trajanja *T = 24 ms,* gde je α = 150 s^{-1} usvojeno kao fiktivni faktor prigušenja. Vremenski inkrement je Δt = *0.8 ms*. Bezdimenzionalni ugib sračunava se kao $w_0 = w \cdot E_{1,f} \cdot h^3/q_0/a^4$. Dinamički odgovor sračunat je za centralno pozicioniranu delaminaciju konstantne površine (videti Sliku 5) na pozicijama 1–2 i prikazan je na Slici 6.

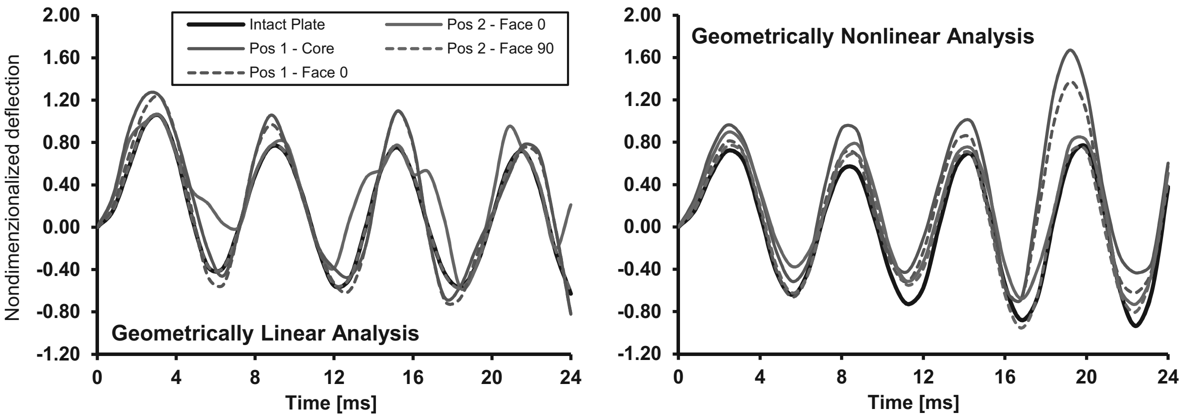
$G_{12,f}$ = $G_{23,f}$ = 6.895 *GPa*, $G_{13,f}$ = 6.205 *GPa*, $\upsilon_{12,f}$ = $\upsilon_{13,f}$ = 0.22, $\upsilon_{23,f}$ = 0.49, ρ_f = 1627 kg/m^3. For the isotropic soft-core the following material parameters are adopted: E_c = 6.89 *MPa*, G_c = 6.895 *MPa*, υ_c = 0, ρ_c = 1550 kg/m^3. The plate is discretized using a 10×10 mesh of 9-node layered finite elements with reduced integration. Uniformly distributed transverse loading q_0 = 1.0 kN/m^2 is prescribed as an exponential pulse $q(t) = q_0 \times e^{-\alpha t}$, with the duration of *T = 24 ms,* using α = 150 s^{-1} as a fictitious damping factor. The time increment is Δt = *0.8 ms*. The normalized centre transverse deflection is calculated as $w_0 = w \cdot E_{1,f} \cdot h^3/q_0/a^4$. The transient response is obtained for centrally located constant-area-delamination (see Figure 5) in positions 1-2 and plotted in Figure 6.



Slika 5. Sendvič-ploča s mekim jezgrom s delaminacijom na različitim pozicijama
Figure 5. Soft-core sandwich plate with different positions of an embedded delamination

Na Slici 6 ilustrovan je uticaj pozicije ranije ubačene delaminacije na rezultate dinamičkog proračuna sendvič-ploče sa oštećenjem. Sendvič-ploča veoma je osetljiva na oštećenje veze između obloge i jezgra, kada je opterećena eksponencijalnim opterećenjem usled eksplozije (plave linije na Slici 6). Ukoliko delaminacija postoji u okviru obloge panela, globalne amplitude pomeranja približno su iste kao i u slučaju neoštećene ploče, jer neoštećen deo ploče ima približno istu krutost na savijanja kao i neoštećena ploča (crvene linije na Slici 6). Razdvojeni segment osciluje lokalno svojom sopstvenom frekvencijom, izazivajući velike otvore prsline. U slučaju geometrijski nelinearne analize, dodatna krutost na savijanje dovodi do redukcije otvaranja prsline u poređenju s linearnom analizom.

Figure 6 illustrates the influence of the position of the previously prescribed delaminated zone on the results of transient analysis of the delaminated sandwich plate. The sandwich plate is highly vulnerable to face-core debonding when subjected to exponential blast pulse loading (blue lines in Figure 6). If the delamination occurs within the face sheets the global amplitudes are nearly the same as for the intact plate because the undamaged part has more or less the same bending stiffness as the intact plate (red lines in Figure 6). The delaminated segment oscillates locally with its local frequency, causing large crack opening displacements. In the geometrically nonlinear case the added bending stiffness leads to the reduction of the crack opening displacements as compared to the linear case.



Slika 6. Promena ugiba u centru dva susedna sloja sendvič-ploče kroz vreme, za različite položaje zone oštećenja
Figure 6. Temporal evolution of the central transverse deflection of two adjacent delaminated interfaces of a sandwich plate considering different positions of the delaminated area

6 ZAKLJUČCI

Na osnovu opšte laminatne teorije ploča izvedeni su slojeviti konačni elementi koji su u stanju da uvedu nezavisno kretanje razdvojenih slojeva. Međusobno prodiranje sloja u sloj sprečeno je uvođenjem kontaktnih uslova između pojedinih slojeva. Geometrijska nelinearnost je uzeta u obzir na osnovu Von Kármán-ovih pretpostavki. Predloženi numerički model primenjen je u numeričkoj analizi slobodnih vibracija i dinamičkog odgovora laminatnih kompozitnih i sendvič-ploča i ljuski s delaminacijama. Mnogim primerima ilustrovano je na koji način delaminacija utiče na fundamentalne dinamičke osobine laminatnih konstrukcija. U narednim radovima predloženi model će biti proširen kako bi se uzela u obzir propagacija delaminacije, pomoću proračuna brzine oslobađanja energije. Iz numeričke analize može se zaključiti:

1. Predloženi model blago (2–4%) precenjuje vrednosti sopstvenih frekvencija kružnih laminatnih kompozitnih ploča, za sve razmatrane šeme laminacije, zbog primene četvorougaonih konačnih elemenata za opisivanje kružne geometrije.

2. Predloženi model je u stanju da precizno predvidi sopstvene frekvencije neoštećenih ili oštećenih kompozitnih ploča. Uvođenjem smičuće deformacije u proračun smanjuju se sopstvene frekvencije neoštećenih i oštećenih kompozitnih ploča. Smanjenje frekvencije zbog prisustva delaminacija izraženije je u višim tonovima, za razmatrane uslove oslanjanja (CC i SS). Sopstvene frekvencije očigledno se povećavaju s porastom stepena ortotropije.

3. Progušćenjem mreže dobijaju se niže vrednosti sopstvenih frekvencija laminatnih kompozitnih cilindričnih ljuski (konvergencija ka tačnom rešenju). Za **0/90/0** šemu laminacije dobijeno je odlično poklapanje u svim tonovima, dok je za **90/90/90** šemu laminacije u svim tonovima dobijena nešto niža sopstvena frekvencija, zbog uticaja deformacije smicanja, kao i zbog idealizacija u pogledu geometrije ljuske.

4. Predloženi model je u stanju da precizno predvidi relativna pomeranja susednih slojeva u oštećenoj zoni. Na dinamički odgovor sendvič-ploča više utiče delaminacija koja se nalazi između mekog jezgra i obloge panela, u poređenju s delaminacijom koja se nalazi između pojedinih slojeva obloge panela. Ovo potvrđuje činjenicu da je čvrsta veza jezgra za oblogu panela od presudnog značaja u projektovanju sendvič-panela.

5. Ukoliko dođe do delaminacija u okviru obloge panela, u geometrijski linearnoj analizi odvojeni segment osciluje lokalno visokom frekvencijom i izaziva kompleksne kontaktne mehanizme između odvojenog sloja i neoštećenog ostatka ploče. U geometrijski nelinearnoj analizi ovaj složeni mehanizam ne postoji zbog prisustva dodatne krutosti na savijanje.

6 CONCLUSIONS

Layered finite plate elements, capable of incorporating the independent motion of delaminated interfaces between layers, have been derived based upon the Generalized Laminated Plate Theory. Interlaminar penetration between delaminated layers was prevented by considering contact conditions between the individual layers. Geometrical nonlinearity is accounted for based upon the Von Kármán assumptions. The proposed numerical model has been applied to the numerical analysis of the free vibrations and the transient response of delaminated composite and sandwich plates and shells. Through the variety of examples it is illustrated how the embedded delamination affect the fundamental dynamic properties of laminated structures. Future work includes the extension of the proposed model to account for propagation of the delamination using Energy Release Rate calculations. From the numerical analyses, the following conclusions are drawn:

1. The proposed model slightly (2-4%) overpredicts the natural frequencies of circular laminated composite plates, for all considered stacking sequences, because of the application of quadrilateral finite elements to describe the circular geometry.

2. The proposed model is capable to accurately predict the fundamental frequencies of intact and delaminated composite plates. The incorporation of the transverse shear deformation reduces the fundamental frequency both for the intact and delaminated composite plates. The reduction caused by the presence of the embedded delamination is more pronounced for higher modes, for examined types of boundary conditions (CC and SS). Natural frequencies obviously increase with the level of orthotropy.

3. The mesh refinement leads to the lower values of natural frequency of laminated composite cylindrical shells. For the **0/90/0** lamination scheme, excellent agreement is obtained for all modes, while for the **90/90/90** scheme, the slightly lower frequency parameters are obtained for all modes, because of the influence of the transverse shear deformation and the idealizations regarding the shell geometry.

4. The proposed model is capable to accurately predict the relative displacements of adjacent laminas in the damaged area. The transient response of sandwich plates is affected more if the delamination is positioned between the soft-core and the rigid face sheets as compared to the delamination within laminas of the face sheet, which confirms that the strong bonding of the soft core to the face sheet is the most critical aspect in the design of sandwich plates.

5. If delamination occurs within the rigid face sheet, when geometrically linear analysis is performed, the delaminated segment oscillates locally with a high frequency, causing complex contact closure and delamination mechanisms between the sheet layer and the intact rest of the plate. In the geometrically nonlinear analysis this complex mechanism is absent due to the added bending stiffness.

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REZIME

PRIMENA SLOJEVITIH KONAČNIH ELEMENATA U NUMERIČKOJ ANALIZI LAMINATNIH KOMPOZITNIH I SENDVIČ-PLOČA I LJUSKI S DELAMINACIJAMA

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Laminatni kompoziti su moderni materijali koji se široko primenjuju u mašinstvu i građevinarstvu. U ovom radu prikazani su odgovarajući moderni pristupi u numeričkoj analizi laminatnih kompozitnih i sendvič-ploča i ljuski s delaminacijom u pojedinim delovima konstrukcije. Za određivanje numeričkog rešenja različitih problema primenjeni su slojeviti konačni elementi bazirani na Reddy-evoj opštoj laminatnoj teoriji ploča. Nakon verifikacije postojećeg modela za konstrukcije bez oštećenja (primenom postojećih podataka iz literature), putem različitih numeričkih primera, analizirani su efekti veličine i položaja delaminacija na odgovor oštećenih laminatnih konstrukcija.

Ključne reči: Laminatni kompozit, Sendvič-ploča, Metod konačnih elemenata, Delaminacija, Kontakt

SUMMARY

APPLICATION OF LAYERED FINITE ELEMENTS IN THE NUMERICAL ANALYSIS OF LAMINATED COMPOSITE AND SANDWICH STRUCTURES WITH DELAMINATIONS

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Miroslav MARJANOVIC

Laminar composites are modern engineering materials widely used in the mechanical and civil engineering. In the paper, some recent advances in a numerical analysis of laminated composite and sandwich plates and shells of different shapes, with existing zones of partial delamination, are presented. The layered finite elements, based on the extended version of the Generalized Laminated Plate Theory of Reddy, are applied for the numerical solution of several structural problems. After the verification of the proposed model for intact structures using the existing data from the literature, the effects of the size and the position of embedded delamination zones on the structural response of laminated structures are investigated numerically by means of a variety of numerical applications.

Key words: Laminar Composite, Sandwich Plate, Finite Element Method, Delamination, Contact