ψ-type stability of reaction-diffusion neural networks with time-varying discrete delays and bounded distributed delays

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Abstract

In this paper, the ψ-type stability and robust ψ-type stability for reaction-diffusion neural networks (RDNNs) with Dirichlet boundary conditions, time-varying discrete delays and bounded distributed delays are investigated, respectively. Firstly, we analyze the ψ-type stability and robust ψ-type stability of RDNNs with time-varying discrete delays by means of ψ-type functions combined with some inequality techniques, and put forward several ψ-type stability criteria for the considered networks. Additionally, the models of RDNNs with bounded distributed delays are established and some sufficient conditions to guarantee the ψ-type stability and robust ψ-type stability are given. Lastly, two examples are provided to confirm the effectiveness of the derived results.

Key words: Bounded distributed delays, ψ-type stability, Time-varying

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discrete delays, Reaction-diffusion neural networks

1. Introduction

In the past several decades, the study of neural networks (NNs) has been receiving extensive attentions because of their potential applications in various disciplines, such as associative memory, pattern recognition, parameter estimation, optimization [1–8]. As a matter of fact, these applications are mainly dependent upon the dynamical behaviors of NNs. Especially, as one of the important dynamical properties, stability of NNs has been widely studied [9–19]. In [12], several new conditions for the exponential stability of delayed second-order memristive NNs were obtained. The authors considered the stability of discrete-time NN with time-varying delays, and a delay-variation-dependent stability criterion was established in [15]. In [16], a new Lyapunov-Krasovskii functional approach was established for ensuring delay-dependent stability of NNs.

Nevertheless, it is noteworthy that most of results about the stability of NNs now available appear the following natures. On the one side, the NN model is usually limited to a model with precise parameters. However, the model with parametric uncertainties is more suitable due to external disturbance and parameter fluctuation. On the other side, the perturbation and parameters of NNs are highly demanding for the asymptotic stability of Lyapunov, which makes it difficult for designing network performance. In addition, the convergence rate of the system is very hard to estimate in many practical applications, which motivates some scholars to study a new type of stability, i.e., general decay stability, which is also called to be ψ-type stability. Actually, ψ-type stability is an extension of the traditional stability, e.g., exponential stability, log-stability, power-rate stability and μ-stability [20–23]. In [21], the ψ-type stability for recurrent NNs was discussed by exploiting the differential inequality. The ψ-type stability of delayed chaotic NNs with discontinuous activations was considered in [23].

As well as we know, reaction-diffusion phenomenon is unavoidable in NNs once the electrons transport in inhomogeneous magnetic field. Hence, taking the reaction-diffusion terms into consideration in NN is necessary, and some researchers have devoted themselves to studying the stability of reaction-diffusion neural networks (RDNNs) [24–31]. A sufficient condition for the stability of interval RDNNs was obtained in [24]. In [25], the stability of RDNN was investigated by making use of the Lyapunov functional method. Moreover, time-varying delays are inevitable during the implementation of artificial NNs due to the finite switching speed of amplifiers and the inherent communication time between neurons, which often result in undesired dynamics like oscillation, instability, and divergence. Therefore, it is important and necessary to take the time-varying delays into account and assess the effect of delays during studying the stability of NNs [32–34]. In [32], the delay-dependent stability problem of NNs with time-varying discrete delays was addressed. The exponential stability of recurrent NNs with time-varying discrete delays was considered in [33]. In addition, it usually has a spatial nature because of the presence of a very large number of parallel path ways with a variety of axon sizes and lengths when implementing a neural network by VLSI in reality. However, the distribution of propagation is not instantaneous, which cannot be modeled by discrete time delays. Therefore, it is requisite to introduce distributed delays in NNs' modeling [20, 35–40]. The globally asymptotic stability of stochastic NNs with distributed delays was investigated in [35]. In [20], the authors considered the ψ-type stability for Cohen-Grossberg NNs with distributed and discrete delays. However, the ψ-type stability of RDNN with distributed delays and discrete delays has never been studied.

Based on the discussion aforementioned, we first construct the models of RDNN with time-varying discrete delays and bounded distributed delays respectively in this paper. Then, several ψ-type stability and robust ψ-type stability criteria for these considered networks are established respectively.

The rest of this paper is organized as follows. In Section 2, several important definitions and lemmas are provided. The network models of RDNN with time-varying discrete delays are firstly presented in Section 3, and then the ψ-type stability and robust ψ-type stability for this kind of network are investigated. Section 4 is devoted to analyzing ψ-type stability and robust ψ-type stability for RDNNs with bounded distributed delays. Several examples with simulation results are given in Section 5 to demonstrate the validity of the obtained theoretical results. Finally, we conclude this paper in Section 6.

2. Preliminaries

Definition 2.1. *(see [41]) If the function $\psi(t)$: $\mathbb{R}_+ \rightarrow (0, +\infty)$ satisfies the following conditions:*

1) $\psi(t)$ is nondecreasing and differentiable;

2) $\psi(0) = 1$ and $\psi(+\infty) = +\infty$;

3) $\overline{\psi}(t) := \frac{\dot{\psi}(t)}{\psi(t)}$ *is decreasing;*

4) $\forall p, q \geqslant 0,\ \psi(p+q) \leqslant \psi(p)\psi(q)$*;*

then it is called to be ψ*-type function.*

Definition 2.2. *For* $\mathbb{R}^n \ni \psi(t) = (\psi_1(t), \psi_2(t), \cdots, \psi_n(t))^T$*, define*

$$\|\psi(t)\|_{\{\eta,\infty\}} = \min_{\iota=1,2,\cdots,n}\{|\eta_\iota^{-1}\psi_\iota(t)|\};$$

for $y(\chi, t) = (y_1(\chi, t), y_2(\chi, t), \cdots, y_n(\chi, t))^T$*, define*

$$\|y(\cdot, t)\|^{\Omega}_{\{\eta,\infty\}} = \min_{\iota=1,2,\cdots,n}\{\eta_\iota^{-1}\int_\Omega y_\iota^2(\chi, t)d\chi\},$$

in which $(\chi, t) \in \Omega \times \mathbb{R}$*,* $\Omega = \{\chi = (\chi_1, \chi_2, \cdots, \chi_q)^T|\ |\chi_k| < \beta_k,\ k = 1, 2, \cdots, q\} \subset \mathbb{R}^q$*,* $\eta = (\eta_1, \eta_2, \cdots, \eta_n)^T$ *and* $\eta_\iota > 0$*.*

Lemma 2.1. *(see [42]) Let* Ω *be a cube* $|\chi_k| < \beta_k(k = 1, 2, \cdots, q)$ *and real-valued function* $Z(\chi) \in C^1(\Omega)$ *satisfies* $Z(\chi)|_{\partial\Omega} = 0$*. Then*

$$\int_\Omega Z^2(\chi)d\chi \leqslant \beta_k^2 \int_\Omega \left(\frac{\partial Z(\chi)}{\partial \chi_k}\right)^2 d\chi,$$

where $\chi = (\chi_1, \chi_2, \cdots, \chi_q)^T$*.*

Lemma 2.2. *Given function* $h(\chi) : [\omega_1, \omega_2] \to \mathbb{R}$ *provide the integral are well defined, then*

$$\left(\int_{\omega_1}^{\omega_2} |h(\chi)|d\chi\right)^2 \leqslant (\omega_2 - \omega_1)\int_{\omega_1}^{\omega_2} h^2(\chi)d\chi.$$

Proof. *From the Hölder inequality integral form (see [43]), we can obtain*

$$\int_{w_1}^{w_2} |h(\chi)g(\chi)|d\chi \leqslant \left(\int_{w_1}^{w_2} |h(\chi)|^p d\chi\right)^{1/p} \left(\int_{w_1}^{w_2} |g(\chi)|^q d\chi\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$, $h\in L^p[w_1,w_2]$, $g\in L^q[w_1,w_2]$ *and* $1<p<\infty$. *Particularly, take* $p=q=2$ *and* $g(\chi)=1$, *then we can deduce*

$$\int_{w_1}^{w_2}|h(\chi)|d\chi \leqslant (w_2-w_1)^{1/2}\left(\int_{w_1}^{w_2}h^2(\chi)d\chi\right)^{1/2}.$$

Equivalently,

$$\left(\int_{w_1}^{w_2}|h(\chi)|d\chi\right)^2 \leqslant (w_2-w_1)\int_{w_1}^{w_2}h^2(\chi)d\chi.$$

The proof is completed.

3. ψ-type stability of RDNN with time-varying discrete delays

3.1. ψ-type stability analysis

The class of considered RDNN with time-varying discrete delays is described by:

$$\begin{aligned}\frac{\partial Y_\iota(\chi,t)}{\partial t} =&\sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial Y_\iota(\chi,t)}{\partial \chi_k}\right)-b_\iota Y_\iota(\chi,t)+\sum_{j=1}^{n}c_{\iota j}f_j(Y_j(\chi,t))+P_\iota(t)\\ &+\sum_{j=1}^{n}d_{\iota j}f_j(Y_j(\chi,t-\tau_{\iota j}(t))), \end{aligned} \tag{1}$$

where $\iota=1,2,\cdots,n$, $\chi=(\chi_1,\chi_2,\cdots,\chi_q)\in\Omega$, $\mathbb{R}\ni a_{\iota k}>0$ symbols the transmission diffusion coefficient along the ιth neuron; $\mathbb{R}\ni Y_\iota(\chi,t)$ is the state of the ιth neuron at time t in space χ; $\mathbb{R}\ni b_\iota>0$ is the rate at which the ιth neuron resets its potential to rest when it disconnects the external inputs in network; $c_{\iota j}$ and $d_{\iota j}$ are the connection strengths of the jth neuron on the ιth neuron; $f_j(\cdot)$ signifies the activation function; the transmission delay $\tau_{\iota j}(t)$ satisfies $0\leqslant\tau_{\iota j}(t)\leqslant\tau$ ($\iota,\ j=1,2,\cdots,n$); $P_\iota(t)$ is the input of ιth neuron at time t.

The boundary condition and initial conditions subject to network (1) are as follows:

$$Y_\iota(\chi,t)=0,\quad (\chi,t)\in\partial\Omega\times[t_0-\tau,+\infty), \tag{2}$$
$$Y_\iota(\chi,t)=\phi_\iota(\chi,t),\quad (\chi,t)\in\Omega\times[t_0-\tau,t_0],$$

where $\phi_\iota(\chi,t)$ $(\iota=1,2,\cdots,n)$ is bounded and continuous on $\Omega\times[t_0-\tau,t_0]$.

Throughout this paper, we assume that the activation function $f_k(\cdot)$ satisfies

$$0\leqslant\frac{f_k(\alpha_1)-f_k(\alpha_2)}{\alpha_1-\alpha_2}\leqslant F_k,$$

for any $\alpha_1,\alpha_2\in\mathbb{R},\ \alpha_1\neq\alpha_2$, where $0\leqslant F_k$, $k=1,2,\cdots,n$.

Suppose that $Y^*(\chi)=(Y_1^*(\chi),Y_2^*(\chi),\cdots,Y_n^*(\chi))^T\in\mathbb{R}^n$ is an equilibrium solution of network (1), then it satisfies

$$\sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{\iota k}\frac{\partial Y_\iota^*(\chi)}{\partial\chi_k}\right)-b_\iota Y_\iota^*(\chi)+\sum_{j=1}^{n}c_{\iota j}f_j(Y_j^*(\chi))+\sum_{j=1}^{n}d_{\iota j}f_j(Y_j^*(\chi))+P_\iota(t)=0.$$

Take $e_\iota(\chi,t)=Y_\iota(\chi,t)-Y_\iota^*(\chi)$, we can get

$$\begin{aligned}\frac{\partial e_\iota(\chi,t)}{\partial t}=&\sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{\iota k}\frac{\partial e_\iota(\chi,t)}{\partial\chi_k}\right)-b_\iota e_\iota(\chi,t)+\sum_{j=1}^{n}c_{\iota j}\left(f_j(Y_j(\chi,t))-f_j(Y_j^*(\chi))\right)\\&+\sum_{j=1}^{n}d_{\iota j}\left(f_j(Y_j(\chi,t-\tau_{\iota j}(t)))-f_j(Y_j^*(\chi))\right),\end{aligned} \tag{3}$$

where $\iota=1,2,\cdots,n$.

Remark 1. As we all know, time delays often inevitable appear in practical applications, such as communication, information conversion and biological systems. Especially, it is usual to expect that time delays exist during the processing and transmission of signals in most circuits. In addition, the existence of time delays may lead to some poor performances, including instability, oscillation, chaos and so on. Hence, it is important to evaluate the effect of delays on stability analysis of NNs, which has become a research hotspot in recent decades [10–17, 20–24, 26–29, 32–40, 42, 44, 45]. Furthermore, formulating the NNs with time-varying discrete delays is essential for the engineering applications because the discretization may not preserve the dynamics of the continuous time counter part even for a small sampling period [34], which motivates the investigation directly for NNs with time-varying discrete delays [15, 20, 32–34, 38]. As we mentioned before, reaction-diffusion phenomenon cannot be avoided in NNs once the electrons transport in inhomogeneous magnetic field. Therefore, we investigate a class of RDNN with time-varying discrete delays in this section.

Definition 3.1. *If there exists a scalar* $\mathbb{R} \ni \lambda > 0$ *such that*

$$\limsup_{t\to\infty} \frac{\ln\|e(\cdot,t)\|^{\Omega}_{\{\eta,\infty\}}}{\ln\|\psi(t)\|_{\{\eta,\infty\}}} \leqslant -\lambda,$$

where $e(\chi,t) = (e_1(\chi,t), e_2(\chi,t), \cdots, e_n(\chi,t))^T$, $\psi(t) = (\psi_1(t), \psi_2(t), \cdots, \psi_n(t))^T$, $\psi_\iota(t)(\iota = 1, 2, \cdots, n)$ *is* ψ*-type function as defined in Definition 2.1, then the network* (1) *is called to be* ψ*-type stable with regard to* $Y^*(\chi)$*.*

Remark 2. In the past several decades, NNs have been extensively applied to various fields, e.g., associative memory, image processing, parameter estimation, signal processing and optimization [1–8]. In fact, most of these applications depend heavily on the dynamic behaviors of NNs. For instance, in order to solve optimization problems by using NNs, it is necessary that each trajectory of the NNs converges to a unique equilibrium point, that is, the NNs are stable. Hence, many researchers have devoted themselves to studying the stability of NNs and obtained numerous results, see [9–19] for instances and the references therein. It is universally known that stability and convergence are prior conditions for theoretical analysis and design. As pointed out in [46], it is extremely interesting subject to estimate the solution's convergence rate of nonlinear systems. However, the convergence time or speed of the system is hard to acquire in many practical cases. Due to this, some new type of convergence rate should be defined, such as convergence with general decay. In recent years, a new type of stability, i.e. μ-stability, is proposed, which combines the concepts of exponential stability, log-stability and power-rate stability of NNs [44, 45]. In 2016, Wang et al. [23] firstly presented the definition of general decay stability based on ψ-type function, which is also said to be ψ-type stability. It extends the concept of μ-stability. Indeed, when NNs possess ψ-type stability, it is helpful to solve the optimization problem and implement content-addressable memories [22]. Since then, a great quantity of literatures of ψ-type stability have been reported [20–23]. Unfortunately, the network models in above-mentioned results about ψ-type stability do not take the diffusion effects into consideration. Therefore, we investigate the ψ-type stability of NNs with reaction-diffusion terms in this paper.

Remark 3. It is obvious that functions $\psi(t) = e^{\mu t}$, $\psi(t) = (1 + t)^{\mu}$ and $\psi(t) = 1 + \mu \ln(1 + t)$ for any $\mu > 0$ satisfy the conditions 1)-4) given in Definition 2.1, thus they are all ψ-type functions. Moreover, ψ-type function offers a basis for the assortment of abstract functions. By introducing ψ-type function, the ψ-type stability of RDNNs is defined in Definition 3.1. It follows from Definition 3.1 that exponential stability and polynomial stability can be regarded as special cases of the ψ-type stability when $\psi(t) = e^{\mu t}$ and $\psi(t) = (1+t)^{\mu}$ for any $\mu > 0$, respectively. Therefore, the ψ-type stability given in Definition 3.1 is a generalization of other stability definitions.

Theorem 3.1. *For $\iota = 1, 2, \cdots, n$ and $\forall t \geqslant t_0 \geqslant 0$, the network* (1) *with respect to $Y^*(\chi)$ is ψ-type stable, if there exists some positive numbers r_ι and functions $\psi_\iota(t)(\iota = 1, 2, \cdots, n)$ such that*

$$\left(\sum_{j=1}^{n}(|c_{\iota j}| + |d_{\iota j}|)F_j - 2\sum_{k=1}^{q}\frac{a_{\iota k}}{\beta_k^2} + r_\iota \overline{\psi}_\iota(t) - 2b_\iota\right)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}$$
$$+\sum_{j=1}^{n}|c_{\iota j}|F_j\left(\frac{\psi_j(t)}{\psi_j(t_0)}\right)^{-r_j} + \sum_{j=1}^{n}|d_{\iota j}|F_j G_{\iota j}(t) < 0,$$

where

$$G_{\iota j}(t) = \begin{cases} 1, & \text{for } t_0 \leqslant t < t_0 + \tau_{\iota j}(t), \\ \left(\dfrac{\psi_j(t - \tau_{\iota j}(t))}{\psi_j(t_0)}\right)^{-r_j}, & \text{for } t \geqslant t_0 + \tau_{\iota j}(t). \end{cases}$$

Proof. Denote

$$V_\iota(t) = \int_\Omega e_\iota^2(\chi, t)d\chi,$$
$$\overline{V}(t_0) = \sum_{\iota=1}^{n}\sup_{t_0-\tau\leqslant\varepsilon\leqslant t_0}\{V_\iota(\varepsilon)\} < +\infty,$$

and

$$H_\iota(t) = \begin{cases} V_\iota(t) - \overline{V}(t_0)\left(\dfrac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}, \ \forall t \geqslant t_0 \geqslant 0, \\ V_\iota(t) - \overline{V}(t_0), \ \forall t_0 - \tau \leqslant t < t_0, \end{cases}$$

where $\iota = 1, 2, \cdots, n$.

Obviously, $H_\iota(t)$ is continuous and $H_\iota(\varepsilon) \leqslant 0$ for $\forall \varepsilon \in [t_0 - \tau, t_0]$. Then, we will prove $H_\iota(t) \leqslant 0$ for $\forall t \geqslant t_0$ and $\iota = 1, 2, \cdots, n$. Otherwise, there exists i and $t_1 (t_1 \geqslant t_0)$ satisfying

$$\begin{cases} H_i(t_1) = 0, \\ D^+ H_i(t)|_{t=t_1} \geqslant 0, \\ H_j(\varepsilon) \leqslant 0, \ \forall \varepsilon \in [t_0 - \tau, t_1], \ j = 1, 2, \cdots, n. \end{cases}$$

Then,

$$\begin{aligned} D^+ H_i(t)|_{t=t_1} =& \dot{V}_i(t)|_{t=t_1} + r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)} \right) \\ =& 2 \int_\Omega e_i(\chi, t) \frac{\partial e_i(\chi, t)}{\partial t} d\chi \bigg|_{t=t_1} + r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)} \right) \\ =& 2 \int_\Omega e_i(\chi, t) \bigg[\sum_{j=1}^n d_{ij} \left(f_j(Y_j(\chi, t - \tau_{ij}(t))) - f_j(Y_j^*(\chi)) \right) - b_i e_i(\chi, t) \\ & + \sum_{k=1}^q \frac{\partial}{\partial \chi_k} \left(a_{ik} \frac{\partial e_i(\chi, t)}{\partial \chi_k} \right) + \sum_{j=1}^n c_{ij} \left(f_j(Y_j(\chi, t)) - f_j(Y_j^*(\chi)) \right) \bigg] d\chi \bigg|_{t=t_1} \\ & + r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)} \right). \end{aligned} \tag{4}$$

According to Dirichlet boundary condition and Green's formula, one can derive

$$\begin{aligned} & \int_\Omega e_i(\chi, t) \sum_{k=1}^q \frac{\partial}{\partial \chi_k} \left(a_{ik} \frac{\partial e_i(\chi, t)}{\partial \chi_k} \right) d\chi \\ = & - \sum_{k=1}^q \int_\Omega a_{ik} \left(\frac{\partial e_i(\chi, t)}{\partial \chi_k} \right)^2 d\chi. \end{aligned}$$

From Lemma 2.1, we can get

$$\sum_{k=1}^q \int_\Omega a_{ik} \left(\frac{\partial e_i(\chi, t)}{\partial \chi_k} \right)^2 d\chi \geqslant \sum_{k=1}^q \frac{a_{ik}}{\beta_k^2} \int_\Omega e_i^2(\chi, t) d\chi. \tag{5}$$

From (4) and (5), we have

$$
\begin{aligned}
D^+H_i(t)|_{t=t_1} \leqslant & -2\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}\int_\Omega e_i^2(\chi,t_1)d\chi + 2\int_\Omega |e_i(\chi,t_1)|\Bigg[\sum_{j=1}^{n}|c_{ij}|F_j|e_j(\chi,t_1)| \\
& + \sum_{j=1}^{n}|d_{ij}|F_j|e_j(\chi,t_1-\tau_{ij}(t_1))|\Bigg]d\chi - 2b_i\int_\Omega e_i^2(\chi,t_1)d\chi \\
& + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
\leqslant & -2(\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}+b_i)\int_\Omega e_i^2(\chi,t_1)d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
& + 2\sum_{j=1}^{n}|d_{ij}|F_j\sqrt{\int_\Omega e_i^2(\chi,t_1)d\chi}\sqrt{\int_\Omega e_j^2(\chi,t_1-\tau_{ij}(t_1))d\chi} \\
& + 2\sum_{j=1}^{n}|c_{ij}|F_j\sqrt{\int_\Omega e_i^2(\chi,t_1)d\chi}\sqrt{\int_\Omega e_j^2(\chi,t_1)d\chi} \\
\leqslant & -2(\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}+b_i)\int_\Omega e_i^2(\chi,t_1)d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
& + \sum_{j=1}^{n}|d_{ij}|F_j\left(\int_\Omega e_i^2(\chi,t_1)d\chi + \int_\Omega e_j^2(\chi,t_1-\tau_{ij}(t_1))d\chi\right) \\
& + \sum_{j=1}^{n}|c_{ij}|F_j\left(\int_\Omega e_i^2(\chi,t_1)d\chi + \int_\Omega e_j^2(\chi,t_1)d\chi\right) \\
= & \left(\sum_{j=1}^{n}(|c_{ij}|+|d_{ij}|)F_j - 2(\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}+b_i)\right)V_i(t_1) + \sum_{j=1}^{n}|c_{ij}|F_jV_j(t_1) \\
& + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) + \sum_{j=1}^{n}|d_{ij}|F_jV_j(t_1-\tau_{ij}(t_1)).
\end{aligned}
$$

By $H_\iota(t)\leqslant 0(\iota=1,2,\cdots,n)$ for any $t\in[t_0-\tau,t_1]$, we can obtain

$$
D^+H_i(t)|_{t=t_1} \leqslant \left(\sum_{j=1}^{n}(|c_{ij}|+|d_{ij}|)F_j - 2\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2} - 2b_i\right)\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}|c_{ij}|F_j\overline{V}(t_0)\left(\frac{\psi_j(t_1)}{\psi_j(t_0)}\right)^{-r_j}+\sum_{j=1}^{n}|d_{ij}|F_j\overline{V}(t_0)G_{ij}(t_1)\\
&+r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\overline{\psi}_i(t_1)\\
=&\overline{V}(t_0)\Bigg[\left(\sum_{j=1}^{n}(|c_{ij}|+|d_{ij}|)F_j-2\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}+r_i\overline{\psi}_i(t_1)-2b_i\right)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\\
&+\sum_{j=1}^{n}|c_{ij}|F_j\left(\frac{\psi_j(t_1)}{\psi_j(t_0)}\right)^{-r_j}+\sum_{j=1}^{n}|d_{ij}|F_jG_{ij}(t_1)\Bigg]\\
<&0,
\end{aligned}
$$

which is unreasonable. Thus

$$
V_\iota(t)\leqslant\overline{V}(t_0)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota},\ \iota=1,2,\cdots,n,\ \forall t\geqslant t_0\geqslant 0.
$$

Moreover, there exist $M(t_0)$ and r such that $V_\iota(t)\leqslant M(t_0)\psi_\iota^{-r}(t)$, where $M(t_0)=\max_{\iota=1,2,\cdots,n}\{\overline{V}(t_0)\psi_\iota^{r_\iota}(t_0)\}$ and $r=\min_{\iota=1,2,\cdots,n}\{r_\iota\}$. Denote $V(t)=(V_1(t),V_2(t),\cdots,V_n(t))^T$ and $\psi(t)=(\psi_1(t),\psi_2(t),\cdots,\psi_n(t))^T$, we have

$$
\|V(t)\|_{\{\xi,\infty\}}=\min_{\iota=1,2,\cdots,n}\{|\xi_\iota^{-1}V_\iota(t)|\}\leqslant M(t_0)\|\psi(t)\|_{\{\xi,\infty\}}^{-r},
$$

where $\xi=(\xi_1,\xi_2,\cdots,\xi_n)^T=(1,1,\cdots,1)^T$. Obviously,

$$
\ln(M^{-1}(t_0)\|V(t)\|_{\{\xi,\infty\}})\leqslant -r\ln(\|\psi(t)\|_{\{\xi,\infty\}}).
$$

According to Definition 2.1, $\ln(\|\psi(t)\|_{\{\xi,\infty\}})>0$ for $t>t_0\geqslant 0$ and $\ln(\|\psi(t)\|_{\{\xi,\infty\}})\to+\infty$ as time $t\to+\infty$. Therefore, one has

$$
\limsup_{t\to+\infty}\frac{\ln(\|V(t)\|_{\{\xi,\infty\}})}{\ln(\|\psi(t)\|_{\{\xi,\infty\}})}\leqslant -r.
$$

Equivalently,

$$
\limsup_{t\to+\infty}\frac{\ln(\|e(\cdot,t)\|_{\{\xi,\infty\}}^{\Omega})}{\ln(\|\psi(t)\|_{\{\xi,\infty\}})}\leqslant -r.
$$

In other words, $e(\chi,t)$ is ψ-type stable. This completes the proof.

3.2. Robust ψ-type stability analysis

As we all know, the limitation of equipment and the existence of external interference in the modeling process of NN may lead to parameter deviations and these deviations are bounded. Therefore, we consider an uncertain reaction-diffusion neural network (URDNN) with time-varying discrete delays in this section, which can be characterized as follows:

$$\begin{aligned}\frac{\partial Y_\iota(\chi,t)}{\partial t} =& \sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial Y_\iota(\chi,t)}{\partial \chi_k}\right) - b_\iota Y_\iota(\chi,t) + \sum_{j=1}^{n} c_{\iota j} f_j(Y_j(\chi,t)) + P_\iota(t)\\ &+ \sum_{j=1}^{n} d_{\iota j} f_j(Y_j(\chi,t-\tau_{\iota j}(t))), \qquad \iota = 1,2,\cdots,n,\end{aligned} \tag{6}$$

where $Y_\iota(\chi,t),\ f_j(\cdot),\ P_\iota(t),\ \tau_{\iota j}(t)$ have the same definitions as in subsection 3.1. The quantities $a_{\iota k},\ b_\iota,\ c_{\iota j},\ d_{\iota j}$ may be intervalized as follows:

$$\begin{cases} A_I :=\{A = (a_{\iota k})_{n\times q} : A^- \leqslant A \leqslant A^+, i.e., 0 < a_{\iota k}^- \leqslant a_{\iota k} \leqslant a_{\iota k}^+,\\ \qquad \iota = 1,2,\cdots,n, k = 1,2,\cdots,q, \forall A \in A_I\},\\ B_I :=\{B = \mathrm{diag}(b_\iota) : B^- \leqslant B \leqslant B^+, i.e., 0 < b_\iota^- \leqslant b_\iota \leqslant b_\iota^+,\\ \qquad \iota = 1,2,\cdots,n, \forall B \in B_I\},\\ C_I :=\{C = (c_{\iota j})_{n\times n} : C^- \leqslant C \leqslant C^+, i.e., c_{\iota j}^- \leqslant c_{\iota j} \leqslant c_{\iota j}^+, \iota,\\ \qquad j = 1,2,\cdots,n, \forall C \in C_I\},\\ D_I :=\{D = (d_{\iota j})_{n\times n} : D^- \leqslant D \leqslant D^+, i.e., d_{\iota j}^- \leqslant d_{\iota j} \leqslant d_{\iota j}^+, \iota,\\ \qquad j = 1,2,\cdots,n, \forall D \in D_I\}. \end{cases} \tag{7}$$

For convenience, we denote

$$c_{\iota j}^* = \max\{|c_{\iota j}^+|, |c_{\iota j}^-|\},\ d_{\iota j}^* = \max\{|d_{\iota j}^+|, |d_{\iota j}^-|\}.$$

For the network (6),

$$Y_\iota(\chi,t)=0,\quad (\chi,t)\in\partial\Omega\times[t_0-\tau,+\infty),$$
$$Y_\iota(\chi,t)=\phi_\iota(\chi,t),\quad (\chi,t)\in\Omega\times[t_0-\tau,t_0],$$

where $\phi_\iota(\chi,t)$ $(\iota=1,2,\cdots,n)$ is bounded and continuous on $\Omega\times[t_0-\tau,t_0]$.

Let $Y^*(\chi)=(Y_1^*(\chi),Y_2^*(\chi),\cdots,Y_n^*(\chi))^T\in\mathbb{R}^n$ be an equilibrium solution of network (6), then

$$\sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{\iota k}\frac{\partial Y_\iota^*(\chi)}{\partial\chi_k}\right)-b_\iota Y_\iota^*(\chi)+\sum_{j=1}^{n}c_{\iota j}f_j(Y_j^*(\chi))+\sum_{j=1}^{n}d_{\iota j}f_j(Y_j^*(\chi))+P_\iota(t)=0,$$

where $a_{\iota j},\ b_\iota,\ c_{\iota j},\ d_{\iota j}$ belong to the parameter ranges defined by (7).

Take $e_\iota(\chi,t)=Y_\iota(\chi,t)-Y_\iota^*(\chi)$, we can obtain

$$\begin{aligned}\frac{\partial e_\iota(\chi,t)}{\partial t}=&\sum_{j=1}^{n}c_{\iota j}\left(f_j(Y_j(\chi,t))-f_j(Y_j^*(\chi))\right)+\sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{\iota k}\frac{\partial e_\iota(\chi,t)}{\partial\chi_k}\right)\\&-b_\iota e_\iota(\chi,t)+\sum_{j=1}^{n}d_{\iota j}\left(f_j(Y_j(\chi,t-\tau_{\iota j}(t)))-f_j(Y_j^*(\chi))\right),\end{aligned}$$

where $\iota=1,2,\cdots,n,\ a_{\iota k},\ b_\iota,\ c_{\iota j},\ d_{\iota j}$ belong to the parameter ranges defined by (7).

Definition 3.2. *If for all $A\in A_I,\ B\in B_I,\ C\in C_I$ and $D\in D_I$, there exists a constant $\lambda>0$ such that*

$$\limsup_{t\to\infty}\frac{\ln\|e(\cdot,t)\|_{\{\eta,\infty\}}^{\Omega}}{\ln\|\psi(t)\|_{\{\eta,\infty\}}}\leqslant-\lambda,$$

where $e(\chi,t)=(e_1(\chi,t),e_2(\chi,t),\cdots,e_n(\chi,t))^T$, $\psi(t)=(\psi_1(t),\psi_2(t),\cdots,\psi_n(t))^T$, $\psi_\iota(t)(\iota=1,2,\cdots,n)$ is a ψ-type function, then the network (6) is called to be robustly ψ-type stable with regard to $Y^*(\chi)$.

Theorem 3.2. *The network (6) with respect to $Y^*(\chi)$ is robustly ψ-type stable, if there exists some positive numbers r_ι and ψ-type functions $\psi_\iota(t)(\iota = 1, 2, \cdots, n)$ such that for $\iota = 1, 2, \cdots, n$ and $\forall t \geqslant t_0 \geqslant 0$*

$$\left(\sum_{j=1}^{n}(c^*_{\iota j} + d^*_{\iota j})F_j - 2\sum_{k=1}^{q}\frac{a^-_{\iota k}}{\beta_k^2} + r_\iota\overline{\psi}_\iota(t) - 2b^-_\iota\right)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota} + \sum_{j=1}^{n}c^*_{\iota j}F_j\left(\frac{\psi_j(t)}{\psi_j(t_0)}\right)^{-r_j} + \sum_{j=1}^{n}d^*_{\iota j}F_jG_{\iota j}(t) < 0,$$

where

$$G_{\iota j}(t) = \begin{cases} 1, & \text{for } t_0 \leqslant t < t_0 + \tau_{\iota j}(t), \\ \left(\dfrac{\psi_j(t - \tau_{\iota j}(t))}{\psi_j(t_0)}\right)^{-r_j}, & \text{for } t \geqslant t_0 + \tau_{\iota j}(t). \end{cases}$$

Proof. Denote

$$V_\iota(t) = \int_\Omega e^2_\iota(\chi, t)d\chi,$$
$$\overline{V}(t_0) = \sum_{\iota=1}^{n}\sup_{t_0-\tau\leqslant\varepsilon\leqslant t_0}\{V_\iota(\varepsilon)\} < +\infty,$$

and

$$H_\iota(t) = \begin{cases} V_\iota(t) - \overline{V}(t_0)\left(\dfrac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}, \ \forall t \geqslant t_0 \geqslant 0, \\ V_\iota(t) - \overline{V}(t_0), \ \forall t_0 - \tau \leqslant t < t_0, \end{cases}$$

where $\iota = 1, 2, \cdots, n$.

Obviously, $H_\iota(t)$ is continuous and $H_\iota(\varepsilon) \leqslant 0$ for $\forall\varepsilon \in [t_0 - \tau, t_0]$. Then, we will prove $H_\iota(t) \leqslant 0$ for $\forall t \geqslant t_0$ and $\iota = 1, 2, \cdots, n$. Otherwise, there exists i and $t_1(t_1 \geqslant t_0)$ satisfying

$$\begin{cases} H_i(t_1) = 0, \\ D^+H_i(t)|_{t=t_1} \geqslant 0, \\ H_j(\varepsilon) \leqslant 0, \ \forall\varepsilon \in [t_0 - \tau, t_1], \ j = 1, 2, \cdots, n. \end{cases}$$

Then,

$$
\begin{aligned}
D^+H_i(t)|_{t=t_1} =& \dot{V}_i(t)|_{t=t_1} + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
=& 2\int_\Omega e_i(\chi,t)\Bigg[\sum_{j=1}^{n} d_{ij}\left(f_j(Y_j(\chi,t-\tau_{ij}(t))) - f_j(Y_j^*(\chi))\right) - b_i e_i(\chi,t) \\
&+ \sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{ik}\frac{\partial e_i(\chi,t)}{\partial\chi_k}\right) + \sum_{j=1}^{n} c_{ij}\left(f_j(Y_j(\chi,t)) - f_j(Y_j^*(\chi))\right)\Bigg]d\chi\Bigg|_{t=t_1} \\
&+ r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right).
\end{aligned} \tag{8}
$$

According to Dirichlet boundary condition, Lemma 2.1 and Green's formula, one has

$$
\begin{aligned}
&\int_\Omega e_i(\chi,t)\sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{ik}\frac{\partial e_i(\chi,t)}{\partial\chi_k}\right)d\chi \\
\leqslant& -\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}\int_\Omega e_i^2(\chi,t)d\chi \\
\leqslant& -\sum_{k=1}^{q}\frac{a_{ik}^-}{\beta_k^2}\int_\Omega e_i^2(\chi,t)d\chi.
\end{aligned} \tag{9}
$$

From (8) and (9), we have

$$
\begin{aligned}
D^+H_i(t)|_{t=t_1} \leqslant& -2\Big(\sum_{k=1}^{q}\frac{a_{ik}^-}{\beta_k^2} + b_i^-\Big)\int_\Omega e_i^2(\chi,t_1)d\chi + 2\int_\Omega |e_i(\chi,t_1)|\Bigg[\sum_{j=1}^{n} c_{ij}^* F_j|e_j(\chi,t_1)| \\
&+ \sum_{j=1}^{n} d_{ij}^* F_j|e_j(\chi,t_1-\tau_{ij}(t_1))|\Bigg]d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
\leqslant& -2\Big(\sum_{k=1}^{q}\frac{a_{ik}^-}{\beta_k^2} + b_i^-\Big)\int_\Omega e_i^2(\chi,t_1)d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
&+ \sum_{j=1}^{n} d_{ij}^* F_j\left(\int_\Omega e_i^2(\chi,t_1)d\chi + \int_\Omega e_j^2(\chi,t_1-\tau_{ij}(t_1))d\chi\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} c_{ij}^{*} F_j \left(\int_{\Omega} e_i^2(\chi, t_1) d\chi + \int_{\Omega} e_j^2(\chi, t_1) d\chi\right) \\
=&\left(\sum_{j=1}^{n}(c_{ij}^{*}+d_{ij}^{*})F_j - 2(\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}+b_i^{-})\right)V_i(t_1)+\sum_{j=1}^{n} c_{ij}^{*} F_j V_j(t_1) \\
&+r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right)+\sum_{j=1}^{n} d_{ij}^{*} F_j V_j(t_1-\tau_{ij}(t_1)) \\
\leqslant&\left(\sum_{j=1}^{n}(c_{ij}^{*}+d_{ij}^{*})F_j - 2\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}-2b_i^{-}\right)\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i} \\
&+\sum_{j=1}^{n} c_{ij}^{*} F_j \overline{V}(t_0)\left(\frac{\psi_j(t_1)}{\psi_j(t_0)}\right)^{-r_j}+\sum_{j=1}^{n} d_{ij}^{*} F_j \overline{V}(t_0) G_{ij}(t_1) \\
&+r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\overline{\psi}_i(t_1) \\
=&\overline{V}(t_0)\Bigg[\left(\sum_{j=1}^{n}(c_{ij}^{*}+d_{ij}^{*})F_j - 2\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}+r_i\overline{\psi}_i(t_1)-2b_i^{-}\right)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i} \\
&+\sum_{j=1}^{n} c_{ij}^{*} F_j \left(\frac{\psi_j(t_1)}{\psi_j(t_0)}\right)^{-r_j}+\sum_{j=1}^{n} d_{ij}^{*} F_j G_{ij}(t_1)\Bigg] \\
<&0,
\end{aligned}
$$

which is unreasonable. Thus

$$
V_\iota(t) \leqslant \overline{V}(t_0)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}, \ \iota = 1, 2, \cdots, n, \ t \geqslant t_0 \geqslant 0.
$$

Similar to the proof of Theorem 3.1, we can get

$$
\limsup_{t\to+\infty} \frac{\ln(\|e(\cdot,t)\|_{\{\xi,\infty\}}^{\Omega})}{\ln(\|\psi(t)\|_{\{\xi,\infty\}})} \leqslant -r.
$$

Therefore, $e(\chi, t)$ is robustly ψ-type stable. The proof is completed.

4. ψ-type stability of RDNN with bounded distributed delays

4.1. ψ-type stability analysis

The class of considered RDNN with bounded distributed delays is described by:

$$\begin{aligned}\frac{\partial Y_\iota(\chi,t)}{\partial t} =&\sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial Y_\iota(\chi,t)}{\partial \chi_k}\right)-b_\iota Y_\iota(\chi,t)+\sum_{j=1}^{n}c_{\iota j}f_j(Y_j(\chi,t))+P_\iota(t)\\ &+\sum_{j=1}^{n}d_{\iota j}\int_{t-\upsilon_j(t)}^{t}f_j(Y_j(\chi,\varsigma))d\varsigma, \qquad (10)\end{aligned}$$

where $\iota=1,2,\cdots,n$, $Y_\iota(\chi,t)$, $f_j(\cdot)$, $P_\iota(t)$, $a_{\iota k}$, b_ι, $c_{\iota j}$, $d_{\iota j}$ have the same definitions as in subsection 3.1, $\upsilon_j(t)$ is the distributed delays which satisfies $0\leqslant \upsilon_j(t)\leqslant \upsilon$ $(j=1,2,\cdots,n)$.

For the network (10),

$$\begin{aligned}&Y_\iota(\chi,t)=0, \quad (\chi,t)\in\partial\Omega\times[t_0-\upsilon,+\infty),\\ &Y_\iota(\chi,t)=\phi_\iota(\chi,t), \quad (\chi,t)\in\Omega\times[t_0-\upsilon,t_0],\end{aligned}$$

where $\phi_\iota(\chi,t)$ $(\iota=1,2,\cdots,n)$ is bounded and continuous on $\Omega\times[t_0-\upsilon,t_0]$.

Suppose that $Y^0(\chi)=(Y_1^0(\chi),Y_2^0(\chi),\cdots,Y_n^0(\chi))^T\in\mathbb{R}^n$ is an equilibrium solution of network (10), then it satisfies

$$\begin{aligned}&\sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial Y_\iota^0(\chi)}{\partial \chi_k}\right)-b_\iota Y_\iota^0(\chi)+\sum_{j=1}^{n}d_{\iota j}\int_{t-\upsilon_j(t)}^{t}f_j(Y_j^0(\chi))d\varsigma\\ &+\sum_{j=1}^{n}c_{\iota j}f_j(Y_j^0(\chi))+P_\iota(t)=0.\end{aligned}$$

Take $e_\iota(\chi,t)=Y_\iota(\chi,t)-Y_\iota^0(\chi)$, we can obtain

$$\frac{\partial e_\iota(\chi,t)}{\partial t} =\sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial e_\iota(\chi,t)}{\partial \chi_k}\right)-b_\iota e_\iota(\chi,t)+\sum_{j=1}^{n}c_{\iota j}\left(f_j(Y_j(\chi,t))-f_j(Y_j^0(\chi))\right)$$

$$+\sum_{j=1}^{n} d_{\iota j} \int_{t-\upsilon_j(t)}^{t} \left(f_j(Y_j(\chi,\varsigma)) - f_j(Y_j^0(\chi))\right) d\varsigma,$$

where $\iota = 1, 2, \cdots, n$.

Remark 4. Due to the existence of a lot of parallel pathways of varying axon size and lengths, NNs often have a spatial extent. Then, a distribution of conduction velocities along these pathways or a distribution of propagation delays over a period of time may exist in some situations, which lead to another kind of time delays, that is, distributed delays in NNs. Therefore, it is necessary to take the distributed delays into account in the study of NNs, and many literatures on NNs with distributed delays have been published recently [20, 24, 26, 27, 29, 35–40]. As far as we know, the ψ-type stability of RDNN with bounded distributed delays has never been considered. Therefore, we concern this topic and derive several ψ-type stability criteria for the RDNNs with bounded distributed delays in this section.

Theorem 4.1. *The network* (10) *with respect to* $Y^0(\chi)$ *is* ψ*-type stable, if there exists some positive numbers* r_ι *and functions* $\psi_\iota(t)(\iota = 1, 2, \cdots, n)$ *such that for* $\iota = 1, 2, \cdots, n$ *and* $\forall t \geqslant t_0 \geqslant 0$

$$\left(\sum_{j=1}^{n}(|c_{\iota j}| + |d_{\iota j}|)F_j - 2\sum_{k=1}^{q}\frac{a_{\iota k}}{\beta_k^2} + r_\iota \overline{\psi}_\iota(t) - 2b_\iota\right)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}$$
$$+\sum_{j=1}^{n}|c_{\iota j}|F_j\left(\frac{\psi_j(t)}{\psi_j(t_0)}\right)^{-r_j} + \upsilon\sum_{j=1}^{n}|d_{\iota j}|F_j W_j(t) < 0,$$

where

$$W_j(t) = \begin{cases} \displaystyle\int_{t_0}^{t}\left(\frac{\psi_j(\varsigma)}{\psi_j(t_0)}\right)^{-r_j} d\varsigma + t_0 + \upsilon - t, \text{ for } t_0 \leqslant t \leqslant t_0 + \upsilon, \\ \displaystyle\int_{t-\upsilon}^{t}\left(\frac{\psi_j(\varsigma)}{\psi_j(t_0)}\right)^{-r_j} d\varsigma, \text{ for } t \geqslant t_0 + \upsilon. \end{cases}$$

Proof. Denote

$$V_\iota(t) = \int_\Omega e_\iota^2(\chi,t)d\chi,$$
$$\overline{V}(t_0) = \sum_{\iota=1}^{n} \sup_{t_0-\upsilon\leqslant\varepsilon\leqslant t_0}\{V_\iota(\varepsilon)\} < +\infty,$$

and

$$H_\iota(t) = \begin{cases} V_\iota(t) - \overline{V}(t_0)\left(\dfrac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}, \ \forall t \geqslant t_0 \geqslant 0, \\ V_\iota(t) - \overline{V}(t_0), \ \forall t_0-\upsilon \leqslant t < t_0, \end{cases}$$

where $\iota = 1, 2, \cdots, n$.

Obviously, $H_\iota(t)$ is continuous and $H_\iota(\varepsilon) \leqslant 0$ for $\forall \varepsilon \in [t_0-\upsilon, t_0]$. We will prove the inequality $H_\iota(t) \leqslant 0$ for $\forall t \geqslant t_0$ and $\iota = 1, 2, \cdots, n$. Otherwise, there exists i and $t_1 (t_1 \geqslant t_0)$ satisfying

$$\begin{cases} H_i(t_1) = 0, \\ D^+H_i(t)|_{t=t_1} \geqslant 0, \\ H_j(\varepsilon) \leqslant 0, \ \forall \varepsilon \in [t_0-\upsilon, t_1], \ j = 1, 2, \cdots, n. \end{cases}$$

Then,

$$\begin{aligned} D^+H_i(t)|_{t=t_1} =& \dot{V}_i(t)|_{t=t_1} + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\ =& 2\int_\Omega e_i(\chi,t)\frac{\partial e_i(\chi,t)}{\partial t}d\chi\bigg|_{t=t_1} + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\ =& 2\int_\Omega e_i(\chi,t)\bigg[\sum_{k=1}^{q}\frac{\partial}{\partial\chi_k}\left(a_{ik}\frac{\partial e_i(\chi,t)}{\partial\chi_k}\right) + \sum_{j=1}^{n}c_{ij}\left(f_j(Y_j(\chi,t)) - f_j(Y_j^0(\chi))\right) \\ & - b_ie_i(\chi,t) + \sum_{j=1}^{n}d_{ij}\int_{t-\upsilon_j(t)}^{t}\left(f_j(Y_j(\chi,\varsigma)) - f_j(Y_j^0(\chi))\right)d\varsigma\bigg]d\chi\bigg|_{t=t_1} \end{aligned}$$

$$
\begin{aligned}
&+ r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right)\\
\leqslant &- 2\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2}\int_\Omega e_i^2(\chi,t_1)d\chi + 2\int_\Omega |e_i(\chi,t_1)|\Bigg[\sum_{j=1}^{n}|c_{ij}|F_j|e_j(\chi,t_1)|\\
&+ \sum_{j=1}^{n}|d_{ij}|F_j\int_{t_1-\upsilon}^{t_1}|e_j(\chi,\varsigma)|d\varsigma\Bigg]d\chi - 2b_i\int_\Omega e_i^2(\chi,t_1)d\chi\\
&+ r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right)\\
\leqslant &- 2(\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2} + b_i)\int_\Omega e_i^2(\chi,t_1)d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right)\\
&+ 2\sum_{j=1}^{n}|d_{ij}|F_j\sqrt{\int_\Omega e_i^2(\chi,t_1)d\chi}\sqrt{\int_\Omega\left(\int_{t_1-\upsilon}^{t_1}|e_j(\chi,\varsigma)|d\varsigma\right)^2 d\chi}\\
&+ 2\sum_{j=1}^{n}|c_{ij}|F_j\sqrt{\int_\Omega e_i^2(\chi,t_1)d\chi}\sqrt{\int_\Omega e_j^2(\chi,t_1)d\chi}.
\end{aligned}
$$

From Lemma 2.2, we have

$$
\begin{aligned}
D^+H_i(t)|_{t=t_1} \leqslant &- 2(\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2} + b_i)\int_\Omega e_i^2(\chi,t_1)d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right)\\
&+ 2\sum_{j=1}^{n}|d_{ij}|F_j\sqrt{\int_\Omega e_i^2(\chi,t_1)d\chi}\sqrt{\int_\Omega \upsilon\int_{t_1-\upsilon}^{t_1}e_j^2(\chi,\varsigma)d\varsigma d\chi}\\
&+ 2\sum_{j=1}^{n}|c_{ij}|F_j\sqrt{\int_\Omega e_i^2(\chi,t_1)d\chi}\sqrt{\int_\Omega e_j^2(\chi,t_1)d\chi}\\
\leqslant &- 2(\sum_{k=1}^{q}\frac{a_{ik}}{\beta_k^2} + b_i)\int_\Omega e_i^2(\chi,t_1)d\chi + r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right)\\
&+ \sum_{j=1}^{n}|d_{ij}|F_j\left(\int_\Omega e_i^2(\chi,t_1)d\chi + \upsilon\int_{t_1-\upsilon}^{t_1}\int_\Omega e_j^2(\chi,\varsigma)d\chi d\varsigma\right)\\
&+ \sum_{j=1}^{n}|c_{ij}|F_j\left(\int_\Omega e_i^2(\chi,t_1)d\chi + \int_\Omega e_j^2(\chi,t_1)d\chi\right)
\end{aligned}
$$

$$
= \left(\sum_{j=1}^{n} (|c_{ij}| + |d_{ij}|) F_j - 2(\sum_{k=1}^{q} \frac{a_{ik}}{\beta_k^2} + b_i) \right) V_i(t_1) + \sum_{j=1}^{n} |c_{ij}| F_j V_j(t_1)
+ r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)} \right) + \upsilon \sum_{j=1}^{n} |d_{ij}| F_j \int_{t_1-\upsilon}^{t_1} V_j(\varsigma) d\varsigma.
$$

By $H_\iota(t) \leqslant 0 (\iota = 1, 2, \cdots, n)$ for any $t \in [t_0 - \upsilon, t_1]$, we can obtain

$$
\begin{aligned}
D^+ H_i(t)|_{t=t_1} \leqslant & \left(\sum_{j=1}^{n} (|c_{ij}| + |d_{ij}|) F_j - 2 \sum_{k=1}^{q} \frac{a_{ik}}{\beta_k^2} - 2b_i \right) \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \\
& + \sum_{j=1}^{n} |c_{ij}| F_j \overline{V}(t_0) \left(\frac{\psi_j(t_1)}{\psi_j(t_0)} \right)^{-r_j} + \upsilon \sum_{j=1}^{n} |d_{ij}| F_j \overline{V}(t_0) W_j(t_1) \\
& + r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \overline{\psi}_i(t_1) \\
= & \overline{V}(t_0) \Bigg[\left(\sum_{j=1}^{n} (|c_{ij}| + |d_{ij}|) F_j - 2 \sum_{k=1}^{q} \frac{a_{ik}}{\beta_k^2} + r_i \overline{\psi}_i(t_1) - 2b_i \right) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)} \right)^{-r_i} \\
& + \sum_{j=1}^{n} |c_{ij}| F_j \left(\frac{\psi_j(t_1)}{\psi_j(t_0)} \right)^{-r_j} + \upsilon \sum_{j=1}^{n} |d_{ij}| F_j W_j(t_1) \Bigg] \\
< & 0,
\end{aligned}
$$

which is unreasonable. Thus

$$
V_\iota(t) \leqslant \overline{V}(t_0) \left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)} \right)^{-r_\iota}, \ \iota = 1, 2, \cdots, n, \ \forall t \geqslant t_0 \geqslant 0.
$$

Similar to the proof of Theorem 3.1, we can obtain

$$
\limsup_{t \to +\infty} \frac{\ln(\|e(\cdot, t)\|_{\{\xi,\infty\}}^{\Omega})}{\ln(\|\psi(t)\|_{\{\xi,\infty\}})} \leqslant -r.
$$

In other words, $e(\chi, t)$ is ψ-type stable. This completes the proof.

4.2. Robust ψ-type stability analysis

The RDNN with parametric uncertainties and bounded distributed delays is described by:

$$\begin{aligned}\frac{\partial Y_\iota(\chi,t)}{\partial t} =&\sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial Y_\iota(\chi,t)}{\partial \chi_k}\right) - b_\iota Y_\iota(\chi,t) + \sum_{j=1}^{n} c_{\iota j} f_j(Y_j(\chi,t)) + P_\iota(t)\\ &+\sum_{j=1}^{n} d_{\iota j}\int_{t-\upsilon_j(t)}^{t} f_j(Y_j(\chi,\varsigma))d\varsigma, \qquad (11)\end{aligned}$$

where $\iota = 1,2,\cdots,n$, $Y_\iota(\chi,t)$, $f_j(\cdot)$, $P_\iota(t)$, $\upsilon_j(t)$, have the same definitions in subsection 4.1, and the parameters $a_{\iota k}$, b_ι, $c_{\iota j}$, $d_{\iota j}$ are defined by (7).

Take $e_\iota(\chi,t) = Y_\iota(\chi,t) - Y^0_\iota(\chi)$, we can obtain

$$\begin{aligned}\frac{\partial e_\iota(\chi,t)}{\partial t} =&\sum_{k=1}^{q}\frac{\partial}{\partial \chi_k}\left(a_{\iota k}\frac{\partial e_\iota(\chi,t)}{\partial \chi_k}\right) - b_\iota e_\iota(\chi,t) + \sum_{j=1}^{n} c_{\iota j}\left(f_j(Y_j(\chi,t)) - f_j(Y^0_j(\chi))\right)\\ &+\sum_{j=1}^{n} d_{\iota j}\int_{t-\upsilon_j(t)}^{t}\left(f_j(Y_j(\chi,\varsigma)) - f_j(Y^0_j(\chi))\right)d\varsigma,\end{aligned}$$

where $a_{\iota k}$, b_ι, $c_{\iota j}$, $d_{\iota j}$ belong to the parameter ranges defined by (7).

Theorem 4.2. *The network* (11) *with respect to $Y^s(\chi)$ is ψ-type stable, if there exists some positive numbers r_ι and ψ-type functions $\psi_\iota(t)(\iota = 1,2,\cdots,n)$ such that for $\iota = 1,2,\cdots,n$ and $\forall t \geqslant t_0 \geqslant 0$*

$$\begin{aligned}&\left(\sum_{j=1}^{n}(c^*_{\iota j} + d^*_{\iota j})F_j - 2\sum_{k=1}^{q}\frac{a^-_{\iota k}}{\beta_k^2} + r_\iota\overline{\psi}_\iota(t) - 2b^-_\iota\right)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}\\ &+\sum_{j=1}^{n} c^*_{\iota j}F_j\left(\frac{\psi_j(t)}{\psi_j(t_0)}\right)^{-r_j} + \upsilon\sum_{j=1}^{n} d^*_{\iota j}F_j W_j(t) < 0,\end{aligned}$$

where

$$W_j(t) = \begin{cases}\displaystyle\int_{t_0}^{t}\left(\frac{\psi_j(\varsigma)}{\psi_j(t_0)}\right)^{-r_j}d\varsigma + t_0 + \upsilon - t, \text{ for } t_0 \leqslant t \leqslant t_0+\upsilon,\\ \displaystyle\int_{t-\upsilon}^{t}\left(\frac{\psi_j(\varsigma)}{\psi_j(t_0)}\right)^{-r_j}d\varsigma, \text{ for } t \geqslant t_0+\upsilon.\end{cases}$$

Proof. Denote

$$V_\iota(t) = \int_\Omega e_\iota^2(\chi, t) d\chi,$$
$$\overline{V}(t_0) = \sum_{\iota=1}^{n} \sup_{t_0 - \upsilon \leqslant \varepsilon \leqslant t_0} \{V_\iota(\varepsilon)\} < +\infty,$$

and

$$H_\iota(t) = \begin{cases} V_\iota(t) - \overline{V}(t_0) \left(\dfrac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota}, \ \forall t \geqslant t_0 \geqslant 0, \\ V_\iota(t) - \overline{V}(t_0), \ \forall t_0 - \upsilon \leqslant t < t_0, \end{cases}$$

where $\iota = 1, 2, \cdots, n$.

Obviously, $H_\iota(t)$ is continuous and $H_\iota(\varepsilon) \leqslant 0$ for $\forall \varepsilon \in [t_0 - \upsilon, t_0]$. We will prove the inequality $H_\iota(t) \leqslant 0$ for $\forall t \geqslant t_0$ and $\iota = 1, 2, \cdots, n$. Otherwise, there exists i and $t_1 (t_1 \geqslant t_0)$ satisfying

$$\begin{cases} H_i(t_1) = 0, \\ D^+ H_i(t)|_{t=t_1} \geqslant 0, \\ H_j(\varepsilon) \leqslant 0, \ \forall \varepsilon \in [t_0 - \upsilon, t_1], \ j = 1, 2, \cdots, n. \end{cases}$$

Then,

$$\begin{aligned} D^+ H_i(t)|_{t=t_1} =& \dot{V}_i(t)|_{t=t_1} + r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i} \left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\ =& 2 \int_\Omega e_i(\chi, t) \Bigg[\sum_{j=1}^{n} d_{ij} \int_{t - \upsilon_j(t)}^{t} \left(f_j(Y_j(\chi, \varsigma)) - f_j(Y_j^0(\chi))\right) d\varsigma - b_i e_i(\chi, t) \\ & + \sum_{k=1}^{q} \frac{\partial}{\partial \chi_k} \left(a_{ik} \frac{\partial e_i(\chi, t)}{\partial \chi_k}\right) + \sum_{j=1}^{n} c_{ij} \left(f_j(Y_j(\chi, t)) - f_j(Y_j^0(\chi))\right) \Bigg] d\chi \Bigg|_{t=t_1} \\ & + r_i \overline{V}(t_0) \left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i} \left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \end{aligned}$$

$$
\begin{aligned}
\leqslant &- 2(\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}+b_i^{-})\int_{\Omega}e_i^2(\chi,t_1)d\chi+2\int_{\Omega}|e_i(\chi,t_1)|\Bigg[\sum_{j=1}^{n}c_{ij}^{*}F_j|e_j(\chi,t_1)| \\
&+\sum_{j=1}^{n}d_{ij}^{*}F_j\int_{t_1-\upsilon}^{t_1}|e_j(\chi,\varsigma)|d\varsigma\Bigg]d\chi+r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
\leqslant &- 2(\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}+b_i^{-})\int_{\Omega}e_i^2(\chi,t_1)d\chi+r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\left(\frac{\dot{\psi}_i(t_1)}{\psi_i(t_1)}\right) \\
&+\sum_{j=1}^{n}d_{ij}^{*}F_j\left(\int_{\Omega}e_i^2(\chi,t_1)d\chi+\upsilon\int_{t_1-\upsilon}^{t_1}\int_{\Omega}e_j^2(\chi,\varsigma)d\chi d\varsigma\right) \\
&+\sum_{j=1}^{n}c_{ij}^{*}F_j\left(\int_{\Omega}e_i^2(\chi,t_1)d\chi+\int_{\Omega}e_j^2(\chi,t_1)d\chi\right) \\
\leqslant &\left(\sum_{j=1}^{n}(c_{ij}^{*}+d_{ij}^{*})F_j-2\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}-2b_i^{-}\right)\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i} \\
&+\sum_{j=1}^{n}c_{ij}^{*}F_j\overline{V}(t_0)\left(\frac{\psi_j(t_1)}{\psi_j(t_0)}\right)^{-r_j}+\upsilon\sum_{j=1}^{n}d_{ij}^{*}F_j\overline{V}(t_0)W_j(t_1) \\
&+r_i\overline{V}(t_0)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i}\overline{\psi}_i(t_1) \\
= &\overline{V}(t_0)\Bigg[\left(\sum_{j=1}^{n}(c_{ij}^{*}+d_{ij}^{*})F_j-2\sum_{k=1}^{q}\frac{a_{ik}^{-}}{\beta_k^2}+r_i\overline{\psi}_i(t_1)-2b_i^{-}\right)\left(\frac{\psi_i(t_1)}{\psi_i(t_0)}\right)^{-r_i} \\
&+\sum_{j=1}^{n}c_{ij}^{*}F_j\left(\frac{\psi_j(t_1)}{\psi_j(t_0)}\right)^{-r_j}+\upsilon\sum_{j=1}^{n}d_{ij}^{*}F_jW_j(t_1)\Bigg] \\
< &0,
\end{aligned}
$$

which is unreasonable. Thus

$$
V_\iota(t)\leqslant\overline{V}(t_0)\left(\frac{\psi_\iota(t)}{\psi_\iota(t_0)}\right)^{-r_\iota},\ \iota=1,2,\cdots,n,\ t\geqslant t_0\geqslant 0.
$$

Similar to the proof of Theorem 3.1, we can obtain

$$
\limsup_{t\to+\infty}\frac{\ln(\|e(\cdot,t)\|_{\{\xi,\infty\}}^{\Omega})}{\ln(\|\psi(t)\|_{\{\xi,\infty\}})}\leqslant -r.
$$

Therefore, $e(\chi, t)$ is robustly ψ-type stable. The proof is completed.

5. Numerical Examples

Example 5.1. Given the following RDNN with time-varying discrete delays and parametric uncertainties:

$$\begin{aligned}\frac{\partial Y_\iota(\chi,t)}{\partial t} =& a_\iota \frac{\partial Y_\iota(\chi,t)}{\partial \chi^2} - b_\iota Y_\iota(\chi,t) + \sum_{j=1}^{3} c_{\iota j} f_j(Y_j(\chi,t)) + P_\iota(t) \\ &+ \sum_{j=1}^{3} d_{\iota j} f_j(Y_j(\chi, t-\tau_{\iota j}(t))), \end{aligned} \tag{12}$$

where $\iota = 1,2,3$, $-1 < \chi < 1$, $f_j(\epsilon) = \frac{|\epsilon+1|-|\epsilon-1|}{8}$ $(j = 1,2,3)$, $\tau_{\iota j}(t) = \frac{1}{\iota+j}(1-e^{-t})$, $\tau = 0.5$, $P_1(t) = P_2(t) = P_3(t) = 0$.

Obviously, $F_1 = F_2 = F_3 = 0.25$. In particular, we choose $t_0 = 0$, $r_1 = r_2 = r_3 = 1$ and $\psi_1(t) = \psi_2(t) = \psi_3(t) = e^{0.02t}$. The parameters $a_{\iota 1}$, b_ι, $c_{\iota j}$, $d_{\iota j}$ in the network (12) can be changed in the following given precisions:

$$\begin{cases} A_I :=\{A = (a_\iota)_{3\times 1} : 0.7 \leqslant a_1 \leqslant 0.8, 0.8 \leqslant a_2 \leqslant 0.9, 0.9 \leqslant a_3 \leqslant 1\}, \\ B_I :=\{B = \mathrm{diag}(b_1,b_2,b_3) : 0.8 \leqslant b_1 \leqslant 0.9, 0.9 \leqslant b_2 \leqslant 1, 1 \leqslant b_3 \leqslant 1.1\}, \\ C_I :=\{C = (c_{\iota j})_{3\times 3} : \dfrac{1}{2(\iota+j)} + 0.005 \leqslant c_{\iota j} \leqslant \dfrac{1}{2(\iota+j)} + 0.01\}, \\ D_I :=\{D = (d_{\iota j})_{3\times 3} : \dfrac{1}{2(\iota+j)} + 0.015 \leqslant d_{\iota j} \leqslant \dfrac{1}{2(\iota+j)} + 0.02\}. \end{cases} \tag{13}$$

Then,

$$\begin{aligned}&\left(\sum_{j=1}^{n}(c_{1j}^{*} + d_{1j}^{*})F_j - 2a_1^{-} + \overline{\psi}_1(t) - 2b_1^{-}\right)\left(\frac{\psi_1(t)}{\psi_1(0)}\right)^{-1} + \sum_{j=1}^{n} c_{1j}^{*} F_j \left(\frac{\psi_j(t)}{\psi_j(0)}\right)^{-1} \\ &+ \sum_{j=1}^{n} d_{1j}^{*} F_j G_{1j}(t) < -2.3918\frac{1}{e^{0.02t}} < 0,\end{aligned}$$

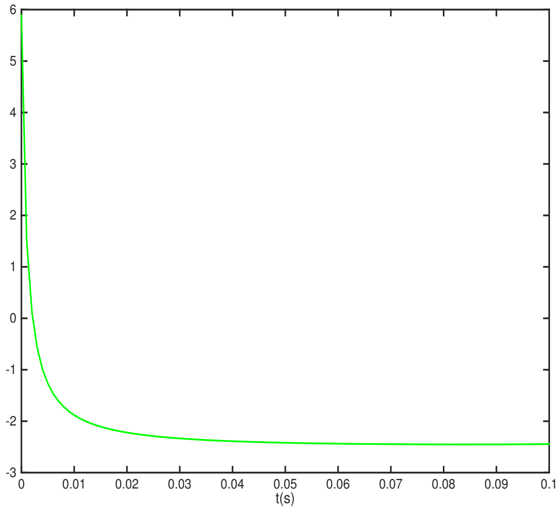

Figure 1: Trajectory of $\frac{\ln\|e(\cdot,t)\|^{\Omega}_{\{1,\infty\}}}{\ln\|\psi(t)\|_{\{1,\infty\}}}$ with respect to the relative convergence rate $\lambda = 1$.

$$\left(\sum_{j=1}^{n}(c^*_{2j}+d^*_{2j})F_j - 2a^-_2 + \overline{\psi}_2(t) - 2b^-_2\right)\left(\frac{\psi_2(t)}{\psi_2(0)}\right)^{-1} + \sum_{j=1}^{n} c^*_{2j}F_j\left(\frac{\psi_j(t)}{\psi_j(0)}\right)^{-1}$$
$$+\sum_{j=1}^{n} d^*_{2j}F_jG_{2j}(t) < -2.9422\frac{1}{e^{0.02t}} < 0,$$
$$\left(\sum_{j=1}^{n}(c^*_{3j}+d^*_{3j})F_j - 2a^-_3 + \overline{\psi}_3(t) - 2b^-_3\right)\left(\frac{\psi_3(t)}{\psi_3(0)}\right)^{-1} + \sum_{j=1}^{n} c^*_{3j}F_j\left(\frac{\psi_j(t)}{\psi_j(0)}\right)^{-1}$$
$$+\sum_{j=1}^{n} d^*_{3j}F_jG_{3j}(t) < -3.4257\frac{1}{e^{0.02t}} < 0.$$

According to Theorem 3.2, the network (12) with the given parameters defined in (13) is robust ψ-type stable with regard to zero solution. The simulation results are displayed in Figures 1 and 2.

Example 5.2. Consider a RDNN with bounded distributed delays and

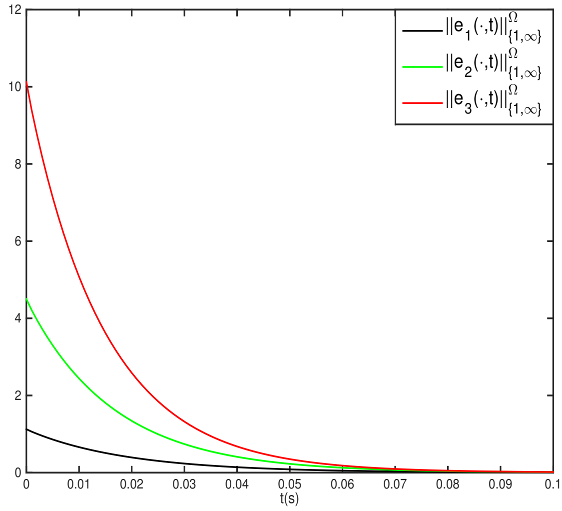

Figure 2: $\|e_\iota(\cdot,t)\|^{\Omega}_{\{1,\infty\}}$, $\iota = 1,2,3$.

parametric uncertainties which can be described as follows:

$$\begin{aligned}\frac{\partial Y_\iota(\chi,t)}{\partial t} =& a_\iota \frac{\partial Y_\iota(\chi,t)}{\partial \chi^2} - b_\iota Y_\iota(\chi,t) + \sum_{j=1}^{3} c_{\iota j} f_j(Y_j(\chi,t)) + P_\iota(t) \\ &+ \sum_{j=1}^{3} d_{\iota j} \int_{t-v_j(t)}^{t} f_j(Y_j(\chi,\varsigma))d\varsigma, \end{aligned} \tag{14}$$

where $\iota = 1,2,3$, $-1 < \chi < 1$, $f_j(\epsilon) = 0.2$ $(j = 1,2,3)$, $v_j(t) = \frac{j}{50}(1-e^{-t})$, $v = 0.06$, $P_\iota(t) = -0.2\sum_{j=1}^{3}(d_{\iota j} v_j(t) + c_{\iota j})$.

Obviously, $F_1 = F_2 = F_3 = 0$. In particular, we choose $t_0 = 0$, $r_1 = r_2 = r_3 = 1$ and $\psi_1(t) = \psi_2(t) = \psi_3(t) = 1 + t$. The parameters a_ι, b_ι, $c_{\iota j}$, $d_{\iota j}$ in the network (14) are defined by (13). Then,

$$\left(\sum_{j=1}^{n}(c^*_{1j} + d^*_{1j})F_j - 2a_1^- + \overline{\psi}_1(t) - 2b_1^-\right)\left(\frac{\psi_1(t)}{\psi_1(0)}\right)^{-1} + \sum_{j=1}^{n} c^*_{1j} F_j \left(\frac{\psi_j(t)}{\psi_j(0)}\right)^{-1}$$

$$+ v\sum_{j=1}^{n} d_{1j}^{*}F_jW_j(t) < -2\frac{1}{1+t} < 0,$$

$$\left(\sum_{j=1}^{n}(c_{2j}^{*}+d_{2j}^{*})F_j - 2a_2^{-} + \overline{\psi}_2(t) - 2b_2^{-}\right)\left(\frac{\psi_2(t)}{\psi_2(0)}\right)^{-1} + \sum_{j=1}^{n} c_{2j}^{*}F_j\left(\frac{\psi_j(t)}{\psi_j(0)}\right)^{-1}$$

$$+ v\sum_{j=1}^{n} d_{2j}^{*}F_jW_j(t) < -2.4\frac{1}{1+t} < 0,$$

$$\left(\sum_{j=1}^{n}(c_{3j}^{*}+d_{3j}^{*})F_j - 2a_3^{-} + \overline{\psi}_3(t) - 2b_3^{-}\right)\left(\frac{\psi_3(t)}{\psi_3(0)}\right)^{-1} + \sum_{j=1}^{n} c_{3j}^{*}F_j\left(\frac{\psi_j(t)}{\psi_j(0)}\right)^{-1}$$

$$+ v\sum_{j=1}^{n} d_{3j}^{*}F_jW_j(t) < -2.8\frac{1}{1+t} < 0.$$

According to Theorem 4.2, the network (14) with the given parameters defined in (13) is robust ψ-type stable with regard to zero solution. The simulation results are displayed in Figures 3 and 4.

Remark 5. Generally speaking, the ψ-type stability is related to the selection of ψ-type functions. Moreover, the ψ-type stability criteria are slightly different because of the different selection of ψ-type function. If exponential functions or polynomial functions are chosen as ψ-type functions, then exponential stability or polynomial stability as the special cases of ψ-type stability can be obtained. As in Example 5.1, the function $\psi(t)$ is given by exponential function, some analogous results have been studied in [24] and [27], in which equilibrium points are exponentially convergent for their considered networks. Therefore, our results can be regarded as the extension of previous results on other type stability (e.g., exponential stability, polynomial stability and μ-stability) of RDNN [12, 24, 27, 29]. To illustrate the ψ-type stability is different from the exponentially stability, we also provide Example 5.2, in which equilibrium points are polynomially convergent for the network.

Remark 6. Due to the difficulty of estimating the convergence rate of the system in practical applications, some researchers have devoted themselves to investigating a new type of stability, namely ψ-type stability, which generalizes some traditional stability definitions, e.g., exponential stability, log-stability, power-rate stability and μ-stability [20–23]. In [21], the multiple ψ-type stability of recurrent NNs with time-varying delays was investigated. Wang et al. [23] studied the ψ-type synchronization problem of NNs by using the conception of ψ-type stability. However, the reaction-diffusion phenomena of NNs has been neglected in the above literatures. In a strict sense, reaction-diffusion effects are unavoidable in NNs once the electrons transport in inhomogeneous magnetic field. Therefore, taking the reaction-diffusion terms into consideration in NNs is necessary and meaningful, and some researchers have studied the traditional stability of RDNNs [12, 24–31, 40, 42]. To our knowledge, the ψ-type stability of RDNNs has not yet been considered until now and this is the first paper toward to investigating ψ-type stability and robust ψ-type stability for RDNNs with time-varying discrete delays and bounded distributed delays.

6. Conclusion

This paper has investigated the ψ-type stability and robust ψ-type stability for RDNNs with and without parametric uncertainties, respectively. By utilizing several new inequality techniques, several ψ-type stability and

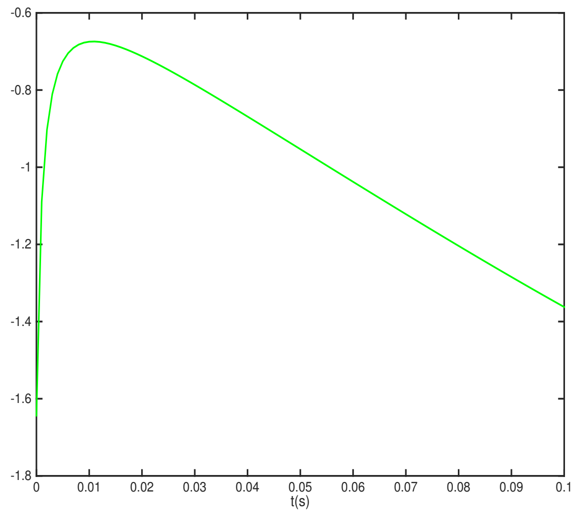

Figure 3: Trajectory of $\frac{\ln\|e(\cdot,t)\|^{\Omega}_{\{1,\infty\}}}{\ln\|\psi(t)\|_{\{1,\infty\}}}$ with respect to the relative convergence rate $\lambda = 1$.

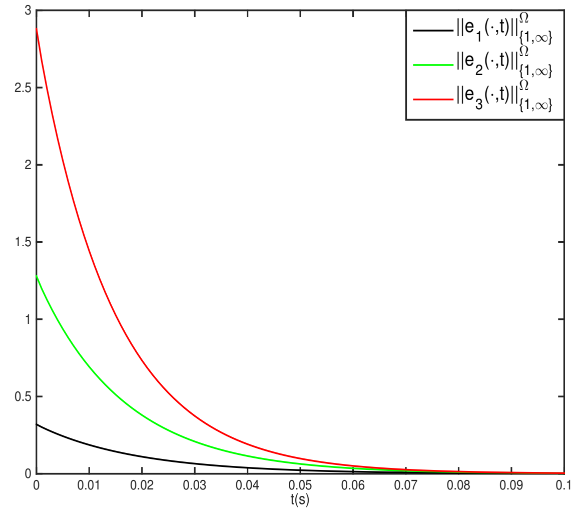

Figure 4: $\|e_\iota(\cdot,t)\|^{\Omega}_{\{1,\infty\}},\ \iota = 1, 2, 3.$

robust ψ-type stability criteria have been proposed for RDNN and URDNN with time-varying discrete delays. Then, the models of RDNNs with bounded distributed delays have been studied and several sufficient conditions to guarantee the ψ-type stability and robust ψ-type stability for these networks have been given. Finally, the validity of these obtained results has been verified through some examples with simulation results.

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