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INFLUENCE OF BOUNDARY CONDITIONS ON NONLINEAR RESPONSE OF LAMINATED COMPOSITE PLATES

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In this paper the influence of different boundary conditions on geometrically nonlinear response of laminated composite plates is analyzed. Mathematical model, based on layer-wise displacement field of Reddy [1], is formulated using the von Karman, small strain large deflection theory. The principle if virtual displacements (PVD) is used to obtain the weak form of the problem. The weak form is discretized utilizing isoparametric finite element approximation. The originally coded MATLAB program is used to investigate the influence of different boundary conditions on geometrically nonlinear response of laminate composite plates. The accuracy of the numerical model is verified by comparison with the available results from the literature.

Keywords: geometrical nonlinearity, boundary conditions, composite plates, finite element

INTRODUCTION

The low mass density and high tensile strength made composite materials lighter and stronger compared to most traditional materials and increased their use in almost all engineering applications. Due to high specific strength design thickness of composite materials is often small [2], the reason why they are usually produced in the form of laminated panels or plates, applied to reduce overall weight of structures. However, during their service life, under specific loading and boundary conditions, these panels may undergo large deflections. Large deflections imply that geometry of structure is continually changing during deformation and geometrical nonlinear analysis should be adopted.

Usual mechanisms of plate geometrically nonlinear response to a transverse load assumes that, when deflection of plate reaches approximately value of plate thickness, coupling between in-plane and out-of plane deformation is activated, giving plate extra stiffness in resisting external loading. However, amount of this extra stiffness to be activated is influenced not only by level of loading, but also by types of boundary conditions. Namely, it is straight forward to assume that restrained edges compare to free edges, would more constrain free deflection of plate, thus giving response which less deviates from linear one. Even more, influence of boundary conditions on geometrically nonlinear response is much pronounced for laminated composite materials, due to their complex anisotropic material behaviour. Unlike in isotropic materials, where the boundary conditions depends only on the type of mechanical loading (bending, buckling, vibrations etc.), which may require natural or geometric, homogeneous or no homogeneous boundary conditions, nature of boundary conditions in composite laminates depend also on level of analysis (linear, nonlinear), as well as on lamination scheme. Depending on lamination scheme different bending-stretching coupling for antisymmetric cross-ply and antisymmetric angle-ply laminates is observed, demanding appropriate simply-supported boundary conditions. Regarding level of analysis, it is noticed that quarter plate symmetry boundary conditions for antisymmetric angle-ply laminates hold for linear analysis, but does not hold for geometrically nonlinear analysis [6].

In order to mathematically describe complex anisotropic nature of composite laminates and find most computationally efficient solution, different approaches are reported in literature. Most of them are restricted to simply supported boundary conditions, specific lamination schemes and

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linear mechanical problem, which enable use of analytical methods in finding appropriate solution [3]. However, when solution of nonlinear mathematical model for different lamination schemes and different boundary conditions is needed, approximate methods should be used. Indeed, literature lacks relevant studies for three dimensional analyses involving boundary conditions which are different from simply supported ones along with multilayered architecture [8]. More ever it is worth to mentioning that a simply supported boundary condition is afar from an easy realization in laboratory. The real model often needs an identification of true boundary conditions which are usually in the middle of the other classical boundary conditions, such as simply supported, clamped, free or their combination [8].

The aim of this paper is to present the influence of different boundary conditions on geometrically nonlinear response of laminated composite plates. Mathematical model, based on layer-wise displacement field of Reddy [1], is formulated using von Karman, small strain large deflection theory, to include geometrically nonlinearity. Principle if virtual displacements (PVD) is used to obtain the weak form of the problem. The weak form or nonlinear integral equilibrium equations are discretized using isoparametric finite element approximation. The nonlinear algebric equations are then solved using direct iteration procedure. The originally coded MATLAB program is used to investigate the influence of different boundary condition on geometrically nonlinear response of cross ply and angle ply laminates [4]. The accuracy of the numerical model is verified by comparison with the available results from the literature.

THEORETICAL FORMULATION

Displacement field

In the LW theory of Reddy [1] or Generalized Layerwise Plate Theory (GLPT), in-plane displacements components (*U,V*) are interpolated through the thickness using 1D linear Lagrangian interpolation function $\Phi^I(z)$, while transverse displacement component w is assumed to be constant through the plate thickness.

$$u_1(x,y,z) = u(x,y) + \sum_{I=1}^{N+1} U^I(x,y)\cdot\Phi^I(z)$$
$$u_2(x,y,z) = v(x,y) + \sum_{I=1}^{N+1} V^I(x,y)\cdot\Phi^I(z),$$
$$u_3(x,y,z) = w(x,y) \tag{1}$$

STRAIN-DISPLACEMENT RELATIONS

The Green Lagrange strain tensor associated with the displacement field Eq.(1) is computed using von Karman strain-displacement relation to include geometric nonlinearities as follows [7]:

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \sum_{I=1}^{N+1}\frac{\partial U^I}{\partial x}\Phi^I + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \sum_{I=1}^{N+1}\frac{\partial V^I}{\partial y}\Phi^I + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N+1}\left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x}\right)\Phi^I + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \sum_{I=1}^{N+1} U^I\frac{d\Phi^I}{dz} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \sum_{I=1}^{N+1} V^I\frac{d\Phi^I}{dz} + \frac{\partial w}{\partial y} \tag{2}$$

Constitutive equations

For Hook's elastic material, the stress-strain relations for k-th orthotropic lamina have the following form:

$$\{\boldsymbol{\sigma}\}^{(k)} = [\mathbf{Q}]^{(k)}\cdot\{\boldsymbol{\varepsilon}\}^{(k)} \tag{3}$$

Where $\boldsymbol{\sigma}^{(k)} = \{\sigma_{xx}\ \sigma_{yy}\ \tau_{xy}\ \tau_{xz}\ \tau_{yz}\}^{(k)T}$ and are stress and strain component srespectively,and $\boldsymbol{\varepsilon}^{(k)} = \{\varepsilon_{xx}\ \varepsilon_{yy}\ \gamma_{xy}\ \gamma_{xz}\ \gamma_{yz}\}^{(k)T}$ are transformed elastic coefficients, of k-th lamina in global coordinates [4].

EQUILIBRIUM EQUATIONS

Equilibrium equations may be obtained from the Principle of Virtual Displacements (PVD), in which sum of external virtual work done on the body and internal virtual work stored in the body should be equal zero [11]:

$$0 = \int_\Omega \left[\left(\{\delta\varepsilon^0\}^T + \{\delta\varepsilon^m\}^T\right)\{N^0\} + \{\delta\varepsilon^I\}^T\{N^I\} + \delta u_0 q_x^0 + \delta v_0 q_y^0 + \delta w_0 q_z^0\right]dxdy$$
$$-\oint_\Gamma \delta u_n N_{nn}\,ds - \oint_\Gamma \delta u_s N_{ns}\,ds - \oint_\Gamma \delta w_0 (Q_n + P_n)ds - \oint_\Gamma \delta U_n^I N_{nn}^I\,ds - \oint_\Gamma \delta U_s^I N_{ns}^I ds \tag{4}$$

where $\{q_x^0, q_y^0, q_z^0\}$ is distributed load in directions, while internal forces are:

$$\begin{Bmatrix} \{\mathbf{N}^0\} \\ \{\mathbf{N}^I\} \end{Bmatrix} = \begin{bmatrix} [\mathbf{A}] & [\mathbf{B}^I] \\ [\mathbf{B}^I] & \sum_{J=1}^{N} [\mathbf{D}^{JI}] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{\varepsilon}^0\} + \{\boldsymbol{\varepsilon}^m\} \\ \{\boldsymbol{\varepsilon}^I\} \end{Bmatrix} \tag{5}$$

where A, B, B^I, D^{JI} matrices are given in [5].

FINITE ELEMENT MODEL

The GLPT finite element consists of middle surface plane and N+1 planes through the plate thickness Figure 1. The element requires only the C^0 continuity of major unknowns, thus in each node only displacement components are adopted, that are (*u,v,w)*) in the middle surface element nodes and (U^I, V^I) in the I-th plane element nodes. The generalized displacements over element Ω^e can be expressed as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} u_j \Psi_j \\ \sum_{j=1}^{m} v_j \Psi_j \\ \sum_{j=1}^{m} w_j \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\boldsymbol{\Psi}_j]^e \{\mathbf{d}_j\}^e \quad \begin{Bmatrix} U^I \\ V^I \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} U_j^I \Psi_j \\ \sum_{j=1}^{m} V_j^I \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\overline{\boldsymbol{\Psi}}_j]^e \{\mathbf{d}_j^I\}^e \tag{6}$$

where $\{\mathbf{d}_j\}^e = \{u_j^e \quad v_j^e \quad w_j^e\}^T$ are displacement vectors, in the middle plane and I-th plane, respectively, are interpolation functions, while $[\boldsymbol{\Psi}_j]^e, [\overline{\boldsymbol{\Psi}}_j]^e$ are interpolation function matrix for the j-th node of the element Ω^e, given in [4].

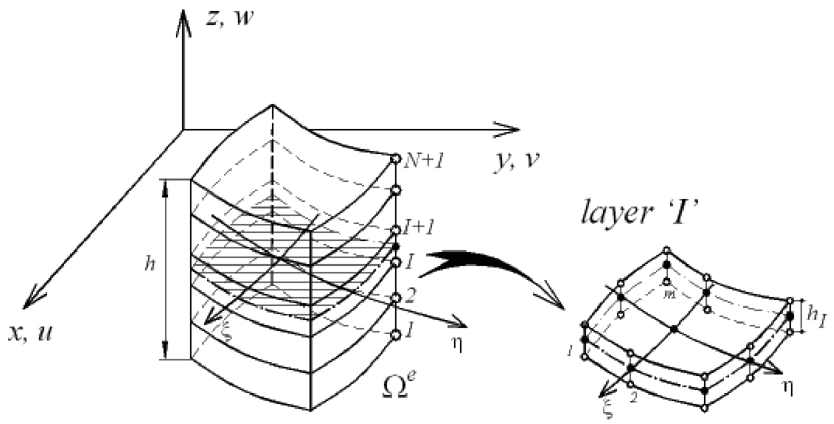

Figure 1. Plate finite element with n layers and m nodes

Substituting element displacement field Eq.(6) in to weak form Eq.(4), the nonlinear laminated finite element is obtained:

$$[\mathbf{K}_{NL}]^e \cdot \{\mathbf{d}\}^e = \{\mathbf{f}\}^e$$

where secant stiffness matrix is: $[\mathbf{K}_{NL}]^e = \begin{bmatrix} [\mathbf{K}^{11}]^e & [\mathbf{K}^{12}]^e \\ [\mathbf{K}^{12}]^e & [\mathbf{K}^{22}]^e \end{bmatrix}$ (7)

$$[\mathbf{K}^{11}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} [\mathbf{H}_i^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_j^e] + \underline{[\mathbf{H}_i^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_{j\,NL}^e]} + \underline{2[\mathbf{H}_{i\,NL}^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_j^e]} + \underline{2[\mathbf{H}_{i\,NL}^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_{j\,NL}^e]}\, d\Omega^e$$

$$[\mathbf{K}^{12}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} \left([\mathbf{H}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\overline{\mathbf{H}}_j^e] + \underline{2[\mathbf{H}_{i\,NL}^e]^T \cdot [\mathbf{B}^I] \cdot [\overline{\mathbf{H}}_j^e]}\right) d\Omega^e$$

$$[\mathbf{K}^{21}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} \left([\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\mathbf{H}_j^e] + \underline{[\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\mathbf{H}_{j\,NL}^e]}\right) d\Omega^e$$

$$[\mathbf{K}^{22}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e} [\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{D}^{IJ}] \cdot [\overline{\mathbf{H}}_j^e]\, d\Omega^e \qquad (8)_{1,2,3,4}$$

and external force vectors $\{\mathbf{f}\}^e = \left\{ \begin{matrix} \{\mathbf{f}^0\}^e \\ \{\mathbf{f}^I\}^e \end{matrix} \right\}$ are:

$$\{\mathbf{f}^0\}^e = \sum_{i=1}^{m} \left[\int_{\Omega^e} [\mathbf{\Psi}_i^e]^T \begin{Bmatrix} q_x^0 \\ q_y^0 \\ q_z^0 \end{Bmatrix} d\Omega^e + \oint_{\Gamma^e} [\mathbf{\Psi}_i^e]^T \begin{Bmatrix} N_{nn} \\ N_{ns} \\ Q_n + P_n \end{Bmatrix} d\Gamma^e \right]$$

$$\{\mathbf{f}^I\}^e = \sum_{i=1}^{m} \left[\int_{\Omega^e} [\mathbf{\Psi}_i^e]^T \begin{Bmatrix} q_x^I \\ q_y^I \end{Bmatrix} d\Omega^e + \oint_{\Gamma^e} [\mathbf{\Psi}_i^e]^T \begin{Bmatrix} N_{nn}^I \\ N_{ns}^I \end{Bmatrix} d\Gamma^e \right] \tag{9}$$

EXAMPLE

A nonlinear bending of square cross ply 0/90 plate, with *a=b=1* and *h=0.1*, with three different boundary conditions (SS, HH and CC, Eqs. 13, 14, 15), made of material [9]:

$$E_1/E_2 = 40, G_{12}/E_2 = 0.6, G_{13}/E_2 = 0.6, G_{23}/E_2 = 0.5, \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{10}$$

subjected to uniform transverse pressure $\bar{q} = q(x,y) \cdot \left(\frac{a}{h}\right)^4 \cdot \frac{1}{E_2}$ are analyzed. The incre-mental load vector is:

$$\{\Delta\bar{q}\} = \{-100, -80, -60, -40, -20, 0, 20, 40, 60, 80, 100\} \tag{11}$$

The displacements and stresses are shown on Figures 2, 3 and are given in following nondimensional form:

$$\bar{w}_{NL} = w \times \frac{1}{h}, \quad (\bar{\sigma}_{xx}) = (\sigma_{xx}) \times \left(\frac{a}{h}\right)^2 \cdot \frac{1}{E_2} \tag{12}$$

Following boundary conditions are analyzed:
Simply supported (SS):

$$\text{SS:} \quad \begin{cases} x = 0, a: & v_0 = w_0 = V^I = N_{xx} = N_{xx}^I = 0 \\ y = 0, b: & u_0 = w_0 = U^I = N_{yy} = N_{yy}^I = 0 \end{cases} \quad I = 1, \ldots N+1 \tag{13}$$

Simply supported-hinged (HH):

$$\text{HH:} \quad \begin{cases} x = 0, a: & u_0 = v_0 = w_0 = V^I = N_{xx}^I = 0 \\ y = 0, b: & u_0 = v_0 = w_0 = U^I = N_{yy}^I = 0 \end{cases} \quad I = 1, \ldots N+1 \tag{14}$$

Clamped (CC):

$$\textbf{CC:} \quad \begin{cases} x = 0, a: & u_0 = v_0 = w_0 = U^I = V^I = 0 \\ y = 0, b: & u_0 = v_0 = w_0 = U^I = V^I = 0 \end{cases} \quad I = 1, \ldots N+1 \tag{15}$$

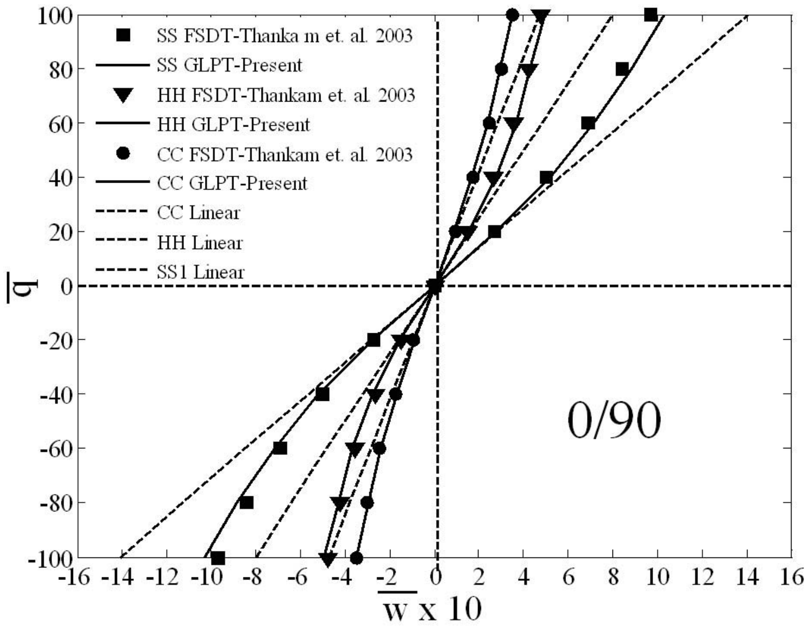

Figure 2. Nonlinear bending of square cross ply 0/90 plate with different boundary conditions and a/h=10; central displacement versus load parameter

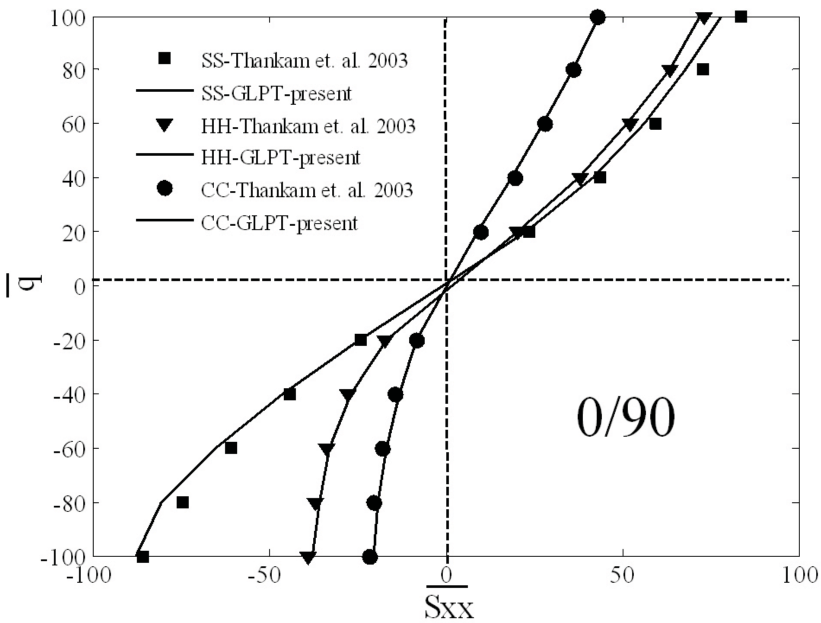

Figure 3. Nonlinear bending of square cross ply 0/90 plate with different boundary conditions and a/h=10; normal stresses versus load parameter

CONCLUSION

In this paper laminated layerwise finite element model for geometrically nonlinear small strain large deflection analysis of laminated composite plates is derived using the PVD. The PVD is utilized to formulate isoparametric finite element model. Finite element solution is incorporated into a original MATLAB computer program. The accuracy of numerical model is verified calculating nonlinear response of plates with different boundary conditions and different load direction (unloading/loading). The analysis has shown that the discrepancy of nonlinear from linear response is greater for flexible plates, such as plates with SS compared to hinged (HH) and clamped (CC) boundary conditions. It is verified that the change of load direction (unloading/loading) has no influence on displacement field, while stress field is load direction dependent. Finally, a close agreement with results from literature is achived.

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