Families of Bianchi modular symbols: critical base-change p-adic L-functions and p-adic Artin formalism

Daniel Barrera Salazar and Chris Williams
with an appendix by Carl Wang-Erickson

Abstract

Let K be an imaginary quadratic field. In this article, we study the eigenvariety for GL_2/K, proving an étaleness result for the weight map at non-critical classical points and a smoothness result at base-change classical points. We give three main applications of this. (1) We construct three-variable p-adic L-functions over the eigenvariety interpolating the (two-variable) p-adic L-functions of classical Bianchi cusp forms in families. (2) Let f be a p-stabilised newform of weight k at least 2 without CM by K. We construct a two-variable p-adic L-function attached to the base-change of f to K under assumptions on f that we conjecture always hold, in particular making no assumption on the slope of f. (3) We prove that these base-change p-adic L-functions satisfy a p-adic Artin formalism result, that is, they factorise in the same way as the classical L-function under Artin formalism.

1. Introduction

The study of p-adic L-functions has proved invaluable for approaching many important problems in arithmetic number theory, playing a major role in the proof of cases of the Birch and Swinnerton-Dyer and Bloch–Kato conjectures and the non-vanishing of certain central L-values (see [JSW17], [Cas17], [DR17], [BDR15], [BC04], [BC09], [DJR18]). Another common theme in the above papers is the notion of varying automorphic representations in p-adic families. Such families are captured geometrically in the theory of *eigenvarieties*. Eigenvarieties and p-adic L-functions are very closely related, and indeed their constructions often use the same tools, such as p-adic automorphic forms, completed cohomology or overconvergent cohomology. It is also natural to try to construct 'many variabled' p-adic L-functions varying in p-adic families over eigenvarieties; such functions are ubiquitous in the works above. In this paper, we study the eigenvariety parametrising automorphic forms for GL_2 over an imaginary quadratic field K and give a number of applications, including an extension of known constructions of p-adic L-functions in this setting and a construction of (three-variable) p-adic L-functions in families.

In general, for every complex L-function attached to a cohomological automorphic representation of a reductive group G, we expect there to be a p-adic analogue. In practice, p-adic L-functions can be very hard to construct, and we are far from achieving this goal. Even in cases where constructions are well-established – such as for classical modular forms, when $G = \mathrm{GL}_2/\mathbb{Q}$ – there are subtleties; for example, when the modular form is 'critical' at p, the usual conditions a p-adic L-function satisfies do not determine it uniquely. In this case, the study of eigenvarieties has provided a much more complete picture; for example, the Coleman–Mazur eigencurve is pivotal in [Bel12], where Bellaïche constructs canonical 'analytic' p-adic L-functions for critical slope modular forms, and in [Han16], where (under a non-vanishing hypothesis) Hansen shows that these are equal to 'algebraic' critical p-adic L-functions constructed by Kato using Euler systems (see [Kat04]). Bellaïche's construction of analytic critical p-adic L-functions has since been generalised by Bergdall and Hansen, using the Hilbert eigenvariety, to the case of Hilbert

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modular forms (see [BH17]).

In light of this, to obtain a more complete theory of p-adic L-functions attached to automorphic representations of G we should:

(a) get a good understanding of the local behaviour (at classical points) of the eigenvariety attached to G, and

(b) find a generic construction of p-adic L-functions (at single points of the eigenvariety) that is well-adapted for variation in families.

In the setting that interests us – the *Bianchi* case, where $G = \mathrm{GL}_2/K$ – the generic construction of (two-variable) p-adic L-functions for single points was done by the second author in [Wil17], using overconvergent modular symbols (the first degree compactly supported overconvergent cohomology of the relevant locally symmetric space). The eigenvariety in question was constructed by Hansen in [Han17], also using overconvergent cohomology. Its further study, however, is challenging; $\mathrm{GL}_2(\mathbb{C})$ does not admit discrete series representations, and consequently many nice properties of the Coleman–Mazur eigencurve (and, more generally, Hilbert eigenvarieties) fail to hold in the Bianchi setting. Strikingly, the classical points are not dense, and there exist classical Bianchi cusp forms that do not vary in classical families. In fact, in [CM09], Calegari and Mazur conjectured that *ordinary* classical families arise only through cases of Langlands functoriality; in particular, such a family should be a (twist) of a base-change family from $\mathrm{GL}_2/\mathbb{Q}$ or a CM family from GL_1/L, where L/K is a quadratic extension. The situation is further complicated as Bianchi modular forms appear in more than one cohomological degree, unlike in (for example) the case of Hilbert modular forms. Our understanding of eigenvarieties in this so-called '$\ell > 0$' case[1] is significantly less developed than when $\ell = 0$.

Many of the methods used in the literature (for example, in [Bel12], [BSDJ17] and [BH17]) for constructing p-adic L-functions over eigenvarieties make essential use of properties that do not hold for the Bianchi eigenvariety. Accordingly, to prove the technical results we need in the construction – namely, an étaleness result at non-critical points and a smoothness result for base-change points – we develop new arguments for working in the $\ell > 0$ case. We hope these methods can be more easily adapted to eigenvarieties for more general reductive groups, such as those for GL_n with $n \geq 3$, where we get similar 'bad' behaviour.

1.1. Main Results

Let K be an imaginary quadratic number field, let $\mathcal{O}_K$ be its ring of integers, and fix p a rational prime. Let $\mathcal{F}$ be a cuspidal Bianchi eigenform of weight $\lambda := (k, k)$ and level $\mathfrak{n} \subset \mathcal{O}_K$, where $\mathfrak{n}$ is divisible by all of the primes of K above p; throughout, we will assume that $\mathcal{F}$ is either[2] new at p or the stabilisation of a newform at primes above p. Suppose that $\mathcal{F}$ either:

(NC) is non-critical in the sense of Definition 2.7, or

(BC) has finite slope at p, and is the base-change of a p-stabilised decent[3] classical newform f of weight $k+2 \geq 2$ and level prime to p, or a twist of such a base-change by a finite order Hecke character of K of conductor prime to p.

If $\mathcal{F}$ is in case (BC), and p is split in K, then it automatically also satisfies condition (NC) if $v_p(a_p(f)) < k+1$, where $a_p(f)$ is the U_p-eigenvalue of f. We say such forms have *small slope*. If p is inert or ramified, however, the small slope condition becomes $v_p(a_p(f)) < (k+1)/2$. In particular, in the latter case, the 'critical slope region' covers fully *half* of the possible range, and classical forms of non-critical slope can base-change to have critical slope over K. There were previously no proven constructions of p-adic L-functions in this substantial case.

[1] Here ℓ denotes the difference between the top and bottom cohomological degrees in which forms appear.
[2] For uniformity, we will call forms satisfying either condition *p-stabilised newforms*; see Definition 2.16.
[3] See Definition 5.4. Conjecturally, every modular form is decent.

We also assume $\mathcal{F}$ satisfies a (mild) multiplicity one result[4] (Definition 4.6). In [Wil17] the author constructs a p-adic L-function $L_p(\mathcal{F})$ for $\mathcal{F}$ in case (NC). This is a distribution on $\mathrm{Cl}_K(p^\infty)$, the ray class group of K of level p^∞, and is constructed as follows. For L a sufficiently large finite extension of $\mathbb{Q}_p$, one can realise $\mathcal{F}$ in the classical cohomology $\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{V}_\lambda(L)^*)$, where $Y_1(\mathfrak{n})$ is the relevant Bianchi locally symmetric space of level $\mathfrak{n}$ and $\mathscr{V}_\lambda^*$ is the local system attached to the algebraic representation of highest weight λ. One can then exhibit a *canonical* class $\Psi_{\mathcal{F}} \in \mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$ in the overconvergent cohomology lifting this classical class, where $\mathscr{D}_\lambda(L)$ is the local system of L-valued locally analytic distributions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We then define $L_p(\mathcal{F})$ to be the Mellin transform (see §2.7) of this canonical class. It is shown *op. cit.* that this distribution interpolates the critical L-values of $\mathcal{F}$.

1.1.1. Critical base-change p-adic L-functions

The first main result of this paper is an extension of this construction to case (BC), allowing critical slope forms. To prove strong analogues of the results of [Wil17], constructing a canonical class $\Psi_{\mathcal{F}}$ in the overconvergent cohomology, we require the following hypothesis.

Hypothesis 1.1. *Suppose $\mathcal{F}$ in case (BC) is critical. Then there is precisely one Bianchi family through $\mathcal{F}$ (the base-change of the Coleman family through f) that admits a Zariski-dense set of classical points. If $\mathcal{F}$ satisfies this condition, we say it is* Σ-smooth.

Theorem 1.2. *Let $\mathcal{F}$ in case (BC) be Σ-smooth. Then the $\mathcal{F}$-eigenspace $\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))[\mathcal{F}]$ (that is, the eigenspace in $\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$ where the Hecke operators act with the same eigenvalues as on $\mathcal{F}$) is one-dimensional over L.*

Choosing $\Psi_{\mathcal{F}}$ to be a generator, and taking the Mellin transform, this result allows us to define the p-adic L-function $L_p(\mathcal{F}) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), L)$ as a locally analytic distribution on $\mathrm{Cl}_K(p^\infty)$ for each base-change $\mathcal{F}$ as above. Moreover, we prove that $L_p(\mathcal{F})$ satisfies the expected growth and interpolation[5] properties.

We conjecture that Hypothesis 1.1 always holds, but if we do *not* assume it, then we can still construct a canonical candidate for $L_p(\mathcal{F})$; see §1.1.3 for more details.

1.1.2. Three-variable p-adic L-functions

Our second main result is that the p-adic L-functions above naturally live in p-adic families. Let $\mathcal{W}_K$ be the *null Bianchi weight space*, which is a rigid space of dimension 2 (see Definition 3.1). Let $\mathcal{E}$ be the Bianchi eigenvariety, together with the weight map $w : \mathcal{E} \to \mathcal{W}_K$, as constructed in [Han17]. The points of $\mathcal{E}$ above a weight λ are in bijection with systems of eigenvalues that appear in the (total) weight λ overconvergent cohomology of $Y_1(\mathfrak{n})$. Hansen's work also gives a finite 'base-change p-adic functoriality' map $\mathrm{BC} : \mathcal{C} \to \mathcal{E}$, where $\mathcal{C}$ is the Coleman–Mazur eigencurve, interpolating the base-change lifts on classical points.

Assume first that $\mathcal{F}$ is in case (NC) or in case (BC) and Σ-smooth; in this case, we prove the stronger result that the canonical overconvergent classes vary in families over $\mathcal{E}$. Let $x_{\mathcal{F}}$ be the point in $\mathcal{E}(L)$ attached to $\mathcal{F}$. Any irreducible component of $\mathcal{E}$ containing $x_{\mathcal{F}}$ is one-dimensional (see Theorem 3.9). In case (NC), we choose $\mathcal{E}'$ to be any such irreducible component, and in case (BC), we choose it to be a (twist of a) base-change component, which is possible by Theorem 3.5. Let V be a neighbourhood of $x_{\mathcal{F}}$ inside $\mathcal{E}'$, with $\Sigma = w(V)$ an affinoid curve inside $\mathcal{W}_K$. We define $\mathcal{D}_\Sigma$ to be the space of $\mathcal{O}(\Sigma)$-valued locally analytic distributions on $\mathcal{O}_K \otimes \mathbb{Z}_p$. This gives rise to a local system $\mathscr{D}_\Sigma$ on $Y_1(\mathfrak{n})$, and for any $h \geq 0$, it is possible to shrink V and Σ so that the overconvergent cohomology $\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)$ admits a slope $\leq h$ decomposition with respect to U_p. At any point $y \in V(L)$, there exists a specialisation map $\mathrm{sp}_y : \mathcal{O}(V) \to L$ given by evaluation at y. We prove (see Proposition 7.5):

[4] In practice, this involves the standard assumption that roots of Hecke polynomials at p are distinct.
[5] In the critical case, the interpolation property is that every critical value vanishes.

Theorem 1.3. *After possibly further shrinking V, there exists a Hecke eigenclass*

$$\Phi_V \in \mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} \mathcal{O}(V)$$

such that:

(i) $\mathrm{sp}_{x_\mathcal{F}}(\Phi_V) = \Psi_\mathcal{F}$, *and*

(ii) for any classical point $y \in V(L)$, with $w(y) = \kappa$, the specialisation $\mathrm{sp}_y(\Phi_V)$ is a generator of the (one-dimensional) L-vector eigenspace $\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\kappa(L))[\mathcal{F}_y]$ (where $\mathcal{F}_y$ is the Bianchi form corresponding to y).

Here $\mathbb{T}_{\Sigma,h}$ denotes the submodule of $\mathrm{End}_{\mathcal{O}(\Sigma)}(\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h})$ generated by the Hecke operators away from $\mathfrak{n}$ and the Hecke operators at p. The class Φ_V is canonical up to an element of $\mathcal{O}(V)^\times$. Taking the Mellin transform of this class we obtain a distribution $\mathcal{L}_p(V) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), \mathcal{O}(V))$. This distribution is the *three-variable p-adic L-function*, characterised by the following interpolation property.

Corollary 1.4. *For each classical point $y \in V(L)$ we have*

$$\begin{aligned}\mathcal{L}_p(y) &:= \mathrm{sp}_y(\mathcal{L}_p(V)) \\ &= c_y L_p(\mathcal{F}_y) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), L),\end{aligned}$$

where $c_y \in L^\times$.

The numbers $c_y \in L^\times$ are *p-adic periods* analogous to those obtained in [GS93]. In case (BC), the corollary can be phrased more explicitly. Let $\mathscr{X}(\mathrm{Cl}_K(p^\infty))$ denote the rigid space of p-adic characters on $\mathrm{Cl}_K(p^\infty)$ (see [BH17, Defn. 5.1.4]); functions on this space are naturally two-variabled, so the following also explains the terminology 'three-variable p-adic L-function', since we add a single weight variable.

Corollary 1.5. *Let f be a decent cuspidal eigenform (for $\mathrm{GL}_2/\mathbb{Q}$) of weight $k+2 \geq 2$ and level $N = Mp$, let ϕ be a finite-order Hecke character of K that has conductor prime to $p\mathcal{O}_K$, and let V be a neighbourhood of f in the Coleman–Mazur eigencurve that is étale over the weight space. Suppose that f does not have CM by K, and that f is either a newform or the p-stabilisation of a newform at level M prime to p. If f is critical, suppose further that its base-change $\mathcal{F}$ is Σ-smooth. Then there exists a unique rigid-analytic function*

$$\mathcal{L}_p : V \times \mathscr{X}(\mathrm{Cl}_K(p^\infty)) \longrightarrow \mathbb{C}_p$$

such that, for any classical point $y \in V$ and any Hecke character φ of K of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $0 \leq (q,r) \leq (k,k)$, we have

$$\mathcal{L}_p(y, \varphi_{p-\mathrm{fin}}) = \begin{cases} c_y\left(\prod_{\mathfrak{p}|p} Z_\mathfrak{p}(\varphi)\right) A(\mathcal{F}_y,\varphi)\cdot\Lambda(\mathcal{F}_y,\varphi\phi) & : y \textit{ is non-critical} \\ 0 & : y \textit{ is critical} \end{cases},$$

where $\mathcal{F}_y$ is the base-change of the classical modular form corresponding to y, $\varphi_{p-\mathrm{fin}}$ is the p-adic avatar of φ, $Z_\mathfrak{p}(\varphi)$ is an exceptional factor, $A(\mathcal{F}_y,\varphi)$ is an explicit non-zero scalar, and $\Lambda(\mathcal{F}_y,*)$ is the (completed) L-function of $\mathcal{F}_y$, all of which are defined in §2.7.

Note that we make no assumption on the splitting behaviour of p in K or the slope of f. The proof of this corollary can be deduced after taking the Amice transform (see [BH17, §5.1]) of the distribution in Corollary 1.4; the interpolation property then follows from Theorem 2.14, Proposition 6.7 and Corollaries 6.9 and 8.12. This interpolation property ensures that the specialisations of $\mathcal{L}_p$ at a Zariski-dense set of classical non-critical points in V are determined uniquely by a growth condition, which gives the required uniqueness property for $\mathcal{L}_p$.

1.1.3. The Σ-smoothness condition

We conjecture that the Σ-smoothness condition always holds. In particular, it would follow from the natural generalisation of the conjecture of Calegari and Mazur, in [CM09], mentioned above. In the non-ordinary case, there are no CM families, so the analogous statement becomes the following, which we make more precise in the sequel (Conjecture 5.14).

Conjecture 1.6. *The only non-ordinary families of Bianchi modular forms which admit a Zariski-dense set of classical points arise from base-change.*

Combined with our later results – namely, a smoothness result in the 'base-change eigenvariety' – this is enough to prove that classical base-change points are Σ-smooth.

We can prove slightly weaker analogues of the results above without assuming this, however. Suppose $\mathcal{F}$ is in case (BC) but is *not* Σ-smooth. By making a non-canonical choice, we construct an analogue of the function $\mathcal{L}_p$ on $V \times \mathscr{X}(\mathrm{Cl}_K(p^\infty))$ in Corollary 1.5 even in this case, and show that it satisfies the same interpolation property at all *non-critical* points. In this case, we *define* the p-adic L-function of $\mathcal{F}$ to be the specialisation of $\mathcal{L}_p$ at x. Since the interpolation at non-critical points gives good control on $\mathcal{L}_p$, we show that for all possible choices made, the resulting distributions in $\mathcal{D}(\mathrm{Cl}_K(p^\infty), L)$ lie in (at most) a one-dimensional L-vector space, showing that as usual $L_p(\mathcal{F})$ is well-defined up to scalar multiple. By general results of Stevens, it has the expected growth property, and by construction, it varies in a canonical three-variable p-adic L-function. We also prove a partial interpolation result (see below).

1.1.4. P-adic Artin formalism

Our third main result uses the three-variable p-adic L-function to prove a p-adic analogue of Artin formalism for complex L-functions. Let f be as above, let $\mathcal{F}$ denote its base-change to K, and let $\chi_{K/\mathbb{Q}}$ be the quadratic character associated to K. We do not assume $\mathcal{F}$ is Σ-smooth. Artin formalism says that $L(\mathcal{F}, s) = L(f, s)L(f, \chi_{K/\mathbb{Q}}, s)$. Now let $\mathrm{Cl}_{\mathbb{Q}}^+(p^\infty) \cong \mathbb{Z}_p^\times$ be the narrow ray class group at (p^∞) over $\mathbb{Q}$, and let $L_p(f)$ and $L_p^{\chi_{K/\mathbb{Q}}}(f)$ be the p-adic L-functions attached to f and its quadratic twist by $\chi_{K/\mathbb{Q}}$ respectively, which are both distributions on $\mathrm{Cl}_{\mathbb{Q}}^+(p^\infty)$. We denote by $L_p^{\mathrm{cyc}}(\mathcal{F})$ the restriction of $L_p(\mathcal{F})$ to the cyclotomic line, which is again a distribution on $\mathrm{Cl}_{\mathbb{Q}}^+(p^\infty)$. Using the three-variable p-adic L-function we obtain the following p-adic Artin formalism result.

Theorem 1.7. *Suppose $L_p^{\mathrm{cyc}}(\mathcal{F})$ and $L_p(f)L_p^{\chi_{K/\mathbb{Q}}}(f)$ are both non-zero. Then we have $L_p^{\mathrm{cyc}}(\mathcal{F}) = L_p(f)L_p^{\chi_{K/\mathbb{Q}}}(f)$ as distributions on $\mathrm{Cl}_{\mathbb{Q}}^+(p^\infty)$.*

We remark that the non-vanishing condition is automatically satisfied when f and $\mathcal{F}$ are non-critical. A conjecture of Greenberg, which says that all critical elliptic modular forms are CM, would imply that $L_p(f)$ and $L_p^{\chi_{K/\mathbb{Q}}}(f)$ are always non-zero by work of Bellaïche (see [Bel]), and we conjecture that $L_p^{\mathrm{cyc}}(\mathcal{F})$ is similarly never zero.

A case of particular interest where this theorem applies is the following. Let $E/\mathbb{Q}$ be an elliptic curve with good supersingular reduction at p, let f be the corresponding weight 2 classical modular form, and let f_α denote a p-stabilisation of f. Suppose p splits in K. Then the base-change $\mathcal{F}_\alpha$ has slope 1/2 at each of the primes above p. Since the L-function of $\mathcal{F}_\alpha$ corresponds to a p-depleted L-function for E/K, in this case we get a factorisation formula

$$L_{p,\alpha}^{\mathrm{cyc}}(E/K, *) = L_{p,\alpha}(E/\mathbb{Q}, *)L_{p,\alpha}(E^{\chi_{K/\mathbb{Q}}}/\mathbb{Q}, *)$$

of the p-adic L-function of E/K in terms of the p-adic L-functions of E and its quadratic twist by $\chi_{K/\mathbb{Q}}$. In the ordinary case, such a factorisation plays a role in Skinner and Urban's proof of the Iwasawa main conjecture (see [SU14]). We hope our results can have applications to more general cases of the conjecture.

Another interesting consequence of this result comes when $\mathcal{F}$ is not Σ-smooth. In this case, the theorem allows us to prove an interpolation property for $L_p(\mathcal{F})$ at critical Hecke characters of K which factor through the norm to $\mathbb{Q}$. We are, however, unable to prove the interpolation property at more general characters.

1.2. Other results

Anticyclotomic p-adic Artin formalism: Modulo the existence of anticyclotomic p-adic L-functions in families, the same methods can be applied to the restriction to the anticyclotomic line as well. In this case, under the same non-vanishing hypothesis, the result we obtain is that $L_p^{\mathrm{anti}}(\mathcal{F}) = L_p^{\mathrm{anti}}(f)^2$ (where the anticyclotomic p-adic L-function exists). We leave the details in this case to the interested reader. Note that anticyclotomic p-adic L-functions do not yet exist in the case where f is critical. The above suggests that a good candidate for (the square of) an anticyclotomic p-adic L-function in this case is the restriction to the anticyclotomic line of the p-adic L-function attached to $\mathcal{F}$ in this paper.

Base-change respecting criticality: The work that goes into proving the above allows two further results as soft consequences. Let f be a decent classical cusp form without CM by K, let $\mathcal{F}$ be its base-change to K, and assume $\mathcal{F}$ is Σ-smooth. Our results on the eigenvariety give Corollaries 6.9 and 8.12, which say that $\mathcal{F}$ is itself critical if and only if f is. This is not *a priori* obvious (for our definition) without assuming further conjectures on equivalent definitions of critical points.

Secondary critical p-adic L-functions: Finally, following Bellaïche, we also construct *secondary p-adic L-functions* for such forms that non-trivially interpolate the classical special L-values. When $L_p(f)$ and $L_p(\mathcal{F})$ are both non-zero, we explicitly relate the secondary p-adic L-functions of $\mathcal{F}$ to those of f constructed by Bellaïche.

1.3. Proofs: local structure of the eigenvariety

The proofs of all of the above results rest on the local properties of the Bianchi eigenvariety. We recall Hansen's construction of the local pieces. For a (two-dimensional) affinoid $\Omega \subset \mathcal{W}_K$, we define $\mathcal{D}_\Omega$ to be the space of $\mathcal{O}(\Omega)$-valued locally analytic distributions on $\mathcal{O}_K \otimes \mathbb{Z}_p$. We say Ω is *h-slope adapted* if the corresponding cohomology groups $\mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)$ admit slope $\leq h$ decompositions. For such a pair (Ω, h), we define $\mathbb{T}_{\Omega,h}$ to be the image of the Hecke operators in $\mathrm{End}_{\mathcal{O}(\Omega)}(\mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega))$. The local piece of the eigenvariety is then the rigid space $\mathcal{E}_{\Omega,h} := \mathrm{Sp}\,\mathbb{T}_{\Omega,h}$, together with a weight map induced from the algebra map $\mathcal{O}(\Omega) \to \mathbb{T}_{\Omega,h}$. Such local pieces can be patched into the global eigenvariety $\mathcal{E}$.

The construction uses the total cohomology. Since the construction of p-adic L-functions in this setting uses $\mathrm{H}^1_{\mathrm{c}}$, it is important to pin down families in the first degree. As mentioned above, this is complicated by the fact that a classical Bianchi cusp form $\mathcal{F}$ contributes to the cohomology in two degrees, namely 1 and 2. An argument due to Hansen, using the spectral sequences introduced in [Han17], shows that in fact the only degree for which the system of eigenvalues attached to our classical Bianchi cusp form $\mathcal{F}$ arises in $\mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}$ is $i = 2$. We show, however, that if $\Sigma \subset \Omega$ is a one-dimensional affinoid over which $\mathcal{F}$ varies in a family[6], then $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}$ is non-zero at $\mathcal{F}$. We define $\mathbb{T}_{\Sigma,h}$ to be the image of the Hecke operators in $\mathrm{End}_{\mathcal{O}(\Sigma)}(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h})$. Note that now we restrict to the degree one cohomology. We show that there is a maximal ideal $\mathfrak{m}_{\mathcal{F}} \subset \mathbb{T}_{\Sigma,h}$ corresponding to $x_{\mathcal{F}}$, and localising the $\mathbb{T}_{\Sigma,h}$-module $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}$ at this ideal gives the generalised eigenspace at $\mathcal{F}$. Writing $\mathfrak{m}_\lambda \subset \mathcal{O}(\Sigma)$ for the corresponding maximal ideal for the weight space, we then prove:

Theorem 1.8. *If $\mathcal{F}$ is critical, assume it is Σ-smooth. The module $\mathrm{H}^1_c(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_{\mathcal{F}}}$ is free of rank one over $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_{\mathcal{F}}}$, which (possibly after a finite extension of the base field) is itself free of finite rank e over $\mathcal{O}(\Sigma)_{\mathfrak{m}_\lambda}$. If $\mathcal{F}$ is non-critical, then $e = 1$.*

[6] That is, there is a connected component of $\mathcal{E}_{\Omega,h}$ containing $x_{\mathcal{F}}$ and mapping to Σ under the weight map.

We first prove this theorem in case (NC). The first important step in the proof, showing that the space is non-zero, was mentioned above. By assumption we have a multiplicity one condition for the generalised eigenspace of $\mathcal{F}$ in classical cohomology, and then the non-criticality condition ensures that the space $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}_{\mathfrak{m}_\mathcal{F}}$ is itself one-dimensional. Two applications of Nakayama's lemma allow us to prove that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_\mathcal{F}}$ is generated by one element. We conclude by using the concrete description of $\mathrm{H}^1_{\mathrm{c}}$ as modular symbols to prove the space is torsion-free, hence free of rank one. In case (NC), after possibly shrinking Σ, this theorem allows the construction of the canonical class Φ_V and the three-variable p-adic L-function.

In case (BC), the proofs are more involved. We introduce the *parallel weight eigenvariety* $\mathcal{E}_{\mathrm{par}}$, built out of only the groups $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)$ for Σ a subset of the one-dimensional parallel weight line in $\mathcal{W}_K$. We prove that this eigenvariety has a Zariski-dense set of classical points and contains the image of the p-adic base-change functoriality map. Let $\mathcal{E}_{\mathrm{bc}}$ denote this image as a closed subspace of $\mathcal{E}_{\mathrm{par}}$. The key geometric input is:

Proposition 1.9. *Let $\mathcal{F}$ be in case (BC). Then $\mathcal{E}_{\mathrm{bc}}$ is smooth and reduced at $x_\mathcal{F}$.*

We prove this using deformation theory. In particular, in an appendix to this paper, Carl Wang-Erickson develops a precise 'base-change' deformation condition that makes this argument possible. It is important to note that these methods work in this case (but not, at present, in the general Bianchi setting) as we can 'port' the necessary properties of $\mathcal{E}_{\mathrm{bc}}$ – notably, a Zariski-dense set of crystalline points – from the analogous properties for the Coleman–Mazur eigencurve. If we have Σ-smoothness, $x_\mathcal{F}$ is also smooth and reduced in $\mathcal{E}_{\mathrm{par}}$. Using a strategy of Bellaïche from [Bel12], this is enough to show Theorem 1.8 in case (BC), assuming Σ-smoothness, and ultimately to prove Theorems 1.2 and 1.3 in this case too.

When we do *not* have Σ-smoothness, we at least still have smoothness in $\mathcal{E}_{\mathrm{bc}}$. This is enough to prove a non-canonical analogue of Theorem 1.8 in this case, giving the partial results stated above at such points.

Finally, we turn to the proof of p-adic Artin formalism (Theorem 1.7). For sufficiently small slope, the product $L_p(f)L_p^{\chi_{K/\mathbb{Q}}}(f)$ is uniquely determined by its critical values, and thus we show that – after normalising the periods appropriately – the result follows from classical Artin formalism. In the general finite slope case, we can take a neighbourhood $V_\mathbb{Q}$ of f in the Coleman–Mazur eigencurve, and attach two-variable p-adic L-functions $\mathcal{L}_p(V_\mathbb{Q}), \mathcal{L}_p^{\chi_{K/\mathbb{Q}}}(V_\mathbb{Q})$ to f over V. Let V_K be the image of $V_\mathbb{Q}$ in the Bianchi eigenvariety under the base-change map; we can restrict the resulting three-variable p-adic L-function of Corollary 1.5 to a two-variable p-adic L-function $\mathcal{L}_p^{\mathrm{cyc}}(V_K)$ over the cyclotomic line. The slope of such a family is constant, so factorisation holds (up to scalars) at a Zariski-dense set of points. We can control the scalars under the non-vanishing hypothesis of Theorem 1.7.

Proposition 1.10. *Suppose $L_p(f)L_p^{\chi_{K/\mathbb{Q}}}(f)$ and $L_p^{\mathrm{cyc}}(\mathcal{F})$ are both non-zero. After possibly shrinking $V_\mathbb{Q}$ and V_K, there is a factorisation (of two-variable p-adic L-functions)*

$$\mathcal{L}_p^{\mathrm{cyc}}(V_K) = \mathcal{L}_p(V_\mathbb{Q})\mathcal{L}_p^{\chi_{K/\mathbb{Q}}}(V_\mathbb{Q}),$$

the equality up to multiplication by an element of $\mathcal{O}(V_\mathbb{Q})^\times$.

(Note that this indeterminacy is expected, since the two-variable p-adic L-functions in question are themselves only well-defined up to scalar multiplication by $\mathcal{O}(V_\mathbb{Q})^\times$). To prove Theorem 1.7, we now specialise to f. Similarly, if a two-variable anticyclotomic p-adic L-function exists in Coleman families, under the same hypotheses the same arguments show that necessarily we have $\mathcal{L}_p^{\mathrm{anti}}(V_K) = \mathcal{L}_p^{\mathrm{anti}}(V_\mathbb{Q})^2$, where we have restricted the three-variable base-change p-adic L-function to the anticyclotomic line.

1.4. Further remarks

For immediate applications, Theorem 1.3 and Corollary 1.4 are of most interest when Σ is contained in the parallel weight line in $\mathcal{W}_K$. In this case, classical non-critical points are Zariski-dense in V, and we have shown that the p-adic L-functions of such forms vary analytically over neighbourhoods in the eigenvariety. When Σ is *not* contained in this line, then it is possible that $x_{\mathcal{F}}$ is the *only* classical point in V. In this case, we have shown instead that the overconvergent eigensymbol $\Psi_{\mathcal{F}}$ varies in a family of eigensymbols parametrised by points of V, in the sense that at any point $y \in V(L)$ with $w(y) = \kappa$, the class

$$\mathrm{sp}_y(\Phi_V) =: \Psi_y \in \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\kappa(L))^{\leq h}$$

is an eigensymbol for the Hecke operators. By analogy with classical points, one might define the p-adic L-function $L_p(\Psi_y, *)$ of this symbol to be the Mellin transform of Ψ_y, which obviously agrees with the previous construction when Ψ_y is associated to a classical form. Outside of classical points, these 'p-adic L-functions' have no obvious link to L-values, but the above does show that they vary in families over the eigencurve.

The following is a possible arithmetic application of these non-parallel families. Suppose $\mathcal{F}$ is new at a prime $\mathfrak{p}$ of K above p. In [BSW17] we showed the existence of an $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}$ attached to $\mathcal{F}$, depending only on $\mathfrak{p}$, arising from *exceptional zeros* of the p-adic L-function of $\mathcal{F}$. The Hecke eigenvalue of Ψ_V at $\mathfrak{p}$ is an analytic function $a_{\mathfrak{p}}$ on V. If V is smooth at $x_{\mathcal{F}}$, or equivalently if Σ is smooth at $w(x_{\mathcal{F}})$, then one can differentiate $a_{\mathfrak{p}}$ (along the curve V) and evaluate at $x_{\mathcal{F}}$. We expect that, at least in the ordinary case, this gives the Benois–Greenberg $\mathcal{L}$-invariant of $\mathcal{F}$ at $\mathfrak{p}$ (see [Ben11]). Using methods introduced by Greenberg–Stevens in [GS93] (see also [BSDJ17]), one should be able to show that the Benois–Greenberg $\mathcal{L}$-invariant is equal to $\mathcal{L}_{\mathfrak{p}}$.

1.5. Structure of the paper

In §2, we recall some aspects of the construction of p-adic L-functions attached to non-critical Bianchi cusp forms. In §3, we introduce the Bianchi eigenvariety $\mathcal{E}$ constructed by Hansen, and state some of its basic properties, including its dimension and the base-change map. In §4, we prove Theorem 1.8 in the case (NC). In §5, we introduce the parallel weight eigenvariety and its properties, proving smoothness of the base-change eigenvariety at classical points. In §6, we use the parallel weight eigenvariety to complete the proof of Theorem 1.8 and to prove Theorem 1.2. In §7, we construct families of p-adic L-functions, proving Theorem 1.3. Finally, §8 is devoted to proving Theorem 1.7 and developing the phenomenon of secondary p-adic L-functions for critical base-change Bianchi cusp forms. An appendix, by Carl Wang-Erickson, provides the technical footing for the deformation theory arguments in §5.

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2. The p-adic L-function of a Bianchi modular form

In this section, we fix notation and briefly recap the results of [Wil17], which will be used heavily in the sequel. Fix, once and for all, embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ for each prime ℓ.

2.1. Basic notation

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, different $\mathfrak{d}$ and discriminant $-d$. Let p be a rational prime.

Denote the adele ring of K by $\mathbb{A}_K = \mathbb{C} \times \mathbb{A}_K^f$, where $\mathbb{A}_K^f$ denotes the finite adeles. Throughout, we work at level $\mathfrak{n} \subset \mathcal{O}_K$ divisible by each prime of K above p. For an ideal $\mathfrak{f} \subset \mathcal{O}_K$, let $\mathrm{Cl}_K(\mathfrak{f})$ denote the ray class group of K modulo $\mathfrak{f}$. We write $U_1(\mathfrak{n})$ for the standard open compact subgroup of $\mathrm{GL}_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$ of matrices congruent to $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo $\mathfrak{n}$, and $\mathcal{K}_\infty = \mathrm{SU}_2(\mathbb{C})\mathbb{C}^\times$, and we define the associated locally symmetric space by

$$Y_1(\mathfrak{n}) := \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}_K)/\mathcal{K}_\infty U_1(\mathfrak{n}).$$

We define $\mathcal{H}_3 := \mathbb{C} \times \mathbb{R}_{>0}$; the space $Y_1(\mathfrak{n})$ can be written as a finite disjoint union of quotients of $\mathcal{H}_3$. Let $j \geq 0$ be an integer, and for any ring R, let $V_j(R)$ denote the ring of polynomials over R of degree at most j. Throughout, we will denote modules of rigid analytic distributions by $\mathcal{D}^0(*)$ and locally analytic distributions by $\mathcal{D}(*)$. These spaces will be equipped with a group action, and the corresponding local systems on $Y_1(\mathfrak{n})$ will be denoted by $\mathscr{D}^0(*)$ and $\mathscr{D}(*)$ respectively.

We will typically reserve f to mean a classical modular form and $\mathcal{F}$ a Bianchi modular form when we work at a fixed weight. If V is an affinoid in a rigid space, we will write $\mathcal{O}(V)$ for the ring of rigid functions on V, so that $V = \mathrm{Sp}(\mathcal{O}(V))$. We will decorate V with a subscript $\mathbb{Q}$ or K to clarify that we are working with spaces defined for GL_2 over $\mathbb{Q}$ and K respectively, unless this is clear from context. If y is a classical point in an eigenvariety, we will write f_y or $\mathcal{F}_y$ for the corresponding modular form (if it is classical or Bianchi respectively).

2.2. Bianchi modular forms and L-functions

A *Bianchi modular form* is an automorphic form for GL_2 over an imaginary quadratic field. The conventions we follow are those of see [Wil17, §1], and we refer the reader there for the precise definitions.

Let $\lambda = (\mathbf{k}, \mathbf{v})$ be a weight, where $\mathbf{k} = (k_1, k_2)$ and $\mathbf{v} = (v_1, v_2)$ are two elements of $\mathbb{Z}[\Sigma_K]$. There is a finite-dimensional $\mathbb{C}$-vector space $S_\lambda(U_1(\mathfrak{n}))$ of *Bianchi cusp forms* of weight λ and level $U_1(\mathfrak{n})$, which are functions

$$\mathcal{F} : \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}_K)/U_1(\mathfrak{n}) \longrightarrow V_{2k+2}(\mathbb{C})$$

transforming appropriately under the subgroup $\mathcal{K}_\infty$, and satisfying suitable harmonicity and growth conditions.

Remark 2.1: If $k_1 \neq k_2$, then $S_\lambda(U_1(\mathfrak{n})) = 0$ (see [Har87]), so henceforth when talking about classical cusp forms we will set $k_1 = k_2 = k$. In this case, we can always twist the central character by a power of the norm to assume that $v_1 = v_2 = 0$ as well. For the rest of this section, we fix $\lambda = [(k, k), (0, 0)]$, and we will write this as $\lambda = (k, k)$ without further comment.

There is a good theory of Hecke operators (indexed by ideals of $\mathcal{O}_K$) on Bianchi modular forms. Let $\mathcal{F}$ be a cuspidal Bianchi modular form that is an eigenform for all of the Hecke operators, and for any non-zero ideal $I \subset \mathcal{O}_K$, write $\mathcal{F}|T_I = \alpha_I f$.

Definition 2.2. Let Λ denote the (completed) L-function of $\mathcal{F}$, normalised so that if φ is a Hecke character of infinity type (q, r), where $q, r \gg 0$, then

$$\Lambda(\mathcal{F}, \varphi) = \frac{\Gamma(q+1)\Gamma(r+1)}{(2\pi i)^{q+r+2}} \sum_{I \subset \mathcal{O}_K, I \neq 0} \alpha_I \varphi(I) N(I)^{-1}.$$

This admits an analytic continuation to all such characters.

The 'critical' values of this L-function can be controlled; in particular, by [Hid94, Thm. 8.1], we see that there exists a period $\Omega_{\mathcal{F}} \in \mathbb{C}^\times$ and a number field E such that, if φ is a Hecke character of infinity type $0 \leq (q,r) \leq (k,k)$, with $q, r \in \mathbb{Z}$, we have

$$\frac{\Lambda(\mathcal{F},\varphi)}{\Omega_{\mathcal{F}}} \in E(\varphi), \tag{2.1}$$

where $E(\varphi)$ is the number field over E generated by the values of φ.

2.3. The cohomology class attached to $\mathcal{F}$

The Bianchi modular forms we consider in this paper are *cohomological* in the following sense.

Definition 2.3. For a ring R, let $V_\lambda = V_{k,k}(R) \coloneqq V_k(R) \otimes_R V_k(R)$. (We think of V_λ as polynomials on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that have degree at most k in each variable). This space has a natural left action of $\mathrm{GL}_2(R)^2$ induced by the action of $\mathrm{GL}_2(R)$ on each factor by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(z) = (a+cz)^k P\left(\frac{b+dz}{a+cz}\right),$$

inducing a right action on the dual space $V_\lambda(R)^* \coloneqq \mathrm{Hom}(V_\lambda(R), R)$. When R is an K-algebra, this then gives rise to a local system, which we denote by $\mathscr{V}_\lambda(R)^*$, on the locally symmetric space $Y_1(\mathfrak{n})$.

Theorem 2.4. *There is an isomorphism*

$$S_\lambda(U_1(\mathfrak{n})) \cong \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{V}_\lambda(\mathbb{C})^*)$$

that is equivariant with respect to the Hecke operators. Let $\mathcal{F} \in S_\lambda(U_1(\mathfrak{n}))$ be a newform or the p-stabilisation of a newform which has distinct Hecke eigenvalues at each prime above p. Then the generalised eigenspace $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{V}_\lambda(\mathbb{C})^*)_{(\mathcal{F})}$ for the Hecke operators is one dimensional, and $\phi_{\mathcal{F}}/\Omega_{\mathcal{F}}$ has coefficients in $\mathscr{V}_\lambda(E)^*$, where $\Omega_{\mathcal{F}} \in \mathbb{C}^\times$ and E are as in equation (2.1.)

Proof. See [Hid94, §3] for the isomorphism, which was initially due to Eichler–Shimura–Harder, and [Hid94, §8] for the dimension result and algebraicity. ☐

2.4. Overconvergent cohomology

Throughout the following, R will denote an $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-algebra, whilst L will be a finite extension of $\mathbb{Q}_p$. We also assume that the level $\mathfrak{n}$ is divisible by every prime of K above p.

Definition 2.5. Let $\mathcal{A}(R)$ (resp. $\mathcal{A}^0(R)$) denote the space of locally analytic (resp. rigid analytic) functions $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to R$. When $R = L$ is a finite extension of $\mathbb{Q}_p$, we equip these spaces with a weight λ-action of the semigroup

$$\Sigma_0(p) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) : v_{\mathfrak{p}}(c) > 0 \ \forall \mathfrak{p}|p, a \in (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times, ad - bc \neq 0 \right\}$$

by setting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \zeta(z) = (a+cz)^k \zeta\left(\frac{b+dz}{a+cz}\right)$. Since $\mathfrak{n}$ is divisible by each prime above p, this gives an action of $U_1(\mathfrak{n})$ by projection to the components at p.

Definition 2.6. Let $\mathcal{D}(R) \coloneqq \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}(R), R)$ denote the space of R-valued locally analytic distributions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. When $R = L$ as above, we write $\mathcal{D}_\lambda(L)$ for this space equipped with the weight λ right action of $\Sigma_0(p)$ given by $\mu|\gamma(\zeta) = \mu(\gamma \cdot \zeta)$. The spaces $\mathcal{D}^0(R)$ and $\mathcal{D}^0_\lambda(L)$ of rigid analytic distributions are defined similarly to be the continuous duals of $\mathcal{A}^0(R)$ and $\mathcal{A}^0(L)$, the latter with the dual weight λ action of $\Sigma_0(p)$. Both $\mathcal{D}_\lambda(L)$ and $\mathcal{D}^0_\lambda(L)$ give rise to local systems on $Y_1(\mathfrak{n})$ via their actions of $\Sigma_0(p)$, which we denote by $\mathscr{D}_\lambda(L)$ and $\mathscr{D}^0_\lambda(L)$ respectively. Define the *overconvergent cohomology of weight λ and level $U_1(\mathfrak{n})$* to be the cohomology group $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$.

There is a natural map $\mathcal{D}_\lambda(L) \to V_\lambda(L)^*$ given by dualising the inclusion of $V_\lambda(L)$ into $\mathcal{A}(L)$. This induces a *specialisation map*

$$\rho_\lambda : \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L)) \longrightarrow \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{V}_\lambda(L)^*).$$

Definition 2.7. Let $\mathcal{F} \in S_\lambda(U_1(\mathfrak{n}))$ be an eigenform. We say $\mathcal{F}$ is *non-critical* if ρ_λ becomes an isomorphism upon restriction to the generalised eigenspaces of the Hecke operators at $\mathcal{F}$.

The following gives a large supply of non-critical forms (see [BSW16, Theorem 8.7]). First, we need a further definition.

Definition 2.8. There are natural valuations on K, considered as a subset of L, corresponding to the primes above p; if p is inert or ramified, then the valuation at $\mathfrak{p}|p$ is simply the p-adic valuation on L, normalised so that $v(p) = 1$. If p is split, then note our fixed choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ singles out a choice of prime $\mathfrak{p}|p$, and we denote the other prime by $\overline{\mathfrak{p}}$. Given $\alpha \in K$, the valuation at $\mathfrak{p}$ is $v_{\mathfrak{p}}(\alpha) := v_p(\alpha)$ (the usual p-adic valuation), and the valuation at $\overline{\mathfrak{p}}$ is $v_{\overline{\mathfrak{p}}}(\alpha) := v_p(\alpha^c)$, where c is conjugation.

Theorem 2.9 (Control theorem)**.** *For each prime* $\mathfrak{p}$ *above* p*, let* $\alpha_{\mathfrak{p}} \in K^\times$*. If* $v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$ *for all* $\mathfrak{p}|p$*, then the restriction of the specialisation map*

$$\rho_\lambda : \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))^{\{U_{\mathfrak{p}}=\alpha_{\mathfrak{p}}:\mathfrak{p}|p\}} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{V}_\lambda(L)^*)^{\{U_{\mathfrak{p}}=\alpha_{\mathfrak{p}}:\mathfrak{p}|p\}}$$

to the simultaneous $\alpha_{\mathfrak{p}}$*-eigenspaces of the* $U_{\mathfrak{p}}$ *operators is an isomorphism. Here recall that* $e_{\mathfrak{p}}$ *is the ramification index of* $\mathfrak{p}|p$*.*

Definition 2.10. If $\mathcal{F} \in S_\lambda(U_1(\mathfrak{n}))$ is an eigenform with eigenvalues α_I, we say $\mathcal{F}$ has *small slope* if $v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$ for all $\mathfrak{p}|p$. (Such a form is non-critical by the control theorem).

Thus to a non-critical form $\mathcal{F}$, we can associate a class $\Psi_{\mathcal{F}} \in \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$ by lifting the corresponding classical class.

2.5. Bianchi modular symbols

Whilst the definitions and results above go through for more general number fields (see, for example, [BSW16]), in the Bianchi setting the first degree compactly supported cohomology admits a considerably more explicit definition in terms of modular symbols. To describe this, let $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(K))$ denote the space of 'paths between cusps' in $\mathcal{H}_3$, and let V be any right $\mathrm{SL}_2(K)$-module. For a subgroup $\Gamma \subset \mathrm{SL}_2(K)$, define the space of *$V$-valued modular symbols for* Γ to be the space

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V)$$

of functions satisfying the Γ-invariance property that

$$(\phi|\gamma)(\delta) := \phi(\gamma\delta)|\gamma = \phi(\delta) \quad \forall \delta \in \Delta_0, \gamma \in \Gamma,$$

where Γ acts on the cusps by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot r = (ar+b)/(cr+d)$.

Now let Γ be a discrete subgroup of $\mathrm{GL}_2(K)$. This Γ also acts naturally on $\mathcal{H}_3$, and we have (see [BSW17, Prop. 8.2]):

Proposition 2.11. *Let* V *be a right* Γ*-module, giving rise to a local system* $\mathscr{V}$ *on* $\Gamma\backslash\mathcal{H}_3$*. There is an isomorphism* $\mathrm{H}^1_{\mathrm{c}}(\Gamma\backslash\mathcal{H}_3, \mathscr{V}) \cong \mathrm{Symb}_\Gamma(V)$*.*

Via strong approximation, the locally symmetric space $Y_1(\mathfrak{n})$ decomposes as a disjoint union of spaces $\Gamma_i\backslash\mathcal{H}_3$, for $\Gamma_i \subset \mathrm{GL}_2(K)$ discrete subgroups indexed by Cl_K (for further details, see [BSW16, §3.2.2]). This induces a decomposition

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{V}) \cong \bigoplus_{i \in \mathrm{Cl}_K} \mathrm{Symb}_{\Gamma_i}(V),$$

using Proposition 2.11. This decomposition is non-canonical, depending on the choice of class group representatives. When V is $V_\lambda(R)^*$ or $\mathcal{D}_\lambda(L)$, there is a natural Hecke action on the direct sum, and the isomorphism is equivariant with respect to this action.

2.6. Ray class groups, distributions and Mellin transforms

In this section we explain how to canonically associate a ray class distribution to an overconvergent cohomology class in $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\lambda)$. The class group in question is the ray class group of K of level p^∞, defined by

$$\mathrm{Cl}_K(p^\infty) \coloneqq K^\times\backslash\mathbb{A}_K^\times/\mathbb{C}^\times\prod_{v\nmid p}\mathcal{O}_v^\times = \bigsqcup_{i\in\mathrm{Cl}_K}(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times/\mathcal{O}_K^\times,$$

where the decomposition is non-canonical, depending once more on our choice of class group representatives.

Let R be an $(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{Z}_p)$-algebra such that $\mathcal{D}(R)$ carries a right action of $U_1(\mathfrak{n})$, hence giving rise to a local system on $Y_1(\mathfrak{n})$.

Definition 2.12. Let $\Psi\in\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}(R))$, and write

$$\Psi = (\Psi^1,...,\Psi^h)\in\bigoplus_{i\in\mathrm{Cl}_K}\mathrm{Symb}_{\Gamma_i}(\mathcal{D}(R))\cong\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}(R)).$$

Define, for each i, a distribution

$$\mu_i(\Psi)\coloneqq\Psi^i(\{0\}-\{\infty\})|_{(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times}$$

on $(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{Z}_p)^\times$, which also gives rise to a distribution on the quotient of this space by $\mathcal{O}_K^\times$. Then define the *Mellin transform* of Ψ to be the (R-valued) locally analytic distribution on $\mathrm{Cl}_K(p^\infty)$ given by

$$\mathrm{Mel}(\Psi)\coloneqq\sum_{i=1}^h\mu_i(\Psi)\mathbf{1}_i\in\mathcal{D}(\mathrm{Cl}_K(p^\infty),R),$$

where here $\mathbf{1}_i$ is the indicator function for the component of $\mathrm{Cl}_K(p^\infty)$ corresponding to I_i. A simple check identical to the arguments given in [BSW16, Prop. 9.7] shows that the distribution $\mathrm{Mel}(\Psi)$ is independent of the choice of class group representatives.

2.7. The p-adic L-function of a Bianchi modular form

Let $\mathcal{F}$ be a non-critical Bianchi modular form of level $U_1(\mathfrak{n})$ and weight $\lambda=(k,k)$, where $(p)|\mathfrak{n}$, and let $\Psi_{\mathcal{F}}\in\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\lambda(L))^{\leq h}$ be the associated overconvergent class.

Definition 2.13. Define the *p-adic L-function of* $\mathcal{F}$ to be the Mellin transform $L_p(\mathcal{F},*)\coloneqq\mathrm{Mel}(\Psi_{\mathcal{F}})$.

We describe the interpolation property satisfied by $L_p(\mathcal{F},*)$. Given an algebraic Hecke character φ of K whose conductor divides (p^∞), there is a natural associated character $\varphi_{p-\mathrm{fin}}$ of $\mathrm{Cl}_K(p^\infty)$ associated to φ (see [Wil17, §7.3]). The main theorem of [Wil17] is the following (Theorem 7.4 *op. cit.*):

Theorem 2.14. *For any Hecke character* φ *of* K *of conductor* $\mathfrak{f}|(p^\infty)$ *and infinity type* $0\leq(q,r)\leq(k,k)$, *we have*

$$L_p(\mathcal{F},\varphi_{p-\mathrm{fin}}) = \left(\prod_{\mathfrak{p}|p}Z_{\mathfrak{p}}(\varphi)\right)A(\mathcal{F},\varphi)\Lambda(\mathcal{F},\varphi),$$

for

$$A(\mathcal{F},\varphi)\coloneqq\left[\frac{\varphi(x_{\mathfrak{f}})d\widetilde{\tau}(\varphi^{-1})\#\mathcal{O}_K^\times}{(-1)^{k+q+r}2\varphi_{\mathfrak{f}}(x_{\mathfrak{f}})\alpha_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right],$$

where $x_{\mathfrak{f}}$ is an explicit idele representing $\mathfrak{f}$, $\varphi_{\mathfrak{f}}$ is the restriction of φ to $\prod_{v|\mathfrak{f}} K_v^\times$, $\widetilde{\tau}(\varphi^{-1})$ is a Gauss sum, $\alpha_{\mathfrak{f}}$ is the eigenvalue of $\mathcal{F}$ at $\mathfrak{f}$ and

$$Z_{\mathfrak{p}}(\varphi) := \begin{cases} 1 - \alpha_{\mathfrak{p}}^{-1}\psi(\mathfrak{p})^{-1} & : \mathfrak{p} \nmid \mathfrak{f}, \\ 1 & : \textit{otherwise}. \end{cases}$$

Writing $h_{\mathfrak{p}} = v_{\mathfrak{p}}(\alpha_{\mathfrak{p}})$, the distribution $L_p(\mathcal{F}, *)$ is $(h_{\mathfrak{p}})_{\mathfrak{p}|p}$-admissible in the sense of [Wil17, Defns. 5.10,6.14]. When $\mathfrak{f}$ has small slope, this ensures it is unique with this interpolation property.

2.8. Stabilisations at p

Suppose now one starts with a form $\mathcal{F}$ of level $\mathfrak{n}$, and that $\mathfrak{n}$ is not divisible by one or more primes $\mathfrak{p}$ above p. To define a p-adic L-function, one must take $\mathfrak{p}$-stabilisations (or $\mathfrak{p}$-refinements) until all primes above p divide $\mathfrak{n}$.

Definition 2.15. Let $a_{\mathfrak{p}}(\mathcal{F})$ denote the $T_{\mathfrak{p}}$ eigenvalue of $\mathcal{F}$, and let $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ denote the roots of the Hecke polynomial $X^2 - a_{\mathfrak{p}}(\mathcal{F})X + N(\mathfrak{p})^{k+1}$. Let $\pi_{\mathfrak{p}}$ denote an idelic representative of $\mathfrak{p}$. We define the $\mathfrak{p}$-*stabilisations of* $\mathcal{F}$ to be

$$\mathcal{F}_{\alpha_{\mathfrak{p}}} := \mathcal{F}(g) - \beta_{\mathfrak{p}}\mathcal{F}\left(\left(\begin{smallmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right),$$

$$\mathcal{F}_{\beta_{\mathfrak{p}}} := \mathcal{F}(g) - \alpha_{\mathfrak{p}}\mathcal{F}\left(\left(\begin{smallmatrix} \pi_{\mathfrak{p}} & 0 \\ 0 & 1 \end{smallmatrix}\right) g\right).$$

The form $\mathcal{F}_{\alpha_{\mathfrak{p}}}$ (resp. $\mathcal{F}_{\beta_{\mathfrak{p}}}$) is an eigenform of level $U_1(\mathfrak{n}\mathfrak{p})$ with $U_{\mathfrak{p}}$-eigenvalue $\alpha_{\mathfrak{p}}$ (resp. $\beta_{\mathfrak{p}}$).

For each prime $\mathfrak{p}$ above p not dividing $\mathfrak{n}$, suppose we can choose a root $\alpha_{\mathfrak{p}}$ of the Hecke polynomial at $\mathfrak{p}$ such that $v_p(\alpha_{\mathfrak{p}}) < (k+1)/e_{\mathfrak{p}}$. Then we can attach a p-adic L-function to the form $\mathcal{F}_\alpha$ obtained by taking the $\alpha_{\mathfrak{p}}$-stabilisation at each of these primes, which thus has level divisible by each of the primes above p.

Examples: (i) Suppose p splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$. Let $\mathcal{F}$ have weight $\lambda = (k,k)$ and level $\mathfrak{N}$ prime to p with $a_{\mathfrak{p}}(\mathcal{F}) = a_{\overline{\mathfrak{p}}}(\mathcal{F}) = 0$. Then the Hecke polynomials at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ coincide, and their roots α, β both have p-adic valuation $(k+1)/2$. Assuming $\alpha \neq \beta$, there are four choices of stabilisations to level $\mathfrak{N}p$, and each is small slope, giving rise to four p-adic L-functions attached to $\mathcal{F}$. In the case where $k = 0$, these are precisely the p-adic L-functions seen in [Loe14, §5].

(ii) Suppose instead that p is inert in K and again that $a_p(\mathcal{F}) = 0$. Then both roots α, β of the Hecke polynomial have p-adic valuation $k+1$, and hence *neither* p-stabilisation has small slope, and the methods of [Wil17] do not necessarily allow the construction of a p-adic L-function attached to $\mathcal{F}$.

We introduce some notation. Forms satisfying the conditions below, and a mild additional regularity condition, satisfy a multiplicity one condition for the Hecke algebra $\mathbb{H}_{\mathfrak{n},p}$ defined in the sequel.

Definition 2.16. Let $\mathcal{F}$ be a Bianchi modular form of level $\mathfrak{n}$ divisible by each prime above p. We say $\mathcal{F}$ is a *p-stabilised newform* if there exists a subset S of primes above p and a newform $\mathcal{F}'$ of level $\mathfrak{N}$, with $\mathfrak{N}$ prime to S and $\mathfrak{n} = \mathfrak{N}\prod_{\mathfrak{p}\in S}\mathfrak{p}$, such that $\mathcal{F}$ can be obtained from $\mathcal{F}'$ by successively stabilising at each prime in S. (Note newforms of level $\mathfrak{n}$ themselves satisfy this with $S = \varnothing$).

3. The Bianchi eigenvariety

In this section, we summarise some of the relevant results from David Hansen's beautiful paper [Han17] and show that if $\mathcal{F}$ is a cuspidal Bianchi modular form that is either non-critical or base-change, then the system of eigenvalues attached to $\mathcal{F}$ varies in a one-dimensional family of overconvergent modular symbols. Hansen uses overconvergent cohomology to construct *universal eigenvarieties* in very wide generality. Whilst his results are stated mainly for singular cohomology, he also gives the tools to produce (essentially identical) proofs in the case of cohomology with compact support, which is the setting that interests us. We will use his results for compactly supported cohomology with little further comment.

The key idea in Hansen's work is the construction of spectral sequences converging to the cohomology groups $\mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}$ that moreover behave well under the action of the Hecke operators. The general theory of spectral sequences then gives a filtration on each of these groups as well as explicit descriptions of the graded pieces in terms of overconvergent cohomology in families.

3.1. Distributions over the weight space

3.1.1. Bianchi weight space and null weights

Definition 3.1. Let $T(\mathbb{Z}_p) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right) : a, d \in (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times\}$. Define the *(full) Bianchi weight space* to be the rigid analytic space whose L-points, for $L \subset \mathbb{C}_p$ any sufficiently large extension of $\mathbb{Q}_p$, are given by

$$\mathscr{W}_K(L) = \mathrm{Hom}_{\mathrm{cts}}(T(\mathbb{Z}_p), L^\times).$$

We will typically restrict to a smaller space of weights. In particular, we can 'twist away' some of this space; for any $\lambda \in \mathscr{W}_K(L)$, write $\lambda = \mathrm{diag}(\lambda_1, \lambda_2)$, where each λ_i is a character $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to L^\times$. Then

$$\lambda = \mathrm{diag}(\lambda_1\lambda_2^{-1}, 1) \cdot (\lambda_2 \circ \det).$$

Variation in this determinant factor is well-understood in the sense that any Bianchi modular form varies in a family in this direction via twisting. As such, we will, without loss of generality, henceforth assume that λ_2 is trivial and focus only on the smaller space cut out by this condition. We make one further restriction; since we only care about weights which give rise to non-trivial local systems on $Y_1(\mathfrak{n})$, we demand that our weights are trivial on the subgroup

$$E(\mathfrak{n}) := K^\times \cap U_1(\mathfrak{n}) = \{\epsilon \in \mathcal{O}_K^\times : \epsilon \equiv 1 \ (\mathrm{mod}\ \mathfrak{n})\}.$$

With this in mind, we define:

Definition 3.2. Define the *(null) Bianchi weight space of level* $\mathfrak{n}$ to be the rigid analytic space whose L-points, for $L \subset \mathbb{C}_p$ any sufficiently large extension of $\mathbb{Q}_p$, are given by

$$\mathcal{W}_{K,\mathfrak{n}}(L) = \mathrm{Hom}_{\mathrm{cts}}((\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / E(\mathfrak{n}), L^\times).$$

Since the level will typically be clear from context, we will usually drop the superscript $\mathfrak{n}$ from the notation.

The (null) Bianchi weight space is then a 2-dimensional space analogous to the 1-dimensional (null) weight space for $\mathrm{GL}_2/\mathbb{Q}$, whose L-points are given by $\mathcal{W}_{\mathbb{Q}}(L) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L^\times)$. By passing to this smaller space of weights, we hope that later comparison with the Coleman–Mazur eigencurve is more clear (compare [Han17, §4.6]). From now on, we will refer to $\mathcal{W}_K$ simply as the Bianchi weight space (without specifying that it is null).

We say a weight $\lambda \in \mathcal{W}_K(L)$ is *classical* if it can be written in the form $\epsilon\lambda^{\mathrm{alg}}$, where ϵ is a finite order character and $\lambda^{\mathrm{alg}}(z) = z^{\mathbf{k}}$, where $\mathbf{k} = (k_1, k_2)$ is a pair of integers. Such a λ represents the weight of a Bianchi modular form of weight $(\mathbf{k}, \mathbf{0})$ with nebentypus character ϵ.

3.1.2. Distributions in families

For each weight $\lambda \in \mathcal{W}_K(L)$, one can define a weight λ action of $\Sigma_0(p)$ on the space of locally analytic functions on $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$ by

$$\gamma \cdot_\lambda f(z) = \lambda(a + cz) f\left(\frac{b + dz}{a + cz}\right),$$

and hence a dual action on $\mathcal{D}(L)$. We can vary these action in families over $\mathcal{W}_K$. In particular, let $\Omega \subset \mathcal{W}_K$ be an affinoid, giving rise to a universal character

$$\chi_\Omega : (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \longrightarrow \mathcal{O}(\Omega)^\times,$$

which has the property that for any $\lambda \in \Omega(L)$, the corresponding homomorphism $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to L^\times$ factors as

$$(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \xrightarrow{\chi_\Omega} \mathcal{O}(\Omega)^\times \xrightarrow{\text{eval. at } \lambda} L^\times.$$

We can thus equip $\mathcal{A}_\Omega := \mathcal{A}(\mathcal{O}(\Omega))$ with a 'weight Ω' action of $\Sigma_0(p)$ given by

$$\gamma \cdot_\Omega f(z) = \chi_\Omega(a + cz) f\left(\frac{b + dz}{a + cz}\right),$$

and dually we get an action on $\mathcal{D}_\Omega := \mathcal{D}(\mathcal{O}(\Omega))$.

Remark 3.3: From the remarks at the end of [Han17, §2.2], we can identify

$$\mathcal{D}_\Omega := \mathcal{D}(\mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(\Omega),$$

where $\widehat{\otimes}$ denotes completed tensor product. In this formulation, it is easier to see that if $\Sigma \subset \Omega$ is some closed subset, then we have an isomorphism $\mathcal{D}_\Omega \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Sigma) \cong \mathcal{D}_\Sigma$. In particular, if $\lambda \in \Omega(L)$ is some weight corresponding to a maximal ideal $\mathfrak{m}_\lambda \subset \mathcal{O}(\Omega)$, then $\mathcal{D}_\Omega \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega)/\mathfrak{m}_\lambda \cong \mathcal{D}_\lambda(L)$.

As $\Sigma_0(p)$-modules, these spaces give rise to associated local systems on $Y_1(\mathfrak{n})$. For more details on these spaces of distributions, see [Han17, §2].

3.2. Hansen's universal eigenvariety for GL_2/K

One of the main results of [Han17] specialises, in our setting, to the following.

Theorem 3.4 (Hansen)**.** *There exists a separated rigid analytic space $\mathscr{E}_\mathfrak{n}$, together with a morphism $w : \mathscr{E}_\mathfrak{n} \to \mathscr{W}_K$, with the property that for each finite extension L of $\mathbb{Q}_p$, the L-points of $\mathscr{E}_\mathfrak{n}$ lying above a weight λ in $\mathscr{W}_K(L)$ are in bijection with finite slope eigenclasses in $\mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$.*

We call this space *the (full) Bianchi eigenvariety*. For singular cohomology, this is [Han17, §4.3]. We need a compactly supported version, but this goes through using identical arguments with Hansen's compactly supported spectral sequence. The level $\mathfrak{n}$ will usually be clear from context, so we usually drop the subscript in the sequel.

We define $\mathcal{E} := \mathscr{E} \cap w^{-1}(\mathcal{W}_K)$ to be the subspace lying over the smaller 2-dimensional weight space defined above. Any point of $\mathscr{E}(L)$ lying above a weight $\mathrm{diag}(\lambda_1, \lambda_2)$ corresponds to a unique point of $\mathcal{E}(L)$ lying above $\lambda_1\lambda_2^{-1}$, and the two points differ by twisting their central characters. On classical points, this corresponds to the same operation discussed in Remark 2.1, where we forced $v_1 = v_2 = 0$.

3.3. Base-change functoriality

Recall classical base-change functoriality; let f be a classical cuspidal eigenform, and let π denote the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ it generates. Then there is an automorphic representation $\mathrm{BC}(\pi)$ of $\mathrm{GL}_2(\mathbb{A}_K)$ with the property that there is an equality $\rho_{\mathrm{BC}(\pi)} = \rho_\pi|_{G_K}$ of the associated Galois representations (as representations of $G_K := \mathrm{Gal}(\overline{K}/K)$). If f is a newform, we will define its *base-change to* K to be a new vector in $\mathrm{BC}(\pi)$. Note that if f has level N, the level of its base-change $\mathcal{F}$ is an ideal $\mathfrak{n}$ with $(N/d)|\mathfrak{n}$, where $-d$ is the discriminant of K, and $\mathfrak{n}|(N)$ as ideals of $\mathcal{O}_K$ (see [Fri83, §2.1]). More generally, if f is a p-stabilised newform, we define its base-change to be the p-stabilisation of the corresponding base-changed newform with the 'correct' eigenvalues (see §8.2). In our applications, such a choice will be unique.

Let $\mathcal{W}_{\mathbb{Q}}$ denote the (null) weight space for $\mathrm{GL}_2/\mathbb{Q}$, that is, the rigid analytic space whose L-points are

$$\mathcal{W}_{\mathbb{Q}}(L) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, L^\times)$$

for $L/\mathbb{Q}_p$ inside $\mathbb{C}_p$. The space $\mathcal{W}_{\mathbb{Q}}$ has no level dependence. There is (for any level $\mathfrak{n}$) a closed immersion $\mathcal{W}_{\mathbb{Q}} \hookrightarrow \mathcal{W}_{K,\mathfrak{n}}$ induced by the norm map $(\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to \mathbb{Z}_p^\times$.

Recall that the *Coleman–Mazur eigencurve (of level N, with $p|N$)* is a separated reduced rigid analytic curve $\mathcal{C} = \mathcal{C}_N$, together with a morphism $w : \mathcal{C} \to \mathcal{W}_{\mathbb{Q}}$, such that the points x of $\mathcal{C}(L)$ with $w(x) = \lambda$ are in bijection with finite slope systems of Hecke eigenvalues arising in the overconvergent cohomology[7] at level N of weight λ. The level N will be implicit and we usually drop it from the notation. The classical points, corresponding to systems of Hecke eigenvalues attached to classical modular forms, are Zariski-dense in $\mathcal{C}$.

Theorem 3.5. *There is a finite morphism $\mathrm{BC}_N : \mathcal{C}_N \longrightarrow \mathcal{E}_{N\mathcal{O}_K}$ of rigid spaces interpolating base-change functoriality on classical points. More precisely, if $x \in \mathcal{C}(L)$ corresponds to a classical modular form f, then $\mathrm{BC}_N(x) \in \mathcal{E}(L)$ corresponds to the (stabilisation to level $N\mathcal{O}_K$ of the) system of eigenvalues attached to the base-change of f to GL_2/K.*

Proof. This is a special case of [Han17, Theorem 5.1.6]. The conditions given *op. cit.* are shown to hold in the case of cyclic base-change from $\mathbb{Q}$ in [JN16a]. □

In particular, if $\mathscr{F}$ is a classical Coleman family over an affinoid $\Sigma_{\mathbb{Q}} \subset \mathcal{W}_{\mathbb{Q}}$, then the base-changes of the classical specialisations $\mathscr{F}_k$ vary in a 'base-change family' $\mathscr{F}_K$ over the image of $\Sigma_{\mathbb{Q}}$ in $\mathcal{W}_K$.

We actually require a more refined version of this result, defined locally, that gives more precise control over the level. It may be true that if x corresponds to a p-stabilised point in $\mathcal{C}_N$, then $\mathrm{BC}_N(x) \in \mathcal{E}_{N\mathcal{O}_K}$ corresponds to a Bianchi form that has been further stabilised, and is thus *not* itself a p-stabilised newform. It is, however, always possible to force a Zariski-dense set of classical p-stabilised points to remain p-stabilised by passing to a lower level in K.

Proposition 3.6. *Let $x \in \mathcal{C}$ correspond to a p-stabilised newform of level Np. There exists an ideal $\mathfrak{n}' \subset \mathcal{O}_K$ such that $\mathfrak{n}\mathfrak{d}^{-1}|\mathfrak{n}'|\mathfrak{n}$, a neighbourhood $V_{\mathbb{Q}}$ of x in $\mathcal{C}_N$, and a finite morphism $\mathrm{BC}' : V_{\mathbb{Q}} \longrightarrow \mathcal{E}_{\mathfrak{n}'}$ interpolating base-change functoriality on classical points and such that the p-stabilised classical points in $\mathrm{BC}'(V_{\mathbb{Q}})$ are Zariski-dense.*

Proof. By [Bel12, Lem. 2.7], there exists a neighbourhood $V_{\mathbb{Q}}$ of x in which every classical point is a p-stabilised newform. By definition, the base-change of a p-stabilised newform is a p-stabilised newform (of some level). There exists a minimal $\mathfrak{n}'$ such that a Zariski-dense set of classical points in $V_{\mathbb{Q}}$ base-change directly to level $\mathfrak{n}'$, and the theorem follows by applying [Han17, Thm. 5.1.6] to interpolate base-change on these points. □

Remark 3.7: For clarity of argument, in the remainder of the paper, we will assume that if $x \in \mathcal{C}_N$ corresponds to a classical p-stabilised newform, then there is a neighbourhood $V_{\mathbb{Q}}$ of x in $\mathcal{C}_N$ such that $\mathrm{BC}_N(V_{\mathbb{Q}})$ contains a Zariski-dense set of classical points corresponding to p-stabilised newforms. This is always the case, for example, if N is coprime to d. Since the proofs in the sequel are all local in nature, all of the results can be proved without this assumption by working in $\mathcal{E}_{\mathfrak{n}'}$ for some $\mathfrak{n}'|N\mathcal{O}_K$ and using Proposition 3.6.

3.4. The dimensions of irreducible components

Let $\mathcal{F}$ be a finite slope cuspidal Bianchi modular form that is an eigenform for the Hecke operators.

[7] More typically, the Coleman–Mazur eigencurve is constructed using overconvergent modular forms. In [PS13, Thm. 7.1] and [Bel12, Thm. 3.30], however, the eigencurve of modular symbols is shown to be essentially the same as the one of modular forms.

Proposition 3.8. *There is a point $x_{\mathcal{F}} \in \mathcal{E}(L)$ corresponding to $\mathcal{F}$.*

Proof. If $\mathcal{F}$ is non-critical, then by definition there exists an overconvergent eigenclass $\Psi \in \mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))$ with the same Hecke eigenvalues as $\mathcal{F}$ (for $i = 1, 2$ and some sufficiently large $L/\mathbb{Q}_p$). Necessarily Ψ also has finite slope, and by Theorem 3.4, there is a point $x_{\mathcal{F}} \in \mathcal{E}(L)$ corresponding to $\mathcal{F}$. If $\mathcal{F}$ is critical, then we instead study the long exact sequence of cohomology attached to specialisation $\mathcal{D}_\lambda \to V_\lambda$. The cokernel of the map $\mathrm{H}^2_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L)) \to \mathrm{H}^2_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{V}_\lambda(L))$ can be identified as a subspace of a degree 3 overconvergent cohomology group (see [BSW16, §9.3]); but an analysis as in [Bel12, Lem. 3.9] shows that cuspidal eigenpackets do not appear in such spaces. In particular, after restricting to the generalised eigenspace at $\mathcal{F}$, the specialisation map is surjective in degree 2. □

For our purposes, if $\mathcal{F}$ is critical it suffices to assume $\mathcal{F}$ is base-change, whence the existence of the point $x_{\mathcal{F}} \in \mathcal{E}(L)$ follows much more simply, as such a point arises in the image of BC.

Theorem 3.9 (Hansen–Newton)**.** *Suppose $\mathcal{F}$ is non-critical. Any irreducible component $\mathcal{E}'$ of $\mathcal{E}$ passing through $x_{\mathcal{F}}$ has dimension 1.*

Proof. Newton proves that the component has dimension at least 1 in Proposition B.1 of the appendix of [Han17], noting that $l(x_{\mathcal{F}}) = 1$ since $\mathcal{F}$ is non-critical.

The following beautiful proof that the component is at most 1-dimensional was communicated to us by David Hansen. Let Z be any 2-dimensional irreducible component passing through $x_{\mathcal{F}}$, and let $\Omega = w(Z)$. In [JN16b], Johansson and Newton construct a two-dimensional Galois determinant (or pseudocharacter) ρ_Z over Z. Let $\mathcal{Z}$ be the set of classical points y of Z such that:

(i) $w(y)$ is non-parallel in Ω, and

(ii) y has small slope.

This set is Zariski-dense in Z. Each point $y \in \mathcal{Z}$ necessarily corresponds to a classical form by the appropriate analogue of Stevens' control theorem (see, for example, [BSW16, Thm. 8.7]), and this classical form must be Eisenstein, since classical cuspidal forms exist only at parallel weights. Hence the specialisation of ρ_Z at y is reducible. But reducibility on a Zariski-dense set of points forces reducibility everywhere, and hence the specialisation of ρ_Z at $x_{\mathcal{F}}$ is reducible. But as $\mathcal{F}$ is cuspidal, the Galois representation attached to $x_{\mathcal{F}}$ is irreducible, which is a contradiction. □

Remark 3.10: This is a slightly stronger formulation than can be obtained by specialising the (very general) results of [Han17], where the analogous result is proved for non-critical classical points that are *strongly interior*, that is, that satisfy a vanishing condition on their overconvergent boundary cohomology. In this case, however, Hansen has shown considerably more. Indeed, an analysis of the Tor spectral sequence constructed *op. cit.* shows that the Bianchi eigenvariety is naturally the union $\mathcal{E}^{\mathrm{punc}} \cup \mathcal{E}^{\mathrm{cusp}} \cup \mathcal{E}^{\mathrm{Eis}}$, where $\mathcal{E}^{\mathrm{punc}}$ is zero-dimensional and supported only above the trivial weight, $\mathcal{E}^{\mathrm{cusp}}$ is equidimensional of dimension 1, and $\mathcal{E}^{\mathrm{Eis}}$ is finite flat over $\mathcal{W}_K$. Such a result was proved in an unpublished preprint [Han12] that eventually became [Han17]. We do not require this stronger formulation for our purposes.

4. Families of modular symbols

The above results can be described explicitly in terms of modular symbols in families, which allows a more concrete link to the p-adic L-functions of [Wil17]. Hansen's construction of eigenvarieties uses the *total* cohomology, but in this section we refine his results to show that p-adic families can be realised as families of modular symbols (that is, in $\mathrm{H}^1_{\mathrm{c}}$). We then study the structure of the space of modular symbols over the eigenvariety, and show that – in a neighbourhood of any p-stabilised non-critical classical cuspidal point – such symbols are free of rank one over a Hecke algebra, which is key to our later applications.

4.1. Families in $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)$

The eigenvariety is 'glued' together from a collection of local pieces, each given by the spectrum of a Hecke algebra acting on overconvergent cohomology. Indeed, one can start from a covering of the weight space $\mathcal{W}_K$ by 'slope-adapted affinoids,' indexed by pairs (Ω, h), where $\Omega = \mathrm{Sp}(R) \subset \mathcal{W}_K$ is a two-dimensional affinoid in weight space and $h \geq 0$ is some real number such that there exists a slope decomposition (with respect to the Hecke operator U_p)

$$\mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega) \cong \mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h} \oplus \mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{>h}.$$

Definition 4.1. Let $\mathbb{H}_{\mathfrak{n},p}$ denote the (abstract) Hecke algebra, that is, the free $\mathbb{Z}_p$-algebra generated by the Hecke operators $\{T_I : (I, \mathfrak{n}) = 1\}, \{U_{\mathfrak{p}} : \mathfrak{p}|p\}$ and $\{\langle v\rangle : v|\mathfrak{n}\}$.

The algebra $\mathbb{H}_{\mathfrak{n},p}$ acts in the usual way on overconvergent cohomology groups, and this action preserves slope decompositions. For a slope-adapted affinoid (Ω, h), we define

$$\mathbb{T}_{\Omega,h} := \text{Image of } \mathbb{H}_{\mathfrak{n},p} \text{ in } \mathrm{End}_{\mathcal{O}(\Omega)}(\mathrm{H}^*_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h})$$

to be the corresponding Hecke algebra. The *local piece of the eigenvariety* is then defined as

$$\mathcal{E}_{\Omega,h} := \mathrm{Sp}(\mathbb{T}_{\Omega,h})$$

with the natural rigid structure. We can naturally view $\mathcal{E}_{\Omega,h}$ as an affinoid subspace of $\mathcal{E}$, and analogously to the global eigenvariety, we have a bijection between L-points of $\mathcal{E}_{\Omega,h}$ lying above a weight $\lambda \in \Omega(L)$ and systems of Hecke eigenvalues arising in $\mathrm{H}^*_{\mathrm{c}}(Y, \mathscr{D}_\lambda(L))^{\leq h}$.

We fix the following notation and terminology.

Definition 4.2. Let (Ω, h) be a slope adapted pair, let $x \in \mathcal{E}_{\Omega,h}(L)$ be any point, and write $\mathfrak{m}_x$ for the corresponding maximal ideal in $\mathbb{T}_{\Omega,h}$. Let $\mathscr{P}_x$ be a minimal prime of $\mathbb{T}_{\Omega,h}$ contained in $\mathfrak{m}_x$, and write $\mathscr{P}_\lambda$ for the contraction of $\mathscr{P}_x$ to $\mathcal{O}(\Omega)$. Define $\Lambda = \mathcal{O}(\Omega)/\mathscr{P}_\lambda$ and let $\Sigma = \mathrm{Sp}(\Lambda)$ be the corresponding closed subset inside Ω, which is a rigid curve by Theorem 3.9. If such a curve $\Sigma \subset \Omega$ arises in this way, say that *x varies in a family over Σ*.

Whilst in the above we worked with the total cohomology, the following result allows us to pin down families in $\mathrm{H}^1_{\mathrm{c}}$.

Proposition 4.3. *Let x be a non-critical cuspidal classical point of $\mathcal{E}_{\Omega,h}(L)$ that varies in a family over Σ. Then, after possibly shrinking Σ,*

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} \neq 0.$$

We need a lemma.

Lemma 4.4. *(i) The spaces $\mathrm{H}^0_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}^0_\lambda)$ and $\mathrm{H}^0_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}^0_\Omega)$ are both 0.*

(ii) The spaces $\mathrm{H}^0_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}$ and $\mathrm{H}^0_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}$ are both 0.

(iii) Let x be a cuspidal classical point of $\mathcal{E}_{\Omega,h}$. The system of eigenvalues for x occurs in $\mathrm{H}^i_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}$ if and only if $i = 2$.

Proof. Write $\mathcal{D}^0 = \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}^0, L)$ for either $\mathcal{D}^0_\lambda$ or $\mathcal{D}^0_\Omega$. For part (i), first note that in the case of singular cohomology, we have $\mathrm{H}^0(Y_1(\mathfrak{n}), \mathscr{D}^0) = \mathrm{H}^0(U_1(\mathfrak{n}), \mathscr{D}^0) = (\mathcal{D}^0)^{U_1(\mathfrak{n})}$. For $b \in \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$, the matrix $\gamma_b := \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ acts on $\mathcal{A}^0$ by sending $f(z)$ to $f(z+b)$. Let $\mu \in (\mathcal{D}^0)^{U_1(\mathfrak{n})}$; then

$$\mu(z \mapsto z) = \mu|\gamma_b(z \mapsto z) = \mu(z \mapsto z + b) = \mu(z \mapsto z) + \mu(z \mapsto b),$$

so that μ is zero on the constant functions. Suppose μ is zero on functions that are polynomial of degree less than $r - 1$. Then

$$\mu(z \mapsto z^{r+1}) = \mu(z \mapsto (z+b)^{r+1}) = \mu(z \mapsto z^{r+1}) + b(r+1)\mu(z \mapsto z^r)$$

for all b, where the lower terms vanish by assumption, so $\mu(z^r) = 0$ and we conclude that $\mu = 0$ by induction. The case with compact support follows since the excision exact sequence for the Borel–Serre compactification of $Y_1(\mathfrak{n})$ starts $0 \to \mathrm{H}^0_{\mathrm{c}} \to \mathrm{H}^0$, so $\mathrm{H}^0_{\mathrm{c}}$ injects into a trivial module. To see part (ii), it is enough to note that after passing to the small slope parts, overconvergent cohomology with coefficients in rigid and locally analytic distributions agree, whence the result by part (i).

For part (iii), we first claim that x does not appear as an eigenpacket in $\mathrm{H}^3_{\mathrm{c}}$, for which we follow an argument of Pollack–Stevens (see [PS13, Lem. 5.2], and also [Bel12, Lem. 3.9]). We identify $\mathrm{H}^3_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega) \cong \mathrm{H}_0(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)$ using Poincaré duality. This decomposes into a direct sum $\oplus_{i \in \mathrm{Cl}_K} \mathrm{H}_0(\Gamma_i, \mathscr{D}_\Omega)$ using the same techniques as in §2.5. Each of these factors is then identified with the coinvariants $\mathcal{D}_\Omega/\Gamma_i\mathcal{D}_\Omega$. An analysis as *op. cit.* shows that this is non-zero only when Ω contains the trivial weight $(0,0)$, and the spectrum of any Hecke algebra on this space is supported at this trivial weight. Further analysis, as in [Bel12], then gives an explicit description of the corresponding systems of eigenvalues, and shows that they are attached to critical (overconvergent) weight $(0,0)$ Eisenstein series. They are thus not cuspidal, and x does not appear in $\mathrm{H}^3_{\mathrm{c}}$. (See also the remark following this proof).

In light of part (ii), it now suffices to prove that it does not appear in $\mathrm{H}^1_{\mathrm{c}}$. We exploit Hansen's Tor spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\mathcal{O}(\Omega)}(\mathrm{H}^j_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}, k_\lambda) \implies \mathrm{H}^{i+j}_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h},$$

where $\mathfrak{m}_\lambda$ is any maximal ideal of $\mathcal{O}(\Omega)$ and k_λ denotes its residue field. Since $\mathcal{O}(\Omega)$ is regular of dimension 2, the $\mathrm{Tor}_i^{\mathcal{O}(\Omega)}$ groups vanish for $i \geq 3$, so that $E_2^{-3,2} = 0$. As $E_2^{1,0} = 0$ as well, we see that

$$E_3^{-1,1} = \ker(E_2^{-1,1} \to 0)/\mathrm{Image}(0 \to E_2^{-1,1}) = E_2^{-1,1},$$

and continuing, that $E_\infty^{-1,1} = \mathrm{Tor}_1^{\mathcal{O}(\Omega)}(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}, k_\lambda)$. This contributes to the grading on $\mathrm{H}^0_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}$, which is zero by the above; hence this Tor term vanishes. A similar analysis, using that $E_2^{0,0} = E_2^{-4,2} = 0$, shows that

$$E_\infty^{-2,1} = E_2^{-2,1} = \mathrm{Tor}_2^{\mathcal{O}(\Omega)}(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}, k_\lambda) = 0$$

as well. We then have vanishing of $\mathrm{Tor}_i^{\Omega(\mathfrak{m})}(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}, k_\lambda)$ for this module for all $i > 0$, and for any maximal ideal $\mathfrak{m}_\lambda$, so by [Han17, Prop. A.3], the $\mathcal{O}(\Omega)$-module $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}$ is either zero or projective. As it is torsion by [Han17, Thm. 4.4.1], it cannot be projective, so it vanishes, as required. □

Remark 4.5: For singular cohomology, Lemma 4.4 is an unpublished result of David Hansen (see also Remark 3.10), and we thank him sincerely for allowing us to reproduce his proof here. We have provided the statement in the cleanest setting, but actually all we really need is the fact that if x varies in a one-dimensional family, then the *minimal* degree x appears in is $i = 2$. This is simpler to prove, as one only needs to check that the modules $\mathrm{H}^3_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)$ are supported only over the trivial weight. For x to appear in a family, then, it has to appear in a lower degree.

Proof. (Proposition 4.3) We defined $\mathscr{P}_\lambda$ to be the contraction of $\mathscr{P}_x$ to $\mathcal{O}(\Omega)$; it has height one, and still has height one in the localisation $\mathcal{O}(\Omega)_{\mathfrak{m}_\lambda}$. This localisation is a regular local ring, and hence a unique factorisation domain, so all height one primes are principal, and we can take some generator r of $\mathscr{P}_\lambda\mathcal{O}(\Omega)_{\mathfrak{m}_\lambda}$. After possibly shrinking Ω, and scaling by a unit in $\mathcal{O}(\Omega)_{\mathfrak{m}_\lambda}$, we may assume $r \in \mathcal{O}(\Omega)$. We obtain a short exact sequence $0 \to \mathcal{D}_\Omega \to \mathcal{D}_\Omega \to \mathcal{D}_\Sigma \to 0$, where the first map is multiplication by r. By truncating the associated long exact sequence at the first degree 2 term, and localising at x, we obtain a short exact sequence

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}_{\mathfrak{m}_x} \to \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} \to \mathrm{H}^2_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Omega)^{\leq h}_{\mathfrak{m}_x}[r] \to 0.$$

By Lemma 4.4, the first term is trivial; and as the system of eigenvalues corresponding to x is r-torsion in $\mathrm{H}^2_{\mathrm{c}}$, this shows that the second map is an isomorphism of non-trivial modules, from which we conclude. □

In the above, we considered $\mathfrak{m}_x$ as a maximal ideal in $\mathbb{T}_{\Omega,h}$, which acted on $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}$ via its image in $\mathbb{T}_{\Omega,h}\otimes_{\mathcal{O}(\Omega)}\Lambda$. The proof shows slightly more, however; define

$$\mathbb{T}_{\Sigma,h} := \text{image of } \mathbb{H}_{\mathfrak{n},p} \text{ in } \mathrm{End}_\Lambda(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}).$$

Then from the isomorphism obtained in the proof, we deduce that there is a maximal ideal of $\mathbb{T}_{\Sigma,h}$ corresponding to x, which we continue to call $\mathfrak{m}_x$ in a slight abuse of notation. It is the image of the corresponding maximal ideal in $\mathbb{T}_{\Omega,h}$ under the natural map $\mathbb{T}_{\Omega,h}\to\mathbb{T}_{\Sigma,h}$. As a $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$-module, we then have $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}\neq 0$.

4.2. Structure of overconvergent cohomology over the weight space

Let $x\in\mathcal{E}_{\Omega,h}(L)$ correspond to a cuspidal non-critical classical Bianchi eigenform $\mathcal{F}$, varying in a family over the curve $\Sigma\subset\Omega$, and let $\lambda = w(x)$. We use the notation of Definition 4.2. In Proposition 4.3 and the remark following it, we showed that x also gives rise to a maximal ideal, denoted $\mathfrak{m}_x$, in $\mathbb{T}_{\Sigma,h}$, and that the $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$-module $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is non-trivial.

Definition 4.6. Let $\mathcal{F}$ be a Bianchi eigenform of level $\mathfrak{n}$, with $(p)|\mathfrak{n}$. We say $\mathcal{F}$ satisfies *multiplicity one for* $\mathbb{H}_{\mathfrak{n},p}$ if the $\mathbb{H}_{\mathfrak{n},p}$-eigenspace $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{V}_\lambda(L)^*)[\mathcal{F}]$, that is the eigenspace on which $\mathbb{H}_{\mathfrak{n},p}$ acts with the same eigenvalues as on $\mathcal{F}$, is one-dimensional, recalling that $\mathbb{H}_{\mathfrak{n},p}$ is generated by the Hecke operators away from $\mathfrak{n}$ and the Hecke operators at primes above p. This is satisfied, for example, when $\mathcal{F}$ is a newform, or the p-stabilisation of a newform whose Hecke polynomials at primes above p have distinct roots[8].

The main result of this section is the following.

Theorem 4.7. *Let $\mathcal{F}$ be non-critical with multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$. Then $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is free of rank 1 over $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$, which (after a finite extension of the base field[9] of Λ) is itself free of rank 1 over $\Lambda_{\mathfrak{m}_\lambda}$.*

The first step is the following proposition[10].

Proposition 4.8. *(i) There is an isomorphism*

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}\otimes_{\Lambda_{\mathfrak{m}_\lambda}}\Lambda_{\mathfrak{m}_\lambda}/\mathfrak{m}_\lambda\cong\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\lambda)^{\leq h}_{\mathfrak{m}_x}.$$

(ii) The module $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is generated by one element over $\Lambda_{\mathfrak{m}_\lambda}$.

Proof. Write $\mathbb{T} := \mathbb{T}_{\Sigma,h}$ for ease of notation. By general facts about slope decompositions, the module $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}$ is a finite Λ-module. We localise at $\mathfrak{m}_x$, obtaining a $\mathbb{T}_{\mathfrak{m}_x}$ module

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} := \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}\otimes_{\mathbb{T}_{\Sigma,h}}\mathbb{T}_{\mathfrak{m}_x}. \tag{4.1}$$

As $\mathfrak{m}_\lambda$ is the contraction of $\mathfrak{m}_x$ to Λ, the module $\mathbb{T}_{\mathfrak{m}_x}$ has a natural $\Lambda_{\mathfrak{m}_\lambda}$-module structure. Hence $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ inherits a $\Lambda_{\mathfrak{m}_\lambda}$-module structure from the second factor of the tensor product in (4.1). Since $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}$ has finite type over $\mathbb{T}$, and $\mathbb{T}$ has finite type over Λ, this implies that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is a $\Lambda_{\mathfrak{m}_\lambda}$-module of finite type.

From the short exact sequence of distribution spaces given by the natural surjection $\mathrm{sp}_\lambda:\mathcal{D}_\Sigma\to\mathcal{D}_\lambda$, we obtain a long exact sequence of cohomology, which we truncate to a short exact sequence

$$0\to\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}\otimes_\Lambda\Lambda/\mathfrak{m}_\lambda\to\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\lambda)^{\leq h}\to\mathrm{H}^2_{\mathrm{c}}(Y_1(\mathfrak{n}),D)[\mathfrak{m}_\lambda]\to 0.$$

[8]This is conjectured to always be the case when p is split; in any case, it can happen only if both roots have valuation $v_p(N(\mathfrak{p})^{k+1})/2$, where $\mathcal{F}$ has weight (k,k), and in particular, only at isolated cuspidal points of the eigenvariety.

[9]Precisely, we replace Λ with $\Lambda\otimes_{\mathbb{Q}_p}L$. Without this base change, $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$ is an étale $\Lambda_{\mathfrak{m}_\lambda}$-algebra.

[10]We are grateful to Adel Betina for his contribution to the proof of this proposition.

where D is the kernel of sp_λ in $\mathcal{D}_\Sigma$ and the last term is the $\mathfrak{m}_\lambda$-torsion. Since localising preserves short exact sequences, we deduce the existence of a short exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} \otimes_{\Lambda_{\mathfrak{m}_\lambda}} \Lambda_{\mathfrak{m}_\lambda}/\mathfrak{m}_\lambda \to \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}_{\mathfrak{m}_x} \to \mathrm{H}^2_{\mathrm{c}}(Y_1(\mathfrak{n}), D)[\mathfrak{m}_\lambda]_{\mathfrak{m}_x} \to 0.$$

The middle term is the generalised eigenspace corresponding to the system of eigenvalues attached to x. (At this point, we are assuming that we have extended the base field of Λ so that x is defined over $\Lambda/\mathfrak{m}_\lambda$). As x is non-critical, this is isomorphic to the generalised eigenspace of $\mathcal{F}$ for $\mathbb{H}_{\mathfrak{n},p}$ in the classical cohomology, and by assumption, this is one-dimensional. Thus either the first or last term is 0. Suppose that the first term is 0; then by Nakayama's lemma, we must have $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} = 0$, which contradicts Proposition 4.3. Hence the last term is 0, and we find that

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} \otimes_{\Lambda_{\mathfrak{m}_\lambda}} \Lambda_{\mathfrak{m}_\lambda}/\mathfrak{m}_\lambda \cong \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}_{\mathfrak{m}_x}$$

as one-dimensional $\Lambda_{\mathfrak{m}_\lambda}/\mathfrak{m}_\lambda$-vector spaces, proving (i).

Now we use Nakayama again. A generator of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} \otimes_{\Lambda_{\mathfrak{m}_\lambda}} \Lambda_{\mathfrak{m}_\lambda}/\mathfrak{m}_\lambda$ lifts to a generator of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ over $\Lambda_{\mathfrak{m}_\lambda}$, which completes the proof. □

Lemma 4.9. *The space* $\mathrm{H}^1_{\mathrm{c}}(Y, \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ *is torsion-free as a* $\Lambda_{\mathfrak{m}_\lambda}$*-module.*

Proof. We use the identification with modular symbols. For fixed i, let $\{\delta_j : j \in J\}$ be a finite set of generators for Δ_0 as a $\mathbb{Z}[\Gamma_i]$-module (see [Wil17, Lem. 3.8]). Then for any R, the map $\mathrm{Symb}_{\Gamma_i}(\mathcal{D}(R)) \hookrightarrow \mathcal{D}(R)^J$ is an injective R-module map. By passing to the direct sum over all i in the class group, we see that there is a Λ-module embedding of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)$ into a finite direct sum of copies of $\mathcal{D}_\Sigma$. But $\mathcal{D}_\Sigma$ is a torsion-free Λ-module since Λ is a domain. The result follows after localising. □

This is enough to complete the proof of Theorem 4.7. Proposition 4.8 and Lemma 4.9 imply that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is free of rank 1 over $\Lambda_{\mathfrak{m}_\lambda}$. Then as

$$\mathbb{T}_{\mathfrak{m}_x} \subset \mathrm{End}_{\Lambda_{\mathfrak{m}_\lambda}}\left(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}\right) \cong \Lambda_{\mathfrak{m}_\lambda}$$

is non-zero by our assumption on Σ, we must have $\mathbb{T}_{\mathfrak{m}_x} \cong \Lambda_{\mathfrak{m}_\lambda}$, and since the actions of $\mathbb{T}$ and Λ on overconvergent cohomology are compatible, we see that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is free of rank 1 over $\mathbb{T}_{\mathfrak{m}_x}$, as required.

4.3. Freeness in families

Let $\mathcal{F}$ be a cuspidal non-critical Bianchi eigenform with multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$, as above, varying in a family over Σ, and write $x = x_{\mathcal{F}}$ for the corresponding point in $\mathcal{E}$. By Theorem 4.7, we have that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is free of rank one over $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$, which is free of rank one over $\Lambda_{\mathfrak{m}_\lambda}$ after a finite base extension of Λ.

Proposition 4.10. *After possibly shrinking* Σ*, there exists a connected component* $V = \mathrm{Sp}\, T \subset \mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ *containing* x *such that* $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} T$ *is free of rank one over* T*, which is free of rank one over* Λ*. In particular, the weight map is étale on* V*.*

Proof. (Compare [BSDJ17, 2.19]). The localisations are defined by

$$\begin{aligned}
\Lambda_{\mathfrak{m}_\lambda} &= \varinjlim_{\lambda \in U \subset \Sigma} \mathcal{O}(U),\\
(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x} &= \varinjlim_{x \in V \subset \mathrm{Sp}(\mathbb{T}_{\Sigma,h})} \mathcal{O}(V),\\
\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x} &= \varinjlim_{x \in V \subset \mathrm{Sp}(\mathbb{T}_{\Sigma,h})} \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} \mathcal{O}(V).
\end{aligned}$$

Comparing the first two equations, we find we are exactly in the situation of [BSDJ17, Lem. 2.13] (working over the rigid space Σ), so that – possibly replacing Σ with some smaller affinoid subset – we may choose some $V \subset \mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ such that $T = \mathcal{O}(V)$ is free of rank one over $\Lambda = \mathcal{O}(\Sigma)$. A second application of the same lemma to the second and third equations, over the rigid space $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$, now shows that, after potentially shrinking Σ and V again, we have $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} T$ free of rank one over T, as required. □

Remark 4.11: This proposition shows in particular that if x is a classical non-critical point of $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$, corresponding to a form with multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$, then the natural weight map $w : \mathrm{Sp}(\mathbb{T}_{\Sigma,h}) \to \Sigma$ is étale in a neighbourhood of x. We cannot conclude that $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ is smooth at x without further work, however, as it is not at all clear that Σ is smooth at $w(x)$.

5. The parallel weight eigenvariety

In this section, we describe a closed subspace $\mathcal{E}_{\mathrm{par}}$ of $\mathcal{E}$ lying over the parallel weight line that is much better behaved than the whole space $\mathcal{E}$. This 'parallel-weight eigenvariety' bears comparison with the 'middle-degree eigenvariety' of [BH17], which plays a similar role in the Hilbert case. We also show that there is a base-change map from the Coleman–Mazur eigencurve into $\mathcal{E}_{\mathrm{par}}$ and use it to show smoothness of (suitably well-behaved, but possibly critical) classical points in the image.

5.1. Definition and basic properties

Recall: Hansen's eigenvariety $\mathcal{E}$ is built from an *eigenvariety datum* $\mathfrak{D} = (\mathcal{W}_K, \mathscr{L}, \mathscr{M}, \mathbb{H}_{\mathfrak{n},p}, \psi)$, where $\mathcal{W}_K$ and $\mathbb{H}_{\mathfrak{n},p}$ are as before, $\mathscr{L}$ is a Fredholm hypersurface in $\mathcal{W}_K \times \mathbb{A}^1$ cut out by the U_p operator, $\mathscr{M}$ is a coherent sheaf on $\mathscr{L}$ given by (total) overconvergent cohomology, and $\psi : \mathbb{H}_{\mathfrak{n},p} \to \mathrm{End}_{\mathcal{O}(\mathscr{L})}(\mathscr{M})$ is the natural map. Define now another eigenvariety datum

$$\mathfrak{D}_{\mathrm{par}} := (\mathcal{W}_{K,\mathrm{par}}, \mathscr{L}_{\mathrm{par}}, \mathscr{M}^1_{\mathrm{par}}, \mathbb{H}_{\mathfrak{n},p}, \psi_{\mathrm{par}}),$$

where:

(i) $\mathcal{W}_{K,\mathrm{par}}$ is the parallel weight line in $\mathcal{W}_K$, or equivalently the image of $\mathcal{W}_{\mathbb{Q}}$ under the natural closed immersion;

(ii) $\mathscr{L}^{\mathrm{par}}$ is the union of the irreducible components of $\mathscr{L}$ that lie above $\mathcal{W}_{K,\mathrm{par}}$, which is itself a Fredholm hypersurface;

(iii) $\mathscr{M}^1_{\mathrm{par}}$ is the coherent sheaf on $\mathscr{L}^{\mathrm{par}}$ such that for any slope adapted affinoid $\mathscr{L}^{\mathrm{par}}_{\Sigma,h}$ lying above $\Sigma \subset \mathcal{W}_{K,\mathrm{par}}$, we have $\mathscr{M}^1_{\mathrm{par}}(\mathscr{L}^{\mathrm{par}}_{\Sigma,h}) = \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}$;

(iv) $\mathbb{H}_{\mathfrak{n},p}$ is as before, and ψ_{par} is the map obtained by gluing the natural map given by the action of Hecke operators.

That this does give a well-defined eigenvariety datum is a simple check using the machinery developed in [Han17, §3,§4].

Proposition 5.1. *The eigenvariety $\mathcal{E}_{\mathrm{par}}$ attached to the datum $\mathfrak{D}_{\mathrm{par}}$ contains a Zariski-dense set of classical points, and its nilreduction admits a closed immersion into $\mathcal{E}$.*

Proof. First we prove the statement about classical points. It is clear that classical weights, which correspond to classical weights for the weight space of $\mathrm{GL}_2/\mathbb{Q}$, are Zariski-dense in $\mathcal{W}_{K,\mathrm{par}}$. Now let (Σ, h) be a slope adapted affinoid in $\mathcal{W}_{K,\mathrm{par}}$; we see that in the local piece $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$, each point lying above a weight λ induces an eigenpacket in $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h}$, and all but finitely many of the points lying above classical weights correspond to small slope eigenpackets. Such points are classical by the control theorem. It follows that the classical points are dense in $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ and hence in $\mathcal{E}_{\mathrm{par}}$.

That there exists a closed immersion $\mathcal{E}_{\mathrm{par}}^{\mathrm{red}} \hookrightarrow \mathcal{E}$ is a consequence of [Han17, Thm. 5.1.2], which essentially says that it suffices to check an inclusion of a Zariski-dense set of points. Since every classical point $x \in \mathcal{E}_{\mathrm{par}}$ corresponds to a system of eigenvalues that appears in $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)$ for some $\lambda \in \mathcal{W}_{K,\mathrm{par}}$, the conditions of the theorem – namely, divisibility of characteristic power series of the U_p operator – are satisfied, and we get the required closed immersion, completing the proof. □

5.2. The base-change eigenvariety and smoothness

Recall the finite morphism $\mathrm{BC} : \mathcal{C} \to \mathcal{E}$ introduced in Theorem 3.5. By using [Han17, Thm. 5.1.6] again, we see that BC factors through

$$\mathcal{C} \xrightarrow{\mathrm{BC}} \mathcal{E}_{\mathrm{par}}.$$

Definition 5.2. Let $\mathcal{E}_{\mathrm{bc}}$ denote the image of $\mathcal{C}$ in $\mathcal{E}_{\mathrm{par}}$ under BC. It is a rigid curve that admits a closed immersion into $\mathcal{E}_{\mathrm{par}}$.

Importantly, $\mathcal{E}_{\mathrm{bc}}$ is reduced at the points we are interested in.

Proposition 5.3. *Let $f \in S_{k+2}(\Gamma_1(N))$ be a p-stabilised newform that does not have CM by K. Let $x_f \in \mathcal{C}(L)$ be the corresponding point in the eigencurve (for some sufficiently large $L/\mathbb{Q}_p$). There exists a neighbourhood V_{bc} of $\mathrm{BC}(x_f)$ in $\mathcal{E}_{\mathrm{bc}}$ that is reduced.*

Proof. Let $V_{\mathbb{Q}}$ be a neighbourhood of x_f in $\mathcal{C}$, lying above Σ, such that $V_{\mathrm{bc}} := \mathrm{BC}(V_{\mathbb{Q}})$ contains a Zariski-dense set of points corresponding to p-stabilised newforms (see Remark 3.7). We claim that $\mathcal{O}(V_{\mathrm{bc}})$ is reduced. We appeal to [BH17, Lem. 6.4.7], which shows that this reducedness is equivalent to $\mathcal{O}(V_{\mathrm{bc}})$ being *generically étale* over $\Lambda = \mathcal{O}(\Sigma)$, in the sense that there exists an open dense subset $\mathcal{V}$ of V_{bc} such that the map $V_{\mathrm{bc}} \times_\Sigma \mathcal{V} \to \Sigma$ is finite étale. Let x be a classical non-critical point of V_{bc} with multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$ (Definition 4.6); then by Proposition 4.10, we can pick some open affinoid neighbourhod $V_x \subset V_{\mathrm{bc}}$ lying over $U_x \subset \Sigma$ such that $\mathcal{O}(V_x)$ is free of rank one over $\mathcal{O}(U_x)$, and hence the weight map is étale on V_x. Now define $\mathcal{V} = \cup_x V_x$, where the union is over all classical non-critical x that satisfy multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$, which form a Zariski-dense set in V_{bc} (footnote 8 and Remark 3.7). By construction, $\mathcal{V}$ is open and dense, and the weight map is étale on $\mathcal{V}$. But it is finite on V_{bc}, so we are done. □

Definition 5.4. Let $f \in S_{k+2}(\Gamma_1(N))$, recalling $p|N$. We say that f is *p-regular* if the roots of the Hecke polynomial $X^2 - a_p(f)X + p^{k+1}$ are distinct. We say that f is *decent*, following [Bel12, §1.4], if it is p-regular and:

(i) f is new (hence non-critical),

(ii) or f is the p-stabilisation of a newform g of level prime to p that either:

  (a) is non-critical,

  (b) or has vanishing adjoint Selmer group $\mathrm{H}^1_f(\mathbb{Q}, \mathrm{ad}\,\rho_f) = 0$, where $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ is the Galois representation attached to f (using the choices of field embeddings at the start of §2) and L is some sufficiently large finite subextension of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$.

The roots of the Hecke polynomial are conjecturally always distinct. All CM forms are decent (see [Bel12, §2.2.4]), and it is conjectured that all cuspidal non-CM forms satisfy (i/iia) and (iib) independently. In the critical case, we make one further definition.

Definition 5.5. Let f be critical and decent. Then necessarily f is the p-stabilisation of a newform g, of level prime to p, corresponding to a choice of root of the Hecke polynomial at p for g. We will henceforth denote this root by α_p (noting that $v_p(\alpha_p) = k+1$). The base-change $\mathcal{F}$ of f has $U_{\mathfrak{p}}$-eigenvalue $\alpha_{\mathfrak{p}}$ depending explicitly on α_p for each $\mathfrak{p}|p$.

Most of the remainder of this section will be dedicated to proving the following result.

Proposition 5.6. *Let $f \in S_{k+2}(\Gamma_1(N))$ be decent, and suppose f does not have CM by K. Let $x_f \in \mathcal{C}(L)$ be the corresponding point in the eigencurve. Then $\mathrm{BC}(x_f)$ is smooth in $\mathcal{E}_{\mathrm{bc}}$.*

First, we treat the case where f is non-critical by looking more closely at the construction of BC.

Proposition 5.6, f non-critical. The Coleman–Mazur eigencurve arises from an eigenvariety datum $(\mathcal{W}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}, \mathscr{M}_{\mathbb{Q}}, \mathbb{H}_{\mathbb{Q},N,p}, \psi_{\mathbb{Q}})$ (specialising [Han17]). There is a natural map $\phi : \mathbb{H}_{\mathfrak{n},p} \to \mathbb{H}_{\mathbb{Q},N,p}$ (see [JN16a, §4.3]). We define a new eigenvariety datum $(\mathcal{W}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}}, \mathscr{M}_{\mathbb{Q}}, \mathbb{H}_{\mathfrak{n},p}, \psi_{\mathbb{Q}} \circ \phi)$, giving rise to an intermediate eigenvariety $\mathcal{C}^K$. Let $\Sigma = \mathrm{Sp}(\Lambda)$ be an affinoid in $\mathcal{W}_{\mathbb{Q}}$ which is slope-h adapted for $\mathscr{M}_{\mathbb{Q}}$; then there is a map $\mathrm{BC}' : \mathcal{C}_{\Sigma,h} \to \mathcal{C}^K_{\Sigma,h}$ arising from the inclusion $\phi(\mathbb{H}_{\mathfrak{n},p}) \subset \mathbb{H}_{\mathbb{Q},N,p}$, which induces an inclusion of the local rings $\mathcal{O}(\mathcal{C}^K_{\Sigma,h}) \subset \mathcal{O}(\mathcal{C}_{\Sigma,h})$. By [Han17, Thm. 5.1.2], there is a closed immersion $\mathcal{C}^K \hookrightarrow \mathcal{E}_{\mathrm{par}}$, and the map BC is the composition $\mathcal{C} \to \mathcal{C}^K \hookrightarrow \mathcal{E}_{\mathrm{par}}$. It suffices, then, to show that $\mathcal{C}^K$ is smooth at $\mathrm{BC}'(x_f)$.

Since f is non-critical, after localising and base-extending Λ, by [Bel12] we know that $\mathcal{O}(\mathcal{C}_{\Sigma,h})_{\mathfrak{m}_x}$ is free of rank one over $\Lambda_{\mathfrak{m}_\lambda}$. Since $\mathcal{O}(\mathcal{C}^K_{\Sigma,h})_{\mathfrak{m}_{\mathrm{BC}'(x)}}$ is a $\Lambda_{\mathfrak{m}_\lambda}$-subalgebra containing 1, it must be isomorphic to $\mathcal{O}(\mathcal{C}_{\Sigma,h})_{\mathfrak{m}_x}$, and BC' is locally an isomorphism at x_f. As $\mathcal{C}$ is smooth at x_f (see [Bel12, Thm. 2.16]), we deduce that $\mathcal{C}^K$ is smooth at $\mathrm{BC}'(x_f)$, as required. ☐

Proposition 5.6, f critical. Suppose f is critical. Let $x \coloneqq \mathrm{BC}(x_f)$ and denote by $\mathfrak{t}_x$ the tangent space of $\mathcal{E}_{\mathrm{bc}}$ at x. As $\mathcal{E}_{\mathrm{bc}}$ is a curve, we know $\dim_L \mathfrak{t}_x \geq 1$, so to prove the proposition we need to show $\dim_L \mathfrak{t}_x \leq 1$.

In order to prove this inequality we use deformations of Galois representations. First we specify the Galois groups involved.

Definition 5.7. Let S be the union of the infinite place with the set of places of $\mathbb{Q}$ supporting N and S_K the set of places of K lying over S. We let $G_{\mathbb{Q},S}$ and G_{K,S_K} be the Galois groups of the maximal algebraic extension of $\mathbb{Q}$ (resp. the imaginary quadratic field K) ramified only at the places S (resp. S_K).

Note that ρ_f factors through $G_{\mathbb{Q},S}$; from now on we consider ρ_f as defined on $G_{\mathbb{Q},S}$. Let $\rho_x = \rho_f|_{G_{K,S_K}}$, the Galois representation attached to x. Here and throughout, we use decomposition groups

$$G_{K_{\mathfrak{q}}} \to G_{K,S_K}, \qquad G_{\mathbb{Q}_q} \to G_{\mathbb{Q},S} \tag{5.1}$$

and complex conjugation $c \in G_{\mathbb{Q},S}$ arising from the choices of embeddings at the beginning of §2. Likewise, $I_{\mathfrak{q}} \subset G_{K_{\mathfrak{q}}}$ denotes an inertia subgroup; similarly, we use I_q over $\mathbb{Q}$.

We adapt the argument used in [Bel12, Thm. 2.16]. By [Bel12, Prop. 2.11], the restriction of ρ_f to $G_{\mathbb{Q}_p}$ decomposes as a direct sum of two characters χ_1 and χ_2. Moreover, both are crystalline, and we order them so that the Hodge–Tate weights of χ_1 and χ_2 are 0 and $k+1$ respectively. (By convention, the cyclotomic character has Hodge–Tate weight -1). Moreover as f is decent, we have $\mathrm{H}^1_f(\mathbb{Q}, \mathrm{ad}\,\rho_f) = 0$. For each $\mathfrak{p} \mid p$, write $\rho_x|_{G_{K_{\mathfrak{p}}}} = \chi_{1,\mathfrak{p}} \oplus \chi_{2,\mathfrak{p}}$ where $\chi_{i,\mathfrak{p}} = \chi_i|_{G_{K_{\mathfrak{p}}}}$ for $i = 1, 2$. We consider the following deformation problems.

Definition 5.8. Let $\mathcal{A}_L$ denote the category of Artinian local L-algebras A with residue field L, and for each $A \in \mathcal{A}_L$, let $X^{\mathrm{ref}}(A)$ be the set of deformations (under strict equivalence) ρ_A of ρ_x to A satisfying the following.

(i) If $\mathfrak{q}$ is a prime of K dividing $\mathfrak{n}$ but coprime to p, then $\rho_A|_{I_{\mathfrak{q}}}$ is constant.

(ii) For each $\mathfrak{p} \mid p$ in K, we have:

  (1) (*null weights*) for each embedding $\tau : K_{\mathfrak{p}} \hookrightarrow L$, one of the τ-Hodge–Sen–Tate weights of $\rho_A|_{G_{K_{\mathfrak{p}}}}$ is 0;

(2) (*crystalline periods/weakly refined*) there exists $\widetilde{\alpha}_{\mathfrak{p}} \in A$ such that the $K_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} A$-module $D_{\mathrm{crys}}(\rho_A|_{G_{K_{\mathfrak{p}}}})^{\varphi^{f_{\mathfrak{p}}}=\widetilde{\alpha}_{\mathfrak{p}}}$ is free of rank 1 and $(\widetilde{\alpha}_{\mathfrak{p}} \mod \mathfrak{m}_A) = \alpha_{\mathfrak{p}}$, where $f_{\mathfrak{p}}$ is the inertia degree of $\mathfrak{p}$.

Define also $X^{\mathrm{ref,bc}}(A)$ to be the set of deformations $\rho_A \in X^{\mathrm{ref}}(A)$ also satisfying:

(iii) (*base-change*) ρ_A admits an extension to $G_{\mathbb{Q},S}$ deforming ρ_f.

Write $\mathfrak{t}^{\mathrm{ref}} := X^{\mathrm{ref}}(L[\varepsilon])$ and $\mathfrak{t}^{\mathrm{ref,bc}} := X^{\mathrm{ref,bc}}(L[\varepsilon])$ for the corresponding tangent spaces where, as usual, $L[\varepsilon] = L[X]/(X^2)$.

We can evaluate ρ_f at complex conjugation c, and note that the operation

$$\iota : \rho_A \longmapsto \left[\mathrm{ad}\,\rho_f(c) \cdot \rho_A^c : g \mapsto \rho_f(c)\rho_A(cgc)\rho_f(c)\right]$$

is a functorial involution on X^{ref}. We thank Carl Wang-Erickson for explaining the utility of this involution, and for supplying the appendix that proves the following.

Proposition 5.9. *(i) The fixed point functor $(X^{\mathrm{ref}})^{\iota}$ is canonically isomorphic to $X^{\mathrm{ref,bc}}$.*

(ii) The deformation problems $X^{\mathrm{ref,bc}}, X^{\mathrm{ref}}$ on $\mathcal{A}_L$ are pro-represented by complete Noetherian local rings $R^{\mathrm{ref,bc}}, R^{\mathrm{ref}} \in \mathcal{A}_L$. The involution ι induces an automorphism $\iota^* : R^{\mathrm{ref}} \to R^{\mathrm{ref}}$, and there is a natural surjection

$$R^{\mathrm{ref}} \twoheadrightarrow \frac{R^{\mathrm{ref}}}{((1-\iota^*)(R^{\mathrm{ref}}))} \cong R^{\mathrm{ref,bc}}.$$

(iii) There is a canonical injection $\mathfrak{t}^{\mathrm{ref,bc}} \hookrightarrow \mathfrak{t}^{\mathrm{ref}}$ of tangent spaces. The image of this injection is the subspace $(\mathfrak{t}^{\mathrm{ref}})^{\iota}$ fixed by the involution $\iota_* : \mathfrak{t}^{\mathrm{ref}} \to \mathfrak{t}^{\mathrm{ref}}$ induced by ι.

We now construct examples of such deformations over the eigenvariety. Let $\mathcal{O}_x$ be the local ring of $\mathcal{E}_{bc}$ at x. Firstly, we observe the following.

Lemma 5.10. *There exists a neighborhood V of x in $\mathcal{E}_{\mathrm{bc}}$ and a Galois representation $\rho_V : G_{K,S_K} \to \mathrm{GL}_2(\mathcal{O}(V))$ such that for each classical point $z \in V$, the specialisation $\rho_{V,z}$ of ρ_V at z is the Galois representation attached to z.*

Indeed, by a theorem of Rouquier and Nyssen (see [Rou96] or [Nys96]), one obtains such a representation from the Galois pseudorepresentation on $\mathcal{E}_{\mathrm{bc}} \subset \mathcal{E}$ constructed in [JN16b]. One can check that if $V_{\mathbb{Q}}$ is a suitable neighbourhood of x_f in $\mathcal{C}$, then ρ_V is the restriction of $\rho_{V_{\mathbb{Q}}}$ to G_{K,S_K}, where $\rho_{V_{\mathbb{Q}}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O}(V_{\mathbb{Q}}))$ lifts ρ_f. (This restriction can be seen to take values in the subring $\mathcal{O}(V) \subset \mathcal{O}(V_{\mathbb{Q}})$ by using the explicit description of this inclusion in [JN16a]).

After localising ρ_V at x, we obtain a representation $\rho_{V,x} : G_{K,S_K} \to \mathrm{GL}_2(\mathcal{O}_x)$. Now if I is a cofinite length ideal of $\mathcal{O}_x$, then from the interpolation property of ρ_V and [Liu15, Prop. 4.1.13] we deduce that $\rho_{V,x} \otimes \mathcal{O}_x/I$ satisfies condition (ii,2) defining $X^{\mathrm{ref,bc}}$, with $\widetilde{\alpha}_{\mathfrak{p}}$ the image of the $U_{\mathfrak{p}}$ operator in $\mathcal{O}_x/I$ (see also Lemma A.4 of the appendix). Using the same argument as in the proof of [Bel12, Thm. 2.16], or using the fact that $\rho_V = \rho_{V_{\mathbb{Q}}}|_{G_{K,S_K}}$, we deduce conditions (i) and (ii,1). We have a given extension to $G_{\mathbb{Q},S}$, giving (iii). Thus the strict class of $\rho_{V,x} \otimes \mathcal{O}_x/I$ is an element of $X^{\mathrm{ref,bc}}(\mathcal{O}_x/I)$. Considering the universal property, and taking the limit with respect to I, we obtain a morphism $R^{\mathrm{ref,bc}} \to \widehat{\mathcal{O}}_x$, the target being the completed local ring at x. A standard argument (see [Ber17, Prop. 4.5]) shows that this morphism is surjective. It follows that $\dim_L \mathfrak{t}_x \leq \dim_L \mathfrak{t}^{\mathrm{ref,bc}}$. Hence it suffices to bound the dimension of $\mathfrak{t}^{\mathrm{ref,bc}}$.

We reduce the argument to a result of Bellaïche. Indeed, in [Bel12, Thm. 2.16], he defines a deformation functor D on $G_{\mathbb{Q}}$ representations deforming ρ_f, satisfying the $G_{\mathbb{Q}}$ analogues of the conditions defining X^{ref}. Using the hypothesis that $\mathrm{H}^1_f(\mathbb{Q}, \mathrm{ad}\,\rho_f) = 0$, he bounds the dimension of the Zariski tangent space of D, which he denotes t_D, by 1. We will show there exists an isomorphism $t_D \cong \mathfrak{t}^{\mathrm{ref,bc}}$. Indeed, by ignoring all the deformation conditions, we can view $\mathfrak{t}^{\mathrm{ref,bc}}$ as a subspace of the tangent space without conditions, which we identify with $\mathrm{H}^1(K, \mathrm{ad}\,\rho_x)$. Using condition (iii) and Proposition 5.9 it is moreover a subspace of $\mathrm{H}^1(K, \mathrm{ad}\,\rho_x)^\iota \cong \mathrm{H}^1(\mathbb{Q}, \mathrm{ad}\,\rho_f)$.

Claim 5.11. *Under the isomorphism* $\phi : \mathrm{H}^1(K, \mathrm{ad}\,\rho_x)^\iota \xrightarrow{\sim} \mathrm{H}^1(\mathbb{Q}, \mathrm{ad}\,\rho_f)$*, the tangent space* $\mathfrak{t}^{\mathrm{ref,bc}}$ *is mapped isomorphically onto the tangent space* t_D *considered in [Bel12, Thm. 2.16].*

Proof of claim: If $\rho_x^\varepsilon \in \mathfrak{t}^{\mathrm{ref,bc}}$, then it admits an extension ρ_f^ε to $G_\mathbb{Q}$ deforming ρ_f. By Lemma A.4 of the appendix, ρ_f^ε satisfies precisely the conditions required to be in $t_D = D(L[\varepsilon])$ in [Bel12]. Hence $\phi(\mathfrak{t}^{\mathrm{ref,bc}}) \subset t_D$. If conversely we take a deformation $\rho_f^\varepsilon \in t_D$, then again by Lemma A.4 we have $\rho_f^\varepsilon|_{G_K} \in X^{\mathrm{ref}}(L[\varepsilon])$. But by definition this restriction also lies in $X^{\mathrm{bc}}(L[\varepsilon])$, so in fact in $\mathfrak{t}^{\mathrm{ref,bc}}$. This is enough to show that $t_D \subset \phi(\mathfrak{t}^{\mathrm{ref,bc}})$, completing the proof of the claim and, by Bellaïche's result, the proof of Proposition 5.6 in the critical case. □

Finally, there is another class of Bianchi families – that arise from base-change, but are not themselves base-change – living in the parallel weight eigenvariety. Let f be as above, let $\mathcal{F}$ be its base-change to K, and let φ be any finite order Hecke character of K with conductor prime to p. Let $\mathcal{C}'$ be the unique irreducible component of the Coleman–Mazur eigencurve through x_f. Then there exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ and a Zariski-dense set Y of points $y \in \mathcal{C}'$ such that $\mathcal{F}_y \otimes \varphi$ has level $\mathfrak{N}$, where $\mathcal{F}_y$ is the base-change of f_y. By applying [Han17, Thm. 5.1.2] to the $\mathrm{BC}(Y)$, we obtain a closed immersion

$$[\varphi] : \mathrm{BC}(\mathcal{C}') \hookrightarrow \mathcal{E}_{\mathrm{par},\mathfrak{N}}$$

interpolating the twist by φ on classical points. Combining with the smoothness and reducedness results above, we've shown the following.

Proposition 5.12. *If* $x = x_{\mathcal{F}\otimes\varphi}$ *is the point corresponding to* $\mathcal{F} \otimes \varphi$*, then there exists an irreducible component* $\mathcal{E}'_{\mathfrak{N}}$ *of* $\mathcal{E}_{\mathrm{par},\mathfrak{N}}$ *through* x *and a smooth reduced neighbourhood* V *of* x *in* $\mathcal{E}'_{\mathfrak{n}'}$*.*

5.3. The Σ-smoothness condition

We would like to conclude that $\mathcal{E}_{\mathrm{par}}$ is smooth at base-change points (or twists thereof). However, without additional work we are unable to rule out the possibility that there exist other irreducible components of $\mathcal{E}_{\mathrm{par}}$, not contained in $\mathcal{E}_{\mathrm{bc}}$, that meet $\mathcal{E}_{\mathrm{bc}}$ at such points. With this in mind, we make the following definition.

Definition 5.13. Let $x \in \mathcal{E}_{\mathrm{bc}}$ be as above. We say that x is Σ-*smooth* if it is smooth in $\mathcal{E}_{\mathrm{par}}$ or, equivalently, if there does not exist another component of $\mathcal{E}_{\mathrm{par}}$ not contained in $\mathcal{E}_{\mathrm{bc}}$ that intersects $\mathcal{E}_{\mathrm{bc}}$ at x.

We conjecture that every classical base-change point is Σ-smooth. At non-critical points, this holds by Proposition 4.10, which shows that $\mathcal{E}_{\mathrm{par}}$ is étale over Σ at x. In general, this is implied by the following more precise version of Conjecture 1.6.

Conjecture 5.14. *Let* $\mathcal{E}'$ *be an irreducible component of* $\mathcal{E}_{\mathrm{par}}$*. There exists an integer* M*, an irreducible component* $\mathcal{C}'_M$ *of* $\mathcal{C}_M$*, and a finite order Hecke character* φ *of* K*, with conductor prime to* p*, such that* $\mathcal{E}' = [\varphi] \circ \mathrm{BC}(\mathcal{C}'_M)$*.*

Proof. (Conjecture 5.14 implies Σ-smoothness). Assume this conjecture, let $x_\mathcal{F} \in \mathcal{E}_{\mathrm{par}}$ be a classical point, and suppose that there exist components $\mathcal{E}', \mathcal{E}''$ passing through $x_\mathcal{F}$. After twisting, we can without loss of generality assume that $x_\mathcal{F} \in \mathcal{E}_{\mathrm{bc}}$, and write $\mathcal{F} = \mathrm{BC}(f)$, with $x_f \in \mathcal{C}'_N \subset \mathcal{C}_N$ and $\mathcal{E}' = \mathrm{BC}(\mathcal{C}'_N)$. By the conjecture, there exists some M, an irreducible component $\mathcal{C}'_M \subset \mathcal{C}_M$ and a Hecke character φ of K such that $\mathcal{E}'' = [\varphi]\mathrm{BC}(\mathcal{C}'_M)$. In particular, there exists some classical modular form g such that $[\varphi]\mathrm{BC}(x_g) = \mathrm{BC}(x_f)$, and we have an equality of Galois representations $\rho_f|_{G_K} = \varphi \otimes \rho_g|_{G_K}$, identifying φ with the Galois character associated to it via class field theory. Define an extension $\varphi_\mathbb{Q}$ of φ to $G_\mathbb{Q}$ by setting $\varphi(cg) = \varphi(g)$

for any $g \in G_K$, where c is any choice of lift of the generator of $\mathrm{Gal}(K/\mathbb{Q})$ to $G_{\mathbb{Q}}$. Then one can check that $\rho_f = \varphi_{\mathbb{Q}} \otimes \rho_g$ as $G_{\mathbb{Q}}$-representations, and there is a map $[\varphi_{\mathbb{Q}}] : \mathcal{C}'_M \to \mathcal{C}_N$ making the diagram

$$\begin{array}{ccc} \mathcal{C}'_M & \xrightarrow{[\varphi_{\mathbb{Q}}]} & \mathcal{C}_N \\ \downarrow \mathrm{BC} & & \downarrow \mathrm{BC} \\ \mathrm{BC}(\mathcal{C}'_M) & \xrightarrow{[\varphi]} & \mathcal{E}_{N,\mathrm{par}} \end{array}$$

commute (since it commutes on classical points). In particular, $[\varphi_{\mathbb{Q}}](x_g) = x_f$. Since x_f is smooth in $\mathcal{C}_N$ (see [Bel12, Thm. 2.16]), we must have $[\varphi_{\mathbb{Q}}](\mathcal{C}'_M) = \mathcal{C}'_N$, and hence $\mathcal{E}'' = \mathcal{E}'$. It follows that $x_{\mathcal{F}}$ is Σ-smooth. □

6. Critical slope base-change p-adic L-functions

We now extend the above results to define (two-variable) p-adic L-functions for *critical* base-change Bianchi modular forms. Throughout this section we will assume Σ-smoothness, as in Definition 5.13, where we can prove stronger results[11]. We closely follow the methods of Bellaïche from [Bel12]. The key, as above, is to show that the space of overconvergent Bianchi modular symbols in a family through the critical slope point is one-dimensional over the corresponding Hecke algebra. Bellaïche's arguments require a density of classical points in the family and as such, there is a fundamental barrier to generalising this method to arbitrary critical slope Bianchi modular forms, where we need not have such density results.

We work with the following set-up. Let f be a decent p-stabilised newform of weight $k \geq 2$ without CM by K. Let $\mathcal{F}$ be its base-change[12] to K, which has weight $\lambda = (k, k)$. Assume $\mathcal{F}$ is critical. Let $x = x_{\mathcal{F}}$ denote the corresponding point of $\mathcal{E}_{\mathrm{bc}}$, the base-change eigenvariety of the previous section, with $w(x) = \lambda$, and let (Σ, h) be a slope-adapted affinoid with $\lambda \in \Sigma = \mathrm{Sp}(\Lambda) \subset \mathcal{W}_{K,\mathrm{par}}$ such that we have $\mathfrak{m}_x \subset \mathbb{T}_{\Sigma,h}$. Assume x is Σ-smooth.

There is some connected component V of $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ containing x. We shrink Σ and V multiple times to obtain a freeness condition. Following Bellaïche, we say an affinoid $\Sigma = \mathrm{Sp}(\Lambda) \subset \mathcal{W}_{K,\mathrm{par}}$ is *nice* if Λ is a principal ideal domain. Every classical weight has a basis of nice affinoid neighbourhoods (see the discussion following [Bel12, Defn. 3.5]), so we now shrink Σ and V so that Σ is nice. We then shrink further so that we have the following, which is key to the whole construction.

Proposition 6.1. *Suppose Σ is a nice affinoid neighbourhood of λ.*

(i) The localisation $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_x}$ is free of finite rank over $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$.

(ii) After possibly shrinking Σ to a smaller nice affinoid, there exists a connected component $V = \mathrm{Sp}\, T \subset \mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ of x such that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} T$ is free of rank one over T.

Proof. To prove part (i), we use a lemma of commutative algebra due to Bellaïche (see [Bel12, Lem. 4.1]). This says that if R and T are discrete valuation rings, with T a finite free R-algebra and M a finitely generated T-module that is free as an R-module, then M is finite free over T. We will use this with R, T and M the localisations of Λ, $\mathbb{T}_{\Sigma,h}$ and $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}$ respectively.

The module $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}$ is finite over Λ (by general properties of slope decompositions) and torsion-free (by Lemma 4.9). As Λ is a principal ideal domain, by the structure theory for such modules $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}$ is a finite free Λ-module. It follows that $\mathbb{T}_{\Sigma,h}$ is also finite and torsion-free over Λ, and hence also a finite free Λ-module. Since $\mathcal{W}_{K,\mathrm{par}}$ and $\mathcal{E}_{\mathrm{par}}$ are reduced rigid curves that are smooth at λ and x respectively, the local rings $\Lambda_{\mathfrak{m}_\lambda}$ and $(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x}$ are

[11] Whilst we conjecture that this always holds, we will give partial results in the general case in the next section.

[12] Everything we say will also hold without modification in the case of (Σ-smooth) twists of $\mathcal{F}$ by a finite order Hecke character of K of conductor prime to p.

discrete valuation rings. Thus we are exactly in the situation of Bellaïche's lemma, giving (i).

An identical argument to that used in the proof of Proposition 4.10 now shows that we can shrink Σ to a nice affinoid and find $V = \mathrm{Sp}\,T$ over Σ such that $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} T$ is free of finite rank over T. This rank is preserved by localising at any point of V, so to evaluate it we check at a suitably nice point. Let $y \in V(L)$ be a non-critical classical cuspidal point satisfying multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$, which must exist as such points are Zariski-dense. Then by Theorem 4.7, $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}_{\mathfrak{m}_y}$ is free of rank one over $T_{\mathfrak{m}_y}$, which completes the proof. □

With this local freeness condition in hand, we have the essential results that make Bellaïche's construction possible. The proofs of the following statements follow in an identical way to those in [Bel12] with the appropriate substitutions, so we give only the statements.

Proposition 6.2. *(See [Bel12, Cor. 4.4]). There is an isomorphism*

$$(\mathbb{T}_{\Sigma,h})_{\mathfrak{m}_x} \otimes_{\Lambda_{\mathfrak{m}_\lambda},\lambda} L \cong (\mathbb{T}_{\lambda,h})_{\mathfrak{m}_x},$$

where $\mathbb{T}_{\lambda,h}$ is the image of $\mathbb{H}$ in $\mathrm{End}_L(\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)^{\leq h})$ and we again write $\mathfrak{m}_x$ for the maximal ideal of this space corresponding to x.

Proposition 6.3. *(See [Bel12, Prop. 4.6]). After a finite base extension of Λ, there exists a uniformiser u of $\Lambda_{\mathfrak{m}_\lambda}$ such that there is an isomorphism of $\Lambda_{\mathfrak{m}_\lambda}$-algebras*

$$\Lambda_{\mathfrak{m}_\lambda}[X]/(X^e - u) \cong (\mathbb{T}_{\Sigma,v})_{\mathfrak{m}_x}$$

that sends X to a uniformiser of $(\mathbb{T}_{\Sigma,v})_{\mathfrak{m}_x}$, where e is the ramification index of the weight map $w : \mathcal{E}_{\mathrm{par}} \to \mathcal{W}_{K,\mathrm{par}}$ at x.

Theorem 6.4. *(See [Bel12, Thm. 4.7]). The generalised eigenspace $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))_{(\mathcal{F})} = \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))_{\mathfrak{m}_x}$ has dimension e over L and is free of rank one over the algebra*

$$(\mathbb{T}_{\lambda,h})_{\mathfrak{m}_x} \cong L[X]/(X^e). \tag{6.1}$$

Corollary 6.5. *(See [Bel12, Cor. 4.8]). Under the isomorphism (6.1), we have an equality of $\mathbb{H}$-eigenspaces*

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))[\mathcal{F}] = X^{e-1}\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda(L))_{(\mathcal{F})},$$

which is one-dimensional over L. Its image under the specialisation map ρ_λ is 0.

Note that this corollary says that, if $\mathcal{F}$ is the base-change of a decent form, then the eigenspace in $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\lambda)$ cut out by $\mathcal{F}$ is *always* one-dimensional. If $\Psi_{\mathcal{F}}$ is any generator, then under the specialisation map $\Psi_{\mathcal{F}}$ maps to the classical cohomology class associated to $\mathcal{F}$ if $\mathcal{F}$ is non-critical and to 0 if $\mathcal{F}$ is critical.

Now let $\Psi_{\mathcal{F}}$ be a generator as above.

Definition 6.6. We define the p-adic L-function of $\mathcal{F}$ to be the Mellin transform

$$L_p(\mathcal{F}, *) := \mathrm{Mel}(\Psi_{\mathcal{F}}) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty)).$$

Proposition 6.7. *The distribution $L_p(\mathcal{F}, *)$ is admissible of order $\mathbf{h} = (v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))_{\mathfrak{p}|p}$ in the sense of [Wil17, Defns. 5.10,6.14]. It satisfies the interpolation property that for any Hecke character φ of K of conductor $\mathcal{F}|(p^\infty)$ and infinity type $0 \leq (q,r) \leq (k,k)$, we have*

$$L_p(\mathcal{F}, \varphi_{p-\mathrm{fin}}) = 0.$$

Proof. The proof of admissibility exactly follows that in [Wil17]. The interpolation property is an immediate consequence of the fact that $\rho_\lambda(\Psi_{\mathcal{F}}) = 0$. □

Remarks 6.8: (i) Unlike in the non-critical case, this admissibility and interpolation property is not sufficient to determine $L_p(\mathcal{F},*)$ uniquely.

(ii) One should see the vanishing of the distribution as an exceptional zero phenomenon. We will show that one can obtain 'secondary p-adic L-functions', that is, distributions with a non-trivial interpolation property, by constructing a three-variable p-adic L-function through $L_p(\mathcal{F},*)$, differentiating in the weight variable, and then evaluating at λ. This bears comparison with the results of [BSW17], where exceptional zeros are removed by differentiating in the cyclotomic variable, up to the introduction of an $\mathcal{L}$-invariant.

We end this section with a soft application of the above. From the definition, it is not clear that base-change respects non-criticality. For example, suppose p is inert in K and f of weight $k+2$ has slope $(k+1)/2$ at p; then the slope of $\mathcal{F}$ is $k+1$, which is not small in the sense of the control theorem.

Corollary 6.9. *Let f be a decent non-critical p-stabilised classical newform of weight $k+2$ that does not have CM by K, and let $\mathcal{F}$ be its base-change to K. Suppose the point $x_{\mathcal{F}}$ is Σ-smooth in $\mathcal{E}_{\mathrm{par}}$. Then $\mathcal{F}$ is non-critical.*

Proof. In [Bel12], it is shown that f is critical if and only if the weight map $\mathcal{C} \to \mathcal{W}_{\mathbb{Q}}$ is ramified at x_f. In this section, we have shown similarly that (if it is Σ-smooth) $\mathcal{F}$ is critical if and only if the weight map $\mathcal{E}_{\mathrm{par}} \to \mathcal{W}_{K,\mathrm{par}} \cong \mathcal{W}_{\mathbb{Q}}$ is ramified at $x_{\mathcal{F}}$. In the proof of Proposition 5.6 in the non-critical case, we showed that if f is non-critical, then the map BC is locally an isomorphism over $\mathcal{W}_{\mathbb{Q}}$ at x_f. Hence as the weight map is unramified at x_f, the weight map $\mathcal{E}_{\mathrm{bc}} \to \mathcal{W}_K$ must also be unramified at $x_{\mathcal{F}}$. By Σ-smoothness, the inclusion $\mathcal{E}_{\mathrm{bc}} \subset \mathcal{E}_{\mathrm{par}}$ is locally an equality at $x_{\mathcal{F}}$. The result follows. □

7. Three-variable p-adic L-functions

In this section, we show that (where they exist) the p-adic L-functions attached to classical Bianchi modular forms can be varied in canonical analytic families over neighbourhoods in the eigenvariety. Let $\mathcal{F}$ be a cuspidal Bianchi eigenform of weight λ satisfying multiplicity one for $\mathbb{H}_{\mathfrak{n},p}$ in the sense of Definition 4.6, and suppose either:

(1) $\mathcal{F}$ is non-critical and varies in a family $V = \mathrm{Sp}\, T \subset \mathcal{E}$ over $\Sigma = \mathrm{Sp}\,\Lambda \subset \mathcal{W}_K$, or

(2) $\mathcal{F}$ is (a twist of) the base-change of a p-stabilised decent classical newform of level prime to p, and varies in a family $V = \mathrm{Sp}\, T \subset \mathcal{E}_{\mathrm{par}}$ over $\Sigma = \mathrm{Sp}\,\Lambda \subset \mathcal{W}_{K,\mathrm{par}}$, with V smooth at $x_{\mathcal{F}}$.

Recall that the Mellin transform gives a map $\mathrm{Mel} : \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \longrightarrow \mathcal{D}(\mathrm{Cl}_K(p^\infty), \Lambda)$. The target of this map can be viewed as a space of analytic functions in three variables – two variables coming from functions on $\mathrm{Cl}_K(p^\infty)$, and one variable on Σ. We want the third variable instead to be on V. Following Bellaïche, we add additional T structure by tensoring by T. The following should be viewed as a specialisation map on the space of modular symbols varying over V, rather than Σ.

Definition 7.1. Let $y \in V(L)$ with $w(y) = \kappa$. Define a map

$$\begin{aligned}\mathrm{sp}_y : \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_\Lambda T \longrightarrow & \ \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_\Lambda T/\mathfrak{m}_y \\ & \subset \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\kappa(L))^{\leq h}.\end{aligned}$$

(Note that this is equivariant for the action of the Hecke operators on the cohomology).

Similarly, we can define a specialisation map at the level of distributions.

Definition 7.2. With y and κ as above, define sp'_y to be the map

$$\begin{aligned}\mathrm{sp}'_y : \mathcal{D}(\mathrm{Cl}_K(p^\infty),\Lambda)\otimes_\Lambda T &\longrightarrow \mathcal{D}(\mathrm{Cl}_K(p^\infty),\Lambda)\otimes_\Lambda T/\mathfrak{m}_y\\ &= \mathcal{D}(\mathrm{Cl}_K(p^\infty),\Lambda)\otimes_\Lambda L \cong \mathcal{D}(\mathrm{Cl}_K(p^\infty),L),\end{aligned}$$

where the last isomorphism is [Han17, Prop. 2.2.1].

We can define the *Mellin transform over* V to be $\mathrm{Mel}_V := \mathrm{Mel}\otimes\mathrm{id}$. A simple check shows that the following diagram commutes:

$$\begin{array}{ccc}\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}\otimes_\Lambda T & \xrightarrow{\ \mathrm{Mel}_V\ } & \mathcal{D}(\mathrm{Cl}_K(p^\infty),\Lambda)\otimes_\Lambda T\\ \downarrow \mathrm{sp}_y & & \downarrow \mathrm{sp}'_y\\ \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\kappa)^{\leq h} & \xrightarrow{\ \mathrm{Mel}\ } & \mathcal{D}(\mathrm{Cl}_K(p^\infty),\Lambda)\end{array}\quad . \tag{7.1}$$

To construct the 3-variable p-adic L-function, our strategy will be to exhibit an element of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}\otimes_\Lambda T$ interpolating eigensymbols in $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\kappa)[\mathfrak{m}_y]$ as y varies in V.

7.1. The case $\mathcal{F}$ non-critical or Σ-smooth

Suppose now that either $\mathcal{F}$ is non-critical or critical (base-change) and Σ-smooth. Let $M_V := \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}\otimes_{\mathbb{T}_{\Sigma,h}} T$. By Σ-smoothness, after possibly shrinking Σ and V, we can choose V to be a connected compoment of $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$, and then M_V is a direct summand of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}$. Possibly shrinking further, we know M_V is free of rank one over T by Propositions 4.10 and 6.1; let Ψ_V denote a generator. In the non-critical case, since T is free of rank one over Λ, the element $\Psi_V\otimes 1$ provides the element we desire, and $\mathrm{Mel}_V(\Psi_V\otimes 1)$ gives the required three-variable p-adic L-function. The critical case is a little more involved. Firstly, we lift Proposition 6.3 to a neighbourhood.

Proposition 7.3. *After possibly shrinking* $\Sigma = \mathrm{Sp}\,\Lambda$ *and* $V = \mathrm{Sp}\,T$, *and after a finite base extension of* Λ, *there is an element* $t\in T$ *with* $t(x_{\mathcal{F}})\neq 0$, *an element* $u\in\Lambda$ *with* $u(\lambda)\neq 0$, *and an isomorphism*

$$\Lambda[X](X^e-u)\cong T$$

that sends X *to* t.

Definition 7.4. Let

$$\Phi_V := \sum_{i=0}^{e-1} t^i\Psi_V\otimes t^{e-1-i}\in M_V\otimes_\Lambda T.$$

This depends on the choices made only up to multiplication by an element of $T^\times$ on the first factor.

The module $M_V\otimes_\Lambda T$ carries the structure of a T-module in two ways (one from each factor). An easy calculation, via a telescoping sum, shows that $(t\otimes 1 - 1\otimes t)$ annihilates Φ_V, that is, T acts on Φ_V the same way for both of these T-structures, and consequently Φ_V is well-defined up to multiplication by an element of $T^\times$ on the *second* factor (compare [Bel12, Lem. 4.13]).

Proposition 7.5. *(See [Bel12, Prop. 4.14]). Let* y *be any classical point in* $V(L)$ *that is non-critical (in case (1)) or base-change (in case (2)), and let* $\kappa = w(y)$. *Then* $\mathrm{sp}_y(\Phi_V)$ *is a generator of the (one-dimensional)* L*-vector eigenspace* $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\kappa(L))[\mathfrak{m}_y]$ *where the Hecke operators act with the same eigenvalues as* y.

Proof. The proof is identical to that *op. cit.*. ☐

Definition 7.6. Recall we can realise M_V as a direct summand of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}),\mathscr{D}_\Sigma)^{\leq h}$. By restriction, Mel_V descends to $M_V\otimes_\Lambda T$, and we define

$$\mathcal{L}_p(V) = \mathrm{Mel}_V(\Phi_V)\in\mathcal{D}(\mathrm{Cl}_K(p^\infty),\Lambda)\otimes_\Lambda T\cong\mathcal{D}(\mathrm{Cl}_K(p^\infty),T).$$

Theorem 7.7. *Let $y \in V(L)$ be a non-critical classical point corresponding to a Bianchi modular form $\mathcal{F}_y$. Then as elements of $\mathcal{D}(\mathrm{Cl}_K(p^\infty), L)$, we have*

$$\mathcal{L}_p(y, *) := \mathrm{sp}'_y(\mathcal{L}_p(V))(*) = c_y L_p(\mathcal{F}_y, *),$$

where $c_y \in L^\times$ is as above.

7.2. Removing the Σ-smoothness condition

Let $\mathcal{F}$ be in case (BC) be critical. We now give a construction of a p-adic L-function attached to $\mathcal{F}$ when we do *not* assume Σ-smoothness. In this case we can still construct a canonical admissible distribution $L_p(\mathcal{F})$, and in the next section we prove it satisfies a partial interpolation property.

In this case, there is a unique connected component $V = \mathrm{Sp}(T)$ of $\mathrm{Sp}(\mathbb{T}_{\Sigma,h})$ passing through x and contained in $\mathcal{E}_{\mathrm{bc}}$ (by smoothness). We can restrict the natural closed immersion $\mathcal{E}_{\mathrm{bc}} \hookrightarrow \mathcal{E}_{\mathrm{par}}$ to obtain a closed immersion $\iota : V \hookrightarrow \mathrm{Sp}(\mathbb{T}_{\Sigma,h})$, cut out by a sheaf of ideals $\mathcal{I}$. We define analogues of overconvergent cohomology over V; note that the eigenvariety machine gives us a coherent sheaf $\mathscr{M}_{\mathrm{par}}$ on $\mathcal{E}_{\mathrm{par}}$ with the property that

$$\mathscr{M}_{\mathrm{par}}(\mathrm{Sp}(\mathbb{T}_{\Sigma,h})) = \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h}.$$

By coherence, $\iota_*\iota^*\mathscr{M}_{\mathrm{par}} \cong \mathscr{M}_{\mathrm{par}}/\mathcal{I}\mathscr{M}_{\mathrm{par}}$ as sheaves on $\mathcal{E}_{\mathrm{par}}$, and hence – from the definition of $\mathscr{M}_{\mathrm{par}}/\mathcal{I}\mathscr{M}_{\mathrm{par}}$ – we have

$$\iota^*\mathscr{M}_{\mathrm{par}}(V) = \left[\mathscr{M}_{\mathrm{par}}/\mathcal{I}\mathscr{M}_{\mathrm{par}}\right]\big(\mathrm{Sp}(\mathbb{T}_{\Sigma,h})\big) = \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_{\mathbb{T}_{\Sigma,h}} T. \tag{7.2}$$

Using smoothness of $\mathcal{E}_{\mathrm{bc}}$ at x, we can shrink V so that it is smooth and intersects $\mathcal{E}_{\mathrm{par}}$ only (possibly) at x.

Proposition 7.8. *It is always possible to shrink Σ to a nice affinoid such that there is a canonical locally free rank one coherent quotient $\mathscr{N}$ of $\iota^*\mathscr{M}_{\mathrm{par}}|_V$ with an equality of stalks*

$$[\iota^*\mathscr{M}_{\mathrm{par}}]_y = \mathscr{N}_y$$

at all p-stabilised non-critical classical points of V.

Proof. Since V is a smooth curve, all the local rings are discrete valuation rings. Using the structure theorem for finitely generated modules over such rings, we can exhibit a torsion coherent sheaf $\mathscr{T}$ on V, supported on a finite set of points (possibly including x). We define $\mathscr{N} := \iota^*\mathscr{M}_{\mathrm{par}}/\mathscr{T}$, which is then locally free on V.
At non-critical classical points other than x, we have

$$[\iota^*\mathscr{M}_{\mathrm{par}}]_y = [\mathscr{M}_{\mathrm{par}}]_y = \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)_{\mathfrak{m}_y},$$

since no other component of $\mathcal{E}_{\mathrm{par}}$ intersects $\mathcal{E}_{\mathrm{bc}}$ at y. If further y corresponds to a non-critical p-stabilised newform with multiplicity one, then by Theorem 4.7, this module is free of rank one over T_y. As this stalk has no torsion part, y is not in the support of $\mathscr{T}$, which shows that this stalk is equal to $\mathscr{N}_y$.

Finally, it remains to determine the rank of $\mathscr{N}$. This rank is preserved after localising at any point in V; by doing so at any classical non-critical point y as above, which is possible by Zariski-density, we conclude that the rank is one. □

Hence we have a canonical quotient M_V of $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma) \otimes_{\mathbb{T}_{\Sigma,h}} T$ that is free of rank one over T. Let Φ_V be any generator. As T is Λ-flat, the natural map

$$\mathrm{sp}_V : \mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\Sigma)^{\leq h} \otimes_\Lambda T \to M_V \otimes_\Lambda T$$

is surjective; let $\widetilde{\Phi}_V$ be any lift of Φ_V under this map. At any classical point y of V, the fibres of $\mathscr{N}$ and $\mathscr{M}_{\mathrm{par}}$ are equal at y, and hence the specialisation map sp_y to $\mathrm{H}^1_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathscr{D}_\kappa)^{\leq h}$ factors through sp_V. Consequently, Proposition 7.5 shows that at such classical points, $\mathrm{sp}_y(\widetilde{\Phi}_V)$ is a generator of $\mathrm{H}^1_{\mathrm{c}}(Y, \mathscr{D}_\kappa)[\mathfrak{m}_y]$.

Definition 7.9. Let $\mathcal{L}_p(V) = \mathrm{Mel}_V(\widetilde{\Phi}_V) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), \Lambda) \otimes_\Lambda T \cong \mathcal{D}(\mathrm{Cl}_K(p^\infty), T)$.

Whilst the lift might not be canonical, the resulting function $\mathcal{L}_p(V)$ is. Indeed, from the above it follows that $\mathcal{L}_p(V)$ satisfies the interpolation property of Corollary 1.4 at all *non-critical* classical points, and as explained in the introduction, this is enough to determine $\mathcal{L}_p(V)$ uniquely; in particular, it is independent of the choice of lift $\widetilde{\Phi}_V$.

Definition 7.10. Let $L_p(\mathcal{F}) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), L)$ be the distribution $\mathrm{sp}_x(\mathcal{L}_p(V))$.

Proposition 7.11. *The distribution $L_p(\mathcal{F}, *)$ is well-defined up to scalar multiple by L. It is admissible of order $\mathbf{h} = (v_{\mathfrak{p}}(\alpha_{\mathfrak{p}}))_{\mathfrak{p}|p}$.*

Proof. The function $\mathcal{L}_p(V)$ can be defined as the Mellin transform of a cohomology class in the slope $\leq \mathbf{h}$ cohomology, hence is $\mathbf{h}$-admissible. Let $\mathcal{L}_p$ and $\mathcal{L}_p'$ be any two analytic functions on $V \times \mathscr{X}(\mathrm{Cl}_K(p^\infty))$ satisfying the interpolation property of Corollary 1.4 at all non-critical classical points, and let $L_p(\mathcal{F})$ and $L_p'(\mathcal{F})$ denote their specialisations to x. If either is zero, then obviously they differ by scaling by L; so assume both are non-zero. Consider, then, the quotient

$$C(y, \phi) := \frac{\mathcal{L}_p'(V)(y, \phi)}{\mathcal{L}_p(V)(y, \phi)} \in \mathrm{Frac}\big(\mathcal{O}(V \times \mathscr{X}(\mathrm{Cl}_K(p^\infty)))\big).$$

We claim that this is well-defined. Indeed, there exists a Zariski-dense set of classical non-critical points y in V of weight (k, k), where $k > 2$. For any Dirichlet character φ of conductor p^r, $r \geq 1$, the quantity $L(\mathcal{F}_y, \varphi, k-1)$ converges absolutely to a non-zero number; it follows that $L_p(\mathcal{F}_y, \varphi|\cdot|^{k-1}) \neq 0$, since the p-adic L-function does not have an exceptional zero there. As every connected component of $\mathcal{O}(\mathscr{X}(\mathrm{Cl}_K(p^\infty)))$ contains a character of the form $\varphi(z)z^{k-1}$, it follows that $L_p(\mathcal{F}_y, *)$ is not a zero-divisor in $\mathcal{O}(\mathscr{X}(\mathbb{Z}_p^\times))$. Now suppose $\mathcal{L}_p(V)$ is is a zero-divisor; then there exists some $D \in \mathcal{D}(\mathrm{Cl}_K(p^\infty), \Lambda)$ such that $D\mathcal{L}_p(V) = 0$. After specialising at any y as above, we see that $D = 0$, as $\mathrm{sp}_y(\mathcal{L}_p(V))$ is not a zero-divisor. As this equality holds at a Zariski-dense set of points, it must hold everywhere, and we see that $D = 0$.

At the specialisation to each classical point $y \neq x_{\mathcal{F}}$ in $V(L)$, we have $C(y, \phi) \in L^\times$ using the interpolation at non-critical points. Again, using the fact that such points are Zariski-dense, we deduce that $C(z, \phi)$ is constant in ϕ for any z, that is, $C \in \mathrm{Frac}(\mathcal{O}(V))$. Since (by assumption) neither $L_p(\mathcal{F})$ nor $L_p'(\mathcal{F})$ is zero, C does not have a zero or pole at $x_{\mathcal{F}}$. Such zeros and poles occur at isolated points, as V is a rigid curve, and hence we may shrink V further so that C has no zeros or poles, that is, $C \in \mathcal{O}(V)^\times$. After specialising to $x_{\mathcal{F}}$, we see that $L_p(\mathcal{F})$ and $L_p'(\mathcal{F})$ differ by scalar multiplication by $L^\times$, hence the result. ∎

8. Factorisation of base-change p-adic L-functions

In this section, we provide an application of the existence of families of p-adic L-functions of Bianchi modular forms. In particular, we prove an 'Artin formalism' style result for p-adic L-functions of cuspidal Bianchi modular forms that are the base-change of a classical modular form. Such a result follows from admissibility in the case where the slope is sufficiently small, and can be extended to arbitrary slope using the three-variable p-adic L-function defined above.

8.1. p-adic L-functions attached to classical eigenforms

We recall the relevant existence of p-adic L-functions for classical modular forms. Let $f \in S_{k+2}(\Gamma_1(N))$ be a decent finite slope classical p-stabilised newform with L-function $\Lambda(f, \varphi)$, normalised to include the Euler factors at infinity, and where φ ranges over Hecke characters of $\mathbb{Q}$. Denote the eigenvalue of f at p by $\alpha_p(f)$ and the periods of f by $\Omega_f^\pm$, which are well-defined up to algebraic numbers. Let $h := v_p(\alpha_p)$, and let η be a Dirichlet character of conductor C prime to p. For any Dirichlet character χ, let $\tau(\chi) := \sum_{a \pmod N} \chi(a)e^{2\pi i a/N}$ be the usual Gauss sum.

Theorem 8.1. *There exists a canonical locally analytic distribution $L_p^\eta(f,*)$ on $\mathbb{Z}_p^\times$ such that, for any Hecke character $\varphi = \chi|\cdot|^j$, where χ is finite order of conductor $p^n > 1$ and $0 \leq j \leq k$, we have*

$$L_p^\eta(f,\varphi_{p-\mathrm{fin}}) = \begin{cases} \frac{(Cp^n)^{j+1}}{\tau((\chi\eta)^{-1})\Omega_f^\pm \alpha_p^n}\Lambda(f,\varphi\eta) & : f \text{ is non-critical,} \\ 0 & : f \text{ is critical.}\end{cases}$$

The sign of the period is given by the sign of $\chi\eta(-1)(-1)^j$. The distribution is admissible of order h, and if $h < k+1$, it is uniquely determined by this interpolation property.

If η is the trivial character, we drop the superscript and write L_p for this distribution.

Proof. First suppose η is the trivial character. When $h < k+1$, this has been proved using different constructions by a number of people (see, for example, [PS11]). Note that in our normalisations, $\Lambda(f,\varphi) = \Lambda(f,\overline{\chi},j+1)$. In the case where $h = k+1$, the most comprehensive version of this result is due to Bellaïche (see [Bel12]). If η is non-trivial, this distribution can be defined via a slight variation of the methods of [PS11] and [Bel12]. In both papers, $L_p(f,*)$ is defined by associating to f a canonical overconvergent modular symbol Ψ_f, then setting $L_p(f,*) := \Psi_f\{0-\infty\}|_{\mathbb{Z}_p^\times}$. To obtain this twisted version, suppose $a \in (\mathbb{Z}/C\mathbb{Z})^\times$; then one defines a distribution $L_p^a(f,*)$ on $\mathbb{Z}_p^\times$ by

$$L_p^a(f,*) := [\Psi_f|\left(\begin{smallmatrix}1 & a\\ 0 & C\end{smallmatrix}\right)]\{0-\infty\},$$

then extends to $L_p^\eta(f,*) = \sum_{a\in(\mathbb{Z}/C\mathbb{Z})^\times}\eta(a)L_p^a(f,*)$. Proving the interpolation result is then a formal calculation. This process is described more thoroughly (in the small slope Bianchi setting) in [BSW17, §3.4]. ☐

It will be important to also vary these p-adic L-functions in families over the Coleman–Mazur eigencurve $\mathcal{C}$. For this, we follow the account of Bellaïche from [Bel12], though in the small slope setting this was known previously. The notation used here is directly analogous to that used above in the Bianchi setting.

Theorem 8.2 (Mazur–Kitagawa, Stevens, Bellaïche)**.** *Let x_f be the point of the Coleman–Mazur eigencurve corresponding to f. There exists an affinoid neighbourhood $\mathcal{V}$ of x_f and a locally analytic distribution*

$$\mathcal{L}_p(\mathcal{V}) \in \mathcal{D}(\mathbb{Z}_p^\times, \mathcal{O}(\mathcal{V}))$$

such that at any classical point $y \in \mathcal{V}$, corresponding to a modular form f_y, we have

$$\mathcal{L}_p(y,\phi) := \mathrm{sp}_y'(\mathcal{L}_p(V))(\phi) = c_y L_p(f_y,\phi),$$

where ϕ is any locally analytic function on $\mathbb{Z}_p^\times$ and c_y is a non-zero scalar depending only on y. This distribution is well-defined up to multiplication by elements of $\mathcal{O}(\mathcal{V})$. Similarly, there exists a distribution $\mathcal{L}_p^\eta(V)$ interpolating the twisted p-adic L-functions L_p^η at classical points.

Implicit in this theorem is a choice of period at each classical point, and the indeterminacy in these choices is measured by multiplication by elements of $\mathcal{O}(\mathcal{V})$. The construction of $\mathcal{L}_p(\mathcal{V})$ uses overconvergent modular symbols in families, and the interpolation follows from a commutative diagram analogous to equation (7.1).

8.2. Statement of p-adic Artin formalism

We assume that the base-change of f remains cuspidal, and denote it by $\mathcal{F} \in S_\lambda(U_1(N\mathcal{O}_K))$ (where $\lambda = (k,k)$, viewing f as an adelic automorphic form for $\mathrm{GL}_2/\mathbb{Q}$ of weight $k+2$ and level $U_1(N) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$). We see that $\mathcal{F}$ is an eigenform and the eigenvalues can be described simply in terms of the eigenvalues of f; in particular, we see that

(i) When p splits as $\mathfrak{p}\overline{\mathfrak{p}}$ in K, we have $\alpha_{\mathfrak{p}}(\mathcal{F}) = \alpha_{\overline{\mathfrak{p}}}(\mathcal{F}) = \alpha_p(f)$.

(ii) When p is inert in K, we have $\alpha_{p\mathcal{O}_K}(\mathcal{F}) = \alpha_p(f)^2$.

(iii) when p is ramified as $\mathfrak{p}^2$ in K, we have $\alpha_{\mathfrak{p}}(\mathcal{F}) = \alpha_p(f)$.

We see that $\mathcal{F}$ has small slope if and only if

$$v_p(\alpha_p(f)) < \begin{cases} k+1 & : p \text{ split}, \\ \frac{k+1}{2} & : p \text{ inert or ramified}. \end{cases}$$

As above, write $L_p(\mathcal{F},*)$ for the p-adic L-function of $\mathcal{F}$ (as a distribution on $\mathrm{Cl}_K(p^\infty)$).

Definition 8.3. We define the *restriction of $L_p(\mathcal{F},*)$ to the cyclotomic line*, denoted by $L_p^{\mathrm{cyc}}(\mathcal{F},*)$, to be the locally analytic distribution on $\mathbb{Z}_p^\times$ given by

$$L_p^{\mathrm{cyc}}(\mathcal{F},\phi) := L_p(\mathcal{F},\phi\circ N_{K/\mathbb{Q}}),$$

where ϕ is any locally analytic function on $\mathbb{Z}_p^\times \cong \mathrm{Cl}_{\mathbb{Q}}^+(p^\infty)$.

We will prove the following p-adic version of Artin formalism:

Theorem 8.4. *Let ϕ be any locally analytic function on $\mathbb{Z}_p^\times$. Choose the periods $\Omega_{\mathcal{F}}$ and $\Omega_f^\pm$ such that*

$$\Omega_{\mathcal{F}} = (-1)^k\frac{\#\mathcal{O}_K^\times}{2}\Omega_f^+\Omega_f^-\tau(\chi_{K/\mathbb{Q}}),$$

where τ denotes the usual Gauss sum of Dirichlet characters, and $\chi_{K/\mathbb{Q}}$ is the quadratic character associated to K. (This convention on the sign is possible since $\chi_{K/\mathbb{Q}}$ is odd). If f is critical, suppose that $L_p^{\mathrm{cyc}}(\mathcal{F})$ and $L_p(f)L_p^{\chi_{K/\mathbb{Q}}}(f)$ are both non-zero. Then we have

$$L_p^{\mathrm{cyc}}(\mathcal{F},\phi) = L_p(f,\phi)L_p^{\chi_{K/\mathbb{Q}}}(f,\phi),$$

In other words, we have $L_p^{\mathrm{cyc}}(\mathcal{F}) = L_p(f)L_p^{\chi_{K/\mathbb{Q}}}(f)$ as distributions on $\mathbb{Z}_p^\times$.

8.3. The case of slope $< (k+1)/2$

First we show the result for forms of sufficiently small slope. Suppose $f \in S_{k+2}(\Gamma_1(N))$ has slope $h < (k+1)/2$ at p, with base-change $\mathcal{F}$ to K. In this case, both the restriction of $L_p(\mathcal{F},*)$ and the product $L_p(f,*)L_p^{\chi_{K/\mathbb{Q}}}(f,*)$ are distributions on $\mathbb{Z}_p^\times$ that are admissible of order $2h < k+1$, and hence it suffices to prove that they agree at the critical Hecke characters, as then the admissibility condition ensures that any two distributions that agree on this set are equal. At the level of classical L-values, Artin formalism says that for any rational Hecke character φ, we have

$$\Lambda(\mathcal{F},\varphi\circ N_{K/\mathbb{Q}}) = \Lambda(f,\varphi)\Lambda(f,\chi_{K/\mathbb{Q}}\varphi),$$

so it suffices to check that the constants in the interpolation formulae agree. In this situation, the interpolating constant of the Bianchi p-adic L-function can be simplified to:

Proposition 8.5. *Let $\phi = \chi|\cdot|^j$ with $\mathrm{cond}(\chi) = p^n > 1$ and $0 \leq j \leq k$, and define $\varphi = \phi\circ N_{K/\mathbb{Q}}$. We have*

$$L_p^{\mathrm{cyc}}(\mathcal{F},\phi) = L_p(\mathcal{F},\varphi) = \left[\frac{d^{j+1}p^{2n(j+1)}\#\mathcal{O}_K^\times}{(-1)^k 2\alpha_p(\mathcal{F})^n\tau_K((\chi\circ N_{K/\mathbb{Q}})^{-1})\Omega_{\mathcal{F}}}\right]\Lambda(\mathcal{F},\varphi),$$

where for a character $\eta : (\mathcal{O}_K/p^n)^\times \to \mathbb{C}^\times$, we define $\tau_K(\eta)$ by

$$\tau_K(\eta) := \sum_{a \pmod{p^n\mathcal{O}_K}} \eta(a)e^{2\pi i \mathrm{Tr}_{K/\mathbb{Q}}\left(\frac{a}{p^n\sqrt{d}}\right)}. \tag{8.1}$$

Proof. This is an exercise in book-keeping, made confusing only by an unfortunate plethora of differing normalisations, which we highlight here. Firstly, by demanding that $p|\mathrm{cond}(\varphi)$, we see that the exceptional factors $Z_{\mathfrak{p}}$ of Theorem 2.14 are equal to 1; the infinity type is (j,j), which simplifies the sign; and the terms $\varphi(x_{\mathfrak{f}})$ and $\varphi_{\mathfrak{f}}(x_{\mathfrak{f}})$, as defined in [Wil17], cancel since the conductor is principal. Finally, the remaining discrepancies between the formulae can be described by renormalisations of Gauss sums. Indeed, the Gauss sums $\widetilde{\tau}(\varphi \circ N_{K/\mathbb{Q}})$ and $\widetilde{\tau}(\chi \circ N_{K/\mathbb{Q}})$ differ by $N_{K/\mathbb{Q}}(p^n\sqrt{-d})^j = d^j p^{2nj}$ (see, for example, [Wil17, §2.6]), whilst the Gauss sum τ_K above (for Dirichlet characters) is naturally inverse to the Gauss sum $\widetilde{\tau}$ used in [Wil17] (for Hecke characters). We have also made use of the standard identity $\tau_K(\varphi)\tau_K(\varphi^{-1}) = N_{K/\mathbb{Q}}(\mathrm{cond}(\varphi))$ to move the Gauss sum to the denominator, in line with Theorem 8.1. □

Recall now that we normalised the periods $\Omega_{\mathcal{F}}$ and $\Omega_f^{\pm}$ such that $\Omega_{\mathcal{F}} = (-1)^k \frac{\#\mathcal{O}_K^\times}{2}\Omega_f^+\Omega_f^-\tau(\chi_{K/\mathbb{Q}})$. From this, and the descriptions of $\alpha_p(\mathcal{F})$ in terms of $\alpha_p(f)$ above, it is immediate that

$$\frac{d^{j+1}p^{2n(j+1)}\#\mathcal{O}_K^\times}{(-1)^k 2\alpha_p(\mathcal{F})^n\Omega_{\mathcal{F}}}\cdot\tau(\chi_{K/\mathbb{Q}}) = \frac{(p^n)^{j+1}}{\alpha_p(f)^n\Omega_f^{\pm}}\cdot\frac{(dp^n)^{j+1}}{\alpha_p(f)\Omega_f^{\mp}}.$$

To complete the proof in the slope $< (k+1)/2$ case, it remains only to check the identity

$$\tau_K((\chi\circ N_{K/\mathbb{Q}}))\tau(\chi_{K/\mathbb{Q}}) = \tau(\chi)\tau(\chi\chi_{K/\mathbb{Q}})$$

of Gauss sums. This is a characteristic 0 version of the classical Hasse–Davenport identity. A simple check shows that it suffices to check this identity locally. In the case p unramified in K, it is [Mar72, §6, Cor. 1]. We've shown:

Corollary 8.6. *Suppose f has slope $v_p(\alpha_p(f)) < \frac{k+1}{2}$. Then Theorem 8.4 holds.*

8.4. The general case

Now suppose f has slope $\frac{k+1}{2} \leq h$. The product $L_p(f,*)L_p^{\chi_{K/\mathbb{Q}}}(f,*)$ and the restriction of $L_p(\mathcal{F},*)$ to the cyclotomic line are both admissible of order $2h \geq k+1$, so we cannot use the methods of §8.3 to prove Theorem 8.4 in this case. To get around this, we use the three variable p-adic L-function through $\mathcal{F}$ over the base-change component of the eigenvariety.

Notation 8.7: Let $V_{\mathbb{Q}}$ be a neighbourhood of x_f in the Coleman–Mazur eigencurve lying over some subset $\Sigma_{\mathbb{Q}} \subset \mathcal{W}_{\mathbb{Q}}$. Let V_K denote the image of $V_{\mathbb{Q}}$ under the p-adic base-change map.

We see that V_K is a neighbourhood of $x_{\mathcal{F}} = \mathrm{BC}(x_f)$ in the Bianchi eigenvariety, containing a Zariski-dense subset of classical points. For any such classical point $y \in V_{\mathbb{Q}}$, write f_y for the corresponding modular form, and write $\mathcal{F}_y$ for its base-change to K (corresponding to $\mathrm{BC}(y) \in V_K$).

Since the slope of a Coleman family is constant, we see that along V_K, the slope at p is also constant, equal to $v_p(\alpha_p(\mathcal{F})) = 2v_p(\alpha_p(f)) = 2h$. Possibly shrinking $V_{\mathbb{Q}}$ if necessary, we can assume that any classical weight $\ell \in \Sigma_{\mathbb{Q}}\backslash\{k\}$ satisfies $\ell > 2(k+1)$. Suppose y is a classical point in $V_{\mathbb{Q}}$ above such a weight ℓ; then we have $v_p(\alpha_p(f_y)) < \frac{\ell+1}{2}$, so that

$$L_p^{\mathrm{cyc}}(\mathcal{F}_y,*) = L_p(f_y,*)L_p^{\chi_{K/\mathbb{Q}}}(f_y,*),$$

where again we normalise the periods appropriately.

After possibly shrinking V_K, let $\mathcal{L}_p(V_K)$ denote the three-variable p-adic L-function over V_K. Again by restricting to functions that factor through the norm to $\mathbb{Q}$, we can restrict this three-variable function to the cyclotomic line, yielding a two-variable function $\mathcal{L}_p^{\mathrm{cyc}}(V_K)$. This two-variable p-adic L-function is only well-defined up to multiplication by elements of $\mathcal{O}(V_K)$, corresponding to renormalising the periods. By composing with the map $\mathrm{BC}^* : \mathcal{O}(V_K) \to \mathcal{O}(V_{\mathbb{Q}})$, we can view the second variable as being over $V_{\mathbb{Q}}$, meaning $\mathcal{L}_p^{\mathrm{cyc}}(V_K)$ lies in the same space as $L_p(V_{\mathbb{Q}})$ and $\mathcal{L}_p^{\chi_{K/\mathbb{Q}}}(V_{\mathbb{Q}})$, the two-variable p-adic L-functions over $V_{\mathbb{Q}}$ interpolating the p-adic L-functions of the classical family.

Proposition 8.8. *Suppose that $L_p^{\mathrm{cyc}}(\mathcal{F},*)$ and $L_p(f,*)L_p^{\chi_{K/\mathbb{Q}}}(f,*)$ are both non-zero. For each classical point $y \in V_{\mathbb{Q}}(L)$, normalise the period of the base-change $\mathcal{F}_y$ so that*

$$\Omega_{\mathcal{F}_y} = (-1)^{\ell}\frac{\#\mathcal{O}_K^\times}{2}\Omega_{f_y}^+\Omega_{f_y}^-\tau(\chi_{K/\mathbb{Q}}).$$

Under these normalisations, and after possibly renormalising $\mathcal{L}_p^{\mathrm{cyc}}(V_K)$ by an element of $\mathcal{O}(V_{\mathbb{Q}})^\times$, the restriction of $\mathcal{L}_p(V_K)$ to the cyclotomic line factors as

$$\mathcal{L}_p^{\mathrm{cyc}}(V_K) = \mathcal{L}_p(V_{\mathbb{Q}})\mathcal{L}_p^{\chi_{K/\mathbb{Q}}}(V_{\mathbb{Q}}).$$

In the general case, Theorem 8.4 follows by specialising this identity at f.

Proof. After taking the Amice transform, we may consider the functions in question as analytic functions on the two-dimensional rigid space $V_{\mathbb{Q}} \times \mathscr{X}(\mathbb{Z}_p^\times)$, where, as in the introduction, we write $\mathscr{X}(\mathbb{Z}_p^\times)$ for the rigid character space of $\mathbb{Z}_p^\times$. Consider, then, the quotient

$$C(z,\phi) := \frac{\mathcal{L}_p^{\mathrm{cyc}}(V_K)}{\mathcal{L}_p(V_{\mathbb{Q}})\mathcal{L}_p^{\chi_{K/\mathbb{Q}}}(V_{\mathbb{Q}})} \in \mathrm{Frac}\big(\mathcal{O}(V_{\mathbb{Q}} \times \mathscr{X}(\mathbb{Z}_p^\times))\big).$$

This is well-defined by a similar argument to that in Proposition 7.11. At the specialisation to each classical point $y \neq x_f$ in $V_{\mathbb{Q}}(L)$, we have $C(y,\phi) = c_y \in L^\times$ using the factorisation at very small slope points. Again, using the fact that such points are Zariski-dense, we deduce that $C(z,\phi)$ is constant in ϕ for any z, that is, $C \in \mathrm{Frac}(\mathcal{O}(V_{\mathbb{Q}}))$. Since (by assumption) neither $L_p^{\mathrm{cyc}}(\mathcal{F}_x,*)$ nor $L_p(f,*)L_p^{\chi_{K.\mathbb{Q}}}(f,*)$ is zero, C does not have a zero or pole at $x_{\mathcal{F}}$. Such zeros and poles occur at isolated points, as $V_{\mathbb{Q}}$ is a rigid curve, and hence we may shrink $V_{\mathbb{Q}}$ further so that C has no zeros or poles, that is, $C \in \mathcal{O}(V_{\mathbb{Q}})^\times$. But this completes the proof. □

In the general case, Theorem 8.4 follows from specialising this result at f.

Remark 8.9: The function $\mathcal{L}_p^{\mathrm{cyc}}(V)$ is, of course, itself only well-defined up to multiplication by elements of $\mathcal{O}(V)^\times$, so this indeterminancy is expected. We note that the non-vanishing condition is always satisfied if f and $\mathcal{F}$ are non-critical by the arguments in the proof (or, in the case of weight 2, by a theorem of Rohrlich; see [Roh84]). When f is critical, it is conjectured that f is CM, and in this case, Bellaïche has shown this non-vanishing property by relating the p-adic L-function to a Katz p-adic L-function (see [Bel]). In light of this, it seems natural to conjecture that when $\mathcal{F}$ is critical, $L_p(\mathcal{F},*)$ is non-zero.

Remark 8.10: Note this result requires only the existence of p-adic L-functions in families, which we constructed under no Σ-smoothness condition. In particular, this implies that in the case where $\mathcal{F}$ is the critical base-change of f and we do *not* have Σ-smoothness, the p-adic L-function $L_p(\mathcal{F})$ satisfies an interpolation property at all Hecke characters φ that factor as $\varphi' \circ N_{K/\mathbb{Q}}$ for a rational Hecke character φ'. If f is non-critical, this interpolation property is the same as that in Theorem 2.14; if f is critical, the interpolation property is that the p-adic L-function vanishes at such characters.

Remark 8.11: Suppose p is split, and that we start with a small slope classical form f of level N prime to p. Let α and β denote the roots of the Hecke polynomial $R_p(X)$, and assume that $\alpha \neq \beta$. There are two possible p-stabilisations f_α, f_β of f to level pN. The base-change $\mathcal{F}$ of f to K, however, has *four* possible p-stabilisations to level $pN\mathcal{O}_K$; we have $R_p(X) = R_{\mathfrak{p}}(X) = R_{\overline{\mathfrak{p}}}(X)$, so we can consider $\mathcal{F}_{\alpha\alpha}, \mathcal{F}_{\alpha\beta}, \mathcal{F}_{\beta\alpha}$ and $\mathcal{F}_{\beta\beta}$. The forms $\mathcal{F}_{\alpha\alpha}$ and $\mathcal{F}_{\beta\beta}$ are the base-changes of f_α and f_β respectively, but the other specialisations cannot be base-change themselves, as they have distinct eigenvalues at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. In this case, Loeffler and Zerbes have recently shown that $L_p^{\mathrm{cyc}}(\mathcal{F}_{\alpha\beta},\phi)$ can be expressed as a linear combination of the two products $L_p(f_\alpha,\phi)L_p^{\chi_{K/\mathbb{Q}}}(f_\beta,\phi)$ and $L_p(f_\beta,\phi)L_p^{\chi_{K/\mathbb{Q}}}(f_\alpha,\phi)$.

We can now prove a converse to Corollary 6.9, which said that (in the Σ-smooth case) the base-change of a non-critical form is non-critical.

Corollary 8.12. *Let f be a decent classical modular form of weight $k+2$ that does not have CM by K, and let $\mathcal{F}$ be its base-change to K. Suppose $\mathcal{F}$ is Σ-smooth. Then $\mathcal{F}$ is non-critical if and only if f is non-critical.*

Proof. We saw that $\mathcal{F}$ is non-critical if f is non-critical in Corollary 6.9. So suppose f is critical. If $\mathcal{F}$ is not Σ-smooth, then it is critical; hence we may assume $\mathcal{F}$ is Σ-smooth without loss of generality. To f, we attach a generator Ψ_f of the one-dimensional eigenspace $\mathrm{H}^1_c(Y_1(N), \mathscr{D}_\lambda(L))[f]$ in the overconvergent cohomology (over $\mathbb{Q}$). Recall from [Bel12, §4] that f is critical if and only if Ψ_f is mapped to zero under the specialisation map to classical cohomology. Let ϕ be any critical Hecke character; then $L_p(f,\phi) = \mathrm{Mel}(\Psi_f)(\phi) = \mathrm{Mel}(\rho_\lambda(\Psi_f))(\phi)$, that is, evaluation at ϕ commutes with the specialisation map. Thus if we have non-vanishing of a critical *p-adic* L-value of f, then $\rho_\lambda(\Psi_f) \neq 0$, that is, f is non-critical.

We claim that there exists a non-trivial Dirichlet character φ of p-power conductor such that

$$L(\mathcal{F}, \varphi \circ N_{K/\mathbb{Q}}, k+1) = L(f, \varphi, k+1)L(f, \varphi\chi_{K/\mathbb{Q}}, k+1) \neq 0.$$

Indeed, if $k > 0$, then for any Dirichlet character φ, the Euler product expressions for $L(f,\varphi,k+1)$ and $L(f,\varphi\chi_{K/\mathbb{Q}},k+1)$ converge to non-zero complex numbers (as we are in the range of absolute convergence). If $k=0$, then this is an easy consequence of the main result of [Roh84].

Since φ has p-power conductor, the p-adic L-functions $L_p(f,*)$, $L_p^{\chi_{K/\mathbb{Q}}}(f,*)$ and $L_p(\mathcal{F},*)$ do not have exceptional zeros at the character $\phi = \varphi|\cdot|^k$. Now suppose $\mathcal{F}$ is non-critical. By the interpolation property, the non-vanishing condition on φ, and the fact that $\mathcal{F}$ is non-critical, we then have

$$0 \neq L_p(\mathcal{F}, \phi \circ N_{K/\mathbb{Q}}) = L_p(f,\phi)L_p^{\chi_{K/\mathbb{Q}}}(f,\phi),$$

the last equality following from p-adic Artin formalism. From the remarks above, we conclude that f is non-critical, which is a contradiction. □

If f is critical, then $\mathcal{F}$ is always critical, since if it were non-critical it would be Σ-smooth. So the only remaining possibility we have not ruled out is f non-critical but $\mathcal{F}$ critical and not Σ-smooth.

8.5. Restriction to the anticyclotomic line

The methods of this section apply in another related case[13], the details of which we leave to the interested reader. Class field theory provides us with an isomorphism $\mathrm{Cl}_K(p^\infty) \cong \mathrm{Gal}(K_\infty/K)$, where K_∞ is the maximal abelian extension of K unramified outside p. Restriction to the cyclotomic line in $\mathrm{Cl}_K(p^\infty)$ is equivalent to looking only at the cyclotomic subextension inside K_∞. We can also naturally restrict to the *anticyclotomic* subextension $K_\infty^{\mathrm{anti}}/K$; a character χ of $\mathrm{Gal}(K_\infty/K)$ is *anticyclotomic* if $\chi(\sigma g \sigma^{-1}) = \chi(g)^{-1}$ for all $g \in \mathrm{Gal}(K_\infty/K)$, where σ is the non-trivial element in $\mathrm{Gal}(K/\mathbb{Q})$. In this setting, anticyclotomic p-adic L-functions were introduced by Bertolini and Darmon in [BD96] for ordinary elliptic curves, and since then the construction has been generalised significantly; see, for example, [Kim17]. In the notation of above, the *anticyclotomic p-adic L-function of f over K* is a distribution $L_p^{\mathrm{anti}}(f,*)$ on $\mathrm{Gal}(K_\infty^{\mathrm{anti}}/K)$, of order h, that satisfies the interpolation property that at a critical anticyclotomic character χ of K, we have

$$\left(L_p^{\mathrm{anti}}(f,\chi)\right)^2 = (*)\Lambda(\mathcal{F},\chi),$$

for a suitable explicit interpolation factor $(*)$. In the case where $h < \frac{k+1}{2}$, this interpolation property, together with admissibility, is enough to show that (after normalising the periods correctly) we have

$$L_p^{\mathrm{anti}}(f)^2 = L_p^{\mathrm{anti}}(\mathcal{F}), \tag{8.2}$$

where $L_p^{\mathrm{anti}}(\mathcal{F})$ is the restriction of $L_p(\mathcal{F})$ to the anticyclotomic line. (See, for example, [Geh17] for this result in the ordinary case). It is widely expected that anticyclotomic p-adic L-functions

[13] We thank Lennart Gehrmann for pointing this out to us.

can be varied in Coleman families; suppose there exists such a two-variable function $\mathcal{L}_p^{\mathrm{anti}}(V_{\mathbb{Q}})$, over a neighbourhood $V_{\mathbb{Q}}$ in the Coleman–Mazur eigencurve, interpolating the p-adic L-functions at classical weights. Then the methods of this section show that, under an analogous non-vanishing condition and up to multiplication by an element of $\mathcal{O}(V_{\mathbb{Q}})^\times$, we have

$$\mathcal{L}_p^{\mathrm{anti}}(V_{\mathbb{Q}})^2 = \mathcal{L}_p^{\mathrm{anti}}(V_K).$$

If $h \geq \frac{k+1}{2}$, we can then specialise this two-variable formula to obtain the identity (8.2) in this case too.

8.6. Secondary p-adic L-functions at critical base-change points

Recall that the p-adic L-function of a critical (Σ-smooth) base-change point vanishes at every special value. Previously, we noted that this could be viewed as an exceptional zero phenonenon in the weight. Indeed, let $\mathcal{F}$ be a critical (Σ-smooth) Bianchi modular form that is the base-change of a decent classical modular form f, and let V_K be some neighbourhood of $x_{\mathcal{F}}$ in $\mathcal{E}_{\mathrm{bc}}(L)$ such that a three-variable p-adic L-function $\mathcal{L}_p(V_K)$ exists over V_K. For $y \in V_K(L)$ and ϕ a locally analytic function on $\mathrm{Cl}_K(p^\infty)$, write

$$\mathcal{L}_p(y,\phi) := \mathrm{sp}_y'(\mathcal{L}_p(V_K))(\phi),$$

which is rigid analytic in y. The neighbourhood V_K is a rigid curve that is smooth at $x_{\mathcal{F}}$, so following Bellaïche, we can define the *ith secondary p-adic L-function* $L_p^{(i)}(\mathcal{F},*) \in \mathcal{D}(\mathrm{Cl}_K(p^\infty),L)$ by

$$L_p^{(i)}(\mathcal{F},\phi) := \frac{\partial^i}{\partial y^i}\mathcal{L}_p(y,\phi)\bigg|_{y=x_{\mathcal{F}}},$$

where $0 \leq i \leq e_{\mathcal{F}}-1$, for $e_{\mathcal{F}}$ the ramification degree of w at $x_{\mathcal{F}}$. An identical argument to that given in [Bel12, §4.4] shows that:

Proposition 8.13. *Let φ be a Hecke character of K of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $0 \leq (q,r) \leq (k,k)$, where (k,k) is the weight of $\mathcal{F}$.*

(i) If $0 \leq i \leq e_{\mathcal{F}}-2$, then

$$L_p^{(i)}(\mathcal{F},\varphi_{p-\mathrm{fin}}) = 0.$$

(ii) If $i = e_{\mathcal{F}}-1$, then (compare Theorem 2.14)

$$L_p^{(e_{\mathcal{F}}-1)}(\mathcal{F},\varphi_{p-\mathrm{fin}}) = (e_{\mathcal{F}}-1)!\left(\prod_{\mathfrak{p}|p} Z_{\mathfrak{p}}(\varphi)\right)\left[\frac{\varphi(x_{\mathfrak{f}})d\widetilde{\tau}(\varphi^{-1})\#\mathcal{O}_K^\times}{(-1)^{k+q+r}2\varphi_{\mathfrak{f}}(x_{\mathfrak{f}})\alpha_{\mathfrak{f}}\Omega_{\mathcal{F}}}\right]\Lambda(\mathcal{F},\varphi).$$

Remarks 8.14: (i) As $\mathcal{L}_p(y,\phi)$ is defined only up to multiplication by an element in $\mathcal{O}(V)^\times$, we get additional indeterminancy for the secondary p-adic L-functions. In particular, $L_p^{(i)}(\mathcal{F},y)$ is only well-defined up to scalar multiplication by $L^\times$ and additional of an element of the L-span of $\{L_p(\mathcal{F},\phi),...,L_p^{(i-1)}(f,\phi)\}$. As in [Bel12, §1.4], however, we see that this gives a well-defined flag $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_e$ in the space of L-valued locally analytic distributions on $\mathrm{Cl}_K(p^\infty)$, with $\dim_L \mathscr{F}_i = i+1$.

(ii) Since differentiating in y and restricting the distribution to the cyclotomic line are independent operations, taking the ith derivative of the identity in Proposition 8.8 (under the same non-vanishing condition) and evaluating at x_f shows that the ith secondary p-adic L-function factors as

$$L_p^{(i),\mathrm{cyc}}(\mathcal{F},*) = \sum_{j=0}^{i}\binom{i}{j}L_p^{(j)}(f,*)L_p^{(i-j),\chi_{K/\mathbb{Q}}}(f,*),$$

where $L_p^{(0)}$ is just L_p.

Appendix: A base-change deformation functor

by Carl Wang-Erickson[14]

The point of this appendix is to supply the proof of Proposition 5.9, regarding deformations of Galois representations. The main idea we will apply here applies under the following running assumptions:

(A) there is an index 2 subgroup $H \subset G$ and a chosen element $c \in G \smallsetminus H$ of order 2. Equivalently, G is expressed as a semi-direct product $H \rtimes \langle c \rangle$.

(B) The characteristic of the base coefficient field L of the deformed representation is not 2.

In the first section we set up the theory of the base change deformation functor. In the second section, we verify that this theory is compatible with arithmetic conditions imposed when G is a Galois group over $\mathbb{Q}$.

A.1. The base change deformation functor

We work under assumptions (A)-(B) above. Let $\rho : G \to \mathrm{GL}_d(L)$ be a representation that is absolutely irreducible after restriction to H. Let $\mathcal{A}_L$ be the category of Artinian local L-algebras $(A, \mathfrak{m}_A)$ with residue field L. We denote by X the deformation functor for $\rho|_H$. This is the functor from $\mathcal{A}_L$ to the category of sets given by

$$A \mapsto \{\tilde{\rho}_A : H \to \mathrm{GL}_d(A) \mid (\tilde{\rho}_A \mod \mathfrak{m}_A) = \rho|_H\}/\sim, \tag{A.1}$$

where $\sim$ is the equivalence relation of "strict equivalence," that is, conjugation by $1 + M_d(\mathfrak{m}_A) \subset \mathrm{GL}_d(A)$. We will let $\rho_A \in X(A)$ denote a *deformation* of $\rho|_H$ with coefficients in A. This is in contrast to the notation $\tilde{\rho}_A$, which we reserve for a *lift* of $\rho|_H$ to A, i.e. a homomorphism $\tilde{\rho}_A \in \rho_A$ as in (A.1).

Let X^{bc} denote the subfunctor of X cut out by the condition that some (equivalently, all) $\tilde{\rho}_A \in \rho_A$ admits an extension to a homomorphism $\tilde{\rho}^G_A : G \to \mathrm{GL}_d(A)$ such that $\tilde{\rho}^G_A|_H = \tilde{\rho}_A$. In this case, we say that ρ_A admits an extension to an A-valued deformation ρ^G_A of ρ.

For $h \in H$, we write $h^c := chc \in H$ for twisting by c. Likewise, for a group homomorphism η with domain H, let $\eta^c(h) := \eta(h^c)$.

Lemma A.1. *Let $A \in \mathcal{A}_L$ and $\rho_A \in X(L)$. Then ρ_A admits an extension to G deforming ρ if and only if there exists $\tilde{\rho}_A \in \rho_A$ such that*

$$\mathrm{ad}\,\rho(c) \cdot \tilde{\rho}^c_A = \tilde{\rho}_A. \tag{A.2}$$

Proof. Assume that there exists $\tilde{\rho}_A \in \rho_A$ and $\tilde{\rho}^G_A : G \to \mathrm{GL}_d(A)$ such that $\tilde{\rho}^G_A|_H = \tilde{\rho}_A$. Because the characteristic of L is not 2, the deformation functor for $\rho|_{\langle c\rangle}$ is trivial; compare the proof of [CWE18, Prop. 5.3.2]. Equivalently, there exists some $x \in 1 + M_d(\mathfrak{m}_A) \subset \mathrm{GL}_d(A)$ such that $\mathrm{ad}\, x \cdot \tilde{\rho}^G_A(c) = \rho(c)$. Then one readily observes that $\mathrm{ad}\, x \cdot \tilde{\rho}_A$ is a solution to (A.2).

Next we prove the converse. Assume that we have $\tilde{\rho}_A$ solving (A.2). Then we define $\tilde{\rho}^G_A : G \to \mathrm{GL}_d(A)$ by

$$\tilde{\rho}^G_A(g) := \begin{cases} \tilde{\rho}_A(g) & \text{for } g \in H, \\ \rho(c)\tilde{\rho}_A(h) & \text{for } g = ch, h \in H. \end{cases}$$

It is then straightforward to calculate that $\tilde{\rho}^G_A$ is a group homomorphism such that $\tilde{\rho}^G_A|_H = \tilde{\rho}_A$. $\square$

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Notice that the map of lifts $\tilde\rho_A$ of $\rho|_H$ to A sending

$$\tilde\iota : \tilde\rho_A \mapsto \operatorname{ad}\rho(c) \cdot \tilde\rho_A^c$$

is an involution on lifts of $\rho|_H$. Its fixed points are exactly those lifts satisfying (A.2). This involution descends to an functorial involution of deformations

$$\iota : X(A) \to X(A).$$

To justify this claim, we calculate that for any $x \in \mathrm{GL}_d(A)$,

$$\tilde\iota(\operatorname{ad} x \cdot \tilde\rho_A) = \operatorname{ad}\rho(c) \cdot \operatorname{ad} x \cdot \tilde\rho_A^c = \operatorname{ad} y \cdot (\tilde\iota(\tilde\rho_A)),$$

where $y = \operatorname{ad}\rho(c) \cdot x$.

Let X^ι denote the ι-fixed subfunctor of X, and let $\mathfrak{t}$ (resp. $\mathfrak{t}^{\mathrm{bc}}$) denote the tangent space $X(L[\varepsilon]/(\varepsilon^2))$ (resp. $X^{\mathrm{bc}}(L[\varepsilon]/(\varepsilon^2))$).

Proposition A.2. *(i) There is a canonical isomorphism $X^\iota \cong X^{\mathrm{bc}}$.*

(ii) The deformation problems X^{bc}, X on $\mathcal{A}_L$ are pro-represented by pro-objects $R^{\mathrm{bc}}, R \in \hat{\mathcal{A}}_L$. The involution ι induces an automorphism $\iota^* : R \to R$, and there is a natural surjection

$$R \twoheadrightarrow R^{\mathrm{bc}} := \frac{R}{((1-\iota^*)(R))}.$$

(iii) There is a canonical injection $\mathfrak{t}^{\mathrm{bc}} \hookrightarrow \mathfrak{t}$ of tangent spaces. The image of this injection is the subspace $\mathfrak{t}^\iota \subset \mathfrak{t}$ fixed by the involution $\iota_* : \mathfrak{t} \to \mathfrak{t}$ induced by ι.

Proof. Part (i) follows directly from Lemma A.1.

For Part (ii), it is well-known that X is pro-representable; see e.g. [Maz89]. It is a brief exercise that a homomorphism $R \to A$ kills $(1-\iota^*)(R)$ if and only if the corresponding deformation of $\rho|_H$ is ι-fixed. Then the pro-representability of X^{bc} by R^{bc} follows from (i).

Part (iii) follows from Part (ii) and the perfect L-linear duality of $\mathfrak{m}_R/\mathfrak{m}_R^2$ and $X(L[\varepsilon]/(\varepsilon^2))$. □

A.2. Galois-theoretic conditions

Using the notation of Definition 5.7, we let $G = G_{\mathbb{Q},S}$ and $H = G_{K,S_K}$. We also use the decomposition groups and complex conjugation $c \in G$ given in (5.1). The data (G, H, c) satisfy assumption (A), as $K/\mathbb{Q}$ is imaginary quadratic.

Because the level of the modular form f of Proposition 5.6 is supported by S, and because $p, \infty \in S$, the representation ρ_f of the absolute Galois group of $\mathbb{Q}$ factors through $G_{\mathbb{Q},S}$. We let $\rho := \rho_f : G \to \mathrm{GL}_2(L)$, as in Definition 5.4, with its critical refinement with eigenvalue α_p. It is an L-linear representation, where L is a p-adic field; thus we have satisfied assumption (B).

Deformation theory as in §A.1 can be carried out for *continuous* representations of G and H, using the p-adic topology of L, and the arguments therein make good sense in this setting. This is standard; see e.g. [Kis03, §9]. From now on, we impose continuity without further comment.

Because G and H satisfy the finiteness condition Φ_p of [Maz89, §1.1], it follows that the deformation rings R, R^{bc} of Proposition A.2 representing X, X^{bc} are Noetherian and (equivalently) $\mathfrak{t}, \mathfrak{t}^{\mathrm{bc}}$ have finite L-dimension.

Lemma A.3. *Conditions (i) and (ii) of Definition 5.8 determine a subfunctor $X^{\mathrm{ref}} \subset X$ that is Zariski-closed, hence representable by a quotient ring $R \twoheadrightarrow R^{\mathrm{ref}}$.*

Proof. This is standard – see e.g. [Ber17, p. 26] and [Kis03, Prop. 8.13]. In particular, the important assumption [Kis03, (8.8.1)] is satisfied because f has been critically refined. □

We now prove Proposition 5.9.

Proof of Proposition 5.9. Because both the "ref" and "bc" conditions have been shown to be Zariski-closed conditions on X, their intersection functor $X^{\mathrm{ref,bc}}$ is representable by a quotient $R^{\mathrm{ref}} \twoheadrightarrow R^{\mathrm{ref,bc}}$. Then apply Proposition A.2 and its proof. □

To make Proposition 5.9 useful, we check that the properties of a G-deformation ρ_A^G of ρ_f guaranteeing that $\rho_A^G|_H$ determines a point of X^{ref} (and, consequently, a point of $X^{\mathrm{ref,bc}}$) are what we would naturally expect them to be.

Lemma A.4. *Let ρ_A^G be a deformation of $\rho_f : G \to \mathrm{GL}_2(L)$ to $A \in \mathcal{A}_L$. Then $\rho_A^G|_H \in X^{\mathrm{ref}}(A)$ if and only if ρ_A^G satisfies*

(i) For primes $q \mid N$ such that $q \neq p$, $\rho_A^G|_{I_q} \simeq \rho|_{I_q} \otimes_L A$.

(ii) The restriction $\rho_A^G|_{G_p}$ has

(1) one Hodge–Sen–Tate weight is constant and equal to 0, and

(2) there exists $\widetilde{\alpha}_p \in A$ such that the A-module $D_{\mathrm{crys}}(\rho_A^G|_{G_p})^{\varphi=\widetilde{\alpha}_p}$ is free of rank 1 and $(\widetilde{\alpha}_p \mod \mathfrak{m}_A) = \alpha_p$.

Proof. It is a straightforward exercise about representations and the corresponding Frobenius isocrystals to verify that the statements of (i)-(ii) of Lemma A.4 are equivalent to (i)-(ii) of Definition 5.8 under both extension and restriction. □

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Daniel Barrera Salazar; Universitat Politècnica de Catalunya
Campus Nord, Calle Jordi Girona, 1-3, 08034 Barcelona, Spain
`daniel.barrera.salazar@upc.edu`

Carl Wang-Erickson; Imperial College London
South Kensington Campus, London SW7 2AZ, United Kingdom
`c.wang-erickson@imperial.ac.uk`

Chris Williams; Imperial College London
South Kensington Campus, London SW7 2AZ, United Kingdom
`christopher.williams@imperial.ac.uk`