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by

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A METHOD OF FINDING ALL EQUILIBRIUM SOLUTIONS OF A 2-PERSON MATRIX GAME

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Abstract

It is shown how all the equilibrium solutions of a 2 person non-cooperative game can be derived from the vertices of two polytopes. Such vertices must be orthogonal in a manner described. A numerical example is used to illustrate the method. Two types of games, zero-sum and evolutionary games are shown to be special cases with special properties. Finally some further areas for investigation are considered.

Key words: Games Theory, Linear Programming, Convex Polytopes, Vertices of Polytopes, Equilibria.

1. INTRODUCTION

We consider a 2-person game in which person A has payoff

$$z_A = \sum_{\substack{i \in I \\ j \in J}} a_{ij} \, x_i \, y_j \tag{1}$$

and person B has payoff

$$z_B = \sum_{\substack{i \in I \\ j \in J}} b_{ij} \, x_i \, y_i \tag{2}$$

if A and B play mixed strategies $\underline{x}$ and $\underline{y}$ respectively. The vectors $\underline{x}$ and $\underline{y}$ are probability vectors where

$$\sum_{i \in I} x_i = 1 \; , \quad \sum_{j \in J} y_j = 1 \tag{3}$$

$$x_i \geq 0, \; y_j \geq 0 \qquad \text{for } i \in I, \; j \in J \tag{4}$$

I and J are finite index sets

$$I = \{1,2,\ldots,m\} \; , \quad J = \{1,2,\ldots,n\} \; .$$

Such a game can be defined by presenting 2 m×n payoff matrices. A specific numerical example is given, in section 3, of a game which one player has 3 pure strategies and the other 4 pure strategies. Both players can mix their strategies.

A Nash Equilibrium is defined as a solution

$$S = \left[(\underline{x}^*, \; z_A^*), \; (\underline{y}^*, \; z_B^*) \right] \tag{5}$$

such that

$$z_A^* = \sum_{\substack{i \in I \\ j \in J}} a_{ij} \, x_i^* \, j_j^* \geq \sum_{\substack{i \in I \\ j \in J}} a_{ij} \, x_i \, y_i^* \tag{6}$$

for all $x_i \geq 0$, $i \in I$ such that $\sum_{i \in I} x_i = 1$

and

$$z_B^* = \sum_{\substack{i \in I \\ i \in J}} b_{ij} x_i^* y_j^* \geq \sum_{\substack{i \in I \\ j \in J}} b_{ij} x_i^* y_i \qquad (7)$$

for all $y_j \geq 0$, $j \in J$ such that $\sum_{j \in J} y_j = 1$.

The interpretation of (6) and (7) is that if A and B play the mixture of strategies in S then A cannot do better by deviating to another strategy (so long as B maintains his strategy) nor can B do better by deviating to another strategy (so long as A maintains his strategy).

The concept of a Nash equilibrium was defined by Nash [10] who proved that, allowing mixed strategies, at least one such equilibrium always exists. A discussion can be found in Luce and Raiffa [6] or the more up-to-day references of, for example, Friedman [3] or Thomas [11].

In section 2 we define two polytopes, one corresponding to each player. Then it is shown that if a vertex of one bears an "orthonality" relationship to a vertex of the other these vertices correspond to an equilibrium solution. We reference a number of methods of generating all the vertices of a polytope which can be used. The recognition that the equilibrium solutions of such games arise from the extreme solutions of these polytopes is due to Kuhn [4]. He uses a different method of devising them. A numerical example is given in section 3, together with all the vertices of each of the associated polytopes. The "orthogonal" pairs are then picked out and the corresponding equilibrium solutions presented.

Two particular instances of 2-person games are taken in section 4, as special cases which prove particularly interesting. Firstly we consider the familiar zero-sum game. Secondly we consider evolutionary games. In each case the associated polyhedra bear a special relationship to each other leading to special properties of the solutions.

Finally, in section 5, we give some discussion of other areas of investigation and further references.

2. POLYTOPES ASSOCIATED WITH EACH PLAYER

We define a polytope P_A associated with player A to be the set of feasible solutions $(\underline{y}, z_A)$ to the inequalities and equation

$$P_A: \quad \begin{aligned} &\sum_{j \in J} a_{ij} y_j - z_A \le 0 , && \text{all } i \in I && (8) \\ &\sum_{j \in J} y_j = 1 && && (9) \\ &y_j \ge 0 && \text{all } j \in J && (10) \end{aligned}$$

Similarly we define the polytope P_B associated with player B to be the set of feasible solutions $(\underline{x}, z_A)$ to the inequalities and equation

$$P_B: \quad \begin{aligned} &\sum_{i \in I} b_{ij} x_i - z_B \le 0 && \text{all } j \in J && (11) \\ &\sum_{i \in I} x_i = 1 && && (12) \\ &x_i \ge 0 && \text{all } i \in I && (13) \end{aligned}$$

Theorem 1. An equilibrium solution S is either a pair of vertex solutions of P_A and P_B or a pair of solutions, either or both of, which is a convex linear combination of other equilibrium solutions which are vertex solutions and have the same value of z.

Proof. Suppose S given in (5) is an equilibrium solution then (6) holds. Taking $\underline{x} = (0,0,\ldots,0,1,0,\ldots,0)$ with the 1 in column i we have

$$z_A^* \ge \sum_{j \in J} a_{ij} y_j^* \qquad \text{all } i \in I \qquad (14)$$

Hence $(\underline{y}^*, \underline{z}_A)$ is a feasible solution to P_B .

Also

$$z^*_A \;=\; \sum_{i\in I} x^*_i \, z^*_A \;\geq\; \sum_{i\in I} x^*_i \sum_{j\in J} a_{ij} \, y^*_j \;=\; z_A \tag{15}$$

Therefore

$$x^*_i > 0 \longrightarrow \sum_{j\in J} a_{ij} \, y^*_j \;=\; z^*_A \qquad \text{for all} \quad i \in I \tag{16}$$

We now construct an objective function for P_A in order to show that $(\underline{y}^*, z_A)$ is an optimal solution to P_A with respect to this objective function.

Define

$$c_j \;=\; \sum_{i\in I} a_{ij} \, x^*_i + z_B \qquad \text{if} \quad y^*_j > 0 \tag{17}$$

$$c_j \;=\; \sum_{i\in I} a_{ij} \, x^*_i + z_B - 1 \qquad \text{if} \quad y^*_j = 0 \tag{18}$$

Consider the Linear Programme

LPA:

$$\text{Maximise} \qquad \sum_{j\in J} c_j \, y_j - z_A \tag{19}$$

$$\text{subject to} \qquad (\underline{y}, z_A) \in P_A \tag{20}$$

for which we have shown $(\underline{y}^*, z^*_A)$ is a feasible solution. The dual LP is

LPA':

$$\text{Minimise} \qquad Z_B \tag{21}$$

$$\text{subject to} \qquad \sum_{i\in I} a_{ij} \, x_i + z_B \geq c_j \qquad \text{all } j \in J \tag{22}$$

$$\sum_{i\in I} x_i \;=\; 1 \tag{23}$$

$$x_i \geq 0 \qquad \text{all} \quad i \in I \tag{24}$$

$(\underline{x}^*, z^A_B)$ is a feasible solution to this dual model and also

$$y^*_j > 0 \longrightarrow \sum_{i\in I} a_{ij} \, x^*_i + z_B \;=\; c_j \tag{25}$$

by virtue of the definition of c_j in (19) and (20).

Therefore we have feasible solutions $(\underline{y}^*, z_A^*)$ and $(\underline{x}^*, z_B^*)$ to LPA and LPA′ respectively which satisfy the orthogonality (complementarily) conditions (16) and (25). By the duality theorem of Linear Programming (see e.g. Dantzig [1]) this is sufficient to prove the optimality of $(\underline{y}^*, z_A^*)$ for LP.

If there are no alternate optimal solutions then this solution is a <u>vertex</u> solution of P_A . Again this is a standard result which applies to linear programmes where polytopes are pointed (as is P_A). Should there be alternate optimal solutions then there are alternate optimal <u>vertex</u> solutions which are equilibrium solutions with the same value of Z_A^* . The non-vertex optimal solutions can each be expressed as convex linear combinations of these vertex solutions.

Similarly we define

$$d_i = \sum_{j\in J} b_{ij}\, y_i^* + z_A \qquad \text{if} \quad x_i^* > 0 \tag{26}$$

$$d_i = \sum_{j\in J} b_{ij}\, y_i^* + z_A - 1 \qquad \text{if} \quad x_i^* = 0 \tag{27}$$

and the linear programme associated with P_B

LPB:

$$\text{Maximise} \quad \sum_{i\in I} d_i\, x_i - z_B \tag{28}$$

$$\text{subject to:} \quad (\underline{x}, z_B) \in P_B \tag{29}$$

and its dual

LPB′:

$$\text{Minimize} \quad z_A$$

$$\text{subject to} \quad \sum_{j\in J} b_{ij}\, y_i + z_A \geq d_i \qquad \text{all } i \in I \tag{30}$$

$$\sum_{j\in J} y_i = 1 \tag{31}$$

$$y_i \geq 0 \qquad \text{all } j \in J$$

It can be shown, in the same manner as that above, that $(\underline{x}^*, z_B)$ and $(\underline{y}^*, z_A)$ are feasible solutions to LPB and LPB′ respectively and that they satisfy the orthogonality relations

$$y_j^* > 0 \longrightarrow \sum_{i \in I} b_{ij} x_i^* = z_B^* \qquad \text{all } j \in J \tag{32}$$

$$x_i^* > 0 \longrightarrow \sum_{j \in J} b_{ij} y_i^* + z_A^* = d_i \qquad \text{all } i \in I \tag{33}$$

Therefore $(\underline{x}^*, z_B)$ is either a vertex of P_B or a convex linear combination of vertex solutions which are also equilibrium solutions. □

Theorem 2. If $(\underline{y}^*, z_A)$ and $(\underline{x}^*, z_B)$ are feasible solutions to P_A and P_B respectively and the following orthogonality conditions hold

$$x_i^* > 0 \longrightarrow \sum_{j \in J} a_{ij} y_i^* = z_A^* \qquad \text{all } i \in I \tag{34}$$

and

$$y_i^* > 0 \longrightarrow \sum_{i \in I} b_{ij} x_i^* = z_B^* \qquad \text{all } j \in J \tag{35}$$

then $S = [(\underline{x}^*, z_A^*), (\underline{y}^*, z_B^*)]$ is an equilibrium solution.

Proof.

$$\sum_{i \in I} x_i^* \sum_{j \in J} a_{ij} y_j^* = z_A^* \quad \text{since} \quad x_{ij}^* > 0 \longrightarrow \sum_{i \in I} a_{ij} x_i^* = z_A \tag{36}$$

$$\sum_{j \in J} a_{ij} y_j^* \le z_A^* \quad \text{for all } i \in I \quad \text{since} \quad (\underline{y}^*, z_A) \in P_A \tag{37}$$

Therefore taking any x such that $x \ge 0$ for all $i \in I$ and $\sum_{i \in I} x_i = 1$

$$\sum_{i \in I} x_i \sum_{j \in J} a_{ij} y_j^* \le z_A^* \sum_{i \in I} x_i = z_A^*$$

Hence condition (6) for an equilibrium solution is satisfied.

Similarly we prove that condition (7) is satisfied, and that therefore S is an equilibrium solution. □

We can therefore search for vertex solutions $(\underline{y}^*, z_A^*)$ of P_A and $(\underline{x}^*, z_B^*)$ of P_B which satisfy the orthogonality conditions (16) and (32). Such solutions will therefore be equilibrium solutions. If there are any other equilibrium solutions they will be convex combinations of vertex equilibrium solutions which have been generated having equal values of z.

Kuhn [4] evaluates the extreme solutions of the polytopes by a method of submatrices. We prefer to view the result in a more general context. There are a number of methods of generating all vertices of a polytope defined by inequalities and equations. For the example in section 3 we used the method described by Williams [12] which has been programmed. Other methods are described by Dyer and Proll [2] and Mattheiss and Rubin [7]. It should be pointed out that the number of vertices of a polytope can become very large, even for a modest number of variables and inequalities. McMullen [9] shows that a strict upper bound on the number of vertices of a polytope defined by m inequalities (including non-negativities) and n variables is

$$\binom{m - \left\lfloor \frac{n+1}{2} \right\rfloor}{m - n} + \binom{m - \left\lfloor \frac{n+2}{2} \right\rfloor}{m - n} \qquad (38)$$

where $\binom{p}{q} = \frac{p(p-1)\ldots(p-q+1)}{q(q-1)\ldots 3.2.1}$.

3. A NUMERICAL EXAMPLE

We consider a game in which A has 3 pure strategies and B has 4 pure strategies. Mixtures of these strategies are allowed.

$$[a_{ij}] = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -1 & 2 & 1 \\ -2 & 3 & 1 & 0 \end{bmatrix}, \quad [b_{ij}] = \begin{bmatrix} 4 & 0 & -1 & 3 \\ -1 & 2 & 0 & 1 \\ 2 & 1 & 4 & -2 \end{bmatrix}$$

The associated polytopes are

$$P_A \qquad \begin{aligned} y_1 + 2y_2 - y_3 - 3y_4 - z_A &\le 0 \\ - y_2 + 2y_3 + y_4 - z_A &\le 0 \\ -2y_1 + 3y_2 + y_3 - z_A &\le 0 \\ y_1 + y_2 + y_3 + y_4 &= 1 \end{aligned}$$

$$y_1, y_2, y_3, y_4 \ge 0$$

and

$$P_B \qquad \begin{aligned} 4x_1 - x_2 + 2x_3 - z_B &\le 0 \\ 2x_2 + x_3 - z_B &\le 0 \\ - x_1 + 4x_3 - z_B &\le 0 \\ 3x_1 + x_2 - 2x_3 - z_B &\le 0 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

$$x_1, x_2, x_3 \ge 0$$

It is convenient to express the vertices as 7-tuples. For P_A the first 4 entries will give values for $y_1,\dots,y_4$, and the latter 3 entries values for the slacks in the first 3 constraints. For P_B the first 3 entries will give values for $x_1,\dots,x_3$, and the latter 4 entries values for the slacks in the first 5 constraints. It will then be straightforward to check for orthogonality by comparing the first 4 entries in a vertex for P_A with the latter 4 entries in the vertex for P_B and the latter 3 entries in the vertex for P_A with the first 3 entries in that for P_B. In each case the pairs of vectors must be orthogonal for the condition to apply.

P_A has the following 11 vertices.

$$\left(\tfrac{13}{32},\ \tfrac{9}{32},\ 0,\ \tfrac{5}{16};\ 0,\ 0\ \ 0\right) z_A = \tfrac{1}{32} \qquad (39)$$

$$\left(\tfrac{4}{5},\ 0,\ 0,\ \tfrac{1}{5};\ 0,\ 0,\ \tfrac{9}{5}\right) z_A = \tfrac{1}{5} \qquad (40)$$

$$\left(\tfrac{3}{4},\ 0,\ \tfrac{1}{4},\ 0;\ 0,\ 0,\ \tfrac{7}{4}\right) z_A = \tfrac{1}{2} \qquad (41)$$

$$\left(\tfrac{9}{26},\ \tfrac{7}{26},\ \tfrac{10}{26},\ 0;\ 0,\ 0,\ 0\right) z_A = \tfrac{1}{2} \qquad (42)$$

$$\left(0,\ \tfrac{1}{5},\ 0,\ \tfrac{4}{5};\ \tfrac{13}{5},\ 0,\ 0\right) z_A = \tfrac{3}{5} \qquad (43)$$

$$\left(1,\ 0,\ 0,\ 0;\ 0,\ 1,\ 3\right) z_A = 1 \qquad (44)$$

$$\left(0,\ 0,\ 0,\ 1;\ 4,\ 0,\ 1\right) z_A = 1 \qquad (45)$$

$$\left(0,\ \tfrac{1}{5}\ \ \tfrac{4}{5},\ 0;\ \tfrac{9}{5},\ 0,\ 0\right) z_A = \tfrac{7}{5} \qquad (46)$$

$$\left(\tfrac{1}{4},\ \tfrac{3}{4},\ 0,\ 0;\ 0,\ \tfrac{5}{2},\ 0\right) a_A = \tfrac{7}{4} \qquad (47)$$

$$\left(0,\ 0,\ 1,\ 0;\ 3,\ 0,\ 1\right) z_A = 2 \qquad (48)$$

$$\left(0,\ 1,\ 0,\ 0;\ 1,\ 4,\ 0\right) z_A = 3 \qquad (49)$$

P_B has the following 9 vertices

$$\left(\tfrac{1}{3},\ \tfrac{1}{2},\ \tfrac{1}{6};\ 0;\ 0,\ \tfrac{5}{6},\ 0\right) z_B = \tfrac{7}{6} \qquad (50)$$

$$\left(\tfrac{7}{31},\ \tfrac{13}{31},\ \tfrac{11}{31};\ 0;\ 0,\ 0,\ \tfrac{25}{31}\right) z_B = \tfrac{37}{31} \qquad (51)$$

$$\left(\tfrac{1}{4},\ \tfrac{3}{4},\ 0;\ \tfrac{5}{4},\ 0,\ \tfrac{7}{4},\ 0\right) z_B = \tfrac{3}{2} \qquad (52)$$

$$\left(0,\ \tfrac{3}{5},\ \tfrac{2}{5};\ \tfrac{7}{5},\ 0,\ 0,\ \tfrac{9}{5}\right) z_B = \tfrac{8}{5} \qquad (53)$$

$$\left(0,\ 1,\ 0;\ 3,\ 0,\ 2,\ 1\right) z_B = 2 \qquad (54)$$

$$\left(\tfrac{2}{3}\ \ \tfrac{1}{3},\ 0;\ 0,\ \tfrac{5}{3},\ 3,\ 0\right) z_B = \tfrac{7}{3} \qquad (55)$$

$$\left(\tfrac{2}{7},\ 0,\ \tfrac{5}{7};\ 0,\ \tfrac{13}{7},\ 0,\ \tfrac{23}{7}\right) z_B = \tfrac{18}{7} \qquad (56)$$

$$\left(1,\ 0,\ 0;\ 0,\ 4,\ 5,\ 1\right) z_B = 4 \qquad (57)$$

$$\left(0,\ 0,\ 1;\ 2,\ 3,\ 0,\ 6\right) z_B = 4 \qquad (58)$$

It can be seen that the following are orthogonal pairs

(39) and (50)

(40) and (55)

(44) and (57)

(46) and (53)

Notice that (44) and (57) corresponds to A and B both playing their 1st pure strategies. The other three equilibria arise from mixed strategies.

With hindsight it is possible to construct objective functions for P_A and P_B.

Equilibrium solution (39), (50) arises from the following objectives for P_A and P_B respectively

$$\text{Maximise} \quad \frac{7}{6} y_1 + \frac{11}{6} y_2 + y_3 + \frac{2}{3} y_4 - z_A \tag{59}$$

$$\text{Maximise} \quad \frac{83}{32} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 - z_b \tag{60}$$

Equilibrium solution (40), (55) arises from the following objectives for P_A and P_B respectively

$$\text{Maximise} \quad 3y_1 + \frac{7}{3} y_2 + \frac{4}{3} y_3 + \frac{2}{3} y_4 - z_A \tag{61}$$

$$\text{Maximise} \quad 4x_1 + \frac{2}{5} x_2 + \frac{1}{5} x_3 - z_B \tag{62}$$

Equilibrium solution (44), (57) arises from the following objectives for P_A and P_B respectively

$$\text{Maximise} \quad 5y_1 + 5y_2 + 2y_3 - z_A \tag{63}$$

$$\text{Maximise} \quad 5x_1 - 5x_1 + 2x_3 - z_B \tag{64}$$

Equilibrium solution (46), (53) arises from the following objectives for P_A and P_B respectively

$$\text{Maximise} \quad -\frac{1}{5}y_1 + \frac{11}{5}y_2 + \frac{16}{5}y_3 + \frac{11}{5}y_4 - z_A \tag{65}$$

$$\text{Maximise} \quad -\frac{2}{5}x_1 + \frac{9}{5}x_2 + \frac{24}{5}x_3 - z_B \tag{66}$$

4. SPECIAL CASES

Zero-Sum Games

In this case

$$b_{ij} = -a_{ij} \tag{67}$$

Constraints (11) of P_B can then be written as

$$\sum_{i \in I} a_{ij} x_i + z_B \geq 0 \qquad \text{all} \quad j \in J \tag{68}$$

If c_j is taken as 0 , P_B then gives rise to the constraints (22), (23) and (24) of LPA'. Hence equilibrium solutions will be obtained from solutions to <u>dual</u> linear programming models. An objective function for P_A which generates these equilibrium solutions is (19), which, with $c_j = 0$, is equivalent to

$$\text{Minimise} \quad z_A \tag{69}$$

and the objective function for P_B is (21) i.e. the objective which produces equilibrium solutions is to <u>minimise</u> <u>ones</u> <u>maximum</u>, (minimax) payoff. This is of course a well known result discussed in the references [3], [6] and [11] already given. Orthogonality of A and B's solutions will be automatic as a result of the duality theorem.

Evolutionary Games

These originate with Maynard Smith [8] and the corresponding polytopes derived by Williams [13]. From the discussion in these references it can be seen that

$$b_{ij} = a_{ji} \tag{70}$$

Such games can therefore only be defined for square payoff matrices where $m = n$.

Constraints (11) of P_B become

$$\sum_{i \in I} a_{ji} x_i - z_B \leq 0 \qquad \text{all} \quad j \in J \tag{71}$$

If indices and variables are appropriately renamed these constraints become identical with (8). Therefore P_A and P_B become the <u>same</u> polytopes. We need only examine vertices of P_A and consider those which are "self orthogonal".

A subset of these resultant equilibrium solutions are known as evolutionary stable states and have a significance discussed in [7]. Their derivation is discussed in [13].

5. FURTHER CONSIDERATIONS

It was pointed out in section 2 that the number of vertices of a polytope can grow rapidly with the number of variables and constraints. Therefore the method of finding <u>all</u> equilibrium solutions presented here may become expensive in time and space. It should be observed, however, that the number of equilibrium solutions may itself be very large making the generating of all of them intrinsically expensive.

An alternative approach is to generate only some. This can be done using the Linear Complementarity algorithm of Lemke [5]. The disadvantage of this approach is that those equilibrium solutions which are generated is arbitrary depending on ones starting points. Also there is no way of knowing how many equilibrium solutions should be sought.

It would be interesting to seek an interpretation of the objective function (19) and the coefficients c_j in general. For the zero-sum case there is a clear interpretation.

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